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ON THE STRUCTURE OF HOLOMORPHIC ISOMETRIC EMBEDDINGS OF COMPLEX UNIT BALLS INTO BOUNDED SYMMETRIC DOMAINS

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We study general properties of holomorphic isometric embeddings of complex unit balls $\mathbb{B}^n$ into bounded symmetric domains of rank ≥ 2. In the first part, we study holomorphic isometries from $(\mathbb{B}^n, kg_{\mathbb{B}^n})$ to (Ω, g_Ω) with nonminimal isometric constants k for any irreducible bounded symmetric domain Ω of rank ≥ 2, where g_D denotes the canonical Kähler–Einstein metric on any irreducible bounded symmetric domain D normalized so that minimal disks of D are of constant Gaussian curvature -2. In particular, results concerning the upper bound of the dimension of isometrically embedded $\mathbb{B}^n$ in Ω and the structure of the images of such holomorphic isometries are obtained.

In the second part, we study holomorphic isometries from $(\mathbb{B}^n, g_{\mathbb{B}^n})$ to (Ω, g_Ω) for any irreducible bounded symmetric domains $\Omega \Subset \mathbb{C}^N$ of rank equal to 2 with $2N > N'+1$, where N' is an integer such that $\iota : X_c \hookrightarrow \mathbb{P}^{N'}$ is the minimal embedding (i.e., the first canonical embedding) of the compact dual Hermitian symmetric space X_c of Ω. We completely classify images of all holomorphic isometries from $(\mathbb{B}^n, g_{\mathbb{B}^n})$ to (Ω, g_Ω) for $1 \leq n \leq n_0(\Omega)$, where $n_0(\Omega) := 2N - N' > 1$. In particular, for $1 \leq n \leq n_0(\Omega) - 1$ we prove that any holomorphic isometry from $(\mathbb{B}^n, g_{\mathbb{B}^n})$ to (Ω, g_Ω) extends to some holomorphic isometry from $(\mathbb{B}^{n_0(\Omega)}, g_{\mathbb{B}^{n_0(\Omega)}})$ to (Ω, g_Ω).

1. Introduction

Calabi [1953] studied local holomorphic isometries from Kähler manifolds endowed with real-analytic metrics into complex space forms and their local rigidity. Many results concerning local holomorphic isometric embeddings between bounded symmetric domains were obtained by Mok [2002b; 2011; 2012; 2016] and by Ng [2010; 2011]. In [Chan and Mok 2017], henceforth abbreviated [CM], Mok and the author obtained a general result concerning general properties of the images of holomorphic isometric embeddings from $(\mathbb{B}^n, g_{\mathbb{B}^n})$ to (Ω, g_Ω), where g_D denotes

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the canonical Kähler–Einstein metric on D normalized so that minimal disks of D are of constant Gaussian curvature -2 for any irreducible bounded symmetric domain $D \Subset \mathbb{C}^N$ in its Harish-Chandra realization. In addition, Mok and the author [CM] classified images of all holomorphic isometric embeddings from $(\mathbb{B}^m, g_{\mathbb{B}^m})$ to $(D_n^{\mathrm{IV}}, g_{D_n^{\mathrm{IV}}})$ for $1 \le m \le n-1$ and $n \ge 3$, where D_n^{IV} denotes the type-IV domain (i.e., the Lie ball) of complex dimension n (see Section 2). On the other hand, Xiao and Yuan [2016] and Upmeier, Wang and Zhang [Upmeier et al. 2016] classified all holomorphic isometric embeddings from $(\mathbb{B}^{n-1}, g_{\mathbb{B}^{n-1}})$ to $(D_n^{\mathrm{IV}}, g_{D_n^{\mathrm{IV}}})$, $n \ge 3$, independently with explicit parametrizations. Moreover, Xiao and Yuan [2016, Theorem 1.1] proved that any proper holomorphic map from the complex unit m-ball $\mathbb{B}^m$ to D_n^{IV}, $n \ge 3$ and $m \le n-1$, with certain boundary regularities is a holomorphic isometric embedding provided that the codimension $n-m$ of the image of the m-ball is sufficiently small and $m \ge 4$.

In the present article, we also denote by ds_U^2 the Bergman metric of any bounded domain $U \Subset \mathbb{C}^N$ and we will simply use the term "*holomorphic isometries*" for holomorphic isometric embeddings. In what follows, we will assume that any bounded symmetric domain in a complex Euclidean space is in its Harish-Chandra realization.

Let $f : (\mathbb{B}^n, \lambda' g_{\mathbb{B}^n}) \to (\Omega, g_\Omega)$ be a holomorphic isometry for some positive real constant λ', where Ω is an irreducible bounded symmetric domain. It is well known that any bounded symmetric domain is equivalently a Hermitian symmetric space of the noncompact type and vice versa by the Harish-Chandra embedding theorem; see [Wolf 1972; Mok 1989]. Then, it follows from [CM, Lemma 3] that λ' is a positive integer satisfying $1 \le \lambda' \le r$, where $r := \mathrm{rank}(\Omega)$ is the rank of Ω as a Hermitian symmetric space of the noncompact type. Throughout the present article, we will call λ' the *isometric constant* of any given holomorphic isometry from $(\mathbb{B}^n, \lambda' g_{\mathbb{B}^n})$ to (Ω, g_Ω). In addition, given any holomorphic isometry $F : (\Delta, k ds_\Delta^2) \to (\Delta^p, ds_{\Delta^p}^2)$, we will call k the *isometric constant* of F, where $\Delta \Subset \mathbb{C}$ (resp. $\Delta^p \Subset \mathbb{C}^p$) denotes the open unit disk (resp. open unit polydisk) in the complex plane $\mathbb{C}$ (resp. the complex p-dimensional Euclidean space $\mathbb{C}^p$).

In the present article, we denote by $\widehat{\mathrm{HI}}_k(\mathbb{B}^n, \Omega)$ the space of all holomorphic isometries from $(\mathbb{B}^n, k g_{\mathbb{B}^n})$ to (Ω, g_Ω), where k is any positive integer satisfying $1 \le k \le \mathrm{rank}(\Omega)$. Motivated by [Mok 2016] and [CM], we continue to study the structure of holomorphic isometries from $(\mathbb{B}^n, k g_{\mathbb{B}^n})$ to (Ω, g_Ω) for any irreducible bounded symmetric domain Ω of rank $r \ge 2$ and any positive integer k such that $1 \le k \le r$.

In the first part, we consider the case where $k \ge 2$ is not the minimal isometric constant and obtain a result similar to [CM, Theorem 1] when the isometric constant k is equal to 2. As a corollary of this result, we will also show that given any irreducible bounded symmetric domain Ω of rank at most 3, all holomorphic isometries from $(\mathbb{B}^n, k g_{\mathbb{B}^n})$ to (Ω, g_Ω) arise from linear sections of the minimal embedding of the compact dual Hermitian symmetric space X_c of Ω.

In the second part, the aim is to generalize our results in [CM] for type-IV domains to more general irreducible bounded symmetric domains Ω of rank 2. Let $\Omega \Subset \mathbb{C}^N$ be an irreducible bounded symmetric domain of rank ≥ 2. Mok [2016] proved that if $f : (\mathbb{B}^n, g_{\mathbb{B}^n}) \to (\Omega, g_\Omega)$ is a holomorphic isometry, then $n \leq p(\Omega) + 1$, where $p(\Omega) := p(X_c) = p$ is defined by $c_1(X_c) = (p + 2)\delta$ for the compact dual Hermitian symmetric space X_c of Ω and the positive generator δ of $H^2(X_c, \mathbb{Z}) \cong \mathbb{Z}$; see [Mok 2016] and [CM]. By slicing the complex unit ball $\mathbb{B}^{p(\Omega)+1}$ with affine linear subspaces L of $\mathbb{C}^{p(\Omega)+1}$ such that $L \cap \mathbb{B}^{p(\Omega)+1}$ is nonempty, we obtain many holomorphic isometries in $\widehat{\mathrm{HI}}_1(\mathbb{B}^n, \Omega)$ from any given holomorphic isometry $F \in \widehat{\mathrm{HI}}_1(\mathbb{B}^{p(\Omega)+1}, \Omega)$ for $n \leq p(\Omega)$. It is natural to ask whether all holomorphic isometries in $\widehat{\mathrm{HI}}_1(\mathbb{B}^n, \Omega)$ were obtained in that way for each $n \leq p(\Omega)$. In the case where $\Omega = D^{\mathrm{IV}}_N$ is the type-IV domain for some integer $N \geq 3$, the author and Mok [CM, Theorem 2] have shown that the answer is affirmative. In general, this problem remains open. In [CM], we showed that holomorphic isometries from $(\mathbb{B}^n, g_{\mathbb{B}^n})$ to (Ω, g_Ω) arise from linear sections of the compact dual X_c of Ω, where Ω is an irreducible bounded symmetric domain of rank ≥ 2. In general, we do not know whether this gives any relation between the spaces $\widehat{\mathrm{HI}}_1(\mathbb{B}^n, \Omega)$ and $\widehat{\mathrm{HI}}_1(\mathbb{B}^m, \Omega)$ for $1 \leq n < m \leq p(\Omega) + 1$, except in the case where $\Omega = D^{\mathrm{IV}}_N$, $N \geq 3$, is the type-IV domain; see [CM]. Recall that a type-IV domain is of rank 2. On the other hand, for a rank-r irreducible bounded symmetric domain Ω, any holomorphic isometry from $(\mathbb{B}^n, r g_{\mathbb{B}^n})$ to (Ω, g_Ω) is totally geodesic by the Ahlfors–Schwarz lemma; see [CM, Proposition 1]. In particular, we only need to consider the space $\widehat{\mathrm{HI}}_1(\mathbb{B}^n, \Omega)$ if Ω is of rank 2. Therefore, it is natural to study the problem when the target bounded symmetric domain Ω is of rank 2.

In short, we will generalize the method in [CM] for classifying images of all holomorphic isometries in $\widehat{\mathrm{HI}}_1(\mathbb{B}^n, D^{\mathrm{IV}}_N)$ for $N \geq 3$ and $n \geq 1$ to the study of images of holomorphic isometries in $\widehat{\mathrm{HI}}_1(\mathbb{B}^n, \Omega)$ for $1 \leq n \leq n_0$ and certain irreducible bounded symmetric domains $\Omega \Subset \mathbb{C}^N$ of rank 2, where $n_0 = n_0(\Omega) > 1$ is some integer depending on Ω. One of the key ingredients is the use of the explicit form of the polynomial $h_\Omega(z, z)$, as mentioned in [CM, Remark 1]. On the other hand, the author has found that the relation between $h_\Omega(z, \xi)$ and $\iota|_{\mathbb{C}^N}$ obtained from [Loos 1977] has been written down explicitly by Fang, Huang and Xiao [Fang et al. 2016] for each irreducible bounded symmetric domain Ω, where $\iota : X_c \hookrightarrow \mathbb{P}\big(\Gamma(X_c, \mathcal{O}(1))^*\big) \cong \mathbb{P}^{N'}$ is the minimal embedding, i.e., the first canonical embedding; see [Nakagawa and Takagi 1976]. Here $\mathcal{O}(1)$ is the positive generator of the Picard group $\mathrm{Pic}(X_c) \cong \mathbb{Z}$ of the compact dual X_c of Ω, and $\mathbb{C}^N \subset X_c$ is identified as a dense open subset of X_c by the Harish-Chandra embedding theorem; see [Mok 1989; 2016] and [CM]. In addition, $\Gamma(X_c, \mathcal{O}(1))^*$ denotes the dual of the space $\Gamma(X_c, \mathcal{O}(1))$ of all holomorphic sections of the holomorphic line bundle $\mathcal{O}(1)$ over X_c; see [Mok 2016] and [CM]. We refer the readers to [CM, Section 2.1] for the background of bounded symmetric domains and their compact dual Hermitian symmetric spaces. We will identify $\mathbb{P}\big(\Gamma(X_c, \mathcal{O}(1))^*\big) = \mathbb{P}^{N'}$ and write $N' := \dim_{\mathbb{C}} \mathbb{P}\big(\Gamma(X_c, \mathcal{O}(1))^*\big)$ throughout the present article, where X_c is the compact dual Hermitian symmetric space of the irreducible bounded symmetric domain Ω.

The main results in the first part of the present article are as follows.

Theorem 1.1. *Let $\Omega \Subset \mathbb{C}^N$ be an irreducible bounded symmetric domain of rank ≥ 2 and $\lambda' \geq 2$ be an integer. If $\widehat{\mathrm{HI}}_{\lambda'}(\mathbb{B}^n, \Omega) \neq \varnothing$, then we have $n \leq n_{\lambda'-1}(\Omega)$, where $n_{\lambda'-1}(\Omega)$ is the $(\lambda'-1)$-th null dimension of Ω* (see [Mok 1989, p. 253] and Section 2A).

Theorem 1.2. *Let $\Omega \Subset \mathbb{C}^N$ be an irreducible bounded symmetric domain with* $\mathrm{rank}(\Omega) =: r \geq 2$ *and $f \in \widehat{\mathrm{HI}}_{\lambda'}(\mathbb{B}^n, \Omega)$ for some real constant $\lambda' > 0$. We have the standard embeddings $\Omega \Subset \mathbb{C}^N \subset X_c$ of Ω as a bounded domain and its Borel embedding $\Omega \subset X_c$ as an open subset of its compact dual Hermitian symmetric space X_c* (see [CM, Theorem 1]). *Suppose that either $\lambda' = 2$ or $2 \leq r \leq 3$. Then, $f(\mathbb{B}^n)$ is an irreducible component of $\mathscr{V} := \mathscr{V}' \cap \Omega$ for some affine-algebraic subvariety $\mathscr{V}' \subset \mathbb{C}^N$ such that $\iota(\mathscr{V}) = P \cap \iota(\Omega)$, where $P \subseteq \mathbb{P}^{N'}$ is some projective linear subspace and $\iota : X_c \hookrightarrow \mathbb{P}^{N'}$ is the minimal embedding.*

The main result of the second part is the following.

Theorem 1.3. *Let $\Omega \Subset \mathbb{C}^N$ be an irreducible bounded symmetric domain of rank 2 satisfying $2N > N'+1$, where $N' := \dim_{\mathbb{C}} \mathbb{P}\big(\Gamma(X_c, \mathcal{O}(1))^*\big)$ and X_c is the compact dual Hermitian symmetric space of Ω. Set $n_0(\Omega) := 2N - N'$. For $1 \leq n \leq n_0(\Omega)-1$, if $f : (\mathbb{B}^n, g_{\mathbb{B}^n}) \to (\Omega, g_\Omega)$ is a holomorphic isometric embedding, then $f = F \circ \rho$ for some holomorphic isometric embeddings $F : (\mathbb{B}^{n_0(\Omega)}, g_{\mathbb{B}^{n_0(\Omega)}}) \to (\Omega, g_\Omega)$ and $\rho : (\mathbb{B}^n, g_{\mathbb{B}^n}) \to (\mathbb{B}^{n_0(\Omega)}, g_{\mathbb{B}^{n_0(\Omega)}})$.*

Remark 1.4. (1) Theorem 1.3 actually asserts that any holomorphic isometric embedding $f \in \widehat{\mathrm{HI}}_1(\mathbb{B}^n, \Omega)$, $1 \leq n \leq n_0(\Omega) - 1$, extends to a holomorphic isometric embedding $F \in \widehat{\mathrm{HI}}_1(\mathbb{B}^{n_0(\Omega)}, \Omega)$, where $\Omega \Subset \mathbb{C}^N$ is a rank-2 irreducible bounded symmetric domain satisfying $2N > N' + 1$.

(2) We will show that for such irreducible bounded symmetric domains Ω, we have $n_0(\Omega) = p(\Omega)+1$ only if $\Omega \cong D_N^{\mathrm{IV}}$ is the type-IV domain for some $N \geq 3$. Therefore, one may regard this theorem as a generalization of Theorem 2 in [CM] to holomorphic isometric embeddings from $(\mathbb{B}^n, g_{\mathbb{B}^n})$ to (Ω, g_Ω) for any rank-2 irreducible bounded symmetric domain Ω satisfying $n_0(\Omega) > 1$ and $1 \leq n \leq n_0(\Omega) - 1$.

2. Preliminaries

Denote by $\|\boldsymbol{v}\|_{\mathbb{C}^n}$ the standard complex Euclidean norm of any vector $\boldsymbol{v}$ in $\mathbb{C}^n$. The following lemma is a special case of a well-known result of Calabi [1953, Theorem 2 (local rigidity)]:

Lemma 2.1 [Calabi 1953; Ng 2011, Lemma 3.3]. *Let $g, f : B \to \mathbb{C}^N$ be holomorphic maps defined on some open subset $B \subset \mathbb{C}^n$ such that $\|f(w)\|^2_{\mathbb{C}^N} = \|g(w)\|^2_{\mathbb{C}^N}$ for any $w \in B$. Then, there exists a unitary transformation U in $\mathbb{C}^N$ such that $f = U \circ g$.*

Remark 2.2. Writing $f = (f^1, \ldots, f^N)$ and $g = (g^1, \ldots, g^N)$, there exists an $N \times N$ unitary matrix $\boldsymbol{U}'$ such that

$$\boldsymbol{U}'(f^1(w), \ldots, f^N(w))^T = (g^1(w), \ldots, g^N(w))^T \quad \text{for all } w \in B.$$

Moreover, we have the following fact from linear algebra.

Lemma 2.3 [CM, Lemma 5]. *Let m' and n' be integers such that $1 \le m' < n'$ and let $\boldsymbol{A}'' \in M(m', n'; \mathbb{C})$ be such that $\boldsymbol{A}''\overline{\boldsymbol{A}}''^T = \boldsymbol{I}_{m'}$. Then, there exists $\boldsymbol{U}' \in M(n' - m', n'; \mathbb{C})$ such that*

$$\begin{bmatrix} \boldsymbol{U}' \\ \boldsymbol{A}'' \end{bmatrix} \in U(n').$$

For the complex unit ball $\mathbb{B}^n \subset \mathbb{C}^n$, the Kähler form $\omega_{g_{\mathbb{B}^n}}$ of $(\mathbb{B}^n, g_{\mathbb{B}^n})$ is given by

$$\omega_{g_{\mathbb{B}^n}} = -\sqrt{-1}\partial\bar{\partial}\log(1 - \|w\|^2_{\mathbb{C}^n})$$

so that $(\mathbb{B}^n, g_{\mathbb{B}^n})$ is of constant holomorphic sectional curvature -2. Note that the Bergman metric $K_\Omega(z, \xi)$ of Ω can be expressed as

$$K_\Omega(z, \xi) = \frac{1}{\mathrm{Vol}(\Omega)} h_\Omega(z, \xi)^{-(p(\Omega)+2)},$$

where $\mathrm{Vol}(\Omega)$ is the Euclidean volume of $\Omega \Subset \mathbb{C}^N$, $h_\Omega(z, \xi)$ is some polynomial in $(z, \bar{\xi})$ such that $h_\Omega(z, \mathbf{0}) \equiv 1$ and $p(\Omega)$ is defined as in Section 1. It follows from [CM] that the Kähler form ω_{g_Ω} of (Ω, g_Ω) is given by

$$\omega_{g_\Omega} = -\sqrt{-1}\partial\bar{\partial}\log h_\Omega(z, z)$$

in terms of the Harish-Chandra coordinates $z \in \Omega \Subset \mathbb{C}^N$. The type-IV domain D^{IV}_N, $N \ge 3$, is given by

$$D^{\mathrm{IV}}_N = \left\{z = (z_1, \ldots, z_N) \in \mathbb{C}^N : \textstyle\sum_{j=1}^N |z_j|^2 < 2,\ \sum_{j=1}^N |z_j|^2 < 1 + \left|\frac{1}{2}\sum_{j=1}^N z_j^2\right|^2\right\};$$

see [Mok 1989, p. 83]. Then, the Kähler form $\omega_{g_{D^{\mathrm{IV}}_N}}$ of $(D^{\mathrm{IV}}_N, g_{D^{\mathrm{IV}}_N})$ is given by

$$\omega_{g_{D^{\mathrm{IV}}_N}} = -\sqrt{-1}\partial\bar{\partial}\log\left(1 - \sum_{j=1}^N |z_j|^2 + \left|\frac{1}{2}\sum_{j=1}^N z_j^2\right|^2\right).$$

As mentioned in Section 1, we have the following: for any irreducible bounded symmetric domain $\Omega \Subset \mathbb{C}^N$ of rank $r \ge 2$, we may suppose that the Harish-Chandra coordinates $z = (z_1, \ldots, z_N)$ on $\Omega \Subset \mathbb{C}^N$ are chosen so that there are homogeneous polynomials $G_l(z)$ in z of degree $\deg(G_l)$, $1 \le l \le N'$, such that

(i) $2 \le \deg(G_l) \le r$ for $N+1 \le l \le N'$ and $G_j(z) = z_j$ for $1 \le j \le N$,

(ii) $h_\Omega(z, \xi) = 1 + \sum_{j=1}^{N'} (-1)^{\deg(G_l)} G_l(z)\overline{G_l(\xi)}$ and the restriction of the minimal embedding $\iota : X_c \hookrightarrow \mathbb{P}^{N'}$ to the dense open subset $\mathbb{C}^N \subset X_c$ may be written as
$$\iota(z) = [1, G_1(z), \dots, G_{N'}(z)]$$
in terms of the Harish-Chandra coordinates $z = (z_1, \dots, z_N) \in \mathbb{C}^N$,

(iii) For any integer μ, $2 \le \mu \le r$, there exists l, $N+1 \le l \le N'$, such that $\deg(G_l) = \mu$.

For instance, if $\Omega = D_N^{\mathrm{IV}} \Subset \mathbb{C}^N$, $N \ge 3$, is the type-IV domain, then
$$h_\Omega(z, z) = 1 - \sum_{j=1}^N |z_j|^2 + \left|\frac{1}{2}\sum_{j=1}^N z_j^2\right|^2 \quad \text{and} \quad \iota(z) = \left[z_1, \dots, z_N, 1, \frac{1}{2}\sum_{j=1}^N z_j^2\right]$$
for $z = (z_1, \dots, z_N) \in \mathbb{C}^N$; see [Mok 1989, p. 83]. We refer the readers to [Loos 1977; Fang et al. 2016] for details of the above facts.

Let $f : (\mathbb{B}^n, kg_{\mathbb{B}^n}) \to (\Omega, g_\Omega)$ be a holomorphic isometry such that $f(\mathbf{0}) = \mathbf{0}$, where Ω is an irreducible bounded symmetric domain of rank $r \ge 2$ and k is an integer such that $1 \le k \le r$. Then, we have the functional equation
$$h_\Omega(f(w), f(w)) = (1 - \|w\|^2_{\mathbb{C}^n})^k$$
for $w \in \mathbb{B}^n$; see [Mok 2012] and [CM].

2A. ***On higher-characteristic bundles over irreducible bounded symmetric domains.*** Let $\Omega \Subset \mathbb{C}^N$ be an irreducible bounded symmetric domain of rank r and X_c be the compact dual of Ω. Throughout this section, we follow [Wolf 1972; Mok 1989, pp. 251–253]. We always identify the base point $o \in X_0$ with $\mathbf{0} \in \Omega = \xi^{-1}(X_0)$, where $\xi : \mathfrak{m}^+ \cong \mathbb{C}^N \to G^{\mathbb{C}}/P \cong X_c$ is the embedding defined by $\xi(v) = \exp(v) \cdot P$; see [Wolf 1972; Mok 1989, p. 94]. Let $\Psi = \{\psi_1, \dots, \psi_r\} \subset \Delta_M^+$ be a maximal strongly orthogonal set of noncompact positive roots; see [Wolf 1972]. Then, we have the corresponding root vectors e_{ψ_j}, $1 \le j \le r$. Moreover, we have $\mathfrak{g}_{\psi_j} = \mathbb{C}e_{\psi_j}$ for $1 \le j \le r$ and the maximal polydisk $\Delta^r \cong \Pi \subset \Omega$ is given by $\Pi = \left(\bigoplus_{j=1}^r \mathfrak{g}_{\psi_j}\right) \cap \Omega$; see [Wolf 1972; Mok 2014]. From [Mok 1989, p. 252], for any $v \in \mathfrak{m}^+ \cong T_{\mathbf{0}}(\Omega)$, there exists $k \in \mathfrak{k}$ such that $\mathrm{ad}(k) \cdot v = \sum_{j=1}^r a_j e_{\psi_j}$ with $a_j \in \mathbb{R}$ $(1 \le j \le r)$ and $a_1 \ge \cdots \ge a_r \ge 0$. Then, $\eta = \sum_{j=1}^r a_j e_{\psi_j}$ is said to be the normal form of v and is uniquely determined by v. The cardinality of the set $\{j \in \{1, \dots, r\} : a_j \ne 0\}$ is called the rank of v, which is denoted by $r(v)$. For $1 \le k \le r = \mathrm{rank}(\Omega)$, we define
$$\mathcal{S}_{k,x}(\Omega) := \{[v] \in \mathbb{P}(T_x(\Omega)) : 1 \le r(v) \le k\} \subseteq \mathbb{P}(T_x(\Omega)),$$
called the k-th characteristic projective subvariety at $x \in \Omega$. Then, $\mathcal{S}_k(\Omega) := \bigcup_{x \in \Omega} \mathcal{S}_{k,x}(\Omega) \subset \mathbb{P}T(\Omega)$ is called the k-th characteristic bundle over Ω. We simply

call $\mathcal{S}_x(\Omega) := \mathcal{S}_{1,x}(\Omega)$ the characteristic variety at $x \in \Omega$. From [Mok 1989], $\mathcal{S}_x(\Omega) \subset \mathbb{P}(T_x(\Omega))$ is a connected complex submanifold, while $\mathcal{S}_{k,x}(\Omega) \subset \mathbb{P}(T_x(\Omega))$ is singular for $2 \le k \le r-1$ provided that $r = \mathrm{rank}(\Omega) \ge 3$. In addition, $\mathcal{S}_{r,x}(\Omega) = \mathbb{P}(T_x(\Omega))$ for $x \in \Omega$ and we have the inclusions $\mathcal{S}_{1,x}(\Omega) \subset \cdots \subset \mathcal{S}_{r,x}(\Omega)$. Furthermore, for $r \ge 2$, $k \ge 2$ and $x \in \Omega$, we know $\mathcal{S}_{k,x}(\Omega) \subseteq \mathbb{P}(T_x(\Omega))$ is an irreducible projective subvariety because $\mathcal{S}_{k,x}(\Omega) \setminus \mathcal{S}_{k-1,x}(\Omega) = P \cdot [v]$ is an orbit for any $[v]$ such that $v \in T_x(\Omega) \setminus \{\mathbf{0}\}$ is a rank-k vector, see [Mok 2002a], and $\mathcal{S}_{k,x}(\Omega) \setminus \mathcal{S}_{k-1,x}(\Omega)$ is dense in $\mathcal{S}_{k,x}(\Omega)$.

Proposition 2.4 [Mok 1989, p. 252]. *The k-th characteristic bundle $\mathcal{S}_k(\Omega) \to \Omega$ is holomorphic. In addition, in terms of the Harish-Chandra embedding $\Omega \hookrightarrow \mathbb{C}^N$, $\mathcal{S}_k(\Omega)$ is parallel on Ω in the Euclidean sense; i.e., identifying $\mathbb{P}T(\Omega)$ with $\Omega \times \mathbb{P}^{N-1}$ using the Harish-Chandra coordinates, we have $\mathcal{S}_k(\Omega) \cong \Omega \times \mathcal{S}_{k,\mathbf{0}}(\Omega)$.*

Remark 2.5. For any nonzero vector $v \in T_{\mathbf{0}}(\Omega)$, we let $\mathcal{N}_v := \{\xi \in T_{\mathbf{0}}(\Omega) : R_{v\bar{v}\xi\bar{\xi}}(\Omega, g_\Omega) = 0\}$ be the null space of v. From [Mok 1989], the k-th null dimension of Ω is defined by $n_k(\Omega) := \dim_{\mathbb{C}} \mathcal{N}_v = \dim_{\mathbb{C}} \mathcal{N}_\eta$, where $\eta = \sum_{j=1}^{k} a_j e_{\psi_j}$ ($a_j > 0$ for $1 \le j \le k$) is the normal form of some vector $v \in T_{\mathbf{0}}(\Omega)$ of rank k. Here $n_k(\Omega) := \dim_{\mathbb{C}} \mathcal{N}_v$ only depends on the rank $k = r(v)$ of v. Then, Mok [1989] proved that $\dim_{\mathbb{C}} \mathcal{S}_k(\Omega) = 2N - n_k(\Omega) - 1$. In particular, $\mathcal{S}_{k,x}(\Omega)$ is of dimension $N - n_k(\Omega) - 1$ as an irreducible projective subvariety of $\mathbb{P}(T_x(\Omega))$ for any $x \in \Omega$. Moreover, we have $n(\Omega) := n_1(\Omega) \ge \cdots \ge n_r(\Omega) = 0$ and $n(\Omega)$ is called the null dimension of Ω. From [Mok 1989], we define $p(\Omega) = \dim_{\mathbb{C}} \mathcal{S}_{\mathbf{0}}(\Omega)$. Then, we have $\dim_{\mathbb{C}} \Omega = N = p(\Omega) + n(\Omega) + 1$.

For $x \in \Omega$, under the identification $T_x(\Omega) = T_x(X_c)$, we have $\mathcal{S}_x(\Omega) = \mathscr{C}_x(X_c)$, where $\mathscr{C}_y(X_c) \subset \mathbb{P}(T_y(X_c))$ is the variety of minimal rational tangents (VMRT) of the compact dual X_c of Ω at $y \in X_c$. We define $p(X_c) := \dim_{\mathbb{C}} \mathscr{C}_o(X_c)$ for the base point $o \in X_c$, which is identified with $\mathbf{0} \in \mathfrak{m}^+$, i.e., $\xi(\mathbf{0}) = o \in X_c \cong G^{\mathbb{C}}/P$. For the notion of the VMRTs of Hermitian symmetric spaces of the compact type, we refer the reader to [Hwang and Mok 1999]. Note that $\dim_{\mathbb{C}} \mathscr{C}_y(X_c)$ does not depend on the choice of $y \in X_c$. Then, we have $p(X_c) = p(\Omega) = \dim_{\mathbb{C}} \mathscr{C}_x(X_c)$ for any $x \in \Omega \subset X_c$.

2A1. *Holomorphic sectional curvature.* Let $\Omega \Subset \mathbb{C}^N$ be an irreducible bounded symmetric domain of rank r and X_c be its compact dual Hermitian symmetric space. Recall that g_Ω is the canonical Kähler–Einstein metric on Ω normalized so that minimal disks are of constant Gaussian curvature -2. Then, the Bergman kernel on Ω is given by

$$K_\Omega(z, \xi) = \frac{1}{\mathrm{Vol}(\Omega)} h_\Omega(z, \xi)^{-(p(\Omega)+2)},$$

where $\mathrm{Vol}(\Omega)$ is the Euclidean volume of Ω in $\mathbb{C}^N$, $h_\Omega(z, \xi)$ is a polynomial in $(z, \bar{\xi})$ and $p(\Omega) := p(X_c)$ is the complex dimension of the VMRT of X_c at the base point $o \in X_c$; see [Mok 2016]. For $z \in \Omega \cong G_0/K$, there exists $k \in K$ such that $k \cdot z = \sum_{j=1}^r a_j e_{\psi_j} \in \big(\bigoplus_{j=1}^r \mathfrak{g}_{\psi_j}\big) \cap \Omega = \Pi$ for $|a_j|^2 < 1$, $1 \le j \le r$, and

$$h_\Omega(z, z) = \prod_{j=1}^{r} (1 - |a_j|^2),$$

where r is the rank of the irreducible bounded symmetric domain Ω, $\Pi \cong \Delta^r$ is a maximal polydisk in Ω which satisfies $(\Pi, g_\Omega|_\Pi) \cong \big(\Delta^r, \frac{1}{2} ds^2_{\Delta^r}\big)$; see [Mok 2014]. In particular, it follows from the polydisk theorem, see [Mok 1989, p. 88], that

$$-2 \le R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}(\Omega, g_\Omega) \le -\frac{2}{r}$$

for any unit vector $\alpha \in T_x(\Omega)$ and $x \in \Omega$. Let $x \in \Omega$ and $\beta \in T_x(\Omega)$ be such that $\|\beta\|^2_{g_\Omega} = 1$. If β is of rank $r(\beta) = s$, then we have $R_{\beta\bar{\beta}\beta\bar{\beta}}(\Omega, g_\Omega) \le -2/s$ because there exists $g \in G_0 \cong \mathrm{Aut}_0(\Omega)$ such that $g \cdot \beta \in T_{\mathbf{0}}(\Pi_s)$ for some totally geodesic submanifold $(\Pi_s, g_\Omega|_{\Pi_s}) \subset (\Pi, g_\Omega|_\Pi)$ which is holomorphically isometric to $\big(\Delta^s, \frac{1}{2} ds^2_{\Delta^s}\big)$.

3. On holomorphic isometries of complex unit balls into bounded symmetric domains with nonminimal isometric constants

Let $\Omega \Subset \mathbb{C}^N$ be an irreducible bounded symmetric domain of rank ≥ 2. Mok [2016] studied the space $\widehat{\mathrm{HI}}_1(\mathbb{B}^n, \Omega)$ and provided a sharp upper bound on dimensions of isometrically embedded complex unit balls $(\mathbb{B}^n, g_{\mathbb{B}^n})$ in the irreducible bounded symmetric domain (Ω, g_Ω) equipped with the canonical Kähler–Einstein metric g_Ω. Recall that given any $f \in \widehat{\mathrm{HI}}_k(\mathbb{B}^n, \Omega)$ with $k > 0$ being a real constant, k is a positive integer satisfying $1 \le k \le \mathrm{rank}(\Omega)$; see [CM]. It is natural to ask whether some results in Mok's study [2016] could be generalized to the study of the space $\widehat{\mathrm{HI}}_k(\mathbb{B}^n, \Omega)$ for $k \ge 2$.

In the first part of this section (see Section 3A), we provide an upper bound of n whenever $\widehat{\mathrm{HI}}_k(\mathbb{B}^n, \Omega) \ne \varnothing$, where $k \ge 2$. Note that such an upper bound is not sharp in general. For instance, if $\Omega = D^{\mathrm{I}}_{p,q}$ with $q \ge p \ge 2$ and $k = \mathrm{rank}(\Omega) = p$, then $\widehat{\mathrm{HI}}_k(\mathbb{B}^n, \Omega) \ne \varnothing$ implies $n \le q/p$; see [Koziarz and Maubon 2008, Proposition 3.2]. On the other hand, our general result will imply that $n \le n_{p-1}(D^{\mathrm{I}}_{p,q}) = q - p + 1$ whenever $\widehat{\mathrm{HI}}_p(\mathbb{B}^n, D^{\mathrm{I}}_{p,q}) \ne \varnothing$ with $q \ge p \ge 2$. In the case where $q = 3$ and $p = 2$, we have $n \le 2$ from our general result. But then it follows from [Koziarz and Maubon 2008, Proposition 3.2] that $n = 1$ whenever $\widehat{\mathrm{HI}}_2(\mathbb{B}^n, D^{\mathrm{I}}_{2,3}) \ne \varnothing$. This explains that the upper bound obtained in our general result is not sharp in general. However, one of the applications of our general result is that if Ω satisfies certain conditions and $\widehat{\mathrm{HI}}_k(\mathbb{B}^n, \Omega) \ne \varnothing$ for some fixed real constant $k > 1$, then $n \le p(\Omega)$. In the second part of this section (see Section 3B), we continue our study in [CM] to the study of the space $\widehat{\mathrm{HI}}_2(\mathbb{B}^n, \Omega)$. In particular, we will obtain an analogue of [CM, Theorem 1] for holomorphic isometries in the space $\widehat{\mathrm{HI}}_2(\mathbb{B}^n, \Omega)$ without using the system of functional equations introduced in [Mok 2012].

3A. ***Upper bounds on dimensions of isometrically embedded complex unit balls in an irreducible bounded symmetric domain.*** Let $\Omega \Subset \mathbb{C}^N$ be an irreducible bounded symmetric domain of rank ≥ 2. Motivated by Mok's study [2016], one may continue to study the space $\widehat{\mathrm{HI}}_{\lambda'}(\mathbb{B}^n, \Omega)$ for $\lambda' > 1$. In this section, we study the upper bound on dimensions of isometrically embedded complex unit balls in an irreducible bounded symmetric domain of rank ≥ 2 when the isometric constant is equal to $\lambda' > 1$. It is natural to ask whether the upper bound $p(\Omega)+1$ obtained in [Mok 2016] is optimal in the sense that $n \leq p(\Omega)+1$ whenever $\widehat{\mathrm{HI}}_{\lambda'}(\mathbb{B}^n, \Omega) \neq \varnothing$ for some real constant $\lambda' > 0$. More specifically, we may ask whether $n \leq p(\Omega)$ whenever $\widehat{\mathrm{HI}}_{\lambda'}(\mathbb{B}^n, \Omega) \neq \varnothing$ for some real constant $\lambda' > 1$.

For any given integer $\lambda' \geq 2$, in order to obtain a sharp upper bound of n such that $\widehat{\mathrm{HI}}_{\lambda'}(\mathbb{B}^n, \Omega) \neq \varnothing$, one should construct a holomorphic isometry $f \in \widehat{\mathrm{HI}}_{\lambda'}(\mathbb{B}^{n_0}, \Omega)$ for some integer $n_0 \geq 1$ such that $\widehat{\mathrm{HI}}_{\lambda'}(\mathbb{B}^n, \Omega) \neq \varnothing$ only if $n \leq n_0$. Note that this problem remains unsolved, but we can provide a (rough) upper bound of n by using the k-th characteristic bundle on Ω. More precisely, for any integer λ' satisfying $2 \leq \lambda' \leq \mathrm{rank}(\Omega)$, we prove that if $\widehat{\mathrm{HI}}_{\lambda'}(\mathbb{B}^n, \Omega) \neq \varnothing$, then $n \leq n_{\lambda'-1}(\Omega)$, where $n_k(\Omega)$ is the k-th null dimension of Ω; see [Mok 1989]. This is precisely the assertion of Theorem 1.1. Moreover, for certain irreducible bounded symmetric domains Ω of rank ≥ 2 (including the two irreducible bounded symmetric domains of the exceptional type) we will show that $n \leq p(\Omega)$ whenever $\widehat{\mathrm{HI}}_{\lambda'}(\mathbb{B}^n, \Omega) \neq \varnothing$ for some integer $\lambda' \geq 2$. Now, we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Let $f \in \widehat{\mathrm{HI}}_{\lambda'}(\mathbb{B}^n, \Omega)$ be a holomorphic isometry. Write $S := f(\mathbb{B}^n)$. If $\mathbb{P}(T_y(S)) \cap \mathcal{S}_{\lambda'-1,y}(\Omega) \neq \varnothing$ for some $y \in S$, then there exists a vector $\alpha \in T_y(S) \subset T_y(\Omega)$ of unit length with respect to g_Ω and of rank $k \leq \lambda'-1$ such that

$$R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}(\Omega, g_\Omega) \leq -\frac{2}{k} \leq -\frac{2}{\lambda'-1}$$

(see Section 2A1). But then we have

$$-\frac{2}{\lambda'} = R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}(S, g_\Omega|_S) \leq R_{\alpha\bar{\alpha}\alpha\bar{\alpha}}(\Omega, g_\Omega) \leq -\frac{2}{\lambda'-1}$$

from the Gauss equation, which is a contradiction. Hence, we have $\mathbb{P}(T_y(S)) \cap \mathcal{S}_{\lambda'-1,y}(\Omega) = \varnothing$ for any $y \in S$. Recall from Section 2A that $\mathcal{S}_{\lambda'-1,y}(\Omega) \subseteq \mathbb{P}(T_y(\Omega))$ is an irreducible projective subvariety of complex dimension $N - n_{\lambda'-1}(\Omega) - 1$. Then, it follows from the inequality

$$\dim_{\mathbb{C}}\big(\mathbb{P}(T_y(S)) \cap \mathcal{S}_{\lambda'-1,y}(\Omega)\big) \geq \dim_{\mathbb{C}} \mathbb{P}(T_y(S)) + \dim_{\mathbb{C}} \mathcal{S}_{\lambda'-1,y}(\Omega) - \dim_{\mathbb{C}} \mathbb{P}(T_y(\Omega))$$

that $n \leq n_{\lambda'-1}(\Omega)$; see [Mumford 1976, p. 57]. $\square$

Lemma 3.1. *Let $\Omega \Subset \mathbb{C}^N$ be an irreducible bounded symmetric domain of rank ≥ 2. Then, $n(\Omega) \leq p(\Omega)$ if and only if Ω is biholomorphic to one of the following*:

(1) $D^{\mathrm{I}}_{p',q'}$, *where p' and q' are integers satisfying $2 = p' < q'$ or $p' = q' = 3$.*

(2) D^{II}_m *for some integer m satisfying $5 \leq m \leq 7$.*

(3) D^{IV}_n *for some integer $n \geq 3$.*

(4) D^{V}.

(5) D^{VI}.

Proof. From [Mok 1989, pp. 105–106], we have $n(\Omega) + p(\Omega) + 1 = N$. Then, the result follows from direct computations by the explicit data provided in [Mok 1989, pp. 249–251]. □

Remark 3.2. We observe that if Ω satisfies $n(\Omega) \leq p(\Omega)$, then $\mathrm{rank}(\Omega) \leq 3$. In addition, Lemma 3.1 implies that any irreducible bounded symmetric domain Ω of rank 2 satisfies $n(\Omega) \leq p(\Omega)$. From [Mok 1989], it is clear that the condition $n(\Omega) \leq p(\Omega)$ is equivalent to $\dim_{\mathbb{C}} \mathbb{P}(T_o(X_c)) \leq 2 \cdot \dim_{\mathbb{C}} \mathscr{C}_o(X_c)$, where X_c is the compact dual Hermitian symmetric space of Ω and $o \in X_c$ is a fixed base point.

The following corollary shows that for certain irreducible bounded symmetric domains Ω of rank ≥ 2 and a fixed real constant $\lambda' > 0$, we have $\widehat{\mathrm{HI}}_{\lambda'}(\mathbb{B}^{p(\Omega)+1}, \Omega) \neq \varnothing$ only if $\lambda' = 1$. On the other hand, Mok [2016, Main Theorem] proved that $\widehat{\mathrm{HI}}_1(\mathbb{B}^{p(\Omega)+1}, \Omega) \neq \varnothing$ for any irreducible bounded symmetric domain Ω of rank ≥ 2. Therefore, combining with [Mok 2016, Main Theorem], we actually have $\widehat{\mathrm{HI}}_{\lambda'}(\mathbb{B}^{p(\Omega)+1}, \Omega) \neq \varnothing$ if and only if $\lambda' = 1$ for certain irreducible bounded symmetric domains Ω of rank ≥ 2.

Corollary 3.3. *Let $\Omega \Subset \mathbb{C}^N$ be an irreducible bounded symmetric domain of rank ≥ 2 such that $n(\Omega) \leq p(\Omega)$. If $f \in \widehat{\mathrm{HI}}_{\lambda'}(\mathbb{B}^n, \Omega)$ for some real constant $\lambda' \geq 2$, then $n \leq p(\Omega)$.*

Proof. Note that λ' is an integer satisfying $2 \leq \lambda' \leq \mathrm{rank}(\Omega)$. By the assumption, it follows from Theorem 1.1 that $n \leq n_{\lambda'-1}(\Omega) \leq n(\Omega) \leq p(\Omega)$. □

Remark 3.4. Actually, Corollary 3.3 together with [Mok 2016, Main Theorem] implies that the upper bound $p(\Omega) + 1$ is optimal when the bounded symmetric domain Ω satisfies $n(\Omega) \leq p(\Omega)$. Moreover, the statement of Corollary 3.3 holds true for any irreducible bounded symmetric domain Ω of rank 2.

3A1. *Holomorphic isometries with the maximal isometric constant and applications.* Let $\Omega \Subset \mathbb{C}^N$ be an irreducible bounded symmetric domain of rank $r \geq 2$. Recall that if $f \in \widehat{\mathrm{HI}}_r(\mathbb{B}^n, \Omega)$, then f is totally geodesic by the Ahlfors–Schwarz lemma. The results obtained in Section 3A can be applied so that we may prove $n \leq p(\Omega)$ without using the total geodesy of holomorphic isometries lying in the space $\widehat{\mathrm{HI}}_r(\mathbb{B}^n, \Omega)$.

Proposition 3.5. *Let $\Omega \Subset \mathbb{C}^N$ be an irreducible bounded symmetric domain of rank $r \geq 2$ such that $\Omega \not\cong D_3^{\mathrm{IV}}$ and let $f \in \widehat{\mathrm{HI}}_r(\mathbb{B}^n, \Omega)$. Then, we have $n < p(\Omega)$. If $F \in \widehat{\mathrm{HI}}_r(\mathbb{B}^n, \Omega)$, where Ω is an irreducible bounded symmetric domain of rank $r \geq 2$ and of tube type, then we have $n = 1$.*

Proof. Under the assumptions, Theorem 1.1 asserts that $n \leq n_{r-1}(\Omega)$, so it remains to check that $n_{r-1}(\Omega) < p(\Omega)$ for any irreducible bounded symmetric domain Ω of rank $r \geq 2$ and $\Omega \not\cong D_3^{\mathrm{IV}}$. Note that if $\Omega \cong D_3^{\mathrm{IV}}$, then $r = 2$ and $n_{r-1}(\Omega) = 1 = p(\Omega)$. It follows from [Mok 2002a] that Ω is of tube type if and only if $n_{r-1}(\Omega) = 1$ due to the dimension formula $\dim_{\mathbb{C}} \mathcal{S}_{r-1,x}(\Omega) = \dim_{\mathbb{C}} \mathbb{P}(T_x(\Omega)) - n_{r-1}(\Omega)$ of [Mok 1989]. It is clear that if Ω is of tube type and $\Omega \not\cong D_3^{\mathrm{IV}}$, then $p(\Omega) > 1$ so that $n_{r-1}(\Omega) = 1 < p(\Omega)$. If Ω is not of tube type, then Ω is biholomorphic to one of the following:

(1) $D^{\mathrm{I}}_{p',q'}$ for some integers p', q' satisfying $2 \leq p' < q'$.

(2) D^{II}_{2m+1} for some integer $m \geq 2$.

(3) D^{V}.

From the classification of the boundary components of bounded symmetric domains and the fact that $n_{r-1}(\Omega)$ is precisely the dimension of rank-1 boundary components of Ω, see [Wolf 1972; Mok 2002a, p. 298], we have

$$\begin{aligned} n_{p'-1}(D^{\mathrm{I}}_{p',q'}) &= q' - p' + 1 < p(D^{\mathrm{I}}_{p',q'}) = p' + q' - 2 && \text{for } 2 \leq p' < q', \\ n_{m-1}(D^{\mathrm{II}}_{2m+1}) &= 3 < p(D^{\mathrm{II}}_{2m+1}) = 2(2m-1) && \text{for } m \geq 2, \\ n_1(D^{\mathrm{V}}) &= 5 < p(D^{\mathrm{V}}) = 10. \end{aligned}$$

Hence, we have $n < p(\Omega)$. On the other hand, given an irreducible bounded symmetric domain Ω of rank $r \geq 2$ and of tube type, if $F \in \widehat{\mathrm{HI}}_r(\mathbb{B}^n, \Omega)$, then we have $n \leq n_{r-1}(\Omega) = 1$, i.e., $n = 1$. □

From the proof of Proposition 3.5, we have $n_{r-1}(\Omega) \leq p(\Omega)$ for any irreducible bounded symmetric domain Ω of rank $r \geq 2$. Given any irreducible bounded symmetric domain Ω of rank $r \geq 2$, we define

$$\lambda_0(\Omega) := \min\{\lambda \in \mathbb{Z} : 1 \leq \lambda \leq r,\ n_\lambda(\Omega) \leq p(\Omega)\}.$$

Then, we have $\lambda_0(\Omega) \leq r - 1$. Note that Ω satisfies $n(\Omega) \leq p(\Omega)$ if and only if $\lambda_0(\Omega) = 1$. Combining with Corollary 3.3, we have the following:

Theorem 3.6. *Let $\Omega \Subset \mathbb{C}^N$ be an irreducible bounded symmetric domain of rank $r \geq 2$ and $\lambda' \geq 2$ be an integer. If $\widehat{\mathrm{HI}}_{\lambda'}(\mathbb{B}^n, \Omega) \neq \varnothing$, then $n \leq p(\Omega)$ provided that one of the following holds true*:

(1) *Ω satisfies $n(\Omega) \leq p(\Omega)$.*

(2) *λ' satisfies $\lambda_0(\Omega) + 1 \leq \lambda' \leq r$.*

Proof. If the bounded symmetric domain Ω satisfies $n(\Omega) \le p(\Omega)$, then the result follows from Corollary 3.3. If λ' satisfies $\lambda_0(\Omega) + 1 \le \lambda' \le r$, then we have $n_{\lambda'-1}(\Omega) \le n_{\lambda_0(\Omega)}(\Omega) \le p(\Omega)$. By Theorem 1.1, we have $n \le n_{\lambda'-1}(\Omega) \le p(\Omega)$. $\square$

Remark 3.7. If Ω satisfies $n(\Omega) \le p(\Omega)$, then $\lambda_0(\Omega) = 1$ so that the condition (2) does not provide an additional restriction on the given isometric constant λ'.

In general, let $\Omega \Subset \mathbb{C}^N$ be an irreducible bounded symmetric domain of rank ≥ 2 such that $n(\Omega) > p(\Omega)$. Then, Lemma 3.1 asserts that Ω is biholomorphic to one of the following:

(1) $D^{\mathrm{I}}_{p,q}$ for some integers p, q satisfying $3 \le p \le q$ and $(p, q) \ne (3, 3)$.

(2) D^{II}_m for some integer $m \ge 8$.

(3) D^{III}_m for some integer $m \ge 3$.

In particular, we are able to compute $\lambda_0(\Omega)$ explicitly for each case.

type	Ω	$\lambda_0(\Omega)$
$\mathrm{I}_{p,q}$ $(3 \le p \le q,\ (p,q) \ne (3,3))$	$D^{\mathrm{I}}_{p,q}$	$\left\lceil \frac{1}{2}\left((p+q) - \sqrt{(q-p)^2 + 4(p+q-2)}\right)\right\rceil$
II_m $(m \ge 8)$	D^{II}_m	$\left\lceil \frac{1}{4}\left((2m-1) - \sqrt{16m-31}\right)\right\rceil$
III_m $(m \ge 3)$	D^{III}_m	$\left\lceil \frac{1}{2}\left((2m+1) - \sqrt{8m-7}\right)\right\rceil$

Here $\lceil x \rceil$ denotes the smallest integer greater than or equal to x for any real number x.

Example 3.8. If $\Omega = D^{\mathrm{III}}_7$, then Ω is of rank 7, $n_k(\Omega) = \frac{1}{2}(7-k)(7-k+1)$ and $p(\Omega) = 6$, see [Mok 1989, p. 86, p. 250], so that $\lambda_0(\Omega) = 4 = \mathrm{rank}(\Omega) - 3$. Given any integer λ' satisfying $5 \le \lambda' \le 7$, Theorem 3.6 asserts that $n \le p(\Omega) = 6$ whenever $\widehat{\mathrm{HI}}_{\lambda'}(\mathbb{B}^n, D^{\mathrm{III}}_7) \ne \varnothing$.

In general, by using the expression of $\lambda_0(D^{\mathrm{III}}_{m+2})$ in terms of m for any integer $m \ge 1$ (see the table above), one observes that the sequence

$$\{\mathrm{rank}(D^{\mathrm{III}}_{m+2}) - (\lambda_0(D^{\mathrm{III}}_{m+2}) + 1)\}_{m=1}^{+\infty}$$

is monotonic increasing and $a_m := \mathrm{rank}(D^{\mathrm{III}}_{m+2}) - (\lambda_0(D^{\mathrm{III}}_{m+2}) + 1) \to +\infty$ as $m \to +\infty$. Moreover, $a_m/\mathrm{rank}(D^{\mathrm{III}}_{m+2}) \to 0$ as $m \to +\infty$. That means $\mathrm{rank}(D^{\mathrm{III}}_{m+2})$ grows much faster than a_m as m is increasing. This shows that in general the range of the isometric constants λ' mentioned in condition (2) of Theorem 3.6 is quite restrictive for a rank-r irreducible bounded symmetric domain Ω, $r \ge 2$, such that $n(\Omega) > p(\Omega)$.

3B. ***Holomorphic isometries with the isometric constant equal to 2 and applications.*** Let $\Omega \Subset \mathbb{C}^N$ be an irreducible bounded symmetric domain of rank ≥ 2 and X_c be the compact dual Hermitian symmetric space of Ω. Then, it follows from the observation in Section 2 that the polynomial $h_\Omega(z, z)$ can be written as

$$h_\Omega(z, z) = 1 - \sum_{l=1}^{m_1(\Omega)} |G_l^{(1)}(z)|^2 + \sum_{l'=1}^{m_2(\Omega)} |G_{l'}^{(2)}(z)|^2,$$

where $G_l^{(1)}(z)$, $G_{l'}^{(2)}(z)$ are homogeneous polynomials in z and $m_1(\Omega)$, $m_2(\Omega)$ are positive integers depending on Ω such that

(1) $m_1(\Omega) + m_2(\Omega) = N'$ and $m_1(\Omega) \geq N$,

(2) $\deg(G_l^{(1)})$ $(1 \leq l \leq m_1(\Omega))$ is odd, while $\deg(G_{l'}^{(2)}) \geq 2$ $(1 \leq l' \leq m_2(\Omega))$ is even,

(3) $G_j^{(1)}(z) = z_j$ for $1 \leq j \leq N$,

(4) when Ω is of rank ≥ 3, we have $m_1(\Omega) > N$ and $\deg(G_l^{(1)}) \geq 3$ for $N + 1 \leq l \leq m_1(\Omega)$.

Moreover, in terms of the Harish-Chandra coordinates $z = (z_1, \ldots, z_N) \in \mathbb{C}^N$, the restriction of ι to the dense open subset $\mathbb{C}^N \subset X_c$ may be written as

$$\iota(z_1, \ldots, z_N) = [1, G_1^{(1)}(z), \ldots, G_{m_1(\Omega)}^{(1)}(z), G_1^{(2)}(z), \ldots, G_{m_2(\Omega)}^{(2)}(z)]$$

up to reparametrizations, where $\iota : X_c \hookrightarrow \mathbb{P}^{N'}$ is the minimal embedding.

Remark 3.9. As mentioned in Section 2, the above observation can be obtained from [Loos 1977] and has been written down explicitly by Fang, Huang and Xiao [Fang et al. 2016].

In [CM], we studied images of holomorphic isometries in $\widehat{\mathrm{HI}}_{\lambda'}(\mathbb{B}^n, \Omega)$ when $\lambda' = 1$. However, it is not obvious how the method in [CM] could be generalized to the study of images of holomorphic isometries in $\widehat{\mathrm{HI}}_{\lambda'}(\mathbb{B}^n, \Omega)$ for $\lambda' > 1$ so as to obtain an analogue of Theorem 1 in [CM] for all holomorphic isometries in $\widehat{\mathrm{HI}}_{\lambda'}(\mathbb{B}^n, \Omega)$ and for any $\lambda' > 0$. After that, we observe that the above explicit form of $h_\Omega(z, z)$ is useful for continuing the study of images of holomorphic isometries in $\widehat{\mathrm{HI}}_{\lambda'}(\mathbb{B}^n, \Omega)$ when the isometric constant λ' equals 2. Recall that the case where $\lambda' = 2$ in Theorem 1.2 is exactly an analogue of Theorem 1 in [CM] for all holomorphic isometries in $\widehat{\mathrm{HI}}_2(\mathbb{B}^n, \Omega)$. We are now ready to prove Theorem 1.2 for the case where $\lambda' = 2$.

Proof of Theorem 1.2 for the case where $\lambda' = 2$. Let $f : (\mathbb{B}^n, 2g_{\mathbb{B}^n}) \to (\Omega, g_\Omega)$ be a holomorphic isometric embedding, where $\Omega \Subset \mathbb{C}^N$ is an irreducible bounded symmetric domain of rank ≥ 2. Assume without loss of generality that $f(\mathbf{0}) = \mathbf{0}$.

Then, we have the functional equation

$$
\begin{aligned}
(3\text{-}1)\quad 1-\sum_{l=1}^{m_1(\Omega)}|G_l^{(1)}(f(w))|^2+\sum_{l=1}^{m_2(\Omega)}|G_l^{(2)}(f(w))|^2 \\
=\Bigl(1-\sum_{\mu=1}^{n}|w_\mu|^2\Bigr)^2=1-\sum_{\mu=1}^{n}|\sqrt{2}w_\mu|^2+\sum_{1\le\mu,\mu'\le n}|w_\mu w_{\mu'}|^2
\end{aligned}
$$

for $w\in\mathbb{B}^n$ and the polarized functional equation

$$
\begin{aligned}
(3\text{-}2)\quad 1-\sum_{l=1}^{m_1(\Omega)}G_l^{(1)}(f(w))\overline{G_l^{(1)}(f(\zeta))}+\sum_{l=1}^{m_2(\Omega)}G_l^{(2)}(f(w))\overline{G_l^{(2)}(f(\zeta))} \\
=\Bigl(1-\sum_{\mu=1}^{n}w_\mu\bar{\zeta}_\mu\Bigr)^2
\end{aligned}
$$

for $w,\zeta\in\mathbb{B}^n$; see equation (14) in [CM, p. 688]. We write

$$\sum_{1\le\mu,\mu'\le n}|w_\mu w_{\mu'}|^2=\sum_{l=1}^{m_0}|\Xi_l(w)|^2$$

for some homogeneous polynomials $\Xi_l(w)$ of degree 2 and $m_0:=\frac{1}{2}n(n+1)$. Moreover, we write $\boldsymbol{G}^{(j)}(z)=(G_1^{(j)}(z),\dots,G_{m_j(\Omega)}^{(j)}(z))^T$ for $j=1,2$. Let $N_0:=\max\{n+m_2(\Omega),m_0+m_1(\Omega)\}$. Then, there exists $\boldsymbol{U}\in U(N_0)$ such that

$$
(3\text{-}3)\qquad \boldsymbol{U}\cdot\begin{pmatrix}\sqrt{2}w_1\\ \vdots\\ \sqrt{2}w_n\\ \boldsymbol{G}^{(2)}(f(w))\\ \boldsymbol{0}_{(N_0-n-m_2(\Omega))\times1}\end{pmatrix}=\begin{pmatrix}\Xi_1(w)\\ \vdots\\ \Xi_{m_0}(w)\\ \boldsymbol{G}^{(1)}(f(w))\\ \boldsymbol{0}_{(N_0-m_1(\Omega)-m_0)\times1}\end{pmatrix}
$$

by Lemma 2.1 and (3-1). We write

$$\boldsymbol{U}=\begin{bmatrix}\boldsymbol{U}_1\\ \boldsymbol{U}_2\end{bmatrix}$$

with $\boldsymbol{U}_1\in M(m_0,N_0;\mathbb{C})$ and $\boldsymbol{U}_2\in M(N_0-m_0,N_0;\mathbb{C})$. We also write $\boldsymbol{U}_2=\begin{bmatrix}\boldsymbol{U}_{21} & \boldsymbol{U}_{22}\end{bmatrix}$ with $\boldsymbol{U}_{21}\in M(N_0-m_0,n;\mathbb{C})$ and $\boldsymbol{U}_{22}\in M(N_0-m_0,N_0-n;\mathbb{C})$. Denote by $(Jf)(w)$ the complex Jacobian matrix of the holomorphic map $f:\mathbb{B}^n\to\Omega\Subset\mathbb{C}^N$ at $w\in\mathbb{B}^n$. Recall that $G_j^{(1)}(z)=z_j$ for $1\le j\le N$, $G_l^{(2)}(z)$, $1\le l\le m_2(\Omega)$, are homogeneous polynomials of degree ≥ 2 in z so that $\frac{\partial}{\partial z_j}G_l^{(2)}(z)|_{z=\boldsymbol{0}}=0$ for $1\le j\le N$, $1\le l\le m_2(\Omega)$. In addition, if the rank of Ω is at least 3 so that $m_1(\Omega)>N$, then $G_l^{(1)}(z)$, $N+1\le l\le m_1(\Omega)$, are homogeneous polynomials of degree ≥ 3 in z, so that $\frac{\partial}{\partial z_j}G_l^{(1)}(z)|_{z=\boldsymbol{0}}=0$ for $1\le j\le N$, $N+1\le l\le m_1(\Omega)$. Then, we have

$$
(3\text{-}4)\qquad \overline{(Jf)(\boldsymbol{0})}^T(f^1(w),\dots,f^N(w))^T=2(w_1,\dots,w_n)^T
$$

by differentiating both sides of (3-2) with respect to $\bar{\zeta}_\mu$ at $\zeta = \mathbf{0}$ for each μ, $1 \le \mu \le n$. In addition, $(Jf)(\mathbf{0}) \in M(N, n; \mathbb{C})$ is of rank n. Moreover, from the above settings and (3-3) we have

$$\boldsymbol{U}_{21} \begin{pmatrix} \sqrt{2}w_1 \\ \vdots \\ \sqrt{2}w_n \end{pmatrix} + \boldsymbol{U}_{22} \begin{pmatrix} \boldsymbol{G}^{(2)}(f(w)) \\ \mathbf{0}_{(N_0-n-m_2(\Omega))\times 1} \end{pmatrix} = \begin{pmatrix} \boldsymbol{G}^{(1)}(f(w)) \\ \mathbf{0}_{(N_0-m_1(\Omega)-m_0)\times 1} \end{pmatrix}. \tag{3-5}$$

Differentiating both sides of (3-5) with respect to w_μ at $w = \mathbf{0}$ for each μ, $1 \le \mu \le n$, we obtain

$$\sqrt{2}\boldsymbol{U}_{21} = \begin{pmatrix} (Jf)(\mathbf{0}) \\ \mathbf{0}_{(N_0-m_0-N)\times n} \end{pmatrix}.$$

In addition, by differentiating both sides of (3-4) with respect to w_μ at $w = \mathbf{0}$ for each μ, $1 \le \mu \le n$, we have $\overline{(Jf)(\mathbf{0})}^T (Jf)(\mathbf{0}) = 2\boldsymbol{I}_n$. Therefore, it follows from (3-5) and (3-4) that

$$\left[\begin{bmatrix} \frac{1}{2}(Jf)(\mathbf{0})\overline{(Jf)(\mathbf{0})}^T \\ \mathbf{0}_{(N_0-m_0-N)\times N} \end{bmatrix} \ \boldsymbol{U}_{22}\right] \begin{pmatrix} \boldsymbol{f}(w) \\ \boldsymbol{G}^{(2)}(f(w)) \\ \mathbf{0}_{(N_0-n-m_2(\Omega))\times 1} \end{pmatrix} = \begin{pmatrix} \boldsymbol{G}^{(1)}(f(w)) \\ \mathbf{0}_{(N_0-m_0-m_1(\Omega))\times 1} \end{pmatrix} \tag{3-6}$$

for any $w \in \mathbb{B}^n$, where $\boldsymbol{f}(w) := (f^1(w), \dots, f^N(w))^T$. Writing $\boldsymbol{B} := \big[\widehat{\boldsymbol{U}}_{21} \ \ \boldsymbol{U}_{22}\big]$ with

$$\widehat{\boldsymbol{U}}_{21} = \begin{bmatrix} \frac{1}{2}(Jf)(\mathbf{0})\overline{(Jf)(\mathbf{0})}^T \\ \mathbf{0}_{(N_0-m_0-N)\times N} \end{bmatrix},$$

we define

$$\mathscr{V}' := \left\{ z \in \mathbb{C}^N : \boldsymbol{B} \begin{pmatrix} z^T \\ \boldsymbol{G}^{(2)}(z) \\ \mathbf{0}_{(N_0-n-m_2(\Omega))\times 1} \end{pmatrix} = \begin{pmatrix} \boldsymbol{G}^{(1)}(z) \\ \mathbf{0}_{(N_0-m_0-m_1(\Omega))\times 1} \end{pmatrix} \right\} \tag{3-7}$$

and $\mathscr{V} := \mathscr{V}' \cap \Omega$. Then, we have $f(\mathbb{B}^n) \subseteq \mathscr{V}$ by (3-6). Note that the tangential dimension $\mathrm{tdim}_{\mathbf{0}}\, \mathscr{V}$ of $\mathscr{V}$ at $\mathbf{0}$ is less than or equal to $N - \mathrm{rank}\big(\frac{1}{2}(Jf)(\mathbf{0})\overline{(Jf)(\mathbf{0})}^T - \boldsymbol{I}_N\big)$. Here we refer the readers to [Gunning 1990] for the notion of the tangential dimension $\mathrm{tdim}_x\, V$ of a complex-analytic variety V at a point $x \in V$. From [Zhang 1999, p. 49], we have

$$\mathrm{rank}\big(\tfrac{1}{2}(Jf)(\mathbf{0})\overline{(Jf)(\mathbf{0})}^T - \boldsymbol{I}_N\big) \ge \big|\mathrm{rank}\big(\tfrac{1}{2}(Jf)(\mathbf{0})\overline{(Jf)(\mathbf{0})}^T\big) - \mathrm{rank}\ \boldsymbol{I}_N\big| = N - n.$$

On the other hand, $\big(\frac{1}{2}(Jf)(\mathbf{0})\overline{(Jf)(\mathbf{0})}^T - \boldsymbol{I}_N\big) \cdot (Jf)(\mathbf{0}) = \mathbf{0}$ so that

$$0 \ge \mathrm{rank}\big(\tfrac{1}{2}(Jf)(\mathbf{0})\overline{(Jf)(\mathbf{0})}^T - \boldsymbol{I}_N\big) + \mathrm{rank}(Jf)(\mathbf{0}) - N$$

and thus $\mathrm{rank}\big(\frac{1}{2}(Jf)(\mathbf{0})\overline{(Jf)(\mathbf{0})}^T - \boldsymbol{I}_N\big) \le N - n$. Therefore, we have

$$\mathrm{rank}\big(\tfrac{1}{2}(Jf)(\mathbf{0})\overline{(Jf)(\mathbf{0})}^T - \boldsymbol{I}_N\big) = N - n.$$

Moreover, $\mathscr{V}$ contains $f(\mathbb{B}^n)$ and $\mathbf{0}\in f(\mathbb{B}^n)$, thus $\dim_{\mathbf{0}}\mathscr{V}\geq n\geq \operatorname{tdim}_{\mathbf{0}}\mathscr{V}$. Note that $\dim_{\mathbf{0}}\mathscr{V}\leq \operatorname{tdim}_{\mathbf{0}}\mathscr{V}$; see [Gunning 1990]. Hence, we have $\dim_{\mathbf{0}}\mathscr{V}=\operatorname{tdim}_{\mathbf{0}}\mathscr{V}=n$ and thus $\mathscr{V}$ is smooth at $\mathbf{0}$. Let S be the irreducible component of $\mathscr{V}$ containing $f(\mathbb{B}^n)$. Then, we have $\dim S=n=\dim f(\mathbb{B}^n)$ and thus $S=f(\mathbb{B}^n)$ because both S and $f(\mathbb{B}^n)$ are irreducible complex-analytic subvarieties of $\mathscr{V}$ containing the smooth point $\mathbf{0}\in\mathscr{V}$ of $\mathscr{V}$. In particular, $f(\mathbb{B}^n)$ is the irreducible component of $\mathscr{V}$ containing $\mathbf{0}$. Moreover, it is clear that $\mathscr{V}'\subset\mathbb{C}^N$ is an affine-algebraic subvariety and $\iota(\mathscr{V})=P\cap\iota(\Omega)$, where

$$P:=\left\{[\xi_0,\xi_1,\dots,\xi_{N'}]\in\mathbb{P}^{N'}:\boldsymbol{Bx}=\boldsymbol{y}\right\},\tag{3-8}$$

with

$$\begin{aligned}\boldsymbol{x}&=\left(\xi_1,\dots,\xi_N,\xi_{m_1(\Omega)+1},\dots,\xi_{N'},\mathbf{0}_{1\times(N_0-n-m_2(\Omega))}\right)^T,\\ \boldsymbol{y}&=\left(\xi_1,\dots,\xi_{m_1(\Omega)},\mathbf{0}_{1\times(N_0-m_0-m_1(\Omega))}\right)^T,\end{aligned}$$

is a projective linear subspace of $\mathbb{P}^{N'}$. □

3B1. *On holomorphic isometries from the Poincaré disk into polydisks.* The author [Chan 2016] and Ng [2010] studied the classification problem of all holomorphic isometries from the Poincaré disk into the p-disk with any isometric constant k, $1\leq k\leq p$, and $p\geq 2$. The classification problem remains unsolved when $p\geq 5$. In this section, we consider the structure of images of such holomorphic isometries for $k\leq 2$ and obtain an analogue of Theorem 1.2 when the domain is the Poincaré disk and the target is the p-disk for some $p\geq 2$.

Note that the restriction ϱ of the Segre embedding $\varsigma:(\mathbb{P}^1)^p\hookrightarrow\mathbb{P}^{2^p-1}$ to the dense open subset $\mathbb{C}^p\subset(\mathbb{P}^1)^p$ is given by

$$\varrho(z_1,\dots,z_p)=\varsigma([1,z_1],\dots,[1,z_p])$$

in terms of the standard holomorphic coordinates $z=(z_1,\dots,z_p)\in\mathbb{C}^p$. Here $\mathbb{C}^p$ is identified with its image $\xi(\mathbb{C}^p)$ in $(\mathbb{P}^1)^p$, where the map $\xi:\mathbb{C}^p\hookrightarrow(\mathbb{P}^1)^p$ is defined by $\xi(z_1,\dots,z_p):=([1,z_1],\dots,[1,z_p])$.

Actually, the author [Chan 2016] observed that the following can be proved by the same method as the proof of Theorem 1 in [CM].

Proposition 3.10 [Chan 2016, Proposition 5.2.4]. *Let $f:(\Delta,ds^2_\Delta)\to(\Delta^p,ds^2_{\Delta^p})$ be a holomorphic isometric embedding, where $p\geq 2$ is an integer. Then, $f(\Delta)$ is an irreducible component of $\mathscr{V}\cap\Delta^p$ for some affine-algebraic subvariety $\mathscr{V}\subset\mathbb{C}^p$ such that $\varrho(\mathscr{V}\cap\Delta^p)=\varrho(\Delta^p)\cap P$, where $P\subseteq\mathbb{P}^{2^p-1}$ is a projective linear subspace.*

Similarly, we observe that the method in the proof of Theorem 1.2 is also valid for any holomorphic isometry from $(\Delta,2ds^2_\Delta)$ to $(\Delta^p,ds^2_{\Delta^p})$, where $p\geq 2$.

Proposition 3.11. *Let $f:(\Delta,2ds^2_\Delta)\to(\Delta^p,ds^2_{\Delta^p})$ be a holomorphic isometric embedding, where $p\geq 2$ is an integer. Then, $f(\Delta)$ is an irreducible component* *of $\mathscr{V} \cap \Delta^p$ for some affine-algebraic subvariety $\mathscr{V} \subset \mathbb{C}^p$ such that $\varrho(\mathscr{V} \cap \Delta^p) = \varrho(\Delta^p) \cap P$, where $P \subseteq \mathbb{P}^{2^p-1}$ is a projective linear subspace.*

Proof. Assume without loss of generality that $f(\mathbf{0}) = \mathbf{0}$. Note that

$$\begin{aligned} h_{\Delta^p}(z,z) &= \prod_{j=1}^{p}(1-|z_j|^2) \\ &= 1 - \sum_{n=1}^{\lfloor (p+1)/2 \rfloor} \sum_{1\le i_1<\cdots<i_{2n-1}\le p} |z_{i_1}\cdots z_{i_{2n-1}}|^2 + \sum_{n=1}^{\lfloor p/2 \rfloor} \sum_{1\le j_1<\cdots<j_{2n}\le p} |z_{j_1}\cdots z_{j_{2n}}|^2. \end{aligned}$$

In the proof of Theorem 1.2, we put $n = 1$ and replace the term $\sum_{l=1}^{m_1(\Omega)} |G_l^{(1)}(z)|^2$ (resp. $\sum_{l=1}^{m_2(\Omega)} |G_l^{(2)}(z)|^2$) by

$$\sum_{n=1}^{\lfloor (p+1)/2 \rfloor} \sum_{1\le i_1<\cdots<i_{2n-1}\le p} \left| \prod_{\mu=1}^{2n-1} z_{i_\mu} \right|^2 \quad \left(\text{resp. } \sum_{n=1}^{\lfloor p/2 \rfloor} \sum_{1\le j_1<\cdots<j_{2n}\le p} \left| \prod_{\mu=1}^{2n} z_{j_\mu} \right|^2 \right). \tag{3-9}$$

Indeed, we may define $m_1(\Delta^p)$ and $m_2(\Delta^p)$. Then, we compute $m_1(\Delta^p) = m_2(\Delta^p) + 1 = 2^{p-1}$. In this situation, the integer N_0 defined in the proof of Theorem 1.2 is equal to $m_1(\Delta^p) + 1 = 2^{p-1} + 1$. Then, the result follows directly from the arguments in the proof of Theorem 1.2. □

3B2. *On holomorphic isometries of complex unit balls into irreducible bounded symmetric domains of rank at most* 3. Given an irreducible bounded symmetric domain $\Omega \Subset \mathbb{C}^N$ of rank ≥ 2, it is natural to ask whether all holomorphic isometries in $\widehat{\mathrm{HI}}(\mathbb{B}^n, \Omega)$ arise from linear sections of the minimal embedding of the compact dual X_c of Ω in general. In [CM], we showed that the answer is affirmative for all holomorphic isometries in $\widehat{\mathrm{HI}}_{\lambda'}(\mathbb{B}^n, \Omega)$ whenever $\widehat{\mathrm{HI}}_{\lambda'}(\mathbb{B}^n, \Omega) \neq \varnothing$ and $\lambda' \in \{1, \mathrm{rank}(\Omega)\}$. On the other hand, Theorem 1.2 asserts that the answer is also affirmative for all holomorphic isometries in $\widehat{\mathrm{HI}}_2(\mathbb{B}^n, \Omega)$ whenever $\widehat{\mathrm{HI}}_2(\mathbb{B}^n, \Omega) \neq \varnothing$. In other words, we may prove Theorem 1.2 for the case where $2 \leq \mathrm{rank}(\Omega) \leq 3$ as follows.

Proof of Theorem 1.2 for the case where $2 \leq \mathrm{rank}(\Omega) \leq 3$. Recall that λ' is an integer satisfying $1 \leq \lambda' \leq r$; see [CM, Lemma 3]. If $r = 2$, then $\lambda' = 1$ or $\lambda' = 2$. In the case of $\lambda' = 1$, the result follows from [CM, Theorem 1]. When $\lambda' = 2$, we may suppose that $f(\mathbf{0}) = \mathbf{0}$. Then, f is totally geodesic by [CM, Proposition 1] and $f(\mathbb{B}^n)$ is indeed an affine linear section of Ω in $\mathbb{C}^N$; see [Mok 2012]. Therefore, the result follows when $r = 2$. Now, we suppose that $r = 3$. If $\lambda' = 1$ or $\lambda' = 3$, then the result follows from Proposition 1 and Theorem 1 in [CM]. If $\lambda' = 2$, then the result follows from the proof of Theorem 1.2 for the case where $\lambda' = 2$. □

The proof of Theorem 1.2 is complete.

Remark 3.12. In general, we expect that Theorem 1 in [CM] holds true for any holomorphic isometry from $(\mathbb{B}^n, kg_{\mathbb{B}^n})$ to (Ω, g_Ω) for $1 \le k \le \operatorname{rank}(\Omega)$. Actually, the case where $2 \le \operatorname{rank}(\Omega) \le 3$ in Theorem 1.2 asserts that our expectation is true when Ω is an irreducible bounded symmetric domain of rank at most 3. Moreover, the statement of Theorem 1.2 for the case where $2 \le \operatorname{rank}(\Omega) \le 3$ also holds true for any holomorphic isometry from (Δ, kds^2_Δ) to $(\Delta^p, ds^2_{\Delta^p})$ for any positive integer k and any integer p such that $2 \le p \le 3$. However, for $2 \le p \le 3$ one may make use of Ng's classification of all holomorphic isometries from (Δ, kds^2_Δ) to $(\Delta^p, ds^2_{\Delta^p})$, see [Ng 2010], to prove such an analogue of Theorem 1.2 for the case where $2 \le \operatorname{rank}(\Omega) \le 3$.

On the other hand, when $\Omega \Subset \mathbb{C}^N$ is an irreducible bounded symmetric domain of rank $r \ge 4$, it is not known whether all holomorphic isometries in $\widehat{\mathrm{HI}}_k(\mathbb{B}^n, \Omega)$ arise from linear sections of the minimal embedding of the compact dual X_c of Ω for $3 \le k \le r-1$. In other words, the problem remains open for the space $\widehat{\mathrm{HI}}_k(\mathbb{B}^n, \Omega)$ when Ω is of rank $r \ge 4$ and $3 \le k \le r-1$.

Now, we would like to emphasize the following consequence of both Theorem 3.6 and Theorem 1.2.

Corollary 3.13. *Let $\Omega \Subset \mathbb{C}^N$ be an irreducible bounded symmetric domain of rank ≥ 2 such that $n(\Omega) \le p(\Omega)$. If $f \in \widehat{\mathrm{HI}}_{\lambda'}(\mathbb{B}^n, \Omega)$ for some real constant $\lambda' > 0$, then we have the following*:

(1) $n \le p(\Omega)$ *when* $\lambda' \ge 2$; $n \le p(\Omega)+1$ *when* $\lambda' = 1$.

(2) $f(\mathbb{B}^n)$ *is an irreducible component of some complex-analytic subvariety $\mathscr{V} \subset \Omega$ satisfying $\iota(\mathscr{V}) = P \cap \iota(\Omega)$, where $\iota : X_c \hookrightarrow \mathbb{P}^{N'}$ is the minimal embedding and $P \subseteq \mathbb{P}^{N'}$ is some projective linear subspace.*

Proof. Note that (1) follows from Theorem 3.6 when $\lambda' \ge 2$. On the other hand, (1) follows from Theorem 2 in [Mok 2016] when $\lambda' = 1$. Moreover, (2) follows from Theorem 1.2 because Ω is of rank at most 3 whenever Ω satisfies $n(\Omega) \le p(\Omega)$. □

Remark 3.14. (1) In particular, Corollary 3.13 holds true when Ω is either of type IV or of the exceptional type by Lemma 3.1. From the method used in this section, it is not known whether both parts (1) and (2) of Corollary 3.13 still hold true in general when the assumption $n(\Omega) \le p(\Omega)$ is removed.

(2) Recently, Yuan (personal communication, 2017) pointed out to the author that one may obtain upper bounds on dimensions of isometrically embedded complex unit balls into irreducible bounded symmetric domains Ω of rank ≥ 2 by using the functional equation for any holomorphic isometry $f : (\mathbb{B}^n, kg_{\mathbb{B}^n}) \to (\Omega, g_\Omega)$, $k \ge 2$, with $f(\mathbf{0}) = \mathbf{0}$ and the signature of the sum of squares; see [Xiao and Yuan 2016, Proposition 2.11]. When the target is $\Omega = D^{\mathrm{I}}_{3,4}$, it suffices to consider the case where $k = 2$ and we compute $m_2(D^{\mathrm{I}}_{3,4}) = \binom{3}{2}\binom{4}{2} = 18$ by [Fang et al. 2016] (noting

that $\Omega = D^{\mathrm{I}}_{3,4}$ does not satisfy $n(\Omega) \le p(\Omega)$). Moreover, one may make use of the signature of the sum of squares, see [Xiao and Yuan 2016, Proposition 2.11], to conclude that $\frac{1}{2}n(n+1) \le m_2(D^{\mathrm{I}}_{3,4}) = \binom{3}{2}\binom{4}{2} = 18$, i.e., $n \le 5 = p(D^{\mathrm{I}}_{3,4})$. In other words, combining with the results of the present article, both parts (1) and (2) of Corollary 3.13 hold true for $\Omega = D^{\mathrm{I}}_{3,4}$. Moreover, in general this method does not imply that $n \le p(\Omega)$ if there exists a holomorphic isometry $f : (\mathbb{B}^n, kg_{\mathbb{B}^n}) \to (\Omega, g_\Omega)$ with $k \ge 2$, where Ω is any irreducible bounded symmetric domain of rank ≥ 2.

4. On holomorphic isometries of complex unit balls into certain irreducible bounded symmetric domains of rank 2

4A. ***Characterization of images of holomorphic isometries.*** We start with the following lemma which identifies those irreducible bounded symmetric domains $\Omega \Subset \mathbb{C}^N$ of rank 2 which carry extra properties.

Lemma 4.1. *Let $\Omega \Subset \mathbb{C}^N$ be an irreducible bounded symmetric domain of rank* 2. *Then*, $2N > N' + 1$ *provided that Ω is not biholomorphic to $D^{\mathrm{I}}_{2,q}$ for any $q \ge 5$.*

Proof. The proof follows from direct computation for any irreducible bounded symmetric domain Ω of rank 2 by using results in [Nakagawa and Takagi 1976, p. 663]. Actually, we obtain from that paper the value of $N' := N(1)$ for any irreducible Hermitian symmetric space X_c of the compact type.

Case 1: When Ω is not biholomorphic to any type-I domains $D^{\mathrm{I}}_{2,q}$ for $q \ge 3$, Ω is either biholomorphic to D^{IV}_m (for some $m \ge 3$), D^{II}_5 or D^{V} because of $D^{\mathrm{IV}}_4 \cong D^{\mathrm{I}}_{2,2}$, $D^{\mathrm{IV}}_6 \cong D^{\mathrm{II}}_4$ and $D^{\mathrm{III}}_2 \cong D^{\mathrm{IV}}_3$. If $\Omega \cong D^{\mathrm{IV}}_m$, $m \ge 3$, then it is clear that $2m > N' + 1 = m + 2$. If $\Omega \cong D^{\mathrm{II}}_5$, then $2\dim_{\mathbb{C}} D^{\mathrm{II}}_5 = 20 > N' + 1 = 2^{5-1} = 16$. If $\Omega \cong D^{\mathrm{V}}$, then $2\dim_{\mathbb{C}} D^{\mathrm{V}} = 32 > N' + 1 = 26 + 1 = 27$, where X_c is the compact dual of D^{V}. Thus, any such Ω satisfies the desired property.

Case 2: When $\Omega \cong D^{\mathrm{I}}_{2,q}$ for some $q \ge 3$, we have

$$4q = 2N > N' + 1 = \binom{2+q}{q} = \tfrac{1}{2}(q+1)(q+2)$$

if and only if $0 > q^2 - 5q + 2 = \left(q - \frac{5}{2}\right)^2 - \frac{17}{4}$, which is equivalent to $q = 3$ or $q = 4$ because $q \ge 3$ is an integer and $\left(q - \frac{5}{2}\right)^2 \ge \frac{25}{4} > \frac{17}{4}$ for $q \ge 5$. The result follows. $\square$

Remark 4.2. We consider rank-2 irreducible bounded symmetric domains Ω because the functional equations of holomorphic isometries from $(\mathbb{B}^n, g_{\mathbb{B}^n})$ to (Ω, g_Ω) are similar to those of holomorphic isometries from $(\mathbb{B}^n, g_{\mathbb{B}^n})$ to $(D^{\mathrm{IV}}_m, g_{D^{\mathrm{IV}}_m})$ for $m \ge 3$ under the assumption that the isometries map $\mathbf{0}$ to $\mathbf{0}$. This is related to the study in [CM]. In addition, we will assume that such a bounded symmetric domain Ω satisfies $2 \cdot \dim_{\mathbb{C}} \Omega > N' + 1$.

Let $\Omega \Subset \mathbb{C}^N$ be an irreducible bounded symmetric domain of rank 2 satisfying $2N > N'+1$, where N' is defined in Section 1. Recall that g_Ω is the canonical Kähler–Einstein metric on Ω normalized so that minimal disks are of constant Gaussian curvature -2. In terms of the Harish-Chandra coordinates $z = (z_1, \ldots, z_N) \in \Omega \subset \mathbb{C}^N$, the Kähler form with respect to g_Ω is equal to $\omega_{g_\Omega} = -\sqrt{-1}\partial\bar{\partial} \log h_\Omega(z, z)$, where

$$h_\Omega(z, \xi) = 1 - \sum_{j=1}^{N} z_j \bar{\xi}_j + \sum_{l=1}^{N'-N} \widehat{G}_l(z)\overline{\widehat{G}_l(\xi)}$$

such that each $\widehat{G}_l(z)$ is a homogeneous polynomial of degree 2 in z so that $\widehat{G}_l(\lambda z) = \lambda^2 \widehat{G}_l(z)$ for any $\lambda \in \mathbb{C}^*$. Note that from Section 2, we have $G_{l+N}(z) = \widehat{G}_l(z)$ for $l = 1, \ldots, N' - N$. Write $\boldsymbol{G}(z) := (\widehat{G}_1(z), \ldots, \widehat{G}_{N'-N}(z))^T$. Let n, N and N' be positive integers satisfying $N' - N + n \leq N$. We also let $\boldsymbol{U}' \in M(N - n, N; \mathbb{C})$ be such that $\operatorname{rank}(\boldsymbol{U}') = N - n$. Then, we define

$$W'_{\boldsymbol{U}} := \left\{ z = (z_1, \ldots, z_N) \in \Omega : \boldsymbol{U}' z^T = \begin{pmatrix} \boldsymbol{G}(z) \\ \boldsymbol{0}_{(2N-n-N')\times 1} \end{pmatrix} \right\}. \tag{4-1}$$

The following generalizes the study of $\widehat{\mathrm{HI}}_1(\mathbb{B}^n, D_N^{\mathrm{IV}})$, $N \geq 3$, in [CM]. Moreover, in the following proposition, the reason of assuming $n \leq 2N - N' =: n_0(\Omega)$ is that there is a certain explicitly defined class of complex-analytic subvarieties of Ω which contains the images of all holomorphic isometries $(\mathbb{B}^n, g_{\mathbb{B}^n}) \to (\Omega, g_\Omega)$ up to composing with elements in $\mathrm{Aut}(\Omega)$, and each of them is contained entirely in $W_{\boldsymbol{U}''}$ for some matrix $\boldsymbol{U}'' \in M(N - n_0(\Omega), N; \mathbb{C})$ satisfying $\boldsymbol{U}''\overline{\boldsymbol{U}}''^T = \boldsymbol{I}_{N-n_0(\Omega)}$. We will show that this gives a relation between the spaces $\widehat{\mathrm{HI}}_1(\mathbb{B}^n, \Omega)$, $1 \leq n \leq n_0(\Omega) - 1$, and $\widehat{\mathrm{HI}}_1(\mathbb{B}^{n_0(\Omega)}, \Omega)$.

Proposition 4.3. *Let $\Omega \Subset \mathbb{C}^N$ be an irreducible bounded symmetric domain of rank 2 such that $2N > N' + 1$, where N' is defined in Section 1. Let n be an integer satisfying $1 \leq n \leq 2N - N'$. If $f \in \widehat{\mathrm{HI}}_1(\mathbb{B}^n, \Omega)$, then $\Psi(f(\mathbb{B}^n))$ is the irreducible component of $W'_{\boldsymbol{U}}$ containing $\boldsymbol{0}$ for some matrix $\boldsymbol{U}' \in M(N - n, N; \mathbb{C})$ satisfying $\boldsymbol{U}'\overline{\boldsymbol{U}}'^T = \boldsymbol{I}_{N-n}$ and some $\Psi \in \mathrm{Aut}(\Omega)$ satisfying $\Psi(f(\boldsymbol{0})) = \boldsymbol{0}$. Conversely, given any matrix $\boldsymbol{U}'' \in M(N - n, N; \mathbb{C})$ satisfying $\boldsymbol{U}''\overline{\boldsymbol{U}}''^T = \boldsymbol{I}_{N-n}$, the irreducible component of $W_{\boldsymbol{U}''}$ containing $\boldsymbol{0}$ is the image of some $\tilde{f} \in \widehat{\mathrm{HI}}_1(\mathbb{B}^n, \Omega)$.*

Proof. Let $f \in \widehat{\mathrm{HI}}_1(\mathbb{B}^n, \Omega)$. Assume without loss of generality that $f(\boldsymbol{0}) = \boldsymbol{0}$. Then, we have

$$1 - \sum_{j=1}^{N} |f^j(w)|^2 + \sum_{l=1}^{N'-N} |\widehat{G}_l(f(w))|^2 = 1 - \sum_{l=1}^{n} |w_l|^2.$$

Note that $2N - 1 \geq N'$ and $2N - N' \geq n$. By Lemma 2.1, there exists $\boldsymbol{U} \in U(N)$ such that

$$\boldsymbol{U}(f^1(w), \ldots, f^N(w))^T = (w_1, \ldots, w_n, \boldsymbol{G}(f(w))^T, \boldsymbol{0}_{1\times(2N-n-N')})^T. \tag{4-2}$$

We write $\boldsymbol{U} = [\boldsymbol{A}'\ \boldsymbol{U}']^T$, where $\boldsymbol{U}' \in M(N-n, N; \mathbb{C})$ is a matrix which satisfies $\boldsymbol{U}'\overline{\boldsymbol{U}}'^T = \boldsymbol{I}_{N-n}$. Then, we have $f(\mathbb{B}^n) \subseteq W'_{\boldsymbol{U}}$ by (4-2). It is clear that the Jacobian matrix of $W'_{\boldsymbol{U}}$ at $\mathbf{0}$ is equal to $\boldsymbol{U}'$, which is of full rank $N-n$ so that $W'_{\boldsymbol{U}}$ is smooth at $\mathbf{0}$ and of dimension n at $\mathbf{0}$. Let S be the irreducible component of $W'_{\boldsymbol{U}}$ containing $f(\mathbb{B}^n)$, which also contains $\mathbf{0}$. Then, we have $\dim S = n$. Since both S and $f(\mathbb{B}^n)$ are irreducible complex-analytic subvarieties of Ω, $f(\mathbb{B}^n) \subseteq S$ and $\dim S = \dim f(\mathbb{B}^n) = n$, we have $S = f(\mathbb{B}^n)$. Thus, the irreducible component of $W'_{\boldsymbol{U}}$ containing $\mathbf{0}$ is the image of some holomorphic isometric embedding $f : (\mathbb{B}^n, g_{\mathbb{B}^n}) \to (\Omega, g_\Omega)$.

Conversely, let n be an integer satisfying $1 \le n \le 2N - N'$ and let $\boldsymbol{U}'' \in M(N-n, N; \mathbb{C})$ be a matrix satisfying $\boldsymbol{U}''\overline{\boldsymbol{U}}''^T = \boldsymbol{I}_{N-n}$. By Lemma 2.3, there exists $\boldsymbol{A}'' \in M(n, N; \mathbb{C})$ such that $[\boldsymbol{A}''\ \boldsymbol{U}'']^T \in U(N)$ so that

$$\begin{bmatrix}\boldsymbol{A}''\\ \boldsymbol{U}''\end{bmatrix}\begin{pmatrix}z_1\\ \vdots\\ z_N\end{pmatrix} = \begin{pmatrix}\boldsymbol{w}(z)\\ \boldsymbol{G}(z)\\ \mathbf{0}_{(2N-n-N')\times 1}\end{pmatrix} \quad \text{for all } z = (z_1, \dots, z_n) \in W_{\boldsymbol{U}''}, \tag{4-3}$$

where $\boldsymbol{w}(z) = (w_1(z), \dots, w_n(z))^T := \boldsymbol{A}''(z_1, \dots, z_N)^T$. Note that the Jacobian matrix of $W_{\boldsymbol{U}''}$ at $\mathbf{0}$ is equal to $\boldsymbol{U}''$, which is of full rank $N-n$ so that $W_{\boldsymbol{U}''}$ is smooth at $\mathbf{0}$ and of dimension n at $\mathbf{0}$. Let S' be the irreducible component of $W_{\boldsymbol{U}''}$ containing $\mathbf{0}$. Then, we have $\dim S' = n$. Actually S' is precisely the point set closure of the connected component of $\mathrm{Reg}(W_{\boldsymbol{U}''})$ containing $\mathbf{0}$ in Ω. Denote by $\mathrm{Reg}(S')$ the regular locus of S'. Then, $\mathrm{Reg}(S')$ is a connected complex manifold lying inside Ω and $\mathbf{0} \in \mathrm{Reg}(S')$. Let $\varphi : B(\mathbf{0}) \to \mathrm{Reg}(S')$ be a biholomorphism onto an open neighborhood of $\mathbf{0}$ in $\mathrm{Reg}(S')$ such that $\varphi(\mathbf{0}) = \mathbf{0}$, where $B(\mathbf{0})$ is some open neighborhood of $\mathbf{0}$ in $\mathbb{C}^n$. Here the image $\varphi(B(\mathbf{0}))$ is a germ of complex submanifold of Ω at $\mathbf{0}$, i.e., a complex submanifold of some open neighborhood of $\mathbf{0}$ in Ω. Note that $h_\Omega(z, z) = 1 - \sum_{l=1}^n |w_l(z)|^2$ for any $z \in S'$ and $\zeta = (\zeta_1, \dots, \zeta_n)$ can be regarded as local holomorphic coordinates on $\mathrm{Reg}(S')$ around $\mathbf{0} \in \mathrm{Reg}(S')$. Then, it follows from (4-3) that for $\zeta \in B(\mathbf{0})$, we have

$$h_\Omega(\varphi(\zeta), \varphi(\zeta)) = 1 - \sum_{l=1}^n |w_l(\varphi(\zeta))|^2 \tag{4-4}$$

and $-\log h_\Omega(\varphi(\zeta), \varphi(\zeta)) = -\log\bigl(1 - \sum_{l=1}^n |w_l(\varphi(\zeta))|^2\bigr)$ is a local Kähler potential on $\mathrm{Reg}(S')$ which is the restriction of the Kähler potential on (Ω, g_Ω) to an open neighborhood of $\mathbf{0}$ in $\mathrm{Reg}(S')$. It follows from (4-4) that the germ of S' at $\mathbf{0}$ is the image of a germ of holomorphic isometry $\tilde{f} : (\mathbb{B}^n, g_{\mathbb{B}^n}; \mathbf{0}) \to (\Omega, g_\Omega; \mathbf{0})$. By the extension theorem of [Mok 2012], $\tilde{f}$ extends to a holomorphic isometric embedding $\tilde{f} : (\mathbb{B}^n, g_{\mathbb{B}^n}) \to (\Omega, g_\Omega)$. Since both $\tilde{f}(\mathbb{B}^n)$ and S' are n-dimensional irreducible complex-analytic subvarieties of Ω and $\tilde{f}(B^n(\mathbf{0}, \varepsilon)) \subset \tilde{f}(\mathbb{B}^n) \cap S'$ for some real number $\varepsilon \in (0, 1)$. It follows that $S' = \tilde{f}(\mathbb{B}^n)$. Hence, the irreducible component of $W_{\boldsymbol{U}''}$ containing $\mathbf{0}$ is the image of some holomorphic isometric embedding $\tilde{f} \in \widehat{\mathrm{HI}}_1(\mathbb{B}^n, \Omega)$. $\square$

Remark 4.4. From the proof of Lemma 4.1, we see that Proposition 4.3 precisely holds true for the space $\widehat{\mathrm{HI}}_1(\mathbb{B}^n, \Omega)$ whenever the integer n and the bounded symmetric domain Ω satisfy one of the following:

(1) $\Omega \cong D^{\mathrm{I}}_{2,3},\ 1 \le n \le 3 = p(D^{\mathrm{I}}_{2,3})$.

(2) $\Omega \cong D^{\mathrm{I}}_{2,4},\ 1 \le n \le 2$.

(3) $\Omega \cong D^{\mathrm{II}}_{5},\ 1 \le n \le 5 = p(D^{\mathrm{II}}_{5}) - 1$.

(4) $\Omega \cong D^{\mathrm{IV}}_{m}$ for some integer $m \ge 3,\ 1 \le n \le m - 1 = p(D^{\mathrm{IV}}_{m}) + 1$.

(5) $\Omega \cong D^{\mathrm{V}},\ 1 \le n \le 6$.

Moreover, Proposition 4.3 actually provides the classification of images of all $f \in \widehat{\mathrm{HI}}_1(\Delta, \Omega)$ whenever Ω is a rank-2 irreducible bounded symmetric domain which is not biholomorphic to $D^{\mathrm{I}}_{2,q}$ for any $q \ge 5$. This also solves part of Problem 3 in [Mok and Ng 2009, p. 2645] theoretically. It is expected that there are many incongruent holomorphic isometries in $\widehat{\mathrm{HI}}_1(\Delta, \Omega)$. However, Proposition 4.3 at least provides a source of constructing explicit examples of holomorphic isometries in $\widehat{\mathrm{HI}}_1(\Delta, \Omega)$. In particular, for the case where the target is an irreducible bounded symmetric domain of rank 2, Problem 3 in [Mok and Ng 2009, p. 2645] remains unsolved precisely in the case where the target Ω is $D^{\mathrm{I}}_{2,q}$ for some $q \ge 5$.

4B. ***Proof of Theorem 1.3.*** As we have mentioned in Section 4A, Proposition 4.3 actually gives a relation between the spaces $\widehat{\mathrm{HI}}_1(\mathbb{B}^n, \Omega),\ 1 \le n \le n_0(\Omega) - 1$, and $\widehat{\mathrm{HI}}_1(\mathbb{B}^{n_0(\Omega)}, \Omega)$. In other words, this yields Theorem 1.3.

Proof of Theorem 1.3. We follow the setting in the proof of Proposition 4.3. Assume without loss of generality that $f(\mathbf{0}) = \mathbf{0}$. Note that $N' - N + n < N$ and thus $f(\mathbb{B}^n)$ is the irreducible component of $W'_{\boldsymbol{U}}$ containing $\mathbf{0}$ for some matrix $\boldsymbol{U}' \in M(N-n, N; \mathbb{C})$ satisfying $\boldsymbol{U}'\overline{\boldsymbol{U}}'^T = \boldsymbol{I}_{N-n}$ by Proposition 4.3. Moreover, we have

$$\begin{bmatrix} \boldsymbol{A}' \\ \boldsymbol{U}' \end{bmatrix} (f^1(w), \dots, f^N(w))^T = \big(w_1, \dots, w_n, \boldsymbol{G}(f(w))^T, \mathbf{0}_{1\times(2N-N'-n)}\big)^T$$

for some $\boldsymbol{A}' \in M(n, N; \mathbb{C})$ such that $[\boldsymbol{A}'\ \boldsymbol{U}']^T \in U(N)$ after composing with some element in the isotropy subgroup of $\mathrm{Aut}(\mathbb{B}^n)$ at $\mathbf{0}$ if necessary (by Lemma 2.3). We write

$$\boldsymbol{U}' = \begin{bmatrix} \boldsymbol{U}'_1 \\ \boldsymbol{U}'_2 \end{bmatrix} \quad \text{for some } \boldsymbol{U}'_1 \in M(N' - N, N; \mathbb{C}),\ \boldsymbol{U}'_2 \in M(2N - N' - n, N; \mathbb{C}).$$

Moreover, we have $\boldsymbol{U}'_1(z_1, \dots, z_N)^T = \boldsymbol{G}(z)$ and $\boldsymbol{U}'_1\overline{\boldsymbol{U}}'^T_1 = \boldsymbol{I}_{N'-N}$ for any $z \in W'_{\boldsymbol{U}}$. It follows from Proposition 4.3 that the irreducible component of $W_{\boldsymbol{U}'_1}$ containing $\mathbf{0}$

is the image of some holomorphic isometric embedding $F : (\mathbb{B}^{n_0}, g_{\mathbb{B}^{n_0}}) \to (\Omega, g_\Omega)$, where $n_0 = n_0(\Omega) := 2N - N'$. We may suppose that $F(\mathbf{0}) = \mathbf{0}$ without loss of generality. Since $f(\mathbb{B}^n) \subset \Omega$ is irreducible and $f(\mathbb{B}^n) \subset W_{U_1'}$, we know $S := f(\mathbb{B}^n)$ lies inside the irreducible component $S' := F(\mathbb{B}^{n_0})$ of $W_{U_1'}$ containing $\mathbf{0}$. Since $(S, g_\Omega|_S) \cong (\mathbb{B}^n, g_{\mathbb{B}^n})$ and $(S', g_\Omega|_{S'}) \cong (\mathbb{B}^{n_0}, g_{\mathbb{B}^{n_0}})$ are of constant holomorphic sectional curvature -2, we have $(S, g_\Omega|_S) \subset (S', g_\Omega|_{S'})$ is totally geodesic and the result follows; see the proof of [CM, Theorem 2]. □

Remark 4.5. (1) It follows from Lemma 4.1 that Theorem 1.3 holds true when the pair $(\Omega, n_0(\Omega))$ is one of the following:

(a) $\Omega \cong D^{\mathrm{I}}_{2,3}$, $n_0(\Omega) = 3$.

(b) $\Omega \cong D^{\mathrm{I}}_{2,4}$, $n_0(\Omega) = 2$.

(c) $\Omega \cong D^{\mathrm{II}}_{5}$, $n_0(\Omega) = 5$.

(d) $\Omega \cong D^{\mathrm{IV}}_{m}$ $(m \geq 3)$, $n_0(\Omega) = m - 1$.

(e) $\Omega \cong D^{\mathrm{V}}$, $n_0(\Omega) = 6$.

(2) It is not known whether Theorem 1.3 still holds true when $n_0(\Omega)$ is replaced by $p(\Omega) + 1$ and $\Omega \ncong D^{\mathrm{IV}}_m$ for any integer $m \geq 3$.

(3) For the particular case where $\Omega = D^{\mathrm{I}}_{2,3}$, it follows from [Mok 2016] that if the space $\widehat{\mathrm{HI}}_1(\mathbb{B}^n, D^{\mathrm{I}}_{2,3})$ is nonempty, then $n \leq p(D^{\mathrm{I}}_{2,3}) + 1 = 4$. In this case, it is motivated by our study in the present article to consider the following problem in order to classify all holomorphic isometries in $\widehat{\mathrm{HI}}_1(\mathbb{B}^n, D^{\mathrm{I}}_{2,3})$:

Given any $f \in \widehat{\mathrm{HI}}_1(\mathbb{B}^3, D^{\mathrm{I}}_{2,3})$, can f be factorized as $f = F \circ \rho$ for some $F \in \widehat{\mathrm{HI}}_1(\mathbb{B}^4, D^{\mathrm{I}}_{2,3})$ and $\rho \in \widehat{\mathrm{HI}}_1(\mathbb{B}^3, \mathbb{B}^4)$?

If the problem were solved and the answer were affirmative, then the classification of all holomorphic isometries in $\widehat{\mathrm{HI}}_1(\mathbb{B}^n, D^{\mathrm{I}}_{2,3})$ would be reduced to the uniqueness problem for nonstandard (i.e., not totally geodesic) holomorphic isometries in $\widehat{\mathrm{HI}}_1(\mathbb{B}^4, D^{\mathrm{I}}_{2,3})$ constructed by Mok [2016].

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