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MONOTONICITY OF EIGENVALUES OF GEOMETRIC OPERATORS ALONG THE RICCI–BOURGUIGNON FLOW

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We study monotonicity of eigenvalues of the Schrödinger-type operator $-\Delta + cR$, where c is a constant, along the Ricci–Bourguignon flow. For $c \neq 0$, we derive monotonicity of the lowest eigenvalue of the Schrödinger-type operator $-\Delta + cR$, which generalizes some results of Cao (2008). As an application, we rule out nontrivial compact steady breathers in the Ricci–Bourguignon flow. For $c = 0$, we derive monotonicity of the first eigenvalue of the Laplacian, which generalizes some results of Ma (2006).

1. Introduction

Let (M, g) be an n-dimensional closed Riemannian manifold and $g(t)$ be a solution to the following Ricci–Bourguignon flow:

(1-1) $$\frac{\partial}{\partial t} g = -2\,\mathrm{Ric} + 2\rho R g = -2(\mathrm{Ric} - \rho R g),$$

where Ric is the Ricci tensor of the manifold, R is scalar curvature and ρ is a real constant. When $\rho = \frac{1}{2}, \frac{1}{n}, \frac{1}{2(n-1)}$ or 0, the tensor $\mathrm{Ric} - \rho R g$ corresponds to the Einstein tensor, the traceless Ricci tensor, the Schouten tensor or the Ricci tensor respectively. Apart from these special values of ρ, for which we will call the associated flows by the same name as the corresponding tensor, in general we will refer to the evolution equation defined by the PDE system (1-1) as the Ricci–Bourguignon flow. Moreover, by a suitable rescaling in time, when ρ is nonpositive, they can be seen as an interpolation between the Ricci flow and the Yamabe flow, see [Brendle 2005; Ye 1994] for instance, obtained as a limit when $\rho \to -\infty$.

The study of these flows was proposed by Jean-Pierre Bourguignon [1981, Question 3.24], building on some unpublished work of Lichnerowicz in the sixties and a paper of Aubin [1970]. Fischer [2004] studied a conformal version of this problem where the scalar curvature is constrained along the flow. Lu, Qing and

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Zheng [Lu et al. 2014] also proved some results on the conformal Ricci–Bourguignon flow. Recently, for suitable values of the scalar parameter involved in these flows, Catino et al. [2017] proved short time existence and provided curvature estimates. Moreover, Catino and Mazzieri [2016] stated some results on the associated solitons.

At present, the eigenvalues of geometric operators have become a powerful tool in the study of geometry and topology of manifolds. Recently, there has been a lot of work on the eigenvalue problems under the Ricci flow. On one hand, Perelman [2002] introduced the so-called $\mathscr{F}$-entropy functional and proved that it is nondecreasing along the Ricci flow coupled to a backward heat-type equation. Since the functional $\mathscr{F}$ is nondecreasing, this implies the monotonicity of the lowest eigenvalue of $-4\Delta + R$ along the Ricci flow.

Cao [2007] extended the operator $-4\Delta + R$ to the new operator $-\Delta + \frac{R}{2}$ on closed Riemannian manifolds, and showed that the eigenvalues of this new operator are nondecreasing along the Ricci flow with nonnegative curvature operator. Shortly thereafter Li [2007] dropped the curvature assumption and also obtained the above result for the operator $-\Delta + \frac{R}{2}$.

At around the same time, Cao [2008] considered the general operator

$$-\Delta + cR, \quad \text{where } c \geq \tfrac{1}{4},$$

and derived the following exact monotonicity formula; thus he showed that the lowest eigenvalue of this operator is nondecreasing along the Ricci flow without any curvature assumption.

Theorem A [Cao 2008]. *Let $(M, g(t))_{t\in[0,T)}$ be a solution of the unnormalized Ricci flow on a closed manifold M. Assume that $\lambda_0(t)$ is the lowest eigenvalue of $-\Delta + cR$, $c \geq \frac{1}{4}$, and $f = f(x,t) > 0$ satisfies*

$$-\Delta f(x,t) + cRf(x,t) = \lambda_0(t) f(x,t)$$

with $\int_M f^2\, d\upsilon = 1$. Then, under the unnormalized Ricci flow, we have

$$\frac{d}{dt}\lambda_0(t) = \frac{1}{2}\int_M |\operatorname{Ric} + \nabla^2\varphi|^2 e^{-\varphi}\, d\upsilon + \frac{4c-1}{2}\int_M |\operatorname{Ric}|^2 e^{-\varphi}\, d\upsilon \geq 0, \tag{1-2}$$

where $e^{-\varphi} = f^2$.

On the other hand, Ma [2006] obtained the monotonicity of the first eigenvalue of the Laplacian operator on a domain with Dirichlet boundary condition along the Ricci flow. Using the differentiability of the eigenvalues and the corresponding eigenfunctions of the Laplace operator under the Ricci flow, he obtained the following result.

Theorem B [Ma 2006]. *Let $g = g(t)$ be the evolving metric along the Ricci–Hamilton flow with $g(0) = g_0$ being the initial metric in M. Let D be a smooth* *bounded domain in* (M, g_0). *Let* $\mu > 0$ *be the first eigenvalue of the Laplace operator of the metric* $g(t)$. *If there is a constant* a *such that the scalar curvature satisfies* $R \geq 2a$ *in* $D \times \{t\}$ *and the Einstein tensor satisfies*

$$E_{ij} \geq -a g_{ij} \quad \text{in } D \times \{t\},$$

where $E_{ij} := R_{ij} - \frac{R}{2} g_{ij}$, *then we have* $\frac{d}{dt}\mu \geq 0$, *that is,* μ *is nondecreasing in* t; *furthermore,* $\frac{d}{dt}\mu(t) > 0$ *when the scalar curvature* R *is not the constant* $2a$. *The same monotonicity result is also true for other eigenvalues.*

Motivated by the above work, we also consider the eigenvalue of $-\Delta + cR$ with c a constant. For $c \neq 0$, inspired by [Cao 2007; 2008; Li 2007], we can derive the following monotonicity of the lowest eigenvalue of $-\Delta + cR$ under the Ricci–Bourguignon flow (1-1). That is, we obtain:

Theorem 1.1. *Let* $(M, g(t))_{t\in[0,T)}$ *be a compact maximal solution of the nontrivial Ricci–Bourguignon flow* (1-1) *and* $\lambda_0(t)$ *be the lowest eigenvalue of the operator* $-\Delta + cR$ *corresponding to the normalized eigenfunction* f, *that is,*

$$(-\Delta + cR) f = \lambda_0 f, \qquad \int_M f^2 \, dv = 1.$$

(1) *If* $\rho \leq 0$,

$$c \in \left[\frac{1}{4}, \frac{n}{4(n-1)}\right] \cup \left[\frac{(1-(n-1)\rho)^2}{4(1-2(n-1)\rho)}, +\infty\right)$$

and the scalar curvature is nonnegative at the initial time, then the lowest eigenvalue of the operator $-\Delta + cR$ *is nondecreasing in* $[0, T)$ *under the Ricci–Bourguignon flow* (1-1). *Furthermore, if* $\rho \neq 0$ *or* $c \neq \frac{1}{4}$, *then the lowest eigenvalue of the operator* $-\Delta + cR$ *is strictly monotone increasing in* $[0, T)$ *under the Ricci–Bourguignon flow* (1-1).

(2) *If* $0 < \rho < \frac{1}{2(n-1)}$,

$$c \geq \frac{3(n-1)^2\sqrt{\rho}}{2(1-2(n-1)\rho)} + \frac{1}{4}$$

and the curvature operator is nonnegative at the initial time, then the quantity

$$(T' - t)^{-\alpha} \lambda_0(t) \tag{1-3}$$

is strictly monotone increasing under the Ricci–Bourguignon flow (1-1) *in* $[0, T')$, *where*

$$T' = \frac{1}{2(1-\rho)\epsilon}, \quad \epsilon = \max_M R(0) \quad \text{and} \quad \alpha = \frac{\rho}{1-\rho} > 0.$$

Remark 1.1. When $\rho = 0$, where the Ricci–Bourguignon flow is the Ricci–Hamilton flow, our (1) reduces to the corresponding result of Cao [2008]. When $\rho \neq 0$, we don't know the differentiability for the lowest eigenvalue, so we show that the lowest eigenvalue is strictly monotone increasing by using the sign-preserving property.

Remark 1.2. According to the proof, it is obvious that (2) will hold whenever the Ricci curvature is nonnegative, but in general, the nonnegativity of the Ricci curvature is not preserved along the Ricci–Bourguignon flow. Nevertheless, the nonnegativity of the Ricci curvature is preserved in dimension three.

Corollary 1.2. *In dimension three, let $g(t)$ and $\lambda_0(t)$ be the same as in Theorem 1.1. But here we assume the Ricci curvature is nonnegative at the initial time. If $0<\rho<\frac{1}{4}$ and*

$$c \geq \frac{6\sqrt{\rho}}{1-4\rho} + \frac{1}{4},$$

then the quantity

$$(T'-t)^{-\alpha}\lambda_0(t) \tag{1-4}$$

is strictly monotone increasing under the Ricci–Bourguignon flow (1-1) in $[0, T')$, where

$$T' = \frac{1}{2(1-\rho)\epsilon}, \quad \epsilon = \max_M R(0) \quad \text{and} \quad \alpha = \frac{\rho}{1-\rho} > 0.$$

Next, as an application of our Theorem 1.1, we rule out nontrivial compact steady breathers. That is, we obtain:

Theorem 1.3. (1) *If $\rho = 0$, $c \geq \frac{1}{4}$, there is no compact steady breather other than the one which is Ricci-flat.*

(2) *If $\rho < 0$,*

$$c \in \left(\frac{1}{4}, \frac{n}{4(n-1)}\right] \cup \left[\frac{(1-(n-1)\rho)^2}{4(1-2(n-1)\rho)}, +\infty\right),$$

there is no compact steady breather with nonnegative scalar curvature other than the one which is Ricci-flat.

For $c = 0$, we derive the following monotonicity of eigenvalues on Laplacian under the Ricci–Bourguignon flow (1-1). That is, we obtain:

Theorem 1.4. *Let $(M, g(t))_{t\in[0,T)}$ be a compact maximal solution of the nontrivial Ricci–Bourguignon flow (1-1) and $\rho < \frac{1}{2(n-1)}$. Let $\lambda_1(t)$ be the first eigenvalue of the Laplace operator of the metric $g(t)$. If there is a nonnegative constant a such that*

$$R_{ij} - \frac{1+(2-n)\rho}{2} R g_{ij} \geq -a g_{ij} \quad \text{in } M \times [0, T), \tag{1-5}$$

$$R \geq \frac{2a}{1-n\rho} \quad \text{in } M \times \{0\}, \tag{1-6}$$

then $\lambda_1(t)$ is strictly monotone increasing and differentiable almost everywhere along the Ricci–Bourguignon flow in $[0, T)$.

Remark 1.3. (1) Wu et al. [2010] proved a similar result about the p-Laplace operator along the Ricci flow, where they assumed $R \geq ap$ and $R \not\equiv ap$ in $M \times \{0\}$, which are a little stronger than (1-6). The key difference is that we use Lemma 2.3.

(2) It should be pointed out that for $\rho = 0$, the above theorem is similar to the main result for the first eigenvalue of the Laplace operator in [Ma 2006]. Moreover, our assumptions are weaker than Ma's.

(3) If $a < 0$, there doesn't exist any scalar curvature which satisfies (1-5) and (1-6) at the same time.

(4) The result may be useful in the study of blow-up models of Ricci–Bourguignon flow on a complete Riemannian manifold (M, g_0).

2. Preliminaries

We begin with the definition for the first eigenvalue (the lowest eigenvalue) of the Laplace operator (the Schrödinger-type operator $-\Delta + cR$) under the Ricci–Bourguignon flow on a closed manifold. Then, we will show that the first eigenvalue of the Laplace operator is a continuous function along the Ricci–Bourguignon flow. Finally, under the Ricci–Bourguignon flow, we show that if $R(g_0) := R(0) \geq \beta$, for some $\beta \in \mathbb{R}$, then either $\max_M R(t) > \beta$ or the flow is trivial (i.e., $g(t) = g(0)$) for every $t \in (0, T)$.

Throughout, M will be taken to be a closed manifold (i.e., compact without boundary). We use moving frames in all calculations and adopt the index convention

$$1 \leq i, j, k, \cdots \leq n$$

throughout this paper.

Now we recall the definition of the first eigenvalue of the Laplace operator on a closed manifold under the Ricci–Bourguignon flow. Let $(M, g(t))$ be a solution of the Ricci–Bourguignon flow on the time interval $[0, T)$. Consider the first nonzero eigenvalue of the Laplace operator at time t, where $0 \leq t < T$,

$$\lambda_1(t) = \inf\left\{ \int_M |\nabla f|^2 \, d\upsilon : f \in W^{1,2}, \int_M f^2 \, d\upsilon = 1 \text{ and } \int_M f \, d\upsilon = 0 \right\},$$

where $d\upsilon$ denotes the volume form of the metric $g = g(t)$. Meanwhile the corresponding eigenfunction f satisfies the equation

$$-\Delta f(t) = \lambda_1(t) f(t),$$

where Δ is the Laplace operator with respect to $g(t)$, given by

$$\Delta_{g(t)} = \frac{1}{\sqrt{|g(t)|}} \partial_i \left(\sqrt{|g(t)|} g(t)^{ij} \partial_j \right),$$

and $g(t)^{ij} = g(t)^{-1}_{ij}$ is the inverse of the matrix $g(t)$ and $|g| = \det(g_{ij})$.

Note that it is not clear whether the first eigenvalue or the corresponding eigenfunction of the Laplace operator is differentiable under the Ricci–Bourguignon flow. When $\rho = 0$, where the Ricci–Bourguignon flow is the Ricci–Hamilton flow, many papers have pointed out that its differentiability under the Ricci–Hamilton flow follows from eigenvalue perturbation theory; e.g., see [Kato 1984; Kleiner and Lott 2008; Reed and Simon 1978]. But for $\rho \neq 0$, as far as we are aware, the differentiability of the first eigenvalue and eigenfunction of the Laplace operator under the Ricci–Bourguignon flow has not been known until now. So we cannot use Ma's trick to derive the monotonicity of the first eigenvalue of the Laplace operator. Although, we do not know the differentiability for $\lambda_1(t)$, following the techniques of [Wu et al. 2010], we will see that $\lambda_1(t)$ in fact is a continuous function along the Ricci–Bourguignon flow on $[0, T)$.

Lemma 2.1 [Wu et al. 2010]. *If g_1 and g_2 are two metrics on M which satisfy*

$$(1+\varepsilon)^{-1} g_1 \le g_2 \le (1+\varepsilon) g_1,$$

then, we have

$$(1+\varepsilon)^{-(n+1)} \le \frac{\lambda_1(g_1)}{\lambda_1(g_2)} \le (1+\varepsilon)^{(n+1)}. \tag{2-1}$$

In particular, $\lambda_1(g(t))$ is a continuous function in the t-variable.

Proof. This can be proved using arguments similar to those for Theorem 2.1 in [Wu et al. 2010]. □

Next we recall the definition of the lowest eigenvalue of $-\Delta + cR$. Let $\lambda_0(t)$ be the lowest eigenvalue of $-\Delta + cR$. Given a metric g on a closed manifold M, we define the functional λ_0 by

$$\lambda_0(t) = \inf\left\{ \mathscr{G}(g, f) : \int_M f^2 \, dv = 1, \ f > 0 \text{ and } f \in W^{1,2} \right\}, \tag{2-2}$$

where

$$\mathscr{G}(g, f) = \int_M (f(-\Delta f) + cRf^2) \, dv = \int_M (|\nabla f|^2 + cRf^2) \, dv.$$

We also do not know the differentiability for $\lambda_0(t)$ and the corresponding eigenfunction. But, following the techniques of [Chow et al. 2008], we will see that $\lambda_0(t)$ in fact is a continuous function along the Ricci–Bourguignon flow on $[0, T)$.

Lemma 2.2 [Chow et al. 2008]. *If g_1 and g_2 are two metrics on M which satisfy*

$$(1+\varepsilon)^{-1} g_1 \le g_2 \le (1+\varepsilon) g_1 \quad \text{and} \quad R(g_1) - \varepsilon \le R(g_2) \le R(g_1) + \varepsilon,$$

then

$$\begin{aligned} \lambda_0(g_2) - \lambda_0(g_1) \le {} & ((1+\varepsilon)^{\frac{n}{2}+1} - (1+\varepsilon)^{-\frac{n}{2}})(1+\varepsilon)^{\frac{n}{2}}(\lambda_0(g_1) - \min_M |c| R(g_1)) \\ & + |c| \big((1+\delta) \max_M |R(g_2) - R(g_1)| + 2\delta \max_M |R(g_1)| \big) (1+\varepsilon)^{\frac{n}{2}}, \end{aligned} \tag{2-3}$$

where $\delta \to 0$ as $\varepsilon \to 0$. In particular, λ_0 is a continuous function with respect to the C^2-topology.

Proof. This can be proved using arguments similar to those for Lemma 5.24 in [Chow et al. 2008]. □

At last, we present the following lemma.

Lemma 2.3. *Let $(M, g_t)_{t\in[0,T)}$ be a compact maximal solution of the Ricci–Bourguignon flow (1-1). If $\rho < \frac{1}{2(n-1)}$ and $R(0) \geq \beta$ for some $\beta \in \mathbb{R}$, then either $\max_M R(t) > \beta$ or $g(t) = g(0)$ for every $t \in (0, T)$.*

Proof. From Lemma 2.6, we know that $R(t) \geq \beta$ for every $t \in [0, T)$. If $\max_M R(t_0) = \beta$ for some $t_0 \in (0, T)$, we have $R(t_0) \equiv \beta$ and $\frac{\partial R}{\partial t}\big|_{t_0} \leq 0$. From (2-7), we have

$$\frac{1}{n}R^2(t_0) \leq |\mathrm{Ric}|^2(t_0) \leq \rho R^2(t_0) < \frac{1}{2(n-1)}R^2(t_0).$$

Obviously, we have $R(t_0) = 0$ and $\mathrm{Ric}(t_0) = 0$. Hence, $\max_M R(t_0) = \beta = 0$. Therefore, if $\beta \neq 0$, we have $\max_M R(t) > \beta$ for every $t \in [0, T)$.

When $\beta = 0$, let $I = \{t > 0 : \max_M R(t) > 0\}$. If $I = \varnothing$, then we have $R(t) \equiv 0$ and $\mathrm{Ric}(t) \equiv 0$. Hence we have $g(t) = g(0)$. When $I \neq \varnothing$ and $t_1 \in I$, for any t_0 and $0 < t_0 < t_1$, if $\max_M R(t_0) = 0$, then $R(t_0) \equiv 0$ and $\mathrm{Ric}(t_0) \equiv 0$. Hence, in $[t_0, T)$, $g(t) = g(t_0)$. So we have $\mathrm{Ric}(t_1) = \mathrm{Ric}(t_0) = 0$, which is in contradiction with $\max_M R(t_1) > 0$. Hence, $t_0 \in I$. Since $t_0 \in (0, t_1)$ is arbitrary, we have $(0, t_1] \subset I$. By the strong maximum principle, we have $(0, T) \subset I$. □

For the reader's convenience, we will recall some basic knowledge about the Ricci–Bourguignon flow.

Lemma 2.4 [Catino et al. 2017]. *Under the Ricci–Bourguignon flow (1-1), we have*

(2-4) $$\frac{\partial}{\partial t}g^{ij} = 2(R^{ij} - \rho R g^{ij}),$$

(2-5) $$\frac{\partial}{\partial t}(d\upsilon) = (n\rho - 1)R\, d\upsilon,$$

(2-6) $$\frac{\partial}{\partial t}(\Gamma^k_{ij}) = -R_{ik,j} - R_{kj,i} + R_{ij,k} + \rho(\delta^i_k R_{,j} + \delta^k_j R_{,i} - g_{ij}R_{,k}),$$

(2-7) $$\frac{\partial}{\partial t}R = [1 - 2(n-1)\rho]\Delta R + 2|\mathrm{Ric}|^2 - 2\rho R^2.$$

Lemma 2.5 (short time existence [Catino et al. 2017]). *Let $\rho < \frac{1}{2(n-1)}$. Then, the evolution equation (1-1) has a unique solution for a positive time interval on any smooth, n-dimensional, compact Riemannian manifold M (without boundary) with any initial metric g_0.*

Lemma 2.6 (preserved curvature conditions [Catino et al. 2017]). *Let $(M, g_t)_{t\in[0,T)}$ be a compact maximal solution of the Ricci–Bourguignon flow (1-1). If $\rho \le \frac{1}{2(n-1)}$, the minimum of the scalar curvature is nondecreasing along the flow. In particular, if $R(g_0) \ge \alpha$, for some $\alpha \in \mathbb{R}$, then $R(g_t) \ge \alpha$ for every $t \in [0, T)$. Moreover, if $\alpha > 0$ then $T \le n/(2(1-n\rho)\alpha)$.*

Lemma 2.7 (conditions preserved in three dimensions [Catino et al. 2017]). *Let $(M, g_t)_{t\in[0,T)}$ be a compact, 3-dimensional, solution of the Ricci–Bourguignon flow (1-1). If $\rho < \frac{1}{4}$, then*

(1) *nonnegative Ricci curvature is preserved along the flow*;

(2) *the pinching inequality* $\mathrm{Ric} \ge \varepsilon R g$ *is preserved along the flow for any $\varepsilon \le \frac{1}{3}$.*

Lemma 2.8 [Catino et al. 2017]. *Let $(M, g_t)_{t\in[0,T)}$ be a compact solution of the Ricci–Bourguignon flow (1-1) with $\rho \le \frac{1}{2(n-1)}$, and such that the initial data g_0 has nonnegative curvature operator. Then $\widetilde{R}_{g(t)} \ge 0$ for every $t \in [0, T)$, where $\widetilde{R} \in$* $\mathrm{End}(\Lambda^2 M)$ *is the Riemann curvature operator.*

Lemma 2.9 [Catino et al. 2017]. *Let $\rho < \frac{1}{2(n-1)}$. If $g(t)$ is a compact solution of the Ricci–Bourguignon flow on a maximal time interval $[0, T)$, $T < +\infty$, then*

$$\limsup_{t\to T} \max_M |\mathrm{Riem}(\,\cdot\,, t)| = +\infty,$$

where $\mathrm{Riem}(\,\cdot\,, t)$ *is Riemann tensor.*

3. Proof of Theorem 1.1

We will now prove Theorem 1.1. In order to achieve this, we first prove the following two lemmas. Our proof uses some tricks from [Cao 2007, 2008].

Let M be an n-dimensional closed Riemannian manifold, and $g(t)$ be a smooth solution of the Ricci–Bourguignon flow on the time interval $[0, T)$. Let $\lambda_0(t)$ be the lowest eigenvalue of the operator $-\Delta + cR$ corresponding to the normalized eigenfunction f, that is,

$$(-\Delta + cR) f = \lambda_0 f, \qquad \int_M f^2 \, dv = 1.$$

From Theorem 7.2 in [Guo et al. 2013], we know that, for any $t_0 \in [0, T)$, there exists a smooth function $\varphi(t) > 0$ satisfying

$$\int_M \varphi^2(t)\, dv = 1 \tag{3-1}$$

and $\varphi(t_0) = f(t_0)$. Let

$$\mu(t) = \int_M \big(\varphi(t)(-\Delta\varphi(t)) + cR\varphi^2(t)\big)\, dv, \tag{3-2}$$

then $\mu(t)$ is a smooth function by definition. And at time t_0, we conclude that

$$\mu(t_0) = \lambda_0(t_0).$$

We first give the following lemma.

Lemma 3.1. *Let $(M, g_t)_{t\in[0,T)}$ be a solution of the Ricci–Bourguignon flow on an n-dimensional closed manifold M, and let $\lambda_0(t)$ be the lowest eigenvalue of $-\Delta + cR$ under the Ricci–Bourguignon flow. Assume that $f(t_0)$ is the corresponding eigenfunction of $\lambda_0(t)$ at time $t_0 \in [0, T)$. Let $\mu(t)$ be a smooth function defined by* (3-2). *Then we have*

$$\begin{aligned}\frac{d}{dt}\mu(t)\Big|_{t=t_0} &= (A-2\rho)c\int_M R^2 f^2\,d\upsilon + (A-2\rho)\int_M R|\nabla f|^2\,d\upsilon \\ &\quad - A\lambda_0\int_M Rf^2\,d\upsilon + 2\int_M \mathrm{Ric}(\nabla f,\nabla f)\,d\upsilon \\ &\quad + 2c\int_M |\mathrm{Ric}|^2 f^2\,d\upsilon,\end{aligned} \tag{3-3}$$

where

$$A = -1 + n\rho + 2c[1-2(n-1)\rho].$$

Proof. The proof is by straightforward computation. Notice that

$$\frac{\partial}{\partial t}(\Delta\varphi) = 2R_{ij}\varphi_{ij} + \Delta(\varphi_t) - 2\rho R\Delta\varphi - (2-n)\rho R_{,k}\varphi_k. \tag{3-4}$$

Using Lemma 2.4, we have

$$\begin{aligned}\frac{d}{dt}\mu(t)\Big|_{t=t_0} &= \int_M \partial_t(-\Delta\varphi + cR\varphi)\varphi\,d\upsilon + \int_M(-\Delta\varphi + cR\varphi)\partial_t(\varphi\,d\upsilon) \\ &= \int_M [\partial_t(-\Delta\varphi) + c\varphi\partial_t R + cR\partial_t\varphi]\varphi\,d\upsilon \\ &\qquad\qquad + \int_M(-\Delta\varphi + cR\varphi)\partial_t(\varphi\,d\upsilon) \\ &= \int_M [-2R_{ij}\varphi_{ij} - \Delta(\varphi_t) + 2\rho R\Delta\varphi + (2-n)\rho R_{,k}\varphi_k]\varphi\,d\upsilon \\ &\quad + \int_M [c\varphi\partial_t R + cR\partial_t\varphi]\varphi\,d\upsilon + \int_M(-\Delta\varphi + cR\varphi)\partial_t(\varphi\,d\upsilon) \\ &= \int_M [-2R_{ij}\varphi_{ij}\varphi + 2\rho R\varphi\Delta\varphi + (2-n)\rho R_{,k}\varphi_k\varphi]\,d\upsilon \\ &\quad + c\int_M \{[1-2(n-1)\rho]\Delta R + 2|\mathrm{Ric}|^2 - 2\rho R^2\}\varphi^2\,d\upsilon \\ &\quad + \int_M(-\Delta\varphi + cR\varphi)[\partial_t(\varphi)d\upsilon + \partial_t(\varphi\,d\upsilon)].\end{aligned} \tag{3-5}$$

From $R_{,i} = 2R_{ij,j}$ and the Stokes formula, we have

$$\int_M \varphi^2 \Delta R\, d\upsilon = \int_M 2R(|\nabla\varphi|^2 + \varphi\Delta\varphi)\, d\upsilon, \tag{3-6}$$

$$\int_M R_{,k}\varphi_k\varphi\, d\upsilon = \int_M 2R(|\nabla\varphi|^2 + \varphi\Delta\varphi)\, d\upsilon, \tag{3-7}$$

and

$$\begin{aligned}\int_M -R_{ij}\varphi_{ij}\varphi\, d\upsilon &= \int_M (R_{ij}\varphi)_j\varphi_i\, d\upsilon \\ &= \int_M R_{ij,j}\varphi\varphi_i\, d\upsilon + \int_M R_{ij}\varphi_j\varphi_i\, d\upsilon \\ &= \frac{1}{2}\int_M R_{,i}\varphi\varphi_i\, d\upsilon + \int_M R_{ij}\varphi_j\varphi_i\, d\upsilon \\ &= -\frac{1}{2}\int_M R(\varphi\varphi_i)_i\, d\upsilon + \int R_{ij}\varphi_j\varphi_i\, d\upsilon \\ &= -\frac{1}{2}\int_M R\Delta\varphi\varphi\, d\upsilon - \frac{1}{2}\int_M R|\nabla\varphi|^2\, d\upsilon + \int_M R_{ij}\varphi_j\varphi_i\, d\upsilon.\end{aligned} \tag{3-8}$$

On the other hand, at time t_0, we know φ is the eigenfunction of $\lambda_0(t_0)$, i.e., $(-\Delta + cR)\varphi = \lambda_0\varphi$, and we have

$$\int_M (-\Delta\varphi + cR\varphi)[\partial_t(\varphi)d\upsilon + \partial_t(\varphi\, d\upsilon)] = \lambda_0 \int_M \varphi[\partial_t(\varphi)d\upsilon + \partial_t(\varphi\, d\upsilon)] = 0. \tag{3-9}$$

The last equality holds because of (3-1). Inserting (3-6)–(3-9) into (3-5), at $t = t_0$, yields

$$\begin{aligned}\frac{d}{dt}\mu(t)\Big|_{t=t_0} &= (-1+n\rho+2c[1-2(n-1)\rho])\int_M R\varphi\Delta\varphi\, d\upsilon \\ &\quad + (-1+(n-2)\rho+2c[1-2(n-1)\rho])\int_M R|\nabla\varphi|^2\, d\upsilon \\ &\quad + 2\int_M R_{ij}\varphi_i\varphi_j\, d\upsilon + 2c\int_M |\mathrm{Ric}|^2\varphi^2\, d\upsilon - 2c\rho\int_M R\varphi^2\, d\upsilon.\end{aligned} \tag{3-10}$$

Inserting $\Delta\varphi = cR\varphi - \lambda_0\varphi$ into (3-10), at $t = t_0$, gives

$$\begin{aligned}\frac{d}{dt}\mu(t)\Big|_{t=t_0} &= (A-2\rho)c\int_M R^2\varphi^2\, d\upsilon + (A-2\rho)\int_M R|\nabla\varphi|^2\, d\upsilon \\ &\quad - A\mu\int_M R\varphi^2\, d\upsilon + 2\int_M R_{ij}\varphi_i\varphi_j\, d\upsilon \\ &\quad + 2c\int_M |\mathrm{Ric}|^2\varphi^2\, d\upsilon.\end{aligned} \tag{3-11}$$

Therefore we finish the proof of Lemma 3.1. □

Now we give the second lemma.

Lemma 3.2. *Let $(M, g_t)_{t\in[0,T)}$ be a solution of the Ricci–Bourguignon flow on an n-dimensional closed manifold M, and let $\lambda_0(t)$ be the lowest eigenvalue of $-\Delta+cR$ under the Ricci–Bourguignon flow. Assume that $f(t_0)$ is the corresponding eigenfunction of $\lambda_0(t)$ at time $t_0\in[0,T)$. Let $\mu(t)$ be a smooth function defined by* (3-2). *Then we have*

$$\begin{aligned}\text{(3-12)}\quad \frac{d}{dt}\mu(t)\Big|_{t=t_0} &= \frac{1}{2k(2-k)}\int_M |R_{ij}-2k(\log f)_{ij}|^2 f^2\,dv \\ &\quad+\left(2c-\frac{1}{2k(2-k)}\right)\int_M |\mathrm{Ric}|^2 f^2\,dv \\ &\quad-\frac{(a-bk)4c-(d-ek)}{2-k}\lambda_0\int_M Rf^2\,dv \\ &\quad+\left(\frac{(a-bk)4c-(d-ek)}{2-k}-2\rho\right)\int_M (cR^2f^2+R|\nabla f|^2)\,dv,\end{aligned}$$

where

$$a=1-2(n-1)\rho,\quad b=1-(n-1)\rho,$$
$$d=1-2n\rho,\quad e=1-n\rho\quad \text{and}\quad 0<k<2.$$

Proof. The proof is by straightforward computation:

$$\text{(3-13)}\quad \int_M |R_{ij}-2k(\log f)_{ij}|^2 f^2\,dv=\int_M |\mathrm{Ric}|^2 f^2\,dv+4k^2\int_M |\nabla^2(\log f)|^2 f^2\,dv \\ -4k\int_M R_{ij}(\log f)_{ij}f^2\,dv.$$

From [Cao 2008], we can get

$$\begin{aligned}\text{(3-14)}\quad 4k^2\int_M |\nabla^2(\log f)|^2 f^2\,dv &= 2k^2c\int_M R\Delta f^2\,dv-4k^2\int_M R_{ij}f_if_j\,dv \\ &= 4k^2c\int_M R(f\Delta f+|\nabla f|^2)\,dv-4k^2\int_M R_{ij}f_if_j\,dv\end{aligned}$$

and

$$\begin{aligned}\text{(3-15)}\quad -4k\int_M R_{ij}(\log f)_{ij}f^2\,dv &= -2k\int_M R(f\Delta f+|\nabla f|^2)\,dv+8k\int_M R_{ij}f_if_j\,dv.\end{aligned}$$

Combining (3-14) and (3-15), we arrive at

$$\begin{aligned}\int_M |R_{ij}-2k(\log f)_{ij}|^2 f^2\,d\upsilon &= \int_M |\mathrm{Ric}|^2 f^2\,d\upsilon+(8k-4k^2)\int_M R_{ij}f_if_j\,d\upsilon\\ &\quad+2k(2kc-1)\Big(c\int_M R^2f^2\,d\upsilon-\lambda_0\int_M Rf^2\,d\upsilon\Big)\\ &\quad+2k(2kc-1)\int_M R|\nabla f|^2\,d\upsilon.\end{aligned}\tag{3-16}$$

Multiplying by $\frac{1}{2k(2-k)}$ on both sides of (3-16), we conclude that

$$\begin{aligned}\frac{1}{2k(2-k)}\int_M |R_{ij}-2k(\log f)_{ij}|^2 f^2\,d\upsilon &= \frac{1}{2k(2-k)}\int_M |\mathrm{Ric}|^2 f^2\,d\upsilon+2\int_M R_{ij}f_if_j\,d\upsilon\\ &\quad+\frac{2kc-1}{2-k}\Big(c\int_M R^2f^2\,d\upsilon-\lambda_0\int_M Rf^2\,d\upsilon\Big)\\ &\quad+\frac{2kc-1}{2-k}\int_M R|\nabla f|^2\,d\upsilon.\end{aligned}\tag{3-17}$$

Subtracting (3-17) from (3-3), we see that

$$\begin{aligned}\frac{d}{dt}\mu(t)\Big|_{t=t_0} &= \frac{1}{2k(2-k)}\int_M |R_{ij}-2k(\log f)_{ij}|^2 f^2\,d\upsilon\\ &\quad+\Big(2c-\frac{1}{2k(2-k)}\Big)\int_M |\mathrm{Ric}|^2 f^2\,d\upsilon\\ &\quad+\Big(A-2\rho-\frac{2kc-1}{2-k}\Big)\int_M (cR^2f^2+R|\nabla f|^2-\lambda_0Rf^2)\,d\upsilon\\ &\quad-2\rho\int_M \lambda_0Rf^2\,d\upsilon,\end{aligned}\tag{3-18}$$

where $A=-1+n\rho+2c[1-2(n-1)\rho]$. Note that

$$\begin{aligned}A-\frac{2kc-1}{2-k}&=\frac{4\{[1-2(n-1)\rho]-[1-(n-1)\rho]k\}c-[(1-2n\rho)-(1-n\rho)k]}{2-k}\\ &:=\frac{(a-bk)c-(d-ek)}{2-k}.\end{aligned}$$

Therefore we finish the proof of Lemma 3.2. □

Proof of Theorem 1.1. We first prove (1). If $\rho<0$, inserting

$$k=\frac{a}{b}=\frac{1-2(n-1)\rho}{1-(n-1)\rho}$$

into (3-12), we obtain

$$\frac{d}{dt}\mu(t)\Big|_{t=t_0} = \frac{(1-(n-1)\rho)^2}{2-4(n-1)\rho}\int_M \left|R_{ij}-2\frac{1-2(n-1)\rho}{1-(n-1)\rho}(\log f)_{ij}\right|^2 f^2\,d\upsilon + \left(2c-\frac{(1-(n-1)\rho)^2}{2(1-2(n-1)\rho)}\right)\int_M |\mathrm{Ric}|^2 f^2\,d\upsilon - \rho\lambda_0\int_M Rf^2\,d\upsilon - \rho\left\{c\int_M R^2 f^2\,d\upsilon + \int_M R|\nabla f|^2\,d\upsilon\right\}. \tag{3-19}$$

If $R \geq 0$ in $M\times\{0\}$, from Lemmas 2.3 and 2.6, we know that either $\max_M R(t) > 0$ or $g(t) = g(0)$ for every $t \in (0, T)$. Assume $\max_M R(t) > 0$ (otherwise the proof is trivial). By (3-19), when

$$c \geq \frac{(1-(n-1)\rho)^2}{4(1-2(n-1)\rho)} > \frac{1}{4},$$

we obtain

$$\frac{d}{dt}\mu(t)\Big|_{t=t_0} > 0. \tag{3-20}$$

Moreover, setting $k = 1$ in (3-12), we obtain

$$\frac{d}{dt}\mu(t)\Big|_{t=t_0} = \frac{1}{2}\int_M |R_{ij}-2(\log f)_{ij}|^2 f^2\,d\upsilon + \left(2c-\frac{1}{2}\right)\int_M |\mathrm{Ric}|^2 f^2\,d\upsilon - \rho[4(n-1)c-n+2]\left\{c\int_M R^2 f^2\,d\upsilon + \int_M R|\nabla f|^2\,d\upsilon\right\} + \rho[4(n-1)c-n]\lambda_0\int_M Rf^2\,d\upsilon. \tag{3-21}$$

Then when $\frac{1}{4} \leq c \leq n/(4(n-1))$, we also obtain (3-20).

Since the eigenfunction of the lowest eigenvalue is not equal to 0 along the Ricci–Bourguignon flow, see Lemma 5.22 in [Chow et al. 2008], and $\mu(t)$ is a smooth function with respect to the t-variable, we have

$$\frac{d}{dt}\mu(t) > 0 \tag{3-22}$$

in $(t_0 - \delta, t_0 + \delta)$, where $\delta > 0$ is sufficiently small. So we get

$$\mu(t_0) > \mu(t_1) \tag{3-23}$$

for any $t_1 \in (t_0 - \delta, t_0 + \delta)$ and $t_1 < t_0$.

Notice that

$$\mu(t_0) = \lambda_0(t_0) \quad \text{and} \quad \mu(t_1) \geq \lambda_0(t_1).$$

This implies $\lambda_0(t_0) > \lambda_0(t_1)$ for any $t_0 > t_1$. Since $\lambda_0(t)$ is continuous and $t_0 \in [0, T)$ is arbitrary, $\lambda_0(t)$ is strictly monotone increasing in $[0, T)$. Therefore we finish the proof of (1).

Next we prove (2). If $0 < \rho < \frac{1}{2(n-1)}$, in (3-18), we pick k such that

$$A - 2\rho - \frac{2kc-1}{2-k} = 0.$$

Then

$$k = \frac{(4c-1)[1-2(n-1)\rho]-2\rho}{(4c-1)[1-(n-1)\rho]-\rho}.$$

Taking $4c-1 = B \geq 0$, we have

$$k = \frac{B[1-2(n-1)\rho]-2\rho}{B[1-(n-1)\rho]-\rho} \quad \text{and} \quad 2-k = \frac{B}{B[1-(n-1)\rho]-\rho}.$$

For $0 < k < 2$, we need

$$B > \frac{2\rho}{1-2(n-1)\rho}. \tag{3-24}$$

Now, we need $2c - \frac{1}{2k(2-k)} \geq 0$, which is

$$B(B+1)(B[1-2(n-1)\rho]-2\rho) \geq (B[1-(n-1)\rho]-\rho)^2. \tag{3-25}$$

It is true when $B \to +\infty$. Next, we will prove that given

$$4c-1 = B \geq \frac{6(n-1)^2\sqrt{\rho}}{1-2(n-1)\rho}, \tag{3-26}$$

both (3-24) and (3-25) are true. Firstly, since $0 < \rho < \frac{1}{2(n-1)} < 1$, we have

$$B \geq \frac{6(n-1)^2\sqrt{\rho}}{1-2(n-1)\rho} > \frac{6(n-1)^2\rho}{1-2(n-1)\rho} > \frac{2\rho}{1-2(n-1)\rho}.$$

Thus, (3-24) holds. Secondly, let's show (3-25):

$$\begin{aligned}
&B(B+1)(B[1-2(n-1)\rho]-2\rho)-(B[1-(n-1)\rho]-\rho)^2\\
&\quad = B^2(B[1-2(n-1)\rho]-2\rho)+B(B[1-2(n-1)\rho]-2\rho)-(B[1-(n-1)\rho]-\rho)^2\\
&\quad = B^2(B[1-2(n-1)\rho]-2\rho)-((n-1)B\rho+\rho)^2\\
&\quad \geq B^2(6(n-1)^2\sqrt{\rho}-2\rho)-((n-1)B\rho+\rho)^2\\
&\quad \geq B^2(4(n-1)^2\sqrt{\rho})-((n-1)B\rho+\rho)^2\\
&\quad = [2B(n-1)\sqrt[4]{\rho}+(n-1)B\rho+\rho][2B(n-1)\sqrt[4]{\rho}-(n-1)B\rho-\rho].
\end{aligned}$$

The first factor is clearly positive. For the second factor, note $\rho^{\frac{1}{4}} > \rho^{\frac{3}{4}} > \rho$,

$$\begin{aligned} 2B(n-1)\sqrt[4]{\rho} - (n-1)B\rho - \rho &\geq B(n-1)\sqrt[4]{\rho} - \rho \\ &\geq \frac{6(n-1)^3\sqrt{\rho}}{1-2(n-1)\rho}\sqrt[4]{\rho} - \rho \\ &\geq 6(n-1)^3\rho^{\frac{3}{4}} - \rho \geq 12(n-1)^3\rho - \rho > 0. \end{aligned}$$

Therefore, given

$$4c - 1 \geq \frac{6(n-1)^2\sqrt{\rho}}{1-2(n-1)\rho},$$

we have

$$\frac{d}{dt}\mu\Big|_{t=t_0} > -2\rho\mu\int_M Rf^2\,d\upsilon. \tag{3-27}$$

By Lemma 2.8, we know that the nonnegativity of the curvature operator is preserved by the Ricci–Bourguignon flow. This implies that the Ricci curvature is also nonnegative, and we have $|\mathrm{Ric}|^2 \leq R^2$. The evolution equation of scalar curvature satisfies

$$\begin{aligned} \frac{\partial}{\partial t}R &= [1-2(n-1)\rho]\Delta R + 2|\mathrm{Ric}|^2 - 2\rho R^2 \\ &\leq [1-2(n-1)\rho]\Delta R + 2(1-\rho)R^2. \end{aligned} \tag{3-28}$$

Let $\sigma(t)$ be the solution of the following ODE with initial value:

$$\begin{cases} \partial\sigma(t)/\partial t = 2(1-\rho)\sigma^2, \\ \sigma(0) = \max_M R(0). \end{cases} \tag{3-29}$$

By the maximum principle, letting $\epsilon = \max_M R(0)$, we can get

$$R(t) \leq \sigma(t) = \Big(-2(1-\rho)t + \frac{1}{\epsilon}\Big)^{-1}$$

on $[0, T'')$, where $T'' = \min\{T', T\}$ and $T' = \frac{1}{2(1-\rho)\epsilon}$. Arguing now as in [Hamilton 1982, Section 14], it follows that the metrics $g(t)$ converge to some limit metric $g(T)$ in the C^∞ topology if $T < T''$; hence, we can restart the flow with this initial metric $g(T)$, obtaining a smooth flow in some larger time interval $[0, T+\delta)$, in contradiction with the fact that T was the maximal time of smooth existence. So we have $T' \leq T$. Hence $R(t) \leq \sigma(t)$ on $[0, T')$. Since the eigenfunction of the lowest eigenvalue is not equal to 0 along the Ricci–Bourguignon flow, see Lemma 5.22 in [Chow et al. 2008], from Lemma 2.3 and (3-27), we have

$$\frac{d}{dt}\mu\Big|_{t=t_0} > -2\rho\mu\int_M Rf^2\,d\upsilon \geq -2\rho\mu\sigma, \tag{3-30}$$

which implies

$$\Big(\frac{d}{dt}\mu + 2\rho\mu\sigma\Big)\Big|_{t=t_0} > 0.$$

By arguments similar to those in the proof of (1), we know that

$$(T' - t)^{-\alpha} \lambda_0(t)$$

is strictly monotone increasing under the Ricci–Bourguignon flow (1-1) on $[0, T')$ and

$$T' = \frac{1}{2(1-\rho)\epsilon},$$

where

$$\epsilon = \max_M R(0) \quad \text{and} \quad \alpha = \frac{\rho}{1-\rho} > 0,$$

which shows (2) holds. Therefore we finish the proof of Theorem 1.1. □

4. Proof of Theorem 1.3

We will now prove Theorem 1.3. First, we recall the definition of breathers.

Definition 4.1. A metric $g(t)$ evolving from the Ricci–Bourguignon flow is called a breather if for some $t_1 < t_2$ and $\alpha > 0$ the metrics $\alpha g(t_1)$ and $g(t_2)$ differ only by a diffeomorphism; the cases $\alpha = 1$, $\alpha < 1$ and $\alpha > 1$ correspond to steady, shrinking and expanding breathers, respectively.

Proof of Theorem 1.3. For a steady breather, let t_1 and t_2 be the same as above; we have

$$\lambda_0(t_1) = \lambda_0(t_2).$$

When $\rho = 0$ and $c \geq \frac{1}{4}$, by (1-2) of Theorem A, we have

$$\lambda_0(t_1) \leq \lambda_0(t_2)$$

provided $t_1 < t_2$. And the equality holds if and only if for any $t_1 \leq t \leq t_2$,

$$\frac{d}{dt} \lambda_0(t) = 0.$$

Since the eigenfunction f cannot be identical to zero, from (1-2) of Theorem A we must have $\mathrm{Ric} \equiv 0$.

But when $\rho < 0$,

$$c \in \left(\frac{1}{4}, \frac{n}{4(n-1)} \right] \cup \left[\frac{(1-(n-1)\rho)^2}{4(1-2(n-1)\rho)}, +\infty \right)$$

and the scalar curvature is nonnegative at the initial time, because of Theorem 1.1(1) for a nontrivial flow we have

$$\lambda_0(t_1) < \lambda_0(t_2)$$

provided $t_1 < t_2$. When $(M, g(t))_{t\in[0,T)}$ is a compact maximal solution of the trivial Ricci–Bourguignon flow (1-1), i.e., $\mathrm{Ric} \equiv 0$, we have $\lambda_0(t_1) = \lambda_0(t_2)$. Hence we have proved Theorem 1.3. □

5. Proof of Theorem 1.4

We will now prove Theorem 1.4. In order to achieve this, we first prove Lemma 5.1. Our proof involves choosing a proper smooth function, which seems to be a delicate trick.

Let M be an n-dimensional closed Riemannian manifold, and $g(t)$ be a smooth solution of the Ricci–Bourguignon flow on the time interval $[0, T)$. Let $\lambda_1(t)$ be the first eigenvalue of the Laplace operator under the Ricci–Bourguignon flow and $f(t_0)$ be the corresponding eigenfunction of $\lambda_1(t)$ at time $t_0 \in [0, T)$, i.e.,

$$-\Delta_{g(t_0)} f(t_0) = \lambda_1(t_0) f(t_0). \tag{5-1}$$

For any $t_0 \in [0, T)$, Wu et al. [2010] pointed out that there exists a smooth function

$$\phi(t) = \frac{\psi(t)}{\left(\int_M \psi(t)^2 \, d\upsilon\right)^{\frac{1}{2}}}, \quad \text{where } \psi(t) = f(t_0)\left(\frac{|g(t_0)|}{|g(t)|}\right)^{\frac{1}{2}}$$

satisfying

$$\int_M \phi(t)^2 \, d\upsilon = 1, \quad \int_M \phi(t) \, d\upsilon = 0, \tag{5-2}$$

and $\varphi(t_0) = f(t_0)$. Now we define a general smooth function

$$\mu(t) = \int_M \phi(t)(-\Delta\phi(t)) \, d\upsilon. \tag{5-3}$$

In general, $\mu(t)$ is not equal to $\lambda_1(t)$. But at time t_0, we conclude that

$$\mu(t_0) = \lambda_1(t_0).$$

Lemma 5.1. *Let $(M, g_t)_{t\in[0,T)}$ be a solution of the Ricci–Bourguignon flow on an n-dimensional closed manifold M and let $\lambda_1(t)$ be the first eigenvalue of the Laplace operator under the Ricci–Bourguignon flow. Assume that $f(t_0)$ is the corresponding eigenfunction of $\lambda_1(t)$ at time $t_0 \in [0, T)$, i.e.,*

$$-\Delta_{g(t_0)} f(t_0) = \lambda_1(t_0) f(t_0).$$

Let $\mu(t)$ be a smooth function defined by (5-3). Then we have

$$\frac{d}{dt}\mu(t)\Big|_{t=t_0} = \int_M \{2R_{ij} f_i f_j + (1-n\rho)\lambda_1 R f^2 - [(2-n)\rho+1]R|\nabla f|^2\} \, d\upsilon. \tag{5-4}$$

Proof. The proof is by direct computation:

$$\begin{aligned}
(5\text{-}5)\quad \frac{d}{dt}\mu(t)\Big|_{t=t_0} &= \int_M \partial_t(-\phi\Delta\phi)\,d\upsilon + \int_M (-\phi\Delta\phi)\partial_t(d\upsilon)\\
&= \int_M [-2R_{ij}\phi_{ij} - \Delta(\partial_t\phi) + 2\rho R\Delta\phi + (2-n)\rho R_{,k}\phi_k]\phi\,d\upsilon\\
&\qquad + \int_M (-\Delta\phi)\partial_t\phi\,d\upsilon + \int_M (-\Delta\phi)\phi(n\rho-1)R\,d\upsilon\\
&= \int_M -2R_{ij}\phi_{ij}\phi\,d\upsilon + \int_M -2(\Delta\phi)\partial_t\phi\,d\upsilon\\
&\quad + (2-n)\rho\int_M R_{,k}\phi_k\phi\,d\upsilon + [1+(2-n)\rho]\int_M R(\Delta\phi)\phi\,d\upsilon.
\end{aligned}$$

From (3-7) and (3-8), we have

$$\begin{aligned}
(5\text{-}6)\quad \frac{d}{dt}\mu(t)\Big|_{t=t_0} &= -\int_M R\Delta\phi\phi\,d\upsilon - \int_M R|\nabla\phi|^2\,d\upsilon + 2\int_M R_{ij}\phi_j\phi_i\,d\upsilon\\
&\quad + \int_M -2(\Delta\phi)\partial_t\phi\,d\upsilon - (2-n)\rho\int_M R\Delta\phi\phi\,d\upsilon\\
&\quad - (2-n)\rho\int_M R|\nabla\phi|^2\,d\upsilon\\
&\quad + [1+(2-n)\rho]\int_M R(\Delta\phi)\phi\,d\upsilon\\
&= 2\int_M R_{ij}\phi_j\phi_i\,d\upsilon + 2\mu\int_M \phi\partial_t\phi\,d\upsilon\\
&\quad - [1+(2-n)\rho]\int_M R|\nabla\phi|^2\,d\upsilon.
\end{aligned}$$

Under the Ricci–Bourguignon flow, from the constraint condition (5-2), we get

$$(5\text{-}7)\qquad 2\int_M \phi\partial_t\phi\,d\upsilon = -(n\rho-1)\int_M \phi^2 R\,d\upsilon.$$

Hence, at time t_0, the desired lemma follows from substituting (5-7) into (5-6). □

Proof of Theorem 1.4. We assume that for any time $t_0 \in [0, T)$, if $f(t_0)$ is the corresponding eigenfunction of the first eigenvalue $\lambda_1(t_0)$, then we have $\lambda_1(t_0) = \mu(t_0)$. By Lemma 5.1, we have

$$\begin{aligned}
(5\text{-}8)\quad \frac{d}{dt}\mu(t)\Big|_{t=t_0} &= \int_M \{(1-n\rho)\lambda_1 Rf^2 + 2R_{ij}f_if_j - [(2-n)\rho+1]R|\nabla f|^2\}\,d\upsilon\\
&= \int_M \{2R_{ij} - [(2-n)\rho+1]Rg_{ij}\}f_if_j\,d\upsilon + \int_M (1-n\rho)\lambda_1 Rf^2\,d\upsilon\\
&\geq \int_M (1-n\rho)\lambda_1 Rf^2\,d\upsilon - 2a\int_M |\nabla f|^2\,d\upsilon\\
&= \int_M (1-n\rho)\lambda_1 Rf^2\,d\upsilon - 2a\lambda_1 = \lambda_1\int_M f^2\{(1-n\rho)R - 2a\}\,d\upsilon,
\end{aligned}$$

where we used the first assumption of Theorem 1.4.

From Lemma 2.3, we know that either $\max_M R(t) > 0$ or $g(t) = g(0)$ for every $t \in (0, T)$. Assume $\max_M R(t) > 0$ (otherwise the proof is trivial). Since the eigenfunction of the first eigenvalue is not equal to 0 along the Ricci–Bourguignon flow, by (5-8), we obtain

$$\frac{d}{dt}\mu(t)\Big|_{t=t_0} > 0. \tag{5-9}$$

By arguments similar to those in the proof of Theorem 1.1, we have $\lambda_1(t)$ is strictly monotone increasing in $[0, T)$.

As for the differentiability for $\lambda_1(t)$, since $\lambda_1(t)$ is increasing on the time interval $[0, T)$ under curvature conditions of the theorem, by the classical Lebesgue's theorem, see for example Chapter 4 in [Mukherjea and Pothoven 1984], it is easy to see that $\lambda_1(t)$ is differentiable almost everywhere on $[0, T)$. □

Remark 5.1. (1) In the course of proving Theorem 1.4, we do not use any differentiability of the first eigenvalue or the corresponding eigenfunction of the Laplace operator under the Ricci–Bourguignon flow.

(2) Using this method, we cannot get any monotonicity for higher-order eigenvalues of the Laplace operator under the Ricci–Bourguignon flow.

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