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BOUNDARY SCHWARZ LEMMA FOR NONEQUIDIMENSIONAL HOLOMORPHIC MAPPINGS AND ITS APPLICATION

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In this paper, we give a boundary Schwarz lemma for holomorphic mappings between nonequidimensional unit balls. As an application, a new boundary rigidity result is presented.

1. Introduction

Let B^n be the unit ball in $\mathbb{C}^n$ for $n \geq 1$. Denote by $\mathrm{Hol}(B^n, B^N)$ the set of all holomorphic mapping from the unit ball $B^n \subset \mathbb{C}^n$ into $B^N \subset \mathbb{C}^N$. For a bounded domain $V \subset \mathbb{C}^n$, let $C^{1+\alpha}(V)$ be the set of all functions f on V whose first order partial derivatives exist and are Hölder continuous. For $z_0 \in \partial B^n$, the tangent space $T_{z_0}(\partial B^n)$ and holomorphic tangent space $T^{1,0}_{z_0}(\partial B^n)$ at z_0 are defined by

$$T_{z_0}(\partial B^n) = \{\beta \in \mathbb{C}^n \mid \mathrm{Re}(\overline{z_0}^T \beta) = 0\}, \quad T^{(1,0)}_{z_0}(\partial B^n) = \{\beta \in \mathbb{C}^n \mid \overline{z_0}^T \beta = 0\},$$

respectively. In this paper, we give a general boundary Schwarz lemma for holomorphic mappings between unit balls in any dimensions as follows.

Theorem 1.1. *Let $f \in \mathrm{Hol}(B^n, B^N)$ for any $n, N \geq 1$, and denote by $J_f(z)$ the Jacobian matrix of f at z. If f is $C^{1+\alpha}$ at $z_0 \in \partial B^n$ and $f(z_0) = w_0 \in \partial B^N$, then we have*:

(I) *$J_f(z_0)\beta \in T_{w_0}(\partial B^N)$ for any $\beta \in T_{z_0}(\partial B^n)$, and $J_f(z_0)\beta \in T^{(1,0)}_{w_0}(\partial B^N)$ for any $\beta \in T^{(1,0)}_{z_0}(\partial B^n)$.*

(II) *There exists $\lambda \in \mathbb{R}$ such that*

$$\overline{J_f(z_0)}^T w_0 = \lambda z_0$$

with $\lambda \geq |1 - \bar{a}^T w_0|^2/(1 - \|a\|^2) > 0$, where $a = f(0)$.

Remark 1.2. For the case of biholomorphic mapping, item (I) holds; see Chapter 3 of [Krantz 1992]. Here we conclude the same result for holomorphic mappings between unit balls of different dimensions. For $n = N = 1$, the theorem says

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$f'(z_0) > 0$, so the image $f(\partial B^1)$ at w_0 is always smooth. For $n > 1$, if $f(\partial B^n)$ is a smooth manifold, then conclusion (I) is almost trivial. However, we would like to point out that $f(\partial B^n)$ may be not a smooth manifold.

In the special case when $n = N$, Theorem 1.1 reduces to (1) and (2) in Theorem 3.1 of [Liu et al. 2015]. For $n = N = 1$, part (II) of the theorem gives the classical boundary Schwarz lemma in [Garnett 1981].

As an application of Theorem 1.1, we will present a new boundary rigidity result. First, recall the following famous rigidity result for holomorphic self-mappings on B^n.

Theorem 1.3 [Burns and Krantz 1994]. *Let $f \in \mathrm{Hol}(B^n, B^n)$ with $n \geq 1$ such that*

$$f(z) = z + O(|z - \mathbf{1}|^4)$$

as $z \to \mathbf{1}$, where $\mathbf{1} = (1, 0, \ldots, 0)^T \in \partial B^n$. Then $f(z) \equiv z$.

Notice that the order of the estimation $O(|z - \mathbf{1}|^4)$ is sharp in Theorem 1.3, as shown by the example [Burns and Krantz 1994]

$$f(z) = z - \tfrac{1}{10}(z - 1)^3, \quad z \in D,$$

where D is the unit disk.

On the other hand, Huang [1995] shows that if $f \in \mathrm{Hol}(B^n, B^n)$ satisfies $f(z) = z + O(|z - \mathbf{1}|^3)$ as $z \to \mathbf{1}$, and $f(z_0) = z_0$ with $z_0 \in B^n$, then $f(z) = z$ on the unit ball. This result gives a condition under which the order of the estimation $O(|z - \mathbf{1}|^4)$ in [Burns and Krantz 1994] can be lower with a fixed point.

A problem of the boundary rigidity for nonequidimensional mappings was given by Krantz [2011]. Using Theorem 1.1, we give a positive answer to this problem, and provide a new boundary rigidity result for holomorphic mappings between nonequidimensional unit balls. In fact, we find conditions under which the order of the estimation can be lower and is also sharp without internal fixed point. Our result is given as follows.

Theorem 1.4. *Let $f \in \mathrm{Hol}(B^n, B^N)$ for $N \geq n \geq 1$, such that*

$$f(z) = (z^T, 0)^T + O(|z - \mathbf{1}|^3) \tag{1-1}$$

as $z \to \mathbf{1}$. If f is C^2 at $\mathbf{1}$ and $f_1(z) = z_1$, where f_1 is the first component of f, then $f(z) \equiv (z^T, 0)^T$.

Example. Let $f(z_1, z_2) = (z_1, z_2 z_1^k, 0)^T \in \mathrm{Hol}(B^2, B^3)$ for integer $k \geq 1$. Since $f(z) - (z_1, z_2, 0)^T = (0, z_2(z_1^k - 1), 0)^T$, and

$$\frac{|f(z) - (z_1, z_2, 0)|}{|z - \mathbf{1}|^2} = \frac{|z_2(z_1^k - 1)|}{|z_1 - 1|^2 + |z_2|^2} \leq \frac{1}{2} \frac{|z_1^k - 1|^2 + |z_2|^2}{|z_1 - 1|^2 + |z_2|^2} \leq \tfrac{1}{2}(k^2 + 1),$$

it satisfies $f(z) = (z_1, z_2, 0)^T + O(|z - \mathbf{1}|^2)$. However, it is obvious that $f(z) \neq (z_1, z_2, 0)^T$, which indicates that the order of $O(|z - \mathbf{1}|^3)$ is sharp.

2. Proof of Theorem 1.1

To prove the main result, we first give some notation and lemmas. For any $z = (z_1, \dots, z_n)^T$, $w = (w_1, \dots, w_n)^T \in \mathbb{C}^n$, the inner product and the corresponding norm are given by $\langle z, w\rangle = \sum_{j=1}^n z_j \overline{w_j}$ and $\|z\| = \langle z, z\rangle^{\frac{1}{2}}$ respectively. ∂B^n denotes the boundary of B^n.

Lemma 2.1 [Rudin 1980]. *Let* $f \in \mathrm{Hol}(B^n, B^N)$ *with* $n, N \geq 1$. *If* $f(0) = 0$, *then* $\|f(z)\| \leq \|z\|$, $z \in B^n$.

Lemma 2.2 [Dai et al. 2010; Liu et al. 2016]. *For given* $p \in B^n \cup \partial B^n$ *and* $q \in \mathbb{C}^n$ *with* $q \neq 0$, *let* $L(\xi) = p + \xi q$ *for* $\xi \in \mathbb{C}$. *Then*

$$L(D_{p,q}) \subset B^n, \quad L(\partial D_{p,q}) \subset \partial B^n,$$

where $D_{p,q} = \{\xi \in \mathbb{C} \mid |\xi - c_{p,q}| < r_{p,q}\}$, *with*

$$c_{p,q} = -\frac{\langle p, q\rangle}{\|q\|^2}, \quad r_{p,q} = \sqrt{\frac{1 - \|p\|^2}{\|q\|^2} + \left|\frac{\langle p, q\rangle}{\|q\|^2}\right|^2}.$$

Proof. Assume $\|L(D_{p,q})\|^2 < 1$, which means

$$\|p\|^2 + 2\,\mathrm{Re}(\bar{p}^T \xi q) + \|\xi q\|^2 < 1,$$

and

$$\frac{\|p\|^2}{\|q\|^2} + 2\frac{\mathrm{Re}(\bar{p}^T q\xi)}{\|q\|^2} + |\xi|^2 < \frac{1}{\|q\|^2},$$

i.e.,

$$\left|\xi + \frac{\langle p, q\rangle}{\|q\|^2}\right|^2 < \frac{1 - \|p\|^2}{\|q\|^2} + \left|\frac{\langle p, q\rangle}{\|q\|^2}\right|^2. \qquad \square$$

Proof of Theorem 1.1. We prove the theorem in five steps.

Step 1. Denote by e_i^n the i-th column of the $n \times n$ identity matrix. Assume $z_0 = e_1^n = \mathbf{1} \in \partial B^n$, and f is $C^{1+\alpha}$ in a neighborhood V of z_0. Moreover, assume $f(0) = 0$ and $f(z_0) = w_0 = e_1^N$.

We first show that for any $q \in H = \{z \in \mathbb{C}^n \mid \mathrm{Re}\, z_1 < 0\}$, there exists a $r_q > 0$ such that

$$\mathbf{1} + tq \in B^n, \quad 0 < t < r_q. \tag{2-1}$$

Assume $q = (q_1, \dots, q_n)^T \in H$ and $\mathrm{Re}\, q_1 < 0$. Then for $t \in \mathbb{R}$,

$$\mathbf{1} + tq \in B^n \;\Leftrightarrow\; \|\mathbf{1} + tq\|^2 < 1 \;\Leftrightarrow\; |1 + t\,\mathrm{Re}\, q_1|^2 + |t\,\mathrm{Im}\, q_1|^2 + \sum_{j=2}^n |q_j|^2 t^2 < 1,$$

which is equivalent to

$$0 < t < \frac{-2\,\mathrm{Re}\, q_1}{\sum_{j=1}^{n} |q_j|^2}.$$

Letting $r_q = -2\,\mathrm{Re}\, q_1/(\sum_{j=1}^{n} |q_j|^2)$ implies the claim.

Let $p = z_0,\ q = (-1+ik)z_0$ for any given $k \in \mathbb{R}$. Then from (2-1), when $t \to 0^+$, $p + tq \in B^n \cap V$. For such t, taking the Taylor expansion of $f((1-t+ikt)z_0)$ at $t = 0$, we have

$$f((1-t+ikt)z_0) = w_0 + J_f(z_0)(-1+ik)z_0 t + O(t^{1+\alpha}).$$

By Lemma 2.1,

$$\|f((1-t+ikt)z_0)\|^2 = \|w_0 + J_f(z_0)(-1+ik)z_0 t + O(t^{1+\alpha})\|^2 \le \|(1-t+ikt)z_0\|^2,$$

i.e.,

$$1 + 2\,\mathrm{Re}\big(\overline{w_0}^T J_f(z_0)(-1+ik)z_0 t\big) + O(t^{1+\alpha}) \le 1 - 2t + O(t^2).$$

Substituting $w_0 = e_1^N,\ z_0 = e_1^n$ and let $t \to 0^+$, we have

$$\mathrm{Re}\big(\overline{e_1^N}^T J_f(z_0)(-1+ik)e_1^n\big) \le -1,$$

i.e.,

$$\mathrm{Re}\bigg(\frac{\partial f_1(z_0)}{\partial z_1}(-1+ik)\bigg) \le -1.$$

Let $\partial f_1(z_0)/\partial z_1 = \mathrm{Re}(\partial f_1(z_0)/\partial z_1) + i\,\mathrm{Im}(\partial f_1(z_0)/\partial z_1)$. Then from the above inequality, one gets

$$-\,\mathrm{Re}\,\frac{\partial f_1(z_0)}{\partial z_1} - k\,\mathrm{Im}\,\frac{\partial f_1(z_0)}{\partial z_1} \le -1,$$

i.e.,

$$-k\,\mathrm{Im}\,\frac{\partial f_1(z_0)}{\partial z_1} \le \mathrm{Re}\,\frac{\partial f_1(z_0)}{\partial z_1} - 1. \tag{2-2}$$

Since (2-2) is valid for any $k \in \mathbb{R}$, we have

$$\mathrm{Im}\,\frac{\partial f_1(z_0)}{\partial z_1} = 0,$$

which implies

$$0 \le \mathrm{Re}\,\frac{\partial f_1(z_0)}{\partial z_1} - 1,$$

and

$$\frac{\partial f_1(z_0)}{\partial z_1} = \mathrm{Re}\,\frac{\partial f_1(z_0)}{\partial z_1} \ge 1. \tag{2-3}$$

Step 2. Let $p = z_0,\ q = -z_0 + ike_j^n$ for $2 \le j \le n$ and $k \in \mathbb{R}$. Then as $t \to 0^+$, $p + tq \in B^n \cap V$. Similarly, taking the Taylor expansion of $f\big((1-t)z_0 + ikte_j^n\big)$ at $t = 0$, we have

$$f\big((1-t)z_0 + ikte_j^n\big) = w_0 + J_f(z_0)(-z_0 + ike_j^n)t + O(t^{1+\alpha}).$$

By Lemma 2.1,

$$\begin{aligned}\big\| f\big((1-t)z_0 + ikte_j^n\big)\big\|^2 &= \|w_0 + J_f(z_0)(-z_0 + ike_j^n)t + O(t^{1+\alpha})\|^2 \\ &\le \|(1-t)z_0 + ikte_j^n\|^2,\end{aligned}$$

i.e.,

$$1 + 2\,\mathrm{Re}\big(\overline{w_0}^T J_f(z_0)(-z_0 + ike_j^n)t\big) + O(t^{1+\alpha}) \le 1 - 2t + O(t^2).$$

Substituting $w_0 = e_1^N$, $z_0 = e_1^n$ and letting $t \to 0^+$, we have

$$\mathrm{Re}\big(\overline{e_1^N}^T J_f(z_0)(-e_1^n + ike_j^n)\big) \le -1,$$

i.e.,

$$\mathrm{Re}\bigg(-\frac{\partial f_1(z_0)}{\partial z_1} + ik\frac{\partial f_1(z_0)}{\partial z_j}\bigg) \le -1.$$

From the above inequality as well as inequality (2-3), one has

$$-k\,\mathrm{Im}\,\frac{\partial f_1(z_0)}{\partial z_j} \le \frac{\partial f_1(z_0)}{\partial z_1} - 1.$$

With an argument similar to Step 1, we have

$$\mathrm{Im}\,\frac{\partial f_1(z_0)}{\partial z_j} = 0, \quad 2 \le j \le n.$$

Meanwhile, if we assume $p = z_0,\ q = -z_0 + ke_j^n$ for $2 \le j \le n$ and any $k \in \mathbb{R}$. It is easy to find

$$\mathrm{Re}\,\frac{\partial f_1(z_0)}{\partial z_j} = 0, \quad 2 \le j \le n.$$

Therefore,

$$\frac{\partial f_1(z_0)}{\partial z_j} = 0, \quad 2 \le j \le n, \tag{2-4}$$

as well. As a result of (2-3) and (2-4), we have

$$\overline{J_f(z_0)}^T w_0 = \lambda_f z_0 \tag{2-5}$$

for $w_0 = e_1^N,\ z_0 = e_1^n$ and $\lambda_f = \partial f_1(z_0)/\partial z_1 \ge 1$.

Step 3. Now let z_0 be any given point at ∂B^n. Then there exists a unitary matrix U_{z_0} such that $U_{z_0}(z_0) = e_1^n$. Assume $f(0) = 0,\ f(z_0) = w_0$ and w_0 is not necessarily e_1^N at ∂B^N. Similarly, there is a unitary matrix U_{w_0} such that $U_{w_0}(w_0) = e_1^N$. Let

$$g(z) = U_{w_0} \circ f \circ \overline{U_{z_0}}^T;$$

then $g(0) = 0,\ g(e_1^n) = e_1^N$. Moreover,

$$J_g(z) = U_{w_0} J_f(\overline{U_{z_0}}^T z)\overline{U_{z_0}}^T. \tag{2-6}$$

From Steps 1 and 2, we have

$$\overline{J_g(e_1^n)}^T e_1^N = \lambda_g e_1^n$$

for $z_0 = e_1^n$ and $\lambda_g = \partial g_1(e_1^n)/\partial z_1 \geq 1$, which implies

$$\overline{U_{w_0} J_f(\overline{U_{z_0}}^T e_1^n)\overline{U_{z_0}}^T}^T e_1^N = \lambda_g e_1^n,$$

i.e.,

$$U_{z_0} \overline{J_f(z_0)}^T \overline{U_{w_0}}^T e_1^N = \lambda_g e_1^n.$$

After multiplying by $\overline{U_{z_0}}^T$ on both sides of the above equation, we obtain

$$\overline{U_{z_0}}^T U_{z_0} \overline{J_f(z_0)}^T \overline{U_{w_0}}^T e_1^N = \lambda_g \overline{U_{z_0}}^T e_1^n,$$

i.e.,

$$\overline{J_f(z_0)}^T w_0 = \lambda_g z_0, \tag{2-7}$$

where $\lambda_g = \partial g_1(e_1^n)/\partial z_1 \geq 1$.

Step 4. Let $f(z_0) = w_0$ with $z_0 \in \partial B^n$, $w_0 \in \partial B^N$. If $f(0) = a \neq 0$, then we use the automorphism of B^N to get the result. Assume $\phi_a(w)$ is an automorphism of B^N such that $\phi_a(a) = 0$. Then $\phi_a(w_0) \in \partial B^N$ as well. With a similar analysis to Step 3, there exists a $U_{\phi_a(w_0)}$ such that $U_{\phi_a}(\phi_a(w_0)) = w_0$. Let

$$h = U_{\phi_a} \circ \phi_a \circ f,$$

then $h(0) = 0$, $h(z_0) = w_0$. As a result of Step 3, there is a real number $\gamma \geq 1$ such that

$$\overline{J_h(z_0)}^T w_0 = \gamma z_0.$$

Using the expression for h, we obtain

$$\overline{J_h(z_0)}^T w_0 = \overline{U_{\phi_a} J_{\phi_a}(w_0) J_f(z_0)}^T w_0 = \overline{J_f(z_0)}^T\, \overline{J_{\phi_a}(w_0)}^T\, \overline{U_{\phi_a}}^T w_0 = \gamma z_0. \tag{2-8}$$

Since $U_{\phi_a}(\phi_a(w_0)) = w_0$, we have $\overline{U_{\phi_a}}^T w_0 = \phi_a(w_0)$. From the expression for the automorphism ϕ_a given by [Rudin 1980], we have the following equality:

$$\overline{J_{\phi_a}(w_0)}^T\, \overline{U_{\phi_a}}^T w_0 = \overline{J_{\phi_a}(w_0)}^T \phi_a(w_0) = \frac{1 - \|a\|^2}{|1 - \bar{a}^T w_0|^2} w_0.$$

Therefore, combining with (2-8) we get

$$\overline{J_f(z_0)}^T \frac{1-\|a\|^2}{|1-\bar{a}^T w_0|^2} w_0 = \gamma z_0.$$

Consequently,

$$\overline{J_f(z_0)}^T w_0 = \lambda z_0, \tag{2-9}$$

where

$$\lambda = \frac{|1-\bar{a}^T w_0|^2}{1-\|a\|^2}\gamma \geq \frac{|1-\bar{a}^T w_0|^2}{1-\|a\|^2} > 0 \quad \text{and} \quad a = f(0).$$

The proof of (II) is completed.

Step 5. For any $\beta \in T_{z_0}(\partial B^n)$, we have

$$\mathrm{Re}(\overline{z_0}^T \beta) = 0. \tag{2-10}$$

To prove $J_f(z_0)\beta \in T_{w_0}(\partial B^N)$, it is sufficient to verify

$$\mathrm{Re}\big(\overline{w_0}^T J_f(z_0)\beta\big) = 0. \tag{2-11}$$

From (2-9), $\overline{J_f(z_0)}^T w_0 = \lambda z_0$, which means

$$\overline{w_0}^T J_f(z_0) = \overline{\overline{J_f(z_0)}^T w_0}^T = \lambda \overline{z_0}^T. \tag{2-12}$$

Then

$$\mathrm{Re}\big(\overline{w_0}^T J_f(z_0)\beta\big) = \mathrm{Re}(\lambda \overline{z_0}^T \beta) = \lambda\, \mathrm{Re}(\overline{z_0}^T \beta) = 0,$$

where the last equality comes from (2-10). Therefore, (2-11) is proved and hence

$$J_f(z_0)\beta \in T_{w_0}(\partial B^N).$$

On the other hand, for any $\beta \in T_{z_0}^{(1,0)}(\partial B^n)$, we have

$$\overline{z_0}^T \beta = 0. \tag{2-13}$$

To prove $J_f^{(1,0)}(z_0)\beta \in T_{w_0}^{(1,0)}(\partial B^N)$, it is sufficient to get

$$\overline{w_0}^T J_f(z_0)\beta = 0.$$

From (2-12) and (2-13),

$$\overline{w_0}^T J_f(z_0)\beta = \lambda \overline{z_0}^T \beta = \lambda \overline{z_0}^T \beta = 0,$$

Therefore, $J_f(z_0)\beta \in T_{w_0}^{(1,0)}(\partial B^N)$. The proof of (I) is completed. □

3. Proof of Theorem 1.4

For any fixed point $b \in B^n$, let $\mathcal{L}_b$ be the complex (straight) line joining b and $\mathbf{1}$:

$$\mathcal{L}_b = \{z \in \mathbb{C}^n \mid z = \mathbf{1} + \xi(\mathbf{1} - b),\ \forall \xi \in \mathbb{C}\},$$

and let d_b be the complex disc given by $\mathcal{L}_b \cap B^n$. In particular,

$$d_0 = \{z \in B^n \mid z_2 = \cdots = z_n = 0\}.$$

From Lemma 2.2, it is found that $d_b = L(D_{\mathbf{1},\mathbf{1}-b})$.

Lemma 3.1. *Let $f = (f_1, \dots, f_N)^T \in \mathrm{Hol}(B^n, B^N)$ with $N \geq n \geq 1$, and $f_1(z) = z_1$, $z \in B^n$. Then*

$$f(z_1, 0, \dots, 0) = (z_1, 0, \dots, 0)^T, \quad z \in d_0.$$

Proof. Restricting $f(z) = (z_1, f_2, \dots, f_N)^T$ on d_0, then $f|_{d_0}$ can be regarded as a holomorphic mapping from D into B^N, which implies $|z_1|^2 + \sum_{j=2}^N |f_j(z)|^2 < 1$, $z \in d_0$ and then $\sum_{j=2}^N |f_j(z)|^2 < 1 - |z_1|^2$, $z \in d_0$. By $z_1 \to 1$, the maximum principle of subharmonic function guarantees $f_j|_{d_0} \equiv 0$ for any $2 \leq j \leq N$. Therefore, $f|_{d_0} = (z_1, 0, \dots, 0)^T$. □

Proof of Theorem 1.4. Step 1. Given $f = (f_1, \dots, f_N)^T \in \mathrm{Hol}(B^n, B^N)$ such that (1-1) holds and $f_1(z) \equiv z_1$ on B^n. From Lemma 3.1, one gets $f|_{d_0} = (z_1, 0, \dots, 0)^T$. We aim to prove $f_j(z) = z_j$ for $2 \leq j \leq n$ and $f_j(z) = 0$ for $n+1 \leq j \leq N$ on the unit ball.

Represent f_j by

$$f_j(z) = \sum_{k=2}^{n} \phi_{jk}(z) z_k, \qquad z \in B^n, \quad 2 \leq j \leq N, \tag{3-1}$$

where $\phi_{jk}(z)$ are all holomorphic functions on the unit ball. In fact, taking the Taylor expansion for $f_j(z)$ at 0 for $2 \leq j \leq N$, one gets

$$f_j(z) = f_j(0) + \sum_{k=1}^{\infty} \sum_{|v|=k} C_v z^v, \quad z \in B^n.$$

Let $\phi_{j1}(z_1) = \sum_{i=1}^{\infty} C_i z_1^i$. Then there are holomorphic functions $\phi_{jk}(z)$ satisfying

$$f_j(z) = f_j(0) + \sum_{k=1}^{\infty} \sum_{|v|=k} C_v z^v = f_j(0) + \phi_{j1}(z_1) + \sum_{k=2}^{n} \phi_{jk}(z) z_k, \quad z \in B^n.$$

We notice that, for $2 \leq k \leq n$, the $\phi_{jk}(z)$ are not necessarily unique in this expression for $f_j(z)$. Since $f_j(z_1, 0, \dots, 0) = 0$ for any $(z_1, 0, \dots, 0)^T \in B^n \cup \{\mathbf{1}\}$, we have $f_j(0) = 0$ and $\phi_{j1}(z_1) \equiv 0$, $z \in B^n \cup \{\mathbf{1}\}$, so that (3-1) holds.

In particular, if

$$\phi_{jk}(z) \equiv \delta_{jk}, \qquad 2 \le j \le N, \quad 2 \le k \le n, \tag{3-2}$$

then the theorem is proved. If not, due to $f(z) \in B^N$,

$$|z_1|^2 + \sum_{j=2}^{N} \left| \sum_{k=2}^{n} \phi_{jk}(z) z_k \right|^2 < 1, \quad z \in B^n. \tag{3-3}$$

Given a $b \in B^n$ with $\tilde{b} = (b_2, \dots, b_n)^T \neq 0$, there at least exists one $b_j \neq 0$ for $2 \le j \le n$; without loss of generality, let $b_2 \neq 0$. We consider $d_b = L(D_{\mathbf{1},\mathbf{1}-b})$ from Lemma 2.2, where the expression for $D_{\mathbf{1},\mathbf{1}-b}$ can be given by

$$D_{\mathbf{1},\mathbf{1}-b} = \left\{ \xi \in \mathbb{C} \,\middle|\, \left| \xi + \frac{1-\bar{b}_1}{\|\mathbf{1}-b\|^2} \right| < \frac{|1-b_1|}{\|\mathbf{1}-b\|^2} \right\}. \tag{3-4}$$

Notice that $\xi = 0 \in \partial D_{\mathbf{1},\mathbf{1}-b}$ and $z = \mathbf{1} \in \partial d_b$. Furthermore, for any $z \in d_b$, $z = L(\xi) = \mathbf{1} + \xi(\mathbf{1}-b) \in d_b$, $\xi \in D_{\mathbf{1},\mathbf{1}-b}$, i.e.,

$$(z_1, z_2, \dots, z_n)^T = (1 + \xi(1-b_1), -\xi b_2, \dots, -\xi b_n)^T, \quad \xi \in D_{\mathbf{1},\mathbf{1}-b},$$

which gives that for $z \in d_b \cup \partial d_b$, the following inequality holds:

$$\frac{1-|z_1|^2}{|z_2|^2} \ge \sum_{j=2}^{n} \frac{|z_j|^2}{|z_2|^2} = \sum_{j=2}^{n} \left| \frac{b_j}{b_2} \right|^2. \tag{3-5}$$

The equality is available only for $z \in \partial d_b$ and $z \neq \mathbf{1}$, i.e., $z_2 \neq 0$ ($\xi \neq 0$).

Step 2. Since (1-1) holds as $z \to \mathbf{1}$, it follows that

$$f(z) - (z_1, \dots, z_n, 0, \dots, 0)^T = O(|z-\mathbf{1}|^3).$$

Restricting $z \in d_b$, we obtain

$$\begin{aligned} &f(z) - (z_1, \dots, z_n, 0, \dots, 0)^T|_{z \in d_b} \\ &= \Bigg(0, \sum_{k=2}^{n} \phi_{2k}(z) z_k - z_2, \dots, \sum_{k=2}^{n} \phi_{nk}(z) z_k - z_n, \\ &\qquad \sum_{k=2}^{n} \phi_{(n+1)k}(z) z_k, \dots, \sum_{k=2}^{n} \phi_{Nk}(z) z_k \Bigg)^T \\ &= \Bigg(0, \Big(\sum_{k=2}^{n} \phi_{2k}(z) \frac{b_k}{b_2} - \frac{b_2}{b_2} \Big) z_2, \dots, \Big(\sum_{k=2}^{n} \phi_{nk}(z) \frac{b_k}{b_2} - \frac{b_n}{b_2} \Big) z_2, \\ &\qquad \Big(\sum_{k=2}^{n} \phi_{(n+1)k}(z) \frac{b_k}{b_2} \Big) z_2, \dots, \Big(\sum_{k=2}^{n} \phi_{Nk}(z) \frac{b_k}{b_2} \Big) z_2 \Bigg)^T, \end{aligned} \tag{3-6a}$$

and

$$O(|z-\mathbf{1}|^3)|_{z\in d_b} = O\left(\left(\left|\frac{1-b_1}{b_2}\right|^2 + \sum_{j=2}^{n}\left|\frac{b_j}{b_2}\right|^2\right)^{\frac{3}{2}}|z_2|^3\right) = O(|z_2|^3). \tag{3-6b}$$

Setting

$$\begin{aligned}\Gamma(z) &= (\Gamma_2(z),\dots,\Gamma_N(z))^T\\ &\triangleq \left(\sum_{k=2}^{n}\phi_{2k}(z)\frac{b_k}{b_2},\dots,\sum_{k=2}^{n}\phi_{nk}(z)\frac{b_k}{b_2},\sum_{k=2}^{n}\phi_{(n+1)k}(z)\frac{b_k}{b_2},\dots,\sum_{k=2}^{n}\phi_{Nk}(z)\frac{b_k}{b_2}\right)^T,\end{aligned}$$

we have from (3-6a) and (3-6b),

$$\Gamma(z) - \left(\frac{b_2}{b_2},\dots,\frac{b_n}{b_2},0\dots,0\right)^T = O(|z_2|^2), \quad z\in d_b. \tag{3-7}$$

Letting $z\to\mathbf{1}\in\partial d_b$, gives $z_2\to 0$ and hence (3-7) yields the following equalities:

$$\begin{aligned}\sum_{k=2}^{n}\phi_{jk}(\mathbf{1})\frac{b_k}{b_2}-\frac{b_j}{b_2}&=0, \quad 2\le j\le n,\\ \sum_{k=2}^{n}\phi_{jk}(\mathbf{1})\frac{b_k}{b_2}&=0, \quad n+1\le j\le N.\end{aligned} \tag{3-8}$$

We consider the first order derivative of (3-7) at $\mathbf{1}$ and obtain

$$\sum_{k=2}^{n}\phi'_{jk}(\mathbf{1})\frac{b_k}{b_2}=0, \quad 2\le j\le N. \tag{3-9}$$

We now set

$$A_0 = (\phi_{ij}(\mathbf{1}))_{(N-1)\times(n-1)}, \quad A_1 = (\phi'_{ij}(\mathbf{1}))_{(N-1)\times(n-1)},$$

so (3-8) and (3-9) are equivalent to

$$A_0\tilde{b} = (\tilde{b},0,\dots,0)^T, \quad A_1\tilde{b}=0, \tag{3-10}$$

where $\tilde{b}=(b_2,\dots,b_n)^T$. Since (3-10) is valid for any $\tilde{b}\neq 0$, we have $A_0 = (I_{n-1},0)^T$ and $A_1=0$, which implies that

$$\phi_{ij}(\mathbf{1})=\delta_{ij}, \quad \phi'_{ij}(\mathbf{1})=0, \qquad 2\le i\le N, \quad 2\le j\le n, \tag{3-11}$$

Step 3. Restricting f on d_b, from (3-3), we have

$$\sum_{j=2}^{N}\left|\sum_{k=2}^{n}\phi_{jk}(z)z_k\right|^2 = \sum_{j=2}^{N}\left|\sum_{k=2}^{n}\phi_{jk}(z)\frac{b_k}{b_2}\right|^2|z_2|^2 < 1-|z_1|^2, \quad z\in d_b.$$

Then

$$\sum_{j=2}^{N}\left|\sum_{k=2}^{n}\phi_{jk}(z)\frac{b_k}{b_2}\right|^2 < \frac{1-|z_1|^2}{|z_2|^2}, \quad z\in d_b.$$

From (3-5),

$$\sum_{j=2}^{N}\left|\sum_{k=2}^{n}\phi_{jk}(z)\frac{b_k}{b_2}\right|^2 \le \frac{1-|z_1|^2}{|z_2|^2} = \sum_{j=2}^{n}\left|\frac{b_j}{b_2}\right|^2, \qquad z\in\partial d_b, \quad z\neq\mathbf{1}. \tag{3-12}$$

For $z=\mathbf{1}$, i.e., $z_2=0$, it follows from (3-11) that

$$\sum_{j=2}^{N}\left|\sum_{k=2}^{n}\phi_{jk}(\mathbf{1})\frac{b_k}{b_2}\right|^2 = \sum_{j=2}^{n}\left|\frac{b_j}{b_2}\right|^2. \tag{3-13}$$

Combining (3-12) and (3-13), we have

$$\sum_{j=2}^{N}\left|\sum_{k=2}^{n}\phi_{jk}(z)\frac{b_k}{b_2}\right|^2 \le \sum_{j=2}^{n}\left|\frac{b_j}{b_2}\right|^2, \quad z\in\partial d_b. \tag{3-14}$$

Since $d_b = L(D_{\mathbf{1},\mathbf{1}-b})$, (3-14) is equivalent to

$$\sum_{j=2}^{N}\left|\sum_{k=2}^{n}\phi_{jk}(L(\xi))\frac{b_k}{b_2}\right|^2 \le \sum_{j=2}^{n}\left|\frac{b_j}{b_2}\right|^2, \quad \xi\in\partial D_{\mathbf{1},\mathbf{1}-b}. \tag{3-15}$$

Considering the maximum principle for the subharmonic function

$$\sum_{j=2}^{N}\left|\sum_{k=2}^{n}\phi_{jk}(L(\xi))\frac{b_k}{b_2}\right|^2$$

on $D_{\mathbf{1},\mathbf{1}-b}$, we have

$$\sum_{j=2}^{N}\left|\sum_{k=2}^{n}\phi_{jk}(L(\xi))\frac{b_k}{b_2}\right|^2 \le \sum_{j=2}^{n}\left|\frac{b_j}{b_2}\right|^2, \quad \xi\in D_{\mathbf{1},\mathbf{1}-b},$$

which means that

$$\sum_{j=2}^{N}\left|\sum_{k=2}^{n}\phi_{jk}(z)\frac{b_k}{b_2}\right|^2 \le \sum_{j=2}^{n}\left|\frac{b_j}{b_2}\right|^2, \quad z\in d_b. \tag{3-16}$$

Step 4. Consider the mapping $\Gamma(z)$ on d_b, which is a holomorphic mapping from d_b to the closure of the ball in $\mathbb{C}^{n-1}$ with the center 0 and radius $\left(\sum_{j=2}^{n}|b_j/b_2|^2\right)^{\frac{1}{2}}$ from (3-16). From the expression of $D_{\mathbf{1},\mathbf{1}-b}$ given by (3-4), let

$$\eta_1(\xi) = \frac{\xi + (1-\bar{b}_1)/\|\mathbf{1}-b\|^2}{|1-b_1|/\|\mathbf{1}-b\|^2} : \bar{D}_{\mathbf{1},\mathbf{1}-b} \to \bar{D},$$

and

$$\eta_2(\xi) = \frac{|1-b_1|}{1-\bar{b}_1}\xi : \bar{D} \to \bar{D},$$

where $\bar{D}_{\mathbf{1},\mathbf{1}-b}$ and $\bar{D}$ denote the closures of $D_{\mathbf{1},\mathbf{1}-b}$ and D, respectively. Constructing a mapping

$$\Psi(\xi) = \left(\sum_{j=2}^{n}\left|\frac{b_j}{b_2}\right|^2\right)^{-\frac{1}{2}} \cdot \Gamma \circ \eta_1^{-1} \circ \eta_2^{-1} : D \to \bar{B}^{N-1},$$

we have from (3-11) that

$$\Psi(1) = \left(\sum_{j=2}^{n}\left|\frac{b_j}{b_2}\right|^2\right)^{-\frac{1}{2}} \cdot \left(\frac{b_2}{b_2}, \dots, \frac{b_n}{b_2}, 0, \dots, 0\right)^T \in \partial B^{N-1}.$$

Moreover, the mapping f is holomorphic on B^n and satisfies (1-1) as $z \to \mathbf{1}$; from the construction, Ψ is holomorphic on D and C^2 at 1. In addition $\Psi(1) = w_0 \in \partial B^{N-1}$. According to Theorem 1.1, there exists a $\lambda > 0$ such that

$$\overline{J_\Psi(1)}^T w_0 = \lambda \cdot 1 > 0$$

unless Ψ is a constant mapping. In other words, the above inequality means that

$$\left(\sum_{j=2}^{n}\left|\frac{b_j}{b_2}\right|^2\right)^{-1} \cdot \frac{|1-b_1|}{\|\mathbf{1}-b\|^2} \cdot \frac{\overline{1-\bar{b}_1}}{|1-b_1|} \cdot \overline{\Gamma'(\mathbf{1})} \cdot \left(\frac{b_2}{b_2}, \dots, \frac{b_n}{b_2}\right)^T > 0.$$

However, from (3-11), it is found that $\Gamma'(\mathbf{1}) = 0$, which is a contradiction and forces Ψ to be a constant mapping such that Γ satisfies (3-11), i.e.,

$$\phi_{ij}(z) = \phi_{ij}(\mathbf{1}) \equiv \delta_{ij}, \qquad 2 \le i \le N, \quad 2 \le j \le n.$$

Consequently, from the expression for $f_j(z)$ in (3-1), one gets $f_j(z) = z_j$ for $2 \le j \le n$ and $f_j(z) = 0$ for $n+1 \le j \le N$. Therefore, we have $f(z) \equiv (z^T, 0)^T$ on the unit ball. □

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