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IGOR CIGANOVIĆ

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COMPOSITION SERIES OF A CLASS OF INDUCED REPRESENTATIONS, A CASE OF ONE HALF CUSPIDAL REDUCIBILITY

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We determine the composition series of the induced representation

$$\delta([\nu^{-b}\rho, \nu^{c}\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \rtimes \sigma,$$

where $a, b, c \in \frac{1}{2}(2\mathbb{Z}+1)$ satisfy $\frac{1}{2} \le a < b < c$, ρ is an irreducible cuspidal unitary representation of a general linear group and σ is an irreducible cuspidal representation of a classical group.

Introduction

In this paper we determine the composition series of a class of standard representations in terms of Mœglin–Tadić classification of discrete series [Mœglin 2002; Mœglin and Tadić 2002]. Interesting on its own, this result should also prove valuable for extending results about Jacquet modules of segment type representations obtained in [Matić and Tadić 2015].

To describe our results we introduce some notation. Fix a local nonarchimedean field F of characteristic different from 2. Let ρ be an irreducible cuspidal unitary representation of $GL(m_\rho, F)$ (this defines m_ρ) and $x, y \in \mathbb{R}$, such that $y-x+1 \in \mathbb{Z}_{\geq 0}$. The set $[\nu^x\rho, \nu^y\rho] = \{\nu^x\rho, \dots, \nu^y\rho\}$ is called a segment. The parabolically induced representation $\nu^y\rho \times \cdots \times \nu^x\rho$ has a unique irreducible subrepresentation; it is essentially square integrable and we denote it by $\delta([\nu^x\rho, \nu^y\rho])$. Also we denote $e([\nu^x\rho, \nu^y\rho]) = e(\delta([\nu^x\rho, \nu^y\rho]) = \frac{1}{2}(x+y)$. If δ is an essentially square integrable representation of $GL(m_\delta, F)$, there exists a segment Δ such that $\delta = \delta(\Delta)$.

Let G_n be a symplectic or (full) orthogonal group having split rank n. Given a sequence of segments $\Delta_1, \dots, \Delta_k$, $e(\Delta_i) > 0$, $i = 1, \dots, k$ and an irreducible tempered representation τ of some $G_{n'}$ we denote by $\mathrm{Lang}(\delta(\Delta_1) \times \cdots \times \delta(\Delta_k) \rtimes \tau)$ the unique irreducible quotient, called the Langlands quotient, of the parabolically induced representation $\delta(\Delta_{\varphi(1)}) \times \cdots \times \delta(\Delta_{\varphi(k)}) \rtimes \tau$ where φ is a permutation of the

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set $\{1, \dots, k\}$ such that $e(\Delta_{\varphi(1)}) \geq \cdots \geq e(\Delta_{\varphi(k)})$. These induced representations are called standard representations and are important because by the Langlands classification every irreducible representation of G_n can be described as a Langlands quotient. Further if τ is a discrete series representation then by the Mœglin–Tadić classification of discrete series it is described by an admissible triple (Jord, τ_{cusp}, ϵ). Here Jord is a set Jordan blocks, τ_{cusp} a partial cuspidal support and ϵ a function from a subset of Jord $\cup$(Jord $\times$ Jord) into $\{\pm 1\}$. Results of Muić about reducibility of the generalized principal series $\delta([\nu^x \rho, \nu^y \rho]) \rtimes \tau$ [Muić 2004; 2005] are stated case by case depending on Jord and x and y where the case $x = \frac{1}{2}$ plays an important role. In our situation, we provide some additional information, see Proposition 2.4. These results are used to compute composition series of the induced representation

$$\delta([\nu^{-b}\rho, \nu^c \rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^a \rho]) \rtimes \sigma,$$

where $a, b, c \in \frac{1}{2}(2\mathbb{Z}+1)$ such that $\frac{1}{2} \leq a < b < c$, ρ is an irreducible unitary cuspidal representation of $GL(m_\rho, F)$ and σ is an irreducible cuspidal representation of G_n such that $\nu^{\frac{1}{2}}\rho \rtimes \sigma$ reduces.

1. Preliminaries

Let F be a local nonarchimedean field of characteristic different from 2. Groups that we consider are as follows. As in [Mœglin and Tadić 2002] we fix a tower of symplectic or orthogonal nondegenerate F vector spaces V_n, $n \geq 0$ where n is the Witt index. We denote by G_n the group of isometries of V_n. It has split rank n. Also we fix the set of standard parabolic subgroups in the usual way. Standard parabolic proper subgroups of G_n are in bijection with the set of ordered partitions of positive integers $m \leq n$. Given positive integers $n_1, \dots, n_k$ such that $m = n_1 + \cdots + n_k \leq n$ the corresponding standard parabolic subgroup P_s, $s = (n_1, \dots, n_k)$ has the Levi factor M_s isomorphic to

$$GL(n_1, F) \times \cdots \times GL(n_k, F) \times G_{n-m}.$$

Further, if δ_i is a smooth representation of $GL(n_i, F)$, $i = 1, \dots, k$ and τ a smooth representation of G_{n-m}, denote by $\pi = \delta_1 \otimes \cdots \otimes \delta_k \otimes \tau$ the representation of M_s and by

$$\delta_1 \times \cdots \times \delta_k \rtimes \tau = \mathrm{Ind}_{M_s}^{G_n}(\pi)$$

the representation induced from π using normalized parabolic induction. If σ is a smooth representation of G_n we denote by $\mathrm{r}_s(\sigma) = \mathrm{r}_{M_s}(\sigma) = \mathrm{Jacq}_{M_s}^{G_n}(\sigma)$ the normalized Jacquet module of σ. We have the Frobenius reciprocity

$$\mathrm{Hom}_{G_n}(\sigma, \mathrm{Ind}_{M_s}^{G_n}(\pi)) = \mathrm{Hom}_{M_s}(\mathrm{Jacq}_{M_s}^{G_n}(\sigma), \pi).$$

Let ρ be an irreducible cuspidal unitary representation of $GL(m_\rho, F)$ (this defines m_ρ) and $x, y \in \mathbb{R}$, such that $y - x + 1 \in \mathbb{Z}_{\geq 0}$. The set $[\nu^x \rho, \nu^y \rho] = \{\nu^x \rho, \dots, \nu^y \rho\}$ is called a segment. The induced representation $\nu^y \rho \times \cdots \times \nu^x \rho$ has the unique irreducible subrepresentation; it is essentially square integrable, and we denote it by $\delta([\nu^x\rho, \nu^y\rho])$. We also denote

$$e([\nu^x\rho, \nu^y\rho]) = e(\delta([\nu^x\rho, \nu^y\rho]) = \frac{x+y}{2}.$$

For $y-x+1 \in \mathbb{Z}_{<0}$ define $[\nu^x\rho, \nu^y\rho] = \varnothing$ and $\delta(\varnothing)$ is the irreducible representation of the trivial group. Let $\Delta = [\nu^x\rho, \nu^y\rho]$ and $\widetilde{\Delta} = [\nu^{-y}\tilde{\rho}, \nu^{-x}\tilde{\rho}]$ where $\tilde{\rho}$ denotes the contragredient of ρ. We have $\delta(\Delta)\tilde{} = \delta(\widetilde{\Delta})$. By [Zelevinsky 1980] if δ is an essentially square integrable representation of $GL(m_\delta, F)$, there exists a segment Δ such that $\delta = \delta(\Delta)$. If Δ' and Δ'' are segments such that $\Delta'' \subseteq \Delta'$ then $\delta(\Delta') \times \delta(\Delta'')$ is irreducible and $\delta(\Delta') \times \delta(\Delta'') \cong \delta(\Delta'') \times \delta(\Delta')$.

Given a sequence of segments $\Delta_1, \dots, \Delta_k$, $e(\Delta_i) > 0$, $i = 1, \dots, k$ and an irreducible tempered representation τ of some $G_{n'}$, we denote by

$$\mathrm{Lang}(\delta(\Delta_1) \times \cdots \times \delta(\Delta_k) \rtimes \tau)$$

the unique irreducible quotient, called the Langlands quotient, of

$$\delta(\Delta_{\varphi(1)}) \times \cdots \times \delta(\Delta_{\varphi(k)}) \rtimes \tau,$$

where φ is a permutation of the set $\{1, \dots, k\}$ such that $e(\Delta_{\varphi(1)}) \geq \cdots \geq e(\Delta_{\varphi(k)})$. It appears with multiplicity 1 in the induced representation and is the unique irreducible subrepresentation of $\delta(\widetilde{\Delta}_{\varphi(1)}) \times \cdots \times \delta(\widetilde{\Delta}_{\varphi(k)}) \rtimes \tau$. By the Langlands classification every irreducible representation of G_n can be written as a Langlands quotient.

If σ is a discrete series representation of G_n then by the Mœglin–Tadić classification of discrete series [Mœglin 2002; Mœglin and Tadić 2002] it is described by an admissible triple $(\mathrm{Jord}, \sigma_{\mathrm{cusp}}, \epsilon)$. We note that the classification, written under a natural hypothesis, is now unconditional; see page 3160 of [Matić 2016]. Here Jord is a set of pairs (a, ρ) where ρ is an irreducible self-dual cuspidal representation of $GL(m_\rho, F)$, a is a positive integer of parity depending on ρ and $\delta([\nu^{-(a-1)/2}\rho, \nu^{(a-1)/2}\rho]) \rtimes \sigma$ is irreducible. We write $\mathrm{Jord}_\rho = \{a : (a, \rho) \in \mathrm{Jord}\}$ and for $a \in \mathrm{Jord}_\rho$ let a_- be the largest element of Jord_ρ strictly less than a, if such exists. Next, σ_{cusp} is the unique irreducible cuspidal representation of some $G_{n'}$ such that there exists an irreducible representation π of $GL(m_\pi, F)$ such that $\sigma \hookrightarrow \pi \rtimes \sigma_{\mathrm{cusp}}$. It is called the partial cuspidal support of σ. Finally, ϵ is a function from a subset of $\mathrm{Jord} \cup (\mathrm{Jord} \times \mathrm{Jord})$ into $\{\pm 1\}$. It is defined on a pair $(a, \rho), (a', \rho') \in \mathrm{Jord}$ if and only if $\rho \cong \rho'$ and $a \neq a'$. In such a case we formally denote the value on the pair by $\epsilon(a, \rho)\epsilon(a', \rho)^{-1}$ and it is equal to the product of $\epsilon(a, \rho)$ and $\epsilon(a', \rho)^{-1}$ if they are defined. Suppose that $(a, \rho) \in \mathrm{Jord}$ and a_- is defined. Then

$$\epsilon(a, \rho)\epsilon(a_-, \rho)^{-1} = 1 \iff \text{there exists a representation } \pi' \text{ of some } G_{n_{\pi'}} \text{ such that } \sigma \hookrightarrow \delta([\nu^{(a_-+1)/2}\rho, \nu^{(a-1)/2}\rho]) \rtimes \pi'.$$

If $(a,\rho)\in\mathrm{Jord}$ and a is even then $\epsilon(a,\rho)$ is defined. Additionally, if $a=\min(\mathrm{Jord}_\rho)$,

$$\epsilon(a,\rho)=1 \iff \text{there exists a representation } \pi'' \text{of some } G_{n_{\pi''}} \text{ such that } \sigma \hookrightarrow \delta([\nu^{1/2}\rho, \nu^{(a-1)/2}\rho]) \rtimes \pi''.$$

Now we recall the Tadić formula for computing Jacquet modules. Let $R(G_n)$ be the Grothendieck group of the category of smooth representations of G_n of finite length. It is the free abelian group generated by classes of irreducible representations of G_n. If σ is a smooth finite length representation of G_n denote by $\mathrm{s.s.}(\sigma)$ the semisimplification of σ, that is the sum of classes of composition series of σ. Put $R(G)=\oplus_{n\geq 0}R(G_n)$. For $\pi_1,\pi_2\in R(G)$ we define $\pi_1\leq\pi_2$ if $\pi_2-\pi_1$ is a linear combination of classes of irreducible representations with nonnegative coefficients. Similarly we have $R(GL)=\oplus_{n\geq 0}R(GL(n,F))$. We have the map $\mu^*: R(G)\to R(GL)\otimes R(G)$ defined by

$$\mu^*(\sigma)=1\otimes\sigma+\sum_{k=1}^{n}\mathrm{s.s.}(r_{(k)}(\sigma)),\quad \sigma\in R(G_n).$$

The following result is derived from Theorems 5.4 and 6.5 of [Tadić 1995]; see also Section 1 in [Mœglin and Tadić 2002]. They are based on Bernstein and Zelevinsky's geometrical lemma [1977, Lemma 2.11].

Theorem 1.1. *Let σ be a smooth representation of a finite length of G_n, ρ an irreducible unitary cuspidal representation of $GL(m_\rho,F)$ and $x,y\in\mathbb{R}$, such that $y-x+1\in\mathbb{Z}_{\geq 0}$. Then*

$$(1\text{-}1)\quad \mu^*(\delta([\nu^x\rho,\nu^y\rho])\rtimes\sigma)= \sum_{\delta'\otimes\sigma'\leq\mu^*(\sigma)}\sum_{i=0}^{y-x+1}\sum_{j=0}^{i}\delta([\nu^{i-y}\tilde\rho,\nu^{-x}\tilde\rho])\times\delta([\nu^{y+1-j}\rho,\nu^y\rho]) \times\delta'\otimes\delta([\nu^{y+1-i}\rho,\nu^{y-j}\rho])\rtimes\sigma',$$

where $\delta'\otimes\sigma'$ denotes an irreducible subquotient in the appropriate Jacquet module.

We also note that in the appropriate Grothendieck group

$$(1\text{-}2)\qquad \delta([\nu^x\rho,\nu^y\rho])\rtimes\sigma=\delta([\nu^{-y}\tilde\rho,\nu^{-x}\tilde\rho])\rtimes\sigma.$$

2. Basic reducibilities

In this section we fix the notation and prepare some reducibility results. Let ρ be an irreducible unitary cuspidal representation of $GL(m_\rho,F)$ and σ an irreducible cuspidal representation of G_n such that $\nu^{\frac{1}{2}}\rho\rtimes\sigma$ reduces. By Proposition 2.4 of [Tadić 1998] ρ is self-dual. Let $a,b,c\in\frac{1}{2}(2\mathbb{Z}+1)$ such that $\frac{1}{2}\leq a<b<c$.

The following result is Theorem 2.3 from [Muić 2004] proved using Jacquet module computation.

Theorem 2.1. (i) *The induced representation* $\delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma$ *is of length* 2. *Besides its Langlands quotient it has the unique irreducible subrepresentation, the discrete series* σ_1. *In the appropriate Grothendieck group we have*

$$\delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma = \sigma_1 + \mathrm{Lang}(\delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma).$$

Here $\mathrm{Jord}(\sigma_1) = \{(2a+1, \rho)\}$ *and* $\epsilon_{\sigma_1}(2a+1, \rho) = 1$.

(ii) *The induced representation* $\delta([\nu^{-b}\rho, \nu^c\rho]) \rtimes \sigma$ *is of length* 3. *Besides its Langlands quotient it has two nonisomorphic irreducible subrepresentations* σ_2 *and* σ_3. *In the appropriate Grothendieck group we have*

$$\delta([\nu^{-b}\rho, \nu^c\rho]) \rtimes \sigma = \sigma_2 + \sigma_3 + \mathrm{Lang}(\delta([\nu^{-b}\rho, \nu^c\rho]) \rtimes \sigma).$$

Here,

$$\mathrm{Jord}(\sigma_2) = \mathrm{Jord}(\sigma_3) = \{(2b+1, \rho), (2c+1, \rho)\}$$
$$\epsilon_{\sigma_2}(2b+1, \rho) = \epsilon_{\sigma_2}(2c+1, \rho) = 1$$
$$\epsilon_{\sigma_3}(2b+1, \rho) = \epsilon_{\sigma_3}(2c+1, \rho) = -1.$$

The next proposition follows from Theorem 2.1 of [Muić 2004].

Proposition 2.2. *The induced representation* $\delta([\nu^{-b}\rho, \nu^c\rho]) \rtimes \sigma_1$ *is of length* 3. *Besides its Langlands quotient it has two nonisomorphic irreducible subrepresentations, the discrete series* σ_4 *and* σ_5. *In the appropriate Grothendieck group we have*

$$\delta([\nu^{-b}\rho, \nu^c\rho]) \rtimes \sigma_1 = \sigma_4 + \sigma_5 + \mathrm{Lang}(\delta([\nu^{-b}\rho, \nu^c\rho]) \rtimes \sigma_1).$$

Here,

$$\mathrm{Jord}(\sigma_4) = \mathrm{Jord}(\sigma_5) = \{(2a+1, \rho), (2b+1, \rho), (2c+1, \rho)\},$$
$$\epsilon_{\sigma_4}(2a+1, \rho) = \epsilon_{\sigma_4}(2b+1, \rho) = \epsilon_{\sigma_4}(2c+1, \rho) = 1,$$
$$\epsilon_{\sigma_5}(2a+1, \rho) = 1, \epsilon_{\sigma_5}(2b+1, \rho) = \epsilon_{\sigma_5}(2c+1, \rho) = -1.$$

Proposition 2.3. *The representation* $\delta([\nu^{-b}\rho, \nu^c\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma$ *has two irreducible subrepresentations* σ_4 *and* σ_5 *and they appear with multiplicity* 1.

Proof. By Theorem 2.1 and Proposition 2.2 we have

$$\sigma_4 \oplus \sigma_5 \hookrightarrow \delta([\nu^{-b}\rho, \nu^c\rho]) \rtimes \sigma_1 \hookrightarrow \delta([\nu^{-b}\rho, \nu^c\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma.$$

To see that there are no other irreducible subrepresentations let

$$\pi \hookrightarrow \delta([\nu^{-b}\rho, \nu^c\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma$$

be an irreducible subrepresentation. Frobenius reciprocity implies

$$\mu^*(\pi) \geq \delta([\nu^{-b}\rho, \nu^c\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \otimes \sigma.$$

We show that $\delta([\nu^{-b}\rho, \nu^{c}\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \otimes \sigma$ appears with multiplicity 2 in $\mu^*(\delta([\nu^{-b}\rho, \nu^{c}\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \rtimes \sigma)$. Looking for possible occurrences, formula (1-1) implies that there exist $i, j, k, l \in \mathbb{Z}$ such that $0 \leq l \leq k \leq a + \frac{1}{2}$, $0 \leq j \leq i \leq b+c+1$ and

$$\begin{aligned}\delta([\nu^{-b}\rho, \nu^{c}\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \leq \delta([\nu^{k-a}\rho, \nu^{-\frac{1}{2}}\rho]) \\ \times \delta([\nu^{a+1-l}\rho, \nu^{a}\rho]) \times \delta([\nu^{i-c}\rho, \nu^{b}\rho]) \times \delta([\nu^{c+1-j}\rho, \nu^{c}\rho]), \\ \sigma \leq \delta([\nu^{a+1-k}\rho, \nu^{a-l}\rho]) \times \delta([\nu^{c+1-i}\rho, \nu^{c-j}\rho]) \rtimes \sigma.\end{aligned}$$

Comparing cuspidal support in the first equation we see $i-c=-b$ or $c+1-j=-b$. The second inequality implies $k = l$ and $i = j$. So we have $i = j = c - b$ or $i = j = c + b + 1$. Now $k = l = a + \frac{1}{2}$. This shows that there are at most two irreducible subrepresentations in $\delta([\nu^{-b}\rho, \nu^{c}\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \otimes \sigma$, so there are no others than σ_4 and σ_5. □

Proposition 2.4. *In the appropriate Grothendieck group we have*

$$\begin{aligned}\delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \rtimes \sigma_2 = \sigma_4 + \mathrm{Lang}(\delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \rtimes \sigma_2), \\ \delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \rtimes \sigma_3 = \sigma_5 + \mathrm{Lang}(\delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \rtimes \sigma_3).\end{aligned}$$

Proof. By Lemma 6.1 of [Muić 2005] the induced representations on the left side of the equations reduce. The proof of that lemma claims that all irreducible subquotients of the induced representations other than Langlands quotients are discrete series. The argument as in the proof of Theorem 2.1 of [Muić 2004] implies that they are all subrepresentations.

Let π_4 be a discrete series subrepresentation of $\delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \rtimes \sigma_2$ and π_5 a discrete series subrepresentation of $\delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \rtimes \sigma_3$. By Theorem 2.1, $\sigma_2 \oplus \sigma_3 \hookrightarrow \delta([\nu^{-b}\rho, \nu^{c}\rho]) \rtimes \sigma$ so we have

$$\begin{aligned}\pi_4 \oplus \pi_5 &\hookrightarrow \delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \rtimes \sigma_2 \oplus \delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \rtimes \sigma_3 \\ &\cong \delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \rtimes (\sigma_2 \oplus \sigma_3) \\ &\hookrightarrow \delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \times \delta([\nu^{-b}\rho, \nu^{c}\rho]) \rtimes \sigma \\ &\cong \delta([\nu^{-b}\rho, \nu^{c}\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \rtimes \sigma.\end{aligned} \tag{2-1}$$

By Proposition 2.3 π_4 and π_5 are not isomorphic and we have

$$\delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \rtimes \sigma_2 = \pi_4 + \mathrm{Lang}(\delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \rtimes \sigma_2), \tag{2-2}$$

$$\delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \rtimes \sigma_3 = \pi_5 + \mathrm{Lang}(\delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \rtimes \sigma_3), \tag{2-3}$$

where $\{\pi_4, \pi_5\} = \{\sigma_4, \sigma_5\}$.

We now prove that $\pi_4 = \sigma_4$ and $\pi_5 = \sigma_5$. It is enough to see that

$$\epsilon_{\pi_4}(2a+1, \rho)\epsilon_{\pi_4}(2b+1, \rho)^{-1} = 1.$$

Since $\epsilon_{\sigma_2}(2b+1, \rho) = 1$ and $\min(\mathrm{Jord}_\rho(\sigma_2)) = 2b+1 \in 2\mathbb{Z}$ there exists an irreducible representation τ of $G_{n+(c+\frac{1}{2})m_\rho}$ such that $\sigma_2 \hookrightarrow \delta([\nu^{\frac{1}{2}}\rho, \nu^b\rho]) \rtimes \tau$. Now we have

$$\begin{aligned}\pi_4 &\hookrightarrow \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma_2 \hookrightarrow \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^b\rho]) \rtimes \tau \\ &\cong \delta([\nu^{\frac{1}{2}}\rho, \nu^b\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \tau \\ &\hookrightarrow \delta([\nu^{a+1}\rho, \nu^b\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \tau.\end{aligned}$$

By Lemma 3.2 of [Mœglin and Tadić 2002] there exists an irreducible representation τ' of $G_{n+(2a+c+\frac{3}{2})m_\rho}$ such that

$$\pi_4 \hookrightarrow \delta([\nu^{a+1}\rho, \nu^b\rho]) \rtimes \tau'.$$

Now $\epsilon_{\pi_4}(2a+1, \rho)\epsilon_{\pi_4}(2b+1, \rho)^{-1} = 1$. As we proved that $\pi_4 = \sigma_4$ and $\pi_5 = \sigma_5$, (2-2) and (2-3) give the claim of the proposition. □

3. The main theorem

Theorem 3.1. *The induced representation $\delta([\nu^{-b}\rho, \nu^c\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma$ is of length* 6, *and it has two nonisomorphic irreducible subrepresentations. They are discrete series. In the appropriate Grothendieck group we have*

$$\begin{aligned}&\delta([\nu^{-b}\rho, \nu^c\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma = \\ &\quad \sigma_4 + \sigma_5 + \mathrm{Lang}(\delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma_2) + \mathrm{Lang}(\delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma_3) \\ &\quad + \mathrm{Lang}(\delta([\nu^{-b}\rho, \nu^c\rho]) \rtimes \sigma_1) \\ &\quad + \mathrm{Lang}(\delta([\nu^{-b}\rho, \nu^c\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma).\end{aligned}$$

Moreover,

$$\begin{aligned}&\mathrm{Lang}(\delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma_2) \oplus \mathrm{Lang}(\delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma_3) \oplus \mathrm{Lang}(\delta([\nu^{-b}\rho, \nu^c\rho]) \rtimes \sigma_1) \\ &\quad \hookrightarrow (\delta([\nu^{-b}\rho, \nu^c\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma)/(\sigma_4 \oplus \sigma_5).\end{aligned}$$

Proof. Suppose that $-b + c \geq \frac{1}{2} + a$. Otherwise we have a similar proof. We look at the composition of some intertwining operators:

$$\begin{aligned}\delta([\nu^{-b}\rho, \nu^c\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma &\to \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \times \delta([\nu^{-b}\rho, \nu^c\rho]) \rtimes \sigma \\ &\to \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \times \delta([\nu^{-c}\rho, \nu^b\rho]) \rtimes \sigma \\ &\to \delta([\nu^{-c}\rho, \nu^b\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma \\ &\to \delta([\nu^{-c}\rho, \nu^b\rho]) \times \delta([\nu^{-a}\rho, \nu^{-\frac{1}{2}}\rho]) \rtimes \sigma.\end{aligned}$$

Since $\frac{1}{2} \leq a < b < c$ the first and the third map are isomorphisms. By Theorem 2.1 the kernel of the second map is in the appropriate Grothendieck group

$$\delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \rtimes \sigma_2 + \delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \rtimes \sigma_3.$$

By Proposition 2.4 this equals

$$\sigma_4 + \sigma_5 + \mathrm{Lang}(\delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \rtimes \sigma_2) + \mathrm{Lang}(\delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \rtimes \sigma_3).$$

By Theorem 2.1 and equation (1-2), the kernel of the last map is in the appropriate Grothendieck group

$$\delta([\nu^{-c}\rho, \nu^{b}\rho]) \rtimes \sigma_1 = \delta([\nu^{-b}\rho, \nu^{c}\rho]) \rtimes \sigma_1$$

which is, by Proposition 2.2, equal to

$$\sigma_4 + \sigma_5 + \mathrm{Lang}(\delta([\nu^{-b}\rho, \nu^{c}\rho]) \rtimes \sigma_1).$$

The image of the composition is

$$\mathrm{Lang}(\delta([\nu^{-b}\rho, \nu^{c}\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \rtimes \sigma).$$

We see that σ_4 and σ_5 appear in two kernels, but by Proposition 2.3 they appear with multiplicity 1 in the induced representation, so we have proved the first formula of the theorem.

To prove the second formula of the theorem, observe that by Theorem 2.1 and Propositions 2.2 and 2.3 we have

$$\sigma_4 \oplus \sigma_5 \hookrightarrow \delta([\nu^{-b}\rho, \nu^{c}\rho]) \rtimes \sigma_1 \hookrightarrow \delta([\nu^{-b}\rho, \nu^{c}\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \rtimes \sigma$$

and

$$\text{(3-1)} \quad \mathrm{Lang}(\delta([\nu^{-b}\rho, \nu^{c}\rho]) \rtimes \sigma_1) \hookrightarrow (\delta([\nu^{-b}\rho, \nu^{c}\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \rtimes \sigma)/(\sigma_4 \oplus \sigma_5).$$

Additionally, Proposition 2.4 and (2-1) imply

$$\begin{aligned}\sigma_4 \oplus \sigma_5 &\hookrightarrow \delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \rtimes \sigma_2 \oplus \delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \rtimes \sigma_3 \\ &\hookrightarrow \delta([\nu^{-b}\rho, \nu^{c}\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \rtimes \sigma\end{aligned}$$

and

$$\text{(3-2)} \quad \mathrm{Lang}(\delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \rtimes \sigma_2) \oplus \mathrm{Lang}(\delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \rtimes \sigma_3) \hookrightarrow (\delta([\nu^{-b}\rho, \nu^{c}\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \rtimes \sigma)/(\sigma_4 \oplus \sigma_5).$$

Now equations (3-1) and (3-2) prove the second formula of the theorem. □

4. Consequences

We have the following result:

Corollary 4.1. *In the appropriate Grothendieck group we have*

$$\delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \rtimes \mathrm{Lang}(\delta([\nu^{-b}\rho, \nu^{c}\rho]) \rtimes \sigma) = \mathrm{Lang}(\delta([\nu^{-b}\rho, \nu^{c}\rho]) \rtimes \sigma_1) + \mathrm{Lang}(\delta([\nu^{-b}\rho, \nu^{c}\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \rtimes \sigma),$$

$$\delta([\nu^{-b}\rho, \nu^{c}\rho]) \rtimes \mathrm{Lang}(\delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \rtimes \sigma) = \mathrm{Lang}(\delta([\nu^{-b}\rho, \nu^{c}\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \rtimes \sigma) + \mathrm{Lang}(\delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \rtimes \sigma_2) + \mathrm{Lang}(\delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \rtimes \sigma_3).$$

Except for $\mathrm{Lang}(\delta([\nu^{-b}\rho, \nu^{c}\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \rtimes \sigma)$ *all irreducible subquotients of induced representations on the left-hand side appear as subrepresentations.*

Proof. Using the exactness of the parabolic induction, Theorem 2.1, Proposition 2.4, (2-1) and Theorem 3.1 we have

$$\begin{aligned}
&\delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \rtimes \mathrm{Lang}(\delta([\nu^{-b}\rho, \nu^{c}\rho]) \rtimes \sigma) \\
&\cong (\delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \times \delta([\nu^{-b}\rho, \nu^{c}\rho]) \rtimes \sigma)/(\delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \rtimes (\sigma_2 \oplus \sigma_3)) \\
&\cong (\delta([\nu^{-b}\rho, \nu^{c}\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \rtimes \sigma)/(\delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \rtimes \sigma_2 \oplus \delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \rtimes \sigma_3).
\end{aligned}$$

Comparing this with the result of the main theorem gives the first formula of the corollary. Similarly, for the second formula use Proposition 2.2 and observe that

$$\delta([\nu^{-b}\rho, \nu^{c}\rho]) \rtimes \mathrm{Lang}(\delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \rtimes \sigma) \cong (\delta([\nu^{-b}\rho, \nu^{c}\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \rtimes \sigma)/(\delta([\nu^{-b}\rho, \nu^{c}\rho]) \rtimes \sigma_1). \quad \square$$

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IGOR CIGANOVIĆ
DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
UNIVERSITY OF ZAGREB
ZAGREB
CROATIA
igor.ciganovic@math.hr

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