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CLASSIFICATION OF POSITIVE SMOOTH SOLUTIONS TO THIRD-ORDER PDES INVOLVING FRACTIONAL LAPLACIANS

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In this paper, we are concerned with the third-order equations

$$\begin{cases} (-\Delta)^{\frac{3}{2}} u = u^{\frac{d+3}{d-3}}, & x \in \mathbb{R}^d, \\ u \in C^3(\mathbb{R}^d), & u(x) > 0, \ x \in \mathbb{R}^d, \end{cases}$$

and

$$\begin{cases} (-\Delta)^{\frac{3}{2}} u = \left(\frac{1}{|x|^6} * |u|^2\right) u, & x \in \mathbb{R}^d, \\ u \in C^3(\mathbb{R}^d), & u(x) > 0, \ x \in \mathbb{R}^d, \ d \geq 7, \end{cases}$$

with $\dot{H}^{\frac{3}{2}}$-critical nonlinearity. By showing the equivalence between the PDEs and the corresponding integral equations and using results from Chen et al. (2006) and Dai et al. (2018), we prove that positive classical solutions u to the above equations are radially symmetric about some point $x_0 \in \mathbb{R}^d$ and derive the explicit forms for u.

1. Introduction

In this paper, we mainly consider the positive classical solutions to the following third-order conformal invariant equation with $\dot{H}^{\frac{3}{2}}$-critical nonlinearity:

$$\begin{cases} (-\Delta)^{\frac{3}{2}} u = u^{\frac{d+3}{d-3}}, & x \in \mathbb{R}^d, \\ u \in C^3(\mathbb{R}^d), & u(x) > 0, \ x \in \mathbb{R}^d, \end{cases} \tag{1-1}$$

where $d \geq 4$ and the nonlocal fractional Laplacian $(-\Delta)^{\frac{1}{2}}$ can be defined by Fourier transform, that is,

$$\widehat{(-\Delta)^{\frac{1}{2}} f}(\xi) := (2\pi|\xi|)\hat{f}(\xi), \tag{1-2}$$

with $\hat{f}(\xi) := \int_{\mathbb{R}^d} f(x) e^{-2\pi i x \cdot \xi}\, dx$. If f is in the Schwartz space $\mathcal{S}$ of rapidly decreasing C^∞ functions in $\mathbb{R}^d$, then $(-\Delta)^{\frac{1}{2}} f$ can also be defined equivalently by

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$$(-\Delta)^{\frac{\alpha}{2}} f(x) = C_{\alpha,d}\,\text{P.V.} \int_{\mathbb{R}^d} \frac{f(x)-f(y)}{|x-y|^{d+\alpha}}\,dy \tag{1-3}$$
$$:= C_{\alpha,d} \lim_{\epsilon\to 0} \int_{|y-x|\geq\epsilon} \frac{f(x)-f(y)}{|x-y|^{d+\alpha}}\,dy$$

with $\alpha = 1$, where the constant $C_{\alpha,d} = \bigl(\int_{\mathbb{R}^d}(1-\cos(2\pi\zeta_1))/|\zeta|^{d+\alpha}d\zeta\bigr)^{-1}$. For general $0<\alpha<2$, the definition (1-3) for $(-\Delta)^{\frac{\alpha}{2}} f$ can be extended and it is well defined for $f \in C^{1,1}_{\text{loc}} \cap \mathcal{L}_\alpha(\mathbb{R}^d)$ (see [Chen et al. 2015; 2017; Dai et al. 2017; Zhuo et al. 2014]) with

$$\mathcal{L}_\alpha(\mathbb{R}^d) := \left\{ f:\mathbb{R}^d \to \mathbb{R} \;\Big|\; \int_{\mathbb{R}^d} \frac{|f(x)|}{1+|x|^{d+\alpha}}\,dx < \infty \right\}.$$

Throughout this paper, we define

$$(-\Delta)^{\frac{3}{2}} u := (-\Delta)^{\frac{1}{2}}(-\Delta u)$$

by definition (1-3) (with $f=-\Delta u$) provided that $\Delta u \in C^{1,1}_{\text{loc}} \cap \mathcal{L}_1(\mathbb{R}^d)$ (i.e., (c) and (d) in Theorems 1.1 and 1.3), otherwise we will define $(-\Delta)^{\frac{3}{2}} u$ by Fourier transform (i.e., (a) and (b) in Theorems 1.1 and 1.3). See the extension method of defining $(-\Delta)^{\frac{\alpha}{2}}$ in [Caffarelli and Silvestre 2007]. The equation (1-1) is $\dot{H}^{\frac{3}{2}}$-critical in the sense that both it and the $\dot{H}^{\frac{3}{2}}$ norm are invariant under the same scaling

$$u_\rho(x) = \rho^{(d-3)/2} u(\rho x),$$

where the homogeneous Sobolev norm is defined as

$$\|u\|_{\dot{H}^{\frac{3}{2}}(\mathbb{R}^d)} := \|(-\Delta)^{\frac{3}{4}} u\|_{L^2(\mathbb{R}^d)} = \left(\int_{\mathbb{R}^d} |\xi|^3 |\hat{u}(\xi)|^2 d\xi \right)^{\frac{1}{2}}.$$

The quantitative and qualitative properties of solutions to fractional order or higher order conformally invariant equations of the form

$$(-\Delta)^{\frac{\alpha}{2}} u = u^{\frac{d+\alpha}{d-\alpha}} \tag{1-4}$$

have been extensively studied. In the special case $\alpha = 2$, (1-4) becomes the well-known Yamabe problem (for related results, please see Gidas, Ni and Nirenberg [Gidas et al. 1979] and Caffarelli, Gidas and Spruck [Caffarelli et al. 1989]); for $d=2$, Chen and Li [2010] classified all the positive smooth solutions with finite total curvature of the equation

$$\begin{cases} -\Delta u = e^{2u}, & x \in \mathbb{R}^2, \\ \int_{\mathbb{R}^2} e^{2u}\,dx < \infty. \end{cases} \tag{1-5}$$

In general, when $\alpha = d$, under some assumptions, Chang and Yang [1997] classified the smooth solutions to

$$(-\Delta)^{\frac{d}{2}} u = (d-1)!\,e^{du}. \tag{1-6}$$

For $\alpha = 4$, Lin [1998] proved the classification results for all the positive smooth solutions of (1-4) ($d \geq 5$) and all the smooth solutions of

$$\begin{cases} \Delta^2 u = 6e^{4u}, & x \in \mathbb{R}^4, \\ \int_{\mathbb{R}^4} e^{4u}\, dx < \infty, & u(x) = o(|x|^2) \text{ as } |x| \to \infty. \end{cases} \tag{1-7}$$

Xu [2006] obtained similar results to Chang and Yang [1997] and Lin [1998] for (1-7) under the assumption $\Delta u(x) \to 0$ as $|x| \to \infty$. For $\alpha \in (0, d]$ an even integer, Wei and Xu [1999] classified the positive smooth solutions of (1-4), they also established the classification results for the smooth solutions of (1-6) with finite total curvature under the assumption $u(x) = o(|x|^2)$ as $|x| \to \infty$. Zhu [2004] classified all the smooth solutions with finite total curvature of the problem

$$\begin{cases} (-\Delta)^{\frac{3}{2}} u = 2e^{3u}, & x \in \mathbb{R}^3, \\ \int_{\mathbb{R}^3} e^{3u}\, dx < \infty, & u(x) = o(|x|^2) \text{ as } |x| \to \infty. \end{cases} \tag{1-8}$$

In [Chen et al. 2006], by developing the method of moving planes in integral forms, Chen, Li and Ou classified all the positive $L^{2d/(d-\alpha)}_{\text{loc}}$ solutions to the equivalent integral equation of PDE (1-4). As a consequence, they obtained the classification results for positive weak solutions to PDE (1-4), moreover, they also derived classification results for positive smooth solutions to (1-4) provided $\alpha \in (0, d)$ is an even integer. For more literature on the quantitative and qualitative properties of solutions to fractional order or higher order conformally invariant PDE and IE problems, please refer to [Chen and Li 2010; Chen et al. 2017; Dai et al. 2017; Xu 2005]. One should observe that, when $\alpha \in (0, d)$ is an odd integer, the classification for positive smooth solutions to (1-4) is still open.

By proving the equivalence between PDE (1-1) and the integral equation

$$u(x) = \int_{\mathbb{R}^d} \frac{R_{3,d}}{|x-y|^{d-3}} u^{\frac{d+3}{d-3}}(y)\, dy \tag{1-9}$$

and using the results for IE (1-9) from [Chen et al. 2006], we will study the classification of positive smooth solutions to the third-order equation (1-1) under assumptions which are similar to (or even weaker than) those in [Chen et al. 2017; Lin 1998; Xu 2006; Zhu 2004].

Our classification result for (1-1) is the following theorem.

Theorem 1.1. *Assume $d \geq 4$ and u is a positive solution of (1-1). If u satisfies one of the four assumptions*

(a) $\int_{\mathbb{R}^d} u^{\frac{2d}{d-3}}\, dx < \infty$ *and* $\Delta u(x) \to 0$ *as* $|x| \to \infty$,

(b) $\int_{\mathbb{R}^d} u^{\frac{2d}{d-3}}\, dx < \infty$ *and there exists some* $\tau < 3$ *such that* $u(x) = O(|x|^{\tau})$ *as* $|x| \to \infty$,

(c) $\Delta u \in C^{1,1}_{\mathrm{loc}} \cap \mathcal{L}_1(\mathbb{R}^d)$ *and* $\Delta u \leq 0$ *in* $\mathbb{R}^d$,

(d) $\Delta u \in C^{1,1}_{\mathrm{loc}} \cap \mathcal{L}_1(\mathbb{R}^d)$, $\int_{\mathbb{R}^d} \frac{u^{(d+3)/(d-3)}}{|x|^{d-3}}\, dx < \infty$ *and* $u(x) = o(|x|^2)$ *as* $|x| \to \infty$,

then u is radially symmetric and monotone decreasing about some point $x_0 \in \mathbb{R}^d$; *in particular*, *the positive solution u must assume the form*

$$u(x) = \left(\frac{1}{R_{3,d} I\left(\frac{d-3}{2}\right)}\right)^{\frac{d-3}{6}} \left(\frac{\lambda}{1+\lambda^2|x-x_0|^2}\right)^{\frac{d-3}{2}} \quad \textit{for some } \lambda > 0,$$

where $R_{m,d} := \Gamma(\frac{d-m}{2})/(\pi^{\frac{d}{2}} 2^m \Gamma(\frac{m}{2}))$ *with* $0 < m < d$ *and*

$$I(s) := \frac{\pi^{\frac{d}{2}} \Gamma(\frac{1}{2}(d-2s))}{\Gamma(d-s)}$$

for $0 < s < \frac{d}{2}$.

Remark 1.2. In Theorem 1.1, we should observe that the integrable condition

$$\int_{\mathbb{R}^d} \frac{u^{\frac{d+3}{d-3}}}{|x|^{d-3}}\, dx < \infty$$

in (d) is much weaker than the condition $\int_{\mathbb{R}^d} u^{\frac{2d}{d-3}}\, dx < \infty$ in (a) and (b). In fact, one immediately has

$$\int_{|x|\geq 1} \frac{u^{\frac{d+3}{d-3}}(x)}{|x|^{d-3}}\, dx \leq \left(\int_{|x|\geq 1} u^{\frac{2d}{d-3}}\, dx\right)^{\frac{d+3}{2d}} \left(\int_{|x|\geq 1} \frac{1}{|x|^{2d}}\, dx\right)^{\frac{d-3}{2d}} < \infty,$$

provided that $\int_{\mathbb{R}^d} u^{\frac{2d}{d-3}}\, dx < \infty$. The assumption $\Delta u \in C^{1,1}_{\mathrm{loc}}$ in (c) and (d) in Theorem 1.1 can also be replaced by weaker assumptions $\Delta u \in C^{1,\epsilon}_{\mathrm{loc}}$ or $u \in C^{3,\epsilon}_{\mathrm{loc}}$ for arbitrarily small $\epsilon > 0$.

We also consider the classification of positive classical solutions to the following third-order $\dot{H}^{\frac{3}{2}}$-critical static Hartree equation with nonlocal nonlinearity:

$$\begin{cases} (-\Delta)^{\frac{3}{2}} u = \left(\frac{1}{|x|^6} * |u|^2\right) u, & x \in \mathbb{R}^d, \\ u \in C^3(\mathbb{R}^d), & u(x) > 0,\ x \in \mathbb{R}^d,\ d \geq 7. \end{cases} \tag{1-10}$$

The solution u to problem (1-10) is also a stationary solution to the $\dot{H}^{\frac{3}{2}}$-critical focusing fractional order dynamic Schrödinger–Hartree equation

$$i\partial_t u + (-\Delta)^{\frac{3}{2}} u = \left(\frac{1}{|x|^6} * |u|^2\right) u, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^d, \tag{1-11}$$

where $d \geq 7$. The Hartree equation has many interesting applications in the quantum theory of large systems of nonrelativistic bosonic atoms and molecules (see, e.g., [Fröhlich and Lenzmann 2004]). PDEs of the type (1-10) also arise in the Hartree–Fock theory of the nonlinear Schrödinger equations (see [Lieb and Simon 1977]).

There is lots of literature on the quantitative and qualitative properties of solutions to fractional order or higher order Hartree equations of the form

$$(-\Delta)^{\frac{\alpha}{2}} u = \left(\frac{1}{|x|^{2\alpha}} * |u|^2 \right) u \tag{1-12}$$

and various related Choquard equations, please see [Cao and Dai 2017; Dai et al. 2018; Liu 2009; Ma and Zhao 2010]. Cao and Dai [2017] classified all the positive C^4 solutions to the $\dot{H}^2$-critical biharmonic equation (1-12) with $\alpha = 4$; they also derived Liouville theorems in the subcritical cases. For general $0 < \alpha < \frac{d}{2}$, Dai et al. [2018] classified all the positive $L^{2d/(d-\alpha)}$ integrable solutions to the equivalent integral equation of PDE (1-12). As a consequence, they obtained the classification results for positive weak solutions to PDE (1-12).

By proving the equivalence between PDE (1-10) and the integral equation

$$u(x) = \int_{\mathbb{R}^d} \frac{R_{3,d}}{|x-y|^{d-3}} \left(\int_{\mathbb{R}^d} \frac{1}{|y-z|^6} |u(z)|^2 \, dz \right) u(y) \, dy \tag{1-13}$$

and using the results for IE (1-13) from [Dai et al. 2018], we establish the following classification theorem for positive smooth solutions of PDE (1-10) under similar assumptions as in Theorem 1.1.

Theorem 1.3. *Assume u is a positive solution of (1-10) such that $\int_{\mathbb{R}^d} u^{\frac{2d}{d-3}} \, dx < \infty$. If u satisfies one of the four assumptions*

(a) $\Delta u(x) \to 0$ *as* $|x| \to \infty$,

(b) *there exists some $\tau < 3$ such that $u(x) = O(|x|^{\tau})$ as $|x| \to \infty$*,

(c) $\Delta u \in C^{1,1}_{\mathrm{loc}} \cap \mathcal{L}_1(\mathbb{R}^d)$ *and* $\Delta u \le 0$ *in* $\mathbb{R}^d$,

(d) $\Delta u \in C^{1,1}_{\mathrm{loc}} \cap \mathcal{L}_1(\mathbb{R}^d)$ *and* $u(x) = o(|x|^2)$ *as* $|x| \to \infty$,

then u is radially symmetric and monotone decreasing about some point $x_0 \in \mathbb{R}^d$; in particular, the positive solution u must assume the following form:

$$u(x) = \sqrt{\frac{1}{R_{3,d} I(3) I\left(\frac{d-3}{2}\right)}} \left(\frac{\lambda}{1 + \lambda^2 |x - x_0|^2} \right)^{\frac{d-3}{2}} \quad \textit{for some } \lambda > 0.$$

The rest of our paper is organized as follows. In Section 2, we carry out our proof for Theorem 1.1. Section 3 is devoted to proving Theorem 1.3.

In the following, we will use C to denote a general positive constant that may depend on d and u, and whose value may differ from line to line.

2. Proof of Theorem 1.1

Lemma 2.1 (Hardy–Littlewood–Sobolev inequality, [Lieb 1983]). *Letting* $d \geq 1$, $0 < s < d$ *and* $1 < p < q < \infty$ *be such that* $\frac{d}{q} = \frac{d}{p} - s$, *we have*

$$\left\| \int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{d-s}} \, dy \right\|_{L^q(\mathbb{R}^d)} \leq C_{d,s,p,q} \|f\|_{L^p(\mathbb{R}^d)}$$

for all $f \in L^p(\mathbb{R}^d)$.

Define

$$v(x) := -\int_{\mathbb{R}^d} \frac{R_{3,d}}{|x-y|^{d-3}} u^{\frac{d+3}{d-3}}(y) \, dy, \qquad w(x) := u(x) + v(x), \tag{2-1}$$

where the Riesz potential's constants $R_{m,d} = \Gamma((d-m)/2)/(\pi^{\frac{d}{2}} 2^m \Gamma(m/2))$ with $0 < m < d$. Since u is a solution to (1-1), we get immediately $(-\Delta)^{\frac{3}{2}} w \equiv 0$ and hence $\Delta^2 w \equiv 0$ in $\mathbb{R}^d$.

Under the following four entirely different assumptions (a), (b), (c) and (d) on u, we will prove that the solution u to PDE (1-1) always satisfies the equivalent integral equation.

(a) Suppose $\Delta u \to 0$ as $|x| \to \infty$. By the Hardy–Littlewood–Sobolev inequality,

$$\begin{aligned} \|\Delta v\|_{L^{2d/(d+1)}(\mathbb{R}^d)} &= \\ & C_d \left\| \int_{\mathbb{R}^d} \frac{1}{|x-y|^{d-1}} u^{\frac{d+3}{d-3}}(y) \, dy \right\|_{L^{2d/(d+1)}(\mathbb{R}^d)} \leq \widetilde{C}_d \|u\|^{\frac{d+3}{d-3}}_{L^{2d/(d-3)}(\mathbb{R}^d)}. \end{aligned} \tag{2-2}$$

Now assume $z \in \mathbb{R}^d$ is arbitrary. We can infer from $\Delta v \in L^{2d/(d+1)}(\mathbb{R}^d)$ that there exists a sequence of radii $r_k \to \infty$ such that

$$r_k \cdot \int_{\partial B_{r_k}(z)} |\Delta v(x)|^{\frac{2d}{d+1}} \, d\sigma \to 0, \quad \text{as } k \to \infty. \tag{2-3}$$

Since Δw is harmonic in $\mathbb{R}^d$, the mean value property yields that

$$\Delta w(z) = \fint_{\partial B_{r_k}(z)} \Delta w(x) \, d\sigma, \tag{2-4}$$

where $\fint_{\partial B_{r_k}(z)} \Delta w(x) \, d\sigma$ is the integral average of Δw over the sphere $|x-z| = r_k$. Therefore, by the Jensen inequality and (2-4), we get

$$\begin{aligned} |\Delta w(z)|^{\frac{2d}{d+1}} &\leq \left(\fint_{\partial B_{r_k}(z)} \big(|\Delta u(x)| + |\Delta v(x)| \big) \, d\sigma \right)^{\frac{2d}{d+1}} \\ &\leq C_d \left\{ \fint_{\partial B_{r_k}(z)} |\Delta u(x)|^{\frac{2d}{d+1}} \, d\sigma + \fint_{\partial B_{r_k}(z)} |\Delta v(x)|^{\frac{2d}{d+1}} \, d\sigma \right\}. \end{aligned} \tag{2-5}$$

Letting $k \to \infty$ in (2-5), we can deduce from (2-3) and the assumption $\Delta u \to 0$ as $|x| \to \infty$ that

$$\Delta w(z) = 0. \tag{2-6}$$

Since $z \in \mathbb{R}^d$ is arbitrarily chosen, we actually have $\Delta w \equiv 0$ in $\mathbb{R}^d$.

Applying Hardy–Littlewood–Sobolev inequality again, we deduce that

$$\|v\|_{L^{2d/(d-3)}(\mathbb{R}^d)} \le C_d \|u^{\frac{d+3}{d-3}}\|_{L^{2d/(d+3)}(\mathbb{R}^d)} \le C_d \|u\|^{\frac{d+3}{d-3}}_{L^{2d/(d-3)}(\mathbb{R}^d)}. \tag{2-7}$$

Since $w \in L^{2d/(d-3)}(\mathbb{R}^d)$ is harmonic in $\mathbb{R}^d$, the Gagliardo–Nirenberg interpolation inequality implies that

$$\|\nabla w\|_{L^{2d/(d-1)}(\mathbb{R}^d)} \le C_d \|w\|^{\frac{1}{2}}_{L^{2d/(d-3)}(\mathbb{R}^d)} \|\Delta w\|^{\frac{1}{2}}_{L^{2d/(d+1)}(\mathbb{R}^d)} = 0, \tag{2-8}$$

thus we arrive at $w \equiv 0$ in $\mathbb{R}^d$. That is, u also satisfies the integral equation

$$u(x) = \int_{\mathbb{R}^d} \frac{R_{3,d}}{|x-y|^{d-3}} u^{\frac{d+3}{d-3}}(y)\, dy. \tag{2-9}$$

(b) Suppose there exists some $\tau < 3$ such that $u(x) = O(|x|^\tau)$ as $|x| \to \infty$. Without loss of generality, we may assume $\tau > 2$. By the Hölder inequality, we have for $|x|$ sufficiently large,

$$\begin{aligned} |v(x)| \le C_d \bigg[\int_{|x-y|\ge 1} \frac{1}{|x-y|^{d-3}} u^{\frac{d+3}{d-3}}(y)\, dy \\ + \int_{|x-y|\le 1} \frac{1}{|x-y|^{d-3}} u^{\frac{d+3}{d-3}}(y)\, dy \bigg] \\ \le C_d + C_{d,\delta} \Big(\sup_{\overline{B}_1(x)} u \Big)^{1+\delta} \le C|x|^{(1+\delta)\tau}, \end{aligned}$$

where $\delta > 0$ is fixed sufficiently small such that $\tau < (1+\delta)\tau < 3$. It follows that $w(x) = O(|x|^{\tilde{\tau}})$ with $\tilde{\tau} := (1+\delta)\tau < 3$.

Since Δw is harmonic in $\mathbb{R}^d$, from the mean value property, we get that, for any $x \in \mathbb{R}^d$ and $s > 0$,

$$\Delta w(x) = \frac{d}{\omega_{d-1} s^d} \int_{|y-x|\le s} \Delta w(y)\, dy = \frac{d}{\omega_{d-1} s^d} \int_{|y-x|=s} \frac{\partial w}{\partial s}(y)\, d\sigma, \tag{2-10}$$

where ω_{d-1} is the area of the unit sphere in $\mathbb{R}^d$. By integrating with respect to s from 0 to r in (2-10), we have

$$\frac{r^2}{2d} \Delta w(x) = \frac{1}{\omega_{d-1} r^{d-1}} \int_{|y-x|=r} w(y)\, d\sigma - w(x). \tag{2-11}$$

Therefore, we can deduce from $w(x) = O(|x|^{\tilde{\tau}})$ and (2-11) that, for any $x \in \mathbb{R}^d$ with $|x|$ sufficiently large and $r = |x|/2$,

$$|\Delta w(x)| \leq \frac{2d}{r^2} \left\{ \sup_{\frac{1}{2}|x| \leq |y| \leq \frac{3}{2}|x|} |w(y)| + |w(x)| \right\} \leq C|x|^{\tilde{\tau}-2}, \tag{2-12}$$

that is, $\Delta w(x) = O(|x|^{\tilde{\tau}-2})$ as $|x| \to \infty$. Thus, by gradient estimates for harmonic functions, we have

$$\Delta w(x) \equiv C \quad \text{for all } x \in \mathbb{R}^d, \tag{2-13}$$

which implies that $w(x) - C/(2d)|x|^2$ is harmonic in $\mathbb{R}^d$. Since $w(x) - C/(2d)|x|^2 = O(|x|^{\tilde{\tau}})$, by gradient estimates for harmonic functions, w must be a quadratic polynomial, that is,

$$w(x) = \sum_{i,j} a_{ij} x_i x_j + \sum_i b_i x_i + c. \tag{2-14}$$

Since $w \in L^{2d/(d-3)}(\mathbb{R}^d)$, all the coefficients a_{ij}, b_i and c in (2-14) must be zero, that is $w(x) \equiv 0$ in $\mathbb{R}^d$, thus u also satisfies the equivalent integral equation (2-9).

(c) Suppose $\Delta u \in C^{1,1}_{\text{loc}} \cap \mathcal{L}_1(\mathbb{R}^d)$ and $\Delta u \leq 0$ in $\mathbb{R}^d$. We will prove the classical solution u to PDE (1-1) also satisfies the equivalent integral equation (2-9) using the ideas from [Chen et al. 2015; Zhuo et al. 2014]. To this end, we will need the following two lemmas established in [Chen et al. 2017; Silvestre 2007; Zhuo et al. 2014].

Lemma 2.2 (maximum principle, [Chen et al. 2017; Silvestre 2007]). *Let Ω be a bounded domain in $\mathbb{R}^d$ and $0 < \alpha < 2$. Assume that $u \in \mathcal{L}_\alpha \cap C^{1,1}_{\text{loc}}(\Omega)$ and is lower semicontinuous on $\bar{\Omega}$. If $(-\Delta)^{\frac{\alpha}{2}} u \geq 0$ in Ω and $u \geq 0$ in $\mathbb{R}^d \setminus \Omega$, then $u \geq 0$ in $\mathbb{R}^d$. Moreover, if $u = 0$ at some point in Ω, then $u = 0$ almost everywhere in $\mathbb{R}^d$. These conclusions also hold for an unbounded domain Ω if we assume further that*

$$\liminf_{|x| \to \infty} u(x) \geq 0.$$

Lemma 2.3 (Liouville theorem, [Zhuo et al. 2014]). *Assume $d \geq 2$ and $0 < \alpha < 2$. Let u be a strong solution of*

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u = 0, & x \in \mathbb{R}^d, \\ u(x) \geq 0, & x \in \mathbb{R}^d, \end{cases}$$

then $u \equiv C \geq 0$.

Remark 2.4. Lemma 2.2 has been established first by Silvestre [2007] without the assumption $u \in C^{1,1}_{\text{loc}}(\Omega)$. In [Chen et al. 2017], Chen, Li and Li provided a much more elementary and simpler proof for Lemma 2.2 under the assumption $u \in C^{1,1}_{\text{loc}}(\Omega)$.

First, assume u is a positive solution to (1-1) satisfying $\Delta u \in C^{1,1}_{\mathrm{loc}} \cap \mathcal{L}_1(\mathbb{R}^d)$ and $\Delta u \leq 0$ in $\mathbb{R}^d$; we will show that $-\Delta u$ also satisfies the integral equation

$$-\Delta u = \int_{\mathbb{R}^d} \frac{R_{1,d}}{|x-y|^{d-1}} u^{\frac{d+3}{d-3}}(y)\, dy + C_1, \tag{2-15}$$

where $C_1 \geq 0$ is a constant.

For arbitrary $R > 0$, let

$$\tilde{v}_R(x) = \int_{B_R(0)} G^1_R(x, y) u^{\frac{d+3}{d-3}}(y)\, dy, \tag{2-16}$$

where the Green's function for $(-\Delta)^{\frac{1}{2}}$ on $B_R(0)$ is given by

$$G^1_R(x, y) = \frac{C_d}{|x-y|^{d-1}} \int_0^{\frac{t_R}{s_R}} \frac{1}{b^{\frac{1}{2}}(1+b)^{\frac{d}{2}}}\, db, \quad \text{if } x, y \in B_R(0) \tag{2-17}$$

with $s_R = |x-y|^2/R^2$, $t_R = (1-|x|^2/R^2)(1-|y|^2/R^2)$, and $G^1_R(x, y) = 0$ if x or $y \in \mathbb{R}^d \setminus B_R(0)$ (see [Kulczycki 1997]).

Then, we can derive

$$\begin{cases} (-\Delta)^{1/2} \tilde{v}_R(x) = u^{\frac{d+3}{d-3}}(x), & x \in B_R(0), \\ \tilde{v}_R(x) = 0, & x \in \mathbb{R}^d \setminus B_R(0). \end{cases} \tag{2-18}$$

Letting $\tilde{w}_R(x) = -\Delta u(x) - \tilde{v}_R(x)$, by (1-1) and (2-18), we have

$$\begin{cases} (-\Delta)^{1/2} \tilde{w}_R(x) = 0, & x \in B_R(0), \\ \tilde{w}_R(x) \geq 0, & x \in \mathbb{R}^d \setminus B_R(0). \end{cases} \tag{2-19}$$

By Lemma 2.2, we deduce that for any $R > 0$,

$$\tilde{w}_R(x) = -\Delta u(x) - \tilde{v}_R(x) \geq 0, \quad \text{for all } x \in \mathbb{R}^d. \tag{2-20}$$

Now, for each fixed $x \in \mathbb{R}^d$, letting $R \to \infty$ in (2-20), we have

$$-\Delta u(x) \geq \int_{\mathbb{R}^d} \frac{R_{1,d}}{|x-y|^{d-1}} u^{\frac{d+3}{d-3}}(y)\, dy =: \tilde{v}(x) > 0. \tag{2-21}$$

Taking $x = 0$ in (2-21), we get

$$\int_{\mathbb{R}^d} \frac{u^{\frac{d+3}{d-3}}(y)}{|y|^{d-1}}\, dy < \infty, \tag{2-22}$$

and it follows easily that $\int_{\mathbb{R}^d} |u(x)|/(1+|x|^d)\, dx < \infty$, and hence $u \in \mathcal{L}_\alpha$ for any $\alpha > 0$. One can easily observe that $\tilde{v}$ is a solution of

$$(-\Delta)^{\frac{1}{2}} \tilde{v}(x) = u^{\frac{d+3}{d-3}}(x), \quad x \in \mathbb{R}^d. \tag{2-23}$$

Define $\tilde{w}(x) = -\Delta u(x) - \tilde{v}(x)$, then it satisfies

$$\begin{cases} (-\Delta)^{\frac{1}{2}}\tilde{w}(x) = 0, & x \in \mathbb{R}^d, \\ \tilde{w}(x) \geq 0 & x \in \mathbb{R}^d. \end{cases} \tag{2-24}$$

From Lemma 2.3, we can deduce that

$$\tilde{w}(x) = -\Delta u(x) - \tilde{v}(x) \equiv C_1 \geq 0. \tag{2-25}$$

Therefore, we have proved (2-15), that is,

$$-\Delta u = \int_{\mathbb{R}^d} \frac{R_{1,d}}{|x-y|^{d-1}} u^{\frac{d+3}{d-3}}(y)\,dy + C_1 =: f(u) \geq C_1 \geq 0. \tag{2-26}$$

Next, we will prove u also satisfies the equivalent integral equation (2-9). For arbitrary $R > 0$, let

$$v_R(x) = \int_{B_R(0)} G_R^2(x,y) f(u)(y)\,dy, \tag{2-27}$$

where the Green's function for $-\Delta$ on $B_R(0)$ is given by

$$G_R^2(x,y) = C_d\left[\frac{1}{|x-y|^{d-2}} - \frac{1}{\big(|x|\cdot|Rx/|x|^2 - y/R|\big)^{d-2}}\right] \quad \text{if } x, y \in B_R(0),$$

and $G_R^2(x,y) = 0$ if x or $y \in \mathbb{R}^d \setminus B_R(0)$. Then, we can get

$$\begin{cases} -\Delta v_R(x) = f(u)(x), & x \in B_R(0), \\ v_R(x) = 0, & x \in \mathbb{R}^d \setminus B_R(0). \end{cases} \tag{2-28}$$

Let $w_R(x) = u(x) - v_R(x)$, by (2-26) and (2-28), we have

$$\begin{cases} -\Delta w_R(x) = 0, & x \in B_R(0), \\ w_R(x) > 0, & x \in \mathbb{R}^d \setminus B_R(0). \end{cases} \tag{2-29}$$

By the maximum principle, we deduce that for any $R > 0$,

$$w_R(x) = u(x) - v_R(x) > 0, \qquad \text{for all } x \in \mathbb{R}^d. \tag{2-30}$$

Now, for each fixed $x \in \mathbb{R}^d$, letting $R \to \infty$ in (2-30), we have

$$u(x) \geq \int_{\mathbb{R}^d} \frac{R_{2,d}}{|x-y|^{d-2}} f(u)(y)\,dy =: V(x) > 0. \tag{2-31}$$

Taking $x = 0$ in (2-31), we get

$$\int_{\mathbb{R}^d} \frac{C_1}{|y|^{d-2}}\,dy \leq \int_{\mathbb{R}^d} \frac{f(u)(y)}{|y|^{d-2}}\,dy < \infty, \tag{2-32}$$

and it follows easily that $C_1 = 0$, and hence

$$-\Delta u = f(u) = \int_{\mathbb{R}^d} \frac{R_{1,d}}{|x-y|^{d-1}} u^{\frac{d+3}{d-3}}(y)\,dy.$$

One can easily observe that V is a solution of

$$-\Delta V(x) = f(u)(x), \quad x \in \mathbb{R}^d. \tag{2-33}$$

Define $W(x) = u(x) - V(x)$, then it satisfies

$$\begin{cases} -\Delta W(x) = 0, & x \in \mathbb{R}^d, \\ W(x) \geq 0 & x \in \mathbb{R}^d. \end{cases} \tag{2-34}$$

From the Liouville theorem for harmonic functions, we can deduce that

$$W(x) = u(x) - V(x) \equiv C_2 \geq 0. \tag{2-35}$$

Therefore, we have proved that

$$u(x) = \int_{\mathbb{R}^d} \frac{R_{2,d}}{|x-y|^{d-2}} f(u)(y)\,dy + C_2 \geq C_2 \geq 0. \tag{2-36}$$

Now (2-22) implies that

$$\int_{\mathbb{R}^d} \frac{C_2^{\frac{d+3}{d-3}}(y)}{|y|^{d-1}}\,dy \leq \int_{\mathbb{R}^d} \frac{u^{\frac{d+3}{d-3}}(y)}{|y|^{d-1}}\,dy < \infty, \tag{2-37}$$

from which we can infer that $C_2 = 0$. Thus, by using the formula

$$\int_{\mathbb{R}^d} \frac{1}{|x-y|^{d-2}} \cdot \frac{1}{|y|^{d-1}}\,dy = \frac{R_{3,d}}{R_{1,d} R_{2,d}} \cdot \frac{1}{|x|^{d-3}} \tag{2-38}$$

(see [Stein 1970]) and direct calculations, we finally deduce from (2-36) that

$$\begin{aligned} u(x) &= \int_{\mathbb{R}^d} \frac{R_{2,d}}{|x-y|^{d-2}} \int_{\mathbb{R}^d} \frac{R_{1,d}}{|y-z|^{d-1}} u^{\frac{d+3}{d-3}}(z)\,dz\,dy \\ &= \int_{\mathbb{R}^d} \frac{R_{3,d}}{|x-z|^{d-3}} u^{\frac{d+3}{d-3}}(z)\,dz, \end{aligned} \tag{2-39}$$

that is, u also satisfies the equivalent integral equation (2-9).

(d) Suppose $\Delta u \in C^{1,1}_{\mathrm{loc}} \cap \mathcal{L}_1(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} \frac{u^{\frac{d+3}{d-3}}}{|x|^{d-3}}\,dx < \infty$$

and $u(x) = o(|x|^2)$ as $|x| \to \infty$. By the above proof under assumption (c), we only need to prove the super-harmonic property $-\Delta u \geq 0$ under assumption (d).

For that purpose, we will first estimate the upper bound for $-v(x)$. Since one can verify that

$$\Delta v(x)=\int_{\mathbb{R}^d}\frac{(d-3)R_{3,d}}{|x-y|^{d-1}}u^{\frac{d+3}{d-3}}(y)\,dy\geq 0, \tag{2-40}$$

we deduce that, for $|x|$ sufficiently large,

$$\begin{aligned}0\leq -v(x)&=\int_{\mathbb{R}^d}\frac{R_{3,d}}{|x-y|^{d-3}}u^{\frac{d+3}{d-3}}(y)\,dy\\&=\int_{|y-x|\geq\frac{|x|}{6}}\frac{R_{3,d}}{|x-y|^{d-3}}u^{\frac{d+3}{d-3}}(y)\,dy+\int_{|y-x|<\frac{|x|}{6}}\frac{R_{3,d}}{|x-y|^{d-3}}u^{\frac{d+3}{d-3}}(y)\,dy\\&\leq 7^{d-3}R_{3,d}\int_{|y-x|\geq\frac{|x|}{6}}\frac{u^{\frac{d+3}{d-3}}(y)}{|y|^{d-3}}\,dy+\frac{|x|^2}{36}\int_{|y-x|<\frac{|x|}{6}}\frac{R_{3,d}}{|x-y|^{d-1}}u^{\frac{d+3}{d-3}}(y)\,dy\\&\leq C_d\int_{\mathbb{R}^d}\frac{u^{\frac{d+3}{d-3}}(y)}{|y|^{d-3}}\,dy+\frac{|x|^2}{36(d-3)}\Delta v(x).\end{aligned}$$

As a consequence, we deduce from the assumption

$$\int_{\mathbb{R}^d}\frac{u^{\frac{d+3}{d-3}}}{|x|^{d-3}}\,dx<\infty$$

that, as $|x|\to\infty$,

$$0\leq -v(x)\leq O(1)+\frac{|x|^2}{36(d-3)}\Delta v(x). \tag{2-41}$$

Next, we can deduce from (2-11) that, for any $x\in\mathbb{R}^d$ with $|x|$ sufficiently large and $r=|x|/2$,

$$\begin{aligned}\Delta w(x)&\leq\frac{2d}{r^2}\left\{\sup_{\frac{1}{2}|x|\leq|y|\leq\frac{3}{2}|x|}w(y)-u(x)-v(x)\right\}\\&\leq\frac{2d}{r^2}\left\{\sup_{\frac{1}{2}|x|\leq|y|\leq\frac{3}{2}|x|}u(y)-v(x)\right\}.\end{aligned} \tag{2-42}$$

Therefore, we get from (2-40), (2-41), (2-42) and the assumption $u(x)=o(|x|^2)$ as $|x|\to\infty$ that, as $|x|\to\infty$,

$$\begin{aligned}\Delta w(x)=\Delta u(x)+\Delta v(x)&\leq\frac{8d}{|x|^2}\left\{o(|x|^2)+O(1)+\frac{|x|^2}{36(d-3)}\Delta v(x)\right\}\\&\leq o(1)+\frac{d}{4(d-3)}\Delta v(x).\end{aligned} \tag{2-43}$$

We can deduce from (2-43) that

$$\limsup_{|x|\to\infty} \Delta u(x) \le 0, \qquad \text{that is ,} \quad \liminf_{|x|\to\infty}(-\Delta u(x)) \ge 0. \tag{2-44}$$

Therefore, from (1-1), (2-44) and the maximum principle (Lemma 2.2), we can infer

$$-\Delta u \ge 0 \quad \text{in } \mathbb{R}^d. \tag{2-45}$$

In conclusion, we have proved respectively under the four different assumptions (a), (b), (c) and (d) on u that the classical solution u to PDE (1-1) always satisfies the equivalent integral equation (2-9). Applying [Chen et al. 2006, Theorem 1.1] ($u \in L^{2d/(d-3)}_{\mathrm{loc}}(\mathbb{R}^d)$ was assumed therein) to integral equation (2-9), we deduce immediately that u is radially symmetric and monotone decreasing about some point $x_0 \in \mathbb{R}^d$ and thus assumes the form

$$u(x) = \left(\frac{1}{R_{3,d} I\left(\frac{d-3}{2}\right)}\right)^{\frac{d-3}{6}} \left(\frac{\lambda}{1+\lambda^2|x-x_0|^2}\right)^{\frac{d-3}{2}} \tag{2-46}$$

for some positive constant λ, where

$$I(s) := \frac{\pi^{\frac{d}{2}}\Gamma\left(\frac{d-2s}{2}\right)}{\Gamma(d-s)}$$

for $0 < s < \frac{d}{2}$. This concludes the proof of Theorem 1.1.

Remark 2.5. In the proof of Theorem 1.1 under assumption (d), one crucial step is to deduce $\Delta u \le 0$ from the assumptions

$$\int_{\mathbb{R}^d} \frac{u^{\frac{d+3}{d-3}}}{|x|^{d-3}}\, dx < \infty$$

and $u(x) = o(|x|^2)$ as $|x| \to \infty$, where the fractional Laplacian $(-\Delta)^{\frac{1}{2}}$ is given by definition (1-3). Suppose $(-\Delta)^{\frac{1}{2}}$ can be defined in terms of the Fourier transform, that is,

$$\widehat{(-\Delta)^{\frac{1}{2}} f}(\xi) := (2\pi|\xi|)\hat{f}(\xi)$$

with $\hat{f}(\xi) := \int_{\mathbb{R}^d} f(x)e^{-2\pi i x\cdot\xi}\, dx$, then the super-harmonic property $\Delta u \le 0$ can be deduced directly from $\int_{\mathbb{R}^d} u^{(d+3)/(d-3)}/|x|^{d-1}\, dx < \infty$. Indeed, we only need to show that $\int_{\mathbb{R}^d}(-\Delta u)\phi dx \ge 0$ for any nontrivial $0 \le \phi \in C_0^\infty(\mathbb{R}^d)$. To this end, we define

$$\psi(x) := (-\Delta)^{-\frac{1}{2}}\phi(x) = \int_{\mathbb{R}^d} \frac{R_{1,d}}{|x-y|^{d-1}}\phi(y)\, dy \ge 0,$$

then $\psi \in C^\infty(\mathbb{R}^d)$ and satisfy $(2\pi|\xi|)\hat{\psi}(\xi) = \hat{\phi}(\xi)$ (see [Stein 1970]). Moreover, one can easily verify that $\psi(x) \sim 1/|x|^{d-1}$ for $|x|$ large enough, thus we have

$\int_{\mathbb{R}^d} u^{\frac{d+3}{d-3}}\psi\, dx < \infty$ provided $\int_{\mathbb{R}^d} u^{\frac{d+3}{d-3}}/|x|^{d-1}\, dx < \infty$. Therefore, we may multiply both sides of the PDE (1-1) by ψ and integrate, then by Parseval's formula, we get

$$\infty > \int_{\mathbb{R}^d} u^{\frac{d+3}{d-3}}\psi\, dx = \int_{\mathbb{R}^d} (-\Delta)^{\frac{3}{2}} u\cdot\psi\, dx = \int_{\mathbb{R}^d} (2\pi|\xi|)\widehat{-\Delta u}\cdot\bar{\hat{\psi}}\, d\xi = \int_{\mathbb{R}^d} (-\Delta u)\cdot\phi\, dx \geq 0.$$

3. Proof of Theorem 1.3

We define

$$\begin{aligned} v(x) &:= -\int_{\mathbb{R}^d} \frac{R_{3,d}}{|x-y|^{d-3}}\left(\int_{\mathbb{R}^d} \frac{1}{|y-z|^6}|u(z)|^2\, dz\right)u(y)\, dy, \\ w(x) &:= u(x) + v(x). \end{aligned} \tag{3-1}$$

Since u is a solution to (1-10), we get immediately $(-\Delta)^{\frac{3}{2}} w \equiv 0$ and hence $\Delta^2 w \equiv 0$ in $\mathbb{R}^d$.

Our goal is to show under the following four entirely different assumptions (a), (b), (c) and (d) that the solution u to PDE (1-10) always satisfies the equivalent integral equation

$$u(x) = -v(x) = \int_{\mathbb{R}^d} \frac{R_{3,d}}{|x-y|^{d-3}}\left(\int_{\mathbb{R}^d} \frac{1}{|y-z|^6}|u(z)|^2\, dz\right)u(y)\, dy. \tag{3-2}$$

(a) Suppose $\Delta u \to 0$ as $|x| \to \infty$. The key ingredients are showing $v \in L^{2d/(d-3)}(\mathbb{R}^d)$ and $\Delta v \in L^{2d/(d+1)}(\mathbb{R}^d)$.

Indeed, let $P(x) := 1/|x|^6 * |u|^2$, then by the Hardy–Littlewood–Sobolev inequality, one has

$$\|P\|_{L^{d/3}(\mathbb{R}^d)} \leq C\|u^2\|_{L^{d/(d-3)}(\mathbb{R}^d)} \leq C\|u\|^2_{L^{2d/(d-3)}(\mathbb{R}^d)}. \tag{3-3}$$

Therefore, by using Hardy–Littlewood–Sobolev inequality again, we get

$$\begin{aligned} \|v\|_{L^{2d/(d-3)}(\mathbb{R}^d)} &\leq C_d\|Pu\|_{L^{2d/(d+3)}(\mathbb{R}^d)} \leq C_d\|P\|_{L^{d/(3)}(\mathbb{R}^d)}\|u\|_{L^{2d/(d-3)}(\mathbb{R}^d)} \\ &\leq C_d\|u\|^3_{L^{\frac{2d}{d-3}}(\mathbb{R}^d)}, \end{aligned} \tag{3-4}$$

$$\begin{aligned} \|\Delta v\|_{L^{\frac{2d}{d+1}}(\mathbb{R}^d)} &= C_d\left\|\int_{\mathbb{R}^d} \frac{P(y)u(y)}{|x-y|^{d-1}}\, dy\right\|_{L^{\frac{2d}{d+1}}(\mathbb{R}^d)} \\ &\leq \widetilde{C}_d\|Pu\|_{L^{\frac{2d}{d+3}}(\mathbb{R}^d)} \leq \widetilde{C}_d\|u\|^3_{L^{\frac{2d}{d-3}}(\mathbb{R}^d)}. \end{aligned} \tag{3-5}$$

The rest of the proof is similar to the proof of Theorem 1.1 under assumption (a) in Section 2.

(b) Suppose there exists some $\tau < 3$ such that $u(x) = O(|x|^\tau)$ as $|x| \to \infty$. Without loss of generality, we may assume $\tau > 2$. The key ingredient is proving $w(x) = O(|x|^{\tilde{\tau}})$ for some $\tau < \tilde{\tau} < 3$.

In fact, using Hölder's inequality, one can verify that for $|x|$ large enough,

$$P(x) \le \int_{|x-y|\ge 1} \frac{1}{|x-y|^6}|u(y)|^2\,dy + \int_{|x-y|\le 1} \frac{1}{|x-y|^6}|u(y)|^2\,dy \\ \le C_d + C_d\big(\sup_{\overline{B}_1(x)} u\big)^2 \le C|x|^{2\tau}. \tag{3-6}$$

Therefore, by $P \in L^{\frac{d}{3}}(\mathbb{R}^d)$ and the Hölder inequality, we have for $|x|$ sufficiently large,

$$|v(x)| \le C_d\bigg[\int_{|x-y|\ge 1} \frac{1}{|x-y|^{d-3}} P(y)u(y)\,dy \\ + \int_{|x-y|\le 1} \frac{1}{|x-y|^{d-3}} P(y)u(y)\,dy\bigg] \\ \le C_d + C_{d,\delta}\Big(\sup_{\overline{B}_1(x)} u\Big)\Big(\sup_{\overline{B}_1(x)} P\Big)^{\delta} \le C|x|^{(1+2\delta)\tau}, \tag{3-7}$$

where $\delta > 0$ is fixed sufficiently small such that $\tau < (1+2\delta)\tau < 3$. It follows that $w(x) = O(|x|^{\tilde{\tau}})$ with $\tilde{\tau} := (1+2\delta)\tau < 3$. The rest of the proof is similar to the proof of Theorem 1.1 under assumption (b) in Section 2.

(c) The proof is similar to the proof of Theorem 1.1 under assumption (c) in Section 2.

(d) Suppose $\Delta u \in C^{1,1}_{\mathrm{loc}} \cap \mathcal{L}_1(\mathbb{R}^d)$ and $u(x) = o(|x|^2)$ as $|x| \to \infty$. The key ingredient is proving $\int_{\mathbb{R}^d} P(x)u(x)/|x|^{d-3}\,dx < \infty$. Indeed, we have

$$\int_{\mathbb{R}^d} \frac{P(x)u(x)}{|x|^{d-3}}\,dx \le \int_{|x|\le 1} \frac{1}{|x|^{d-3}}\,dx \cdot \|Pu\|_{L^\infty(\overline{B}_1)} \\ + \bigg(\int_{|x|>1} \frac{1}{|x|^{2d}}\,dx\bigg)^{\frac{d-3}{2d}} \|P\|_{L^{d/3}} \|u\|_{L^{2d/(d-3)}} < \infty.$$

The rest of the proof is similar to the proof of Theorem 1.1 under assumption (d) in Section 2.

In conclusion, we have proved respectively under the four different assumptions (a), (b), (c) and (d) on u that the classical solution u to PDE (1-10) always satisfies the equivalent integral equation (3-2). Applying [Dai et al. 2018, Theorem 1.4] ($u \in L^{\frac{2d}{d-3}}(\mathbb{R}^d)$ was assumed therein) to integral equation (3-2), we deduce immediately that u is radially symmetric and monotone decreasing about some point $x_0 \in \mathbb{R}^d$ and thus assumes the form

$$u(x) = \sqrt{\frac{1}{R_{3,d}\, I(3)\, I\big(\frac{d-3}{2}\big)}} \bigg(\frac{\lambda}{1+\lambda^2|x-x_0|^2}\bigg)^{\frac{d-3}{2}} \tag{3-8}$$

for some positive constant λ. This concludes the proof of Theorem 1.3.

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