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We establish a maximum principle for a two-point function in order to analyze the convexity of level sets of harmonic functions. We show that this can be used to prove a strict convexity result involving the smallest principal curvature of the level sets.

1. Introduction

The study of the convexity of level sets of solutions to elliptic PDEs has a long history, starting with the well-known result that the level curves of the Green's function of a convex domain Ω in $\mathbb{R}^2$ are convex [Ahlfors 1973]. Gabriel [1957] proved the analogous result in three dimensions and this was extended by Lewis [1977] and later Caffarelli and Spruck [1982] to higher dimensions and more general elliptic PDEs. These results show that for a large class of PDEs, there is a principle that convexity properties of the boundary of the domain Ω imply convexity of the level sets of the solution u.

There are several approaches to these kinds of convexity results; see for example [Kawohl 1985, Section III.11]. One is the "macroscopic" approach, which uses a globally defined function of two points x, y (which could be far apart) such as $u\big(\frac{1}{2}(x+y)\big) - \min(u(x), u(y))$. Another is the "microscopic" approach, which computes with functions of the principal curvatures of the level sets at a single point. This is often used together with a constant rank theorem. There is now a vast literature on these and closely related results, see for example [Alvarez et al. 1997; Bian and Guan 2009; Bianchini et al. 2009; Borell 1982; Brascamp and Lieb 1976; Caffarelli and Friedman 1985; Caffarelli et al. 2007; Diaz and Kawohl 1993; Hamel et al. 2016; Korevaar 1983; 1990; Korevaar and Lewis 1987; Rosay and Rudin 1989; Shiffman 1956; Singer et al. 1985; Székelyhidi and Weinkove 2016; Wang 2014].

It is natural to ask whether these ideas can be extended to cases where the boundary of the domain is *not* convex. Are the level sets of the solution at least as

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convex as the boundary in some appropriate sense? In this short note we introduce a global "macroscopic" function of two points which gives a kind of measure of convexity and makes sense for nonconvex domains. Our function

$$(Du(y) - Du(x)) \cdot (y - x) \tag{1-1}$$

is evaluated at two points x, y, which are constrained to lie on the same level set of u. Under suitable conditions, a level set of u is convex if and only if this quantity has the correct sign on that level set. We prove a maximum principle for this function using the method of Rosay and Rudin [1989], who considered a different two-point function

$$\tfrac{1}{2}(u(x) + u(y)) - u\Big(\frac{x+y}{2}\Big). \tag{1-2}$$

In addition, we show that our "macroscopic" approach can be used to prove a "microscopic" result. Namely, we localize our function and show that it gives another proof of a result of Chang, Ma, and Yang [Chang et al. 2010] on the principal curvatures of the level sets of a harmonic function u. In this paper, we consider only the case of harmonic functions. However, we expect that our techniques extend to some more general types of PDEs.

We now describe our results more precisely. Let Ω_0 and Ω_1 be bounded domains in $\mathbb{R}^n$ with $\overline{\Omega}_1 \subset \Omega_0$. Define $\Omega = \Omega_0 \setminus \Omega_1$. Assume that $u \in C^1(\overline{\Omega})$ satisfies

$$\Delta u = 0 \text{ in } \Omega = \Omega_0 \setminus \overline{\Omega}_1, \quad u = 0 \text{ on } \partial\Omega_0, \quad u = 1 \text{ on } \partial\Omega_1, \tag{1-3}$$

and

$$Du \text{ is nowhere vanishing in } \Omega. \tag{1-4}$$

It is well known that (1-4) is satisfied if Ω_0 and Ω_1 are both starshaped with respect to some point $p \in \Omega_1$. A special case of interest is when both Ω_0 and Ω_1 are convex, but this is not required for our main result.

To introduce our two-point function, first fix a smooth function $\psi : [0, \infty) \to \mathbb{R}$ satisfying

$$\psi'(t) - 2|\psi''(t)|t \geq 0. \tag{1-5}$$

For example, we could take $\psi(t) = at$ for $a \geq 0$. Then define

$$Q(x, y) = (Du(y) - Du(x)) \cdot (y - x) + \psi(|y - x|^2) \tag{1-6}$$

restricted to (x, y) in

$$\Sigma = \{(x, y) \in \overline{\Omega} \times \overline{\Omega} \mid u(x) = u(y)\}.$$

Comparing with the Rosay–Rudin function (1-2), note that the function $Q(x, y)$ does not require $\frac{1}{2}(x + y) \in \overline{\Omega}$ and makes sense whether or not $\partial\Omega_0$ or $\partial\Omega_1$ are

convex. Taking $\psi = 0$, the level set $\{u = c\}$ is convex if and only if the quantity Q is nonpositive on $\{u = c\}$. If $\psi(t) = at$ for $a > 0$ then $Q \le 0$ implies strict convexity of the level set. More generally Q gives quantitative information about the convexity of the level sets $\{u = c\}$, relative to the gradient Du.

We also remark that the function (1-6) looks formally similar to the two-point function of Andrews and Clutterbuck [2011], a crucial tool in their proof of the fundamental gap conjecture. However, here x and y are constrained to lie on the same level set of u and so the methods of this paper are quite different.

Our main result is the following:

Theorem 1.1. *Q does not attain a strict maximum at a point in the interior of Σ.*

Roughly speaking, this result says that the level sets $\{u = c\}$ for $0 \le c \le 1$ are "the least convex" when $c = 0$ or $c = 1$. As mentioned above, the result holds even in the case that $\partial\Omega_0$ and $\partial\Omega_1$ are nonconvex.

The proof of Theorem 1.1 follows quite closely the paper of Rosay and Rudin [1989]. Indeed a key tool of [Rosay and Rudin 1989] is Lemma 2.1 below, which gives a map from points x to points y with the property that x, y lie on the same level set.

Next we localize our function (1-6) to prove a strict convexity result on the level sets of u. If we assume now that $\partial\Omega_0$ and $\partial\Omega_1$ are strictly convex, we can apply the technique of Theorem 1.1 to obtain an alternative proof of the following result of Chang, Ma, and Yang [Chang et al. 2010].

Theorem 1.2. *Assume in addition that $\partial\Omega_0$ and $\partial\Omega_1$ are strictly convex and C^2. Then the quantity $|Du|\kappa_1$ attains its minimum on the boundary of Ω, where κ_1 is the smallest principal curvature of the level sets of u.*

Note that many other strict convexity results of this kind are proved in [Chang et al. 2010; Jost et al. 2012; Longinetti 1983; Ma et al. 2010; 2011; Ortel and Schneider 1983; Zhang and Zhang 2013].

2. Proof of Theorem 1.1

First we assume that n is even. We suppose for a contradiction that Q attains a maximum at an interior point, and assume that $\sup_\Sigma Q > \sup_{\partial\Sigma} Q$. Then we may choose $\delta > 0$ sufficiently small so that

$$Q_\delta(x, y) = Q(x, y) + \delta|x|^2$$

still attains a maximum at an interior point.

We use a lemma from [Rosay and Rudin 1989]. Suppose (x_0, y_0) is an interior point with $u(x_0) = u(y_0)$. We may assume that $Du(x_0)$ and $Du(y_0)$ are nonzero

vectors. Let L be an element of $\mathrm{O}(n)$ with the property that

$$L(Du(x_0)) = cDu(y_0) \quad \text{for } c = |Du(x_0)|/|Du(y_0)|. \tag{2-1}$$

Note that there is some freedom in the definition of L. We will make a specific choice later. Rosay and Rudin [1989, Lemma 1.3] show the following — it is a special case of the lemma:

Lemma 2.1. *There exists a real analytic function $\alpha(w) = O(|w|^3)$ such that for all $w \in \mathbb{R}^n$ sufficiently close to the origin,*

$$u(x_0 + w) = u(y_0 + cLw + f(w)\xi + \alpha(w)\xi), \quad \text{where } \xi = \frac{Du(y_0)}{|Du(y_0)|}, \tag{2-2}$$

where f is a harmonic function defined in a neighborhood of the origin in $\mathbb{R}^n$, given by

$$f(w) = \frac{1}{|Du(y_0)|}(u(x_0 + w) - u(y_0 + cLw)). \tag{2-3}$$

Proof of Lemma 2.1. We include the brief argument here for the sake of completeness. Define a real analytic map G which takes $(w, \alpha) \in \mathbb{R}^n \times \mathbb{R}$ sufficiently close to the origin to

$$G(w, \alpha) = u(y_0 + cLw + f(w)\xi + \alpha\xi) - u(x_0 + w),$$

for c, L, ξ, and f defined by (2-1), (2-2), and (2-3). Note that $G(0, 0) = 0$ and, by the definition of ξ,

$$\frac{\partial G}{\partial \alpha}(0, 0) = D_i u(y_0)\xi_i = |Du(y_0)| > 0,$$

where here and henceforth we are using the convention of summing repeated indices.

Hence by the implicit function theorem there exists a real analytic map $\alpha = \alpha(w)$ defined in a neighborhood U of the origin in $\mathbb{R}^n$ to $\mathbb{R}$ with $\alpha(0) = 0$ such that $G(w, \alpha(w)) = 0$ for all $w \in U$. It only remains to show that $\alpha(w) = O(|w|^3)$.

Write $y = y_0 + cLw + f(w)\xi + \alpha(w)\xi$, $x = x_0 + w$, and $L = (L_{ij})$ so that $L_{ij}D_j u(x_0) = cD_i u(y_0)$ and $cL_{ij}D_i u(y_0) = D_j u(x_0)$. Then at $w \in U$,

$$\begin{aligned} 0 &= \frac{\partial G}{\partial w_j} \\ &= D_i u(y)\left(cL_{ij} + \frac{(D_j u(x) - cD_k u(y_0 + cLw)L_{kj})}{|Du(y_0)|}\xi_i + \frac{\partial \alpha}{\partial w_j}\xi_i\right) - D_j u(x), \end{aligned} \tag{2-4}$$

and evaluating at $w = 0$ gives $0 = |Du(y_0)|\, \partial\alpha/\partial w_j(0)$ and hence $\partial\alpha/\partial w_j(0) = 0$ for all j.

Differentiating (2-4) and evaluating at $w = 0$, we obtain for all j, ℓ,

$$\begin{aligned}
0 &= \frac{\partial^2 G}{\partial w_\ell \partial w_j} \\
&= D_k D_i u(y_0) c^2 L_{ij} L_{k\ell} - D_\ell D_j u(x_0) \\
&\qquad + D_i u(y_0)\left(\frac{(D_\ell D_j u(x_0) - c^2 D_m D_k u(y_0) L_{kj} L_{m\ell})}{|Du(y_0)|}\xi_i + \frac{\partial^2 \alpha}{\partial w_\ell \partial w_j}(0)\xi_i \right) \\
&= |Du(y_0)| \frac{\partial^2 \alpha}{\partial w_\ell \partial w_j}(0).
\end{aligned}$$

Hence $\alpha(w) = O(|w|^3)$, as required. □

Now assume that Q_δ achieves a maximum at the interior point (x_0, y_0). Write $x = x_0 + w = (x_1, \ldots, x_n)$ and $y = y_0 + cLw + f(w)\xi + \alpha(w)\xi = (y_1, \ldots, y_n)$ and

$$F(w) = Q_\delta(x, y) = Q(x_0 + w, y_0 + cLw + f(w)\xi + \alpha(w)\xi) + \delta|x_0 + w|^2.$$

To prove the lemma it suffices to show that $\Delta_w F(0) > 0$, where we write $\Delta_w = \sum_j \partial^2/\partial w_j^2$. Observe that

$$\Delta_w x(0) = 0 = \Delta_w y(0).$$

Hence, evaluating at 0, we get

$$\begin{aligned}
\Delta_w F = &\sum_j \left(\frac{\partial^2}{\partial w_j^2}(D_i u(y) - D_i u(x)) \right)(y_i - x_i) \\
&+ 2\frac{\partial}{\partial w_j}(D_i u(y) - D_i u(x)) \frac{\partial}{\partial w_j}(y_i - x_i) + \sum_j \frac{\partial^2}{\partial w_j^2}\psi(|y - x|^2) + 2n\delta.
\end{aligned}$$

First we compute

$$\begin{aligned}
\sum_j \frac{\partial^2}{\partial w_j^2}\psi(|y-x|^2) &= 2\psi' \sum_{i,j}(cL_{ij} - \delta_{ij})^2 + 4\psi'' \sum_j \left(\sum_i (y_i - x_i)(cL_{ij} - \delta_{ij}) \right)^2 \\
&\geq 2\psi' \sum_{i,j}(cL_{ij} - \delta_{ij})^2 - 4|\psi''||y-x|^2 \sum_{i,j}(cL_{ij} - \delta_{ij})^2 \geq 0
\end{aligned}$$

using the Cauchy–Schwarz inequality and the condition (1-5).

Next, at $w = 0$,

$$\frac{\partial}{\partial w_j} D_i u(y) = D_k D_i u(y) \frac{\partial y_k}{\partial w_j} = c D_k D_i u(y) L_{kj},$$

$$\sum_j \frac{\partial^2}{\partial w_j^2} D_i u(y) = D_\ell D_k D_i u(y) \frac{\partial y_k}{\partial w_j} \frac{\partial y_\ell}{\partial w_j} = c^2 D_\ell D_k D_i u(y) L_{kj} L_{\ell j} = 0,$$

$$\frac{\partial}{\partial w_j} D_i u(x) = D_j D_i u(x), \quad \sum_j \frac{\partial^2}{\partial w_j^2} D_i u(x) = D_j D_j D_i u(x) = 0,$$

where for the second line we used the fact that $\Delta_w y(0) = 0$ and $L_{kj} L_{\ell j} D_\ell D_k u = \Delta u = 0$. Hence, combining the above,

$$\begin{aligned} \Delta_w F &> 2(c D_k D_i u(y) L_{kj} - D_j D_i u(x))(c L_{ij} - \delta_{ij}) \\ &= 2c^2 \Delta u(y) - 2c L_{ki} D_k D_i u(y) - 2c L_{ij} D_j D_i u(x) + 2\Delta u(x) \\ &= -2c L_{ki} D_k D_i u(y) - 2c L_{ij} D_j D_i u(x). \end{aligned}$$

Now we use the fact that n is even, and we make an appropriate choice of L following [Rosay and Rudin 1989, Lemma 4.1(a)]. Namely, after making an orthonormal change of coordinates, we may assume, without loss of generality that $Du(x_0)/|Du(x_0)|$ is e_1, and

$$Du(y_0)/|Du(y_0)| = \cos\theta\, e_1 + \sin\theta\, e_2,$$

for some $\theta \in [0, 2\pi)$. Here we are writing $e_1 = (1, 0, \ldots 0)$ and $e_2 = (0, 1, 0, \ldots)$, etc., for the standard unit basis vectors in $\mathbb{R}^n$. Then define the isometry L by

$$L(e_i) = \begin{cases} \cos\theta\, e_i + \sin\theta\, e_{i+1} & \text{for } i = 1, 3, \ldots, n-1, \\ -\sin\theta\, e_{i-1} + \cos\theta\, e_i & \text{for } i = 2, 4, \ldots, n. \end{cases}$$

In terms of entries of the matrix (L_{ij}), this means that $L_{kk} = \cos\theta$ for $k = 1, \ldots, n$ and for $\alpha = 1, 2, \ldots, \frac{1}{2}n$, we have

$$L_{2\alpha-1,2\alpha} = -\sin\theta, \quad L_{2\alpha,2\alpha-1} = \sin\theta,$$

with all other entries zero. Then

$$\begin{aligned} \text{(2-5)} \quad &\sum_{i,k} L_{ki} D_k D_i u(y) \\ &= \sum_{k=1}^{n} L_{kk} D_k D_k u(y) + \sum_{\alpha=1}^{n/2} (L_{2\alpha-1,2\alpha} + L_{2\alpha,2\alpha-1}) D_{2\alpha-1} D_{2\alpha} u(y) \\ &= (\cos\theta) \Delta u(y) = 0. \end{aligned}$$

Similarly $\sum_{i,k} L_{ki} D_k D_i u(x) = 0$. This completes the proof of Theorem 1.1 in the case of n even.

For n odd, we argue in the same way as in [Rosay and Rudin 1989]. Let L be an isometry of the even-dimensional $\mathbb{R}^{n+1}$, defined in the same way as above, but now

$$L(Du(x_0), 0) = (c(Du)(y_0), 0).$$

In Lemma 2.1, replace $w \in \mathbb{R}^n$ by $w \in \mathbb{R}^{n+1}$. Define $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ to be the projection $(w_1, \ldots, w_{n+1}) \mapsto (w_1, \ldots, w_n)$ and replace (2-2) and (2-3) by

$$u(x_0 + \pi(w)) = u\big(y_0 + c\pi(Lw) + f(w)\xi + \alpha(w)\xi\big), \tag{2-6}$$

where $\xi = Du(y_0)/|Du(y_0)|$ and f is given by

$$f(w) = \frac{1}{|Du(y_0)|}\big(u(x_0 + \pi(w)) - u(y_0 + c\pi(Lw))\big). \tag{2-7}$$

As in [Rosay and Rudin 1989], note that if $g : \mathbb{R}^n \to \mathbb{R}$ is harmonic in $\mathbb{R}^n$ then $w \mapsto g(\pi(Lw))$ is harmonic in $\mathbb{R}^{n+1}$. In particular, f is harmonic in a neighborhood of the origin in $\mathbb{R}^{n+1}$. The function G above becomes $G(w, \alpha) = u(y_0 + c\pi(Lw) + f(w)\xi + \alpha\xi) - u(x_0 + \pi(w))$ with $w \in \mathbb{R}^{n+1}$, and we make similar changes to F. It is straightforward to check that the rest of the proof goes through. □

Remark 2.2. The proof of Theorem 1.1 also shows that when $\psi = 0$ the quantity $Q(x, y)$ does not attain a strict interior *minimum*.

3. Global to infinitesimal

Here we give a proof of Theorem 1.2 using the quantity Q. We first claim that, for $x \in \Omega$ and $a > 0$,

$$(Du(y) - Du(x)) \cdot (y - x) + a|y - x|^2 \le O(|y - x|^3) \quad \text{for } y \sim x,\ u(x) = u(y)$$

if and only if

$$(\kappa_1 |Du|)(x) \ge a.$$

Indeed, to see this, first choose coordinates such that at x we have $Du = (0, \ldots, 0, D_n u)$ and $(D_i D_j u)_{1 \le i, j \le n-1}$ is diagonal with

$$D_1 D_1 u \ge \cdots \ge D_{n-1} D_{n-1} u.$$

For the "if" direction of the claim, choose $y(t) = x + te_1 + O(t^2)$ such that $u(x) = u(y(t))$, for t small. By Taylor's theorem,

$$\big(Du(y(t)) - Du(x)\big) \cdot (y(t) - x) + a|y(t) - x|^2 = t^2 D_1 D_1 u(x) + at^2 + O(t^3),$$

giving $D_1 D_1 u(x) \le -a$, which is the same as $|Du|\kappa_1 \ge a$. Indeed, from a well-known and elementary calculation (see for example [Chang et al. 2010, § 2]),

$$\kappa_1 = \frac{-D_1 D_1 u}{|Du|}$$

at x. Hence $|Du|\kappa_1 \geq a$. The "only if" direction of the claim follows similarly.

We will make use of this correspondence in what follows.

Proof of Theorem 1.2. By assumption, $\kappa_1|Du| \geq a > 0$ on $\partial\Omega$. It follows from Theorem 1.1 and the discussion above that the level sets of u are all strictly convex. Assume for a contradiction that $\kappa_1|Du|$ achieves a strict (positive) minimum at a point x_0 in the interior of Ω, say

$$(\kappa_1|Du|)(x_0) = a - \eta > 0 \text{ for some } \eta > 0. \tag{3-1}$$

We may assume without loss of generality that $\eta < \frac{1}{6}a$. Indeed, if not then if x_0 lies on the level set $\{u = c\}$ for some $c \in (0, 1)$ we can replace Ω by a convex ring $\{c_0 < u < c_1\}$ for c_0, c_1 with $0 \leq c_0 < c < c_1 \leq 1$. We still denote by a the minimum value of $\kappa_1|Du|$ on the boundary of this new Ω. For appropriately chosen c_0, c_1 we have (3-1) and $\eta < \frac{1}{6}a$. This changes the boundary conditions on $\partial\Omega_0$ and $\partial\Omega_1$ to $u = c_0$ and $u = c_1$, but this will not affect any of the arguments.

Pick $\varepsilon > 0$ sufficiently small, so that the distance from x_0 to the boundary of Ω is much larger than ε, and in addition, so that $\varepsilon^{1/3} \ll \eta$.

Consider the quantity

$$Q(x, y) = (Du(y) - Du(x)) \cdot (y - x) + a|y - x|^2 - \frac{a}{6\varepsilon^2}|y - x|^4,$$

and restrict to the set

$$\Sigma^\varepsilon = \{(x, y) \in \bar{\Omega} \times \bar{\Omega} \mid u(x) = u(y), |y - x| \leq \varepsilon\}.$$

Suppose that Q attains a maximum on Σ^ε at a point (x, y). First assume that (x, y) lies in the boundary of Σ^ε. There are two possible cases:

(1) If $x, y \in \Sigma^\varepsilon$ with x and y in $\partial\Omega$ (note that since $u(x) = u(y)$, if one of x, y is a boundary point then so is the other), then since $\kappa_1|Du| \geq a$ on $\partial\Omega$ we have

$$(Du(y) - Du(x)) \cdot (y - x) + a|y - x|^2 \leq O(\varepsilon^3).$$

Hence in this case $Q(x, y) \leq O(\varepsilon^3)$.

(2) If $|y - x| = \varepsilon$ then since $\kappa_1|Du| \geq a - \eta$ everywhere,

$$Q(x, y) \leq -(a - \eta)\varepsilon^2 + O(\varepsilon^3) + a\varepsilon^2 - \tfrac{1}{6}a\varepsilon^2 = \left(\eta - \tfrac{1}{6}a\right)\varepsilon^2 + O(\varepsilon^3) < 0,$$

by the assumption $\eta < \frac{1}{6}a$.

We claim that neither case can occur. Indeed, consider $y = x_0 + tv + O(t^2)$ for t small, where v is vector in the direction of the smallest curvature of the level set of

u and x_0 satisfies (3-1). Then since $(|Du|\kappa_1)(x_0) = a - \eta$,

$$\begin{aligned} Q(x, y) &= -(a-\eta)|y-x_0|^2 + O(|y-x_0|^3) + a|y-x_0|^2 - \frac{a}{6\varepsilon^2}|y-x_0|^4 \\ &= \eta|y-x_0|^2 - \frac{a}{6\varepsilon^2}|y-x_0|^4 + O(|y-x_0|^3). \end{aligned}$$

If $|y-x_0| \sim \varepsilon^{4/3}$ say then $Q(x_0, y) \sim \eta\varepsilon^{8/3} + O(\varepsilon^3) \gg \varepsilon^3$ since we assume $\eta \gg \varepsilon^{1/3}$. Since Q here is larger than in (1) or (2), this rules out (1) or (2) as being possible cases for the maximum of Q.

This implies that Q must attain an interior maximum, contradicting the argument of Theorem 1.1. Here we use the fact that if $\psi(t) = at - a/(6\varepsilon^2)\, t^2$ then for t with $0 \le t \le \varepsilon^2$,

$$\psi'(t) - 2|\psi''(t)|t = a(1 - t/\varepsilon^2) \ge 0. \qquad \square$$

Remark 3.1. In [Chang et al. 2010] and also [Ma et al. 2011] it was shown that when $n = 3$ the smallest principal curvature κ_1 also satisfies a minimum principle. It would be interesting to know whether a modification of the quantity (1-6) can give another proof of this.

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Volume 295 No. 2 August 2018

Nonsmooth convex caustics for Birkhoff billiards 257
MAXIM ARNOLD and MISHA BIALY

Certain character sums and hypergeometric series 271
RUPAM BARMAN and NEELAM SAIKIA

On the structure of holomorphic isometric embeddings of complex unit balls into bounded symmetric domains 291
SHAN TAI CHAN

Hamiltonian stationary cones with isotropic links 317
JINGYI CHEN and YU YUAN

Quandle theory and the optimistic limits of the representations of link groups 329
JINSEOK CHO

Classification of positive smooth solutions to third-order PDEs involving fractional Laplacians 367
WEI DAI and GUOLIN QIN

The projective linear supergroup and the SUSY-preserving automorphisms of $\mathbb{P}^{1|1}$ 385
RITA FIORESI and STEPHEN D. KWOK

The Gromov width of coadjoint orbits of the symplectic group 403
IVA HALACHEVA and MILENA PABINIAK

Minimal braid representatives of quasipositive links 421
KYLE HAYDEN

Four-dimensional static and related critical spaces with harmonic curvature 429
JONGSU KIM and JINWOO SHIN

Boundary Schwarz lemma for nonequidimensional holomorphic mappings and its application 463
YANG LIU, ZHIHUA CHEN and YIFEI PAN

Theta correspondence and the Prasad conjecture for SL(2) 477
HENGFEI LU

Convexity of level sets and a two-point function 499
BEN WEINKOVE

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