

*Pacific
Journal of
Mathematics*

**MONOTONICITY OF EIGENVALUES OF GEOMETRIC
OPERATORS ALONG THE RICCI–BOURGUIGNON FLOW**

BIN CHEN, QUN HE AND FANQI ZENG

Volume 296 No. 1

September 2018

MONOTONICITY OF EIGENVALUES OF GEOMETRIC OPERATORS ALONG THE RICCI–BOURGUIGNON FLOW

BIN CHEN, QUN HE AND FANQI ZENG

We study monotonicity of eigenvalues of the Schrödinger-type operator $-\Delta + cR$, where c is a constant, along the Ricci–Bourguignon flow. For $c \neq 0$, we derive monotonicity of the lowest eigenvalue of the Schrödinger-type operator $-\Delta + cR$, which generalizes some results of Cao (2008). As an application, we rule out nontrivial compact steady breathers in the Ricci–Bourguignon flow. For $c = 0$, we derive monotonicity of the first eigenvalue of the Laplacian, which generalizes some results of Ma (2006).

1. Introduction

Let (M, g) be an n -dimensional closed Riemannian manifold and $g(t)$ be a solution to the following Ricci–Bourguignon flow:

$$(1-1) \quad \frac{\partial}{\partial t} g = -2 \operatorname{Ric} + 2\rho Rg = -2(\operatorname{Ric} - \rho Rg),$$

where Ric is the Ricci tensor of the manifold, R is scalar curvature and ρ is a real constant. When $\rho = \frac{1}{2}, \frac{1}{n}, \frac{1}{2(n-1)}$ or 0, the tensor $\operatorname{Ric} - \rho Rg$ corresponds to the Einstein tensor, the traceless Ricci tensor, the Schouten tensor or the Ricci tensor respectively. Apart from these special values of ρ , for which we will call the associated flows by the same name as the corresponding tensor, in general we will refer to the evolution equation defined by the PDE system (1-1) as the Ricci–Bourguignon flow. Moreover, by a suitable rescaling in time, when ρ is nonpositive, they can be seen as an interpolation between the Ricci flow and the Yamabe flow, see [Brendle 2005; Ye 1994] for instance, obtained as a limit when $\rho \rightarrow -\infty$.

The study of these flows was proposed by Jean-Pierre Bourguignon [1981, Question 3.24], building on some unpublished work of Lichnerowicz in the sixties and a paper of Aubin [1970]. Fischer [2004] studied a conformal version of this problem where the scalar curvature is constrained along the flow. Lu, Qing and

The research of authors is supported by the NNSFC (grant nos. 11471246 and 11671361) and the NSFAP (grant no. 1608085MA03). Fanqi Zeng is the corresponding author.

MSC2010: primary 53C21; secondary 53C44.

Keywords: eigenvalue, Laplacian, monotonicity, Ricci–Bourguignon flow, breathers.

Zheng [Lu et al. 2014] also proved some results on the conformal Ricci–Bourguignon flow. Recently, for suitable values of the scalar parameter involved in these flows, Catino et al. [2017] proved short time existence and provided curvature estimates. Moreover, Catino and Mazzieri [2016] stated some results on the associated solitons.

At present, the eigenvalues of geometric operators have become a powerful tool in the study of geometry and topology of manifolds. Recently, there has been a lot of work on the eigenvalue problems under the Ricci flow. On one hand, Perelman [2002] introduced the so-called \mathcal{F} -entropy functional and proved that it is nondecreasing along the Ricci flow coupled to a backward heat-type equation. Since the functional \mathcal{F} is nondecreasing, this implies the monotonicity of the lowest eigenvalue of $-4\Delta + R$ along the Ricci flow.

Cao [2007] extended the operator $-4\Delta + R$ to the new operator $-\Delta + \frac{R}{2}$ on closed Riemannian manifolds, and showed that the eigenvalues of this new operator are nondecreasing along the Ricci flow with nonnegative curvature operator. Shortly thereafter Li [2007] dropped the curvature assumption and also obtained the above result for the operator $-\Delta + \frac{R}{2}$.

At around the same time, Cao [2008] considered the general operator

$$-\Delta + cR, \quad \text{where } c \geq \frac{1}{4},$$

and derived the following exact monotonicity formula; thus he showed that the lowest eigenvalue of this operator is nondecreasing along the Ricci flow without any curvature assumption.

Theorem A [Cao 2008]. *Let $(M, g(t))_{t \in [0, T)}$ be a solution of the unnormalized Ricci flow on a closed manifold M . Assume that $\lambda_0(t)$ is the lowest eigenvalue of $-\Delta + cR$, $c \geq \frac{1}{4}$, and $f = f(x, t) > 0$ satisfies*

$$-\Delta f(x, t) + cRf(x, t) = \lambda_0(t)f(x, t)$$

with $\int_M f^2 d\nu = 1$. Then, under the unnormalized Ricci flow, we have

$$(1-2) \quad \frac{d}{dt} \lambda_0(t) = \frac{1}{2} \int_M |\text{Ric} + \nabla^2 \varphi|^2 e^{-\varphi} d\nu + \frac{4c-1}{2} \int_M |\text{Ric}|^2 e^{-\varphi} d\nu \geq 0,$$

where $e^{-\varphi} = f^2$.

On the other hand, Ma [2006] obtained the monotonicity of the first eigenvalue of the Laplacian operator on a domain with Dirichlet boundary condition along the Ricci flow. Using the differentiability of the eigenvalues and the corresponding eigenfunctions of the Laplace operator under the Ricci flow, he obtained the following result.

Theorem B [Ma 2006]. *Let $g = g(t)$ be the evolving metric along the Ricci–Hamilton flow with $g(0) = g_0$ being the initial metric in M . Let D be a smooth*

bounded domain in (M, g_0) . Let $\mu > 0$ be the first eigenvalue of the Laplace operator of the metric $g(t)$. If there is a constant a such that the scalar curvature satisfies $R \geq 2a$ in $D \times \{t\}$ and the Einstein tensor satisfies

$$E_{ij} \geq -ag_{ij} \quad \text{in } D \times \{t\},$$

where $E_{ij} := R_{ij} - \frac{R}{2}g_{ij}$, then we have $\frac{d}{dt}\mu \geq 0$, that is, μ is nondecreasing in t ; furthermore, $\frac{d}{dt}\mu(t) > 0$ when the scalar curvature R is not the constant $2a$. The same monotonicity result is also true for other eigenvalues.

Motivated by the above work, we also consider the eigenvalue of $-\Delta + cR$ with c a constant. For $c \neq 0$, inspired by [Cao 2007; 2008; Li 2007], we can derive the following monotonicity of the lowest eigenvalue of $-\Delta + cR$ under the Ricci–Bourguignon flow (1-1). That is, we obtain:

Theorem 1.1. *Let $(M, g(t))_{t \in [0, T]}$ be a compact maximal solution of the nontrivial Ricci–Bourguignon flow (1-1) and $\lambda_0(t)$ be the lowest eigenvalue of the operator $-\Delta + cR$ corresponding to the normalized eigenfunction f , that is,*

$$(-\Delta + cR)f = \lambda_0 f, \quad \int_M f^2 dv = 1.$$

(1) If $\rho \leq 0$,

$$c \in \left[\frac{1}{4}, \frac{n}{4(n-1)} \right] \cup \left[\frac{(1 - (n-1)\rho)^2}{4(1 - 2(n-1)\rho)}, +\infty \right)$$

and the scalar curvature is nonnegative at the initial time, then the lowest eigenvalue of the operator $-\Delta + cR$ is nondecreasing in $[0, T)$ under the Ricci–Bourguignon flow (1-1). Furthermore, if $\rho \neq 0$ or $c \neq \frac{1}{4}$, then the lowest eigenvalue of the operator $-\Delta + cR$ is strictly monotone increasing in $[0, T)$ under the Ricci–Bourguignon flow (1-1).

(2) If $0 < \rho < \frac{1}{2(n-1)}$,

$$c \geq \frac{3(n-1)^2 \sqrt{\rho}}{2(1 - 2(n-1)\rho)} + \frac{1}{4}$$

and the curvature operator is nonnegative at the initial time, then the quantity

$$(1-3) \quad (T' - t)^{-\alpha} \lambda_0(t)$$

is strictly monotone increasing under the Ricci–Bourguignon flow (1-1) in $[0, T')$, where

$$T' = \frac{1}{2(1 - \rho)\epsilon}, \quad \epsilon = \max_M R(0) \quad \text{and} \quad \alpha = \frac{\rho}{1 - \rho} > 0.$$

Remark 1.1. When $\rho = 0$, where the Ricci–Bourguignon flow is the Ricci–Hamilton flow, our (1) reduces to the corresponding result of Cao [2008]. When $\rho \neq 0$, we

don't know the differentiability for the lowest eigenvalue, so we show that the lowest eigenvalue is strictly monotone increasing by using the sign-preserving property.

Remark 1.2. According to the proof, it is obvious that (2) will hold whenever the Ricci curvature is nonnegative, but in general, the nonnegativity of the Ricci curvature is not preserved along the Ricci–Bourguignon flow. Nevertheless, the nonnegativity of the Ricci curvature is preserved in dimension three.

Corollary 1.2. *In dimension three, let $g(t)$ and $\lambda_0(t)$ be the same as in Theorem 1.1. But here we assume the Ricci curvature is nonnegative at the initial time. If $0 < \rho < \frac{1}{4}$ and*

$$c \geq \frac{6\sqrt{\rho}}{1-4\rho} + \frac{1}{4},$$

then the quantity

$$(1-4) \quad (T' - t)^{-\alpha} \lambda_0(t)$$

is strictly monotone increasing under the Ricci–Bourguignon flow (1-1) in $[0, T')$, where

$$T' = \frac{1}{2(1-\rho)\epsilon}, \quad \epsilon = \max_M R(0) \quad \text{and} \quad \alpha = \frac{\rho}{1-\rho} > 0.$$

Next, as an application of our Theorem 1.1, we rule out nontrivial compact steady breathers. That is, we obtain:

Theorem 1.3. (1) *If $\rho = 0$, $c \geq \frac{1}{4}$, there is no compact steady breather other than the one which is Ricci-flat.*

(2) *If $\rho < 0$,*

$$c \in \left(\frac{1}{4}, \frac{n}{4(n-1)} \right] \cup \left[\frac{(1-(n-1)\rho)^2}{4(1-2(n-1)\rho)}, +\infty \right),$$

there is no compact steady breather with nonnegative scalar curvature other than the one which is Ricci-flat.

For $c = 0$, we derive the following monotonicity of eigenvalues on Laplacian under the Ricci–Bourguignon flow (1-1). That is, we obtain:

Theorem 1.4. *Let $(M, g(t))_{t \in [0, T)}$ be a compact maximal solution of the nontrivial Ricci–Bourguignon flow (1-1) and $\rho < \frac{1}{2(n-1)}$. Let $\lambda_1(t)$ be the first eigenvalue of the Laplace operator of the metric $g(t)$. If there is a nonnegative constant a such that*

$$(1-5) \quad R_{ij} - \frac{1+(2-n)\rho}{2} R g_{ij} \geq -a g_{ij} \quad \text{in } M \times [0, T),$$

$$(1-6) \quad R \geq \frac{2a}{1-n\rho} \quad \text{in } M \times \{0\},$$

then $\lambda_1(t)$ is strictly monotone increasing and differentiable almost everywhere along the Ricci–Bourguignon flow in $[0, T)$.

Remark 1.3. (1) Wu et al. [2010] proved a similar result about the p -Laplace operator along the Ricci flow, where they assumed $R \geq ap$ and $R \not\equiv ap$ in $M \times \{0\}$, which are a little stronger than (1-6). The key difference is that we use Lemma 2.3.

(2) It should be pointed out that for $\rho = 0$, the above theorem is similar to the main result for the first eigenvalue of the Laplace operator in [Ma 2006]. Moreover, our assumptions are weaker than Ma's.

(3) If $a < 0$, there doesn't exist any scalar curvature which satisfies (1-5) and (1-6) at the same time.

(4) The result may be useful in the study of blow-up models of Ricci–Bourguignon flow on a complete Riemannian manifold (M, g_0) .

2. Preliminaries

We begin with the definition for the first eigenvalue (the lowest eigenvalue) of the Laplace operator (the Schrödinger-type operator $-\Delta + cR$) under the Ricci–Bourguignon flow on a closed manifold. Then, we will show that the first eigenvalue of the Laplace operator is a continuous function along the Ricci–Bourguignon flow. Finally, under the Ricci–Bourguignon flow, we show that if $R(g_0) := R(0) \geq \beta$, for some $\beta \in \mathbb{R}$, then either $\max_M R(t) > \beta$ or the flow is trivial (i.e., $g(t) = g(0)$) for every $t \in (0, T)$.

Throughout, M will be taken to be a closed manifold (i.e., compact without boundary). We use moving frames in all calculations and adopt the index convention

$$1 \leq i, j, k, \dots \leq n$$

throughout this paper.

Now we recall the definition of the first eigenvalue of the Laplace operator on a closed manifold under the Ricci–Bourguignon flow. Let $(M, g(t))$ be a solution of the Ricci–Bourguignon flow on the time interval $[0, T)$. Consider the first nonzero eigenvalue of the Laplace operator at time t , where $0 \leq t < T$,

$$\lambda_1(t) = \inf \left\{ \int_M |\nabla f|^2 dv : f \in W^{1,2}, \int_M f^2 dv = 1 \text{ and } \int_M f dv = 0 \right\},$$

where dv denotes the volume form of the metric $g = g(t)$. Meanwhile the corresponding eigenfunction f satisfies the equation

$$-\Delta f(t) = \lambda_1(t) f(t),$$

where Δ is the Laplace operator with respect to $g(t)$, given by

$$\Delta_{g(t)} = \frac{1}{\sqrt{|g(t)|}} \partial_i (\sqrt{|g(t)|} g(t)^{ij} \partial_j),$$

and $g(t)^{ij} = g(t)_{ij}^{-1}$ is the inverse of the matrix $g(t)$ and $|g| = \det(g_{ij})$.

Note that it is not clear whether the first eigenvalue or the corresponding eigenfunction of the Laplace operator is differentiable under the Ricci–Bourguignon flow. When $\rho = 0$, where the Ricci–Bourguignon flow is the Ricci–Hamilton flow, many papers have pointed out that its differentiability under the Ricci–Hamilton flow follows from eigenvalue perturbation theory; e.g., see [Kato 1984; Kleiner and Lott 2008; Reed and Simon 1978]. But for $\rho \neq 0$, as far as we are aware, the differentiability of the first eigenvalue and eigenfunction of the Laplace operator under the Ricci–Bourguignon flow has not been known until now. So we cannot use Ma’s trick to derive the monotonicity of the first eigenvalue of the Laplace operator. Although, we do not know the differentiability for $\lambda_1(t)$, following the techniques of [Wu et al. 2010], we will see that $\lambda_1(t)$ in fact is a continuous function along the Ricci–Bourguignon flow on $[0, T)$.

Lemma 2.1 [Wu et al. 2010]. *If g_1 and g_2 are two metrics on M which satisfy*

$$(1 + \varepsilon)^{-1}g_1 \leq g_2 \leq (1 + \varepsilon)g_1,$$

then, we have

$$(2-1) \quad (1 + \varepsilon)^{-(n+1)} \leq \frac{\lambda_1(g_1)}{\lambda_1(g_2)} \leq (1 + \varepsilon)^{(n+1)}.$$

In particular, $\lambda_1(g(t))$ is a continuous function in the t -variable.

Proof. This can be proved using arguments similar to those for Theorem 2.1 in [Wu et al. 2010]. \square

Next we recall the definition of the lowest eigenvalue of $-\Delta + cR$. Let $\lambda_0(t)$ be the lowest eigenvalue of $-\Delta + cR$. Given a metric g on a closed manifold M , we define the functional λ_0 by

$$(2-2) \quad \lambda_0(t) = \inf \left\{ \mathcal{G}(g, f) : \int_M f^2 dv = 1, f > 0 \text{ and } f \in W^{1,2} \right\},$$

where

$$\mathcal{G}(g, f) = \int_M (f(-\Delta f) + cRf^2) dv = \int_M (|\nabla f|^2 + cRf^2) dv.$$

We also do not know the differentiability for $\lambda_0(t)$ and the corresponding eigenfunction. But, following the techniques of [Chow et al. 2008], we will see that $\lambda_0(t)$ in fact is a continuous function along the Ricci–Bourguignon flow on $[0, T)$.

Lemma 2.2 [Chow et al. 2008]. *If g_1 and g_2 are two metrics on M which satisfy*

$$(1 + \varepsilon)^{-1}g_1 \leq g_2 \leq (1 + \varepsilon)g_1 \quad \text{and} \quad R(g_1) - \varepsilon \leq R(g_2) \leq R(g_1) + \varepsilon,$$

then

$$(2-3) \quad \lambda_0(g_2) - \lambda_0(g_1) \leq ((1 + \varepsilon)^{\frac{n}{2}+1} - (1 + \varepsilon)^{-\frac{n}{2}})(1 + \varepsilon)^{\frac{n}{2}}(\lambda_0(g_1) - \min_M |c|R(g_1)) \\ + |c| \left((1 + \delta) \max_M |R(g_2) - R(g_1)| + 2\delta \max_M |R(g_1)| \right) (1 + \varepsilon)^{\frac{n}{2}},$$

where $\delta \rightarrow 0$ as $\varepsilon \rightarrow 0$. In particular, λ_0 is a continuous function with respect to the C^2 -topology.

Proof. This can be proved using arguments similar to those for Lemma 5.24 in [Chow et al. 2008]. \square

At last, we present the following lemma.

Lemma 2.3. *Let $(M, g_t)_{t \in [0, T]}$ be a compact maximal solution of the Ricci–Bourguignon flow (1-1). If $\rho < \frac{1}{2(n-1)}$ and $R(0) \geq \beta$ for some $\beta \in \mathbb{R}$, then either $\max_M R(t) > \beta$ or $g(t) = g(0)$ for every $t \in (0, T)$.*

Proof. From Lemma 2.6, we know that $R(t) \geq \beta$ for every $t \in [0, T]$. If $\max_M R(t_0) = \beta$ for some $t_0 \in (0, T)$, we have $R(t_0) \equiv \beta$ and $\frac{\partial R}{\partial t} \Big|_{t_0} \leq 0$. From (2-7), we have

$$\frac{1}{n} R^2(t_0) \leq |\text{Ric}|^2(t_0) \leq \rho R^2(t_0) < \frac{1}{2(n-1)} R^2(t_0).$$

Obviously, we have $R(t_0) = 0$ and $\text{Ric}(t_0) = 0$. Hence, $\max_M R(t_0) = \beta = 0$. Therefore, if $\beta \neq 0$, we have $\max_M R(t) > \beta$ for every $t \in [0, T)$.

When $\beta = 0$, let $I = \{t > 0 : \max_M R(t) > 0\}$. If $I = \emptyset$, then we have $R(t) \equiv 0$ and $\text{Ric}(t) \equiv 0$. Hence we have $g(t) = g(0)$. When $I \neq \emptyset$ and $t_1 \in I$, for any t_0 and $0 < t_0 < t_1$, if $\max_M R(t_0) = 0$, then $R(t_0) \equiv 0$ and $\text{Ric}(t_0) \equiv 0$. Hence, in $[t_0, T)$, $g(t) = g(t_0)$. So we have $\text{Ric}(t_1) = \text{Ric}(t_0) = 0$, which is in contradiction with $\max_M R(t_1) > 0$. Hence, $t_0 \in I$. Since $t_0 \in (0, t_1)$ is arbitrary, we have $(0, t_1) \subset I$. By the strong maximum principle, we have $(0, T) \subset I$. \square

For the reader's convenience, we will recall some basic knowledge about the Ricci–Bourguignon flow.

Lemma 2.4 [Catino et al. 2017]. *Under the Ricci–Bourguignon flow (1-1), we have*

$$(2-4) \quad \frac{\partial}{\partial t} g^{ij} = 2(R^{ij} - \rho R g^{ij}),$$

$$(2-5) \quad \frac{\partial}{\partial t} (d\nu) = (n\rho - 1)R d\nu,$$

$$(2-6) \quad \frac{\partial}{\partial t} (\Gamma_{ij}^k) = -R_{ik,j} - R_{kj,i} + R_{ij,k} + \rho(\delta_k^i R_{,j} + \delta_j^k R_{,i} - g_{ij} R_{,k}),$$

$$(2-7) \quad \frac{\partial}{\partial t} R = [1 - 2(n-1)\rho]\Delta R + 2|\text{Ric}|^2 - 2\rho R^2.$$

Lemma 2.5 (short time existence [Catino et al. 2017]). *Let $\rho < \frac{1}{2(n-1)}$. Then, the evolution equation (1-1) has a unique solution for a positive time interval on any smooth, n -dimensional, compact Riemannian manifold M (without boundary) with any initial metric g_0 .*

Lemma 2.6 (preserved curvature conditions [Catino et al. 2017]). *Let $(M, g_t)_{t \in [0, T]}$ be a compact maximal solution of the Ricci–Bourguignon flow (1-1). If $\rho \leq \frac{1}{2(n-1)}$, the minimum of the scalar curvature is nondecreasing along the flow. In particular, if $R(g_0) \geq \alpha$, for some $\alpha \in \mathbb{R}$, then $R(g_t) \geq \alpha$ for every $t \in [0, T]$. Moreover, if $\alpha > 0$ then $T \leq n/(2(1 - n\rho)\alpha)$.*

Lemma 2.7 (conditions preserved in three dimensions [Catino et al. 2017]). *Let $(M, g_t)_{t \in [0, T]}$ be a compact, 3-dimensional, solution of the Ricci–Bourguignon flow (1-1). If $\rho < \frac{1}{4}$, then*

- (1) *nonnegative Ricci curvature is preserved along the flow;*
- (2) *the pinching inequality $\text{Ric} \geq \varepsilon Rg$ is preserved along the flow for any $\varepsilon \leq \frac{1}{3}$.*

Lemma 2.8 [Catino et al. 2017]. *Let $(M, g_t)_{t \in [0, T]}$ be a compact solution of the Ricci–Bourguignon flow (1-1) with $\rho \leq \frac{1}{2(n-1)}$, and such that the initial data g_0 has nonnegative curvature operator. Then $\tilde{R}_{g(t)} \geq 0$ for every $t \in [0, T]$, where $\tilde{R} \in \text{End}(\Lambda^2 M)$ is the Riemann curvature operator.*

Lemma 2.9 [Catino et al. 2017]. *Let $\rho < \frac{1}{2(n-1)}$. If $g(t)$ is a compact solution of the Ricci–Bourguignon flow on a maximal time interval $[0, T)$, $T < +\infty$, then*

$$\limsup_{t \rightarrow T} \max_M |\text{Riem}(\cdot, t)| = +\infty,$$

where $\text{Riem}(\cdot, t)$ is Riemann tensor.

3. Proof of Theorem 1.1

We will now prove Theorem 1.1. In order to achieve this, we first prove the following two lemmas. Our proof uses some tricks from [Cao 2007, 2008].

Let M be an n -dimensional closed Riemannian manifold, and $g(t)$ be a smooth solution of the Ricci–Bourguignon flow on the time interval $[0, T)$. Let $\lambda_0(t)$ be the lowest eigenvalue of the operator $-\Delta + cR$ corresponding to the normalized eigenfunction f , that is,

$$(-\Delta + cR)f = \lambda_0 f, \quad \int_M f^2 dv = 1.$$

From Theorem 7.2 in [Guo et al. 2013], we know that, for any $t_0 \in [0, T)$, there exists a smooth function $\varphi(t) > 0$ satisfying

$$(3-1) \quad \int_M \varphi^2(t) dv = 1$$

and $\varphi(t_0) = f(t_0)$. Let

$$(3-2) \quad \mu(t) = \int_M (\varphi(t)(-\Delta\varphi(t)) + cR\varphi^2(t)) dv,$$

then $\mu(t)$ is a smooth function by definition. And at time t_0 , we conclude that

$$\mu(t_0) = \lambda_0(t_0).$$

We first give the following lemma.

Lemma 3.1. *Let $(M, g_t)_{t \in [0, T]}$ be a solution of the Ricci–Bourguignon flow on an n -dimensional closed manifold M , and let $\lambda_0(t)$ be the lowest eigenvalue of $-\Delta + cR$ under the Ricci–Bourguignon flow. Assume that $f(t_0)$ is the corresponding eigenfunction of $\lambda_0(t)$ at time $t_0 \in [0, T]$. Let $\mu(t)$ be a smooth function defined by (3-2). Then we have*

$$(3-3) \quad \begin{aligned} \frac{d}{dt}\mu(t)\Big|_{t=t_0} &= (A - 2\rho)c \int_M R^2 f^2 d\nu + (A - 2\rho) \int_M R |\nabla f|^2 d\nu \\ &\quad - A\lambda_0 \int_M R f^2 d\nu + 2 \int_M \text{Ric}(\nabla f, \nabla f) d\nu \\ &\quad + 2c \int_M |\text{Ric}|^2 f^2 d\nu, \end{aligned}$$

where

$$A = -1 + n\rho + 2c[1 - 2(n - 1)\rho].$$

Proof. The proof is by straightforward computation. Notice that

$$(3-4) \quad \frac{\partial}{\partial t}(\Delta\varphi) = 2R_{ij}\varphi_{ij} + \Delta(\varphi_t) - 2\rho R\Delta\varphi - (2 - n)\rho R_{,k}\varphi_k.$$

Using Lemma 2.4, we have

$$(3-5) \quad \begin{aligned} \frac{d}{dt}\mu(t)\Big|_{t=t_0} &= \int_M \partial_t(-\Delta\varphi + cR\varphi)\varphi d\nu + \int_M (-\Delta\varphi + cR\varphi)\partial_t(\varphi d\nu) \\ &= \int_M [\partial_t(-\Delta\varphi) + c\varphi\partial_t R + cR\partial_t\varphi]\varphi d\nu \\ &\quad + \int_M (-\Delta\varphi + cR\varphi)\partial_t(\varphi d\nu) \\ &= \int_M [-2R_{ij}\varphi_{ij} - \Delta(\varphi_t) + 2\rho R\Delta\varphi + (2 - n)\rho R_{,k}\varphi_k]\varphi d\nu \\ &\quad + \int_M [c\varphi\partial_t R + cR\partial_t\varphi]\varphi d\nu + \int_M (-\Delta\varphi + cR\varphi)\partial_t(\varphi d\nu) \\ &= \int_M [-2R_{ij}\varphi_{ij}\varphi + 2\rho R\varphi\Delta\varphi + (2 - n)\rho R_{,k}\varphi_k\varphi] d\nu \\ &\quad + c \int_M \{[1 - 2(n - 1)\rho]\Delta R + 2|\text{Ric}|^2 - 2\rho R^2\}\varphi^2 d\nu \\ &\quad + \int_M (-\Delta\varphi + cR\varphi)[\partial_t(\varphi)d\nu + \partial_t(\varphi d\nu)]. \end{aligned}$$

From $R_{,i} = 2R_{ij,j}$ and the Stokes formula, we have

$$(3-6) \quad \int_M \varphi^2 \Delta R \, d\nu = \int_M 2R(|\nabla\varphi|^2 + \varphi\Delta\varphi) \, d\nu,$$

$$(3-7) \quad \int_M R_{,k}\varphi_k\varphi \, d\nu = \int_M 2R(|\nabla\varphi|^2 + \varphi\Delta\varphi) \, d\nu,$$

and

$$(3-8) \quad \begin{aligned} \int_M -R_{ij}\varphi_i\varphi \, d\nu &= \int_M (R_{ij}\varphi)_j\varphi_i \, d\nu \\ &= \int_M R_{ij,j}\varphi\varphi_i \, d\nu + \int_M R_{ij}\varphi_j\varphi_i \, d\nu \\ &= \frac{1}{2} \int_M R_{,i}\varphi\varphi_i \, d\nu + \int_M R_{ij}\varphi_j\varphi_i \, d\nu \\ &= -\frac{1}{2} \int_M R(\varphi\varphi_i)_i \, d\nu + \int_M R_{ij}\varphi_j\varphi_i \, d\nu \\ &= -\frac{1}{2} \int_M R\Delta\varphi\varphi \, d\nu - \frac{1}{2} \int_M R|\nabla\varphi|^2 \, d\nu + \int_M R_{ij}\varphi_j\varphi_i \, d\nu. \end{aligned}$$

On the other hand, at time t_0 , we know φ is the eigenfunction of $\lambda_0(t_0)$, i.e., $(-\Delta + cR)\varphi = \lambda_0\varphi$, and we have

$$(3-9) \quad \int_M (-\Delta\varphi + cR\varphi)[\partial_t(\varphi)d\nu + \partial_t(\varphi \, d\nu)] = \lambda_0 \int_M \varphi[\partial_t(\varphi)d\nu + \partial_t(\varphi \, d\nu)] = 0.$$

The last equality holds because of (3-1). Inserting (3-6)–(3-9) into (3-5), at $t = t_0$, yields

$$(3-10) \quad \begin{aligned} \left. \frac{d}{dt}\mu(t) \right|_{t=t_0} &= (-1+n\rho+2c[1-2(n-1)\rho]) \int_M R\varphi\Delta\varphi \, d\nu \\ &\quad + (-1+(n-2)\rho+2c[1-2(n-1)\rho]) \int_M R|\nabla\varphi|^2 \, d\nu \\ &\quad + 2 \int_M R_{ij}\varphi_i\varphi_j \, d\nu + 2c \int_M |\text{Ric}|^2\varphi^2 \, d\nu - 2c\rho \int_M R\varphi^2 \, d\nu. \end{aligned}$$

Inserting $\Delta\varphi = cR\varphi - \lambda_0\varphi$ into (3-10), at $t = t_0$, gives

$$(3-11) \quad \begin{aligned} \left. \frac{d}{dt}\mu(t) \right|_{t=t_0} &= (A-2\rho)c \int_M R^2\varphi^2 \, d\nu + (A-2\rho) \int_M R|\nabla\varphi|^2 \, d\nu \\ &\quad - A\mu \int_M R\varphi^2 \, d\nu + 2 \int_M R_{ij}\varphi_i\varphi_j \, d\nu \\ &\quad + 2c \int_M |\text{Ric}|^2\varphi^2 \, d\nu. \end{aligned}$$

Therefore we finish the proof of Lemma 3.1. □

Now we give the second lemma.

Lemma 3.2. *Let $(M, g_t)_{t \in [0, T]}$ be a solution of the Ricci–Bourguignon flow on an n -dimensional closed manifold M , and let $\lambda_0(t)$ be the lowest eigenvalue of $-\Delta + cR$ under the Ricci–Bourguignon flow. Assume that $f(t_0)$ is the corresponding eigenfunction of $\lambda_0(t)$ at time $t_0 \in [0, T)$. Let $\mu(t)$ be a smooth function defined by (3-2). Then we have*

$$\begin{aligned}
 (3-12) \quad & \frac{d}{dt} \mu(t) \Big|_{t=t_0} \\
 &= \frac{1}{2k(2-k)} \int_M |R_{ij} - 2k(\log f)_{ij}|^2 f^2 dv \\
 &+ \left(2c - \frac{1}{2k(2-k)} \right) \int_M |\text{Ric}|^2 f^2 dv \\
 &- \frac{(a-bk)4c - (d-ek)}{2-k} \lambda_0 \int_M R f^2 dv \\
 &+ \left(\frac{(a-bk)4c - (d-ek)}{2-k} - 2\rho \right) \int_M (cR^2 f^2 + R|\nabla f|^2) dv,
 \end{aligned}$$

where

$$\begin{aligned}
 a &= 1 - 2(n-1)\rho, & b &= 1 - (n-1)\rho, \\
 d &= 1 - 2n\rho, & e &= 1 - n\rho \quad \text{and} \quad 0 < k < 2.
 \end{aligned}$$

Proof. The proof is by straightforward computation:

$$\begin{aligned}
 (3-13) \quad & \int_M |R_{ij} - 2k(\log f)_{ij}|^2 f^2 dv = \int_M |\text{Ric}|^2 f^2 dv + 4k^2 \int_M |\nabla^2(\log f)|^2 f^2 dv \\
 & - 4k \int_M R_{ij}(\log f)_{ij} f^2 dv.
 \end{aligned}$$

From [Cao 2008], we can get

$$\begin{aligned}
 (3-14) \quad & 4k^2 \int_M |\nabla^2(\log f)|^2 f^2 dv \\
 &= 2k^2 c \int_M R \Delta f^2 dv - 4k^2 \int_M R_{ij} f_i f_j dv \\
 &= 4k^2 c \int_M R(f \Delta f + |\nabla f|^2) dv - 4k^2 \int_M R_{ij} f_i f_j dv
 \end{aligned}$$

and

$$\begin{aligned}
 (3-15) \quad & -4k \int_M R_{ij}(\log f)_{ij} f^2 dv \\
 &= -2k \int_M R(f \Delta f + |\nabla f|^2) dv + 8k \int_M R_{ij} f_i f_j dv.
 \end{aligned}$$

Combining (3-14) and (3-15), we arrive at

$$\begin{aligned}
 (3-16) \quad \int_M |R_{ij} - 2k(\log f)_{ij}|^2 f^2 d\nu &= \int_M |\text{Ric}|^2 f^2 d\nu + (8k - 4k^2) \int_M R_{ij} f_i f_j d\nu \\
 &+ 2k(2kc - 1) \left(c \int_M R^2 f^2 d\nu - \lambda_0 \int_M R f^2 d\nu \right) \\
 &+ 2k(2kc - 1) \int_M R |\nabla f|^2 d\nu.
 \end{aligned}$$

Multiplying by $\frac{1}{2k(2-k)}$ on both sides of (3-16), we conclude that

$$\begin{aligned}
 (3-17) \quad \frac{1}{2k(2-k)} \int_M |R_{ij} - 2k(\log f)_{ij}|^2 f^2 d\nu &= \frac{1}{2k(2-k)} \int_M |\text{Ric}|^2 f^2 d\nu + 2 \int_M R_{ij} f_i f_j d\nu \\
 &+ \frac{2kc - 1}{2-k} \left(c \int_M R^2 f^2 d\nu - \lambda_0 \int_M R f^2 d\nu \right) \\
 &+ \frac{2kc - 1}{2-k} \int_M R |\nabla f|^2 d\nu.
 \end{aligned}$$

Subtracting (3-17) from (3-3), we see that

$$\begin{aligned}
 (3-18) \quad \frac{d}{dt} \mu(t) \Big|_{t=t_0} &= \frac{1}{2k(2-k)} \int_M |R_{ij} - 2k(\log f)_{ij}|^2 f^2 d\nu \\
 &+ \left(2c - \frac{1}{2k(2-k)} \right) \int_M |\text{Ric}|^2 f^2 d\nu \\
 &+ \left(A - 2\rho - \frac{2kc - 1}{2-k} \right) \int_M (cR^2 f^2 + R|\nabla f|^2 - \lambda_0 R f^2) d\nu \\
 &- 2\rho \int_M \lambda_0 R f^2 d\nu,
 \end{aligned}$$

where $A = -1 + n\rho + 2c[1 - 2(n-1)\rho]$. Note that

$$\begin{aligned}
 A - \frac{2kc - 1}{2-k} &= \frac{4\{[1 - 2(n-1)\rho] - [1 - (n-1)\rho]k\}c - [(1 - 2n\rho) - (1 - n\rho)k]}{2-k} \\
 &:= \frac{(a - bk)c - (d - ek)}{2-k}.
 \end{aligned}$$

Therefore we finish the proof of Lemma 3.2. \square

Proof of Theorem 1.1. We first prove (1). If $\rho < 0$, inserting

$$k = \frac{a}{b} = \frac{1 - 2(n-1)\rho}{1 - (n-1)\rho}$$

into (3-12), we obtain

$$(3-19) \quad \left. \frac{d}{dt} \mu(t) \right|_{t=t_0} = \frac{(1-(n-1)\rho)^2}{2-4(n-1)\rho} \int_M \left| R_{ij} - 2 \frac{1-2(n-1)\rho}{1-(n-1)\rho} (\log f)_{ij} \right|^2 f^2 d\nu \\ + \left(2c - \frac{(1-(n-1)\rho)^2}{2(1-2(n-1)\rho)} \right) \int_M |\text{Ric}|^2 f^2 d\nu - \rho \lambda_0 \int_M R f^2 d\nu \\ - \rho \left\{ c \int_M R^2 f^2 d\nu + \int_M R |\nabla f|^2 d\nu \right\}.$$

If $R \geq 0$ in $M \times \{0\}$, from Lemmas 2.3 and 2.6, we know that either $\max_M R(t) > 0$ or $g(t) = g(0)$ for every $t \in (0, T)$. Assume $\max_M R(t) > 0$ (otherwise the proof is trivial). By (3-19), when

$$c \geq \frac{(1-(n-1)\rho)^2}{4(1-2(n-1)\rho)} > \frac{1}{4},$$

we obtain

$$(3-20) \quad \left. \frac{d}{dt} \mu(t) \right|_{t=t_0} > 0.$$

Moreover, setting $k = 1$ in (3-12), we obtain

$$(3-21) \quad \left. \frac{d}{dt} \mu(t) \right|_{t=t_0} = \frac{1}{2} \int_M |R_{ij} - 2(\log f)_{ij}|^2 f^2 d\nu + \left(2c - \frac{1}{2} \right) \int_M |\text{Ric}|^2 f^2 d\nu \\ - \rho [4(n-1)c - n + 2] \left\{ c \int_M R^2 f^2 d\nu + \int_M R |\nabla f|^2 d\nu \right\} \\ + \rho [4(n-1)c - n] \lambda_0 \int_M R f^2 d\nu.$$

Then when $\frac{1}{4} \leq c \leq n/(4(n-1))$, we also obtain (3-20).

Since the eigenfunction of the lowest eigenvalue is not equal to 0 along the Ricci–Bourguignon flow, see Lemma 5.22 in [Chow et al. 2008], and $\mu(t)$ is a smooth function with respect to the t -variable, we have

$$(3-22) \quad \frac{d}{dt} \mu(t) > 0$$

in $(t_0 - \delta, t_0 + \delta)$, where $\delta > 0$ is sufficiently small. So we get

$$(3-23) \quad \mu(t_0) > \mu(t_1)$$

for any $t_1 \in (t_0 - \delta, t_0 + \delta)$ and $t_1 < t_0$.

Notice that

$$\mu(t_0) = \lambda_0(t_0) \quad \text{and} \quad \mu(t_1) \geq \lambda_0(t_1).$$

This implies $\lambda_0(t_0) > \lambda_0(t_1)$ for any $t_0 > t_1$. Since $\lambda_0(t)$ is continuous and $t_0 \in [0, T)$ is arbitrary, $\lambda_0(t)$ is strictly monotone increasing in $[0, T)$. Therefore we finish the proof of (1).

Next we prove (2). If $0 < \rho < \frac{1}{2(n-1)}$, in (3-18), we pick k such that

$$A - 2\rho - \frac{2kc - 1}{2 - k} = 0.$$

Then

$$k = \frac{(4c - 1)[1 - 2(n - 1)\rho] - 2\rho}{(4c - 1)[1 - (n - 1)\rho] - \rho}.$$

Taking $4c - 1 = B \geq 0$, we have

$$k = \frac{B[1 - 2(n - 1)\rho] - 2\rho}{B[1 - (n - 1)\rho] - \rho} \quad \text{and} \quad 2 - k = \frac{B}{B[1 - (n - 1)\rho] - \rho}.$$

For $0 < k < 2$, we need

$$(3-24) \quad B > \frac{2\rho}{1 - 2(n - 1)\rho}.$$

Now, we need $2c - \frac{1}{2k(2-k)} \geq 0$, which is

$$(3-25) \quad B(B + 1)(B[1 - 2(n - 1)\rho] - 2\rho) \geq (B[1 - (n - 1)\rho] - \rho)^2.$$

It is true when $B \rightarrow +\infty$. Next, we will prove that given

$$(3-26) \quad 4c - 1 = B \geq \frac{6(n - 1)^2 \sqrt{\rho}}{1 - 2(n - 1)\rho},$$

both (3-24) and (3-25) are true. Firstly, since $0 < \rho < \frac{1}{2(n-1)} < 1$, we have

$$B \geq \frac{6(n - 1)^2 \sqrt{\rho}}{1 - 2(n - 1)\rho} > \frac{6(n - 1)^2 \rho}{1 - 2(n - 1)\rho} > \frac{2\rho}{1 - 2(n - 1)\rho}.$$

Thus, (3-24) holds. Secondly, let's show (3-25):

$$\begin{aligned} & B(B + 1)(B[1 - 2(n - 1)\rho] - 2\rho) - (B[1 - (n - 1)\rho] - \rho)^2 \\ &= B^2(B[1 - 2(n - 1)\rho] - 2\rho) + B(B[1 - 2(n - 1)\rho] - 2\rho) - (B[1 - (n - 1)\rho] - \rho)^2 \\ &= B^2(B[1 - 2(n - 1)\rho] - 2\rho) - ((n - 1)B\rho + \rho)^2 \\ &\geq B^2(6(n - 1)^2 \sqrt{\rho} - 2\rho) - ((n - 1)B\rho + \rho)^2 \\ &\geq B^2(4(n - 1)^2 \sqrt{\rho}) - ((n - 1)B\rho + \rho)^2 \\ &= [2B(n - 1)\sqrt[4]{\rho} + (n - 1)B\rho + \rho][2B(n - 1)\sqrt[4]{\rho} - (n - 1)B\rho - \rho]. \end{aligned}$$

The first factor is clearly positive. For the second factor, note $\rho^{\frac{1}{4}} > \rho^{\frac{3}{4}} > \rho$,

$$\begin{aligned} 2B(n-1)\sqrt[4]{\rho} - (n-1)B\rho - \rho &\geq B(n-1)\sqrt[4]{\rho} - \rho \\ &\geq \frac{6(n-1)^3\sqrt{\rho}}{1-2(n-1)\rho}\sqrt[4]{\rho} - \rho \\ &\geq 6(n-1)^3\rho^{\frac{3}{4}} - \rho \geq 12(n-1)^3\rho - \rho > 0. \end{aligned}$$

Therefore, given

$$4c - 1 \geq \frac{6(n-1)^2\sqrt{\rho}}{1-2(n-1)\rho},$$

we have

$$(3-27) \quad \left. \frac{d}{dt}\mu \right|_{t=t_0} > -2\rho\mu \int_M Rf^2 dv.$$

By Lemma 2.8, we know that the nonnegativity of the curvature operator is preserved by the Ricci–Bourguignon flow. This implies that the Ricci curvature is also nonnegative, and we have $|\text{Ric}|^2 \leq R^2$. The evolution equation of scalar curvature satisfies

$$(3-28) \quad \begin{aligned} \frac{\partial}{\partial t} R &= [1 - 2(n-1)\rho]\Delta R + 2|\text{Ric}|^2 - 2\rho R^2 \\ &\leq [1 - 2(n-1)\rho]\Delta R + 2(1-\rho)R^2. \end{aligned}$$

Let $\sigma(t)$ be the solution of the following ODE with initial value:

$$(3-29) \quad \begin{cases} \partial\sigma(t)/\partial t = 2(1-\rho)\sigma^2, \\ \sigma(0) = \max_M R(0). \end{cases}$$

By the maximum principle, letting $\epsilon = \max_M R(0)$, we can get

$$R(t) \leq \sigma(t) = \left(-2(1-\rho)t + \frac{1}{\epsilon}\right)^{-1}$$

on $[0, T'')$, where $T'' = \min\{T', T\}$ and $T' = \frac{1}{2(1-\rho)\epsilon}$. Arguing now as in [Hamilton 1982, Section 14], it follows that the metrics $g(t)$ converge to some limit metric $g(T)$ in the C^∞ topology if $T < T''$; hence, we can restart the flow with this initial metric $g(T)$, obtaining a smooth flow in some larger time interval $[0, T + \delta)$, in contradiction with the fact that T was the maximal time of smooth existence. So we have $T' \leq T$. Hence $R(t) \leq \sigma(t)$ on $[0, T')$. Since the eigenfunction of the lowest eigenvalue is not equal to 0 along the Ricci–Bourguignon flow, see Lemma 5.22 in [Chow et al. 2008], from Lemma 2.3 and (3-27), we have

$$(3-30) \quad \left. \frac{d}{dt}\mu \right|_{t=t_0} > -2\rho\mu \int_M Rf^2 dv \geq -2\rho\mu\sigma,$$

which implies

$$\left. \left(\frac{d}{dt}\mu + 2\rho\mu\sigma \right) \right|_{t=t_0} > 0.$$

By arguments similar to those in the proof of (1), we know that

$$(T' - t)^{-\alpha} \lambda_0(t)$$

is strictly monotone increasing under the Ricci–Bourguignon flow (1-1) on $[0, T']$ and

$$T' = \frac{1}{2(1-\rho)\epsilon},$$

where

$$\epsilon = \max_M R(0) \quad \text{and} \quad \alpha = \frac{\rho}{1-\rho} > 0,$$

which shows (2) holds. Therefore we finish the proof of Theorem 1.1. \square

4. Proof of Theorem 1.3

We will now prove Theorem 1.3. First, we recall the definition of breathers.

Definition 4.1. A metric $g(t)$ evolving from the Ricci–Bourguignon flow is called a breather if for some $t_1 < t_2$ and $\alpha > 0$ the metrics $\alpha g(t_1)$ and $g(t_2)$ differ only by a diffeomorphism; the cases $\alpha = 1$, $\alpha < 1$ and $\alpha > 1$ correspond to steady, shrinking and expanding breathers, respectively.

Proof of Theorem 1.3. For a steady breather, let t_1 and t_2 be the same as above; we have

$$\lambda_0(t_1) = \lambda_0(t_2).$$

When $\rho = 0$ and $c \geq \frac{1}{4}$, by (1-2) of Theorem A, we have

$$\lambda_0(t_1) \leq \lambda_0(t_2)$$

provided $t_1 < t_2$. And the equality holds if and only if for any $t_1 \leq t \leq t_2$,

$$\frac{d}{dt} \lambda_0(t) = 0.$$

Since the eigenfunction f cannot be identical to zero, from (1-2) of Theorem A we must have $\text{Ric} \equiv 0$.

But when $\rho < 0$,

$$c \in \left(\frac{1}{4}, \frac{n}{4(n-1)} \right] \cup \left[\frac{(1-(n-1)\rho)^2}{4(1-2(n-1)\rho)}, +\infty \right)$$

and the scalar curvature is nonnegative at the initial time, because of Theorem 1.1(1) for a nontrivial flow we have

$$\lambda_0(t_1) < \lambda_0(t_2)$$

provided $t_1 < t_2$. When $(M, g(t))_{t \in [0, T]}$ is a compact maximal solution of the trivial Ricci–Bourguignon flow (1-1), i.e., $\text{Ric} \equiv 0$, we have $\lambda_0(t_1) = \lambda_0(t_2)$. Hence we have proved Theorem 1.3. \square

5. Proof of Theorem 1.4

We will now prove Theorem 1.4. In order to achieve this, we first prove Lemma 5.1. Our proof involves choosing a proper smooth function, which seems to be a delicate trick.

Let M be an n -dimensional closed Riemannian manifold, and $g(t)$ be a smooth solution of the Ricci–Bourguignon flow on the time interval $[0, T]$. Let $\lambda_1(t)$ be the first eigenvalue of the Laplace operator under the Ricci–Bourguignon flow and $f(t_0)$ be the corresponding eigenfunction of $\lambda_1(t)$ at time $t_0 \in [0, T]$, i.e.,

$$(5-1) \quad -\Delta_{g(t_0)} f(t_0) = \lambda_1(t_0) f(t_0).$$

For any $t_0 \in [0, T]$, Wu et al. [2010] pointed out that there exists a smooth function

$$\phi(t) = \frac{\psi(t)}{\left(\int_M \psi(t)^2 d\nu\right)^{\frac{1}{2}}}, \quad \text{where } \psi(t) = f(t_0) \left(\frac{|g(t_0)|}{|g(t)|}\right)^{\frac{1}{2}}$$

satisfying

$$(5-2) \quad \int_M \phi(t)^2 d\nu = 1, \quad \int_M \phi(t) d\nu = 0,$$

and $\phi(t_0) = f(t_0)$. Now we define a general smooth function

$$(5-3) \quad \mu(t) = \int_M \phi(t)(-\Delta\phi(t)) d\nu.$$

In general, $\mu(t)$ is not equal to $\lambda_1(t)$. But at time t_0 , we conclude that

$$\mu(t_0) = \lambda_1(t_0).$$

Lemma 5.1. *Let $(M, g_t)_{t \in [0, T]}$ be a solution of the Ricci–Bourguignon flow on an n -dimensional closed manifold M and let $\lambda_1(t)$ be the first eigenvalue of the Laplace operator under the Ricci–Bourguignon flow. Assume that $f(t_0)$ is the corresponding eigenfunction of $\lambda_1(t)$ at time $t_0 \in [0, T]$, i.e.,*

$$-\Delta_{g(t_0)} f(t_0) = \lambda_1(t_0) f(t_0).$$

Let $\mu(t)$ be a smooth function defined by (5-3). Then we have

$$(5-4) \quad \left. \frac{d}{dt} \mu(t) \right|_{t=t_0} = \int_M \{2R_{ij} f_i f_j + (1 - n\rho)\lambda_1 R f^2 - [(2 - n)\rho + 1]R|\nabla f|^2\} d\nu.$$

Proof. The proof is by direct computation:

$$\begin{aligned}
(5-5) \quad \frac{d}{dt}\mu(t)\Big|_{t=t_0} &= \int_M \partial_t(-\phi\Delta\phi) dv + \int_M (-\phi\Delta\phi)\partial_t(dv) \\
&= \int_M [-2R_{ij}\phi_{ij} - \Delta(\partial_t\phi) + 2\rho R\Delta\phi + (2-n)\rho R_{,k}\phi_k]\phi dv \\
&\quad + \int_M (-\Delta\phi)\partial_t\phi dv + \int_M (-\Delta\phi)\phi(n\rho-1)R dv \\
&= \int_M -2R_{ij}\phi_{ij}\phi dv + \int_M -2(\Delta\phi)\partial_t\phi dv \\
&\quad + (2-n)\rho \int_M R_{,k}\phi_k\phi dv + [1+(2-n)\rho] \int_M R(\Delta\phi)\phi dv.
\end{aligned}$$

From (3-7) and (3-8), we have

$$\begin{aligned}
(5-6) \quad \frac{d}{dt}\mu(t)\Big|_{t=t_0} &= - \int_M R\Delta\phi\phi dv - \int_M R|\nabla\phi|^2 dv + 2 \int_M R_{ij}\phi_j\phi_i dv \\
&\quad + \int_M -2(\Delta\phi)\partial_t\phi dv - (2-n)\rho \int_M R\Delta\phi\phi dv \\
&\quad - (2-n)\rho \int_M R|\nabla\phi|^2 dv \\
&\quad + [1+(2-n)\rho] \int_M R(\Delta\phi)\phi dv \\
&= 2 \int_M R_{ij}\phi_j\phi_i dv + 2\mu \int_M \phi\partial_t\phi dv \\
&\quad - [1+(2-n)\rho] \int_M R|\nabla\phi|^2 dv.
\end{aligned}$$

Under the Ricci–Bourguignon flow, from the constraint condition (5-2), we get

$$(5-7) \quad 2 \int_M \phi\partial_t\phi dv = -(n\rho-1) \int_M \phi^2 R dv.$$

Hence, at time t_0 , the desired lemma follows from substituting (5-7) into (5-6). \square

Proof of Theorem 1.4. We assume that for any time $t_0 \in [0, T)$, if $f(t_0)$ is the corresponding eigenfunction of the first eigenvalue $\lambda_1(t_0)$, then we have $\lambda_1(t_0) = \mu(t_0)$. By Lemma 5.1, we have

$$\begin{aligned}
(5-8) \quad \frac{d}{dt}\mu(t)\Big|_{t=t_0} &= \int_M \{(1-n\rho)\lambda_1 R f^2 + 2R_{ij}f_i f_j - [(2-n)\rho+1]R|\nabla f|^2\} dv \\
&= \int_M \{2R_{ij} - [(2-n)\rho+1]Rg_{ij}\} f_i f_j dv + \int_M (1-n\rho)\lambda_1 R f^2 dv \\
&\geq \int_M (1-n\rho)\lambda_1 R f^2 dv - 2a \int_M |\nabla f|^2 dv \\
&= \int_M (1-n\rho)\lambda_1 R f^2 dv - 2a\lambda_1 = \lambda_1 \int_M f^2 \{(1-n\rho)R - 2a\} dv,
\end{aligned}$$

where we used the first assumption of Theorem 1.4.

From Lemma 2.3, we know that either $\max_M R(t) > 0$ or $g(t) = g(0)$ for every $t \in (0, T)$. Assume $\max_M R(t) > 0$ (otherwise the proof is trivial). Since the eigenfunction of the first eigenvalue is not equal to 0 along the Ricci–Bourguignon flow, by (5-8), we obtain

$$(5-9) \quad \left. \frac{d}{dt} \mu(t) \right|_{t=t_0} > 0.$$

By arguments similar to those in the proof of Theorem 1.1, we have $\lambda_1(t)$ is strictly monotone increasing in $[0, T)$.

As for the differentiability for $\lambda_1(t)$, since $\lambda_1(t)$ is increasing on the time interval $[0, T)$ under curvature conditions of the theorem, by the classical Lebesgue’s theorem, see for example Chapter 4 in [Mukherjea and Pothoven 1984], it is easy to see that $\lambda_1(t)$ is differentiable almost everywhere on $[0, T)$. \square

Remark 5.1. (1) In the course of proving Theorem 1.4, we do not use any differentiability of the first eigenvalue or the corresponding eigenfunction of the Laplace operator under the Ricci–Bourguignon flow.

(2) Using this method, we cannot get any monotonicity for higher-order eigenvalues of the Laplace operator under the Ricci–Bourguignon flow.

References

- [Aubin 1970] T. Aubin, “Métriques riemanniennes et courbure”, *J. Differential Geometry* **4** (1970), 383–424. MR Zbl
- [Bourguignon 1981] J.-P. Bourguignon, “Ricci curvature and Einstein metrics”, pp. 42–63 in *Global differential geometry and global analysis* (Berlin, 1979), edited by D. Ferus et al., Lecture Notes in Math. **838**, Springer, 1981. MR Zbl
- [Brendle 2005] S. Brendle, “Convergence of the Yamabe flow for arbitrary initial energy”, *J. Differential Geom.* **69**:2 (2005), 217–278. MR Zbl
- [Cao 2007] X. Cao, “Eigenvalues of $(-\Delta + \frac{R}{2})$ on manifolds with nonnegative curvature operator”, *Math. Ann.* **337**:2 (2007), 435–441. MR Zbl
- [Cao 2008] X. Cao, “First eigenvalues of geometric operators under the Ricci flow”, *Proc. Amer. Math. Soc.* **136**:11 (2008), 4075–4078. MR Zbl
- [Catino and Mazzieri 2016] G. Catino and L. Mazzieri, “Gradient Einstein solitons”, *Nonlinear Anal.* **132** (2016), 66–94. MR Zbl
- [Catino et al. 2017] G. Catino, L. Cremaschi, Z. Djadli, C. Mantegazza, and L. Mazzieri, “The Ricci–Bourguignon flow”, *Pacific J. Math.* **287**:2 (2017), 337–370. MR Zbl
- [Chow et al. 2008] B. Chow, S.-C. Chu, D. Glickenstein, C. Guenther, J. Isenberg, T. Ivey, D. Knopf, P. Lu, F. Luo, and L. Ni, *The Ricci flow: techniques and applications, II: Analytic aspects*, Mathematical Surveys and Monographs **144**, American Mathematical Society, Providence, RI, 2008. MR Zbl
- [Fischer 2004] A. E. Fischer, “An introduction to conformal Ricci flow”, *Classical Quantum Gravity* **21**:3 (2004), S171–S218. MR Zbl

- [Guo et al. 2013] H. Guo, R. Philipowski, and A. Thalmaier, “Entropy and lowest eigenvalue on evolving manifolds”, *Pacific J. Math.* **264**:1 (2013), 61–81. MR Zbl
- [Hamilton 1982] R. S. Hamilton, “Three-manifolds with positive Ricci curvature”, *J. Differential Geom.* **17**:2 (1982), 255–306. MR Zbl
- [Kato 1984] T. Kato, *Perturbation theory for linear operators*, 2nd ed., Die Grundlehren der mathematischen Wissenschaften **132**, Springer, 1984. Zbl
- [Kleiner and Lott 2008] B. Kleiner and J. Lott, “Notes on Perelman’s papers”, *Geom. Topol.* **12**:5 (2008), 2587–2855. MR Zbl
- [Li 2007] J.-F. Li, “Eigenvalues and energy functionals with monotonicity formulae under Ricci flow”, *Math. Ann.* **338**:4 (2007), 927–946. MR Zbl
- [Lu et al. 2014] P. Lu, J. Qing, and Y. Zheng, “A note on conformal Ricci flow”, *Pacific J. Math.* **268**:2 (2014), 413–434. MR Zbl
- [Ma 2006] L. Ma, “Eigenvalue monotonicity for the Ricci–Hamilton flow”, *Ann. Global Anal. Geom.* **29**:3 (2006), 287–292. MR Zbl
- [Mukherjea and Pothoven 1984] A. Mukherjea and K. Pothoven, *Real and functional analysis, Part A: Real analysis*, 2nd ed., Mathematical Concepts and Methods in Science and Engineering **27**, Plenum Press, New York, 1984. MR Zbl
- [Perelman 2002] G. Perelman, “The entropy formula for the Ricci flow and its geometric applications”, preprint, 2002. Zbl arXiv
- [Reed and Simon 1978] M. Reed and B. Simon, *Methods of modern mathematical physics, IV: Analysis of operators*, Academic Press, 1978. MR Zbl
- [Wu et al. 2010] J.-Y. Wu, E.-M. Wang, and Y. Zheng, “First eigenvalue of the p -Laplace operator along the Ricci flow”, *Ann. Global Anal. Geom.* **38**:1 (2010), 27–55. MR Zbl
- [Ye 1994] R. Ye, “Global existence and convergence of Yamabe flow”, *J. Differential Geom.* **39**:1 (1994), 35–50. MR Zbl

Received January 23, 2016. Revised March 7, 2017.

BIN CHEN
SCHOOL OF MATHEMATICAL SCIENCES
TONGJI UNIVERSITY
SHANGHAI
CHINA
chenbin@tongji.edu.cn

QUN HE
SCHOOL OF MATHEMATICAL SCIENCES
TONGJI UNIVERSITY
SHANGHAI
CHINA
hequn@tongji.edu.cn

FANQI ZENG
SCHOOL OF MATHEMATICAL SCIENCES
TONGJI UNIVERSITY
SHANGHAI
CHINA
fanzeng10@126.com

PACIFIC JOURNAL OF MATHEMATICS

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

msp.org/pjm

EDITORS

Don Blasius (Managing Editor)
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
blasius@math.ucla.edu

Paul Balmer
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
balmer@math.ucla.edu

Wee Teck Gan
Mathematics Department
National University of Singapore
Singapore 119076
matgwt@nus.edu.sg

Sorin Popa
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
popa@math.ucla.edu

Vyjayanthi Chari
Department of Mathematics
University of California
Riverside, CA 92521-0135
chari@math.ucr.edu

Kefeng Liu
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
liu@math.ucla.edu

Jie Qing
Department of Mathematics
University of California
Santa Cruz, CA 95064
qing@cats.ucsc.edu

Daryl Cooper
Department of Mathematics
University of California
Santa Barbara, CA 93106-3080
cooper@math.ucsb.edu

Jiang-Hua Lu
Department of Mathematics
The University of Hong Kong
Pokfulam Rd., Hong Kong
jhlu@maths.hku.hk

Paul Yang
Department of Mathematics
Princeton University
Princeton NJ 08544-1000
yang@math.princeton.edu

PRODUCTION

Silvio Levy, Scientific Editor, production@msp.org

SUPPORTING INSTITUTIONS

ACADEMIA SINICA, TAIPEI
CALIFORNIA INST. OF TECHNOLOGY
INST. DE MATEMÁTICA PURA E APLICADA
KEIO UNIVERSITY
MATH. SCIENCES RESEARCH INSTITUTE
NEW MEXICO STATE UNIV.
OREGON STATE UNIV.

STANFORD UNIVERSITY
UNIV. OF BRITISH COLUMBIA
UNIV. OF CALIFORNIA, BERKELEY
UNIV. OF CALIFORNIA, DAVIS
UNIV. OF CALIFORNIA, LOS ANGELES
UNIV. OF CALIFORNIA, RIVERSIDE
UNIV. OF CALIFORNIA, SAN DIEGO
UNIV. OF CALIF., SANTA BARBARA

UNIV. OF CALIF., SANTA CRUZ
UNIV. OF MONTANA
UNIV. OF OREGON
UNIV. OF SOUTHERN CALIFORNIA
UNIV. OF UTAH
UNIV. OF WASHINGTON
WASHINGTON STATE UNIVERSITY

These supporting institutions contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its contents or policies.


See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2018 is US \$475/year for the electronic version, and \$640/year for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFlow® from Mathematical Sciences Publishers.

PUBLISHED BY

 **mathematical sciences publishers**
nonprofit scientific publishing

<http://msp.org/>

© 2018 Mathematical Sciences Publishers

PACIFIC JOURNAL OF MATHEMATICS

Volume 296 No. 1 September 2018

Monotonicity of eigenvalues of geometric operators along the Ricci–Bourguignon flow	1
BIN CHEN, QUN HE and FANQI ZENG	
Composition series of a class of induced representations, a case of one half cuspidal reducibility	21
IGOR CIGANOVIĆ	
Higgs bundles over cell complexes and representations of finitely presented groups	31
GEORGIOS DASKALOPOULOS, CHIKAKO MESE and GRAEME WILKIN	
Besov-weak-Herz spaces and global solutions for Navier–Stokes equations	57
LUCAS C. F. FERREIRA and JHEAN E. PÉREZ-LÓPEZ	
Four-manifolds with positive Yamabe constant	79
HAI-PING FU	
On the structure of cyclotomic nilHecke algebras	105
JUN HU and XINFENG LIANG	
Two applications of the Schwarz lemma	141
BINGYUAN LIU	
Monads on projective varieties	155
SIMONE MARCHESI, PEDRO MACIAS MARQUES and HELENA SOARES	
Minimal regularity solutions of semilinear generalized Tricomi equations	181
ZHUOPING RUAN, INGO WITT and HUICHENG YIN	
Temperedness of measures defined by polynomial equations over local fields	227
DAVID TAYLOR, V. S. VARADARAJAN, JUKKA VIRTANEN and DAVID WEISBART	



0030-8730(201809)296:1;1-N