

*Pacific
Journal of
Mathematics*

**NONSMOOTH CONVEX CAUSTICS
FOR BIRKHOFF BILLIARDS**

MAXIM ARNOLD AND MISHA BIALY

NONSMOOTH CONVEX CAUSTICS FOR BIRKHOFF BILLIARDS

MAXIM ARNOLD AND MISHA BIALY

This paper is devoted to the examination of the properties of the string construction for the Birkhoff billiard. Based on purely geometric considerations, string construction is suited to providing a table for the Birkhoff billiard, having the prescribed caustic. Exploiting this framework together with the properties of convex caustics, we give a geometric proof of a result by Innami first proved in 2002 by means of Aubry–Mather theory. In the second part of the paper we show that applying the string construction one can find a new collection of examples of C^2 -smooth convex billiard tables with a nonsmooth convex caustic.

1. Introduction

Let Γ be a simple closed C^1 -smooth convex curve in the Euclidean plane. We consider a Birkhoff billiard inside Γ . This simple dynamical system creates many geometric and dynamical questions and reflects many difficulties appearing in general Hamiltonian systems. Readers may refer to any textbook among the wide variety written on the subject (e.g., [Katok et al. 1986; Kozlov and Treshchëv 1991; Mather and Forni 1994; Tabachnikov 2005]).

We will use the following nonstandard notations: the interior of the set bounded by the simple closed curve γ will be denoted by γ° , while $\bar{\gamma}$ denotes the compact $\gamma^\circ \cup \gamma$. The length of the curve is denoted by $\text{Length}(\gamma)$. The convex hull of γ is denoted by $\text{Conv}(\gamma)$.

Definition 1. A simple closed curve $\gamma \subset \Gamma^\circ$ is called a *convex caustic* for Γ if $\bar{\gamma}$ is a convex set and any supporting line for $\bar{\gamma}$ remains a supporting line for $\bar{\gamma}$ after billiard reflection in Γ .

Every convex caustic γ corresponds to an invariant curve r_γ of the billiard ball map. The curve $r_\gamma \subset \mathbb{R}_+ \times \mathbb{S}^1$ consists of all supporting lines to γ . This curve winds once around the phase cylinder and therefore is called rotational. We shall denote its rotation number by ρ_γ .

MSC2010: primary 37E30, 37E40; secondary 78A05.

Keywords: string construction, convex caustics, Birkhoff billiard.

In the original Birkhoff paper [1917] there was posed a conjecture that the existence of a continuous set of caustics, being a very restrictive property, actually provides an extreme rigidity on the shape of the curve Γ . The first result in this direction was achieved in [Bialy 1993]. Our paper is motivated by recent progress in the Birkhoff conjecture solution achieved in [Avila et al. 2016; Kaloshin and Sorrentino 2016]. The crucial assumption in these papers consists in the existence of convex caustics such that the rotation numbers of the corresponding invariant curves form a rational sequence in the interval $(0; \frac{1}{3}]$, converging to 0. It seems natural to compare such a result with one proved by N. Innami [2002].

Theorem 2 [Innami 2002]. *Assume that there exists a sequence of convex caustics γ_n inside Γ such that the rotation numbers ρ_n of the corresponding invariant curves tend to $\frac{1}{2}$. Then Γ is an ellipse.*

Originally, Innami's arguments were based on the Aubry–Mather variational theory. In the next section we present a simple geometric proof using string construction. Yet, it remains a challenging question whether one can prove a more general statement relaxing the requirement of convexity of the caustics.

Let us recall the string construction framework. Given a convex compact set $\bar{\gamma}$ bounded by γ , and a number $S > \text{Length}(\gamma)$, define the curve Γ as a union of those points P such that the *cap-body* $\text{Conv}(P \cup \bar{\gamma})$ has boundary of length S . Geometrically such a construction gives the set of all points traversed by the tip of a nonelastic string of length $S > \text{Length}(\gamma)$ wrapped around γ and stretched to its full extent. The curve Γ provided by such construction has γ as its billiard caustic. We shall refer to S as a string parameter of the caustic. A closely related so-called Lazutkin parameter is defined as $L = S - \text{Length}(\gamma)$.

The string construction is widely known and can be easily proved to provide Γ for smooth enough γ . In fact it remains valid also in the more general case as it is stated in the following theorem.

Theorem 3 [Stoll 1930; Turner 1982].

- (1) *For a given compact convex set $\bar{\gamma}$ and for every $S > \text{Length}(\gamma)$ the string construction determines a C^1 -smooth convex closed curve Γ such that γ is a billiard caustic for Γ .*
- (2) *If γ is a convex billiard caustic for a C^1 curve Γ then Γ can be obtained from γ by the string construction for some S .*

Let us emphasize that the string construction is highly nonexplicit and makes calculations difficult. A very important consequence of KAM theory, proved by Lazutkin [1973; 1981] and Douady [1982], states the existence of convex caustics near the boundary of a sufficiently smooth (at least C^6) billiard table. On the other hand, applying string construction to the triangle, one gets a billiard table which is

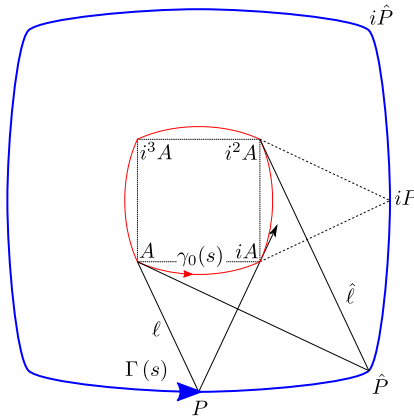


Figure 1. A switched caustic string construction.

piecewise C^2 with jumps of the curvature and hence by [Hubacher 1987] cannot have caustics near the boundary.

The scenario of destruction of caustics when one moves away from the boundary towards the interior could be understood in principle by the analogy with wave front propagation inside a convex curve [Mather and Forni 1994]. For example, take the ellipse and consider the wave fronts as in the famous picture [Arnold 1990, Figure 36]. For small distances the fronts remain smooth, but starting from some critical value they start to develop singularities. However, nobody has observed such a bifurcation in practice for caustics of convex billiards due to the lack of integrable examples. On the other hand, nonconvex caustics exist, for instance, for convex bodies of constant width, and were studied in [Knill 1998].

Motivated by the above discussion, the natural question about the existence of nonsmooth convex caustics arises. More generally, it is natural to study how irregular the convex caustic can be. In [Fetter 2012] a billiard table of class C^2 was constructed which has a caustic of a regular hexagon. In this paper we were able to construct the whole functional family of the examples of C^2 billiard tables having nonsmooth convex caustics.

Theorem 4. *There exist a one-parametric family of strictly convex nonsmooth compact sets $\bar{\gamma}$ and values of the string parameter S such that the curves Γ obtained by the string construction are C^2 -smooth.*

We will use the following geometric idea (we use the complex notation $x + iy$ for points (x, y) in the plane). Start with a curve $\gamma_0(t) : [-1, 1] \rightarrow \mathbb{C}$ such that $\gamma_0(-1) = A = -1 - i$, $\gamma_0(1) = iA = 1 - i$ and $\gamma_0(t)$ is symmetric with respect to the vertical axis (i.e., $i\gamma_0(-t) = \overline{i\gamma_0(t)}$) (see Figure 1). Construct γ as a concatenation of $\{i^k \gamma_0\}_{k=0}^3$. Parametrize γ by the arc-length parameter s and choose the initial

point in such a way that $\gamma(0) = A$. We will denote the total length of γ by $4S$. Then $\gamma(S) = iA$.

The main idea is to choose the curve γ and string parameter S in such a way that the string construction will have the following properties:

- At the beginning (point P in [Figure 1](#)), the left part AP of the string remains fixed at point A while the right part of the string unwinds from the arc (iA, i^2A) .
- At the moment when the left part of the string becomes tangent to γ at the point A (this corresponds to the point \hat{P} on Γ) the right part reaches the point i^2A and remains fixed after that. We will call this moment the *switching of the first kind*.
- While the left part of the string winds around the arc $(\widehat{A, iA})$ the right part remains fixed at i^2A (see [Figure 1](#)) until the moment when the vertex of the string reaches the point iP . We will call this the *switching of the second kind*.
- D_4 symmetry provides the whole picture.

Let us reemphasize, that the string construction, being a nonexplicit procedure, typically does not provide any analytic expression for the table Γ from a given γ . In the example [[Fetter 2012](#)], the construction is made explicit by fixing two end-points on the string. The disadvantage of such a situation is the complete loss of any flexibility, since the corresponding table may consist only of the elliptic arcs. We propose another, more flexible yet explicit construction, fixing only one end-point of the string and allowing another point to slide along the given curve γ .

Structure of the paper. In the next section we will provide geometric arguments for the proof of [Theorem 2](#). [Section 3](#) is devoted to the construction of the C^2 tables with nonsmooth caustics. In [Section 4](#) we will pose some open questions arising in our considerations.

2. Geometric proof of Innami's result

We will start with the following simple remarks.

Remark 5. If the billiard in Γ has a convex caustic γ with $\gamma^\circ = \emptyset$ then Γ is either an ellipse or a circle.

Indeed, the condition $\gamma^\circ = \emptyset$ for convex γ means that γ is either a point or a segment. The rest follows from the string construction.

Remark 6. Recall that for any point P and for any convex body with nonempty interior there exist exactly two supporting lines to the body passing through P . Moreover if the convex caustic γ has nonempty interior, then every supporting line to $\bar{\gamma}$ after reflection in Γ at point P becomes the second supporting line to $\bar{\gamma}$ from P .

Indeed, assume that there exists a supporting line l to $\bar{\gamma}$ which is reflected to itself at a point $P \in \Gamma$. This means that l is orthogonal to Γ at P . Let l' be the other supporting line to $\bar{\gamma}$ passing through P . Then by the definition of convex caustic, the line l' is also reflected to itself at the point P and hence is also orthogonal to Γ at P . Thus l and l' coincide, which contradicts the assumption that l and l' are two different supporting lines to $\bar{\gamma}$.

Lemma 7. *Let γ be a convex caustic for Γ . Then $\gamma^\circ \neq \emptyset$ if and only if the rotation number of the corresponding invariant curve is strictly less than $\frac{1}{2}$.*

Proof. If a convex caustic γ has empty interior then, by the [Remark 5](#), Γ is necessarily an ellipse (or a circle) and the invariant curve corresponding to γ has rotation number $\frac{1}{2}$ since it contains a diameter. Vice versa, any convex caustic with nonempty interior has a rotation number strictly less than $\frac{1}{2}$, since otherwise the invariant curve corresponding to the caustic would have a 2-periodic orbit, i.e., a diameter, which is not possible due to [Remark 6](#). □

Let γ_n be a sequence of convex caustics for Γ with the rotation numbers $\rho_n \in (0; \frac{1}{2}]$ of corresponding invariant curves. By [Lemma 7](#) we may assume that $\rho_n < \frac{1}{2}$ since otherwise γ_n has empty interior and then Γ must be an ellipse by the [Remark 5](#). Passing to a subsequence we can assume with no loss of generality that ρ_n is strictly increasing, $\rho_n \nearrow \frac{1}{2}$.

Lemma 8. *Let γ_1 and γ_2 be two convex caustics for Γ . If the corresponding invariant curves have rotation numbers $\rho_1 < \rho_2$, then $\bar{\gamma}_2 \subset \gamma_1^\circ$.*

Proof. Assume that $\bar{\gamma}_2$ is not a subset of γ_1° . Then there are only three possibilities: (1): $\bar{\gamma}_1 \cap \bar{\gamma}_2 = \emptyset$; (2): $\gamma_1 \cap \gamma_2 \neq \emptyset$ or (3): $\bar{\gamma}_1 \subset \gamma_2^\circ$.

In the third case one obviously has $\rho_1 \geq \rho_2$ contrary to the assumption of the lemma. In the first and the second cases there necessarily exists a supporting line to both $\bar{\gamma}_1$ and $\bar{\gamma}_2$. Therefore, all billiard reflections in Γ of this line are also supporting lines for both $\bar{\gamma}_1$ and $\bar{\gamma}_2$. This means that there exists a whole infinite orbit lying in the intersection of the invariant curves r_1 and r_2 corresponding to γ_1 and γ_2 . But then ρ_1 must be equal to ρ_2 , since the rotation number is completely determined by one orbit. □

Remark 9. The statement of [Lemma 8](#) holds true also in the opposite direction which will not be used below. Namely, $\bar{\gamma}_2 \subset \gamma_1^\circ$ implies $\rho_1 < \rho_2$. As we already mentioned in the proof, it is obvious that $\rho_1 \leq \rho_2$. In addition ρ_1 cannot be equal to ρ_2 . Otherwise there exist two disjoint graphs of r_1 and r_2 with the same rotation number, invariant under the billiard map of the cylinder, which is impossible since a billiard map is a twist map (see [\[Katok and Hasselblatt 1995, p. 428\]](#)).

Let $\{S_n\}$ be the sequence of string parameters corresponding to the caustics γ_n . Then by [Lemma 8](#), S_n is decreasing. Denote $S = \lim_{n \rightarrow \infty} S_n$.

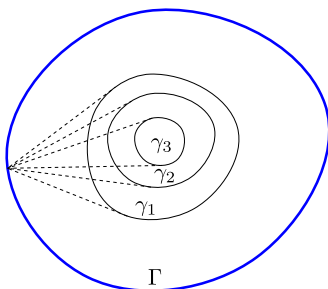


Figure 2. A family of nested convex caustics with decreasing string parameter.

Lemma 10. *The boundary of the intersection set*

$$C = \bigcap_{n=1}^{\infty} \bar{\gamma}_n$$

is a convex caustic for Γ with string parameter S .

Proof. The intersection set C is compact and convex. Moreover, it is easy to see that ∂_C is also a caustic with string parameter S . Indeed, this follows from the following geometric consideration (see Figure 2). Fix a point P on Γ and consider the cap-bodies

$$K_n = \text{Conv}(P \cup \bar{\gamma}_n), \quad K = \text{Conv}(P \cup C).$$

Then, obviously,

$$K_n \subseteq K, \quad K = \bigcap_{n=1}^{\infty} K_n,$$

and moreover

$$\text{Length}(\partial_{K_n}) = S_n \rightarrow S = \text{Length}(\partial_K).$$

In addition, since γ_n is a caustic then S_n does not depend on $P \in \Gamma$ (by Theorem 3). Therefore, S also does not depend on P , and hence C reconstructs Γ via string construction. Thus ∂_C is a caustic by Theorem 3. □

The last step in the proof of Theorem 2 consists in the following Lemma.

Lemma 11. *The limit caustic ∂_C has empty interior.*

Proof. First notice that it follows from continuity of the invariant curves and their rotation numbers that the invariant curve corresponding to C has rotation number $\frac{1}{2}$. Then from Lemma 7 we conclude that ∂_C has empty interior. □

3. Nonsmooth caustic

The main idea of the proof of our result is to carefully choose the Lazutkin parameter and the germ of the function γ at the point A . While a vertex of the string slides in

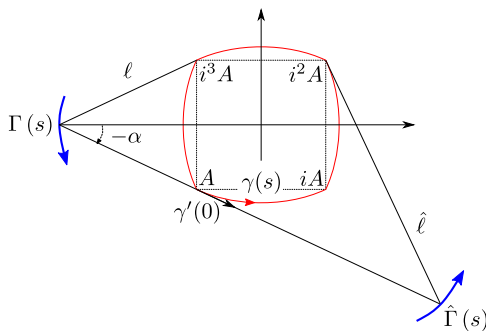


Figure 3. A switched caustic string construction.

the regime corresponding to the unwinding from $\gamma(s)$, its trajectory corresponds to the smooth curve. Thus we have to take care of the smoothness of Γ near only two points corresponding to the switching moments of the first and second kinds respectively. We will denote by $\Gamma(s)$ the part of Γ corresponding to the switching of the second kind about the point A . The part of Γ corresponding to the switching of the first kind about the point A will be denoted by $\hat{\Gamma}$. The smoothness conditions read as follows: all odd terms in the germs of Γ and $\hat{\Gamma}$ have to be orthogonal to the axis of the symmetry while all the even terms must be collinear with the axis of symmetry. Indeed, let $\Gamma(s)$ be the curve symmetric with respect to the line l and intersecting l at the point $\Gamma(0)$. Let R_l be the reflection of the plane in the line l . Differentiating the identity

$$R_l \Gamma(s) = \Gamma(-s)$$

n times, at $s = 0$, we get

$$R_l(\Gamma^{(n)}(0)) = (-1)^n \Gamma^{(n)}(0).$$

Coordinate formulation. Parametrize the curve γ by the arc-length parameter s , so that $|\gamma'(s)| = 1$. Choose the initial point such that $\gamma(0) = A$. Denote by α the angle between $\gamma'(0)$ and the horizontal axis. Then one easily obtains a parametrization for Γ and $\hat{\Gamma}$ (see Figure 3):

$$(1) \quad \begin{aligned} \Gamma(s) &= \gamma(s) - t(s)\gamma'(s), \\ \hat{\Gamma}(s) &= \gamma(s) + \hat{t}(s)\gamma'(s), \end{aligned}$$

where $t(s)$ and $\hat{t}(s)$ are some functions of s denoting the length of the right part of the string near the point $\Gamma(s)$ and the left part of the string near the point $\hat{\Gamma}(s)$ correspondingly. Functions t and \hat{t} can be found from the condition of the string to be unstretchable. We will denote $iA = B$.

$$(2) \quad \begin{aligned} |\Gamma(s) + B| + |t\gamma'(s)| - s &= 2\ell, \\ |\hat{\Gamma}(s) + A| + |\hat{t}\gamma'(s)| + s &= 2\hat{\ell}, \end{aligned}$$

where $\ell = 1/\sin \alpha$ and $\hat{\ell} = \sqrt{2}/\sin(\pi/4 - \alpha)$. Simple computations yield:

$$(3) \quad \begin{aligned} t(s) &= \frac{p(s)}{p'(s)}, \quad \text{with } p(s) = \frac{1}{2}((s + 2\ell)^2 - |\gamma(s) + B|^2), \\ \hat{t}(s) &= -\frac{\hat{p}(s)}{\hat{p}'(s)}, \quad \text{with } \hat{p}(s) = \frac{1}{2}((s - 2\hat{\ell})^2 - |\gamma(s) + A|^2). \end{aligned}$$

Finally, introducing (3) into (1) we get

$$(4) \quad \Gamma(s) = \gamma(s) - \frac{p(s)}{p'(s)}\gamma'(s), \quad \hat{\Gamma}(s) = \gamma(s) - \frac{\hat{p}(s)}{\hat{p}'(s)}\gamma'(s).$$

Orient the curve γ as it is shown in Figure 3. We will use the complex notation for the coordinates of the points. Then smoothness conditions for the n -th derivative of Γ read

$$(5) \quad \Re(i^{n-1}\Gamma^{(n)}(0)) = 0, \quad \Re(i^{n-1}\hat{\Gamma}^{(n)}(0)) = \Im(i^{n-1}\hat{\Gamma}^{(n)}(0)).$$

Here \Re and \Im stand for the real and imaginary part of the complex number. For the curve $\gamma(s)$ we get the following parametrization:

$$(6) \quad \gamma(s) = A + \int_0^s \exp\{i(\varphi(t) - \alpha)\} dt, \quad \text{where } \varphi(t) = \sum_{n=0}^{\infty} \varphi_n t^n.$$

Thus $\varphi_0 = 0$, and φ_n corresponds to the $(n-1)$ -st derivative of the curvature κ .

Lemma 12. *The smoothness conditions in (5) for $n = 1$ are always satisfied.*

This lemma follows from the fact that any C^0 caustic produces a C^1 table via string construction. However, we present a more analytic proof of this result for the sake of completeness.

Proof. Switching of the second kind. From (4) we get

$$\Gamma' = \left(1 - \left(\frac{p}{p'}\right)'\right)\gamma' - \frac{p}{p'}\gamma''.$$

Therefore the conditions in (5) read $\Re(p''\gamma' - p'\gamma'') = 0$. We will denote $z_1 \cdot z_2 := \frac{1}{2}\Re(z_1\bar{z}_2)$. Using (3) we get

$$p' = -(A + B) \cdot \gamma' + 2\ell, \quad p'' = -(A + B) \cdot \gamma''.$$

From (6) it follows that $\gamma'' = i\kappa\gamma'$ thus $p''\gamma' - p'\gamma''$ can be written as

$$\begin{aligned} p''\gamma' - p'\gamma'' &= \frac{1}{2}(-\Re((A + B)\overline{i\kappa\gamma'})\gamma' + \Re((A + B)\bar{\gamma}') (i\kappa\gamma') - 4i\kappa\gamma') \\ &= i\kappa(A + B - 2\ell\gamma'). \end{aligned}$$

Thus

$$\Re(p''\gamma' - p'\gamma'') = \kappa\Im(A + B - 2\ell\gamma').$$

The latter is identically zero since $\ell\gamma'(0) = \Gamma(0) - \gamma(0)$ and so $\Im(\ell\gamma') = \Im(A)$ (see Figure 3).

Switching of the first kind. Similarly, the smoothness conditions in (5) read

$$\Re(\hat{p}''\gamma' - \hat{p}'\gamma'') = \Im(\hat{p}''\gamma' - \hat{p}'\gamma''),$$

where

$$\hat{p}' = -(2A) \cdot \gamma' - 2\hat{\ell}, \quad \hat{p}'' = -(2A) \cdot \gamma''$$

and so

$$\hat{p}''\gamma' - \hat{p}'\gamma'' = (\Re(Ai\kappa\bar{\gamma}')\gamma' + \Re(A\bar{\gamma}')(i\kappa\gamma') + 2\hat{\ell}i\kappa\gamma') = 2i\kappa(A + \hat{\ell}\gamma').$$

The real part of the right-hand side of the latter is always equal to the imaginary part by the definition of $\hat{\ell}$. □

The two conditions in (5) for $n = 2$ provide, via computations similar to the above, two equations for parameters φ_1 and φ_2 with coefficients depending on α :

$$\frac{\varphi_1^2 \sin \alpha - \varphi_1 \sin \alpha \cos \alpha - \varphi_2 \cos \alpha}{\sin \alpha \cos^2 \alpha} = 0,$$

$$\frac{\varphi_1(\cos 2\alpha + 2(\sin \alpha - \cos \alpha)\varphi_1) - 2(\cos \alpha + \sin \alpha)\varphi_2}{(\cos \alpha - \sin \alpha)(1 + \sin 2\alpha)} = 0.$$

The latter system has a solution,

$$(7) \quad \varphi_1 = \frac{1}{2} \cos \alpha(1 + \sin 2\alpha), \quad \varphi_2 = -\frac{1}{8} \cos^2 2\alpha \sin 2\alpha,$$

which provides a family of germs for γ , depending on the parameter α , guaranteeing the C^2 -smoothness for the table Γ .

Next we will need to construct the whole curve γ providing the needed phenomenon in the string construction. Recall that our geometric idea was based on the construction of the curve γ_0 (see Figure 1). Thus we need to present a convex curve of length S , starting at A and ending at iA , having tangent slope $-\alpha$ at the left end and being symmetric with respect to the vertical axis. We define γ from φ through (6). In order to finish the construction we have to prove the following theorem.

Theorem 13. *There exists a strictly monotonically increasing function $\varphi(s)$ satisfying the following three conditions: (1) $\varphi(s)$ has the given germ (7) at $s = 0$, (2) $\varphi_0(S/2) = \alpha$ and $\varphi_{2n}(S/2) = 0$ for $n \geq 1$, and (3) $\int_0^{S/2} \cos \varphi(s) ds = 1$.*

Proof. The Borel theorem states that every power series is the Taylor series of some smooth function. Obviously, using cutting off, one can find a smooth function having a given Taylor series at two given points. Thus there exists a nonempty set Ψ of C^∞ functions having given germs at $s = 0$ and $s = S/2$. Since for $\alpha < \frac{\pi}{2}$ the term φ_1 in (7) is positive, one may assume without loss of generality that Ψ consists of strictly monotonically increasing functions. Therefore the only condition which

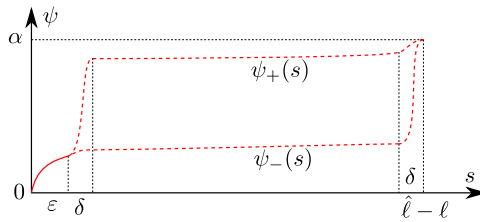


Figure 4. Construction of the solution.

has to be satisfied is [Theorem 13\(3\)](#). Taking a small enough ε -step in s we can ensure $\psi(\varepsilon) < \frac{\alpha}{100}$ for all $\psi \in \Psi$. Next we choose two functions ψ_- and ψ_+ from the set Ψ as in [Figure 4](#). That is, $\psi_+(s)$ is almost equal to α for $s \in (\varepsilon + \delta, S/2 - \delta)$ and $\psi_-(s)$ is almost equal to $\psi(\varepsilon)$ for $s \in (\varepsilon, S/2 - \delta)$ for small enough δ . We will look for φ as a convex combination $\varphi(s) = l\psi_-(s) + (1 - l)\psi_+(s)$. Therefore $\varphi(s)$ obviously satisfies conditions 1 and 2. If we may choose ψ_{\pm} in such a way that

$$(8) \quad (S/2) \cos \alpha < \int_0^{S/2} \cos(\psi_-(s) - \alpha) ds < 1 \quad \text{and} \quad S/2 > \int_0^{S/2} \cos(\psi_+(s) - \alpha) ds > 1$$

then there exists l such that $\int_0^{S/2} \cos(\varphi(s)) ds = 1$, thus satisfying condition [Theorem 13\(3\)](#). Hence it is sufficient to check that the conditions in (8) can be satisfied for an open set of parameters α . Recall that by the construction $S = 2\hat{l} - 2\ell$. From the first inequality in (8) we obtain, since $\alpha < \frac{\pi}{4}$,

$$\hat{l} - \ell = \frac{2}{\cos \alpha - \sin \alpha} - \frac{1}{\sin \alpha} < \frac{1}{\cos \alpha}.$$

This condition can be interpreted as follows: *the length of the curve γ cannot exceed the sum of the lengths of the segments of the two tangent lines from point P to γ* (see [Figure 1](#)). The latter inequality is satisfied whenever $\tan 2\alpha < 1$ or

$$(9) \quad \alpha < \frac{\pi}{8}.$$

The second condition in (8) has the following geometric interpretation: *the length of γ cannot be less than the distance between points A and B* . This yields:

$$3 \sin \alpha - \cos \alpha > \cos \alpha \sin \alpha - \sin^2 \alpha.$$

Since the latter is satisfied for $\alpha = \frac{\pi}{8}$ we have found an open set of α for which one can find appropriate functions ψ_- and ψ_+ shown in [Figure 4](#). □

Remark 14. Since the conditions in (5) provide two conditions on φ_n to obtain C^3 of Γ one gets four equations for $\varphi_1, \varphi_2, \varphi_3$ and α . Although the number of parameters matches the number of equations, the corresponding value of α violates (9). Since (9) arises from the construction based on square symmetry, there

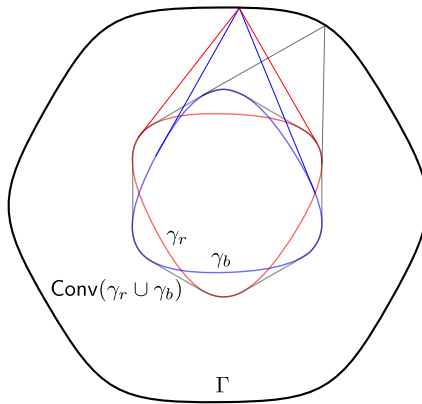


Figure 5. The convex hull of two intersecting caustics is also a caustic.

is a hope that starting from other regular polygons one can obtain an inequality which can be satisfied. However, we haven't found any such examples.

4. Open problems

Here we want to highlight some general questions which are ultimately related to the string construction. Since the string construction is implicit these questions turn out to be nontrivial.

Question 15. Is it possible to have two convex caustics γ_1 and γ_2 of Γ such that neither of them is a subset of the interior of the other?

In such a case γ_1 and γ_2 must have the same rotation number since there is a line tangent to both of the caustics. Moreover it is obvious that $\bar{\gamma}_1$ and $\bar{\gamma}_2$ cannot be disjoint. So the question is if it is possible that two convex caustics have nontrivial intersection. In such a case their convex hull is also a caustic. One can strengthen the question:

Question 16. Is it possible for a Γ which is symmetric with respect to a certain axis to have a convex caustic C which is not symmetric with respect to this axis?

For example one could imagine two caustics forming a rounded Star of David (Figure 5). The answer to the quantum analog of this question is positive: for a symmetric domain the Dirichlet eigenfunction can be nonsymmetric. We could not however decide if such a counterexample would be possible in the original setting.

Question 17. How irregular a convex caustic can be compared to a regular boundary curve Γ ?

Question 18. Let Γ be a billiard table different from a circle and having a convex caustic γ . For every point $P \in \Gamma$, denote by P_- , and P_+ the tangency points of the caustic γ with tangent lines to γ passing through P . Is it possible that the length of the arc of γ between P_- and P_+ does not depend on P ?

Acknowledgements.

The authors are thankful to the anonymous referee for the useful remarks which greatly improved this manuscript. Bialy is also thankful to the participants of the course “Billiards” given in Tel Aviv University for very useful discussions and ideas. He was supported by ISF 162/15.

References

- [Arnold 1990] V. I. Arnold, *Singularities of caustics and wave fronts*, Math. Appl. **62**, Kluwer, Dordrecht, 1990. [MR](#) [Zbl](#)
- [Avila et al. 2016] A. Avila, J. De Simoi, and V. Kaloshin, “An integrable deformation of an ellipse of small eccentricity is an ellipse”, *Ann. Math. (2)* **184**:2 (2016), 527–558. [MR](#) [Zbl](#)
- [Bialy 1993] M. Bialy, “Convex billiards and a theorem by E. Hopf”, *Math. Z.* **214**:1 (1993), 147–154. [MR](#) [Zbl](#)
- [Birkhoff 1917] G. D. Birkhoff, “Dynamical systems with two degrees of freedom”, *Trans. Amer. Math. Soc.* **18**:2 (1917), 199–300. [MR](#) [Zbl](#)
- [Douady 1982] R. Douady, *Applications du théorème des tores invariants*, thèse de 3ème cycle, Paris Diderot Univ., 1982.
- [Fetter 2012] H. L. Fetter, “Numerical exploration of a hexagonal string billiard”, *Phys. D* **241**:8 (2012), 830–846. [MR](#) [Zbl](#)
- [Hubacher 1987] A. Hubacher, “Instability of the boundary in the billiard ball problem”, *Comm. Math. Phys.* **108**:3 (1987), 483–488. [MR](#) [Zbl](#)
- [Innami 2002] N. Innami, “Geometry of geodesics for convex billiards and circular billiards”, *Nihonkai Math. J.* **13**:1 (2002), 73–120. [MR](#) [Zbl](#)
- [Kaloshin and Sorrentino 2016] V. Kaloshin and A. Sorrentino, “On local Birkhoff conjecture for convex billiards”, preprint, 2016. [arXiv](#)
- [Katok and Hasselblatt 1995] A. Katok and B. Hasselblatt, *Introduction to the modern theory of dynamical systems*, Encyclopedia of Math. Appl. **54**, Cambridge Univ. Press, 1995. [MR](#) [Zbl](#)
- [Katok et al. 1986] A. Katok, J.-M. Strelcyn, F. Ledrappier, and F. Przytycki, *Invariant manifolds, entropy and billiards: smooth maps with singularities*, Lecture Notes in Math. **1222**, Springer, 1986. [MR](#) [Zbl](#)
- [Knill 1998] O. Knill, “On nonconvex caustics of convex billiards”, *Elem. Math.* **53**:3 (1998), 89–106. [MR](#) [Zbl](#)
- [Kozlov and Treshchëv 1991] V. V. Kozlov and D. V. Treshchëv, *Billiards: a genetic introduction to the dynamics of systems with impacts*, Translations of Mathematical Monographs **89**, Amer. Math. Soc., Providence, RI, 1991. [MR](#) [Zbl](#)
- [Lazutkin 1973] V. F. Lazutkin, “Existence of caustics for the billiard problem in a convex domain”, *Izv. Akad. Nauk SSSR Ser. Mat.* **37** (1973), 186–216. In Russian; translated in *Math. USSR Izv.* **7**:1 (1973), 185–214. [MR](#) [Zbl](#)
- [Lazutkin 1981] V. F. Lazutkin, Выпуклый бильярд и собственные функции оператора Лапласа, Leningrad. Univ., 1981. [MR](#) [Zbl](#)
- [Mather and Forni 1994] J. N. Mather and G. Forni, “Action minimizing orbits in Hamiltonian systems”, pp. 92–186 in *Transition to chaos in classical and quantum mechanics* (Montecatini Terme, Italy, 1991), edited by S. Graffi, Lecture Notes in Math. **1589**, Springer, 1994. [MR](#) [Zbl](#)

- [Stoll 1930] A. Stoll, “Über den Kappenkörper eines konvexen Körpers”, *Comment. Math. Helv.* **2**:1 (1930), 35–68. [MR](#) [Zbl](#)
- [Tabachnikov 2005] S. Tabachnikov, *Geometry and billiards*, Student Mathematical Library **30**, Amer. Math. Soc., Providence, RI, 2005. [MR](#) [Zbl](#)
- [Turner 1982] P. H. Turner, “Convex caustics for billiards in \mathbb{R}^2 and \mathbb{R}^3 ”, pp. 85–106 in *Convexity and related combinatorial geometry* (Norman, OK, 1980), edited by D. C. Kay and M. Breen, Lecture Notes in Pure and Appl. Math. **76**, Dekker, New York, 1982. [MR](#) [Zbl](#)

Received August 14, 2017. Revised December 29, 2017.

MAXIM ARNOLD
DEPARTMENT OF MATHEMATICAL SCIENCES
UNIVERSITY OF TEXAS AT DALLAS
RICHARDSON, TX
UNITED STATES
maxim.arnold@utdallas.edu

MISHA BIALY
TEL AVIV UNIVERSITY
TEL AVIV
ISRAEL
bialy@post.tau.ac.il

PACIFIC JOURNAL OF MATHEMATICS

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

msp.org/pjm

EDITORS

Don Blasius (Managing Editor)
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
blasius@math.ucla.edu

Paul Balmer
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
balmer@math.ucla.edu

Wee Teck Gan
Mathematics Department
National University of Singapore
Singapore 119076
matgwt@nus.edu.sg

Sorin Popa
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
popa@math.ucla.edu

Vyjayanthi Chari
Department of Mathematics
University of California
Riverside, CA 92521-0135
chari@math.ucr.edu

Kefeng Liu
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
liu@math.ucla.edu

Jie Qing
Department of Mathematics
University of California
Santa Cruz, CA 95064
qing@cats.ucsc.edu

Daryl Cooper
Department of Mathematics
University of California
Santa Barbara, CA 93106-3080
cooper@math.ucsb.edu

Jiang-Hua Lu
Department of Mathematics
The University of Hong Kong
Pokfulam Rd., Hong Kong
jhlu@maths.hku.hk

Paul Yang
Department of Mathematics
Princeton University
Princeton NJ 08544-1000
yang@math.princeton.edu

PRODUCTION

Silvio Levy, Scientific Editor, production@msp.org

SUPPORTING INSTITUTIONS

ACADEMIA SINICA, TAIPEI
CALIFORNIA INST. OF TECHNOLOGY
INST. DE MATEMÁTICA PURA E APLICADA
KEIO UNIVERSITY
MATH. SCIENCES RESEARCH INSTITUTE
NEW MEXICO STATE UNIV.
OREGON STATE UNIV.

STANFORD UNIVERSITY
UNIV. OF BRITISH COLUMBIA
UNIV. OF CALIFORNIA, BERKELEY
UNIV. OF CALIFORNIA, DAVIS
UNIV. OF CALIFORNIA, LOS ANGELES
UNIV. OF CALIFORNIA, RIVERSIDE
UNIV. OF CALIFORNIA, SAN DIEGO
UNIV. OF CALIF., SANTA BARBARA

UNIV. OF CALIF., SANTA CRUZ
UNIV. OF MONTANA
UNIV. OF OREGON
UNIV. OF SOUTHERN CALIFORNIA
UNIV. OF UTAH
UNIV. OF WASHINGTON
WASHINGTON STATE UNIVERSITY

These supporting institutions contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its contents or policies.


See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2018 is US \$475/year for the electronic version, and \$640/year for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by [Mathematical Reviews](#), [Zentralblatt MATH](#), [PASCAL CNRS Index](#), [Referativnyi Zhurnal](#), [Current Mathematical Publications](#) and [Web of Knowledge \(Science Citation Index\)](#).

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW[®] from Mathematical Sciences Publishers.

PUBLISHED BY

 **mathematical sciences publishers**
nonprofit scientific publishing

<http://msp.org/>

© 2018 Mathematical Sciences Publishers

PACIFIC JOURNAL OF MATHEMATICS

Volume 295 No. 2 August 2018

Nonsmooth convex caustics for Birkhoff billiards	257
MAXIM ARNOLD and MISHA BIALY	
Certain character sums and hypergeometric series	271
RUPAM BARMAN and NEELAM SAIKIA	
On the structure of holomorphic isometric embeddings of complex unit balls into bounded symmetric domains	291
SHAN TAI CHAN	
Hamiltonian stationary cones with isotropic links	317
JINGYI CHEN and YU YUAN	
Quandle theory and the optimistic limits of the representations of link groups	329
JINSEOK CHO	
Classification of positive smooth solutions to third-order PDEs involving fractional Laplacians	367
WEI DAI and GUOLIN QIN	
The projective linear supergroup and the SUSY-preserving automorphisms of $\mathbb{P}^{1 1}$	385
RITA FIORESI and STEPHEN D. KWOK	
The Gromov width of coadjoint orbits of the symplectic group	403
IVA HALACHEVA and MILENA PABINIAK	
Minimal braid representatives of quasipositive links	421
KYLE HAYDEN	
Four-dimensional static and related critical spaces with harmonic curvature	429
JONGSU KIM and JINWOO SHIN	
Boundary Schwarz lemma for nonequidimensional holomorphic mappings and its application	463
YANG LIU, ZHIHUA CHEN and YIFEI PAN	
Theta correspondence and the Prasad conjecture for $SL(2)$	477
HENGFEI LU	
Convexity of level sets and a two-point function	499
BEN WEINKOVE	