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JINGYI CHEN AND YU YUAN

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HAMILTONIAN STATIONARY CONES WITH ISOTROPIC LINKS

JINGYI CHEN AND YU YUAN

In memory of Professor Wei-Yue Ding

We show that any closed oriented immersed Hamiltonian stationary isotropic surface Σ with genus g_Σ in $S^5 \subset \mathbb{C}^3$ is (1) Legendrian and minimal if $g_\Sigma = 0$; (2) either Legendrian or with exactly $2g_\Sigma - 2$ Legendrian points if $g_\Sigma \geq 1$. In general, every compact oriented immersed isotropic submanifold $L^{n-1} \subset S^{2n-1} \subset \mathbb{C}^n$ such that the cone $C(L^{n-1})$ is Hamiltonian stationary must be Legendrian and minimal if its first Betti number is zero. Corresponding results for nonorientable links are also provided.

1. Introduction

In this note we study the problem of when a Hamiltonian stationary cone $C(L)$ with isotropic link L on S^{2n-1} in $\mathbb{C}^n$ becomes special Lagrangian. A submanifold $M \subset \mathbb{C}^n$, not necessarily a Lagrangian submanifold, is *Hamiltonian stationary* if

$$\operatorname{div}_M(JH) = 0,$$

where J is the complex structure in $\mathbb{C}^n$ and H is the mean curvature vector of M in $\mathbb{C}^n$. In fact this is the variational equation of the volume of M, when one makes an arbitrary deformation $J\nabla_M\varphi$ with $\varphi \in C_0^\infty(M)$ for M:

$$\int_M \langle H, J\nabla_M\varphi\rangle = \int_M \varphi \operatorname{div}_M(JH) - \operatorname{div}_M(\varphi JH) = \int_M \varphi \operatorname{div}_M(JH).$$

The notion of Hamiltonian stationary Lagrangian submanifolds in a Kähler manifold was introduced in [Oh 1993] as critical points of the volume functional under Hamiltonian variations (known to A. Weinstein, as noted there). Chen and Morvan [1994] generalized it to the isotropic deformations.

As in [Harvey and Lawson 1982], a submanifold M in $\mathbb{C}^n$ is *isotropic* at $p \in M$ if

$$J(T_pM) \perp T_pM,$$

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and it is isotropic if it is isotropic for every p. A submanifold M being isotropic is equivalent to the standard symplectic 2-form on $\mathbb{R}^{2n}$ vanishing on M. The dimension of an isotropic submanifold is at most n, the half real dimension of $\mathbb{C}^n$, and when it is n, the submanifold is Lagrangian.

For an immersed $(n-1)$-dimensional submanifold L in the unit sphere S^{2n-1}, let $u : L \to S^{2n-1}$ be the restriction of the coordinate functions in $\mathbb{R}^{2n}$ to L. A point $u \in L$ is *Legendrian* if $T_u L$ is isotropic in $\mathbb{R}^{2n}$ and

$$J(T_u L) \perp u.$$

L is Legendrian if all the points u are Legendrian. This is equivalent to L being an $(n-1)$-dimensional integral submanifold of the standard contact distribution on S^{2n-1}. The cone

$$C(L) = \{rx : r \geq 0, x \in L\}$$

is said to have *link* L. In this article, all links $L$are assumed to be connected, and we shall use Σ for the 2-dimensional link L.

The Hamiltonian stationary condition is a third-order constraint on the submanifold M, as seen when M is locally written as a graph over its tangent space at a point. The minimal submanifolds, a second-order constraint on the local graphical representation of M, are automatically Hamiltonian stationary. We are particularly interested in the case when M is a Lagrangian submanifold. The existence of (many) compact Hamiltonian stationary Lagrangian submanifolds in $\mathbb{C}^n$ versus the nonexistence of compact minimal submanifolds makes the study of Hamiltonian stationary ones interesting. In this note, we shall not be concerned with the existence of Hamiltonian stationary ones; instead, we shall concentrate on the rigidity property, namely, when the Hamiltonian stationary ones reduce to special Lagrangians, in the case when the submanifold is a cone over a spherical link in $\mathbb{C}^n$.

A well-known fact about a link $L^m \subset S^n$ and the cone $C(L)$ over it is that L is minimal in S^n if and only if $C(L) \setminus \{0\}$ is minimal in $\mathbb{R}^{n+1}$. When $C(L)$ is Hamiltonian stationary and isotropic, possibly away from the cone vertex $0 \in \mathbb{R}^{2n}$, we observe that the Hamiltonian stationary equation for $C(L)$ splits into two equations:

$$\operatorname{div}_L(JH_L) = 0,$$

i.e., the link L is Hamiltonian stationary in $\mathbb{R}^{2n}$ as well, and

$$\langle JH_L, u\rangle = 0,$$

where H_L is the mean curvature vector of L in $\mathbb{R}^{2n}$ and u is the position vector of L. Moreover, if the link L is isotropic in $\mathbb{C}^n$, then

$$\operatorname{div}_L(J\overline{H}_L) = 0,$$

where $\overline{H}_L = H_L - mu$ is the mean curvature vector of L in $\mathbb{S}^{2n-1}$; in fact,

$$\operatorname{div}_L(Ju) = \sum_{i=1}^{m} \langle D_{E_i}(Ju), E_i \rangle = \sum_{i=1}^{m} \langle J D_{E_i} u, E_i \rangle = 0$$

as $D_{E_i} u$ is tangent to L, where D is the derivative in $\mathbb{R}^{2n}$ and $\{E_1, \ldots, E_m\}$ is an orthonormal local frame on TL.

Our observation is that the rigidity statements in [Chen and Yuan 2006] for minimal links generalize to the Hamiltonian stationary setting.

Theorem 1.1. *Let Σ be a closed oriented immersed isotropic surface with genus g_Σ in $S^5 \subset \mathbb{C}^3$ such that the cone $C(\Sigma)$ is Hamiltonian stationary away from its vertex. Then*

(1) *if $g_\Sigma = 0$, the surface Σ is Legendrian and minimal* (*in fact*, *totally geodesic*);

(2) *if $g_\Sigma \geq 1$, the surface Σ is either Legendrian or has exactly $2g_\Sigma - 2$ Legendrian points counting the multiplicity.*

It is known that the immersed minimal Legendrian sphere ($g_\Sigma = 0$) must be a great two-sphere in S^5; see, for example, [Haskins 2004, Theorem 2.7]. Simple isotropic tori ($g_\Sigma = 1$) can be constructed so that the Hamiltonian stationary cone $C(\Sigma)$ is nowhere Lagrangian. A family of Hamiltonian stationary (nonminimal) Lagrangian cones $C(\Sigma)$ with $g_\Sigma = 1$ are presented in [Iriyeh 2005]. Bryant's classification [1985, p. 269] of minimal surfaces with constant curvature in spheres provides examples of flat Legendrian minimal tori, as well as flat non-Legendrian isotropic minimal tori ($g_\Sigma = 1$) in S^5. The constructions of [Haskins 2004; Haskins and Kapouleas 2007] show that there are infinitely many immersed (embedded if $g_\Sigma = 1$) minimal Legendrian surfaces for each odd genus in S^5.

In general dimensions and codimensions, we have:

Theorem 1.2. *Let L^m be a compact isotropic immersed oriented submanifold in the unit sphere $S^{2n-1} \subset \mathbb{C}^n$ such that the cone $C(L^m)$ is Hamiltonian stationary away from its vertex. Suppose that the first Betti number of L^m is* 0. *Then*, *away from its vertex*,

(1) *when m is the top dimension $n-1$, the cone $C(L^{n-1})$ is Lagrangian and minimal* (*or equivalently L^{n-1} is Legendrian and minimal*);

(2) *for $m < n-1$, the cone $C(L^m)$ is isotropic*, *and if the differential* 1*-form $\langle JH_{C(L^m)}, \cdot \rangle$ is closed then the mean curvature $H_{C(L^m)}$ of $C(L^m)$ vanishes on the normal subbundle $JTC(L^m)$.*

We make two remarks when the dimension m of the link is two. First, Theorem 1.2 also implies Theorem 1.1(1). Second, if the first Betti number of L^2 is not zero ($g_{L^2} > 0$) and L is isotropically immersed in S^{2n-1}, with $2n-1 \geq 5$, and $C(L)$ is

Hamiltonian stationary away from its cone vertex, the same argument as in the proof of Theorem 1.1 leads to the same conclusion as in part (2) of Theorem 1.1, that the cone $C(L^2)$ is isotropic either everywhere or along exactly $2g_{L^2}-2=-\chi(L^2)$ lines.

Theorems 1.2 and 1.1 (except the totally geodesic part) remain valid for nonorientable links (note that $\chi(\Sigma)=2-g_\Sigma$ for a compact nonorientable surface Σ); see Remarks 2.1 and 3.1. The nonorientable version of Theorem 1.2 implies that one cannot immerse a compact nonorientable L^{n-1} with first Betti number zero Hamiltonian stationarily and isotropically into $S^{2n-1}\subset\mathbb{C}^n$. Otherwise, the cone $C(L^{n-1})$ would be a special Lagrangian cone; then $C(L^{n-1})$ would be orientable, and L^{n-1} would also be orientable. In particular, there exists no isotropic Hamiltonian stationary immersion of a real projective sphere $\mathbb{R}P^2$ into $S^5\subset\mathbb{C}^3$. In passing, we mention that Lê and Wang [2001] showed that minimal link $L^{n-1}\subset S^{2n-1}$ is Legendrian if and only if $f=\langle Au, Ju\rangle$ satisfies $\triangle_L f=-2nf$ for any $A\in su(n)$.

It is interesting to find out whether there exists an isotropic Hamiltonian stationary surface in S^5 with exactly $2g_\Sigma-2$ Legendrian points for $g_\Sigma>1$.

2. Hopf differentials and proof of Theorem 1.1

To measure how far the isotropic Σ is from being Legendrian, or the deviation of the corresponding is cone from being Lagrangian, we project Ju onto the tangent space of Σ in $\mathbb{C}^3$, where J is the complex structure in $\mathbb{C}^3$. Denote the length of the projection by

$$f=|\operatorname{Pr} Ju|^2.$$

To compute the length, we need some preparation. Locally, take an isothermal coordinate system (t^1,t^2) on the isotropic surface

$$u:\Sigma\to S^5\subset\mathbb{C}^3.$$

Set the complex variable

$$z=t^1+\sqrt{-1}t^2.$$

Then the induced metric has the local expression with the conformal factor φ

$$g=\varphi^2[(dt^1)^2+(dt^2)^2]=\varphi^2\,dz\,d\bar{z}.$$

We project Ju to each of the orthonormal bases $\varphi^{-1}u_1$, $\varphi^{-1}u_2$ with $u_i=\partial u/\partial t^i$. Then the sum of the squares of each projection is

$$f=\frac{|\langle Ju,u_1\rangle|^2+|\langle Ju,u_2\rangle|^2}{\varphi^2}=\frac{4|\langle Ju,u_z\rangle|^2}{\varphi^2},$$

where $u_z=\partial u/\partial z$ and $\langle\cdot,\cdot\rangle$ is the Euclidean inner product on $\mathbb{R}^6$, and in particular $0\le f\le 1$. In fact, f is the square of the norm of the symplectic form ω in $\mathbb{C}^3$

restricted on the cone $C(\Sigma)$ with link Σ:

$$\omega|_{C(\Sigma)} \wedge *\omega|_{C(\Sigma)} = f \cdot \text{volume form of } C(\Sigma).$$

The Hamiltonian stationary condition for the cone $C(\Sigma) = ru(t^1, t^2)$ is

$$\begin{aligned} 0 &= \operatorname{div}_{C(\Sigma)}(JH_{C(\Sigma)}) \\ &= \langle \partial_r(JH_{C(\Sigma)}), \partial_r \rangle + \frac{1}{r^2} \operatorname{div}_\Sigma\Big(J\frac{1}{r}H_\Sigma\Big) \\ &= -\frac{1}{r^2}\langle JH_\Sigma, u\rangle + \frac{1}{r^3}\operatorname{div}_\Sigma(JH_\Sigma). \end{aligned}$$

It follows that

$$\operatorname{div}_\Sigma(JH_\Sigma) = 0$$

and

$$0 = \langle JH_\Sigma, u\rangle = -\Big\langle \frac{4}{\varphi^2} u_{z\bar z}, Ju\Big\rangle.$$

Coupled with the isotropy condition

$$\langle Ju_i, u_j\rangle = 0,$$

we have the holomorphic condition

$$\langle Ju, u_z\rangle_{\bar z} = \langle Ju_{\bar z}, u_z\rangle + \langle Ju, u_{z\bar z}\rangle = \big\langle Ju, -\tfrac{1}{2}\varphi^2 u\big\rangle = 0.$$

The induced metric g yields a compatible conformal structure on the oriented surface Σ, which makes Σ a Riemann surface. We shall consider two cases according to the genus g_Σ.

Case 1: $g_\Sigma = 0$. By the uniformization theorem for Riemann surfaces, see, for example, [Ahlfors and Sario 1960, p. 125, p. 181], there exists a holomorphic covering map

$$\Phi : (S^2, g_{\text{canonical}}) \to (\Sigma, g),$$

or locally

$$\Phi : \Big(\mathbb{C}^1, \frac{1}{(1+|w|^2)^2}\, dw\, d\bar w\Big) \to (\Sigma, g).$$

For $z = \Phi(w)$ one has

$$\frac{1}{(1+|w|^2)^2}\, dw\, d\bar w = \Phi^*(\psi^2 g) = \Phi^*(\psi^2\varphi^2\, dz\, d\bar z) = \psi^2\varphi^2 |z_w|^2\, dw\, d\bar w,$$

where ψ is a positive (real analytic) function on Σ. In particular

$$|z_w|^2 = \frac{1}{\psi^2\varphi^2(1+|w|^2)^2}.$$

Note that

$$\langle Ju, u_w\rangle = \langle Ju, u_z\rangle z_w = \langle Ju, u_z\rangle \frac{1}{w_z}$$

is a holomorphic function of z; in turn it is a holomorphic function of w. Also $\langle Ju, u_w\rangle$ is bounded, approaching 0 as w goes to ∞, because

$$|\langle Ju, u_w\rangle|^2 = \frac{|\langle Ju, u_z\rangle|^2}{\varphi^2}\frac{1}{\psi^2(1+|w|^2)^2}.$$

So $\langle Ju, u_w\rangle \equiv 0$. Therefore $f \equiv 0$ and Σ is Legendrian. We conclude that $C(\Sigma)\setminus\{0\}$ is Lagrangian.

The 1-form $\langle JH_{C(\Sigma)}, \cdot\,\rangle$ on the Lagrangian submanifold $C(\Sigma)\setminus\{0\}$ is closed. (This follows directly either from Theorem 3.4 of [Dazord 1981], or can be verified by local exactness via the local expression

$$H_{C(\Sigma)} = -J\nabla_{C(\Sigma)}\theta$$

given in [Harvey and Lawson 1982]; this will be done in next section.) Its restriction along Σ is therefore a closed 1-form $i^*\langle JH_{C(\Sigma)}, \cdot\,\rangle$ as the pullback by the inclusion $i : \Sigma \to C(\Sigma)$ of a closed 1-form. Since the first Betti number of Σ is zero ($g_\Sigma = 0$), there is a smooth function θ_Σ on Σ such that

$$d\theta_\Sigma = i^*\langle JH_{C(\Sigma)}, \cdot\,\rangle.$$

Then

$$\langle \nabla_\Sigma\theta_\Sigma, \cdot\,\rangle = d\theta_\Sigma = \langle JH_\Sigma, \cdot\,\rangle.$$

As we have seen, the Hamiltonian stationary condition on $C(\Sigma)$ implies

$$0 = \operatorname{div}_\Sigma(JH_\Sigma) = \operatorname{div}_\Sigma(\nabla_\Sigma\theta_\Sigma) = \Delta_g\theta_\Sigma.$$

On the closed surface Σ, we have θ_Σ is constant, and in turn, Σ is minimal.

An immersed minimal Legendrian 2-sphere in $\mathbb{S}^5$ is totally geodesic. This is a known fact; for a proof, see, for example, [Chen and Yuan 2006].

Case 2: $g_\Sigma \geq 1$. As in Case 1, where $g_\Sigma = 0$, the isotropic and Hamiltonian stationary condition gives us a local holomorphic function $\langle Ju, u_z\rangle$ and global holomorphic Hopf 1-differential $\langle Ju, u_z\rangle\, dz$. We only consider the case where $\langle Ju, u_z\rangle\, dz$ is not identically zero. The zeros of $\langle Ju, u_z\rangle$ are therefore isolated and near each of the zeros, we can write

$$\langle Ju, u_z\rangle = h(z)z^k,$$

where h is a local holomorphic function, nonvanishing at the zero point $z = 0$ and k is a positive integer. One can also view

$$\langle Ju, u_z\rangle = \tfrac{1}{2}\big(\langle Ju, u_1\rangle - \sqrt{-1}\langle Ju, u_2\rangle\big)$$

as the tangent vector

$$\tfrac{1}{2}\langle Ju, u_1\rangle u_1 - \tfrac{1}{2}\langle Ju, u_2\rangle u_2 = \tfrac{1}{2}\langle Ju, u_1\rangle\, \partial_1 - \tfrac{1}{2}\langle Ju, u_2\rangle\, \partial_2$$

along the tangent space $T\Sigma$, where $\partial_i = \partial u/\partial t^i$. The projection $\Pr Ju$ on the tangent space of $T\Sigma$ is locally represented as

$$\Pr Ju = \frac{\langle Ju, u_1\rangle\,\partial_1 + \langle Ju, u_2\rangle\,\partial_2}{\varphi^2}.$$

The index of the globally defined vector field $\Pr Ju$ at each of its singular points, i.e., where $\Pr Ju = 0$, is the negative of that for the vector field $\frac{1}{2}\langle Ju, u_1\rangle\,\partial_1 - \frac{1}{2}\langle Ju, u_2\rangle\,\partial_2$. Note that the index of the latter is k.

From the Poincaré–Hopf index theorem, for any vector field V with isolated singularities on Σ, one has

$$\sum_{V=0}\operatorname{index}(V) = \chi(\Sigma) = 2 - 2g_\Sigma \le 0.$$

The zeros of $\Pr Ju$ are just the Legendrian points on Σ. So we conclude that the number of Legendrian points is $2g_\Sigma - 2$ counting the multiplicity. This completes the proof of Theorem 1.1.

Remark 2.1. As mentioned in the Introduction, Theorem 1.1 (except the totally geodesic part) and its generalization to higher codimensions can be extended for the nonorientable links. This can be seen as follows. The Poincaré–Hopf index theorem holds on compact nonorientable surfaces, our count of the indices of the still globally defined $\Pr Ju$ via *local* holomorphic functions is valid too, and the index of a singular point of a vector field is independent of local orientations. Moreover, this index-counting argument yields an alternative proof for Theorem 1.1(1) (except the totally geodesic part) and its generalization.

3. Harmonic forms and proof of Theorem 1.2

Consider an immersed isotropic Hamiltonian stationary submanifold in S^{2n-1}

$$u : L^m \to S^{2n-1} \subset \mathbb{C}^n.$$

The isotropy condition for any local coordinates $(t^1, \dots, t^m)$ on L^m is given by

$$\langle Ju_i, u_j\rangle = 0,$$

where J is the complex structure of $\mathbb{C}^n$ and $u_i = \partial u/\partial t^i$.

The Hamiltonian stationary condition for the cone $C(\Sigma) = ru(t)$ is

$$\begin{aligned}
0 &= \operatorname{div}_{C(L)}(JH_{C(L)})\\
&= \langle \partial_r(JH_{C(L)}), \partial_r\rangle + \frac{1}{r^2}\operatorname{div}_L\Big(J\Big(\frac{1}{r}H_L\Big)\Big)\\
&= -\frac{1}{r^2}\langle JH_L, u\rangle + \frac{1}{r^3}\operatorname{div}_L(JH_L).
\end{aligned}$$

Notice that $\langle JH_L, u\rangle$ and $\operatorname{div}_L(JH_L)$ are independent of r. Therefore, the equation above splits into two equations

$$\operatorname{div}_L(JH_L) = 0$$

and

$$0 = \langle JH_L, u\rangle = -\langle \Delta_g u, Ju\rangle,$$

where g is the induced metric on L and Δ_g is the Laplace–Beltrami operator of (L, g).

To measure the deviation of the corresponding cone $C(u(L^m))$ from being isotropic, we project Ju onto the tangent space of $u(L^m)$ in $\mathbb{C}^n$. Note that the projection is the vector field along $u(L)$

$$\operatorname{Pr} Ju = \sum_{i,j=1}^{m} g^{ij}\langle Ju, u_i\rangle u_j,$$

where $g_{ij} = \langle u_i, u_j\rangle,\ 1 \le i,\ j \le m$. The corresponding 1-form

$$\alpha = \sum_{i=1}^{m}\langle Ju, u_i\rangle\, dt^i$$

is of course globally defined on L^m. In fact it is a harmonic 1-form, because α is closed and coclosed as verified as follows:

$$\begin{aligned} d\alpha &= \sum_{i,j=1}^{m}\langle Ju, u_i\rangle_j\, dt^j \wedge dt^i \\ &= \sum_{i,j=1}^{m}(\langle Ju_j, u_i\rangle + \langle Ju, u_{ij}\rangle)\, dt^j \wedge dt^i \\ &= \sum_{i,j=1}^{m}\langle Ju, u_{ij}\rangle\, dt^j \wedge dt^i = 0, \end{aligned}$$

and

$$\begin{aligned} \delta\alpha &= (-1)^{m\cdot 1+m+1} * d * \alpha \\ &= - * d\Bigg(\sum_{i,j=1}^{m}(-1)^{j+1}\sqrt{g}\, g^{ij}\langle Ju, u_i\rangle\, dt^1\wedge\cdots\wedge\widehat{dt^j}\wedge\cdots\wedge dt^m\Bigg) \\ &= - * \sum_{i,j=1}^{m}\partial_j(\sqrt{g}\, g^{ij}\langle Ju, u_i\rangle)\, dt^1\wedge\cdots\wedge dt^j\wedge\cdots\wedge dt^m \\ &= -\frac{1}{\sqrt{g}}\sum_{i,j=1}^{m}\partial_j(\sqrt{g}\, g^{ij}\langle Ju, u_i\rangle) \\ &= -\sum_{i,j=1}^{m}\Big(\langle Ju_j, g^{ij}u_i\rangle + \Big\langle Ju, \frac{1}{\sqrt{g}}\partial_j(\sqrt{g}\, g^{ij}u_i)\Big\rangle\Big) = -\langle Ju, \Delta_g u\rangle = 0, \end{aligned}$$

where we have used the isotropy condition and the consequence of Hamiltonian stationary condition in the last two steps, respectively.

The Hodge–de Rham theorem implies that the harmonic 1-form α must vanish because the first Betti number of L^m is zero by assumption. It follows that $\Pr Ju$ must vanish. Therefore, the cone $C(L^m)$ is isotropic.

Next, we claim that the differential 1-form

$$\beta = \langle JH_L, \cdot \rangle$$

on L^m is closed. When $m = n-1$, the isotropic cone $C(L^{n-1})$ is Lagrangian. By [Harvey and Lawson 1982], around each point of $C(L^{n-1}) \setminus \{0\}$, there is a locally defined Lagrangian angle θ such that

$$H_{C(L)} = -J\nabla_{C(L)}\theta.$$

Now the globally defined 1-form β on the link L can be expressed locally as

$$\beta = \langle \nabla_{C(L)}\theta, \cdot \rangle = \langle \nabla_L\theta, \cdot \rangle = d_L\theta$$

by noticing that $H_{C(L)} = H_L$ as $r = 1$, where the second equality holds as the two 1-forms are on TL and the tangent vectors to L are orthogonal to ∂_r, and d_L stands for the exterior differentiation on L. We conclude that β is a closed 1-form on L. When $m < n-1$, the 1-form $\langle JH_{C(L)}, \cdot \rangle$ is closed by assumption, so its restriction β on L is closed.

Since the first Betti number of L is zero, there is a smooth function θ_L on L such that $\langle JH_L, \cdot \rangle = d_L\theta_L$. This implies that the projection of JH_L onto TL satisfies

$$\sum_{i=1}^{m} \langle JH_L, E_i \rangle E_i = \nabla_L\theta_L,$$

where $\{E_1, \ldots, E_m\}$ is a local orthonormal frame of TL. The Hamiltonian stationary condition on $C(L)$ asserts, as we have seen earlier, that

$$\Delta_L\theta_L = \operatorname{div}_L \nabla_L\theta_L = \operatorname{div}_L(JH_L) = 0.$$

On the closed submanifold L, we know θ_L is constant. In turn, for $m = n-1$, $C(L^{n-1})$ is minimal, and for $m < n-1$, $C(L^m)$ is partially minimal, namely $H_{C(L^m)}$ vanishes on the normal subbundle $JTC(L^m)$. The proof of Theorem 1.2 is complete.

Remark 3.1. As the projection $\Pr Ju$ and the adjoint operator δ are independent of the local orientations and the Hodge–de Rham theorem holds for compact nonorientable manifolds, see, for example, [Lawson and Michelsohn 1994, p. 125–126], we see that Theorem 1.2 remains true for nonorientable links L^m.

Remark 3.2. For a surface link $L^2 \subset \mathbb{S}^{2n-1}$ with $g_L = 0$ for the case $n > 3$, if it is isotropic and $C(L^2)$ is Hamiltonian stationary, the same argument as in [Chen and Yuan 2006] leads to the conclusion that the second fundamental form of L in $\mathbb{S}^{2n-1}$ vanishes in the normal subbundle $Ju \oplus JTL$. When $n = 3$, L is totally geodesic in $\mathbb{S}^5$ as noted before.

Corollary 3.3. *Let L^m be a compact immersed isotropic submanifold in the unit sphere $S^{2n-1} \subset \mathbb{C}^n$. If the Ricci curvature of L^m is nonnegative, and it is positive somewhere or the Euler characteristic $\chi(L^m)$ is not zero, then the Hamiltonian stationary cone $C(L^m)$ is isotropic; in particular, $C(L^{n-1})$ is Lagrangian (or equivalently L^{n-1} is Legendrian) and minimal when m is the top dimension $n-1$.*

Under the above condition, from [Bochner 1948, p. 381], it follows immediately that the first Betti number of L^m is zero. Then Theorem 1.2 and its nonorientable version imply the corollary.

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JINGYI CHEN
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF BRITISH COLUMBIA
VANCOUVER, BC
CANADA
jychen@math.ubc.ca

YU YUAN
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF WASHINGTON
SEATTLE, WA
UNITED STATES
yuan@math.washington.edu

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