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We use relations between the base change representations and theta lifts, to give a new proof to the local period problems of SL(2) over a nonarchimedean quadratic field extension E/F. Then we verify the Prasad conjecture for SL(2). With a similar strategy, we obtain a certain result for the Prasad conjecture for Sp(4).

1. Introduction

Assume that F is a nonarchimedean local field with characteristic 0. Let G be a connected reductive group defined over F and H be a closed subgroup of G. Given a smooth irreducible representation π of $G(F)$, one may consider the complex vector space $\mathrm{Hom}_{H(F)}(\pi, \mathbb{C})$. If it is nonzero, then we say that π is $H(F)$-distinguished, or has a nonzero $H(F)$-period.

Period problems, which are closely related to harmonic analysis, have been extensively studied for classical groups. The most general situations have been studied in [Sakellaridis and Venkatesh 2017] when G is split. Given a spherical variety $X = H \backslash G$, Sakellaridis and Venkatesh [2017] introduce a certain complex reductive group $\hat{G}_X$ associated with the variety X, to deal with the spectral decomposition of $L^2(H \backslash G)$ under the assumption that G is split. In a similar way, Prasad [2015, §9] introduces a certain quasisplit reductive group G^{op} to deal with the period problem when the subgroup H is the Galois fixed points of G, i.e., $H = G^{\mathrm{Gal}(E/F)}$, where E is a quadratic field extension of F. In this paper, we will mainly focus on the cases $G = R_{E/F}\mathrm{SL}_2$ and $H = \mathrm{SL}_2$, where $R_{E/F}$ denotes the Weil restriction of scalars, i.e., the Prasad conjecture [2015, Conjecture 2] for SL_2.

Let W_F and W_E be the Weil groups of F and E, and let WD_F and WD_E be the Weil–Deligne groups. Let ψ be any additive character of F and $\psi_E = \psi \circ \mathrm{tr}_{E/F}$. Assume that τ is an irreducible smooth representation of $\mathrm{SL}_2(F)$, with a Langlands parameter $\phi_\tau : WD_F \to \mathrm{PGL}_2(\mathbb{C})$ and a character λ of the component group $S_{\phi_\tau} = C(\phi_\tau)/C^\circ(\phi_\tau)$, where $C(\phi_\tau)$ is the centralizer of ϕ_τ in $\mathrm{PGL}_2(\mathbb{C})$ and $C^\circ(\phi_\tau)$ is the connected component of $C(\phi)$. Then $\phi_\tau|_{WD_E}$ gives a Langlands

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parameter of $\mathrm{SL}_2(E)$. The map $\phi_\tau \to \phi_\tau|_{WD_E}$ is called the base change map. Prasad's conjecture for SL(2) predicts the following result, which was shown in [Anandavardhanan and Prasad 2003].

Theorem 1.1. *Let E be a quadratic field extension of a nonarchimedean local field F with associated Galois group $\mathrm{Gal}(E/F) = \{1, \sigma\}$ and associated quadratic character $\omega_{E/F}$ of $F^\times$. Assume that τ is an irreducible smooth admissible representation of $\mathrm{SL}_2(E)$ with central character ω_τ satisfying $\omega_\tau(-1) = 1$. Then the following are equivalent*:

(i) *τ is $\mathrm{SL}_2(F)$-distinguished.*

(ii) *$\phi_\tau = \phi_{\tau'}|_{WD_E}$ for some irreducible representation τ' of $\mathrm{SL}_2(F)$ and τ has a Whittaker model with respect to a nontrivial additive character of E which is trivial on F.*

Anandavardhanan and Prasad [2003] deal with the cases for the principal series and square-integrable representations separately, using the restriction of $\mathrm{GL}_2(F)$-distinguished representations of $\mathrm{GL}_2(E)$. There is a key lemma [Anandavardhanan and Prasad 2003, Lemma 3.1] that if τ is $\mathrm{SL}_2(F)$-distinguished, then τ has a Whittaker model with respect to a nontrivial additive character of E which is trivial on F. Moreover, the multiplicity $\dim \mathrm{Hom}_{\mathrm{SL}_2(F)}(\tau, \mathbb{C})$ is invariant under the $\mathrm{GL}_2(F)$-conjugation action on τ. In [Anandavardhanan and Prasad 2016], they use a similar idea to deal with the case for SL_n, involving the restriction of $\mathrm{GL}_n(F)$-distinguished representations of $\mathrm{GL}_n(E)$. In this paper, we will use the local theta correspondence to give a new proof for a tempered representation of $\mathrm{SL}_2(E)$. Then we use Mackey theory and the double coset decomposition to deal with the principal series, instead of involving representations of GL(2). In order to verify Prasad's conjecture [2015, Conjecture 2] for SL(2), we will list all possible explicit parameter lifts

$$\tilde{\phi} : WD_F \to \mathrm{PGL}_2(\mathbb{C})$$

such that $\tilde{\phi}|_{WD_E} = \phi_\tau$, which are different from Prasad's descriptions in [2015, §18]. Our methods can also be used for the Sp(4)-distinction problems over a quadratic field extension; see Theorem 4.2.

Theorem 1.2. *Assume that τ is an irreducible $\mathrm{SL}_2(F)$-distinguished representation of $\mathrm{SL}_2(E)$, with an enhanced L-parameter (ϕ_τ, λ), where λ is a character of the component group S_{ϕ_τ}, then*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{SL}_2(F)}(\tau, \mathbb{C}) = |F(\phi_\tau)|,$$

where $F(\phi_\tau) = \{\tilde{\phi} : WD_F \to \mathrm{PGL}_2(\mathbb{C}) : \tilde{\phi}|_{WD_E} = \phi_\tau$ and $\lambda|_{S_{\tilde{\phi}}} \supset \mathbf{1}\}$ and $|F(\phi_\tau)|$ denotes its cardinality.

Remark 1.3. The statement in Theorem 1.2 is slightly different from the original Prasad conjecture for SL(2). We have used the fact that the degree of the base change map

$$\Phi : \mathrm{Hom}(WD_F, \mathrm{PGL}_2(\mathbb{C})) \to \mathrm{Hom}(WD_E, \mathrm{PGL}_2(\mathbb{C}))$$

at each parameter $\tilde{\phi}$ is equal to the size of the cokernel

$$\mathrm{coker}\{S_{\tilde{\phi}} \to S_{\phi_\tau}^{\mathrm{Gal}(E/F)}\}$$

for $\tilde{\phi} \in F(\phi_\tau)$ when $G = \mathrm{SL}(2)$, which is easy to check; see [Prasad 2015, §18].

Remark 1.4. Raphael Beuzart-Plessis [2017, Theorem 1] uses the relative trace formula to give an identity for the multiplicity $\dim_{\mathbb{C}} \mathrm{Hom}_{H'(F)}(\pi', \chi_{H'})$, where H' is an inner form of H defined over F, $\chi_{H'}$ is a quadratic character of $H'(F)$ and π' is a stable square-integrable representation of $(R_{E/F}H')(F) = H'(E)$. For example, $H' = \mathrm{SL}_1(D)$ and $H'(E) = \mathrm{SL}_2(E)$, where D is a quaternion division algebra defined over F. We plan to use the local theta correspondence to deal with the distinction problems for the pair $(\mathrm{SL}_2(E), \mathrm{SL}_1(D))$ in a subsequent paper. More precisely, we will figure out the multiplicity $\dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{SL}_1(D)}(\tau, \mathbb{C})$ for a smooth irreducible representation τ of $\mathrm{SL}_2(E)$.

Remark 1.5. Anandavardhanan and Prasad [2006; 2013] discuss the global period problems for SL_2 over a quadratic number field extension $\mathbb{E}/\mathbb{F}$. More generally, there are several results for the global period problems of $\mathrm{SL}_1(D)$ in [Anandavardhanan and Prasad 2013, §9], where $\mathrm{SL}_1(D)$ is an inner form of SL_2 defined over a number field $\mathbb{F}$. We hope that we can also use the global theta correspondence to revisit these questions in future.

Now we briefly describe the contents and the organization of this paper. In §2, we set up the notation about the local theta lifts. In §3, we give the proof of Theorem 1.1, and then we verify Prasad's conjecture for SL(2), i.e., Theorem 1.2 in §4. Finally, we give a partial result for the Prasad conjecture for Sp_4, i.e., Theorem 4.2.

2. The local theta correspondences

In this section, we will briefly recall some results about the local theta correspondence, following [Kudla 1996].

Let F be a local field of characteristic zero. Consider the dual pair $\mathrm{O}(V)\times\mathrm{Sp}(W)$. For simplicity, we may assume that $\dim V$ is even. Fix a nontrivial additive character ψ of F. Let ω_ψ be the Weil representation for $\mathrm{O}(V) \times \mathrm{Sp}(W)$, which can be described as follows. Fix a Witt decomposition $W = X \oplus Y$ and let $P(Y) = \mathrm{GL}(Y)N(Y)$ be the parabolic subgroup stabilizing the maximal isotropic subspace Y. Then

$$N(Y) = \{b \in \mathrm{Hom}(X, Y) \mid b^t = b\},$$

where $b^t \in \mathrm{Hom}(Y^*, X^*) \cong \mathrm{Hom}(X, Y)$. The Weil representation ω_ψ can be realized on the Schwartz space $S(X \otimes V)$ and the action of $P(Y) \times \mathrm{O}(V)$ is given by the usual formula

$$\begin{cases} \omega_\psi(h)\phi(x) = \phi(h^{-1}x), & \text{for } h \in \mathrm{O}(V), \\ \omega_\psi(a)\phi(x) = \chi_V(\det_Y(a))|\det_Y a|^{\frac{1}{2}\dim V}\phi(a^{-1}\cdot x), & \text{for } a \in \mathrm{GL}(Y), \\ \omega_\psi(b)\phi(x) = \psi(\langle bx, x\rangle)\phi(x), & \text{for } b \in N(Y), \end{cases}$$

where χ_V is the quadratic character associated to the disc $V \in F^\times/F^{\times 2}$ and $\langle -, -\rangle$ is the natural symplectic form on $W \otimes V$. To describe the full action of $\mathrm{Sp}(W)$, one needs to specify the action of a Weyl group element, which acts by a Fourier transform.

If π is an irreducible representation of $\mathrm{O}(V)$ (resp. $\mathrm{Sp}(W)$), the maximal π-isotypic quotient has the form

$$\pi \boxtimes \Theta_\psi(\pi)$$

for some smooth representation of $\mathrm{Sp}(W)$ (resp. $\mathrm{O}(V)$). We call $\Theta_\psi(\pi)$ the big theta lift of π. It is known that $\Theta_\psi(\pi)$ is of finite length and hence is admissible. Let $\theta_\psi(\pi)$ be the maximal semisimple quotient of $\Theta_\psi(\pi)$, which is called the small theta lift of π. Then there is a conjecture of Howe which states that

- $\theta_\psi(\pi)$ is irreducible whenever $\Theta_\psi(\pi)$ is nonzero.
- the map $\pi \mapsto \theta_\psi(\pi)$ is injective on its domain.

This has been proved by Waldspurger [1990] when the residual characteristic p of F is not 2. Recently, it has been proved completely in [Gan and Takeda 2016a; 2016b].

Theorem 2.1. *The Howe conjecture holds.*

First occurrence indices for pairs of orthogonal Witt towers. Let W_n be the $2n$-dimensional symplectic vector space with associated symplectic group $\mathrm{Sp}(W_n)$ and consider the two towers of orthogonal groups attached to the quadratic spaces with nontrivial discriminant. Let V_E and ϵV_E be 2-dimensional quadratic spaces with discriminant E and Hasse invariants $+1$ and -1, respectively, and let $\mathbb{H}$ be the 2-dimensional hyperbolic quadratic space over F,

$$V_r^+ = V_E \oplus \mathbb{H}^{r-1} \quad \text{and} \quad V_r^- = \epsilon V_E \oplus \mathbb{H}^{r-1},$$

and denote the orthogonal groups by $\mathrm{O}(V_r^+)$ and $\mathrm{O}(V_r^-)$, respectively. For an irreducible representation π of $\mathrm{Sp}(W_n)$, one may consider the theta lifts $\theta_r^+(\pi)$ and $\theta_r^-(\pi)$ to $\mathrm{O}(V_r^+)$ and $\mathrm{O}(V_r^-)$, respectively, with respect to a fixed nontrivial additive character ψ. Set

$$\begin{cases} r^+(\pi) = \inf\{2r : \theta_r^+(\pi) \neq 0\}; \\ r^-(\pi) = \inf\{2r : \theta_r^-(\pi) \neq 0\}. \end{cases}$$

Then Kudla and Rallis [2005] and B. Sun and C. Zhu [2015] showed the following:

Theorem 2.2 (conservation relation). *For any irreducible representation* π *of* $\mathrm{Sp}(W_n)$, *we have*

$$r^+(\pi)+r^-(\pi)=4n+4=4+2\dim W_n.$$

On the other hand, one may consider the mirror situation, where one fixes an irreducible representation of $\mathrm{O}(V_r^+)$ or $\mathrm{O}(V_r^-)$, and consider its theta lifts $\theta_n(\pi)$ to the tower of the symplectic group $\mathrm{Sp}(W_n)$. Then with $n(\pi)$ defined in the analogous fashion, due to [Sun and Zhu 2015, Theorem 1.10], we have

$$n(\pi)+n(\pi\otimes\det)=\dim V_r^{\pm}.$$

See-saw identities. Let (V,q) be a quadratic vector space over E of even dimension. Let $V'=\mathrm{Res}_{E/F}V$ be the same space V but now thought of as a vector space over F with a quadratic form

$$q'(v)=\tfrac{1}{2}\operatorname{tr}_{E/F}q(v).$$

If W_0 is a symplectic vector space over F, then $W_0\otimes_F E$ is a symplectic vector space over E. Then we have the following isomorphism of symplectic spaces:

$$\mathrm{Res}_{E/F}[(W_0\otimes_F E)\otimes_E V]\cong W_0\otimes V'=\boldsymbol{W}.$$

There is a pair

$$(\mathrm{Sp}(W_0),\mathrm{O}(V'))\quad\text{and}\quad(\mathrm{Sp}(W_0\otimes E),\mathrm{O}(V))$$

of dual reductive pairs in the symplectic group $\mathrm{Sp}(\boldsymbol{W})$. A pair (G_1,H_1) and (G_2,H_2) of dual reductive pairs in a symplectic group is called a see-saw pair if $H_1\subset G_2$ and $H_2\subset G_1$.

Lemma 2.3 [Kudla 1984]. *For a see-saw pair of dual reductive pairs* (G_1,H_1) *and* (G_2,H_2), *let* π_1 *be an irreducible representation of* H_1 *and* π_2 *of* H_2, *then we have the isomorphism*

$$\mathrm{Hom}_{H_1}(\Theta_\psi(\pi_2),\pi_1)\cong\mathrm{Hom}_{H_2}(\Theta_\psi(\pi_1),\pi_2).$$

Quadratic spaces. Let K/E be a quadratic field extension and $V=V_K$ be a 2-dimensional quadratic space over E with the norm map $N_{K/E}$. Set ϖ to be the uniformizer of $\mathcal{O}_F$ and $\mathrm{Gal}(K/E)=\langle s\rangle$. Let u be a unit in $\mathcal{O}_F^\times\setminus\mathcal{O}_F^{\times 2}$. Assume that the Hilbert symbol $(\varpi,u)_F$ is -1.

Example 2.4. Assume that p is odd. Let $L=F(\sqrt{-\varpi})$ be a quadratic field extension over F with associated quadratic character $\omega_{L/F}=\omega_{F(\sqrt{-\varpi})/F}$ by local class field theory. Let K be a quadratic field extension over E, then V_K is a 2-dimensional quadratic space over E with norm map $N_{K/E}$. We may regard V_K as a 4-dimensional quadratic space V' over F with quadratic form $q'(k)=\frac{1}{2}\operatorname{tr}_{E/F}N_{K/E}(k)$ for $k\in K$.

(i) If $E = F(\sqrt{\varpi})$ is ramified, then:

- If $K = E(\sqrt{u})$, then the discriminant $\mathrm{disc}(V') = 1 \in F^\times/F^{\times 2}$ and the Hasse invariant $\epsilon(V') = -1$.
- If $K = E(\sqrt[4]{\varpi})$, then $V' = V_L \oplus \mathbb{H}$ and $\mathrm{disc}(V') = -\varpi \in F^\times/F^{\times 2}$.
- If $K = E(\sqrt[4]{\varpi} \cdot \sqrt{u})$, then $\mathrm{disc}(V') = L$.

(ii) If $E = F(\sqrt{u})$ is unramified, then:

- If $K = E(\sqrt{\varpi})$, then $\mathrm{disc}(V') = 1$ and

$$\epsilon(V') = -(-1, \varpi)_F = \begin{cases} +1 & \text{if } -1 \in uF^{\times 2}; \\ -1 & \text{if } -1 \in F^{\times 2}. \end{cases}$$

- If $K = E(\sqrt{u'})$ and $u' \notin F^\times$, then $\mathrm{disc}(V') = N_{E/F}(u') \in F^\times/F^{\times 2}$.

If $-1 \in (F^\times)^2$ is a square in $F^\times$ and the discriminant of $V' = \mathrm{Res}_{E/F} V_K$ is the same as the discriminant of the 2-dimensional vector space E over F, i.e., $\mathrm{disc}(V') = E$, then $\chi_{V'}$ is $\omega_{E/F}$ and its special orthogonal group, denoted by $\mathrm{SO}(V') = \mathrm{SO}(3, 1)$, is isomorphic to

$$\mathrm{SO}(3,1) = \frac{\{(g, \lambda) \in \mathrm{GL}_2(E) \times F^\times : \lambda^2 N_{E/F}(\det g) = 1\}}{\{(t, N_{E/F}(t)^{-1}) : t \in E^\times\}} \cong \frac{\{g \in \mathrm{GL}_2(E) : \det(g) \in F^\times\}}{F^\times}.$$

Set $K^1 = \{k \in K^\times : k \cdot k^s = 1\}$, then there is a natural embedding

$$\mathrm{O}(V_K) = K^1 \rtimes \mu_2 \subset \mathrm{SO}(3,1) \quad \text{where } K^1 = \mathrm{SO}(V_K) \subset \mathrm{GL}_2(E).$$

In general, the discriminant $\mathrm{disc}(V')$ may not be equal to E. There is a group embedding $K^1 \hookrightarrow \mathrm{GL}_2(L')$ where $L' = F(\delta)$ and $\delta^2 = N_{E/F}(u')$ if $K = E(\sqrt{u'})$.

Remark 2.5. If $V' = \mathrm{Res}_{E/F} V_K$ has discriminant $1 \in F^\times/F^{\times 2}$ and Hasse invariant $+1$, then V' is called a split 4-dimensional quadratic space over F. Set $\mathrm{SO}_{2,2}(F) = \mathrm{SO}(V')$ to be the special orthogonal group.

Degenerate principal series representations. Let V_K be a 2-dimensional quadratic space over E with the norm map $N_{K/E}$. Assume that $V' = \mathrm{Res}_{E/F} V_K$ is a split 4-dimensional quadratic space over F. There is a natural embedding $\mathrm{O}(V_K) \hookrightarrow \mathrm{O}_{2,2}(F)$. Let P be a Siegel parabolic subgroup of $\mathrm{O}_{2,2}(F)$. Assume that $\mathcal{I}(s)$ is the degenerate principal series of $\mathrm{O}_{2,2}(F)$. Let us consider the double coset decomposition $P \backslash \mathrm{O}_{2,2}(F)/\mathrm{O}(V_K)$.

- If K is a field, then there are four open orbits in $P \backslash \mathrm{O}_{2,2}(F)/\mathrm{O}(V_K)$.
- If $K = E \oplus E$, then there are one closed orbit and three open orbits in $P \backslash \mathrm{O}_{2,2}(F)/\mathrm{O}_{1,1}(E)$.

Assume that there is a stratification $P\backslash \mathrm{O}_{2,2}(F)/\mathrm{O}(V_K)=\sqcup_{i=0}^r X_i$ such that $\bigsqcup_{i=0}^k X_i$ is open for each k lying in $\{0, 1, 2, \ldots, r\}$. Then there is an $\mathrm{O}(V_K)$-equivariant filtration $\{I_i\}_{i=0,1,2,\ldots,r}$ of $\mathcal{I}(s)|_{\mathrm{O}(V_K)}$ such that

$$0 = I_{-1} \subset I_0 \subset I_1 \subset \cdots \subset I_r = \mathcal{I}(s)|_{\mathrm{O}(V_K)}$$

and the smooth functions in the quotient I_i/I_{i-1} are supported on a single orbit X_i in $P \backslash \mathrm{O}_{2,2}(F)/\mathrm{O}(V_K)$.

Definition 2.6. Given an irreducible representation π of $\mathrm{O}(V_K)$, if

$$\mathrm{Hom}_{\mathrm{O}(V_K)}(I_{i+1}/I_i, \pi) \neq 0$$

implies that I_{i+1}/I_i is supported on the open orbits in $P \backslash \mathrm{O}_{2,2}(F)/\mathrm{O}(V_K)$, then we say that the representation π does not occur on the boundary of $\mathcal{I}(s)$.

It is well known that only the open orbits can support supercuspidal representations. Due to the Casselman criterion for a tempered representation, only the open orbits can support the tempered representations in our case if $s = \frac{1}{2}$; see [Lu 2017, Lemma 4.2.9].

3. Proof of Theorem 1.1

Before we prove Theorem 1.1, let us recall some facts.

Lemma 3.1. *If the discriminant of $V' = \mathrm{Res}_{E/F} V_K$ is E, then the theta lift of the trivial representation from $\mathrm{SL}_2(F)$ to $\mathrm{SO}(3, 1) = \mathrm{SO}(V')$ is a character, i.e.,*

$$\Theta_\psi(\mathbf{1}) = \mathbf{1} \boxtimes \omega_{E/F}.$$

Proof. Due to [Lu 2017, Theorem 2.4.11], the big theta lift of the Steinberg representation St from $\mathrm{GL}_2^+(F)$ to $\mathrm{GSO}(3, 1)$ is $\Theta_\psi(\mathrm{St}) = \mathrm{St}_E \boxtimes \omega_{E/F}$. By a similar argument, one can get $\Theta_\psi(\mathbf{1}) = \mathbf{1} \boxtimes \omega_{E/F}$. Notice that

$$\Theta_\psi(\mathbf{1}|_{\mathrm{SL}_2}) = \Theta_\psi(\mathbf{1})|_{\mathrm{SO}(3,1)},$$

then we are done. □

Remark 3.2. In fact, the theta lift $\theta'_\psi(\mathbf{1})$ from $\mathrm{SL}_2(F)$ to $\mathrm{O}(3, 1)$ remains irreducible when restricted to $\mathrm{SO}(3, 1)$, see [Prasad 1993, §5].

Now we begin the proof of Theorem 1.1, which we will complete in Section 4.

Proof of Theorem 1.1. According to the representation τ, we separate the proof into four cases:

- τ is a supercuspidal representation; see (A).
- τ is an irreducible principal series representation; see (B).

- τ is a Steinberg representation St_E; see (C).
- τ is a constituent of a reducible principle series $I(\chi)$ with $\chi^2 = 1$; see (D).

These exhaust all irreducible smooth representations of $\mathrm{SL}_2(E)$.

(A) If τ is supercuspidal, then there exists a character $\mu : K^\times \to \mathbb{C}^\times$ such that $\phi_\tau = i \circ (\mathrm{Ind}_{W_K}^{W_E} \mu)$, where

- W_K is the Weil group of K, where K is a quadratic field extension over E;
- μ does not factor through the norm map $N_{K/E}$, so the irreducible Langlands parameter
$$\mathrm{Ind}_{W_K}^{W_E} \mu : W_E \to \mathrm{GL}_2(\mathbb{C})$$
corresponds to a dihedral supercuspidal representation of $\mathrm{GL}_2(E)$ with respect to K;
- $i : \mathrm{GL}_2(\mathbb{C}) \to \mathrm{PGL}_2(\mathbb{C})$ is the projection map, which coincides with the adjoint map
$$\mathrm{Ad} : \mathrm{GL}(2) \to \mathrm{SO}(3).$$

In fact, the Langlands parameter ϕ of the representation Σ of $\mathrm{O}(V_K)$, where $\tau = \theta_\psi(\Sigma)$, is given by

$$\phi(g) = \begin{cases} \begin{pmatrix} \chi_K(g) & \\ & \chi_K^{-1}(g) \end{pmatrix} & \text{if } g \in W_K, \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \text{if } g = s, \end{cases}$$

where $s \in W_E \setminus W_K$ and the character $\chi_K : W_K \to \mathbb{C}^\times$ is the pull back of a nontrivial character μ_1 of K^1 under the map $K^\times \to K^1$ via $k \mapsto k^s k^{-1}$, i.e., $\chi_K(k) = \mu_1(k^s k^{-1})$, see [Kudla 1996, §6.4]. Furthermore, there is an isomorphism between two Langlands parameters of $\mathrm{O}(2)$,

$$\phi \otimes \omega_{K/E} \cong \mathrm{Ind}_{W_K}^{W_E} \frac{\mu^s}{\mu}.$$

In other words, one has $\chi_K = \mu^s \mu^{-1}$ and $\mu_1 = \mu|_{K^1}$ is the restricted character.

Moreover, if $\mu_1^2 \neq \mathbf{1}$, then

$$\tau = \theta_\psi(\mathrm{Ind}_{\mathrm{SO}(V_K)}^{\mathrm{O}(V_K)}(\mu_1)).$$

If $\mu_1^2 = \mathbf{1}$, then there are two extensions of μ_1 from $\mathrm{SO}(V_K)$ to $\mathrm{O}(V_K)$, denoted by $\mu_1^\pm$. For convenience, if $\mu_1^2 \neq \mathbf{1}$, we denote the irreducible representation

$$\mathrm{Ind}_{\mathrm{SO}(V_K)}^{\mathrm{O}(V_K)}(\mu_1)$$

by μ_1^+ as well. Assume that $\tau = \Theta_\psi(\mu_1^+)$ is supercuspidal.

If the discriminant disc $V' = L \in F^\times/(F^\times)^2$ is nontrivial, by the see-saw diagram

$$\begin{array}{cccc} \tau & \mathrm{SL}_2(E) & \mathrm{O}(V') & \Theta_\psi(\mathbf{1}) \\ & & \!\!\!\!\times\!\!\!\! & \\ \mathbf{1} & \mathrm{SL}_2(F) & \mathrm{O}(V_K) & \mu_1{}^+ \end{array}$$

one has an isomorphism

$$\mathrm{Hom}_{\mathrm{SL}_2(F)}(\tau, \mathbb{C}) \cong \mathrm{Hom}_{\mathrm{O}(V_K)}(\mathbf{1} \boxtimes \omega_{L/F}, \mu_1^+)$$

which is nonzero if and only if $\mu_1 = \mathbf{1}$. But $\mathrm{Hom}_{K^1}(\mathbf{1}, \mu_1) = 0$, and therefore $\mathrm{Hom}_{\mathrm{SL}_2(F)}(\tau, \mathbb{C}) = 0$.

If the discriminant of V' is $1 \in F^\times/(F^\times)^2$ and its Hasse invariant is -1, then the theta lift $\theta_\psi(\mathbf{1})$ from $\mathrm{SL}_2(F)$ to $\mathrm{O}(V')$ is zero by the conservation relation, so that

$$\mathrm{Hom}_{\mathrm{SL}_2(F)}(\tau, \mathbb{C}) = \mathrm{Hom}_{\mathrm{O}(V_K)}(\Theta_\psi(\mathbf{1}), \theta_\psi(\tau)) = 0.$$

If $V' \cong \mathbb{H}^2$ is a split 4-dimensional quadratic space over F, we denote by $\mathcal{I}(s)$ the degenerate principal series of $\mathrm{O}_{2,2}(F)$ and we assume that $F^\times/(F^\times)^2 \supset \{1, u, \varpi, u\varpi\}$ and $E = F(\sqrt{u})$ with associated Galois group $\mathrm{Gal}(E/F) = \langle \sigma \rangle$. Then

$$\mathrm{Hom}_{\mathrm{SL}_2(F)}(\tau, \mathbb{C}) = \mathrm{Hom}_{\mathrm{O}(V_K)}\big(\mathcal{I}\big(\tfrac{1}{2}\big), \mu_1^+\big) \cong \bigoplus_{j=1}^{4} \mathrm{Hom}_{\mathrm{O}(V_j)}(\mu_1^+, \mathbb{C}), \tag{3-1}$$

where $K = F(\sqrt{\varpi}, \sqrt{u})$ is a biquadratic field over F, and

- $V_1 = V_{E'}$ (where $E' = F(\sqrt{\varpi})$ is a quadratic field extension over F) is a 2-dimensional quadratic space over F with quadratic form $q(e') = N_{E'/F}(e')$, Hasse invariant $+1$ and quadratic character $\chi_{V_1} = \omega_{E'/F} = \omega_{F(\sqrt{\omega})/F}$;
- $V_2 = \epsilon' V_1 (\epsilon' \in F^\times \setminus N_{E'/F}(E')^\times)$ is the 2-dimensional quadratic space $F(\sqrt{\varpi})$ with quadratic form $\epsilon' N_{E'/F}$, Hasse invariant -1 and quadratic character $\chi_{V_2} = \chi_{V_1}$;
- $V_3 = V_{E''}$ is a 2-dimensional quadratic space over F with quadratic character $\omega_{F(\sqrt{\varpi u})/F}$ and Hasse invariant $+1$, where $E'' = F(\sqrt{\varpi u})$ is a quadratic field extension over F; and
- $V_4 = \epsilon'' V_3$ with Hasse invariant -1, where $\epsilon'' \in F^\times \setminus N_{E''/F}(E'')^\times$.

In the latter case, (3-1) can be rewritten as the identity

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{SL}_2(F)}(\tau, \mathbb{C}) = \sum_{j=1}^{4} \dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{O}(V_j)}(\mu_1^+, \mathbb{C}), \tag{3-2}$$

which is nonzero if and only if one of the following holds:

- $\mu(x - y\sqrt{\varpi}) = \mu(x + y\sqrt{\varpi})$ for $x, y \in F$.
- $\mu(x - y\sqrt{u\varpi}) = \mu(x + y\sqrt{u\varpi})$ for $x, y \in F$.

Remark 3.3. Because $\mu^s \neq \mu$, these two conditions cannot hold at the same time unless $p = 2$.

We would like to highlight a fact about the group embeddings $\mathrm{O}(V_j) \hookrightarrow K^1 \rtimes \langle s\rangle$ for $j \in \{1, 2\}$. There is a natural group embedding $\mathrm{SO}(V_1) \rtimes \langle s\rangle \to K^1 \rtimes \langle s\rangle$. Via the isomorphism between two quadratic E-vector spaces $(V_{E'} \otimes_F E, \epsilon' N_{E'/F}) \cong (V_K, N_{K/E})$, one has an identity

$$\dim \mathrm{Hom}_{\mathrm{O}(\epsilon' V_{E'})}(\mu_1^+, \mathbb{C}) = \dim \mathrm{Hom}_{\mathrm{O}(V_{E'})}((\mu_1^+)^{g_{\epsilon'}}, \mathbb{C}),$$

where $(\mu_1^+)^{g_{\epsilon'}}$ is a representation of $\mathrm{O}(V_K)$ given by

$$(\mu_1^+)^{g_{\epsilon'}}(x) = \mu_1^+(g_{\epsilon'}^{-1} x g_{\epsilon'}),\ \ x \in \mathrm{O}(V_K),\ \ g_{\epsilon'} \in \mathrm{GSO}(V_K) = K^\times \ \text{ with } N_{K/E}(g_{\epsilon'}) = \epsilon'.$$

Further, if the Whittaker datum is fixed, then the enhanced L-parameter of $(\mu_1^+)^{g_{\epsilon'}}$ is known if the enhanced L-parameter of μ_1^+ is given; see [Atobe and Gan 2017, §3.6].

The case p ≠ 2. (i) If $\mu_1^2 \neq \mathbf{1}$, then $\mathrm{Ind}_{\mathrm{SO}(V_K)}^{\mathrm{O}(V_K)}(\mu_1)$ is irreducible and

$$\dim \mathrm{Hom}_{\mathrm{O}(V_2)}(\mathrm{Ind}_{\mathrm{SO}(V_K)}^{\mathrm{O}(V_K)}(\mu_1), \mathbb{C}) = \dim \mathrm{Hom}_{\mathrm{O}(V_1)}(\mathrm{Ind}_{\mathrm{SO}(V_K)}^{\mathrm{O}(V_K)}(\mu_1), \mathbb{C}).$$

(ii) If $\mu_1^2 = \mathbf{1}$, then $\mu^2 = \chi_E \circ N_{K/E}$ and $\mu^s = -\mu$, so

$$\dim \mathrm{Hom}_{\mathrm{O}(V_2)}(\mu_1^+, \mathbb{C}) = \dim \mathrm{Hom}_{\mathrm{O}(V_1)}(\mu_1^-, \mathbb{C}).$$

Hence, if $p \neq 2$, (3-2) implies the following:

- If $\mu_1^2 \neq \mathbf{1}$ and $\mu|_{E'}$ factors through the norm map $N_{E'/F}$ for $E' \neq E$, then
$$\dim \mathrm{Hom}_{\mathrm{SL}_2(F)}(\tau, \mathbb{C}) = 2.$$
- If $\mu_1^2 = \mathbf{1}$ and $\mu|_{E'}$ factors through the norm map $N_{E'/F}$ for $E' \neq E$, then
$$\dim \mathrm{Hom}_{\mathrm{SL}_2(F)}(\tau, \mathbb{C}) = 1.$$

If $\mu_1^2 = \mathbf{1}$ and $\tau = \theta_\psi(\mu_1^+)$ is $\mathrm{SL}_2(F)$-distinguished, then

$$\dim \mathrm{Hom}_{\mathrm{SL}_2(F)}(\theta_\psi(\mu_1^-), \mathbb{C}) = \dim \mathrm{Hom}_{\mathrm{O}(V_K)}\big(\mathcal{I}\big(\tfrac{1}{2}\big), \mu_1^-\big)$$

which is equal to

$$\sum_{j=1}^{4} \dim \mathrm{Hom}_{\mathrm{O}(V_j)}(\mu_1^-, \mathbb{C}) = \dim \mathrm{Hom}_{\mathrm{SL}_2(F)}(\theta_\psi(\mu_1^+), \mathbb{C}).$$

Hence

$$\dim \mathrm{Hom}_{\mathrm{SL}_2(F)}(\theta_\psi(\mu_1^-), \mathbb{C}) = 1$$

if and only if $\mu|_{E'}$ factors through the norm map $N_{E'/F}$ for $E' \neq E$.

The case $p = 2$. (i) Suppose that there are two distinct quadratic fields E' and E'' over F such that $\mu|_{E'} = \chi'_F \circ N_{E'/F}$ and $\mu|_{E''} = \chi''_F \circ N_{E''/F}$. Furthermore, χ'_F/χ''_F is a quadratic character of $F^\times$ that is not trivial restricted on the Weil group W_K of K, i.e., χ'_F/χ''_F is different from three quadratic characters $\omega_{E/F}$, $\omega_{E'/F}$ and $\omega_{E''/F}$, which may happen only when $p = 2$. In this case, $\mu^s(t) = \mu(t) \cdot \chi'_F/\chi''_F(t)$ for $t \in W_K$,

$$\dim \mathrm{Hom}_{\mathrm{O}(V_1)}(\mu_1^+, \mathbb{C}) = \dim \mathrm{Hom}_{\mathrm{O}(V_2)}(\mu_1^+, \mathbb{C}),$$

and $\dim \mathrm{Hom}_{\mathrm{SL}_2(F)}(\tau, \mathbb{C}) = 4$ by the identity (3-2).

(ii) Given a cuspidal representation π of $\mathrm{GL}_2(E)$ with $\pi|_{\mathrm{SL}_2(E)} \supset \tau$, if π is not dihedral with respect to any quadratic extension K over E, then $\pi|_{\mathrm{SL}_2(E)} = \tau$ is irreducible.

We consider a 4-dimensional quadratic space X over F with discriminant E, then the orthogonal group $\mathrm{O}(X) = \mathrm{O}(3, 1)$ can be naturally embedded into the orthogonal group $\mathrm{O}(X \otimes_F E) = \mathrm{O}(2, 2)(E)$. Let $\pi \boxtimes \pi$ be the irreducible representation of the similitude special orthogonal group $\mathrm{GSO}(2, 2)(E)$. By the property of the big theta lift $\Theta(\pi)$ from $\mathrm{GL}_2(E)$ to $\mathrm{GSO}(2, 2)(E)$,

$$(\pi \boxtimes \pi)|_{\mathrm{SO}(2,2)(E)} = \Theta(\pi)|_{\mathrm{SO}(2,2)(E)} = \Theta(\pi|_{\mathrm{SL}_2(E)}) = \Theta(\tau)$$

is irreducible since τ is supercuspidal. Let $\mathfrak{I}(s)$ be the degenerate principal series of $\mathrm{Sp}_4(F)$. Assume that $(\pi \boxtimes \pi)^+$ is the unique extension from $\mathrm{GSO}(2, 2)(E)$ to $\mathrm{GO}(2, 2)(E)$ which participates with the theta correspondence with $\mathrm{GL}_2(E)$. Then $(\pi \boxtimes \pi)^+|_{\mathrm{O}(2,2)(E)}$ is irreducible. Considering the see-saw diagram

$$\begin{array}{cccc} \mathfrak{I}\left(\tfrac{1}{2}\right) & \mathrm{Sp}_4(F) & \mathrm{O}(2,2)(E) & (\pi \boxtimes \pi)^+ \\ & & \times & \\ \pi|_{\mathrm{SL}_2(E)} & \mathrm{SL}_2(E) & \mathrm{O}(3,1)(F) & \mathbb{C} \end{array}$$

due to the structure of $\mathfrak{I}\left(\frac{1}{2}\right)$ in [Gan and Ichino 2014, Proposition 7.2], one can get an equality

$$\dim \mathrm{Hom}_{\mathrm{SL}_2(E)}\left(\mathfrak{I}\left(\tfrac{1}{2}\right), \pi\right) = \dim \mathrm{Hom}_{\mathrm{O}(3,1)(F)}((\pi \boxtimes \pi)^+, \mathbb{C}).$$

The supercuspidal representation $\pi|_{\mathrm{SL}_2(E)}$ does not occur on the boundary of $\mathfrak{I}\left(\frac{1}{2}\right)$, therefore

$$\dim \mathrm{Hom}_{\mathrm{SL}_2(E)}\left(\mathfrak{I}\left(\tfrac{1}{2}\right), \pi\right) = \dim \mathrm{Hom}_{\mathrm{SL}_2(F)}(\pi^\vee, \mathbb{C}).$$

By the conservation relation, the fact that the first occurrence index of the determinant map det of $\mathrm{O}(3, 1)(F)$ is 4 implies that $\Theta_\psi(\det)$ from $\mathrm{O}(3, 1)(F)$ to

$\mathrm{Sp}(W_2) = \mathrm{Sp}_4(F)$ is zero and

$$\begin{aligned}\mathrm{Hom}_{\mathrm{O}(3,1)(F)}((\pi \boxtimes \pi)^-, \mathbb{C}) &\cong \mathrm{Hom}_{\mathrm{O}(3,1)(F)}((\pi \boxtimes \pi)^+, \det) \\ &= \mathrm{Hom}_{\mathrm{SL}_2(E)}(\Theta_\psi(\det), \pi|_{\mathrm{SL}_2(E)}) = 0.\end{aligned}$$

Hence

$$\begin{aligned}\text{(3-3)}\quad & \dim \mathrm{Hom}_{\mathrm{SL}_2(F)}(\pi^\vee, \mathbb{C}) \\ &= \dim \mathrm{Hom}_{\mathrm{O}(3,1)(F)}((\pi \boxtimes \pi)^+, \mathbb{C}) \\ &= \dim \mathrm{Hom}_{\mathrm{O}(3,1)(F)}((\pi \boxtimes \pi)^+, \mathbb{C}) + \dim \mathrm{Hom}_{\mathrm{O}(3,1)(F)}((\pi \boxtimes \pi)^-, \mathbb{C}) \\ &= \dim \mathrm{Hom}_{\mathrm{O}(3,1)(F)}(\mathrm{Ind}_{\mathrm{SO}(2,2)(E)}^{\mathrm{O}(2,2)(E)} (\pi \boxtimes \pi)|_{\mathrm{SO}(2,2)(E)}, \mathbb{C}) \\ &= \dim \mathrm{Hom}_{\mathrm{SO}(3,1)(F)}((\pi \boxtimes \pi), \mathbb{C}) \\ &= \dim \mathrm{Hom}_{\mathrm{GSO}(3,1)(F)}(\pi \boxtimes \pi, \mathbb{C}) \\ &= \dim \mathrm{Hom}_{\mathrm{GL}_2(E)}(\pi^\sigma, \pi^\vee).\end{aligned}$$

Therefore, if π is not dihedral with respect to any quadratic field extension K over E then $\tau = \pi|_{\mathrm{SL}_2(E)}$ is irreducible, and so the following are equivalent:

- $\pi^\sigma \cong \pi^\vee$, i.e., ϕ_π is conjugate-self-dual in the sense of [Gan et al. 2012, §3].
- $\dim \mathrm{Hom}_{\mathrm{SL}_2(F)}(\tau, \mathbb{C}) = 1$.

Remark 3.4. This method can be used to deal with the case when τ is the Steinberg representation St_E of $\mathrm{SL}_2(E)$, which will imply $\dim \mathrm{Hom}_{\mathrm{SL}_2(F)}(\mathrm{St}_E, \mathbb{C}) = 1$ directly. It will appear in the proof of Theorem 4.2 as well.

(B) Let χ be a unitary character of $E^\times$. If $\tau = I(z, \chi) = \mathrm{Ind}_{B(E)}^{\mathrm{SL}_2(E)} \chi|-|_E^z$ (normalized induction) is an irreducible principal series, by the double coset decomposition for $B(E) \backslash \mathrm{SL}_2(E)/\mathrm{SL}_2(F)$

$$\mathrm{SL}_2(E) = B(E)\mathrm{SL}_2(F) \sqcup B(E)\eta_1\mathrm{SL}_2(F) \sqcup B(E)\eta_2\mathrm{SL}_2(F),$$

where

$$\eta_1 = \begin{pmatrix} 1 & \\ \sqrt{d} & 1 \end{pmatrix} \quad \text{and} \quad \eta_2 = \begin{pmatrix} 1 & \\ \epsilon\sqrt{d} & 1 \end{pmatrix},$$

$\epsilon \in F^\times \setminus N_{E/F}(E^\times)$, then there is a short exact sequence

$$\begin{aligned}\text{(3-4)}\quad & \mathrm{Hom}_{F^\times}(|-|_E^z \chi, \mathbb{C}) \hookrightarrow \mathrm{Hom}_{\mathrm{SL}_2(F)}(\tau, \mathbb{C}) \\ &\to \textstyle\prod_{j=1}^2 \mathrm{Hom}_{E^1}(\tau^{\eta_j}, \mathbb{C}) \to \mathrm{Ext}^1_{F^\times}(|-|_E^z \chi, \mathbb{C}),\end{aligned}$$

where $\tau^{\eta_j}\left(\begin{smallmatrix} a & * \\ & \bar{a} \end{smallmatrix}\right) = \chi(a)$ for $a \in E^1 = \ker\{N_{E/F} : E^\times \to F^\times\}$. Then $\mathrm{Hom}_{\mathrm{SL}_2(F)}(\tau, \mathbb{C})$ is not equal to 0 if and only if one of the following conditions holds:

- $\chi|_{F^\times} = \mathbf{1}$ and $z = 0$;
- $\chi = \chi_F \circ N_{E/F}$.

In order to verify the Prasad conjecture, we need to figure out the exact dimension $\dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{SL}_2(F)}(\tau, \mathbb{C})$.

(i) If χ is trivial and $z{=}0$, then $\tau{=}I(\mathbf{1})$ is irreducible and $\dim \mathrm{Hom}_{\mathrm{SL}_2(F)}(\tau,\mathbb{C}){=}2$.

(ii) If $\chi = \chi_F \circ N_{E/F}$ with $\chi^2 = \mathbf{1} \neq \chi$ and $z = 0$, then $I(\chi)$ is reducible, which belongs to the tempered cases and we will discuss later; see (D).

(iii) If $\chi = \chi_F \circ N_{E/F}$ with $\chi^2 \neq \mathbf{1}$, then $\dim \mathrm{Hom}_{\mathrm{SL}_2(F)}(\tau, \mathbb{C}) = 2$.

(iv) If χ does not factor through $N_{E/F}$ but $\chi|_{F^\times} = \mathbf{1}$ and $s = 0$, then

$$\dim \mathrm{Hom}_{\mathrm{SL}_2(F)}(\tau, \mathbb{C}) = 1.$$

(C) If $\tau = \mathrm{St}_E$ is a Steinberg representation of $\mathrm{SL}_2(E)$, then the exact sequence (3-4) implies that

$$\dim \mathrm{Hom}_{\mathrm{SL}_2(F)}(I(|-|_E), \mathbb{C}) = 2,$$

so that $\dim \mathrm{Hom}_{\mathrm{SL}_2(F)}(\mathrm{St}_E, \mathbb{C}) = 2 - 1 = 1$.

(D) Assume that τ is tempered. If $\tau \subset I(\omega_{K/E})$ is an irreducible constituent of a reducible principal series, set $\chi = \omega_{K/E},\ \chi^+(\omega) = 1,\ \omega = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$, then from [Kudla 1996, page 86], we can see that

$$I(\omega_{K/E}) = \theta_\psi(\chi^+) \oplus \theta_\psi(\chi^-) \quad \text{where } \chi^- = \chi^+ \otimes \det$$

and $\tau = \theta_\psi(\chi^+) = \Theta_\psi(\chi^+)$, where $\theta_\psi(\chi^+)$ is the theta lift of χ^+ from $\mathrm{O}_{1,1}(E)$ to $\mathrm{SL}_2(E)$. By the see-saw diagram

$$\begin{array}{cccc} \tau & \mathrm{SL}_2(E) & \mathrm{O}_{2,2}(F) & \mathcal{I}\left(\frac{1}{2}\right) \\ & & \!\!\!\!\!\!\!\!\times\!\!\!\!\!\!\!\! & \\ \mathbb{C} & \mathrm{SL}_2(F) & \mathrm{O}_{1,1}(E) & \chi^+ \end{array}$$

where $\mathcal{I}(s)$ is the principal series of $\mathrm{O}_{2,2}(F)$, we have an identity,

$$\dim \mathrm{Hom}_{\mathrm{SL}_2(F)}(\tau, \mathbb{C}) = \dim \mathrm{Hom}_{\mathrm{O}_{1,1}(E)}\big(\mathcal{I}\left(\tfrac{1}{2}\right), \chi^+\big),$$

which is equal to

$$\dim \mathrm{Hom}_{\mathrm{O}_{1,1}(F)}(\chi^+, \mathbb{C}) + \dim \mathrm{Hom}_{\mathrm{O}(V_E)}(\chi^+, \mathbb{C}) + \dim \mathrm{Hom}_{\mathrm{O}(\epsilon V_E)}(\chi^+, \mathbb{C}).$$

If $\chi|_{F^\times} = 1$, then $\dim \mathrm{Hom}_{\mathrm{O}_{1,1}(F)}(\chi^+, \mathbb{C}) = 1$ and $\dim \mathrm{Hom}_{\mathrm{O}_{1,1}(F)}(\chi^-, \mathbb{C}) = 0$. If $\chi = \chi_F \circ N_{E/F}$, then $\dim \mathrm{Hom}_{\mathrm{O}(V_E)}(\chi^+, \mathbb{C}) = 1$. Hence we have the conclusion:

- If $\chi = \omega_{K/E} = \chi_F \circ N_{E/F}$ with $\chi_F^2 = 1$, then

$$\dim \mathrm{Hom}_{\mathrm{O}(\epsilon V_E)}(\chi^+, \mathbb{C}) = \dim \mathrm{Hom}_{\mathrm{O}(V_E)}(\chi^+, \mathbb{C}) = 1$$

and

$$\dim \mathrm{Hom}_{\mathrm{SL}_2(F)}(\tau, \mathbb{C}) = 3.$$

- If $\chi = \chi_F \circ N_{E/F}$ with $\chi_F^2 = \omega_{E/F}$, then

$$\dim \mathrm{Hom}_{\mathrm{O}(\epsilon V_E)}(\chi^+, \mathbb{C}) = \dim \mathrm{Hom}_{\mathrm{O}(V_E)}(\chi^-, \mathbb{C})$$

and

$$\dim \mathrm{Hom}_{\mathrm{SL}_2(F)}(\theta_\psi(\chi^+), \mathbb{C}) = \dim \mathrm{Hom}_{E^1}(\chi, \mathbb{C}) = 1.$$

- If χ does not factor through the norm map $N_{E/F}$, but $\chi|_{F^\times} = 1$, then

$$\dim \mathrm{Hom}_{\mathrm{SL}_2(F)}(\tau, \mathbb{C}) = 1.$$

In this case, if $\dim \mathrm{Hom}_{\mathrm{SL}_2(F)}(\theta_\psi(\chi^+), \mathbb{C}) \neq 0$, then $\dim \mathrm{Hom}_{\mathrm{SL}_2(F)}(\theta_\psi(\chi^-), \mathbb{C})$ is equal to the sum

$$\dim \mathrm{Hom}_{\mathrm{O}_{1,1}(F)}(\chi^+, \det) + \dim \mathrm{Hom}_{\mathrm{O}(V_E)}(\chi^+, \det) + \dim \mathrm{Hom}_{\mathrm{O}(\epsilon V_E)}(\chi^+, \det),$$

which is nonzero if and only if $\chi = \chi_F \circ N_{E/F}$ with $\chi_F^2 = \omega_{E/F}$.

After the discussions for the parameter side in Section 4, we finish the proof of Theorem 1.1.

4. The Prasad conjecture for SL(2)

Let us recall a well known result for SL_2.

Proposition 4.1 [Shelstad 1979]. *Let $\phi : WD_F \to \mathrm{GL}_2(\mathbb{C})$ be an irreducible representation and $\tau = i(\phi) = \mathrm{Ad}(\phi) : WD_F \to \mathrm{PGL}_2(\mathbb{C})$ be the associated discrete series L-parameter for SL_2, then there is a short exact sequence of component groups,*

$$1 \to S_\phi \to S_\tau \to I(\phi) \to 1,$$

where $I(\phi) = \{\chi : F^\times \to \mathbb{C}^\times \mid \chi^2 = 1 \text{ and } \phi \otimes \chi = \phi\}$.

Assume that τ is $\mathrm{SL}_2(F)$-distinguished and $\ell \in W_F \setminus W_E$, $\omega_{E/F}(\ell) = -1$. We start to verify the Prasad conjecture for SL_2. The main work here is to choose a proper element $A \in \mathrm{PGL}_2(\mathbb{C})$ such that $\tilde{\phi}(\ell) = A$ and $\tilde{\phi}|_{WD_E} = \phi_\tau$ for a certain Langlands parameter $\tilde{\phi} \in \mathrm{Hom}(WD_F, \mathrm{PGL}_2(\mathbb{C}))$ under the assumption that τ is $\mathrm{SL}_2(F)$-distinguished. In accordance with the discussions in Section 3, we separate the possible cases for τ into four parts.

Recall that $F^\times/F^{\times 2} \supset \{1, u, \varpi, u\varpi\}$, $E = F(\sqrt{u})$, $E'' = F(\sqrt{u\varpi})$ and $E' = F(\sqrt{\varpi})$. Let $K = F(\sqrt{u}, \sqrt{\varpi})$ be a biquadratic field extension over F with Galois group $\mathrm{Gal}(K/F) = \langle 1, s, \sigma, s\sigma \rangle$ and Weil group W_K. Suppose that $\mathrm{Gal}(K/E) = \langle s \rangle$, $\mathrm{Gal}(K/E'') = \langle s\sigma \rangle$ and $\mathrm{Gal}(K/E') = \langle \sigma \rangle$.

(A) Assume that $\tau \subset \pi|_{\mathrm{SL}_2(E)}$ is a supercuspidal representation of $\mathrm{SL}_2(E)$. If the Langlands parameter of τ,

$$\phi_\tau = i(\mathrm{Ind}_{W_K}^{W_E} \mu) = \omega_{K/E} \oplus \mathrm{Ind}_{W_K}^{W_E}\left(\frac{\mu^s}{\mu}\right)$$

with $\mu|_{E''} = \chi_F \circ N_{E''/F}$, then $\mu(t)\mu^{s\sigma}(t) = \chi_F(t)$ for $t \in W_K$. So

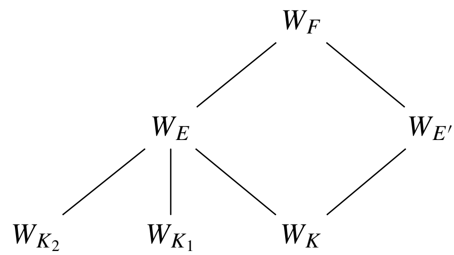
$$\left(\frac{\mu^s}{\mu}\right)^{\sigma}(t) = \frac{\mu^{s\sigma}(t)}{\mu^{\sigma}(t)} = \frac{\chi_F(t)}{\mu(t)\mu^{\sigma}(t)} = \frac{\chi_F(sts^{-1})}{\mu(t)\mu^{\sigma}(t)} = \frac{\mu^s(t)}{\mu(t)} \quad \text{for } t \in W_K,$$

i.e., $\mu^s/\mu = \chi_{E'} \circ N_{K/E'}$ for a character $\chi_{E'}$ of $E'^{\times}$.

The case $p \neq 2$. • If $\mu_1^2 = \mathbf{1}$, then the Langlands parameter satisfies

$$\phi_\tau = \omega_{K/E} \oplus \omega_{K_2/E} \oplus \omega_{K_1/E},$$

where each $K_j \neq K$ is a quadratic field extension over E:



Set

$$\tilde{\phi} = \omega_{E'/F} \oplus \operatorname{Ind}_{W_{E'}}^{W_F} \chi_{E'}, \tag{4-1}$$

where $E' \neq E$ are two distinct quadratic field extensions over F, then $\tilde{\phi}|_{W_E} = \phi_\tau$.

• If $\mu_1^2 \neq \mathbf{1}$, then the Langlands parameter

$$\phi_\tau = \omega_{K/E} \oplus \operatorname{Ind}_{W_K}^{W_E} \frac{\mu^s}{\mu}$$

has a lift $\tilde{\phi}$ defined in (4-1). Moreover, there is one more lift,

$$\tilde{\phi}' = \omega_{E'/F} \oplus \operatorname{Ind}_{W_{E'}}^{W_F} \chi_{E'}^{-1} \quad \text{with} \quad \chi_{E'} \circ N_{K/E'} = \frac{\mu^s}{\mu}$$

since $\operatorname{Ind}_{W_K}^{W_E}(\mu/\mu^s) = \operatorname{Ind}_{W_K}^{W_E}(\mu^s/\mu)$ is irreducible. In the L-packet Π_{ϕ_τ} containing ϕ_τ, set $\phi = \operatorname{Ind}_{W_K}^{W_E} \mu$ and $\phi_\tau = \operatorname{Ad}(\phi)$.

If the component group S_{ϕ_τ} has order 4, then we denote the four characters of S_{ϕ_τ} by $\{\lambda^{++}, \lambda^{--}, \lambda^{-+}, \lambda^{+-}\}$ which corresponds to the L-packet

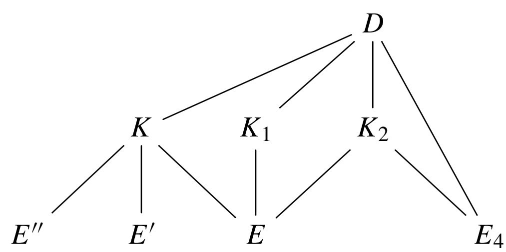
$$\Pi_{\phi_\tau} = \{\tau^{++}, \tau^{--}, \tau^{-+}, \tau^{+-}\}.$$

If the order of S_{ϕ_τ} is 2, then we denote its two characters as $\{\lambda^{+}, \lambda^{-}\}$, which corresponds to $\Pi_{\phi_\tau} = \{\tau^{+}, \tau^{-}\}$.

- If $\mu_1^2 = \mathbf{1}$, then $|I(\phi)| = 4$, two representations in Π_{ϕ_τ} are $\mathrm{SL}_2(F)$-distinguished and of dimension 1, say τ^{++} and τ^{--}. Since the component group $S_{\tilde{\phi}} = \mu_2 \hookrightarrow S_{\phi_\tau}$ is the diagonal embedding, τ^{+-} and τ^{-+} are not $\mathrm{SL}_2(F)$-distinguished, which is compatible with the fact that neither the restricted representation $\lambda^{+-}|_{S_{\tilde{\phi}}}$ nor $\lambda^{-+}|_{S_{\tilde{\phi}}}$ contains the trivial character of $S_{\tilde{\phi}}$, where λ^{+-} and λ^{-+} correspond to the representations τ^{+-} and τ^{-+}, respectively.
- If $\mu_1^2 \neq \mathbf{1}$, then $|I(\phi)| = 2$ and only one of them is $\mathrm{SL}_2(F)$-distinguished, say $\tau^+ = \theta_{\psi, V_K, W}((\mu^s/\mu)^+)$. If $\tau^- = \theta_{\psi, \epsilon V_K, W}((\mu^s/\mu)^+)$ corresponds to the nontrivial character of S_{ϕ_τ}, denoted by λ^-, where ϵV_K is the 2-dimensional quadratic space K over E with a quadratic form $\epsilon N_{K/E}$, $\epsilon \in E^\times \setminus N_{K/E}(K^\times)$ and the Hasse invariant of $\mathrm{Res}_{E/F}(\epsilon V_K)$ is -1, then
$$\dim \mathrm{Hom}_{\mathrm{SL}_2(F)}(\tau^-, \mathbb{C}) = 0.$$
Note that $S_{\tilde{\phi}} = \mu_2 \cong S_{\phi_\tau}$, then $\lambda^-|_{S_{\tilde{\phi}}}$ is nontrivial.

The case p = 2. There are some special cases if $p = 2$.

- If $\dim \mathrm{Hom}_{\mathrm{SL}_2(F)}(\tau, \mathbb{C}) = 4$, then $\mu_1^2 = \mathbf{1}$ and there is a quadratic field extension D over K such that $\chi_K = \omega_{D/K}$ and D is the composite field KE_4, where E_4 is the quadratic field extension of F corresponding to the quadratic character χ'_F/χ''_F where $\mu|_{E'} = \chi'_F \circ N_{E'/F}$, $\mu|_{E''} = \chi''_F \circ N_{E''/F}$ and E' and E'' are two distinct quadratic field extensions over F, which are different from E:



Set $\{1, u, \varpi, d, du, \varpi u, \varpi d, \varpi du\} \subset F^\times/F^{\times 2}$, $E_4 = F(\sqrt{d})$, $K = F(\sqrt{u}, \sqrt{\varpi})$, $K_2 = F(\sqrt{u}, \sqrt{d})$, and $K_1 = F(\sqrt{u}, \sqrt{d\varpi})$. There are four distinct Langlands parameter lifts of ϕ_τ:
$$\tilde{\phi}_1 = \omega_{E_4/F} \oplus \omega_{F(\sqrt{\varpi u})/F} \oplus \omega_{F(\sqrt{d\varpi u})/F},$$
$$\tilde{\phi}_2 = \omega_{E_4/F} \oplus \omega_{F(\sqrt{\varpi})/F} \oplus \omega_{F(\sqrt{d\varpi})/F},$$
$$\tilde{\phi}_3 = \omega_{F(\sqrt{du})/F} \oplus \omega_{F(\sqrt{\varpi u})/F} \oplus \omega_{F(\sqrt{d\varpi})/F},$$
$$\tilde{\phi}_4 = \omega_{F(\sqrt{du})/F} \oplus \omega_{F(\sqrt{\varpi})/F} \oplus \omega_{F(\sqrt{\varpi ud})/F},$$
where $\omega_{F(\sqrt{\varpi})/F}$ is the quadratic character associated to the quadratic field extension $F(\sqrt{\varpi})/F$, and similarly for the other quadratic characters $\omega_{F(\sqrt{du})/F}$ and so on.

Since $S_{\tilde{\phi}_i} = S_{\phi_\tau} \cong \mu_2 \times \mu_2$, only τ^{++} can survive, i.e., the rest of the elements in the L-packet Π_{ϕ_τ} cannot be $\mathrm{SL}_2(F)$-distinguished.

• If $\dim \mathrm{Hom}_{\mathrm{SL}_2(F)}(\tau, \mathbb{C}) = 1$ and π is not dihedral, i.e., $\tau = \pi|_{\mathrm{SL}_2(E)}$ is irreducible, then $\phi_\tau = \phi_\tau^\sigma$. There exists one element $A \in \mathrm{PGL}_2(\mathbb{C})$ such that

$$\phi_\tau(\ell \cdot t \cdot \ell^{-1}) = A \cdot \phi_\tau(t) \cdot A^{-1}$$

for $t \in WD_E$. Set $\tilde{\phi}(\ell) = A$ and $\tilde{\phi}(t) = \phi_\tau(t)$ for $t \in WD_E$. Since ϕ_τ is irreducible, A is unique. Hence ϕ_τ admits a unique lift $\tilde{\phi} : W_F \to \mathrm{PGL}_2(\mathbb{C})$ such that $\tilde{\phi}|_{W_E} = \phi_\tau$.

(B) If $\phi_\tau(t) = \begin{pmatrix} \chi(t)|t|^z & \\ & 1 \end{pmatrix} \in \mathrm{PGL}_2(\mathbb{C})$, then

• if $z = 0$ and χ is trivial, $\tilde{\phi}(\ell)$ can be chosen as $\begin{pmatrix} \omega_{E/F}(\ell) & \\ & 1 \end{pmatrix} = \begin{pmatrix} -1 & \\ & 1 \end{pmatrix}$ or $\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$;

• if $z = 0$, χ does not factor through the norm $N_{E/F}$ but $\chi|_{F^\times} = 1$, set $\chi = \nu^\sigma/\nu$ for a quadratic character ν of $E^\times$, then there is only one lift,

$$\tilde{\phi} = i(\mathrm{Ind}_{W_E}^{W_F} \nu);$$

• if $\chi = \chi_F \circ N_{E/F}, \chi^2 \neq 1$, then there are two lifts

$$\tilde{\phi}(\ell) = \begin{pmatrix} \chi_F(\ell) & \\ & 1 \end{pmatrix} \text{ or } \begin{pmatrix} -\chi_F(\ell) & \\ & 1 \end{pmatrix}.$$

(C) If $\phi_\tau = \mathrm{Ad}(\mathbf{1} \otimes S_2)$ corresponds to the Steinberg representation St_E of $\mathrm{SL}_2(E)$, then there is only one lift $\tilde{\phi} = \mathrm{Ad}(\mathbf{1} \otimes S_2) : WD_F \to \mathrm{PGL}_2(\mathbb{C})$.

(D) If $\phi_\tau(t) = \begin{pmatrix} \omega_{K/E}(t) & \\ & 1 \end{pmatrix} \in \mathrm{PGL}_2(\mathbb{C})$, then there are several subcases.

- If $\omega_{K/E} = \chi_F \circ N_{E/F}$ with $\chi_F^2 = \mathbf{1}$, then

$$\tilde{\phi}(\ell) = \begin{pmatrix} \chi_F(\ell) & \\ & 1 \end{pmatrix} \text{ or } \begin{pmatrix} -\chi_F(\ell) & \\ & 1 \end{pmatrix}.$$

  Moreover, $\omega_{K/E}|_{F^\times} = \chi_F^2 = \mathbf{1}$, and $\omega_{K/E} = \nu^\sigma/\nu$ so for a quadratic character ν of $E^\times$, we may set

$$\tilde{\phi}_3 = i(\mathrm{Ind}_{W_E}^{W_F} \nu) = \omega_{E/F} \oplus \mathrm{Ind}_{W_E}^{W_F}\left(\frac{\nu^\sigma}{\nu}\right).$$

- If $\omega_{K/E} = \chi_F \circ N_{E/F}$ with $\chi_F^2 = \omega_{E/F}$, then there is only one extension

$$\tilde{\phi}(\ell) = \begin{pmatrix} \chi_F(\ell) & \\ & 1 \end{pmatrix}.$$

- If $\omega_{K/E}$ does not factor through the norm map $N_{E/F}$ but $\omega_{K/E}|_{F^\times} = 1$, then

$$\tilde{\phi} = i(\mathrm{Ind}_{W_E}^{W_F} \nu) \quad \text{where } \omega_{K/E} = \nu^\sigma/\nu.$$

Hence, we finish the proof of Theorem 1.1 and Theorem 1.2. □

Further discussion. Inspired by the case that $\tau = \pi|_{\mathrm{SL}_2(E)}$ is an irreducible representation of $\mathrm{SL}_2(E)$, where π is a representation of $\mathrm{GL}_2(E)$, we have a certain result of the Prasad conjecture for $G = \mathrm{Sp}_4$.

Theorem 4.2. *Let E be a quadratic field extension over a nonarchimedean local field F with characteristic zero. Assume that τ is an irreducible representation of* $\mathrm{Sp}_4(E)$. *Let π be an irreducible representation of* $\mathrm{GSp}_4(E)$ *and* $\pi|_{\mathrm{Sp}_4(E)} \supset \tau$, *then*

(i) *if π is tempered and nongeneric, then* $\mathrm{Hom}_{\mathrm{Sp}_4(F)}(\tau, \mathbb{C}) = 0$;

(ii) *if π is a generic square-integrable representation of* $\mathrm{GSp}_4(E)$ *and* $\pi|_{\mathrm{Sp}_4(E)}$ *is irreducible, then the L-packet Π_{ϕ_τ} is a singleton and*

$$\dim \mathrm{Hom}_{\mathrm{Sp}_4(F)}(\tau, \mathbb{C}) = |F(\phi_\tau)|,$$

where $F(\phi_\tau) = \{\tilde{\phi} : WD_F \to \mathrm{SO}_5(\mathbb{C}) \mid \tilde{\phi}|_{WD_E} = \phi_\tau\}$ *and* $|F(\phi_\tau)|$ *denotes its cardinality.*

Proof. (i) If π is tempered and nongeneric, then $\pi = \Theta(\Sigma)$ where Σ is an irreducible representation of $\mathrm{GSO}(V_{D_E})$, where V_{D_E} is the nonsplit 4-dimensional quadratic space over E with trivial discriminant and Hasse invariant -1. Since $\mathrm{Res}_{E/F} V_{D_E}$ is an 8-dimensional quadratic space over F with trivial discriminant and Hasse invariant -1, the conservation relation implies that the theta lift of the trivial representation from $\mathrm{Sp}_4(F)$ to $\mathrm{O}(\mathrm{Res}_{E/F} V_{D_E})$ is zero. Due to the see-saw diagram

$$\begin{array}{cccc} \tau & \mathrm{Sp}_4(E) & \mathrm{O}(\mathrm{Res}_{E/F}(V_{D_E})) & 0 \\ & & \times & \\ \mathbb{C} & \mathrm{Sp}_4(F) & \mathrm{O}(V_{D_E}) & \theta(\tau) \end{array}$$

one has the desired equality, $\mathrm{Hom}_{\mathrm{Sp}_4(F)}(\tau, \mathbb{C}) = 0$.

(ii) By the assumption, $\tau = \pi|_{\mathrm{Sp}_4(E)}$ is a square-integrable representation. Fix $\ell \in W_F \setminus W_E$.

- If the theta lift $\Theta^{2,2}(\pi)$ from $\mathrm{GSp}_4(E)$ to $\mathrm{GSO}(2,2)(E)$ is zero, then one can use a similar method appearing in the proof of [Lu 2017, Theorem 4.2.18(iii)] to obtain the equality

$$\dim \mathrm{Hom}_{\mathrm{Sp}_4(F)}(\pi, \mathbb{C}) = \dim \mathrm{Hom}_{\mathrm{SO}(3,3)(F)}(\Theta^{3,3}(\pi), \mathbb{C}),$$

which is equal to the number

$$\left|\{\chi : F^\times \to \mathbb{C}^\times \mid \mathrm{Hom}_{\mathrm{GSO}(3,3)(F)}(\Theta^{3,3}(\pi), \chi \circ \lambda) \neq 0\}\right|,$$

where $\Theta^{3,3}(\pi)$ is the theta lift of π from $\mathrm{GSp}_4(E)$ to $\mathrm{GSO}(3,3)(E)$ and λ is the similitude character of the group $\mathrm{GSO}(3,3)(F)$. Therefore, the dimension

$\dim \mathrm{Hom}_{\mathrm{Sp}_4(E)}(\tau, \mathbb{C}) = 1$ if and only if the Langlands parameter ϕ_π of π is conjugate-self-dual, i.e., $\phi_\pi^\vee = \phi_\pi^\sigma$.

On the parameter side, $\phi_\tau : WD_E \to \mathrm{PGSp}_4(\mathbb{C}) = \mathrm{SO}_5(\mathbb{C})$ is irreducible and $\phi_\tau \cong \phi_\tau^\vee \cong \phi_\tau^\sigma$. There exists a unique element $A \in \mathrm{SO}_5(\mathbb{C})$ such that

$$\phi_\tau(\ell \cdot t \cdot \ell^{-1}) = A \cdot \phi_\tau(t) \cdot A^{-1}$$

for $t \in WD_E$. Set $\tilde{\phi}(\ell) = A$ and $\tilde{\phi}(t) = \phi_\tau(t)$ for $t \in WD_E$. Then $\tilde{\phi}$ is what we want.

- If $\Theta^{2,2}(\pi) \neq 0$, then $\phi_\pi = \phi_1 \oplus \phi_2$ where $\phi_i : WD_E \to \mathrm{GL}_2(\mathbb{C})$ is irreducible and $\phi_1 \neq \phi_2$. Moreover, $\phi_\tau = \mathbf{1} \oplus (\phi_1^\vee \otimes \phi_2)$; see [Gan and Takeda 2010, page 3008]. Let Σ be the irreducible representation of $\mathrm{GSO}(2,2)(E)$ satisfying $\theta_\psi(\Sigma) = \pi$, then $\Sigma|_{\mathrm{SO}(2,2)(E)}$ is irreducible since $\pi|_{\mathrm{Sp}_4(E)}$ is irreducible. Using a similar method appearing in [Lu 2017, Theorem 4.2.18(ii)], one can get that the dimension

$$\dim \mathrm{Hom}_{\mathrm{Sp}_4(F)}(\tau, \mathbb{C})$$

has an upper bound

$$\dim \mathrm{Hom}_{\mathrm{SO}(3,3)(F)}(\Theta^{3,3}(\pi), \mathbb{C}) + \dim \mathrm{Hom}_{\mathrm{SO}(4,0)(F)}(\Sigma, \mathbb{C}) \tag{4-2}$$

and a lower bound

$$\sum_X \dim \mathrm{Hom}_{\mathrm{SO}(X,F)}(\Sigma, \mathbb{C}), \tag{4-3}$$

where X runs over all elements in the kernel $\ker\{H^1(F, \mathrm{O}(4)) \to H^1(E, \mathrm{O}(4))\}$. We will show that both the lower bound (4-3) and the upper bound (4-2) are equal to 2 if $\pi|_{\mathrm{Sp}_4(E)}$ is an irreducible $\mathrm{Sp}_4(F)$-distinguished representation. Then

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{Sp}_4(F)}(\tau, \mathbb{C}) = 2.$$

There are two subcases.

(a) If $\phi_1^\vee = \phi_1^\sigma$, then $\phi_1^\vee \neq \phi_2^\sigma$, otherwise $\phi_1 = \phi_2$, which contradicts $\phi_1 \neq \phi_2$. Since ϕ_1 is irreducible, the Langlands parameter ϕ_1 is either conjugate-orthogonal or conjugate-symplectic, but cannot be both. Note that there is an equality

$$\dim \mathrm{Hom}_{\mathrm{SO}(3,3)(F)}(\Theta^{3,3}(\pi), \mathbb{C}) = \left|\{\chi : F^\times \to \mathbb{C}^\times \mid \mathrm{Hom}_{\mathrm{GSO}(3,3)(F)}(\Theta^{3,3}(\pi), \chi \circ \lambda) \neq 0\}\right|.$$

We have a similar result for $\dim \mathrm{Hom}_{\mathrm{SO}(4,0)(F)}(\Sigma, \mathbb{C})$ and $\dim \mathrm{Hom}_{\mathrm{SO}(2,2)(F)}(\Sigma, \mathbb{C})$. If $\phi_2^\vee = \phi_2^\sigma$ is conjugate-self-dual with the same sign as ϕ_1, then

$$\dim \mathrm{Hom}_{\mathrm{Sp}_4(E)}(\tau, \mathbb{C}) = 2.$$

Otherwise, τ is not $\mathrm{Sp}_4(F)$-distinguished.

On the parameter side, $1/\det\phi_1 = (\det\phi_1)^\sigma$. Without loss of generality, suppose that ϕ_1 is conjugate-orthogonal, i.e., $\det\phi_1 = \nu^\sigma/\nu = \det\phi_2$, then $\nu\otimes\phi_j$ is $\mathrm{Gal}(E/F)$-invariant. For each j, there exists a parameter $\tilde{\phi}_j : WD_F \to \mathrm{GL}_2(\mathbb{C})$ such that $\tilde{\phi}_j|_{WD_E} = \phi_j \otimes \nu$. Set $\rho_1 = \tilde{\phi}_1 \oplus \tilde{\phi}_2$ and $\rho_2 = \tilde{\phi}_1 \oplus \tilde{\phi}_2\omega_{E/F}$. Let $i : \mathrm{GSp}_4(\mathbb{C}) \to \mathrm{SO}_5(\mathbb{C})$ be the natural projection map. Then the parameters $i(\rho_1)$ and $i(\rho_2)$ are what we want.

(b) If $\phi_1^\vee = \phi_2^\sigma$, then $\dim \mathrm{Hom}_{\mathrm{Sp}_4(F)}(\tau, \mathbb{C}) = 2$ since the upper bound (4-2) is 2 and the lower bound (4-3) is at least 2. On the parameter side, $\phi_\tau = \mathbf{1} \oplus (\phi_2^\sigma \otimes \phi_2)$ is $\mathrm{Gal}(E/F)$-invariant. There exist two natural parameters $\tilde{\phi}_j : WD_F \to \mathrm{GL}_5(\mathbb{C})$ such that $\tilde{\phi}_j|_{WD_E} = \phi_\tau$, which are $\omega_{E/F} \oplus \mathrm{As}^+(\phi_2)$ and $\omega_{E/F} \oplus \mathrm{As}^-(\phi_2)$, where $\mathrm{As}^\pm(\phi_2)$ are the Asai lifts of ϕ_2; see [Gan et al. 2012, §7]. Then the images of $\tilde{\phi}_j$ lie in $\mathrm{SO}_5(\mathbb{C})$. Therefore, we have finished the proof. □

Remark 4.3. If $\tau = \pi|_{\mathrm{Sp}_4(E)}$ is irreducible, one can also use the method appearing in [Anandavardhanan and Prasad 2003] directly to get that the dimension $\dim \mathrm{Hom}_{\mathrm{Sp}_4(F)}(\tau, \mathbb{C})$ equals the sum

$$\sum_{\chi : F^\times/(F^\times)^2 \to \mathbb{C}^\times} \dim \mathrm{Hom}_{\mathrm{GSp}_4(F)}(\pi, \chi). \tag{4-4}$$

Combining this with the results in [Lu 2017, Theorem 4.2.18], we can obtain $\dim \mathrm{Hom}_{\mathrm{Sp}_4(F)}(\tau, \mathbb{C})$ if π is tempered.

Remark 4.4. Let $U_2(D)$ be the unique inner form of $\mathrm{Sp}_4(F)$ defined over F. Suppose that π is a generic representation of $\mathrm{GSp}_4(E)$. Thanks to [Beuzart-Plessis 2017, Theorem 1], if $\pi|_{\mathrm{Sp}_4(E)} = \tau$ is an irreducible square-integrable representation of $\mathrm{Sp}_4(E)$ and $\Theta^{2,2}(\pi)$ is 0, then

$$\dim \mathrm{Hom}_{U_2(D)}(\tau, \mathbb{C}) = 1.$$

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