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MINIMAL BRAID REPRESENTATIVES OF QUASIPOSITIVE LINKS

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We show that every quasipositive link has a quasipositive minimal braid representative, partially resolving a question posed by Orevkov. These quasipositive minimal braids are used to show that the maximal self-linking number of a quasipositive link is bounded below by the negative of the minimal braid index, with equality if and only if the link is an unlink. This implies that the only amphichiral quasipositive links are the unlinks, answering a question of Rudolph's.

1. Introduction

Quasipositive links in S^3 were introduced by Rudolph [1983] and defined in terms of special braid diagrams, the details of which we recall below. These links possess a variety of noteworthy features. Perhaps most strikingly, results from [Rudolph 1983; Boileau and Orevkov 2001] show that quasipositive links are precisely those links which arise as transverse intersections of the unit sphere $S^3 \subset \mathbb{C}^2$ with complex plane curves $\Sigma \subset \mathbb{C}^2$. The hierarchy of braid-positive, positive, strongly quasipositive, and quasipositive links interacts in compelling ways with conditions such as fiberedness [Etnyre and Van Horn-Morris 2011; Hedden 2010], sliceness [Rudolph 1993], homogeneity [Baader 2005], and symplectic or Lagrangian fillability [Boileau and Orevkov 2001; Hayden and Sabloff 2015]. Quasipositive links also have well-understood behavior with respect to invariants such as the four-ball genus, the maximal self-linking number, and the Ozsváth–Szabó concordance invariant τ [Hedden 2010]. For a different perspective, we can view quasipositive braids as a monoid in the mapping class group of a disk with marked points, where they lie inside the contact-geometrically important monoid of right-veering diffeomorphisms; see [Etnyre and Van Horn-Morris 2015] for more details.

The braid-theoretic description of quasipositivity is as follows: A braid is called *quasipositive* if it is the closure of a word

$$\prod_i \omega_i \sigma_{j_i} \omega_i^{-1},$$

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where ω_i is any word in the braid group and σ_{j_i} is a positive standard generator. A link is then called *quasipositive* if it has a quasipositive braid representative. However, an arbitrary braid representative of a quasipositive link need not be a quasipositive braid. Along these lines, Orevkov [2000] posed the following question:

Question 1.1 (Orevkov). Let $\mathcal{L}$ be a quasipositive link and β a minimal braid index representative of $\mathcal{L}$. Is β quasipositive?

Partial resolutions to this question have appeared in [Etnyre and Van Horn-Morris 2011; Feller and Krcatovich 2017]. The first of these showed that the answer to Question 1.1 is "yes" for fibered strongly quasipositive links. (In contrast, the answer to the analogue of Question 1.1 for positive braids is "no", as Stoimenow [2002] has provided examples of braid positive knots that have no positive minimal braid representatives. See also [Stoimenow 2006, §1].) The main purpose of this note is to provide another partial answer to Question 1.1.

Theorem 1.2. *Every quasipositive link has a quasipositive minimal braid index representative.*

This claim follows quickly from the proof of the generalized Jones conjecture in [LaFountain and Menasco 2014] — a substantial result in the theory of braid foliations. Our method of proof is similar to that of [Etnyre and Van Horn-Morris 2011; 2015].

A few simple consequences follow from Theorem 1.2. First, by considering the self-linking number of a quasipositive minimal braid index representative of a quasipositive link, we obtain a lower bound on the maximal self-linking number $\overline{\mathrm{sl}}$ in terms of the minimal braid index b:

Theorem 1.3. *If $\mathcal{L}$ is a quasipositive link, then*

$$\overline{\mathrm{sl}}(\mathcal{L}) \geq -b(\mathcal{L}),$$

with equality if and only $\mathcal{L}$ is an unlink.

The calculation underlying Theorem 1.3 also lets us resolve an earlier question of Rudolph's from [Morton 1988, Problem 9.2]:

Question 1.4 (Rudolph). Are there any amphichiral quasipositive links other than the unlinks?

At the time this question was asked, it was already known that nontrivial strongly quasipositive knots were chiral; see [Rudolph 1999, Remark 4] for a discussion of precedent results. Additional evidence for a negative answer came in the form of strong constraints on invariants of amphichiral quasipositive links (including their being slice [Wu 2011]). We confirm that the answer to Rudolph's question is "no".

Corollary 1.5. *If a link $\mathcal{L}$ and its mirror $m(\mathcal{L})$ are both quasipositive, then $\mathcal{L}$ is an unlink. In particular, the unlinks are the only amphichiral quasipositive links.*

After recalling the necessary background in Section 2, we supply proofs for the above results in Section 3.

2. Background

The generalized Jones conjecture, first confirmed by Dynnikov and Prasolov [2013], relates the writhe w and braid index n of braids with a given link type.

Theorem 2.1 [Dynnikov and Prasolov 2013, generalized Jones conjecture]. *Let β and β_0 be closed braids with the same link type $\mathcal{L}$, where $n(\beta_0)$ is minimal for $\mathcal{L}$. Then there is an inequality*

$$|w(\beta) - w(\beta_0)| \leq n(\beta) - n(\beta_0).$$

Recall Bennequin's formula for the self-linking number of a braid β:

$$\mathrm{sl}(\beta) = w(\beta) - n(\beta).$$

It follows from the generalized Jones conjecture, Bennequin's formula, and the transverse Alexander theorem that a minimal braid index representative of $\mathcal{L}$ achieves the maximal self-linking number among all transverse representatives of $\mathcal{L}$, denoted $\overline{\mathrm{sl}}(\mathcal{L})$. For any braid β representing a link type $\mathcal{L}$, we can plot the pair $(w(\beta), n(\beta))$ in a plane. The *cone* of β is the collection of all pairs (w, n) realized by braids which are stabilizations of β; see Figure 1 for an example. If β_0 is a minimal braid index representative of $\mathcal{L}$, we see that the right edge of its cone consists of all pairs (w, n) corresponding to braids achieving the maximal self-linking number of $\mathcal{L}$.

The other tool central to the proof of Theorem 1.2 is due to Orevkov and concerns braid moves that preserve quasipositivity.

Theorem 2.2 [Orevkov 2000]. *Suppose the braids β and β' are related by positive (de)stabilization. Then β is quasipositive if and only if β' is quasipositive.*

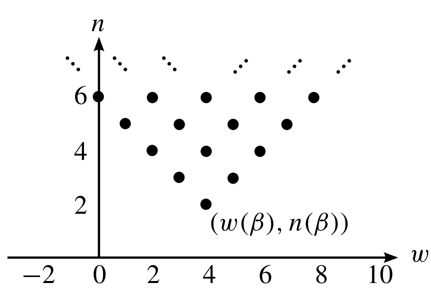

Figure 1. The cone of a braid β with $(w(\beta), n(\beta)) = (4, 2)$.

Remark 2.3. In [Orevkov 2000], an n-stranded braid is viewed as an isotopy class of n-valued functions $f : [0, 1] \to \mathbb{C}$ where $f(0)$ and $f(1)$ equal $\{1, 2, \ldots, n\} \subset \mathbb{C}$. A braid is then quasipositive if one of its representatives can be expressed as a product of conjugates of the standard generators. For us, it is more convenient to study *closed* braids (up to isotopy through closed braids). Two closed braids are equivalent if and only if they can be expressed as closures of conjugate open braids. Since quasipositivity is a property of conjugacy classes of open braids, Theorem 2.2 holds equally well for closed braids.

3. Quasipositive minimal braids

We proceed to the proof of the of the main result, namely that every quasipositive link has a quasipositive minimal braid representative.

Proof of Theorem 1.2. Let $\mathcal{L}$ be a quasipositive link with a minimal braid index representative β_0 and a quasipositive braid representative β_+. Since the slice-Bennequin inequality is sharp for quasipositive links [Rudolph 1993; Hedden 2010], β_+ achieves the maximal self-linking number for $\mathcal{L}$. As noted above, it follows that $(w(\beta_+), n(\beta_+))$ lies along the right edge of the cone of β_0. The braids β_0 and β_+ have the same link type, so [LaFountain and Menasco 2014, Proposition 1.1] implies that there are braids β_0' and β_+' obtained from β_0 and β_+ by negative and positive stabilization, respectively, such that β_0' and β_+' cobound embedded annuli. Note that β_0' and β_+' lie along the left and right edges of the cone, respectively, as depicted on the left side of Figure 2. We also note that β_+' is quasipositive since it is obtained from β_+ by positive stabilization.

Next, as in the proof of [LaFountain and Menasco 2014, Proposition 3.2], we can find braids β_0'' and β_+'' obtained from β_0' and β_+' by braid isotopy, destabilization, and exchange moves such that $w(\beta_+'') = w(\beta_0'')$ and $n(\beta_+'') = n(\beta_0'')$. We claim that β_+'' has minimal braid index (as does β_0''). Indeed, since β_0' and β_+' lie on the left and right edges of the cone of β_0, the destabilizations applied to them must be negative and positive, respectively. Given this and the fact that exchange moves preserve writhe and braid index, we see that β_0'' and β_+'' must also lie on the left and right edges of the cone of β_0, respectively. But since these braids occupy the same (w, n)-point, they must lie where the edges of the cone meet. As depicted on the right side of Figure 2, this implies that β_0'' and β_+'' have minimal braid index.

Finally, we show that the braid β_+'' is quasipositive. As noted above, any destabilizations of β_+' must be positive, and these preserve quasipositivity by Theorem 2.2. An exchange move also preserves quasipositivity, since it can be expressed as a combination of one positive stabilization, one positive destabilization, and a number of conjugations; see [Birman and Wrinkle 2000, Figure 8]. □

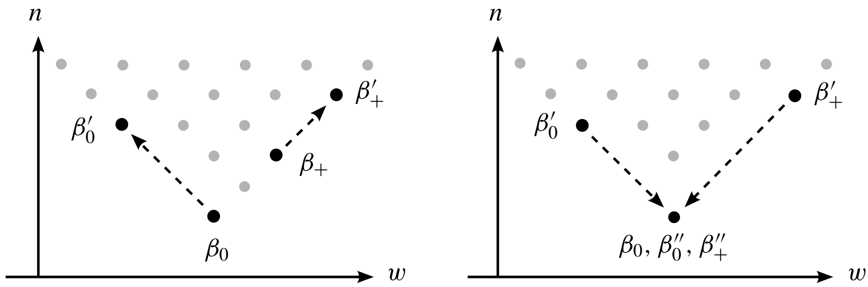

Figure 2. On the left, β_0' and β_+' are obtained from β_0 and β_+ by negative and positive stabilization, respectively. Then, on the right, β_0'' and β_+'' are obtained from β_0' and β_+' by negative and positive destabilization, respectively.

Remark 3.1. The question of whether or not *all* minimal braid index representatives of a quasipositive link are quasipositive remains open. The answer is seen to be "yes" for transversely simple link types: beginning with a quasipositive braid representative of a transversely simple link, the transverse Markov theorem implies that any minimal braid index representative can be related to it by positive (de)stabilization, which preserves quasipositivity. By the same reasoning, the answer to Question 1.1 is "yes" for any link type that has a unique transverse class achieving its maximal self-linking number (but is not necessarily transversely simple). This is the case for fibered strongly quasipositive links, as shown by Etnyre and Van Horn-Morris. But it fails to hold even for nonfibered strongly quasipositive links; as pointed out by Etnyre and Van Horn-Morris, there are infinite families of 3-braids found by Birman and Menasco [2006] which are (strongly) quasipositive and of minimal braid index but not transversely isotopic.

Remark 3.2. As pointed out by Eli Grigsby, the proof of Theorem 1.2 can be mirrored to show that any property of closed braids that is

(1) preserved under transverse isotopy, and
(2) satisfied by at least one braid representative of $\mathcal{L}$ with maximal self-linking number

is also satisfied by at least one minimal braid index representative of $\mathcal{L}$.

Now we obtain the lower bound in Theorem 1.3 by applying Bennequin's formula to a quasipositive minimal braid.

Proof of Theorem 1.3. Recall that a quasipositive braid always achieves the maximal self-linking number of its link type. Thus if β is a quasipositive minimal braid index representative of $\mathcal{L}$, we have

$$\overline{\mathrm{sl}}(\mathcal{L}) = \mathrm{sl}(\beta) = w(\beta) - n(\beta) = w(\beta) - b(\mathcal{L}).$$

The desired inequality now follows from the fact that the writhe of a quasipositive braid is nonnegative, vanishing if and only if the braid is trivial. □

Finally, we prove the corollary that resolves Question 1.4.

Proof of Corollary 1.5. Observe that if β is a minimal braid index representative of $\mathcal{L}$, then its mirror $m(\beta)$ is minimal for $m(\mathcal{L})$. Now suppose $\mathcal{L}$ and $m(\mathcal{L})$ are both quasipositive. The preceding proof implies that $w(\beta)$ and $w(m(\beta)) = -w(\beta)$ are both nonnegative, so $w(\beta)$ must be zero. Since we can choose the braid β to be quasipositive, the vanishing of its writhe implies that the braid itself is trivial. □

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