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FOUR-DIMENSIONAL STATIC AND RELATED CRITICAL SPACES WITH HARMONIC CURVATURE

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We study any four-dimensional Riemannian manifold (M, g) with harmonic curvature which admits a smooth nonzero solution f to the equation

$$\nabla df = f\left(\mathrm{Rc} - \frac{R}{n-1}g\right) + x\mathrm{Rc} + y(R)g,$$

where Rc is the Ricci tensor of g, x is a constant and $y(R)$ a function of the scalar curvature R. We show that a neighborhood of any point in some open dense subset of M is locally isometric to one of the following five types: (i) $\mathbb{S}^2\left(\frac{R}{6}\right) \times \mathbb{S}^2\left(\frac{R}{3}\right)$ with $R > 0$, (ii) $\mathbb{H}^2\left(\frac{R}{6}\right) \times \mathbb{H}^2\left(\frac{R}{3}\right)$ with $R < 0$, where $\mathbb{S}^2(k)$ and $\mathbb{H}^2(k)$ are the two-dimensional Riemannian manifolds with constant sectional curvatures $k > 0$ and $k < 0$, respectively, (iii) the static spaces we describe in Example 3, (iv) conformally flat static spaces described by Kobayashi (1982), and (v) a Ricci flat metric.

We then get a number of corollaries, including the classification of the following four-dimensional spaces with harmonic curvature: static spaces, Miao–Tam critical metrics and V-static spaces.

For the proof we use some Codazzi-tensor properties of the Ricci tensor and analyze the equation displayed above depending on the various cases of multiplicity of the Ricci-eigenvalues.

1. Introduction

In this article we consider an n-dimensional Riemannian manifold (M, g) with constant scalar curvature R which admits a smooth nonzero solution f to the equation

$$\nabla df = f\left(\mathrm{Rc} - \frac{R}{n-1}g\right) + x \cdot \mathrm{Rc} + y(R)g, \tag{1-1}$$

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where Rc is the Ricci curvature of g, x is a constant and $y(R)$ a function of R. There are several well-known classes of spaces which admit such solutions. Below we describe them and briefly explain their geometric significance and recent developments.

A *static space* admits by definition a smooth nonzero solution f to

$$\nabla df = f\left(\mathrm{Rc} - \frac{R}{n-1}g\right). \tag{1-2}$$

A Riemannian geometric interest of a static space comes from the fact that the scalar curvature functional $\mathfrak{S}$, defined on the space $\mathfrak{M}$ of smooth Riemannian metrics on a closed manifold, is locally surjective at $g \in \mathfrak{M}$ if there is no nonzero smooth function satisfying (1-2); see Chapter 4 of [Besse 1987].

This interpretation also holds in a local sense. Roughly speaking, if no nonzero smooth function on a compactly contained subdomain Ω of a smooth manifold satisfies (1-2) for a Riemannian metric g on Ω, then the scalar curvature functional defined on the space of Riemannian metrics on Ω is locally surjective at g in a natural sense; see Theorem 1 of [Corvino 2000]. This local viewpoint has been developed to make remarkable progress in Riemannian and Lorentzian geometry [Chruściel et al. 2005; Corvino 2000; Corvino et al. 2013; Corvino and Schoen 2006; Qing and Yuan 2016].

Kobayashi [1982] studied a classification of conformally flat static spaces. In his study the list of *complete* ones is made. Moreover, all *local* ones are described for all varying parameter conditions and initial values of the static space equation. Indeed, they belong to the cases I–VI in Section 2 of [Kobayashi 1982] and the existence of solutions in each case is thoroughly discussed. Lafontaine [1983] independently proved a classification of closed conformally flat static spaces. Qing and Yuan [2013] classified complete Bach-flat static spaces which contain compact level hypersurfaces.

Next to static spaces we consider a Miao–Tam critical metric [2009; 2011], which is a compact Riemannian manifold (M, g) that admits a smooth nonzero solution f, vanishing at the smooth boundary of M, to

$$\nabla df = f\left(\mathrm{Rc} - \frac{R}{n-1}g\right) - \frac{g}{n-1}. \tag{1-3}$$

In [Miao and Tam 2011], Miao–Tam critical metrics are classified when they are Einstein or conformally flat. In [Barros et al. 2015], Barros, Diógenes and Ribeiro proved that if (M^4, g, f) is a Bach-flat simply connected, compact Miao–Tam critical metric with boundary isometric to a standard sphere $\mathbb{S}^3$, then (M^4, g) is isometric to a geodesic ball in a simply connected space form $\mathbb{R}^4$, $\mathbb{H}^4$ or $\mathbb{S}^4$.

In [Corvino et al. 2013], Corvino, Eichmair and Miao defined a *V-static space* to be a Riemannian manifold (M, g) which admits a nontrivial solution (f, c), for a constant c, to the equation

$$\nabla\, df = f\left(\mathrm{Rc} - \frac{R}{n-1} g\right) - \frac{c}{n-1} g. \tag{1-4}$$

Note that (M, g) is a V-static space if and only if it admits a solution f to (1-2) or (1-3) on M, seen by scaling constants. Under a natural assumption, a V-static metric g is a critical point of a geometric functional, as explained in Theorem 2.3 of [Corvino et al. 2013]. Like static spaces, *local* V-static spaces are still important; see, e.g., Theorems 1.1, 1.6 and 2.3 in [Corvino et al. 2013].

Lastly, one may consider Riemannian metrics (M, g) which admit a nonconstant solution f to

$$\nabla\, df = f\left(\mathrm{Rc} - \frac{R}{n-1} g\right) + \mathrm{Rc} - \frac{R}{n} g. \tag{1-5}$$

If M is a closed manifold, then g is a critical point of the total scalar curvature functional defined on the space of Riemannian metrics with unit volume and with constant scalar curvature on M. By an abuse of terminology we shall call a metric g satisfying (1-5) *a critical point metric* even when M is not closed. There are a number of works on this subject, including [Besse 1987, Section 4.F] and [Lafontaine 1983; Yun et al. 2014; Barros and Ribeiro 2014; Qing and Yuan 2013].

Finally we note that the existence of a nonzero or nonconstant solution to any of (1-2)–(1-5) guarantees the scalar curvature is constant. Indeed, it is shown for (1-2)–(1-4) in [Corvino 2000; Miao and Tam 2009; Corvino et al. 2013] and can be shown similarly for (1-5). But it does not hold true generally for (1-1).

In this paper we study spaces with harmonic curvature having a nonzero solution to (1-1). It is confined to four-dimensional spaces here, but our study may be extendible to higher dimensions. As motivated by Corvino's local deformation theory of scalar curvature, we study local (i.e., not necessarily complete) classification. We completely characterize nonconformally flat spaces, so that together with Kobayashi's work on conformally flat ones we get a full classification as follows.

Theorem 1.1. *Let (M, g) be a four-dimensional (not necessarily complete) Riemannian manifold with harmonic curvature, satisfying (1-1) with nonconstant f. Then for each point p in some open dense subset $\widetilde{M}$ of M, there exists a neighborhood V of p with one of the following properties:*

(i) *(V, g) is isometric to a domain in $\left(\mathbb{S}^2\left(\frac{R}{6}\right) \times \mathbb{S}^2\left(\frac{R}{3}\right), g_{R/6} + g_{R/3}\right)$ with $R > 0$, where $\mathbb{S}^2(k)$ is the two-dimensional sphere with constant sectional curvature $k > 0$ and g_k is the Riemannian metric of constant curvature k, and $f = c_1 \cos\left(\sqrt{\frac{R}{6}} s\right) - x$*

for any constant c_1, where s is the distance from a point on $\mathbb{S}^2\big(\frac{R}{6}\big)$. The constant R equals the scalar curvature of g. It holds that $\frac{1}{3}xR + y(R) = 0$.

(ii) (V, g) *is isometric to a domain in $\big(\mathbb{H}^2\big(\frac{R}{6}\big) \times \mathbb{H}^2(\frac{R}{3}),\, g_{R/6} + g_{R/3}\big)$ with $R < 0$, where $\mathbb{H}^2(k)$ is the hyperbolic plane with constant sectional curvature $k < 0$. The metric $g_{R/6}$ can be written as $g_{R/6} = ds^2 + p(s)^2\, dt^2$ with $p(s) = k_1 \sinh\big(\sqrt{-\frac{R}{6}}s\big) + k_2 \cosh\big(\sqrt{-\frac{R}{6}}s\big)$ for constants k_1, k_2, and then $f = c_2 p'(s) - x$ for any constant c_2. It holds that $\frac{1}{3}xR + y(R) = 0$.*

(iii) (V, g) *is isometric to a domain in one of the static spaces in Example 3 of Section 2A2, which is the Riemannian product* $(\mathbb{R}^1 \times W^3, dt^2 + ds^2 + h(s)^2\tilde{g})$ *of* $(\mathbb{R}^1, dt^2)$ *and some three-dimensional conformally flat static space* $(W^3, ds^2 + h(s)^2\tilde{g})$ *with zero scalar curvature, and $f = c \cdot h'(s) - x$ for any constant c. It holds that $R = 0$ and $y(0) = 0$.*

(iv) (V, g) *is conformally flat. It is one of the metrics whose existence is described in Section 2 of [Kobayashi 1982]; $g = ds^2 + h(s)^2 g_k$, where h is a solution of*

$$h'' + \tfrac{1}{12}Rh = ah^{-3} \quad \textit{for a constant } a. \tag{1-6}$$

For the constant k, the function h satisfies

$$(h')^2 + ah^{-2} + \tfrac{1}{12}Rh^2 = k, \tag{1-7}$$

and f is a nonconstant solution to the following ordinary differential equation for f:

$$h'f' - fh'' = x\big(h'' + \tfrac{1}{3}Rh\big) + y(R)h. \tag{1-8}$$

Conversely, any (V, g, f) from (i)–(iv) *has harmonic curvature and satisfies* (1-1).

Theorem 1.1 only considers the case when f is a nonconstant solution, but the other case of f being a nonzero constant solution is easier, which is described in Section 2A1.

Theorem 1.1 yields a number of classification theorems on four-dimensional spaces with harmonic curvature as follows. Theorem 8.2 classifies complete spaces satisfying (1-1). Then Theorems 9.1, 10.2 and 11.1 state the classification of *local* static spaces, V-static spaces and critical point metrics, respectively. Theorems 9.2 and 11.2 classify complete static spaces and critical point metrics, respectively. Theorem 10.3 gives a characterization of some four-dimensional Miao–Tam critical metrics with harmonic curvature, which is comparable to the aforementioned Bach-flat result [Barros et al. 2015].

To prove Theorem 1.1 we look into the eigenvalues of the Ricci tensor, which is a Codazzi tensor under the harmonic curvature condition. This Codazzi tensor encodes some geometric information, as investigated by Derdziński [1980]. In [Kim 2017], one of us has analyzed it in the Ricci soliton setting. We shall work in the same framework of arguments: we show that all Ricci-eigenvalues λ_i, $i = 1, 2, 3, 4$, locally depend on the function f only, and then analyze case I when the three $\lambda_2, \lambda_3, \lambda_4$ are pairwise distinct and case II when exactly two of them are equal.

Our contribution in this paper is first to show the dependence of all Ricci-eigenvalues on f in the setting of (1-1) by modifying the original soliton proof. Then in analyzing cases I and II, we manage to prove the desired key arguments of Propositions 4.2, 6.3 and 6.4 using involved formulas, which turns out to be fairly different from the soliton proof. Finally in the last five sections we discuss local-to-global results ranging from static spaces to critical point metrics.

This paper is organized as follows. In Section 2, we discuss examples and some properties from (1-1) and harmonic curvature. In Section 3, we prove that all Ricci-eigenvalues locally depend on only one variable. We study in Section 4 the case when the three eigenvalues $\lambda_2, \lambda_3, \lambda_4$ are pairwise distinct. In Sections 5 and 6 we analyze the case when exactly two of the three are equal. In Section 7 we characterize the case when all the three are equal, and then prove the local classification theorem as Theorem 1.1. We discuss the classification of complete spaces in Section 8. In Sections 9, 10 and 11 we treat static spaces, Miao–Tam critical and V-static spaces and critical point metrics respectively.

2. Examples and properties from (1-1) and harmonic curvature

We are going to describe some examples of spaces which satisfy (1-1) in Section 2A and state basic properties of spaces with harmonic curvature satisfying (1-1) in Section 2B.

2A. *Examples of spaces satisfying (1-1).*

2A1. *Spaces with a nonzero constant solution to (1-1).* When (M, g) has a constant solution $f = -x$ to (1-1), then $y(R) + xR/(n-1) = 0$. Conversely, any metric with its scalar curvature satisfying $y(R) + xR/(n-1) = 0$ admits the constant solution $f = -x$ to (1-1) because

$$\nabla\, df = f\left(\mathrm{Rc} - \frac{R}{n-1}g\right) + x\mathrm{Rc} + y(R)g = (f + x)\left(\mathrm{Rc} - \frac{R}{n-1}g\right).$$

This proves the following lemma.

Lemma 2.1. *An n-dimensional Riemannian manifold (M, g) of constant scalar curvature R admits the constant solution $f = -x$ if and only if it satisfies $y(R) + xR/(n-1) = 0$.*

If (M, g) has a constant solution $f = c_0$, which does not equal $-x$, then g is an Einstein metric. Conversely, if g is Einstein, i.e., $\mathrm{Rc} = (R/n)g$ with $R \neq 0$, then any constant c_0 satisfying $c_0 R = (n-1)xR + y(R)n(n-1)$ is a solution to (1-1); but if g is Ricci-flat, then $f = c_0$ is a solution exactly when $y(0) = 0$.

2A2. *Some examples of spaces which satisfy (1-1) with nonconstant f.*

Example 1 (Einstein spaces satisfying (1-1) with nonconstant f). Let (M, g, f) be a four-dimensional space satisfying (1-1), where g is an Einstein metric. We shall show that g has constant sectional curvature. We may use the argument in Section 1 of [Cheeger and Colding 1996]. In fact, the relation (1.6) of that paper corresponds to the equation

$$\nabla\, df = \bigl[-\tfrac{1}{12}Rf + x\tfrac{1}{4}R + y(R)\bigr]g \tag{2-1}$$

in our Einstein case. One can readily see that their argument to get their (1.19) still works; in some neighborhood of any point in M we can write $g = ds^2 + (f'(s))^2\tilde{g}$, where s is a function such that $\nabla s = \nabla f/|\nabla f|$ and $\tilde{g}$ is considered as a Riemannian metric on a level surface of f.

As g is Einstein, so is $\tilde{g}$ from Lemma 4 in [Derdziński 1980]. As $\tilde{g}$ is three-dimensional, it has constant sectional curvature, say k. Moreover, f satisfies $f'' = -\frac{1}{12}Rf + \frac{1}{4}xR + y(R)$, by feeding $(\partial/\partial s, \partial/\partial s)$ to (2-1).

Since g is Einstein, we can readily see that our warped product metric g has constant sectional curvature. In particular, a four-dimensional complete positive Einstein space satisfying (1-1) with nonconstant f is a round sphere; see [Obata 1962; Yano and Nagano 1959].

Example 2. Assume $\frac{1}{3}xR + y(R) = 0$. Then (1-1) reduces to

$$\nabla df = (f + x)\Bigl(\mathrm{Rc} - \frac{R}{n-1}g\Bigr).$$

This is the static space equation for g and $F = f + x$. We recall one example from [Lafontaine 1983]. On the round sphere $\mathbb{S}^2(1)$ of sectional curvature 1, we consider the local coordinates $(s, t) \in (0, \pi) \times \mathbb{S}^1$ so that the round metric is written $ds^2 + \sin^2(s)\, dt^2$. Let $f(s) = c_1 \cos s - x$ for any constant c_1. Then the product metric of $\mathbb{S}^2(1) \times \mathbb{S}^2(2)$ with f satisfies (1-1). This example is neither Einstein nor conformally flat.

Example 3. Here we shall describe some four-dimensional nonconformally flat static space $g_W + dt^2$. We first recall some spaces among Kobayashi's warped product static spaces [1982] on $I \times N(k)$ with the metric $g = ds^2 + r(s)^2\bar{g}$, where I is an interval and $(\bar{g}, N(k))$ is an $(n-1)$-dimensional Riemannian manifold of constant sectional curvature k. Moreover, $f = cr'$ for a nonzero constant c.

In order for g to be a static space, the next equation needs to be satisfied; for a constant α

$$r'' + \frac{R}{n(n-1)}r = \alpha r^{1-n}, \tag{2-2}$$

along with an integrability condition: for a constant k,

$$(r')^2 + \frac{2\alpha}{n-2}r^{2-n} + \frac{R}{n(n-1)}r^2 = k. \tag{2-3}$$

Existence of solutions depends on the values of α, R, k. Here we consider only when $R = 0$. Then there are three cases:

(i) $R = 0$, $\alpha > 0$.

(ii) $R = 0$, $\alpha < 0$.

(iii) $R = 0$, $\alpha = 0$.

The above (i), (ii) and (iii) correspond respectively to the cases IV.1, III.1 and II in Section 2 of [Kobayashi 1982]. The solutions for these cases are discussed in Proposition 2.5, Example 5 and Proposition 2.4 in that paper. In particular, if $R = 0$, $\alpha > 0$ (then $k > 0$) and $n = 3$, we get the warped product metric on $\mathbb{R}^1 \times \mathbb{S}^2(1)$ which contains the spatial slice of a Schwarzschild space-time. Next, if $R = 0$, $\alpha < 0$, then there is an incomplete metric on $I \times N(k)$. If $R = 0$, $\alpha = 0$, then g is readily seen to be a flat metric.

Let (W^3, g_W, f) be one of the three-dimensional static spaces (g, f) in the above paragraph. We now consider the four-dimensional product metric $g_W + dt^2$ on $W^3 \times \mathbb{R}^1$. One can check that $(W^3 \times \mathbb{R}^1, g_W + dt^2, f \circ \mathrm{pr}_1)$ is a static space, where pr_1 is the projection of $W^3 \times \mathbb{R}^1$ onto the first factor. When $R = 0$ and $\alpha \neq 0$ for g_W, the metric $g_W + dt^2$ is not conformally flat and has three distinct Ricci-eigenvalues.

2B. ***Spaces with harmonic curvature.*** A Riemannian metric is said to have harmonic curvature [Besse 1987, Chapter 16] if the divergence of the curvature tensor is zero. The Ricci tensor Rc of a Riemannian metric, when evaluated on two vectors (X, Y), shall be denoted by $R(X, Y)$ rather than $\mathrm{Rc}(X, Y)$, and its components in vector frames shall be written as R_{ij}.

By the differential Bianchi identity, the Ricci tensor of a Riemannian metric with harmonic curvature is a Codazzi tensor, written in local coordinates as $\nabla_k R_{ij} = \nabla_i R_{kj}$. A Riemannian metric with harmonic curvature has constant scalar curvature. We begin with a basic formula.

Lemma 2.2. *For a four-dimensional manifold (M^4, g, f) with harmonic curvature satisfying (1-1), it holds that*

$$-R(X, Y, Z, \nabla f) = -R(X, Z)g(\nabla f, Y) + R(Y, Z)g(\nabla f, X) \\ - \tfrac{1}{3}R\{g(\nabla f, X)g(Y, Z) - g(\nabla f, Y)g(X, Z)\}.$$

Proof. By the Ricci identity, $\nabla_i\nabla_j\nabla_k f - \nabla_j\nabla_i\nabla_k f = -\sum_l R_{ijkl}\nabla_l f$. The equation (1-1) gives

$$\begin{aligned}\sum_l - R_{ijkl}\nabla_l f &= \nabla_i\bigl\{f\bigl(R_{jk} - \tfrac{1}{3}Rg_{jk}\bigr) + xR_{jk} + y(R)g_{jk}\bigr\} \\ &\qquad - \nabla_j\bigl\{f\bigl(R_{ik} - \tfrac{1}{3}Rg_{ik}\bigr) + xR_{ik} + y(R)g_{ik}\bigr\} \\ &= \nabla_i f\bigl(R_{jk} - \tfrac{1}{3}Rg_{jk}\bigr) - \nabla_j f\bigl(R_{ik} - \tfrac{1}{3}Rg_{ik}\bigr),\end{aligned}$$

which yields the lemma. □

A Riemannian manifold with harmonic curvature is real analytic in harmonic coordinates [DeTurck and Goldschmidt 1989]. Equation (1-1) then implies that f is real analytic in harmonic coordinates.

One may mimic arguments in [Cao and Chen 2013] and get the next lemma. We shall often denote the metric $g(X, Y)$ by $\langle X, Y\rangle$.

Lemma 2.3. *Let (M^n, g, f) have harmonic curvature, satisfying (1-1) with non-constant f. Let c be a regular value of f and $\Sigma_c = \{x \mid f(x) = c\}$ be the level surface of f. Then the following hold*:

(i) $E_1 := \nabla f/|\nabla f|$ *is an eigenvector field of* Rc, *where* $\nabla f \neq 0$.

(ii) $|\nabla f|$ *is constant on any connected component of* Σ_c.

(iii) *There is a function s locally defined with $s(x) = \int df/|\nabla f|$, so that $ds = df/|\nabla f|$ and $E_1 = \nabla s$.*

(iv) *$R(E_1, E_1)$ is constant on any connected component of Σ_c.*

(v) *Near a point in Σ_c, the metric g can be written as*

$$g = ds^2 + \sum_{i,j>1} g_{ij}(s, x_2, \dots, x_n)\, dx_i \otimes dx_j,$$

where $x_2, \dots, x_n$ is a local coordinate system on Σ_c.

(vi) $\nabla_{E_1} E_1 = 0$.

Proof. In Lemma 2.2, put $Y = Z = \nabla f$ and $X \perp \nabla f$ to get

$$0 = -R(X, \nabla f, \nabla f, \nabla f) = -R(X, \nabla f) g(\nabla f, \nabla f).$$

So, $R(X, \nabla f) = 0$. Hence $E_1 = \nabla f/|\nabla f|$ is an eigenvector of Rc. By (1-1), $\frac{1}{2}\nabla_X |\nabla f|^2 = \langle \nabla_X \nabla f, \nabla f\rangle = f R(\nabla f, X) = 0$ for $X \perp \nabla f$. This proves (ii). Next

$$d\left(\frac{df}{|\nabla f|}\right) = -\frac{1}{2|\nabla f|^{\frac{3}{2}}} d|\nabla f|^2 \wedge df = 0$$

as $\nabla_X(|\nabla f|^2) = 0$ for $X \perp \nabla f$. So, (iii) is proved. As ∇f and the level surfaces of f are perpendicular, one gets (v). One uses (v) to compute Christoffel symbols and gets (vi).

Now we shall prove (iv). Locally, f is a function of the local variable s only. We can write

$$E_1(f) = df(E_1) = \frac{df}{ds} ds(E_1) = \frac{df}{ds} g(\nabla s, \nabla s) = \frac{df}{ds},$$

which again depends on s only. Similarly we get $E_1 E_1(f) = d^2 f/ds^2$. By (1-1), we have

$$\begin{aligned} E_1 E_1 f &= E_1 E_1 f - (\nabla_{E_1} E_1) f \\ &= \nabla df(E_1, E_1) = (f + x) R(E_1, E_1) - \frac{1}{n-1} R f + y(R). \end{aligned}$$

Since $f + x$ is not zero on an open subset,

$$R(E_1, E_1) = \frac{1}{(f + x)} \left\{ E_1 E_1 f + \frac{1}{n-1} R f - y(R) \right\}$$

depends on s only. So $R(E_1, E_1)$ is constant on any connected component of Σ_c. This proves (iv). □

As (M, g) has harmonic curvature, the Ricci tensor Rc is a Codazzi tensor. Following [Derdziński 1980], for $x \in M$, let $E_{\mathrm{Rc}}(x)$ be the number of distinct eigenvalues of Rc_x, and set $M_{\mathrm{Rc}} = \{x \in M \mid E_{\mathrm{Rc}}$ is constant in a neighborhood of $x\}$. The open subset M_{Rc} is dense in M. To see this, one may argue as follows. For each point $x \in M$, consider any open ball B centered at x. As the range of the map E_{Rc} is finite, there is a point $q \in B$ where $E_{\mathrm{Rc}}(q)$ equals the maximum of E_{Rc} on B. By definition $E_{\mathrm{Rc}} \geq E_{\mathrm{Rc}}(q)$ near q. So, $E_{\mathrm{Rc}} \equiv E_{\mathrm{Rc}}(q)$ near q. Then $q \in M_{\mathrm{Rc}}$. This implies that M_{Rc} is dense.

Now we have:

Lemma 2.4. *For a Riemannian metric g of dimension $n \geq 4$ with harmonic curvature, consider orthonormal vector fields E_i, $i = 1, \ldots, n$, such that $R(E_i, \cdot) = \lambda_i g(E_i, \cdot)$. Then the following hold in each connected component of M_{Rc}*:

(i) $(\lambda_j - \lambda_k) \langle \nabla_{E_i} E_j, E_k \rangle + E_i \{R(E_j, E_k)\} = (\lambda_i - \lambda_k) \langle \nabla_{E_j} E_i, E_k \rangle + E_j \{R(E_k, E_i)\}$, *for any $i, j, k = 1, \ldots, n$.*

(ii) *If $k \neq i$ and $k \neq j$, then $(\lambda_j - \lambda_k) \langle \nabla_{E_i} E_j, E_k \rangle = (\lambda_i - \lambda_k) \langle \nabla_{E_j} E_i, E_k \rangle$.*

(iii) *Given distinct Ricci-eigenvalues λ, μ and local vector fields v, u such that $R(v, \cdot) = \lambda g(v, \cdot)$ and $R(u, \cdot) = \mu g(u, \cdot)$ with $|u| = 1$, it holds that $v(\mu) = (\mu - \lambda) \langle \nabla_u u, v \rangle$.*

(iv) *For each eigenvalue λ, the λ-eigenspace distribution is integrable and its leaves are totally umbilic submanifolds of M.*

Proof. The statement (i) was proved in [Kim 2017]. Parts (ii) and (iii) follow from (i). Parts (iii) and (iv) are from Section 2 of [Derdziński 1980]. □

Given (M^n, g, f) with harmonic curvature satisfying (1-1), f is real analytic in harmonic coordinates, so $\{\nabla f \neq 0\}$ is open and dense in M. Lemma 2.3 gives that for any point p in the open dense subset $M_r \cap \{\nabla f \neq 0\}$ of M^n, there is a neighborhood U of p where there exist orthonormal Ricci-eigenvector fields E_i, $i = 1, \ldots, n$, such that

(i) $E_1 = \nabla f / |\nabla f|$,

(ii) E_i is tangent to smooth level hypersurfaces of f for $i > 1$.

These local orthonormal Ricci-eigenvector fields $\{E_i\}$ shall be called an *adapted frame field* of (M, g, f).

3. Constancy of λ_i on level hypersurfaces of f

For an adapted frame field of (M^n, g, f) with harmonic curvature satisfying (1-1), we set $\zeta_i := -\langle \nabla_{E_i} E_i, E_1\rangle = \langle E_i, \nabla_{E_i} E_1\rangle$ for $i > 1$. Then by (1-1),

$$\nabla_{E_i} E_1 = \nabla_{E_i}\left(\frac{\nabla f}{|\nabla f|}\right) = \frac{\nabla_{E_i}\nabla f}{|\nabla f|}$$
$$= \frac{f R(E_i, \cdot) - f R/(n-1) g(E_i, \cdot) + x R(E_i, \cdot) + y(R) g(E_i, \cdot)}{|\nabla f|}.$$

So we may write

$$\text{(3-1)}\quad \nabla_{E_i} E_1 = \zeta_i E_i, \quad \text{where } \zeta_i = \frac{(f+x)R(E_i, E_i) - f R/(n-1) + y(R)}{|\nabla f|}.$$

Due to Lemma 2.3, in a neighborhood of a point $p \in M_{\mathrm{Rc}} \cap \{\nabla f \neq 0\}$, f may be considered as a function of s only, and we write the derivative in s by a prime: $f' = df/ds$.

Lemma 3.1. *Let (M, g, f) be a four-dimensional space with harmonic curvature, satisfying (1-1) with nonconstant f. Then the Ricci-eigenvalue λ_i associated to an adapted frame field E_i is constant on any connected component of a regular level hypersurface Σ_c of f, and so depend on the local variable s only. Moreover, ζ_i, $i = 2, 3, 4$, in (3-1) also depend on s only. In particular, we have $E_i(\lambda_j) = E_i(\zeta_k) = 0$ for $i, k > 1$ and any j.*

Proof. We denote $\nabla_{E_i} f$ by f_i and $\nabla_{E_j}\nabla_{E_i} f$ by f_{ij}. We have

$$\sum_{j=1}^{4} \tfrac{1}{2}\nabla_{E_j}\nabla_{E_j}(|\nabla f|^2) = \sum_{i,j} \tfrac{1}{2}\nabla_{E_j}\nabla_{E_j}(f_i f_i) = \sum_{i,j} \nabla_{E_j}(f_i f_{ij}).$$

We use $f_{ij} = f\left(R_{ij} - \frac{1}{3} R g_{ij}\right) + x R_{ij} + y(R) g_{ij}$ from (1-1) to compute:

$$\begin{aligned}\sum_{i,j}\nabla_{E_j}(f_i f_{ij}) &= \sum_{i,j}\nabla_{E_j}\left\{f f_i\left(R_{ij} - \tfrac{1}{3}Rg_{ij}\right) + x f_i R_{ij} + y(R) f_i g_{ij}\right\}\\ &= \sum_{i,j} f_j f_i\left(R_{ij} - \tfrac{1}{3}Rg_{ij}\right) + f f_{ij}\left(R_{ij} - \tfrac{1}{3}Rg_{ij}\right) + x f_{ij} R_{ij} + y(R) f_{ij} g_{ij}\\ &= \left(R_{11} - \tfrac{1}{3}R\right)|\nabla f|^2 + \sum_{i,j}(f+x)^2 R_{ij}R_{ij} - \tfrac{2}{9}R^2 f^2 - \tfrac{2}{3}xR^2 f\\ &\qquad + \left(2x - \tfrac{2}{3}f\right) y(R) R + 4y(R)^2,\end{aligned}$$

where in obtaining the second equality we use the Bianchi identity $\nabla_k R_{jk} = \frac{1}{2}\nabla_k R$ and the fact that R is constant.

Meanwhile,

$$\sum_{j=1}^{4} \nabla_{E_j} \nabla_{E_j} (|\nabla f|^2) = \sum_{j=1}^{4} E_j E_j (|\nabla f|^2) - (\nabla_{E_j} E_j)(|\nabla f|^2)$$
$$= (|\nabla f|^2)'' + \sum_{j=2}^{4} \zeta_j (|\nabla f|^2)'.$$

Since R and $\lambda_1 = R_{11}$ depend on s only by Lemma 2.3, the function $\sum_{j=2}^{4} \zeta_j$ depends only on s by (3-1). We compare the above two expressions of

$$\sum_{j=1}^{4} \nabla_{E_j} \nabla_{E_j} (|\nabla f|^2)$$

to see that

$$\sum_{i,j} (f + x)^2 R_{ij} R_{ij}$$

depends only on s. As f is nonconstant real analytic, $\sum_{i,j} R_{ij} R_{ij}$ depends only on s.

We compute

$$\begin{aligned}
&\sum_{i,j,k} \nabla_k (f_i f_{ij} R_{jk}) \\
&= \sum_{i,j,k} \nabla_k \big[f_i R_{jk} \big\{ f (R_{ij} - \tfrac{1}{3} R g_{ij}) + x R_{ij} + y(R) g_{ij} \big\} \big] \\
&= \sum_{i,j,k} \nabla_k \big[f_i \big\{ (f + x) R_{ij} R_{jk} - \big(\tfrac{1}{3} f R - y(R) \big) g_{ij} R_{jk} \big\} \big] \\
&= \sum_{i,j,k} f_{ik} \big\{ (f + x) R_{ij} R_{jk} - \big(\tfrac{1}{3} f R - y(R) \big) g_{ij} R_{jk} \big\} \\
&\qquad + \sum_{i,j,k} f_i \big\{ f_k R_{ij} R_{jk} + (f + x) R_{jk} \nabla_k R_{ij} - \tfrac{1}{3} f_k R g_{ij} R_{jk} \big\} \\
&= \sum_{i,j,k} \big\{ (f+x) R_{ik} - \big(\tfrac{1}{3} f R - y(R) \big) g_{ik} \big\} \big\{ (f+x) R_{ij} R_{jk} - \big(\tfrac{1}{3} f R - y(R) \big) g_{ij} R_{jk} \big\} \\
&\qquad + \sum_{i,j,k} f_i f_k R_{ij} R_{jk} + (f + x) f_i R_{jk} \nabla_k R_{ij} - \tfrac{1}{3} f_i f_k R g_{ij} R_{jk} \\
&= \sum_{i,j,k} (f + x)^2 R_{ik} R_{ij} R_{jk} + (f + x) f_i R_{jk} \nabla_k R_{ij} + L(s),
\end{aligned}$$

where $L(s)$ is a function of s only, and the Bianchi identity $\nabla_k R_{jk} = \frac{1}{2} \nabla_k R = 0$ is used in obtaining the third equality.

Using $\nabla_k R_{ij} = \nabla_i R_{jk}$, we get

$$\sum_{i,j,k} \nabla_k (f_i f_{ij} R_{jk}) = \sum_{i,j,k} (f + x)^2 R_{ik} R_{ij} R_{jk} + \tfrac{1}{2} (f + x) f_i \nabla_i (R_{jk} R_{jk}) + L(s). \tag{3-2}$$

All terms except $(f+x)^2 R_{ij}R_{jk}R_{ik}$ in the right-hand side of (3-2) depend on s only. From the constancy of R and (3-1) we also get

$$
\begin{aligned}
(3\text{-}3)\quad & \sum_{i,j,k} 2\nabla_k(f_i f_{ij} R_{jk}) \\
&= \sum_{i,j,k} \nabla_k(2f_i f_{ij})\cdot R_{jk} = \sum_{i,j,k} \nabla_k\nabla_j(f_i f_i)\cdot R_{jk} \\
&= \sum_{i,j,k} E_kE_j(f_if_i)\cdot R_{jk} - (\nabla_{E_k}E_j)(f_if_i)\cdot R_{jk} \\
&= \sum_{j,i} E_jE_j(f_if_i)\cdot R_{jj} - (\nabla_{E_j}E_j)(f_if_i)\cdot R_{jj} \\
&= \sum_{i} E_1E_1(f_if_i)\cdot R_{11} + \sum_{j=2}^{4} \zeta_j E_1(|\nabla f|^2)\cdot R_{jj} \\
&= (|\nabla f|^2)''\cdot R_{11} + \sum_{j=2}^{4} \frac{(f+x)R_{jj}R_{jj} - \frac{1}{3}RfR_{jj} + y(R)R_{jj}}{|\nabla f|} E_1(|\nabla f|^2),
\end{aligned}
$$

which depends only on s.

So, we compare (3-2) with (3-3) to see that $R_{ij}R_{jk}R_{ik}$ depends only on s. Now λ_1 and $\sum_{i=1}^{4}(\lambda_i)^k$, $k=1,2,3$, depend only on s. This implies that each λ_i, $i=1,2,3,4$, depends only on s. By (3-1), ζ_i, $i=2,3,4$, depends on s only. □

4. Four-dimensional space with distinct $\lambda_2, \lambda_3, \lambda_4$

Let (M, g, f) be a four-dimensional Riemannian manifold with harmonic curvature satisfying (1-1). For an adapted frame field $\{E_j\}$ with its eigenvalue λ_j in an open subset of $M_{\mathrm{Rc}} \cap \{\nabla f \neq 0\}$, we may only consider three cases depending on the distinctiveness of $\lambda_2, \lambda_3, \lambda_4$; the first case is when λ_i, $i=2,3,4$, are all equal (on an open subset), and the second is when exactly two of the three are equal. And the last is when the three λ_i, $i=2,3,4$, are mutually distinct. In this section we shall study the last case. Note that by (3-1) two eigenvalues λ_i and λ_j are distinct if and only if ζ_i and ζ_j are. We set $\Gamma_{ij}^k := \langle \nabla_{E_i}E_j, E_k\rangle$.

Lemma 4.1. *Let (M, g, f) be a four-dimensional Riemannian manifold with harmonic curvature satisfying (1-1) with nonconstant f. Suppose that for an adapted frame field E_j, $j=1,2,3,4$, in an open subset W of $M_{\mathrm{Rc}} \cap \{\nabla f \neq 0\}$, the eigenvalues $\lambda_2, \lambda_3, \lambda_4$ are distinct from each other. Then the following hold in W*:

$$
\begin{aligned}
&R_{1ij1} = 0 \quad \textit{for distinct } i, j > 1,\\
&R_{1ii1} = -\zeta_i' - \zeta_i^2,\\
&R_{1ii1} = -R_{ii} + \tfrac{1}{3}R,
\end{aligned}
$$

where

$$R_{11} = -\zeta_2' - \zeta_2^2 - \zeta_3' - \zeta_3^2 - \zeta_4' - \zeta_4^2,$$
$$R_{22} = -\zeta_2' - \zeta_2^2 - \zeta_2\zeta_3 - \zeta_2\zeta_4 - 2\Gamma_{34}^2\Gamma_{43}^2,$$
$$R_{33} = -\zeta_3' - \zeta_3^2 - \zeta_3\zeta_2 - \zeta_3\zeta_4 + 2\frac{\zeta_2-\zeta_4}{\zeta_3-\zeta_4}\Gamma_{34}^2\Gamma_{43}^2,$$
$$R_{44} = -\zeta_4' - \zeta_4^2 - \zeta_4\zeta_2 - \zeta_4\zeta_3 + 2\frac{\zeta_2-\zeta_3}{\zeta_4-\zeta_3}\Gamma_{34}^2\Gamma_{43}^2,$$

Proof. Now $\nabla_{E_1}E_1 = 0$ from Lemma 2.3(vi) and $\nabla_{E_i}E_1 = \zeta_i E_i$ from (3-1). Let $i, j > 1$ be distinct. From Lemma 2.4(iii) and Lemma 3.1, $\langle \nabla_{E_i}E_i, E_j\rangle = 0$. Since $\langle \nabla_{E_i}E_i, E_1\rangle = -\langle E_i, \nabla_{E_i}E_1\rangle = -\zeta_i$, we get $\nabla_{E_i}E_i = -\zeta_i E_1$. Now,

$$\langle \nabla_{E_i}E_j, E_i\rangle = -\langle \nabla_{E_i}E_i, E_j\rangle = 0,$$
$$\langle \nabla_{E_i}E_j, E_j\rangle = 0,$$
$$\langle \nabla_{E_i}E_j, E_1\rangle = -\langle \nabla_{E_i}E_1, E_j\rangle = 0.$$

So, $\nabla_{E_i}E_j = \Gamma_{ij}^k E_k$, where k is the number such that $\{2, 3, 4\} = \{i, j, k\}$. Clearly $\Gamma_{ij}^k = -\Gamma_{ik}^j$. From Lemma 2.4(ii), $(\lambda_i - \lambda_j)\langle \nabla_{E_1}E_i, E_j\rangle = (\lambda_1 - \lambda_j)\langle \nabla_{E_i}E_1, E_j\rangle$. As $\langle \nabla_{E_i}E_1, E_j\rangle = 0$, we have $\langle \nabla_{E_1}E_i, E_j\rangle = 0$. This gives $\nabla_{E_1}E_i = 0$. Summarizing, we have the following for $i, j > 1$, $i \neq j$:

$$\nabla_{E_1}E_1 = 0, \quad \nabla_{E_i}E_1 = \zeta_i E_i, \quad \nabla_{E_i}E_i = -\zeta_i E_1, \quad \nabla_{E_1}E_i = 0,$$
$$\nabla_{E_i}E_j = \Gamma_{ij}^k E_k, \quad \text{where } k \text{ is the number such that } \{2, 3, 4\} = \{i, j, k\}.$$

One uses Lemma 3.1 in computing curvature components. For $i > 1$, we get $R_{1ii1} = -\zeta_i' - \zeta_i^2$, and for distinct $i, j, k > 1$, we get

$$R_{jiij} = -\zeta_j\zeta_i - \Gamma_{ji}^k\Gamma_{ik}^j - \Gamma_{ji}^k\Gamma_{ki}^j + \Gamma_{ij}^k\Gamma_{ki}^j,$$
$$R_{kijk} = E_k(\Gamma_{ij}^k),$$
$$R_{1ij1} = 0.$$

From Lemma 2.4, for distinct $i, j, k > 1$, we have

$$(\zeta_j - \zeta_k)\Gamma_{ij}^k = (\zeta_i - \zeta_k)\Gamma_{ji}^k, \tag{4-1}$$

which helps to express R_{ii}. Lemma 2.2 gives

$$-R(E_1, E_i, E_i, \nabla f) = \big(R_{ii} - \tfrac{1}{3}R\big)g(\nabla f, E_1)$$

for $i > 1$. From this we get

$$R_{1ii1} = -R_{ii} + \tfrac{1}{3}R. \tag{4-2}$$

□

From the proof of the above lemma, we may write

$$[E_2, E_3] = \alpha E_4, \quad [E_3, E_4] = \beta E_2, \quad [E_4, E_2] = \gamma E_3. \tag{4-3}$$

From the Jacobi identity $[[E_1, E_2], E_3] + [[E_2, E_3], E_1] + [[E_3, E_1], E_2] = 0$, we have

$$E_1(\alpha) = \alpha(\zeta_4 - \zeta_2 - \zeta_3). \tag{4-4}$$

Moreover, (4-1) gives

$$\beta = \frac{(\zeta_3 - \zeta_4)^2}{(\zeta_2 - \zeta_3)^2}\alpha, \quad \gamma = \frac{(\zeta_2 - \zeta_4)^2}{(\zeta_2 - \zeta_3)^2}\alpha. \tag{4-5}$$

We set $a := \zeta_2$, $b := \zeta_3$ and $c := \zeta_4$. Lemma 4.1 states two formulas for R_{1ii1}: $R_{1ii1} = -\zeta_i' - \zeta_i^2$ and $R_{1ii1} = -R_{ii} + \frac{1}{3}R$ for $i > 1$. So we have $R_{22} - R_{33} = a' + a^2 - b' - b^2$. The Ricci curvature formulas in Lemma 4.1 also give

$$R_{22} - R_{33} = -a' - a^2 + b' + b^2 - ac - 2\Gamma_{34}^2\Gamma_{43}^2 + bc - 2\frac{a-c}{b-c}\Gamma_{34}^2\Gamma_{43}^2.$$

Adding the last two equalities, we obtain

$$2(R_{22} - R_{33}) = (b-a)c - 2\Gamma_{34}^2\Gamma_{43}^2 - 2\frac{a-c}{b-c}\Gamma_{34}^2\Gamma_{43}^2.$$

From (1-1), $\zeta_i f' = f\left(R_{ii} - \frac{1}{3}R\right) + xR_{ii} + y(R)$ for $i > 1$. Then we get

$$(a-b)\frac{f'}{f} = \left(1 + \frac{x}{f}\right)(R_{22} - R_{33}) = \frac{1}{2}\left(1 + \frac{x}{f}\right)\left[(b-a)c - 2\left\{1 + \frac{a-c}{b-c}\right\}\Gamma_{34}^2\Gamma_{43}^2\right].$$

So,

$$-\frac{f'}{f} = \frac{1}{2}\left(1 + \frac{x}{f}\right)\left[c + 2\frac{a+b-2c}{(a-b)(b-c)}\Gamma_{34}^2\Gamma_{43}^2\right]. \tag{4-6}$$

Similarly,

$$(a-c)\frac{f'}{f} = \frac{1}{2}\left(1 + \frac{x}{f}\right)\left[(c-a)b - 2\left\{1 + \frac{a-b}{c-b}\right\}\Gamma_{34}^2\Gamma_{43}^2\right].$$

So,

$$-\frac{f'}{f} = \frac{1}{2}\left(1 + \frac{x}{f}\right)\left[b + 2\frac{a+c-2b}{(a-c)(c-b)}\Gamma_{34}^2\Gamma_{43}^2\right]. \tag{4-7}$$

From (4-6) and (4-7), we get

$$4\Gamma_{34}^2\Gamma_{43}^2 = \frac{(a-b)(a-c)(b-c)^2}{(a^2+b^2+c^2-ab-bc-ac)}, \tag{4-8}$$

$$-\frac{f'}{f} = \frac{1}{2}\left(1 + \frac{x}{f}\right)\frac{a^2b + a^2c + ab^2 + ac^2 + b^2c + c^2b - 6abc}{2(a^2+b^2+c^2-ab-bc-ac)}. \tag{4-9}$$

The formula (4-2) gives $R_{1212} - R_{1313} = R_{22} - R_{33}$, which reduces to

$$2(a'-b') = -2(a^2-b^2)+bc-ac+\frac{(a-b)(b-c)(c-a)(a+b-2c)}{2(a^2+b^2+c^2-ab-bc-ac)} \tag{4-10}$$
$$= -2(a^2-b^2)+\frac{a-b}{2P}A,$$

where we set $P := a^2+b^2+c^2-ab-bc-ac$, and $A := 6abc - a^2b - ab^2 - a^2c - ac^2 - b^2c - bc^2$. By symmetry, we get

$$\zeta_i' - \zeta_j' = -(\zeta_i^2 - \zeta_j^2) + \frac{\zeta_i - \zeta_j}{4P}A \quad \text{for } i, j \in \{2, 3, 4\}. \tag{4-11}$$

The formula (4-11) looks different from the corresponding one in the soliton case in [Kim 2017]: $\zeta_i' - \zeta_j' = -(\zeta_i^2 - \zeta_j^2)$. But surprisingly the next proposition still works in resolving (1-1); refer to Proposition 3.4 in [Kim 2017]. Here the formula (4-9) is crucial.

Proposition 4.2. *Let (M, g, f) be a four-dimensional Riemannian manifold with harmonic curvature, satisfying (1-1) with nonconstant f. For any adapted frame field E_j, $j = 1, 2, 3, 4$, in an open dense subset $M_{\mathrm{Rc}} \cap \{\nabla f \neq 0\}$ of M, the three eigenfunctions $\lambda_2, \lambda_3, \lambda_4$ cannot be pairwise distinct, i.e., at least two of the three coincide.*

Proof. Suppose that $\lambda_2, \lambda_3, \lambda_4$ are pairwise distinct. We shall prove then that f should be a constant, a contradiction to the hypothesis.

From (4-8) and (4-1),

$$(\alpha - \gamma + \beta)^2 = 4(\Gamma_{34}^2)^2 = 4\Gamma_{34}^2\Gamma_{43}^2\frac{a-b}{a-c} = \frac{(a-b)^2(b-c)^2}{(a^2+b^2+c^2-ab-bc-ac)}.$$

From (4-5),

$$(\alpha - \gamma + \beta)^2 = \alpha^2\left\{1 - \frac{(a-c)^2}{(a-b)^2} + \frac{(b-c)^2}{(a-b)^2}\right\}^2 = \frac{4\alpha^2(b-c)^2}{(a-b)^2}.$$

So, $\alpha^2 = (a-b)^4/(4P)$. Since a, b, c are all functions of s only, so is α. We compute from (4-11)

$$\begin{aligned}(a-b)(a'-b') + (a-c)(a'-c') + (b-c)(b'-c') \tag{4-12}\\ = -(a-b)(a^2-b^2) - (a-c)(a^2-c^2) - (b-c)(b^2-c^2)\\ + \frac{A}{4P}\{(a-b)^2 + (a-c)^2 + (b-c)^2\}\\ = -2(a^3+b^3+c^3) + a^2b + ab^2 + a^2c + ac^2 + b^2c + bc^2 + \tfrac{1}{2}A\\ = -2(a^3+b^3+c^3-3abc) - \tfrac{1}{2}A.\end{aligned}$$

Differentiating $\alpha^2=(a-b)^4/(4P)$ in s and using (4-11) and (4-12),

$$
\begin{aligned}
2\alpha\alpha' &= \frac{(a-b)^3(a'-b')}{P}-\frac{(a-b)^4(2aa'+2bb'+2cc'-ab'-ba'-ac'-ca'-cb'-bc')}{4P^2}\\
&= \frac{-(a-b)^3(a^2-b^2)}{P}+\frac{(a-b)^4}{4P^2}A\\
&\qquad -\frac{(a-b)^4\{(a-b)(a'-b')+(a-c)(a'-c')+(b-c)(b'-c')\}}{4P^2}\\
&= -\frac{(a-b)^4(a+b)}{P}+\frac{(a-b)^4}{4P^2}A+\frac{(a-b)^4\{2(a^3+b^3+c^3-3abc)\}}{4P^2}+\frac{(a-b)^4\{\frac{1}{2}A\}}{4P^2}\\
&= -\frac{(a-b)^4}{P}\frac{(a+b-c)}{2}+\frac{3(a-b)^4}{8P^2}A.
\end{aligned}
$$

Meanwhile, from (4-4) and $\alpha^2=(a-b)^4/(4P)$,

$$2\alpha\alpha' = 2\alpha^2(c-a-b) = -\frac{(a-b)^4}{2P}(a+b-c).$$

Equating these two expressions for $2\alpha\alpha'$, we get $A=0$. From (4-9), $f'=0$. □

5. Four-dimensional space with $\lambda_2 \neq \lambda_3 = \lambda_4$

In this section we study when exactly two of $\lambda_2, \lambda_3, \lambda_4$ are equal. We may well assume that $\lambda_2 \neq \lambda_3 = \lambda_4$. By (3-1) we then have $\zeta_2 \neq \zeta_3 = \zeta_4$. We use (3-1), Lemma 2.4 and Lemma 3.1 to compute $\nabla_{E_i} E_j$ and get the next lemma.

Lemma 5.1. *Let (M, g, f) be a four-dimensional Riemannian manifold with harmonic curvature satisfying (1-1) with nonconstant f. Suppose that $\lambda_2 \neq \lambda_3 = \lambda_4$ for an adapted frame field E_j, $j = 1, 2, 3, 4$, on an open subset U of $M_{\mathrm{Rc}} \cap \{\nabla f \neq 0\}$. Then we have*

$$[E_1, E_2] = -\zeta_2 E_2,$$
$$\langle \nabla_{E_i} E_j, E_2\rangle = 0 \quad \textit{and} \quad \langle \nabla_{E_i} E_j, E_1\rangle = -\delta_{ij}\zeta_3 \quad \textit{for } i, j \in \{3, 4\}.$$

In particular, the distribution spanned by E_1 and E_2 is integrable. So is that spanned by E_3 and E_4.

Proof. From Lemma 2.4 (ii) and (3-1),

$$(\lambda_2-\lambda_i)\langle \nabla_{E_1} E_2, E_i\rangle = (\lambda_1-\lambda_i)\langle \nabla_{E_2} E_1, E_i\rangle = (\lambda_1-\lambda_i)\langle \zeta_2 E_2, E_i\rangle = 0$$

for $i = 3, 4$. This gives $\nabla_{E_1} E_2 = 0$, and so $[E_1, E_2] = -\zeta_2 E_2$.

From Lemma 2.4 (ii), $(\lambda_2-\lambda_4)\langle \nabla_{E_3} E_2, E_4\rangle = (\lambda_3-\lambda_4)\langle \nabla_{E_2} E_3, E_4\rangle = 0$. So, $\langle \nabla_{E_3} E_2, E_4\rangle = -\langle E_2, \nabla_{E_3} E_4\rangle = 0$. This and (3-1) yield $\nabla_{E_3} E_4 = \beta_3 E_3$ for some function β_3. Similarly, $\nabla_{E_4} E_3 = -\beta_4 E_4$ for some function β_4. Then $[E_3, E_4] =$

$\beta_3 E_3 + \beta_4 E_4$. For $i = 3, 4$, Lemma 2.4(iii) and Lemma 3.1 give $\langle \nabla_{E_i} E_i, E_2\rangle = 0$ and (3-1) gives $\langle \nabla_{E_i} E_j, E_1\rangle = -\delta_{ij}\zeta_3$ for $i, j \in \{3, 4\}$. □

We shall express the metric g in a simple form as in the next lemma.

Lemma 5.2. *Under the same hypothesis as Lemma 5.1, for each point p_0 in U, there exists a neighborhood V of p_0 in U with coordinates (s, t, x_3, x_4) such that $\nabla s = \nabla f/|\nabla f|$ and g can be written on V as*

$$g = ds^2 + p(s)^2\, dt^2 + h(s)^2 \tilde{g}, \tag{5-1}$$

where $p := p(s)$ and $h := h(s)$ are smooth functions of s and $\tilde{g}$ is (a pull-back of) a Riemannian metric of constant curvature, say k, on a two-dimensional domain with x_3, x_4 coordinates.

Proof. Once Lemma 5.1 is in hand, this lemma may follow from the proof of Lemma 4.3 in [Kim 2017]. We produce a simplified proof for the sake of completeness.

We let D^1 be the two-dimensional distribution spanned by $E_1 = \nabla s$ and E_2, and let D^2 be the one spanned by E_3 and E_4. Then D^1 and D^2 are both integrable by Lemma 5.1. We may consider the coordinates (x_1, x_2, x_3, x_4) from Lemma 4.2 of [Kim 2017], so that D^1 is tangent to the two-dimensional level sets $\{(x_1, x_2, x_3, x_4)\,|\,x_3, x_4 \text{ constants}\}$ and D^2 is tangent to the level sets $\{(x_1, x_2, x_3, x_4)\,|\, x_1, x_2 \text{ constants}\}$. We may write g as

$$g = g_{11}\, dx_1^2 + g_{12}\, dx_1 \odot dx_2 + g_{22}\, dx_2^2 + g_{33}\, dx_3^2 + g_{34}\, dx_3 \odot dx_4 + g_{44}\, dx_4^2,$$

where $\odot$ is the symmetric tensor product and g_{ij} are functions of (x_1, x_2, x_3, x_4).

Defining a 1-form $\omega_2(\,\cdot\,) := g(E_2, \cdot\,)$, we can see that

$$ds^2 + \omega_2^2 = g_{11}\, dx_1^2 + g_{12}\, dx_1 \odot dx_2 + g_{22}\, dx_2^2.$$

Setting a function

$$p(s) := e^{\int_{s_0}^{s} \zeta_2(u)\, du}$$

for a constant s_0, we can check that $d(\omega_2/p) = 0$ from Lemma 5.1. So, $\omega_2/p = dt$ for some local function t modulo a constant. The metric g can be now written as

$$g = ds^2 + p(s)^2\, dt^2 + g_{33}\, dx_3^2 + g_{34}\, dx_3 \odot dx_4 + g_{44}\, dx_4^2. \tag{5-2}$$

Writing $\partial_i := \partial/\partial x_i$ in new coordinates $(x_1 := s, x_2 := t, x_3, x_4)$, from Lemma 5.1, we compute $0 = \langle \nabla_{\partial_i} \partial_j, \partial_2\rangle = -\frac{1}{2}\partial_2 g_{ij}$ for $i, j \in \{3, 4\}$.

We consider the second fundamental form of a leaf for D^2 with respect to E_1: $H^{E_1}(u, v) = -\langle \nabla_u v, E_1\rangle$. For $i, j \in \{3, 4\}$, from Lemma 5.1

$$\zeta_3 g_{ij} = H^{E_1}(\partial_i, \partial_j) = -\Big\langle \nabla_{\partial_i} \partial_j, \frac{\partial}{\partial s}\Big\rangle = \frac{1}{2}\frac{\partial}{\partial s} g_{ij}.$$

If $g_{34} > 0$ or $g_{34} < 0$ in a neighborhood of p_0, we can integrate the above and get

$$\ln |g_{ij}| = \int_{c_0}^{s} 2\zeta_3(u)\, du + C_{ij}(x_3, x_4)$$

for $i, j \in \{3, 4\}$ and a constant c_0. Setting

$$h(s) := e^{\int_{c_0}^{s} \zeta_3(u)\, du},$$

we have $|g_{ij}| = (h(s))^2 e^{C_{ij}(x_3,x_4)}$. Then we may write

$$G := g_{33}\, dx_3^2 + g_{34}\, dx_3 \odot dx_4 + g_{44}\, dx_4^2 = (h(s))^2 \tilde{g},$$

where $\tilde{g}$ is a Riemannian metric in a domain of the (x_3, x_4)-plane.

If $g_{34}(p_0) = 0$, by changing coordinates as $x_3 = z_3$ and $x_4 = z_3 + z_4$, we get

$$\begin{aligned} G &= g_{33}\, dz_3^2 + g_{34}\, dz_3 \odot (dz_3 + dz_4) + g_{44}(dz_3 + dz_4)^2 \\ &= a_{33}\, dz_3^2 + a_{34}\, dz_3 \odot dz_4 + a_{44}\, dz_4^2, \end{aligned}$$

where $a_{ij} = g(\partial/\partial z_i, \partial/\partial z_j)$. As $g_{44}(p_0) > 0$, we have $a_{34}(p_0) \neq 0$. So, $a_{34} \neq 0$ in a neighborhood of p_0. In z_i-coordinates we can still have $\partial_2 a_{ij} = 0$ and $\zeta_3 a_{ij} = \frac{1}{2}(\partial/\partial s)a_{ij}$. Arguing as the above paragraph, we can write G in the form $G = (h(s))^2 \tilde{g}$, where

$$h(s) := e^{\int_{c_1}^{s} \zeta_3(u)\, du}$$

for a constant c_1 and $\tilde{g}$ is a Riemannian metric in a domain of the (z_3, z_4)-plane which is also a domain of the (x_3, x_4)-plane.

In any case g can be written as $g = ds^2 + p(s)^2\, dt^2 + h(s)^2 \tilde{g}$, where $\tilde{g}$ can be viewed as a Riemannian metric in a domain of the (x_3, x_4)-plane.

The argument used in the proof of Lemma 4 in [Derdziński 1980] can prove that $\tilde{g}$ has constant curvature, say k. □

6. Analysis of the metric when $\lambda_2 \neq \lambda_3 = \lambda_4$

We continue to suppose that $\lambda_2 \neq \lambda_3 = \lambda_4$ for an adapted frame field E_j, $j = 1, 2, 3, 4$.

The metric $\tilde{g}$ in (5-1) can be written locally: $\tilde{g} = dr^2 + u(r)^2 d\theta^2$ on a domain in $\mathbb{R}^2$ with polar coordinates (r, θ), where $u''(r) = -ku$. We set an orthonormal basis

$$e_3 = \frac{\partial}{\partial r} \quad \text{and} \quad e_4 = \frac{1}{u(r)} \frac{\partial}{\partial \theta}.$$

Lemma 6.1. *For the local metric $g = ds^2 + p(s)^2\, dt^2 + h(s)^2 \tilde{g}$ with harmonic curvature satisfying (1-1) with nonconstant f, obtained in Lemma 5.2, if we set*

$$E_1 = \frac{\partial}{\partial s}, \quad E_2 = \frac{1}{p(s)} \frac{\partial}{\partial t}, \quad E_3 = \frac{1}{h(s)} e_3, \quad E_4 = \frac{1}{h(s)} e_4,$$

where e_3 and e_4 are as in the above paragraph, then we have the following. Here $R_{ij} = R(E_i, E_j)$ and $R_{ijkl} = R(E_i, E_j, E_k, E_l)$:

$$\nabla_{E_1} E_1 = 0,$$

$$\textit{for } i = 2, 3, 4, \quad \nabla_{E_1} E_i = 0, \quad \nabla_{E_i} E_1 = \zeta_i E_i, \quad \textit{where } \zeta_2 = \frac{p'}{p},\ \zeta_3 = \zeta_4 = \frac{h'}{h},$$

$$\nabla_{E_2} E_2 = -\zeta_2 E_1, \quad \nabla_{E_2} E_3 = 0, \quad \nabla_{E_2} E_4 = 0, \quad \nabla_{E_3} E_2 = 0,$$

$$\nabla_{E_3} E_3 = -\zeta_3 E_1, \quad \nabla_{E_3} E_4 = 0, \quad \nabla_{E_4} E_2 = 0, \quad \nabla_{E_4} E_3 = -\beta_4 E_4,$$

$$\nabla_{E_4} E_4 = -\zeta_4 E_1 + \beta_4 E_3 \quad \textit{for some function } \beta_4,$$

and

$$R_{1221} = -\frac{p''}{p} = -\zeta_2' - \zeta_2^2,$$

$$R_{1ii1} = -\zeta_i' - \zeta_i^2 = -\frac{h''}{h} \quad \textit{for } i = 3, 4,$$

$$R_{11} = -\zeta_2' - \zeta_2^2 - 2\zeta_3' - 2\zeta_3^2 = -\frac{p''}{p} - 2\frac{h''}{h},$$

$$R_{22} = -\zeta_2' - \zeta_2^2 - 2\zeta_2\zeta_3 = -\frac{p''}{p} - 2\frac{p'}{p}\frac{h'}{h},$$

$$R_{33} = R_{44} = -\zeta_3' - \zeta_3^2 - \zeta_3\zeta_2 - (\zeta_3)^2 + \frac{k}{h^2} = -\frac{h''}{h} - \frac{p'}{p}\frac{h'}{h} - \frac{(h')^2}{h^2} + \frac{k}{h^2},$$

$$R_{ij} = 0 \quad \textit{for } i \neq j.$$

Proof. Now $\nabla_{E_1} E_1 = 0$ from Lemma 2.3(vi) and $\nabla_{E_i} E_1 = \zeta_i E_i$, $i > 1$, from (3-1). From the proof of Lemma 5.1, we already have $\nabla_{E_1} E_2 = 0$, $\nabla_{E_3} E_4 = \beta_3 E_3$ and $\nabla_{E_4} E_3 = -\beta_4 E_4$.

As $\langle \nabla_{E_1} E_3, E_2\rangle = -\langle E_3, \nabla_{E_1} E_2\rangle = 0$, one can readily get $\nabla_{E_1} E_3 = \rho E_4$ for some function ρ and $\nabla_{E_1} E_4 = -\rho E_3$. We get $\rho = 0$ by computing directly (in coordinates)

$$\nabla_{E_1} E_3 = \nabla_{\partial/\partial s} \frac{1}{h(s)} \frac{\partial}{\partial r} = 0.$$

From Lemma 3.1 and Lemma 2.4(iii), we have

$$(\lambda_2 - \lambda_i)\langle \nabla_{E_2} E_2, E_i\rangle = E_i(\lambda_2) = 0 \quad \text{for } i = 3, 4,$$
$$\langle \nabla_{E_2} E_2, E_1\rangle = -\langle E_2, \nabla_{E_2} E_1\rangle = -\zeta_2(s).$$

So, $\nabla_{E_2} E_2 = -\zeta_2(s) E_1$. By a similar argument, $\nabla_{E_3} E_3 = -\zeta_3 E_1 - \beta_3 E_4$ and $\nabla_{E_4} E_4 = -\zeta_4 E_1 + \beta_4 E_3$. Direct computation of the coordinates gives $\beta_3 = 0$.

Then $\nabla_{E_2} E_3 = q E_4$ for some function q and $\nabla_{E_2} E_4 = -q E_3$. One computes directly that $q = 0$. We similarly get $\nabla_{E_3} E_2 = 0$ and $\nabla_{E_4} E_2 = 0$.

We compute directly that $\nabla_{E_2} E_1 = (p'/p) E_2$ and $\nabla_{E_3} E_1 = (h'/h) E_3$ so that (3-1) gives $\zeta_2 = p'/p$ and $\zeta_3 = \zeta_4 = h'/h$. We now get $\nabla_{E_3} E_4 = 0$ and $\nabla_{E_4} E_3 = -\beta_4 E_4$, where $\beta_4 = u'(r)/(h(s)u(r))$.

With these computations in hand, it is straightforward to compute the curvature components. □

We set $a := \zeta_2$ and $b := \zeta_3$.

Lemma 6.2. *For the local metric $g = ds^2 + p(s)^2\, dt^2 + h(s)^2 \tilde{g}$ with harmonic curvature satisfying (1-1) with nonconstant f, obtained in Lemma 5.2, it holds that*

$$\big(ab + \tfrac{1}{12} R\big) b = 0. \tag{6-1}$$

Proof. Equation (4-2) gives

$$2a' + 2a^2 + 2ab + \tfrac{1}{3} R = 0, \tag{6-2}$$

$$2b' + 3b^2 + ab - \frac{k}{h^2} + \tfrac{1}{3} R = 0. \tag{6-3}$$

From $\nabla df(E_i, E_i) = f\big(\mathrm{Rc} - \frac{1}{3} R g\big)(E_i, E_i) + x R(E_i, E_i) + y(R)$, we get

$$-(\nabla_{E_i} E_i) f = f\big(R_{ii} - \tfrac{1}{3} R\big) + x R_{ii} + y(R) = -f R_{1ii1} + x R_{ii} + y(R)$$

for $i = 2, 3$. From Lemma 6.1 we have

$$f' a = f(a' + a^2) - x(a' + a^2 + 2ab) + y(R), \tag{6-4}$$

$$f' b = f(b' + b^2) - x\left(b' + 2b^2 + ab - \frac{k}{h^2}\right) + y(R). \tag{6-5}$$

From the harmonic curvature condition we have

$$\begin{aligned} 0 &= \nabla_{E_1} R_{22} - \nabla_{E_2} R_{12} = \nabla_{E_1}(R_{22}) + R(\nabla_{E_2} E_1, E_2) + R(\nabla_{E_2} E_2, E_1) \\ &= (R_{22})' + a(R_{22} - R_{11}) \\ &= (-a' - a^2 - 2ab)' + a(-2ab + 2b' + 2b^2) \\ &= -a'' - 2aa' - 2a'b - 2a^2 b + 2ab^2. \end{aligned} \tag{6-6}$$

We differentiate (6-2) to get $a'' + 2aa' + a'b + ab' = 0$. Together with (6-6) we obtain

$$ab' - a'b - 2a^2 b + 2ab^2 = 0. \tag{6-7}$$

Putting (6-2) and (6-3) into (6-7) we get

$$\begin{aligned} 0 &= -a\left(3b^2 + ab - \frac{k}{h^2} + \tfrac{1}{3} R\right) + 2\big(a^2 + ab + \tfrac{1}{6} R\big) b - 4a^2 b + 4ab^2 \\ &= a\frac{k}{h^2} + \tfrac{1}{3} R(b - a) + 3ab(b - a). \end{aligned}$$

Then, as $a \neq b$,

$$\frac{a}{a-b}\frac{k}{h^2} = \tfrac{1}{3}R + 3ab. \tag{6-8}$$

From (6-4) and (6-5) we get

$$\frac{f'}{f}(a-b) = (a'+a^2-b'-b^2) - \frac{x}{f}\left(a'+a^2+2ab-b'-2b^2-ab+\frac{k}{h^2}\right).$$

With (6-3) and (6-2), the above gives

$$2\frac{f'}{f}(a-b) = \left(1+\frac{x}{f}\right)\left(b^2-ab-\frac{k}{h^2}\right).$$

Then by (6-8),

$$2\frac{f'}{f}a = \left(1+\frac{x}{f}\right)\left(-ab-\frac{ka}{h^2(a-b)}\right) = \left(1+\frac{x}{f}\right)\left(-4ab-\tfrac{1}{3}R\right).$$

Meanwhile, (6-4) and (6-2) give $f'a = -f\left(ab+\frac{1}{6}R\right) - x\left(ab-\frac{1}{6}R\right) + y(R)$, so

$$-2\left(ab+\tfrac{1}{6}R\right) - \frac{2x}{f}\left(ab-\tfrac{1}{6}R\right) + \frac{2y(R)}{f} = 2\frac{f'}{f}a = \left(1+\frac{x}{f}\right)\left(-4ab-\tfrac{1}{3}R\right).$$

So we obtain

$$x\left(ab+\tfrac{1}{3}R\right) + y(R) = -fab. \tag{6-9}$$

Differentiating (6-9) and dividing by f,

$$\frac{f'}{f}ab = -\frac{x}{f}(a'b+ab') - (a'b+ab').$$

From (6-4) we get

$$\frac{f'}{f}ab = (a'+a^2)b - \frac{x}{f}(a'+a^2+2ab)b + \frac{yb}{f}.$$

Equating the above and arranging terms, we get

$$\frac{x}{f}(-ab'+a^2b+2ab^2) = 2a'b+ab'+a^2b+\frac{yb}{f}.$$

Using (6-9) we get

$$\frac{x}{f}\left(-ab'+a^2b+3ab^2+\tfrac{1}{3}Rb\right) = 2a'b+ab'+a^2b-ab^2. \tag{6-10}$$

Using (6-7) and (6-2), the left-hand side of (6-10) equals $(x/f)\left(6ab^2+\frac{1}{2}Rb\right)$, while the right-hand side equals $-\left(6ab^2+\frac{1}{2}Rb\right)$.

So we get $(1+x/f)\left(6ab+\frac{1}{2}R\right)b = 0$. Then $(ab+\frac{1}{12}R)b = 0$. □

Proposition 6.3. *For the local metric $g = ds^2 + p(s)^2\,dt^2 + h(s)^2\tilde{g}$ with harmonic curvature satisfying (1-1) with nonconstant f, obtained in Lemma 5.2, suppose that $ab = -\frac{1}{12}R$.*

Then $R = 0$, $y(0) = 0$ and p is a constant. The metric g is locally isometric to a domain in the nonconformally flat static space $(W^3 \times \mathbb{R}^1, g_W + dt^2)$ of Example 3 in Section 2A2. Moreover, $f = ch'(s) - x$.

Proof. As $ab = -\frac{1}{12}R$, (6-9) gives $\frac{1}{4}Rx + y(R) = \frac{1}{12}Rf$.

If $R \neq 0$, then f is a constant, a contradiction to the hypothesis. Therefore $R = 0$. Then $y(0) = 0$ from the preceding equation. From (6-2), $a' + a^2 = 0$ and we have two cases: (i) $a = 1/(s+c)$ for a constant c or (ii) $a = 0$.

Case (i): $a = 1/(s+c)$. From (6-4), $f'a = 0$, so f is a constant, a contradiction to the hypothesis.

Case (ii): $a = 0$, i.e., p is a constant. From (6-5) and (6-3), we get $f'(h'/h) = (f+x)(h''/h)$. If h' vanishes, we get $\lambda_2 = \lambda_3$, a contradiction. So we may assume that h is not constant. Then $ch' = f + x$ for a constant $c \neq 0$. Evaluating (1-1) at (E_1, E_1),

$$f'' = (f+x)R(E_1, E_1) - \tfrac{1}{3}Rf + y(R). \tag{6-11}$$

Here we get $f'' = -2(f+x)(h''/h)$, so $h''' = -2h'(h''/h)$. Hence, for a constant α,

$$h^2h'' = \alpha. \tag{6-12}$$

From (6-3),

$$0 = 2b' + 3b^2 - \frac{k}{h^2} = 2\left(\frac{h''}{h}\right) + \left(\frac{h'}{h}\right)^2 - \frac{k}{h^2} = \frac{2\alpha}{h^3} + \left(\frac{h'}{h}\right)^2 - \frac{k}{h^2}.$$

So we have

$$(h')^2 + \frac{2\alpha}{h} - k = 0. \tag{6-13}$$

We have exactly (2-2) and (2-3) in the case $R = 0$ and $n = 3$. At this point we may write

$$g = ds^2 + dt^2 + h(s)^2\tilde{g} = \left(k - \frac{2\alpha}{h}\right)^{-1} dh^2 + dt^2 + h(s)^2\tilde{g}.$$

When $\alpha = 0$, we have $(h')^2 = k \geq 0$. As h is not constant, $k > 0$. When $h' = \pm\sqrt{k} \neq 0$, we have $h = \pm\sqrt{k}s + c_0$ for a constant c_0. One can see that g is a flat metric, a contradiction to $\lambda_2 \neq \lambda_3$.

When $\alpha > 0$, then $k > 0$ from (6-13). We set $r := h/\sqrt{k}$, and then

$$g = \left(1 - \frac{2\alpha}{k\sqrt{k}r}\right)^{-1} dr^2 + dt^2 + r^2\tilde{g}_1,$$

where $\tilde{g}_1$ is the metric of constant curvature 1 on S^2. When $\alpha < 0$, the three-dimensional metric $(1 - 2\alpha/(k\sqrt{k}r))^{-1}dr^2 + r^2\tilde{g}_1$ corresponds to case III.1 of Kobayashi's conditions [1982, p. 670]. It is incomplete as explained in his Proposition 2.4.

In these two cases of $\alpha > 0$ and $\alpha < 0$, we get the same Riemannian metrics as those of static spaces $(W^3 \times \mathbb{R}^1, g_W + dt^2)$ explained in Example 3, and $f = ch' - x$.

Conversely, these metrics have harmonic curvature and satisfy (1-1) with the above f. Indeed, nontrivial components of (1-1) are (6-4), (6-5) and (6-11), whereas the harmonic curvature condition essentially consists of (6-6) and the equation $\nabla_{E_1} R_{33} - \nabla_{E_3} R_{13} = 0$; all these can be verified from $a = R = y(0) = 0$ and h, f which satisfy (6-12), (6-13) and $f = ch' - x$. □

Proposition 6.4. *For the local metric $g = ds^2 + p(s)^2\,dt^2 + h(s)^2\tilde{g}$ with harmonic curvature satisfying (1-1) with nonconstant f, obtained in Lemma 5.2, suppose that $b = 0$ and that $ab = 0 \neq -\frac{1}{12}R$. Then the following hold*:

(i) $\frac{1}{3}xR + y(R) = 0$.

(ii) *If $R > 0$, then g is locally isometric to the Riemannian product $\big(\mathbb{S}^2\big(\frac{R}{6}\big) \times \mathbb{S}^2\big(\frac{R}{3}\big), g_{R/6} + g_{R/3}\big)$, where g_δ is the two-dimensional Riemannian metric of constant curvature δ, and $f = c_1 \cos\big(\sqrt{\frac{R}{6}}s\big) - x$ for any constant c_1, where s is the distance from a point on $\mathbb{S}^2\big(\frac{R}{6}\big)$.*

(iii) *If $R < 0$, then g is locally isometric to $\big(\mathbb{H}^2\big(\frac{R}{6}\big) \times \mathbb{H}^2\big(\frac{R}{3}\big), g_{R/6} + g_{R/3}\big)$. The metric $g_{R/6}$ can be written as $g_{R/6} = ds^2 + p(s)^2\,dt^2$ with $p(s) = k_1 \sinh\big(\sqrt{-\frac{R}{6}}s\big) + k_2\cosh\big(\sqrt{-\frac{R}{6}}s\big)$ for constants k_1, k_2, and then $f = c_2 p' - x$ for any constant c_2.*

Proof. As $b = 0$, (6-9) gives (i). Next, (6-3) gives $k/h^2 = \frac{1}{3}R$ and (6-2) gives $a' + a^2 + \frac{1}{6}R = p''/p + \frac{1}{6}R = 0$. Along with (6-4) these give

$$f'a = -\tfrac{1}{6}R(f + x). \tag{6-14}$$

Assume $R > 0$. Set $r_0 = \sqrt{\frac{R}{6}}$. For some constants $C_1 \neq 0$ and s_0, we have $p = C_1 \sin(r_0(s + s_0))$ so that $a = r_0 \cot(r_0(s + s_0))$. Then (6-14) and (i) give $f = c_1 \cos(r_0(s + s_0)) - x$. Then $g = ds^2 + \sin^2(r_0(s + s_0))\,dt^2 + \tilde{g}_{R/3}$ by absorbing a constant into dt^2 and using $k/h^2 = \frac{1}{3}R$.

Replacing $s + s_0$ by a new s, we have $g = ds^2 + \sin^2(r_0 s)\,dt^2 + \tilde{g}_{R/3}$. Here s becomes the distance from a point on $\mathbb{S}^2\big(\frac{R}{6}\big)$. And $f = c_1 \cos(r_0 s) - x$.

Assume $R < 0$. From $p''/p + \frac{1}{6}R = 0$ we get $p(s) = k_1 \sinh(r_1 s) + k_2 \cosh(r_1 s)$ for constants k_1, k_2, where $r_1 = \sqrt{-\frac{R}{6}}$, and $f = c_2 p' - x$ for any constant c_2.

Conversely, the above product metrics clearly have harmonic curvature. One can check they satisfy (1-1). Indeed, as in the proof of Proposition 6.3 one may check (6-4), (6-5) and (6-11). □

7. Local four-dimensional space with harmonic curvature

We first treat the remaining case of $\lambda_2 = \lambda_3 = \lambda_4$ and then give the proof of Theorem 1.1.

Proposition 7.1. *Let (M, g, f) be a four-dimensional Riemannian manifold with harmonic curvature satisfying (1-1) with nonconstant f. Suppose that $\lambda_2 = \lambda_3 = \lambda_4 \neq \lambda_1$ for an adapted frame field in an open subset U of $M_{\mathrm{Rc}} \cap \{\nabla f \neq 0\}$.*

Then for each point p_0 in U, there exists a neighborhood V of p_0 in U where g is a warped product,

$$g = ds^2 + h(s)^2 \tilde{g}, \tag{7-1}$$

where h is a positive function and the Riemannian metric $\tilde{g}$ has constant curvature, say k. In particular, g is conformally flat.

As a Riemannian manifold, (M, g) is locally one of Kobayashi's warped product spaces, as described in Sections 2 and 3 of [Kobayashi 1982], so that

$$h'' + \tfrac{1}{12} R h = a h^{-3} \tag{7-2}$$

for a constant a, so that by integration we have for some constant k

$$(h')^2 + a h^{-2} + \tfrac{1}{12} R h^2 = k. \tag{7-3}$$

Moreover, f is a nonconstant solution to

$$h' f' - f h'' = x\big(h'' + \tfrac{1}{3} R h\big) + y(R) h. \tag{7-4}$$

Conversely, any (h, f) satisfying (7-2), (7-3) and (7-4) gives rise to (g, f) which has harmonic curvature and satisfies (1-1).

Proof. To prove that g is in the form of (7-1), we may use Lemma 2.3(v) and Lemma 2.4(iii)–(iv). For a detailed proof we refer to that of Proposition 7.1 of [Kim 2017] since the argument is almost the same as in the gradient Ricci soliton case. To prove that $\tilde{g}$ has constant curvature, we use Lemma 4 in [Derdziński 1980]. It then follows that the metric g in (7-1) is conformally flat.

In the setting of Lemma 2.3, f is a function of s only. For $g = ds^2 + h(s)^2 \tilde{g}$, in a local adapted frame field, we have

$$\begin{gathered} R_{11} = -3\frac{h''}{h}, \quad R_{ii} = -\frac{h''}{h} - 2\frac{(h')^2}{h^2} + 2\frac{k}{h^2}, \\ R_{ij} = 0 \quad \text{for } i \neq j, \\ R = -6\frac{h''}{h} - 6\frac{(h')^2}{h^2} + 6\frac{k}{h^2}. \end{gathered} \tag{7-5}$$

Feeding (E_i, E_i), $i = 1, 2$ to (1-1) we obtain

$$f'' = -3f\frac{h''}{h} - f\tfrac{1}{3}R - 3x\frac{h''}{h} + y(R), \tag{7-6}$$

$$h' f' - f h'' = x\big(h'' + \tfrac{1}{3} R h\big) + y(R) h. \tag{7-7}$$

Differentiating (7-7) and using (7-6), we get

$$(f+x)\left\{h''' + 3\frac{h''h'}{h} + \tfrac{1}{3}Rh'\right\} = 0.$$

As $f \neq -x$, we get

$$h''' + 3\frac{h''h'}{h} + \tfrac{1}{3}Rh' = 0.$$

Multiplying this by h^3, we get $\left(h^3h'' + \frac{1}{12}Rh^4\right)' = 0$. Then we have (7-2) and then (7-3). Kobayashi solved these completely according to each parameter and initial condition.

One can check that any h and f satisfying (7-7), (7-2) and (7-3) satisfy (7-5) and (7-6). $\square$

We are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Recall that we have already discussed the case $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4$ in Example 1 of Section 2A2. The conformally flat spaces in Example 1 belong to the type (iv) of Theorem 1.1; in particular $a = 0$ in (1-6) and (1-7).

As the metrics g and f are real analytic, the Ricci-eigenvalues λ_i are real analytic on $M_{\mathrm{Rc}} \cap \{\nabla f \neq 0\}$. And ζ_i's are real analytic from (3-1). So we can combine Proposition 4.2, Lemma 6.2, Propositions 6.3, 6.4, 7.1 and Example 1 of Section 2A2, to obtain a classification of four-dimensional *local* spaces with harmonic curvature satisfying (1-1) as Theorem 1.1. $\square$

Remark 7.2. In the statement of Theorem 1.1, among the types (i)–(iv), there is possibly only one type of neighborhood V on a *connected* space (M, g, f); this holds by a continuity argument of Riemannian metrics. Then one can prove that $\widetilde{M} = M$ if M is of type (i), (ii) or (iii).

8. Complete four-dimensional space with harmonic curvature

It is not hard to describe complete spaces corresponding to parts (i), (ii), (iii) of Theorem 1.1.

For the complete conformally flat case corresponding to (iv) of Theorem 1.1, we may use Theorem 3.1 of Kobayashi's classification [1982]. Then (M, g) can be either $\mathbb{S}^4$, $\mathbb{H}^4$, a flat space or one of the spaces in Examples 1–5 in [Kobayashi 1982]. Now our task is to determine f, which is described by (1-8).

We first recall the spaces in Examples 3–5 in [Kobayashi 1982]. Any space in Examples 3 and 4 in that paper is a quotient of a warped product $\mathbb{R} \times_h N(1)$ where h is a smooth periodic function on $\mathbb{R}$; recall that $N(k)$ is a Riemannian manifold of constant sectional curvature k. Any space in Example 5 in that paper is a quotient of a warped product $\mathbb{R} \times_h N(k)$ where h is smooth on $\mathbb{R}$. Here $h \geq \rho_1 > 0$.

We verify the following lemma.

Lemma 8.1. *For any one of the spaces in Examples* 3, 4 *and* 5 *in [Kobayashi 1982], the following hold*:

(i) *The solution f to (1-1) can be defined and is smooth on $\mathbb{R}$.*

(ii) *If h is periodic and $\frac{1}{3}xR + y(R) = 0$, then f is periodic.*

Proof. As stated in Proposition 7.1, any (h, f) satisfying (7-2), (7-3) and (7-4) gives rise to (g, f) which satisfies (1-1). So, (h, f) satisfies (7-6).

Choose some point s_0 with $h''(s_0) \neq 0$. For any constant c, we consider the initial-value problem

$$f'' = -f\left(\tfrac{1}{12}R + 3ah^{-4}\right) + 3x\left(\tfrac{1}{12}R - ah^{-4}\right) + y(R), \tag{8-1}$$

with initial conditions $f'(s_0) = c$ and

$$f(s_0) = \frac{ch'(s_0) - \left\{x(h''(s_0) + \frac{1}{3}Rh(s_0)) + y(R)h(s_0)\right\}}{h''(s_0)}$$

so that (1-8) holds at s_0. Note that (8-1) is equivalent to (7-6) since h satisfies (1-6).

As h exists smoothly on $\mathbb{R}$ as a solution of (1-6), by global Lipschitz continuity of the right-hand side of (8-1), the solution f exists globally on $\mathbb{R}$.

From (1-6) we obtain

$$h''' = -\left(\tfrac{1}{12}R + 3ah^{-4}\right)h'. \tag{8-2}$$

Then by (8-1) and (8-2) it satisfies

$$h'f'' - fh''' = x\left(h''' + \tfrac{1}{3}Rh'\right) + y(R)h',$$

which is the derivative of (1-8). So, (1-8) holds on $\mathbb{R}$. As h and f satisfy (1-8), the induced (g, f) satisfies (1-1) on $\mathbb{R}$.

If $\frac{1}{3}xR + y(R) = 0$, then from (1-8) we get $f(s) = -x + Ch'(s)$ for a constant C, which is periodic as h. □

About Lemma 8.1(ii), we note that if $\frac{1}{3}xR + y(R) \neq 0$ and h is periodic, then the periodicity of f should be checked by computation.

We are ready to state the following result.

Theorem 8.2. *Let (M, g) be a four-dimensional complete Riemannian manifold with harmonic curvature satisfying (1-1) with nonconstant f. Then it is one of the following*:

(8.2-i) *(M, g) is isometric to a quotient of $\left(\mathbb{S}^2\left(\frac{R}{6}\right) \times \mathbb{S}^2\left(\frac{R}{3}\right), g_{R/6} + g_{R/3}\right)$ with $R > 0$, where $f = c_1 \cos\left(\sqrt{\frac{R}{6}}s\right) - x$ for any constant c_1, where s is the distance from a point on $\mathbb{S}^2\left(\frac{R}{6}\right)$. It holds that $\frac{1}{3}xR + y(R) = 0$.*

(8.2-ii) (M, g) *is isometric to a quotient of* $\big(\mathbb{H}^2\big(\frac{R}{6}\big) \times \mathbb{H}^2\big(\frac{R}{3}\big), g_{R/6} + g_{R/3}\big)$ *with* $R < 0$. *The metric* $g_{R/6}$ *can be written as* $g_{R/6} = ds^2 + p(s)^2\, dt^2$ *with* $p(s) = k_1 \sinh\big(\sqrt{-\frac{R}{6}}s\big) + k_2 \cosh\big(\sqrt{-\frac{R}{6}}s\big)$ *for constants* k_1, k_2, *and then* $f = c_2 p' - x$ *for any constant* c_2. *It holds that* $\frac{1}{3}xR + y(R) = 0$.

(8.2-iii) (M, g) *is isometric to a quotient of one of the static spaces in Example 3 of Section 2A2, which is the Riemannian product* $(\mathbb{R}^1 \times W^3, dt^2 + ds^2 + h(s)^2\tilde{g})$ *of* $\mathbb{R}^1$ *and some three-dimensional conformally flat static space* $(W^3 = \mathbb{R}^1 \times \mathbb{S}^2(1), ds^2 + h(s)^2\tilde{g})$ *with zero scalar curvature, which contains the spatial slice of the Schwarzschild space-time*

And $f = c \cdot h'(s) - x$ *for a constant c. It holds that* $R = y(0) = 0$.

(8.2-iv) (M, g) *is conformally flat. It is either* $\mathbb{S}^4$, $\mathbb{H}^4$, *a flat space or one of the spaces in Examples* 1–5 *in [Kobayashi 1982]. Below we describe f in each subcase*:

(8.2-iv-1) $\mathbb{S}^4(k^2)$ *with the metric* $g = ds^2 + (\sin(ks)^2/k^2)g_1$ *for any constant c*,

$$f(s) = c \cdot \cos(ks) + 3x + \frac{y(12k^2)}{k^2}.$$

(8.2-iv-2) $\mathbb{H}^4(-k^2)$ *with* $g = ds^2 + (\sinh(ks)^2/k^2)g_1$ *for any constant c*,

$$f(s) = c \cdot \cosh(ks) + 3x - \frac{y(-12k^2)}{k^2}.$$

(8.2-iv-3) *A flat space*, $f = a + \sum_i +b_i x_i + \frac{1}{2}y(0)x_i^2$ *in local Euclidean coordinates* x_i *for constants a and* b_i.

(8.2-iv-4) *Examples* 1 *and* 2 *in [Kobayashi 1982]: the Riemannian product* $(\mathbb{R}\times N(k), ds^2 + g_k)$ *or its quotient*, $k \neq 0$, *where* $N(k)$ *is three-dimensional complete space of constant sectional curvature k*,

$$f = \begin{cases} c_1 \sin\sqrt{\frac{R}{3}}s + c_2 \cos\sqrt{\frac{R}{3}}s - x & \text{when } R > 0,\\ c_1 \sinh\sqrt{-\frac{R}{3}}s + c_2 \cosh\sqrt{-\frac{R}{3}}s - x & \text{when } R < 0.\end{cases}$$

It holds that $\frac{1}{3}xR + y(R) = 0$ *and* $R = 6k$.

(8.2-iv-5) *Examples* 3 *and* 4 *in [Kobayashi 1982]: a warped product* $\mathbb{R} \times_h N(1)$ *or its quotient, where h is a periodic function on* $\mathbb{R}$, f *is on* $\mathbb{R}$, *satisfying* (1-8).

(8.2-iv-6) *Example* 5 *in [Kobayashi 1982]: a warped product* $\mathbb{R} \times_h N(k)$ *where h is defined on* $\mathbb{R}$, f *is on* $\mathbb{R}$, *satisfying* (1-8).

Proof. To obtain (8.2-i), (8.2-ii) and (8.2-iii), we use the continuity argument of Riemannian metrics from Theorem 1.1. To describe f in the subcases of (8.2-iv), we use (1-8) and (7-6). □

9. Four-dimensional static spaces with harmonic curvature

In this section we study static spaces, i.e., those satisfying (1-2). As explained in the Introduction, studying local static spaces is interesting due to Corvino's local deformation theory of scalar curvature. Qing and Yuan's work [2016] on local scalar curvature rigidity arouses another motivation. Here we state a local classification which can be read off from Theorem 1.1:

Theorem 9.1. *Let (M, g, f) be a four-dimensional (not necessarily complete) static space with harmonic curvature and nonconstant f. Then for each point p in some open dense subset $\widetilde{M}$ of M, there exists a neighborhood V of p with one of the following properties:*

(9.1-i) *(V, g) is isometric to a domain in $\left(\mathbb{S}^2\left(\frac{R}{6}\right) \times \mathbb{S}^2\left(\frac{R}{3}\right), g_{R/6} + g_{R/3}\right)$ with $R > 0$. And $f = c_1 \cos\left(\sqrt{\frac{R}{6}}(s + s_0)\right)$, where s is the distance from a point on $\mathbb{S}^2\left(\frac{R}{6}\right)$ and c_1, s_0 are constants.*

(9.1-ii) *(V, g) is isometric to a domain in $\left(\mathbb{H}^2\left(\frac{R}{6}\right) \times \mathbb{H}^2\left(\frac{R}{3}\right), g_{R/6} + g_{R/3}\right)$ with $R < 0$. The metric $g_{R/6}$ can be written as $g_{R/6} = ds^2 + p(s)^2\, dt^2$ with $p(s) = k_1 \sinh\left(\sqrt{-\frac{R}{6}}s\right) + k_2 \cosh\left(\sqrt{-\frac{R}{6}}s\right)$ for constants k_1, k_2, and then $f = c_2 p'$ for any constant c_2.*

(9.1-iii) *(V, g) is isometric to a domain in one of the static spaces in Example 3 of Section 2A2, which is the Riemannian product $\mathbb{R}^1 \times W^3$ of $\mathbb{R}^1$ and some three-dimensional conformally flat static space $(W^3, ds^2 + h(s)^2\tilde{g})$ with zero scalar curvature, and $f = ch'$.*

(9.1-iv) *(V, g) is conformally flat. So, it is one of the warped product metrics of the form $ds^2 + h(s)^2 g_k$ whose existence is described in Section 2 of [Kobayashi 1982]. The function h satisfies (1-6) and (1-7), and we have $f(s) = Ch'(s)$.*

For complete conformally flat case corresponding to (9.1-iv) in Theorem 9.1, if we use Theorem 3.1 of Kobayashi's classification, we get either $\mathbb{S}^4$, $\mathbb{H}^4$, a flat space or one of the spaces in Examples 1–5 in [Kobayashi 1982]. We may thus obtain classification of complete four-dimensional static spaces with harmonic curvature:

Theorem 9.2. *Let (M, g, f) be a complete four-dimensional static space with harmonic curvature. Then it is one of the following:*

(9.2-i) *(M, g) is isometric to a quotient of $\left(\mathbb{S}^2\left(\frac{R}{6}\right) \times \mathbb{S}^2\left(\frac{R}{3}\right), g_{R/6} + g_{R/3}\right)$ with $R > 0$. And $f = c_1 \cos\left(\sqrt{\frac{R}{6}}s\right)$, where s is the distance function from a point on $\mathbb{S}^2\left(\frac{R}{6}\right)$.*

(9.2-ii) *(M, g) is isometric to a quotient of $\left(\mathbb{H}^2\left(\frac{R}{6}\right) \times \mathbb{H}^2\left(\frac{R}{3}\right), g_{R/6} + g_{R/3}\right)$ with $R < 0$. The metric $g_{R/6}$ can be written as $g_{R/6} = ds^2 + p(s)^2\, dt^2$ with $p(s) = k_1 \sinh\left(\sqrt{-\frac{R}{6}}s\right) + k_2 \cosh\left(\sqrt{-\frac{R}{6}}s\right)$ for constants k_1, k_2, and then $f = c_2 p'$ for any constant c_2.*

(9.2-iii) (M, g) *is isometric to a quotient of the Riemannian product* $(\mathbb{R}^1 \times W^3, dt^2 + \tilde{g})$, *where* $(W^3, \tilde{g})$ *denotes the warped product manifold on the smooth product* $\mathbb{R}^1 \times \mathbb{S}^2(1)$ *which contains the spatial slice of the Schwarzschild space-time*; *see Example 3 of Section 2A2.*

(9.2-iv) (M, g, f) *is* $\mathbb{S}^4$, $\mathbb{H}^4$, *a flat space or one of the spaces in Examples* 1–5 *in [Kim 2017].*

(9.2-v) g *is a complete Ricci-flat metric with* f *a constant function.*

Proof. It follows from Theorem 8.2. When f is a nonzero constant, g is clearly Ricci-flat. So we get (v). □

Fischer and Marsden [1974] made the conjecture that any closed static space is Einstein. But it was disproved by conformally flat examples in [Lafontaine 1983; Kobayashi 1982]. Now we ask:

Question 1. Does there exist a closed static space which does not have harmonic curvature?

The space in (9.2-iii) of Theorem 9.2 has three distinct Ricci-eigenvalues. We only know examples of static spaces with at most three distinct Ricci-eigenvalues. So we ask the following:

Question 2. Does there exist a static space with more than three distinct Ricci-eigenvalues? Is there a limit on the number of distinct Ricci-eigenvalues for a static space?

10. Miao–Tam critical metrics and V-critical spaces

In this section we treat Miao–Tam critical metrics. These metrics originate from [Miao and Tam 2009], where they studied the critical points of the volume functional on the space $\mathcal{M}_\gamma^K$ of metrics with constant scalar curvature K on a compact manifold M with a prescribed metric γ at the boundary of M. Miao–Tam critical metrics are precisely described [Miao and Tam 2011] in case they are Einstein or conformally flat.

Here we first describe four-dimensional metrics with harmonic curvature which have a nonzero solution f to (1-3). We do not assume the condition $f_{|\Sigma} = 0$ but still can show that any such metric must be conformally flat;

Theorem 10.1. *Let* (M, g) *be a four-dimensional* (*not necessarily complete*) *Riemannian manifold with harmonic curvature*, *satisfying* (1-3) *with nonconstant* f. *Then* (M, g) *is conformally flat. It is one of the warped product metrics of the form* $ds^2 + h(s)^2 g_k$ *whose existence is described in Section* 2 *of [Kobayashi 1982]. The function* h *satisfies* (1-6) *and* (1-7), *and* f *satisfies* $h' f' - f h'' = -h/(n-1)$.

Proof. The proof is immediate from Theorem 1.1; the cases (i)–(ii) of Theorem 1.1 require $\frac{1}{3}xR + y(R) = 0$ and (iii) requires $y(0) = 0$, which contradict the conditions $x = 0$ and $y(R) = -\frac{1}{3}$ that (1-3) has. The description of Theorem 1.1(iv) holds for g and f of Theorem 10.1, and in particular g is conformally flat. □

Theorem 10.1 shows an advantage of our local approach over [Barros et al. 2015] in analyzing (1-3). In fact, the integration argument of Lemma 5 of that paper only works for compact manifolds, but our analysis can resolve local solutions.

From Theorems 9.1 and 10.1 we can classify local four-dimensional V-static spaces with harmonic curvature:

Theorem 10.2. *Let (M, g, f) be a four-dimensional (not necessarily complete) V-static space with harmonic curvature and nonconstant f. Then for each point p in some open dense subset $\widetilde{M}$ of M, there exists a neighborhood V of p with one of the following properties*:

(10.2-i) (V, g) *is isometric to a domain in* $\big(\mathbb{S}^2\big(\frac{R}{6}\big) \times \mathbb{S}^2\big(\frac{R}{3}\big), g_{R/6} + g_{R/3}\big)$ *with* $R > 0$. *And* $f = c_1 \cos\big(\sqrt{\frac{R}{6}}(s + s_0)\big)$, *where s is the distance function from a point on* $\mathbb{S}^2\big(\frac{R}{6}\big)$ *and c_1, s_0 are constants.*

(10.2-ii) (V, g) *is isometric to a domain in* $\big(\mathbb{H}^2\big(\frac{R}{6}\big) \times \mathbb{H}^2\big(\frac{R}{3}\big), g_{R/6} + g_{R/3}\big)$ *with* $R < 0$. *The metric $g_{R/6}$ can be written as $g_{R/6} = ds^2 + p(s)^2\, dt^2$ with* $p(s) = k_1 \sinh\big(\sqrt{-\frac{R}{6}}s\big) + k_2 \cosh\big(\sqrt{-\frac{R}{6}}s\big)$ *for constants k_1, k_2, and then $f = c_2 p'$ for any constant c_2.*

(10.2-iii) (V, g) *is isometric to a domain in one of the static spaces in Example 3 of Section 2A2 which is the Riemannian product $\mathbb{R}^1 \times W^3$ of $\mathbb{R}^1$ and some three-dimensional conformally flat static space $(W^3, ds^2 + h(s)^2\tilde{g})$ with zero scalar curvature. And $f = ch'$ for any constant c.*

(10.2-iv) (V, g) *is conformally flat. It is one of the warped product metrics of the form $ds^2 + h(s)^2 g_k$ whose existence is described in Section 2 of [Kobayashi 1982]. The function h satisfies (1-6) and (1-7), and we have $f(s) = ch'(s)$ for any constant c.*

(10.2-v) (V, g) *is conformally flat. It is one of the warped product metrics of the form $ds^2 + h(s)^2 g_k$ whose existence is described in Section 2 of [Kobayashi 1982]. The function h satisfies (1-6) and (1-7) and f is any constant multiple of a solution f_0 satisfying $h' f_0' - f_0 h'' = -h/(n-1)$.*

Note that the last equation in (10.2-v) comes from (1-4), which allows any constant multiple of one solution.

As a corollary of Theorem 10.1, we could state an extension of Theorem 1.2 in [Miao and Tam 2011] to the case of harmonic curvature. Instead we choose to state the following version, which is a twin to Corollary 1 of [Barros et al. 2015].

Theorem 10.3. *If (M^4, g, f) is a simply connected, compact Miao–Tam critical metric of harmonic curvature with boundary isometric to a standard sphere S^3, then (M^4, g) is isometric to a geodesic ball in a simply connected space form $\mathbb{R}^4$, $\mathbb{H}^4$ or $\mathbb{S}^4$.*

One can also make classification statements of complete spaces with harmonic curvature satisfying (1-3) or (1-4). We omit them.

Theorem 10.1 gives a speculation that it might hold in general dimension. So, we ask the following:

Question 3. Let (M, g) be an n-dimensional Miao–Tam critical metric with harmonic curvature. Is it conformally flat?

It is also interesting to find examples of nonconformally flat Miao–Tam critical metrics in any dimension.

11. On critical point metrics

In this section we study a critical point metric, i.e., a Riemannian metric g on a manifold M which admits a nonzero solution f to (1-5). According to [Yun et al. 2014], these critical point metrics with harmonic curvature on closed manifolds in any dimension are Einstein.

On a closed manifold, by taking the trace of this equation, R must be positive and f satisfies $\int_M f\,dv = 0$. Here M is not necessarily closed and g may have nonpositive scalar curvature. From Theorem 1.1, we can easily obtain the next theorem.

Theorem 11.1. *Let (M, g) be a four-dimensional (not necessarily complete) Riemannian manifold with harmonic curvature, satisfying (1-5) with nonconstant f. Then one of the following holds:*

(11.1-i) *(M, g) is locally isometric to a domain in one of the static spaces of Example 3 in Section 2A2, which is the Riemannian product $(\mathbb{R}^1 \times W^3, dt^2 + ds^2 + h(s)^2\tilde{g})$ of $\mathbb{R}^1$ and a three-dimensional conformally flat static space $(W^3, ds^2 + h(s)^2\tilde{g})$ with zero scalar curvature. And $f = c \cdot h'(s) - 1$.*

(11.1-ii) *(M, g) is conformally flat and is locally one of the metrics whose existence is described in Section 2 of [Kobayashi 1982]: $g = ds^2 + h(s)^2 g_k$ where h and f satisfy (1-6), (1-7) and (1-8).*

Proof. We have $\frac{1}{3}xR + y(R) = 0$ and $R \neq 0$ in the cases (i), (ii) of Theorem 1.1. This is not compatible with (1-5). $\square$

Complete spaces with harmonic curvature which admit a solution f to (1-5) are described in the next theorem. We obtain nonconformally flat examples with zero scalar curvature in (11.2-i), which is in contrast to the above result of [Yun et al.

2014] for closed manifolds. The case (11.2-v) is also noteworthy; it is conformally flat with positive scalar curvature and the metric g can exist on a compact quotient but the function f can survive on the universal cover $\mathbb{R} \times_h N(1)$.

Theorem 11.2. *Let (M, g) be a four-dimensional complete Riemannian manifold with harmonic curvature, satisfying (1-5) with nonconstant f. Then (M, g) is one of the following:*

(11.2-i) *(M, g) is isometric to a quotient of one of the static spaces of Example 3 in Section 2A2, which is the Riemannian product $(\mathbb{R}^1 \times W^3, dt^2 + ds^2 + h(s)^2\tilde{g})$ of $\mathbb{R}^1$ and a three-dimensional conformally flat static space $(W^3, ds^2 + h(s)^2\tilde{g})$ with zero scalar curvature which contains the spatial slice of the Schwarzschild space-time. And $f = c \cdot h'(s) - 1$ for a constant c.*

(11.2-ii) *$\mathbb{S}^4(k^2)$ with the metric $g = ds^2 + (\sin^2(ks)/k^2)g_1$, with $f(s) = c \cdot \cos(ks)$.*

(11.2-iii) *$\mathbb{H}^4(-k^2)$ with $g = ds^2 + (\sinh(ks)^2/k^2)g_1$, with $f(s) = c \cdot \cosh(ks)$.*

(11.2-iv) *A flat space, $f = a + \sum_i b_i x_i$ in a local Euclidean coordinate x_i and constants a, b_i.*

(11.2-v) *Example* 3 *in [Kobayashi 1982]: a warped product $\mathbb{R} \times_h N(1)$ where h is a periodic function on $\mathbb{R}$, f is smooth on $\mathbb{R}$ but is not periodic. Here $R > 0$.*

(11.2-vi) *Example* 5 *in [Kobayashi 1982]: a warped product $\mathbb{R} \times_h N(k)$ where h is defined on $\mathbb{R}$, f is smooth on $\mathbb{R}$. Here $R \leq 0$.*

Proof. We may check the list in Theorem 8.2. The spaces of (8.2-i) and (8.2-ii) in Theorem 8.2 are excluded as in the proof of Theorem 11.1. The space for (8.2-iv-4) of Theorem 8.2, where $R \neq 0$, does not satisfy the equation $h'f' - fh'' = x\big(h'' + \frac{1}{3}Rh\big) + y(R)h$; when $x = 1$, $y(R) = -\frac{1}{4}R$ and $h = 1$, it reduces to $0 = \frac{1}{12}R$.

On the space of (8.2-iv-5) in Theorem 8.2, f is defined and smooth on $\mathbb{R}$ by Lemma 8.1 (i). As $\frac{1}{3}xR + y(R) \neq 0$, Lemma 8.1(ii) does not apply. According to Section E.2 of [Lafontaine 1983], f cannot be periodic. This yields (11.2-v). □

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