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We prove two transformations for the p-adic hypergeometric series which can be described as p-adic analogues of a Kummer's linear transformation and a transformation of Clausen. We first evaluate two character sums, and then relate them to the p-adic hypergeometric series to deduce the transformations. We also find another transformation for the p-adic hypergeometric series from which many special values of the p-adic hypergeometric series as well as finite field hypergeometric functions are obtained.

1. Introduction and statement of results

For a complex number a, the rising factorial or the Pochhammer symbol is defined as $(a)_0 = 1$ and $(a)_k = a(a+1)\cdots(a+k-1),\ k \geq 1$. For a nonnegative integer r, and $a_i, b_i \in \mathbb{C}$ with $b_i \notin \{\ldots, -3, -2, -1\}$, the classical hypergeometric series ${}_{r+1}F_r$ is defined by

$$ {}_{r+1}F_r\left(\begin{matrix} a_1, & a_2, & \ldots, & a_{r+1} \\ & b_1, & \ldots, & b_r \end{matrix} \,\Big|\, \lambda\right) := \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_{r+1})_k}{(b_1)_k \cdots (b_r)_k} \cdot \frac{\lambda^k}{k!}, $$

which converges for $|\lambda| < 1$. Throughout the paper, p denotes an odd prime and $\mathbb{F}_q$ denotes the finite field with q elements, where $q = p^r,\ r \geq 1$. Greene [1987] introduced the notion of hypergeometric functions over finite fields analogous to the classical hypergeometric series. Finite field hypergeometric series were developed mainly to simplify character sum evaluations. Let $\widehat{\mathbb{F}_q^{\times}}$ be the group of all multiplicative characters on $\mathbb{F}_q^{\times}$. We extend the domain of each $\chi \in \widehat{\mathbb{F}_q^{\times}}$ to $\mathbb{F}_q$ by setting $\chi(0) = 0$ including the trivial character ε. For multiplicative characters A and B on $\mathbb{F}_q$, the binomial coefficient $\binom{A}{B}$ is defined by

$$ \binom{A}{B} := \frac{B(-1)}{q} J(A, \bar{B}) = \frac{B(-1)}{q} \sum_{x \in \mathbb{F}_q} A(x)\bar{B}(1-x), \tag{1-1} $$

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where $J(A, B)$ denotes the usual Jacobi sum and $\bar{B}$ is the character inverse of B. Let n be a positive integer. For characters $A_0, A_1, \ldots, A_n$ and $B_1, B_2, \ldots, B_n$ on $\mathbb{F}_q$, Greene defined the ${}_{n+1}F_n$ finite field hypergeometric functions over $\mathbb{F}_q$ by

$$ {}_{n+1}F_n\left(\begin{matrix} A_0, & A_1, & \ldots, & A_n \\ & B_1, & \ldots, & B_n \end{matrix} \,\Big|\, x\right)_q = \frac{q}{q-1} \sum_{\chi \in \widehat{\mathbb{F}_q^\times}} \binom{A_0\chi}{\chi}\binom{A_1\chi}{B_1\chi}\cdots\binom{A_n\chi}{B_n\chi}\chi(x). $$

Some of the biggest motivations for studying finite field hypergeometric functions have been their connections with Fourier coefficients and eigenvalues of modular forms and with counting points on certain kinds of algebraic varieties. Their links to Fourier coefficients and eigenvalues of modular forms are established by many authors, for example, see [Ahlgren and Ono 2000; Evans 2010; Frechette et al. 2004; Fuselier 2010; Fuselier and McCarthy 2016; Lennon 2011b; McCarthy 2012b; Mortenson 2005]. Very recently, McCarthy and Papanikolas [2015] linked the finite field hypergeometric functions to Siegel modular forms. It is well known that finite field hypergeometric functions can be used to count points on varieties over finite fields. For example, see [Barman and Kalita 2013a; 2013b; Fuselier 2010; Koike 1992; Lennon 2011a; Ono 1998; Salerno 2013; Vega 2011].

Since the multiplicative characters on $\mathbb{F}_q$ form a cyclic group of order $q-1$, a condition like $q \equiv 1 \pmod{\ell}$ must be satisfied where ℓ is the least common multiple of the orders of the characters appearing in the hypergeometric function. Consequently, many results involving these functions are restricted to primes in certain congruence classes. To overcome these restrictions, McCarthy [2012a; 2013] defined a function ${}_nG_n[\cdots]_q$ in terms of quotients of the p-adic gamma function which can best be described as an analogue of hypergeometric series in the p-adic setting (defined in Section 2).

Many transformations exist for finite field hypergeometric functions which are analogues of certain classical results [Greene 1987; McCarthy 2012c]. Results involving finite field hypergeometric functions can readily be converted to expressions involving ${}_nG_n[\cdots]$. However these new expressions in ${}_nG_n[\cdots]$ will be valid for the same set of primes for which the original expressions involving finite field hypergeometric functions existed. It is a nontrivial exercise to then extend these results to almost all primes. There are very few identities and transformations for the p-adic hypergeometric series ${}_nG_n[\cdots]_q$ which exist for all but finitely many primes (see for example [Barman and Saikia 2014; 2015; Barman et al. 2015]. Recently, Fuselier and McCarthy [2016] proved certain transformations for ${}_nG_n[\cdots]_q$, and used them to establish a supercongruence conjecture of Rodriguez-Villegas between a truncated ${}_4F_3$ hypergeometric series and the Fourier coefficients of a certain weight four modular form.

Let χ_4 be a character of order 4. Then a finite field analogue of ${}_2F_1\left(\begin{smallmatrix} 1/4, & 3/4 \\ & 1 \end{smallmatrix} \,|\, x\right)$ is the function ${}_2F_1\left(\begin{smallmatrix} \chi_4, & \chi_4^3 \\ & \varepsilon \end{smallmatrix} \,|\, x\right)$. Using the relation between finite field hypergeometric functions and ${}_nG_n$-functions as given in Proposition 3.5 in Section 3, the function ${}_2G_2\left[\begin{smallmatrix} 1/4, & 3/4 \\ 0, & 0 \end{smallmatrix} \mid \frac{1}{x}\right]_q$ can be described as a p-adic analogue of the classical hypergeometric series ${}_2F_1\left(\begin{smallmatrix} 1/4, & 3/4 \\ & 1 \end{smallmatrix} \mid x\right)$. In this article, we prove the following transformation for the p-adic hypergeometric series which can be described as a p-adic analogue of the Kummer's linear transformation [Bailey 1935, p. 4, Equation (1)]. Let φ be the quadratic character on $\mathbb{F}_q$.

Theorem 1.1. *Let p be an odd prime and $x \in \mathbb{F}_q$. Then, for $x \neq 0, 1$, we have*

$$ {}_2G_2\left[\begin{matrix} \frac{1}{4}, & \frac{3}{4} \\ 0, & 0 \end{matrix} \,\middle|\, \frac{1}{x}\right]_q = \varphi(-2)\, {}_2G_2\left[\begin{matrix} \frac{1}{4}, & \frac{3}{4} \\ 0, & 0 \end{matrix} \,\middle|\, \frac{1}{1-x}\right]_q. $$

We note that the finite field analogue of Kummer's linear transformation was discussed by Greene [1984, p. 109, Equation (7.7)] when $q \equiv 1 \pmod 4$.

We have $\varphi(-2) = -1$ if and only if $p \equiv 5, 7 \pmod 8$. Hence, using Theorem 1.1 for $x = \frac{1}{2}$, we obtain the following special value of the ${}_2G_2$-function.

Corollary 1.2. *Let p be a prime such that $p \equiv 5, 7 \pmod 8$. Then we have*

$$ {}_2G_2\left[\begin{matrix} \frac{1}{4}, & \frac{3}{4} \\ 0, & 0 \end{matrix} \,\middle|\, 2\right]_p = 0. \tag{1-2} $$

If we convert the ${}_2G_2$-function given in (1-2) using Proposition 3.5 in Section 3, then we have ${}_2F_1\left(\begin{smallmatrix} \chi_4, & \chi_4^3 \\ & \varepsilon \end{smallmatrix} \mid \frac{1}{2}\right)_p = 0$ for $p \equiv 5 \pmod 8$ which also follows from [Greene 1987, Equation (4.15)]. The value of ${}_2G_2\left[\begin{smallmatrix} 1/4, & 3/4 \\ 0, & 0 \end{smallmatrix} \mid 2\right]_p$ can be deduced from [Greene 1987, Equation (4.15)] when $p \equiv 1 \pmod 8$. It would be interesting to know the value of ${}_2G_2\left[\begin{smallmatrix} 1/4, & 3/4 \\ 0, & 0 \end{smallmatrix} \mid 2\right]_p$ when $p \equiv 3 \pmod 8$.

The following transformation for classical hypergeometric series is a special case of Clausen's famous classical identity [Bailey 1935, p. 86, Equation (4)]:

$$ {}_3F_2\left(\begin{matrix} \frac{1}{2}, & \frac{1}{2}, & \frac{1}{2} \\ & 1, & 1 \end{matrix} \,\middle|\, x\right) = (1-x)^{-1/2}\, {}_2F_1\left(\begin{matrix} \frac{1}{4}, & \frac{3}{4} \\ & 1 \end{matrix} \,\middle|\, \frac{x}{x-1}\right)^2. \tag{1-3} $$

A finite field analogue of (1-3) was studied by Greene [1984, p. 94, Proposition 6.14]. Evans and Greene [2009a] gave a finite field analogue of the Clausen's classical identity. We prove the following transformation for the ${}_nG_n$-function which can be described as a p-adic analogue of (1-3). Let δ be the function defined on $\mathbb{F}_q$ by

$$ \delta(x) = \begin{cases} 1 & \text{if } x = 0; \\ 0 & \text{if } x \neq 0. \end{cases} $$

Theorem 1.3. *Let p be an odd prime and $x \in \mathbb{F}_p$. Then, for $x \neq 0, 1$, we have*

$$ {}_3G_3\left[\begin{matrix} \frac{1}{2}, & \frac{1}{2}, & \frac{1}{2} \\ 0, & 0, & 0 \end{matrix} \,\middle|\, \frac{1}{x}\right]_p = \varphi(1-x) \cdot {}_2G_2\left[\begin{matrix} \frac{1}{4}, & \frac{3}{4} \\ 0, & 0 \end{matrix} \,\middle|\, \frac{x-1}{x}\right]_p^2 - p \cdot \varphi(1-x). $$

We also prove the following transformation using Theorem 1.1 and [Greene 1987, Theorem 4.16].

Theorem 1.4. *Let p be an odd prime and $x \in \mathbb{F}_q$. Then, for $x \neq 0, \pm 1$, we have*

$$(1\text{-}4)\qquad {}_2G_2\left[\begin{matrix}\frac{1}{4}, & \frac{3}{4}\\ 0, & 0\end{matrix}\,\Big|\,\frac{(1+x)^2}{(1-x)^2}\right]_q = \varphi(-2)\varphi(1+x)\,{}_2G_2\left[\begin{matrix}\frac{1}{2}, & \frac{1}{2}\\ 0, & 0\end{matrix}\,\Big|\,x^{-1}\right]_q.$$

The following transformation is a finite field analogue of (1-4).

Theorem 1.5. *Let p be an odd prime and $q = p^r$ for some $r \geq 1$ such that $q \equiv 1 \pmod 4$. Then, for $x \neq 0, \pm 1$, we have*

$${}_2F_1\left(\begin{matrix}\chi_4, & \chi_4^3\\ & \varepsilon\end{matrix}\,\Big|\,\frac{(1-x)^2}{(1+x)^2}\right)_q = \varphi(-2)\varphi(1+x)\,{}_2F_1\left(\begin{matrix}\varphi, & \varphi\\ & \varepsilon\end{matrix}\,\Big|\,x\right)_q.$$

Using Theorems 1.4 and 1.5, one can deduce many special values of the p-adic hypergeometric series as well as the finite field hypergeometric functions. For example, we have the following special values of a ${}_2G_2$-function and its finite field analogue.

Theorem 1.6. *For any odd prime p, we have*

$$\begin{aligned}&{}_2G_2\left[\begin{matrix}\frac{1}{4}, & \frac{3}{4}\\ 0, & 0\end{matrix}\,\Big|\,9\right]_p\\ &\qquad = \begin{cases} 0 & \text{if } p \equiv 3 \pmod 4;\\ -2x\varphi(6)(-1)^{\frac{x+y+1}{2}} & \text{if } p \equiv 1 \pmod 4,\ x^2+y^2=p, \text{ and } x \text{ odd.}\end{cases}\end{aligned}$$

For $p \equiv 1 \pmod 4$, we have

$${}_2F_1\left(\begin{matrix}\chi_4, & \chi_4^3\\ & \varepsilon\end{matrix}\,\Big|\,\frac{1}{9}\right)_p = \frac{2x\varphi(6)(-1)^{\frac{x+y+1}{2}}}{p},$$

where $x^2 + y^2 = p$ and x is odd.

We also find special values of the following ${}_2G_2$-function.

Theorem 1.7. *For $q \equiv 1 \pmod 8$ we have*

$$(1\text{-}5)\qquad {}_2G_2\left[\begin{matrix}\frac{1}{4}, & \frac{3}{4}\\ 0, & 0\end{matrix}\,\Big|\,\left(\frac{6\sqrt{2}\pm 3}{-2\sqrt{2}\pm 3}\right)^2\right]_q = -q\varphi(6\pm 12\sqrt{2})\left\{\binom{\chi_4}{\varphi}+\binom{\chi_4^3}{\varphi}\right\}.$$

For $q \equiv 11 \pmod{12}$ we have

$$(1\text{-}6)\qquad {}_2G_2\left[\begin{matrix}\frac{1}{4}, & \frac{3}{4}\\ 0, & 0\end{matrix}\,\Big|\,\left(\frac{6\pm\sqrt{3}}{-2\pm\sqrt{3}}\right)^2\right]_q = 0.$$

For $q \equiv 1 \pmod{12}$ *we have*

$$(1\text{-}7)\qquad {}_2G_2\left[\begin{matrix}\frac{1}{4}, & \frac{3}{4}\\ 0, & 0\end{matrix}\;\middle|\;\left(\frac{6\pm\sqrt{3}}{-2\pm\sqrt{3}}\right)^2\right]_q = -q\varphi\left(\frac{8\pm5\sqrt{3}}{12\pm6\sqrt{3}}\right)\left\{\binom{\varphi}{\chi_3}+\binom{\varphi}{\chi_3^2}\right\}.$$

The following theorem is a finite field analogue of Theorem 1.7.

Theorem 1.8. *For* $q \equiv 1 \pmod 8$ *we have*

$$(1\text{-}8)\qquad {}_2F_1\left(\begin{matrix}\chi_4, & \chi_4^3\\ & \varepsilon\end{matrix}\;\middle|\;\left(\frac{-2\sqrt{2}\pm3}{6\sqrt{2}\pm3}\right)^2\right)_q = \varphi(6\pm12\sqrt{2})\left\{\binom{\chi_4}{\varphi}+\binom{\chi_4^3}{\varphi}\right\}.$$

For $q \equiv 1 \pmod{12}$ *we have*

$$(1\text{-}9)\qquad {}_2F_1\left(\begin{matrix}\chi_4, & \chi_4^3\\ & \varepsilon\end{matrix}\;\middle|\;\left(\frac{-2\pm\sqrt{3}}{6\pm\sqrt{3}}\right)^2\right)_q = \varphi\left(\frac{8\pm5\sqrt{3}}{12\pm6\sqrt{3}}\right)\left\{\binom{\varphi}{\chi_3}+\binom{\varphi}{\chi_3^2}\right\}.$$

In Section 3 we prove two character sum identities and then use them to prove Theorems 1.1, 1.3, and 1.4. We also prove Theorem 1.5 in Section 3. In Section 4 we prove Theorems 1.6, 1.7 and 1.8.

2. Notations and preliminaries

Let $\mathbb{Z}_p$ and $\mathbb{Q}_p$ denote the ring of p-adic integers and the field of p-adic numbers, respectively. Let $\overline{\mathbb{Q}}_p$ be the algebraic closure of $\mathbb{Q}_p$ and $\mathbb{C}_p$ the completion of $\overline{\mathbb{Q}}_p$. Let $\mathbb{Z}_q$ be the ring of integers in the unique unramified extension of $\mathbb{Q}_p$ with residue field $\mathbb{F}_q$. We know that $\chi \in \widehat{\mathbb{F}_q^\times}$ takes values in μ_{q-1}, where μ_{q-1} is the group of $(q-1)$-th roots of unity in $\mathbb{C}^\times$. Since $\mathbb{Z}_q^\times$ contains all $(q-1)$-th roots of unity, we can consider multiplicative characters on $\mathbb{F}_q^\times$ to be maps $\chi : \mathbb{F}_q^\times \to \mathbb{Z}_q^\times$. Let $\omega : \mathbb{F}_q^\times \to \mathbb{Z}_q^\times$ be the Teichmüller character. For $a \in \mathbb{F}_q^\times$, the value $\omega(a)$ is just the $(q-1)$-th root of unity in $\mathbb{Z}_q$ such that $\omega(a) \equiv a \pmod p$.

We now introduce some properties of Gauss sums. For further details, see [Berndt et al. 1998]. Let ζ_p be a fixed primitive p-th root of unity in $\overline{\mathbb{Q}}_p$. The trace map $\mathrm{tr} : \mathbb{F}_q \to \mathbb{F}_p$ is given by

$$\mathrm{tr}(\alpha) = \alpha + \alpha^p + \alpha^{p^2} + \cdots + \alpha^{p^{r-1}}.$$

For $\chi \in \widehat{\mathbb{F}_q^\times}$, the *Gauss sum* is defined by

$$g(\chi) := \sum_{x\in\mathbb{F}_q} \chi(x)\zeta_p^{\mathrm{tr}(x)}.$$

Now, we will see some elementary properties of Gauss and Jacobi sums. We let T denote a fixed generator of $\widehat{\mathbb{F}_q^\times}$.

Lemma 2.1 [Greene 1987, Equation 1.12]. *If* $k \in \mathbb{Z}$ *and* $T^k \neq \varepsilon$, *then*

$$g(T^k)g(T^{-k}) = qT^k(-1).$$

Let δ denote the function on multiplicative characters defined by

$$\delta(A)=\begin{cases}1 & \text{if } A \text{ is the trivial character;}\\ 0 & \text{otherwise.}\end{cases}$$

Lemma 2.2 [Greene 1987, Equation 1.14]. *For $A, B \in \widehat{\mathbb{F}_q^\times}$ we have*

$$J(A,B)=\frac{g(A)g(B)}{g(AB)}+(q-1)B(-1)\delta(AB).$$

The following are character sum analogues of the binomial theorem [Greene 1987]. For any $A \in \widehat{\mathbb{F}_q^\times}$ and $x \in \mathbb{F}_q$ we have

$$\overline{A}(1-x)=\delta(x)+\frac{q}{q-1}\sum_{\chi\in\widehat{\mathbb{F}_q^\times}}\binom{A\chi}{\chi}\chi(x), \tag{2-1}$$

$$A(1+x)=\delta(x)+\frac{q}{q-1}\sum_{\chi\in\widehat{\mathbb{F}_q^\times}}\binom{A}{\chi}\chi(x). \tag{2-2}$$

We recall some properties of the binomial coefficients from [Greene 1987]:

$$\binom{A}{B}=\binom{A}{A\overline{B}}, \tag{2-3}$$

$$\binom{A}{\varepsilon}=\binom{A}{A}=\frac{-1}{q}+\frac{q-1}{q}\delta(A). \tag{2-4}$$

Theorem 2.3 [Berndt et al. 1998, Davenport–Hasse relation]. *Let m be a positive integer and let $q=p^r$ be a prime power such that $q\equiv 1 \pmod m$. For multiplicative characters χ and ψ in $\widehat{\mathbb{F}_q^\times}$, we have*

$$\prod_{\chi^m=\varepsilon} g(\chi\psi)=-g(\psi^m)\psi(m^{-m})\prod_{\chi^m=\varepsilon} g(\chi).$$

Now, we recall the p-adic gamma function. For further details, see [Koblitz 1980]. For a positive integer n, the p-adic gamma function $\Gamma_p(n)$ is defined as

$$\Gamma_p(n):=(-1)^n\prod_{0<j<n,\, p\nmid j} j$$

and one extends it to all $x\in\mathbb{Z}_p$ by setting $\Gamma_p(0):=1$ and

$$\Gamma_p(x):=\lim_{x_n\to x}\Gamma_p(x_n)$$

for $x\neq 0$, where x_n runs through any sequence of positive integers p-adically approaching x. This limit exists, is independent of how x_n approaches x, and determines a continuous function on $\mathbb{Z}_p$ with values in $\mathbb{Z}_p^\times$. For $x\in\mathbb{Q}$ we let $\lfloor x\rfloor$ denote the greatest integer less than or equal to x and $\langle x\rangle$ denote the fractional part of x, i.e., $x - \lfloor x \rfloor$, satisfying $0 \le \langle x \rangle < 1$. We now recall the McCarthy's p-adic hypergeometric series ${}_nG_n[\cdots]$ as follows.

Definition 2.4 [McCarthy 2013, Definition 5.1]. Let p be an odd prime and $q = p^r$, $r \ge 1$. Let $t \in \mathbb{F}_q$. For a positive integer n and $1 \le k \le n$, let $a_k,\ b_k \in \mathbb{Q} \cap \mathbb{Z}_p$. Then the function ${}_nG_n[\cdots]$ is defined by

$$
{}_nG_n\left[\begin{matrix} a_1, & a_2, & \dots, & a_n \\ b_1, & b_2, & \dots, & b_n \end{matrix} \,\middle|\, t\right]_q :=
\frac{-1}{q-1}\sum_{a=0}^{q-2}(-1)^{an}\,\overline{\omega}^a(t) \times \prod_{k=1}^{n}\prod_{i=0}^{r-1}(-p)^{-\lfloor\langle a_k p^i\rangle - \frac{ap^i}{q-1}\rfloor - \lfloor\langle -b_k p^i\rangle + \frac{ap^i}{q-1}\rfloor}
\times \frac{\Gamma_p(\langle(a_k - \frac{a}{q-1})p^i\rangle)}{\Gamma_p(\langle a_k p^i\rangle)} \cdot \frac{\Gamma_p(\langle(-b_k + \frac{a}{q-1})p^i\rangle)}{\Gamma_p(\langle -b_k p^i\rangle)}.
$$

Let $\pi \in \mathbb{C}_p$ be the fixed root of $x^{p-1} + p = 0$ which satisfies

$$\pi \equiv \zeta_p - 1 \pmod{(\zeta_p - 1)^2}.$$

Then the Gross–Koblitz formula relates Gauss sums and the p-adic gamma function as follows.

Theorem 2.5 [Gross and Koblitz 1979]. *For $a \in \mathbb{Z}$ and $q = p^r$,*

$$g(\overline{\omega}^a) = -\pi^{(p-1)\sum_{i=0}^{r-1}\langle\frac{ap^i}{q-1}\rangle}\prod_{i=0}^{r-1}\Gamma_p\Big(\langle\frac{ap^i}{q-1}\rangle\Big).$$

The following lemma relates products of values of p-adic gamma function.

Lemma 2.6 [Barman and Saikia 2014, Lemma 3.1]. *Let p be a prime and $q = p^r$. For $0 \le a \le q-2$ and $t \ge 1$ with $p \nmid t$, we have*

$$\omega(t^{-ta})\prod_{i=0}^{r-1}\Gamma_p\Big(\langle\frac{-tp^i a}{q-1}\rangle\Big)\prod_{h=1}^{t-1}\Gamma_p\Big(\langle\frac{hp^i}{t}\rangle\Big) = \prod_{i=0}^{r-1}\prod_{h=0}^{t-1}\Gamma_p\Big(\langle\frac{p^i(1+h)}{t} - \frac{p^i a}{q-1}\rangle\Big).$$

We now prove a lemma that will be used to prove our results.

Lemma 2.7. *Let p be an odd prime and $q = p^r$. Then for $0 \le a \le q-2$ and $0 \le i \le r-1$ we have*

$$-\left\lfloor\frac{-4ap^i}{q-1}\right\rfloor + \left\lfloor\frac{-2ap^i}{q-1}\right\rfloor = -\left\lfloor\left\langle\frac{p^i}{4}\right\rangle - \frac{ap^i}{q-1}\right\rfloor - \left\lfloor\left\langle\frac{3p^i}{4}\right\rangle - \frac{ap^i}{q-1}\right\rfloor. \tag{2-5}$$

Proof. Let

$$\left\lfloor \frac{-4ap^i}{q-1} \right\rfloor = 4k+s,$$

where $k, s \in \mathbb{Z}$ satisfy $0 \le s \le 3$. Then

$$4k+s \le \frac{-4ap^i}{q-1} < 4k+s+1. \tag{2-6}$$

If $p^i \equiv 1 \pmod 4$, then (2-6) yields

$$\left\lfloor \frac{-2ap^i}{q-1} \right\rfloor = \begin{cases} 2k & \text{if } s = 0,\, 1; \\ 2k+1 & \text{if } s = 2,\, 3, \end{cases} \tag{2-7}$$

$$\left\lfloor \left\langle \frac{p^i}{4} \right\rangle - \frac{ap^i}{q-1} \right\rfloor = \begin{cases} k & \text{if } s = 0,\, 1,\, 2; \\ k+1 & \text{if } s = 3, \end{cases} \tag{2-8}$$

$$\left\lfloor \left\langle \frac{3p^i}{4} \right\rangle - \frac{ap^i}{q-1} \right\rfloor = \begin{cases} k & \text{if } s = 0; \\ k+1 & \text{if } s = 1,\, 2,\, 3. \end{cases} \tag{2-9}$$

Putting the above values for different values of s we readily obtain (2-5). The proof of (2-5) is similar when $p^i \equiv 3 \pmod 4$. □

3. Proofs of the main results

We first prove two propositions which enable us to express certain character sums in terms of the p-adic hypergeometric series.

Proposition 3.1. *Let p be an odd prime and $x \in \mathbb{F}_q^\times$. Then we have*

$$\begin{aligned} \sum_{y \in \mathbb{F}_q} \varphi(y)\varphi(1-2y+xy^2) &= \varphi(2x) + \frac{q^2\varphi(-2)}{q-1} \sum_{\chi \in \widehat{\mathbb{F}_q^\times}} \binom{\varphi\chi^2}{\chi}\binom{\varphi\chi}{\chi}\chi\left(\frac{x}{4}\right) \\ &= -\varphi(-2)\,{}_2G_2\left[\begin{matrix} \frac{1}{4}, & \frac{3}{4} \\ 0, & 0 \end{matrix} \,\Big|\, \frac{1}{x}\right]_q. \end{aligned}$$

Proof. Applying (2-3) and then (1-1) we have

$$\begin{aligned}\sum_{\chi\in\widehat{\mathbb{F}_q^\times}}\binom{\varphi\chi^2}{\chi}\binom{\varphi\chi}{\chi}\chi\Big(\frac{x}{4}\Big) &= \sum_{\chi\in\widehat{\mathbb{F}_q^\times}}\binom{\varphi\chi}{\chi}\chi\Big(\frac{x}{4}\Big)\binom{\varphi\chi^2}{\varphi\chi}\\ &= \frac{\varphi(-1)}{q}\sum_{\chi\in\widehat{\mathbb{F}_q^\times}}\binom{\varphi\chi}{\chi}\chi\Big(\frac{-x}{4}\Big)J(\varphi\chi^2,\varphi\overline{\chi})\\ &= \frac{\varphi(-1)}{q}\sum_{\substack{\chi\in\widehat{\mathbb{F}_q^\times}\\ y\in\mathbb{F}_q}}\binom{\varphi\chi}{\chi}\chi\Big(\frac{-x}{4}\Big)\varphi\chi^2(y)\varphi\overline{\chi}(1-y)\\ &= \frac{\varphi(-1)}{q}\sum_{\substack{\chi\in\widehat{\mathbb{F}_q^\times}\\ y\in\mathbb{F}_q,y\neq 1}}\varphi(y)\varphi(1-y)\binom{\varphi\chi}{\chi}\chi\Big(-\frac{xy^2}{4(1-y)}\Big).\end{aligned}$$

Now, (2-1) yields

$$\begin{aligned}&\sum_{\chi\in\widehat{\mathbb{F}_q^\times}}\binom{\varphi\chi^2}{\chi}\binom{\varphi\chi}{\chi}\chi\Big(\frac{x}{4}\Big)\\ &\quad=\frac{\varphi(-1)(q-1)}{q^2}\sum_{y\in\mathbb{F}_q,y\neq 1}\varphi(y)\varphi(1-y)\bigg(\varphi\Big(1+\frac{xy^2}{4(1-y)}\Big)-\delta\Big(\frac{xy^2}{4(1-y)}\Big)\bigg)\\ &\quad=\frac{(q-1)\varphi(-1)}{q^2}\sum_{y\in\mathbb{F}_q,y\neq 1}\varphi(y)\varphi(1-y)\varphi\Big(1+\frac{xy^2}{4(1-y)}\Big).\end{aligned}$$

Since p is an odd prime, taking the transformation $y\mapsto 2y$ we get

$$\begin{aligned}&\sum_{\chi\in\widehat{\mathbb{F}_q^\times}}\binom{\varphi\chi^2}{\chi}\binom{\varphi\chi}{\chi}\chi\Big(\frac{x}{4}\Big)\\ &\qquad=\frac{(q-1)\varphi(-2)}{q^2}\sum_{\substack{y\in\mathbb{F}_q\\ y\neq\frac12}}\varphi(y)\varphi(1-2y)\varphi\Big(1+\frac{xy^2}{1-2y}\Big)\\ &\qquad=\frac{(q-1)\varphi(-2)}{q^2}\sum_{\substack{y\in\mathbb{F}_q\\ y\neq\frac12}}\varphi(y)\varphi(1-2y+xy^2)\\ &\qquad=\frac{(q-1)\varphi(-2)}{q^2}\sum_{y\in\mathbb{F}_q}\varphi(y)\varphi(1-2y+xy^2)-\frac{\varphi(-x)(q-1)}{q^2},\end{aligned}$$

from which we readily obtain the first identity of the proposition.

To complete the proof of the proposition, we relate the above character sums to the p-adic hypergeometric series. From (1-1), Lemma 2.2, and then using the facts that $\delta(\chi)=0$ for $\chi\neq\varepsilon$, $\delta(\varepsilon)=1$ and $g(\varepsilon)=-1$, we deduce that

$$\begin{aligned}
A &:= \sum_{\chi\in\widehat{\mathbb{F}_q^\times}} \binom{\varphi\chi^2}{\chi}\binom{\varphi\chi}{\chi}\chi\left(\frac{x}{4}\right)\\
&=\frac{1}{q^2}\sum_{\chi\in\widehat{\mathbb{F}_q^\times}} J(\varphi\chi^2,\overline{\chi})J(\varphi\chi,\overline{\chi})\chi\left(\frac{x}{4}\right)\\
&=\frac{1}{q^2}\sum_{\chi\in\widehat{\mathbb{F}_q^\times}} \frac{g(\varphi\chi^2)g^2(\overline{\chi})}{g(\varphi)}\chi\left(\frac{x}{4}\right)+\frac{q-1}{q^2}\sum_{\chi\in\widehat{\mathbb{F}_q^\times}}\frac{g(\varphi\chi)g(\overline{\chi})}{g(\varphi)}\chi\left(-\frac{x}{4}\right)\delta(\varphi\chi)\\
&=\frac{1}{q^2}\sum_{\chi\in\widehat{\mathbb{F}_q^\times}} \frac{g(\varphi\chi^2)g^2(\overline{\chi})}{g(\varphi)}\chi\left(\frac{x}{4}\right)-\frac{q-1}{q^2}\varphi(-x).
\end{aligned}$$

Now, taking $\chi=\omega^a$ we have

$$A=\frac{1}{q^2}\sum_{a=0}^{q-2}\frac{g(\varphi\omega^{2a})g^2(\overline{\omega}^a)}{g(\varphi)}\omega^a\left(\frac{x}{4}\right)-\frac{q-1}{q^2}\varphi(-x).$$

Using the Davenport–Hasse relation for $m=2$ and $\psi=\omega^{2a}$ we obtain

$$g(\varphi\omega^{2a})=\frac{g(\omega^{4a})\overline{\omega}^{2a}(4)g(\varphi)}{g(\omega^{2a})}.$$

Thus,

$$A=\frac{1}{q^2}\sum_{a=0}^{q-2}\omega^a(x)\overline{\omega}^{3a}(4)\frac{g(\omega^{4a})g^2(\overline{\omega}^a)}{g(\omega^{2a})}-\frac{q-1}{q^2}\varphi(-x).$$

Applying the Gross–Koblitz formula we deduce that

$$A=\frac{1}{q^2}\sum_{a=0}^{q-2}\omega^a(x)\overline{\omega}^{3a}(4)\pi^{(p-1)\alpha}\prod_{i=0}^{r-1}\frac{\Gamma_p\left(\left\langle\frac{-4ap^i}{q-1}\right\rangle\right)\Gamma_p^2\left(\left\langle\frac{ap^i}{q-1}\right\rangle\right)}{\Gamma_p\left(\left\langle\frac{-2ap^i}{q-1}\right\rangle\right)}-\frac{q-1}{q^2}\varphi(-x),$$

where

$$\alpha=\sum_{i=0}^{r-1}\left\{\left\langle\frac{-4ap^i}{q-1}\right\rangle+2\left\langle\frac{ap^i}{q-1}\right\rangle-\left\langle\frac{-2ap^i}{q-1}\right\rangle\right\}.$$

Using Lemma 2.6 for $t = 4$ and $t = 2$, we deduce that

$$A = \frac{1}{q^2}\sum_{a=0}^{q-2}\omega^a(x)\pi^{(p-1)\alpha}\prod_{i=0}^{r-1}\frac{\Gamma_p\big(\big\langle\big(\frac{1}{4}-\frac{a}{q-1}\big)p^i\big\rangle\big)\Gamma_p\big(\big\langle\big(\frac{3}{4}-\frac{a}{q-1}\big)p^i\big\rangle\big)\Gamma_p^2\big(\big\langle\frac{ap^i}{q-1}\big\rangle\big)}{\Gamma_p\big(\big\langle\frac{p^i}{4}\big\rangle\big)\Gamma_p\big(\big\langle\frac{3p^i}{4}\big\rangle\big)} - \frac{q-1}{q^2}\varphi(-x).$$

Finally, using Lemma 2.7 we have

$$A = -\frac{q-1}{q^2}\cdot {}_2G_2\left[\begin{matrix}\frac{1}{4}, & \frac{3}{4}\\ 0, & 0\end{matrix}\,\Big|\,\frac{1}{x}\right]_q - \frac{q-1}{q^2}\varphi(-x). \qquad \square$$

Proposition 3.2. *Let p be an odd prime and $x \in \mathbb{F}_q$. Then, for $x \neq 1$, we have*

$$\sum_{y\in\mathbb{F}_q}\varphi(y)\varphi(1-2y+xy^2) = 2\varphi(x-1) + \frac{q^2}{q-1}\sum_{\chi\in\widehat{\mathbb{F}_q^\times}}\binom{\varphi\chi^2}{\chi}\binom{\varphi\chi}{\chi^2}\chi(x-1)$$
$$= -{}_2G_2\left[\begin{matrix}\frac{1}{4}, & \frac{3}{4}\\ 0, & 0\end{matrix}\,\Big|\,\frac{1}{1-x}\right]_q.$$

Proof. From (1-1) and then using Lemma 2.2, we have

$$\text{(3-1)}\quad \binom{\varphi\chi^2}{\chi}\binom{\varphi\chi}{\chi^2} = \frac{\chi(-1)}{q^2}J(\varphi\chi^2,\overline{\chi})J(\varphi\chi,\overline{\chi}^2)$$
$$= \frac{\chi(-1)}{q^2}\left[\frac{g(\varphi\chi^2)g(\overline{\chi})}{g(\varphi\chi)} + (q-1)\chi(-1)\delta(\varphi\chi)\right] \times\left[\frac{g(\varphi\chi)g(\overline{\chi}^2)}{g(\varphi\overline{\chi})} + (q-1)\delta(\varphi\overline{\chi})\right].$$

From Lemma 2.1, we have $g(\varphi)^2 = q\varphi(-1)$. Since $\delta(\chi) = 0$ for $\chi \neq \varepsilon$, $\delta(\varepsilon) = 1$ and $g(\varepsilon) = -1$, (3-1) yields

$$\text{(3-2)}\quad B := \sum_{\chi\in\widehat{\mathbb{F}_q^\times}}\binom{\varphi\chi^2}{\chi}\binom{\varphi\chi}{\chi^2}\chi(x-1)$$
$$= \frac{1}{q^2}\sum_{\chi\in\widehat{\mathbb{F}_q^\times}}\frac{g(\varphi\chi^2)g(\overline{\chi})g(\overline{\chi}^2)}{g(\varphi\overline{\chi})}\chi(1-x) - 2\frac{q-1}{q^2}\varphi(x-1).$$

Using Lemma 2.2 and then (1-1) we obtain

$$\text{(3-3)}\quad \frac{g(\varphi\chi^2)g(\overline{\chi}^2)}{g(\varphi)} = q\binom{\varphi\chi^2}{\chi^2},$$

and

$$\frac{g(\varphi)g(\overline{\chi})}{g(\varphi\overline{\chi})}=q\chi(-1)\binom{\varphi}{\chi}-(q-1)\chi(-1)\delta(\varphi\overline{\chi}). \tag{3-4}$$

From (2-4), we have $\binom{\varphi}{\varepsilon}=-\frac{1}{q}$. Hence, (3-3) and (3-4) yield

$$\begin{aligned}\frac{1}{q^2}\sum_{\chi\in\widehat{\mathbb{F}_q^\times}}&\frac{g(\varphi\chi^2)g(\overline{\chi})g(\overline{\chi}^2)}{g(\varphi\overline{\chi})}\chi(1-x)\\&=\sum_{\chi\in\widehat{\mathbb{F}_q^\times}}\binom{\varphi\chi^2}{\chi^2}\binom{\varphi}{\chi}\chi(x-1)-\frac{q-1}{q}\sum_{\chi\in\widehat{\mathbb{F}_q^\times}}\chi(x-1)\binom{\varphi\chi^2}{\chi^2}\delta(\varphi\overline{\chi})\\&=\sum_{\chi\in\widehat{\mathbb{F}_q^\times}}\binom{\varphi\chi^2}{\chi^2}\binom{\varphi}{\chi}\chi(x-1)-\frac{q-1}{q}\binom{\varphi}{\varepsilon}\varphi(x-1)\\&=\sum_{\chi\in\widehat{\mathbb{F}_q^\times}}\binom{\varphi\chi^2}{\chi^2}\binom{\varphi}{\chi}\chi(x-1)+\frac{q-1}{q^2}\varphi(x-1).\end{aligned} \tag{3-5}$$

Applying (1-1) on the right-hand side of (3-5), and then (2-2) we have

$$\begin{aligned}&\frac{1}{q^2}\sum_{\chi\in\widehat{\mathbb{F}_q^\times}}\frac{g(\varphi\chi^2)g(\overline{\chi})g(\overline{\chi}^2)}{g(\varphi\overline{\chi})}\chi(1-x)\\&=\frac{1}{q}\sum_{\substack{\chi\in\widehat{\mathbb{F}_q^\times}\\ y\in\mathbb{F}_q}}\binom{\varphi}{\chi}\chi(x-1)\varphi\chi^2(y)\overline{\chi}^2(1-y)+\frac{q-1}{q^2}\varphi(x-1)\\&=\frac{1}{q}\sum_{\substack{\chi\in\widehat{\mathbb{F}_q^\times}\\ y\in\mathbb{F}_q,y\neq1}}\varphi(y)\binom{\varphi}{\chi}\chi\left(\frac{(x-1)y^2}{(1-y)^2}\right)+\frac{q-1}{q^2}\varphi(x-1)\\&=\frac{q-1}{q^2}\sum_{y\in\mathbb{F}_q,y\neq1}\varphi(y)\left[\varphi\left(1+\frac{(x-1)y^2}{(1-y)^2}\right)-\delta\left(\frac{(x-1)y^2}{(1-y)^2}\right)\right]+\frac{q-1}{q^2}\varphi(x-1)\\&=\frac{q-1}{q^2}\sum_{\substack{y\in\mathbb{F}_q\\ y\neq1}}\varphi(y)\varphi(1-2y+xy^2)+\frac{q-1}{q^2}\varphi(x-1).\end{aligned}$$

Adding and subtracting the term under summation for $y=1$, we have

$$\frac{1}{q^2}\sum_{\chi\in\widehat{\mathbb{F}_q^\times}}\frac{g(\varphi\chi^2)g(\overline{\chi})g(\overline{\chi}^2)}{g(\varphi\overline{\chi})}\chi(1-x)=\frac{q-1}{q^2}\sum_{y\in\mathbb{F}_q}\varphi(y)\varphi(1-2y+xy^2). \tag{3-6}$$

Combining (3-2) and (3-6) we readily obtain the first equality of the proposition.

To complete the proof of the proposition, we relate the character sums given in (3-2) to the p-adic hypergeometric series. Using the Davenport–Hasse relation for $m=2,\ \psi=\chi^2$ and $m=2,\ \psi=\overline{\chi}$, we have

$$g(\varphi\chi^2)=\frac{g(\chi^4)g(\varphi)\overline{\chi}^2(4)}{g(\chi^2)} \quad \text{and} \quad g(\varphi\overline{\chi})=\frac{g(\overline{\chi}^2)g(\varphi)\chi(4)}{g(\overline{\chi})},$$

respectively. Plugging these two expressions into (3-2) we obtain

$$B=\frac{1}{q^2}\sum_{\chi\in\widehat{\mathbb{F}_q^\times}}\frac{g(\chi^4)g^2(\overline{\chi})}{g(\chi^2)}\overline{\chi}^3(4)\chi(1-x)-2\frac{(q-1)}{q^2}\varphi(x-1).$$

Now, considering $\chi=\omega^a$ and then applying the Gross–Koblitz formula we obtain

$$B=\frac{1}{q^2}\sum_{a=0}^{q-2}\omega^a(1-x)\,\overline{\omega}^{3a}(4)\pi^{(p-1)\alpha}\prod_{i=0}^{r-1}\frac{\Gamma_p\big(\big\langle\frac{-4ap^i}{q-1}\big\rangle\big)\Gamma_p^2\big(\big\langle\frac{ap^i}{q-1}\big\rangle\big)}{\Gamma_p\big(\big\langle\frac{-2ap^i}{q-1}\big\rangle\big)}-2\frac{(q-1)}{q^2}\varphi(x-1),$$

where

$$\alpha=\sum_{i=0}^{r-1}\Big\{\Big\langle\frac{-4ap^i}{q-1}\Big\rangle+2\Big\langle\frac{ap^i}{q-1}\Big\rangle-\Big\langle\frac{-2ap^i}{q-1}\Big\rangle\Big\}.$$

Proceeding in a similar way to that shown in the proof of Proposition 3.1, we deduce:

$$B=-\frac{q-1}{q^2}\cdot {}_2G_2\left[\begin{matrix}\frac14, & \frac34\\ 0, & 0\end{matrix}\;\Big|\;\frac{1}{1-x}\right]_q-2\frac{q-1}{q^2}\varphi(x-1). \qquad \square$$

Before we prove our main results, we now recall the following definition of a finite field hypergeometric function introduced by McCarthy [2012c].

Definition 3.3 [McCarthy 2012c, Definition 1.4]. Let $A_0,A_1,\dots,A_n,B_1,B_2,\dots,B_n$ be in $\widehat{\mathbb{F}_q^\times}$. Then the ${}_{n+1}F_n(\cdots)^*$ finite field hypergeometric function over $\mathbb{F}_q$ is defined by

$${}_{n+1}F_n\left(\begin{matrix}A_0, & A_1, & \dots, & A_n\\ & B_1, & \dots, & B_n\end{matrix}\;\Big|\;x\right)_q^*=\frac{1}{q-1}\sum_{\chi\in\widehat{\mathbb{F}_q^\times}}\prod_{i=0}^{n}\frac{g(A_i\chi)}{g(A_i)}\prod_{j=1}^{n}\frac{g(\overline{B_j\chi})}{g(\overline{B_j})}g(\overline{\chi})\chi(-1)^{n+1}\chi(x).$$

The following proposition gives a relation between McCarthy's and Greene's finite field hypergeometric functions when certain conditions on the parameters are satisfied.

Proposition 3.4 [McCarthy 2012c, Proposition 2.5]. *If $A_0 \neq \varepsilon$ and $A_i \neq B_i$ for $1 \leq i \leq n$, then*

$$ {}_{n+1}F_n\left(\begin{matrix} A_0, & A_1, & \dots, & A_n \\ & B_1, & \dots, & B_n \end{matrix} \,\Big|\, x\right)^*_q = \left[\prod_{i=1}^{n} \binom{A_i}{B_i}^{-1}\right] {}_{n+1}F_n\left(\begin{matrix} A_0, & A_1, & \dots, & A_n \\ & B_1, & \dots, & B_n \end{matrix} \,\Big|\, x\right)_q. $$

McCarthy [2013, Lemma 3.3] proved a relation between ${}_{n+1}F_n(\cdots)^*$ and the p-adic hypergeometric series ${}_nG_n[\cdots]$. We note that the relation is true for $\mathbb{F}_q$ though it was proved for $\mathbb{F}_p$ in [McCarthy 2013]. Hence, we obtain a relation between ${}_nG_n[\cdots]$ and the Greene's finite field hypergeometric functions due to Proposition 3.4. In the following proposition, we list three such identities which will be used to prove our main results.

Proposition 3.5. *Let $x \neq 0$. Then*

$$ {}_2G_2\left[\begin{matrix} \frac14, & \frac34 \\ 0, & 0 \end{matrix} \,\Big|\, x\right]_q = -q \cdot {}_2F_1\left(\begin{matrix} \chi_4, & \chi_4^3 \\ & \varepsilon \end{matrix} \,\Big|\, \frac{1}{x}\right)_q; \tag{3-7} $$

$$ {}_2G_2\left[\begin{matrix} \frac12, & \frac12 \\ 0, & 0 \end{matrix} \,\Big|\, x\right]_q = -q \cdot {}_2F_1\left(\begin{matrix} \varphi, & \varphi \\ & \varepsilon \end{matrix} \,\Big|\, \frac{1}{x}\right)_q; \tag{3-8} $$

$$ {}_3G_3\left[\begin{matrix} \frac12, & \frac12, & \frac12 \\ 0, & 0, & 0 \end{matrix} \,\Big|\, x\right]_q = q^2 \cdot {}_3F_2\left(\begin{matrix} \varphi, & \varphi, & \varphi \\ & \varepsilon, & \varepsilon \end{matrix} \,\Big|\, \frac{1}{x}\right)_q. \tag{3-9} $$

We note that (3-7) is valid when $q \equiv 1 \pmod 4$.

Proof. Applying [McCarthy 2013, Lemma 3.3] we have

$$ {}_2F_1\left(\begin{matrix} \chi_4, & \chi_4^3 \\ & \varepsilon \end{matrix} \,\Big|\, \frac{1}{x}\right)^*_q = {}_2G_2\left[\begin{matrix} \frac14, & \frac34 \\ 0, & 0 \end{matrix} \,\Big|\, x\right]_q. \tag{3-10} $$

From (2-4), we have $\binom{\chi_4^3}{\varepsilon} = \frac{-1}{q}$. Using this value and Proposition 3.4 we find that

$$ {}_2F_1\left(\begin{matrix} \chi_4, & \chi_4^3 \\ & \varepsilon \end{matrix} \,\Big|\, \frac{1}{x}\right)_q = -\frac{1}{q} {}_2F_1\left(\begin{matrix} \chi_4, & \chi_4^3 \\ & \varepsilon \end{matrix} \,\Big|\, \frac{1}{x}\right)^*_q. \tag{3-11} $$

Now, combining (3-10) and (3-11) we readily obtain (3-7). Proceeding similarly we deduce (3-8) and (3-9). This completes the proof. □

We now prove our main results.

Proof of Theorem 1.1. From Proposition 3.1 and Proposition 3.2 we have

$$ \sum_{y \in \mathbb{F}_q} \varphi(y)\varphi(1-2y+xy^2) = -\varphi(-2) \cdot {}_2G_2\left[\begin{matrix} \frac14, & \frac34 \\ 0, & 0 \end{matrix} \,\Big|\, \frac{1}{x}\right]_q = -{}_2G_2\left[\begin{matrix} \frac14, & \frac34 \\ 0, & 0 \end{matrix} \,\Big|\, \frac{1}{1-x}\right]_q, $$

which readily gives the desired transformation. □

Proof of Theorem 1.3. From [Greene and Stanton 1986, Equation 4.5] we have

$$\varphi\Big(\frac{1-u}{u}\Big){}_3F_2\Big(\begin{matrix}\varphi, & \varphi, & \varphi\\ & \varepsilon, & \varepsilon\end{matrix}\,\Big|\,\frac{u}{u-1}\Big)_p = \varphi(u)f(u)^2+2\frac{\varphi(-1)}{p}f(u)-\frac{p-1}{p^2}\varphi(u)+\frac{p-1}{p^2}\delta(1-u), \tag{3-12}$$

where $u=x/(x-1),\ x\neq 1$ and

$$f(u):=\frac{p}{p-1}\sum_{\chi\in\widehat{\mathbb{F}_p^\times}}\binom{\varphi\chi^2}{\chi}\binom{\varphi\chi}{\chi}\chi\Big(\frac{u}{4}\Big).$$

From (3-9) and (3-12), we have

$$\frac{\varphi((1-u)/u)}{p^2}\cdot{}_3G_3\Big[\begin{matrix}\frac12, & \frac12, & \frac12\\ 0, & 0, & 0\end{matrix}\,\Big|\,\frac{u-1}{u}\Big]_p = \varphi(u)f(u)^2+2\frac{\varphi(-1)}{p}f(u)-\frac{p-1}{p^2}\varphi(u)+\frac{p-1}{p^2}\delta(1-u). \tag{3-13}$$

Now, Proposition 3.1 gives

$$f(u)=\frac{-\varphi(-u)}{p}-\frac{1}{p}\cdot{}_2G_2\Big[\begin{matrix}\frac14, & \frac34\\ 0, & 0\end{matrix}\,\Big|\,\frac{1}{u}\Big]_p. \tag{3-14}$$

Finally, combining (3-13) and (3-14) and then putting $u=\frac{x}{x-1}$ we obtain the desired result. This completes the proof of the theorem. □

Proof of Theorem 1.4. Let $A=B=\varphi$ and $x\neq 0,\pm1$. Then [Greene 1987, Theorem 4.16] yields

$${}_2F_1\Big(\begin{matrix}\varphi, & \varphi\\ & \varepsilon\end{matrix}\,\Big|\,x\Big)_q=\frac{\varphi(-1)}{q}\varphi(x(1+x)) +\varphi(1+x)\frac{q}{q-1}\sum_{\chi\in\widehat{\mathbb{F}_q^\times}}\binom{\varphi\chi^2}{\chi}\binom{\varphi\chi}{\chi}\chi\Big(\frac{x}{(1+x)^2}\Big). \tag{3-15}$$

Now, using Proposition 3.1 we have

$$\sum_{\chi\in\widehat{\mathbb{F}_q^\times}}\binom{\varphi\chi^2}{\chi}\binom{\varphi\chi}{\chi}\chi\Big(\frac{x}{(1+x)^2}\Big) =-\frac{q-1}{q^2}\varphi\Big(\frac{-4x}{(1+x)^2}\Big)-\frac{q-1}{q^2}\cdot{}_2G_2\Big[\begin{matrix}\frac14, & \frac34\\ 0, & 0\end{matrix}\,\Big|\,\frac{(1+x)^2}{4x}\Big]_q. \tag{3-16}$$

Applying Theorem 1.1 on the right-hand side of (3-16) we obtain

$$\sum_{\chi\in\widehat{\mathbb{F}_q^\times}} \binom{\varphi\chi^2}{\chi}\binom{\varphi\chi}{\chi}\chi\Big(\frac{x}{(1+x)^2}\Big) = -\frac{q-1}{q^2}\varphi(-x) - \frac{q-1}{q^2}\varphi(-2)\cdot {}_2G_2\left[\begin{matrix}\frac14, & \frac34\\ 0, & 0\end{matrix}\;\Big|\; \frac{(1+x)^2}{(1-x)^2}\right]_q. \tag{3-17}$$

Combining (3-15) and (3-17) we have

$$ {}_2G_2\left[\begin{matrix}\frac14, & \frac34\\ 0, & 0\end{matrix}\;\Big|\; \frac{(1+x)^2}{(1-x)^2}\right]_q = -q\varphi(-2)\varphi(1+x)\cdot {}_2F_1\left(\begin{matrix}\varphi, & \varphi\\ & \varepsilon\end{matrix}\;\Big|\; x\right)_q, \tag{3-18}$$

which completes the proof of the theorem due to (3-8). □

Proof of Theorem 1.5. Let $q \equiv 1 \pmod 4$. Then we readily obtain the desired transformation for the finite field hypergeometric functions from (1-4) using (3-7) and (3-8). □

4. Special values of ${}_2G_2[\cdots]$

Finding special values of hypergeometric function is an important and interesting problem. Only a few special values of the ${}_nG_n$-functions are known; see for example [Barman et al. 2015]. Therein, we obtained some special values of ${}_nG_n[\cdots]$ when $n = 2, 3, 4$. From (3-18), for any odd prime p and $x \neq 0, \pm 1$, we have

$$ {}_2G_2\left[\begin{matrix}\frac14, & \frac34\\ 0, & 0\end{matrix}\;\Big|\; \frac{(1+x)^2}{(1-x)^2}\right]_q = -q\varphi(-2)\varphi(1+x)\cdot {}_2F_1\left(\begin{matrix}\varphi, & \varphi\\ & \varepsilon\end{matrix}\;\Big|\; x\right)_q. \tag{4-1}$$

Values of the finite field hypergeometric function ${}_2F_1\left(\begin{smallmatrix}\varphi, & \varphi\\ & \varepsilon\end{smallmatrix}\,\big|\, x\right)_q$ are obtained for many values of x. For example, see [Barman and Kalita 2012; 2013a; Evans and Greene 2009b; Greene 1987; Kalita 2018; Ono 1998].

Proof of Theorem 1.6. Let $\lambda \in \{-1, \frac12, 2\}$. If p is an odd prime, then from [Ono 1998, Theorem 2] we have

$$ {}_2F_1\left(\begin{matrix}\varphi, & \varphi\\ & \varepsilon\end{matrix}\;\Big|\; \lambda\right)_p = \begin{cases} 0 & \text{if } p \equiv 3 \pmod 4;\\ \dfrac{2x}{p}(-1)^{\frac{x+y+1}{2}} & \text{if } p \equiv 1 \pmod 4,\ x^2+y^2 = p, \text{ and } x \text{ odd.}\end{cases}$$

Putting the above values for $\lambda = \frac12, 2$ into (4-1) we readily obtain the required values of the ${}_2G_2$-function.

Let $q \equiv 1 \pmod 4$. Then from (3-7) we have

$$ {}_2F_1\left(\begin{matrix}\chi_4, & \chi_4^3\\ & \varepsilon\end{matrix}\;\Big|\; \frac19\right)_q = -\frac1q\, {}_2G_2\left[\begin{matrix}\frac14, & \frac34\\ 0, & 0\end{matrix}\;\Big|\; 9\right]_q.$$

From the above identity we readily obtain the required value of the finite field hypergeometric function. This completes the proof of the theorem. □

Corollary 4.1. *Let $p \equiv 1 \pmod 4$. We have*

$$\binom{\chi_4}{\varphi} + \binom{\chi_4^3}{\varphi} = \frac{2x(-1)^{\frac{x+y+1}{2}}}{p},$$

where $x^2+y^2=p$ and x is odd.

Proof. From Theorem 1.6 and [Barman and Kalita 2013a, Theorem 1.4(i)] we have

$$\binom{\chi_4}{\varphi} + \binom{\chi_4^3}{\varphi} = \frac{2x\varphi(2)\chi_4(-1)(-1)^{\frac{x+y+1}{2}}}{p},$$

where $x^2+y^2=p$ and x is odd. Let m be the order of $\chi \in \widehat{\mathbb{F}_q^\times}$. We know that $\chi(-1)=-1$ if and only if m is even and $(q-1)/m$ is odd. Since $p \equiv 1 \pmod 4$, therefore, either $p \equiv 1 \pmod 8$ or $p \equiv 5 \pmod 8$. If $p \equiv 1 \pmod 8$, then $\varphi(2) = \chi_4(-1) = 1$. Also, if $p \equiv 5 \pmod 8$, then $\varphi(2) = \chi_4(-1) = -1$. Hence, in both the cases, $\varphi(2)\cdot\chi_4(-1)=1$. This completes the proof. □

Proof of Theorem 1.7. From [Kalita 2018, Theorem 1.1], for $q \equiv 1 \pmod 8$, we have

$$(4\text{-}2) \qquad {}_2F_1\left(\begin{matrix}\varphi, & \varphi\\ & \varepsilon\end{matrix} \,\middle|\, \frac{4\sqrt{2}}{2\sqrt{2}\pm 3}\right)_q = \varphi(3\pm 2\sqrt{2})\left\{\binom{\chi_4}{\varphi} + \binom{\chi_4^3}{\varphi}\right\}.$$

Now, comparing (3-18) and (4-2) for $x = 4\sqrt{2}/(2\sqrt{2}\pm 3)$, we obtain (1-5). Similarly, using [Kalita 2018, Theorem 1.1] and (3-18) for $x = 4/(2\pm\sqrt{3})$ we derive (1-6) and (1-7). □

Proof of Theorem 1.8. From (3-7), we have

$$(4\text{-}3) \qquad {}_2F_1\left(\begin{matrix}\chi_4, & \chi_4^3\\ & \varepsilon\end{matrix} \,\middle|\, \left(\frac{-2\sqrt{2}\pm 3}{6\sqrt{2}\pm 3}\right)^2\right)_q = -\frac{1}{q}\cdot {}_2G_2\left[\begin{matrix}\frac14, & \frac34\\ 0, & 0\end{matrix} \,\middle|\, \left(\frac{6\sqrt{2}\pm 3}{-2\sqrt{2}\pm 3}\right)^2\right]_q.$$

Comparing (1-5) and (4-3) we readily obtain (1-8). Again, we have

$$(4\text{-}4) \qquad {}_2F_1\left(\begin{matrix}\chi_4, & \chi_4^3\\ & \varepsilon\end{matrix} \,\middle|\, \left(\frac{-2\pm\sqrt{3}}{6\pm\sqrt{3}}\right)^2\right)_q = -\frac{1}{q}\cdot {}_2G_2\left[\begin{matrix}\frac14, & \frac34\\ 0, & 0\end{matrix} \,\middle|\, \left(\frac{6\pm\sqrt{3}}{-2\pm\sqrt{3}}\right)^2\right]_q.$$

Now, comparing (1-7) and (4-4) we deduce (1-9). □

Applying Corollary 4.1, from (1-5) and (1-8) we have the following corollary.

Corollary 4.2. *Let $p \equiv 1 \pmod 8$. Then*

$${}_2G_2\left[\begin{matrix}\frac14, & \frac34\\ 0, & 0\end{matrix} \,\middle|\, \left(\frac{6\sqrt{2}\pm 3}{-2\sqrt{2}\pm 3}\right)^2\right]_p = -2x\varphi(6\pm 12\sqrt{2})(-1)^{\frac{x+y+1}{2}},$$

where $x^2+y^2=p$ and x is odd.

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