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HIGGS BUNDLES OVER CELL COMPLEXES AND REPRESENTATIONS OF FINITELY PRESENTED GROUPS

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The purpose of this paper is to extend the Donaldson–Corlette theorem to the case of vector bundles over cell complexes. We define the notions of a vector bundle and a Higgs bundle over a complex, and describe the associated Betti, de Rham and Higgs moduli spaces. The main theorem is that the $\mathrm{SL}(r, \mathbb{C})$ character variety of a finitely presented group Γ is homeomorphic to the moduli space of rank-r Higgs bundles over an admissible complex X with $\pi_1(X) = \Gamma$. A key role is played by the theory of harmonic maps defined on singular domains.

1. Introduction

Higgs bundles were first introduced by Hitchin [1987] as a PDE on a vector bundle over a Riemann surface obtained by the dimensional reduction of the anti-self-dual equations on $\mathbb{R}^4$. Since then, the field has seen a remarkable explosion in different directions, most notably the work of Simpson [1988; 1992] on variations of Hodge structures and applications to Kähler groups. The work of Donaldson [1987] and Corlette [1988] provided links with the theory of flat bundles and character varieties of groups. Higgs bundles have been generalised over noncompact manifolds [Corlette and Simpson 2008; Simpson 1990; Jost et al. 2007; Jost and Zuo 1996] and singular curves [Balaji et al. 2013]. The goal of this paper is to push this even further by considering Higgs bundles over more general singular spaces; namely, finite simplicial complexes.

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As pointed out by Hitchin, Donaldson and Corlette, a key role in the relation between character varieties and Higgs bundles is played by the theory of harmonic maps. Harmonic maps have been used in the study of representations of Kähler manifold groups starting with the work of Siu [1980], also see [Carlson and Toledo 1989], and have seen some remarkable applications in providing new proofs of the celebrated Margulis superrigidity theorem, see [Jost 1997], and the only existing proof of the rank-1 superrigidity theorem due to Corlette [1992] and Gromov and Schoen [1992]. But these directions involved showing that the representations are rigid, in contrast with Hitchin's point of view, which is to study the moduli space of such representations.

In all the above references, one studies representations of fundamental groups of smooth manifolds rather than arbitrary finitely presented groups. In other words, the domain space of the harmonic map is smooth. Chen [1995] and Eells and Fuglede [2001] developed the theory of harmonic maps from a certain class of singular domains including admissible simplicial complexes. By admissible they mean complexes that are dimensionally homogeneous and locally chainable in order to avoid certain analytic pathologies (see the next section for precise definitions). Since any finitely presented group is the fundamental group of an admissible complex, there is no real restriction in considering admissible complexes. The key property of harmonic maps shown in the above references is that they are Hölder continuous but in general they fail to be Lipschitz. In fact, the work of the first two authors [Daskalopoulos and Mese 2008; 2009] shows that Lipschitz harmonic maps often imply that the representations are rigid.

The starting point of this paper is a finitely presented group Γ and a 2-dimensional admissible complex without boundary X with fundamental group $\pi_1(X) \simeq \Gamma$. We also fix a piecewise-smooth vector bundle E over X that admits a flat $\mathrm{SL}(r, \mathbb{C})$ structure. Such bundles are parametrised topologically by the (finitely many) connected components of the $\mathrm{SL}(r, \mathbb{C})$ character variety of $\pi_1(X)$. One can write down Hitchin's equations

$$F_A + \psi \wedge \psi = 0, \tag{1-1}$$

$$d_A \psi = 0 \tag{1-2}$$

for a sufficiently regular unitary connection A and Higgs field ψ. Again, as in the smooth case, the $\mathrm{SL}(r, \mathbb{C})$ connection $d_A + \psi$ is flat and one can ask what the precise condition is so that the pair (d_A, ψ) corresponds to a representation $\rho : \pi_1(X) \to \mathrm{SL}(r, \mathbb{C})$.

Given a representation ρ as above, we can associate as in the smooth case a ρ-equivariant harmonic map from the universal cover $\widetilde{X}$ to the symmetric space $\mathrm{SL}(r, \mathbb{C})/\mathrm{SU}(r)$. The first two authors [Daskalopoulos and Mese 2008] studied harmonic maps from simplicial complexes to smooth manifolds and discovered the following crucial properties:

(1) The harmonic map is smooth away from the codimension-2 skeleton of $\widetilde{X}$.

(2) The harmonic map satisfies a balancing condition at the codimension-1 skeleton of $\widetilde{X}$ in the sense that the sum of the normal derivatives vanishes identically.

(3) The harmonic map blows up in a controlled way at the codimension-2 skeleton of $\widetilde{X}$.

All the above properties are described precisely in Theorem 3.3. This allows us to prove that the derivative of the harmonic map belongs in an appropriate weighted Sobolev space $L^2_{1,\delta}$ (see Proposition 4.5). The definition of weighted Sobolev spaces is given in Section 3B. Finally, the main theorem describing the correspondence between equivalence classes of balanced Higgs pairs of class $L^2_{1,\delta}$ and representations is given in Section 4 (see Theorem 4.3).

We would like to end this introduction with a brief discussion of some motivation and future applications of this paper that we will explore elsewhere. Note that, with the exception of [Balaji et al. 2013], the theory of Higgs bundles on singular varieties is not very well understood. For example, one of the important questions about fundamental groups of singular projective varieties is whether fundamental groups of normal varieties behave more like the ones of smooth manifolds, or in the other extreme, if there are very few restrictions on them [Arapura et al. 2016; Kapovich and Kollár 2014]. The connection with the results of this paper is as follows: By [Eells and Fuglede 2001, Example 8.9], an n-dimensional normal projective variety X admits a bi-Lipschitz triangulation with its singular set as a subcomplex of dimension at most $n-2$. Furthermore, X is admissible in the sense of Definition 2.2. Thus, studying harmonic maps on X, or more generally constructing moduli spaces of bundles on X, could imply restrictions on fundamental groups as in [Carlson and Toledo 1989; Simpson 1992].

2. Vector bundles over complexes

2A. *Basic definitions of smooth bundles.*

Definition 2.1 [Lojasiewicz 1964]. Let $\mathbb{E}^N$ be an N-dimensional affine space. A *cell* of dimension i is a nonempty, open, convex, bounded subset in some $\mathbb{E}^i \subset \mathbb{E}^N$. We will use the notation σ^i to denote a cell of dimension i and call $\mathbb{E}^i$ the *extended plane* defined by σ^i. A *locally finite convex cell complex*, or simply a *complex* X in $\mathbb{E}^N$, is a locally a finite collection $\mathcal{F} = \{\sigma\}$ of disjoint cells in $\mathbb{E}^N$ such that for any $\sigma \in \mathcal{F}$ its closure $\bar{\sigma}$ is a union of cells in $\mathcal{F}$. *The dimension of a complex* X is the maximum dimension of a cell in X.

For example, a simplicial complex is a cell complex whose cells are all simplices.

Definition 2.2. A connected complex X of dimension n is said to be admissible [Chen 1995; Eells and Fuglede 2001] if the following two conditions hold:

(i) X is dimensionally homogeneous, i.e., every cell is contained in a closure of at least one n-cell, and

(ii) X is locally $(n-1)$-chainable, i.e., given any $(n-2)$-cell v, every two n-cells σ and σ' incident to v can be joined by a sequence $\sigma = \sigma_0, \sigma_1, \dots, \sigma_k = \sigma'$ where σ_i and σ_{i+1} are two adjacent n-cells incident to v for $i = 0, 1, \dots, k-1$.

The boundary ∂X of X is the union of the closures of the $(n-1)$-cells contained in the closure of exactly one n-cell. Using a regular barycentric subdivision we obtain that given any locally finite complex there is a locally finite simplicial complex such that any cell is a union of simplices.

Definition 2.3. Let U be a subset of a complex X. A function $f : U \to \mathbb{R}$ is called *smooth* if for any n-cell σ of X, the restriction $f|_{\sigma \cap U}$ can be extended to a smooth function on $\mathbb{E}^i \cap U$ in the extended plane defined by σ. A map $f : U \to Z \subset \mathbb{E}^M$ into a complex Z is called *smooth* if with respect to some affine coordinate system on $\mathbb{E}^M$ we have $f = (f^1, \dots, f^M)$ where f^j is smooth for every $j = 1, \dots, M$.

Definition 2.4. A Riemannian metric g_σ on a cell σ is the restriction to σ of a smooth Riemannian metric on its extended plane. A Riemannian metric g on X is a smooth Riemannian metric g_σ on each n-cell σ of X satisfying the additional property that if τ is a face of σ, then $g_\sigma|_\tau = g_\tau$, where $g_\sigma|_\tau$ denotes the restriction of the extension of g_σ to the extended plane of σ. In particular, the expressions of g_σ with respect to some affine coordinates in the extended plane are smooth functions in the sense of Definition 2.3.

Definition 2.5. A *smooth complex vector bundle of rank r* over a complex X is a topological space E and a continuous, surjective map $\pi : E \to X$ such that:

(1) for each $x \in X$ the fibre $\pi^{-1}(x)$ has the structure of a complex vector space, and

(2) there exists an open cover $\{U_\alpha\}_{\alpha \in I}$ of X such that for each $\alpha \in I$ there exists a homeomorphism $\varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}^r$ such that

  (i) φ_α restricts to a linear isomorphism $\pi^{-1}(x) \cong \{x\} \times \mathbb{C}^r$ for each $x \in U_\alpha$, and

  (ii) if $U_\alpha \cap U_\beta \neq \varnothing$, then the transition function $g_{\alpha\beta} = \varphi_\alpha \circ \varphi_\beta^{-1} : U_\alpha \cap U_\beta \times \mathbb{C}^r \to U_\alpha \cap U_\beta \times \mathbb{C}^r$ induces a smooth map $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(r, \mathbb{C})$.

A *section* of $\pi : E \to X$ is a continuous map $s : X \to E$ satisfying $\pi \circ s = \mathrm{id}_X$. The section is *smooth* if on each local trivialisation $\pi^{-1}(U_\alpha) \cong U_\alpha \times \mathbb{C}^r$ with projection onto the second factor denoted by $p_2 : \pi^{-1}(U_\alpha) \to \mathbb{C}^r$, the composition of $U_\alpha \xrightarrow{s} \pi^{-1}(U_\alpha) \xrightarrow{p_2} \mathbb{C}^r$ is a smooth map as in Definition 2.3. Let $\Omega^0(X, E)$ denote the vector space of all smooth sections of $\pi : E \to X$. If E is a smooth vector bundle, then so is any associated bundle formed by taking the dual, tensor product, etc. In particular, if E is smooth then $\mathrm{End}(E)$ is smooth.

Definition 2.6. A smooth complex p-form on a cell σ is the restriction to σ of a smooth complex p form on the extended plane of the cell. A *smooth p-form* $\omega=\{\omega_\sigma\}_{\sigma\in\mathcal{F}}$ on a complex X with values in a smooth vector bundle E is a collection of smooth p-forms ω_σ with values in E for each cell σ of X, with the additional property that if τ is a face of σ, then $\omega_\sigma|_\tau = \omega_\tau$. In particular, the expressions of ω_σ with respect to some affine coordinates in the extended plane are smooth functions in the sense of Definition 2.3. We define $\Omega^p(X, E)$ as the space of all smooth p-forms with values in E. If E is the trivial line bundle, then we write $\Omega^p(X) = \Omega^p(X, E)$ and this is the space of smooth p-forms on X. Given a smooth p-form $\omega = \{\omega_\sigma\}_{\sigma\in\mathcal{F}} \in \Omega^p(X)$, we define $d\omega = \{d\omega_\sigma\}_{\sigma\in\mathcal{F}}$ and note that this is a well-defined smooth $(p+1)$-form. Clearly, $d^2 = 0$ and the complex $(\Omega^*(X), d)$ denotes the smooth de Rham complex. We denote by $H^p_{\mathrm{dR}}(X)$ the cohomology groups associated with this complex; see [Griffiths and Morgan 1981, Chapter VIII].

Definition 2.7. A *smooth connection* on a smooth vector bundle $\pi : E \to X$ is a $\mathbb{C}$-linear map $D : \Omega^0(X, E) \to \Omega^1(X, E)$ that satisfies the Leibniz rule

$$D(fs) = (df)s + f(Ds), \quad f \in \Omega^0(X), \quad s \in \Omega^0(X, E).$$

We denote the space of all smooth connections by $\mathcal{A}^{\mathbb{C}}(E)$.

The definition of D can be extended to bundle-valued forms in the usual way. More precisely, any element in $\sigma \in \Omega^p(X, E)$ can be written as a linear combination of elements of the form $\sigma = s\omega$ with $\omega \in \Omega^p(X)$ and $s \in \Omega^0(X, E)$, and define

$$D\sigma = s(d\omega) + (Ds) \wedge \omega. \tag{2-1}$$

Remark 2.8. Implicit in the definition of $\Omega^1(X, E)$ is that 1-forms with values in E must agree on the interfaces between the cells in the complex X. Therefore, the definition above implies that a connection must map sections that agree on the interfaces to bundle-valued 1-forms that agree on the interfaces.

As for the case of a smooth vector bundle over a smooth manifold, with respect to a trivialization, $\varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}^r$, $D = d + A_\alpha$, where $(A_\alpha)_{ij}$ is a complex-valued smooth 1-form. A_α is called the *connection form* of D with respect to the trivialization φ_α. In a different trivialization φ_β and with $g_{\alpha\beta} = \varphi_\alpha \circ \varphi_\beta^{-1}$ we have,

$$A_\beta = g_{\alpha\beta}^{-1}\, dg_{\alpha\beta} + g_{\alpha\beta}^{-1} A_\alpha g_{\alpha\beta}. \tag{2-2}$$

Definition 2.9. The *curvature* of a smooth connection D is the matrix-valued 2-form F_D defined by

$$D^2 s = F_D s \quad \text{for all } s \in \Omega^0(X, E).$$

Locally, we have $(F_D)_\alpha = dA_\alpha + A_\alpha \wedge A_\alpha$, where A_α is the connection form of D. Furthermore,

$$(F_D)_\beta = g_{\alpha\beta}^{-1}(F_D)_\alpha g_{\alpha\beta}, \tag{2-3}$$

and so the curvature form F_D is an element of $\Omega^2(X, \mathrm{End}(E))$.

Definition 2.10. The *complex gauge group* is the group $\mathcal{G}^{\mathbb{C}}(E)$ of all smooth automorphisms of E. If D is a smooth connection on E and $g \in \mathcal{G}^{\mathbb{C}}(E)$, then we define $g(D) = g^{-1} \circ D \circ g$. In local coordinates, the action of $\mathcal{G}^{\mathbb{C}}(E)$ on $\mathcal{A}^{\mathbb{C}}(E)$ is

$$g(d + A_\alpha) = d + g^{-1}\, dg + g^{-1} A_\alpha g. \tag{2-4}$$

Definition 2.11. A *smooth Hermitian metric* $h = (h_\sigma)$ on a rank-r complex vector bundle $\pi : E \to X$ is a smooth section h of $\mathrm{End}(E)$ such that for each cell σ its restriction h_σ is a Hermitian metric and if τ is a face of σ, then $h_\sigma|_\tau = h_\tau$. A Hermitian metric in a trivialization $\varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}^r$ is given locally by a smooth map $\tilde{h}_\alpha$ from U_α into the positive definite matrices in $\mathrm{GL}(r, \mathbb{C})$, and the induced inner product on the fibres of E is

$$\langle s_1(x), s_2(x)\rangle = \overline{\varphi_\alpha(s_1(x))}^T\, \tilde{h}_\alpha(x)\, \varphi_\alpha(s_2(x)) \in \mathbb{C}.$$

Definition 2.12. A connection D on a vector bundle E with a Hermitian metric h is a *unitary connection* if the following equation is satisfied:

$$d\langle s_1, s_2\rangle = \langle Ds_1, s_2\rangle + \langle s_1, Ds_2\rangle,$$

where $\langle\,\cdot\,,\cdot\,\rangle$ is the pointwise inner product on the fibres of E induced by the metric h. The space of smooth unitary connections on E is denoted by $\mathcal{A}(E, h)$. If $D \in \mathcal{A}(E, h)$, then the curvature F_D is a section of $\Omega^2(\mathrm{ad}(E))$. In other words, with respect to a unitary frame field the curvature satisfies $F_D^* = -F_D$.

Definition 2.13. The *unitary gauge group* $\mathcal{G}(E)$ is the subgroup of $\mathcal{G}^{\mathbb{C}}(E)$ that preserves the Hermitian metric h on each fibre of E. The action on $\mathcal{G}(E)$ on $\mathcal{A}^{\mathbb{C}}(E)$ preserves the space $\mathcal{A}(E, h)$.

Definition 2.14. A connection D on a vector bundle E is *flat* if $F_D = 0$. Given a flat connection, we can define the twisted de Rham complex $(\Omega^*(X, E), D)$. The cohomology groups will be denoted by $H^p(X, E)$.

Definition 2.15. A *flat structure* on a vector bundle $\pi : E \to X$ is given by an open cover $\{U_\alpha\}_{\alpha\in I}$ and trivialisations $\{\varphi_\alpha\}_{\alpha\in I}$ for which the transition functions $g_{\alpha\beta} = \varphi_\alpha \circ \varphi_\beta^{-1}$ are constant. A vector bundle with a flat structure is also called a *flat bundle*.

Remark 2.16. Equation (2-2) shows that the connection $D = d$ (with zero connection form) is globally defined on a flat bundle. Thus a flat bundle clearly admits a connection of curvature zero. The converse is also true.

Theorem 2.17. *Let X be n-complex, U an open subset of X and E a smooth vector bundle with a smooth flat connection on U. Then E admits a flat structure.*

Proof. Given a flat connection D on E, fix a cell σ, a point $x_0 \in \bar{\sigma} \cap U$ and consider a contractible neighbourhood V_σ of x_0 in the extended plane of σ. Choose a local frame s^0_σ of E on V_σ and let A^σ be the corresponding connection form. We are assuming that the local frames s^0_σ patch together to define a piecewise smooth frame s_0 in a neighbourhood of x_0 in X. We are going to choose a different trivialisation s_σ for which the connection can be written as $D = d$. This can be done by solving the equation

$$g_\sigma^{-1} A^\sigma g_\sigma + g_\sigma^{-1} dg_\sigma = 0 \quad \Longleftrightarrow \quad dg_\sigma = -A^\sigma g_\sigma \tag{2-5}$$

locally for a gauge transformation g_σ. By the result in the smooth case (this is an application of the Frobenius theorem) a solution g_σ exists and by multiplying by a constant matrix we may assume without loss of generality that $g_\sigma(x_0) = \mathrm{id}$. This makes the solution unique and thus if a cell τ is a face of a cell σ then, since $A^\sigma|_\tau = A^\tau$, it must be $g_\sigma|_\tau = g_\tau$. It follows that the new frames $s_\sigma = g_\sigma \circ s^0_\sigma$ patch together to define a piecewise smooth frame s in a neighbourhood of x_0 in X. The flat structure is now defined by the local frames $\{s\}$. □

Definition 2.18. A section $s \in \Omega^0(X, E)$ is *parallel with respect to D* if $Ds = 0$. Given a smooth curve $c : [a, b] \to X$, a section s is *parallel along c with respect to D* if $D_{c'(t)} s = 0$. Given a curve $c : [a, b] \to X$ and $s_a \in \pi^{-1}(c(a))$ the *parallel transport of s along c with respect to D* is the section $s : \pi^{-1}(c([a, b])) \to E$ which is given locally by the solution to the equation

$$\frac{ds(c(t))}{dt} + A_{c(t)}(c'(t))s(c(t)) = 0.$$

Lemma 2.19. *Let $c_1, c_2 : [a, b] \to X$ be two closed smooth curves in X which are homotopy equivalent, and which satisfy $x_0 = c_1(a) = c_1(b) = c_2(a) = c_2(b)$. Let D be a smooth flat connection on a rank-r bundle $\pi : E \to X$, and let s_1 and s_2 be the parallel transport with respect to D along c_1 and c_2 respectively, with initial condition $s_0 \in \pi^{-1}(x_0)$. If $F_D = 0$ then $s_1(c_1(b)) = s_2(c_2(b))$.*

Proof. As usual, note that it suffices to show that the holonomy is trivial around a homotopically trivial loop. If there is a homotopy equivalence between two loops that is constant except on a single cell, then standard theorems for smooth manifolds show that the holonomy around the two loops is the same. Given a homotopically trivial loop γ, there is a sequence of homotopy equivalences $\gamma \sim \gamma_1,\ \gamma_1 \sim \gamma_2,\ \ldots,$

$\gamma_N \sim \mathrm{id}$ between γ and the trivial loop (denoted id), such that each homotopy equivalence is constant except on a single n-cell. For example, one can do this by identifying the fundamental group with the edge group of a simplicial complex; see [Armstrong 1983, Section 6.4]. Therefore, the holonomy of γ is the same as the holonomy of each γ_n along this sequence of homotopy equivalences, and so the holonomy of γ is trivial. □

Definition 2.20. A flat connection D on a rank-r vector bundle $\pi : E \to X$ defines a representation $\rho : \pi_1(X) \to \mathrm{GL}(r, \mathbb{C})$ called the *holonomy representation of* D. A flat connection is called *irreducible* if its holonomy representation is irreducible. The space of irreducible, flat smooth connections is denoted by $\mathcal{A}^{\mathbb{C},\mathrm{irr}}(E)$.

Lemma 2.21. *A representation $\rho : \pi_1(X) \to \mathrm{GL}(r, \mathbb{C})$ defines a flat connection on a bundle $\pi : E_\rho \to X$ with holonomy representation ρ. Moreover, the flat connection on E_ρ depends continuously on the representation ρ.*

Proof. In the standard way, from a representation $\rho : \pi_1(X) \to \mathrm{GL}(r, \mathbb{C})$ we construct a flat vector bundle $E_\rho \to X$, with total space

$$E_\rho = \widetilde{X} \times_\rho \mathbb{C}^r, \tag{2-6}$$

where $\widetilde{X}$ is the universal cover of X, and the equivalence is by deck transformations on the left factor $\widetilde{X}$, and via the representation ρ on the right factor $\mathbb{C}^r$. On each trivialisation we have the trivial connection d, and since the transition functions of E are constant, this connection is globally defined. Since the deck transformations depend continuously on the representation ρ, the flat connection on E_ρ depends continuously on ρ. □

Corollary 2.22. *A flat connection on a vector bundle over a simply connected complex X is complex gauge-equivalent to the trivial connection d on the trivial vector bundle.*

Definition 2.23. The $\mathrm{SL}(r, \mathbb{C})$ *character variety* is the space of irreducible representations $\rho : \pi_1(X) \to \mathrm{SL}(r, \mathbb{C})$ modulo conjugation by $\mathrm{SL}(r, \mathbb{C})$:

$$\mathcal{M}_{\mathrm{char}} = \{\text{irreducible reps } \rho : \pi_1(X) \to \mathrm{SL}(r, \mathbb{C})\} / \mathrm{SL}(r, \mathbb{C}). \tag{2-7}$$

The next lemma is a trivial consequence of the path-lifting property and is standard.

Lemma 2.24. *If two characters defined by the representations ρ and ρ' belong to the same connected component of $\mathcal{M}_{\mathrm{char}}$ then the vector bundles E_ρ and $E_{\rho'}$ are smoothly isomorphic.*

In view of the above, let $\mathcal{C}$ denote the set of connected components of $\mathcal{M}_{\mathrm{char}}$. Then we can write

$$\mathcal{M}_{\mathrm{char}} = \bigsqcup_{c \in \mathcal{C}} \mathcal{M}^c_{\mathrm{char}}$$

and write $E_c = E_\rho$ for any representative in the isomorphism class of bundles defined by $\rho \in \mathcal{M}^c_{\text{char}}$.

Remark 2.25. Since we are interested in the $\mathrm{SL}(r, \mathbb{C})$ character variety instead of the $\mathrm{GL}(r, \mathbb{C})$ character variety, we need to fix determinants in our definitions of connections and gauge transformations. Henceforth *we will impose the condition that all connection forms are traceless and all gauge transformations have determinant 1*. For the sake of notational simplicity we will keep the same notation as before for the various spaces of $\mathrm{SL}(r, \mathbb{C})$ connections and gauge groups.

Proposition 2.26. $\mathcal{A}^{\mathbb{C},\text{irr}}_{\text{flat}}(E_c)/\mathcal{G}^{\mathbb{C}}(E_c) \cong \mathcal{M}^c_{\text{char}}.$

Proof. The holonomy map applied to an irreducible flat connection D gives an irreducible representation $\rho : \pi_1(X, x_0) \to \mathrm{GL}(r, \mathbb{C})$. The action of a complex gauge transformation $g \in \mathcal{G}^{\mathbb{C}}(E_c)$ on D induces the conjugate action of an element $\xi = g(x_0) \in \mathrm{GL}(r, \mathbb{C})$ on ρ. Therefore we have a continuous map $\tau : \mathcal{A}^{\mathbb{C},\text{irr}}_{\text{flat}}(E_c)/\mathcal{G}^{\mathbb{C}}(E_c) \to \mathcal{M}^c_{\text{char}}$. Note that $\tau([D_1]) = \tau([D_2])$ implies that the flat structures associated to D_1 and D_2 by Theorem 2.17 are complex gauge-equivalent, and so D_1 and D_2 are complex gauge-equivalent. Therefore τ is injective.

Similarly, given a representation $\rho : \pi_1(X, x_0) \to \mathrm{GL}(r, \mathbb{C})$ we construct a flat connection d on the flat bundle E_ρ as in the proof of Lemma 2.21. If we conjugate the representation by an element $\xi \in \mathrm{GL}(r, \mathbb{C})$, then the flat connection associated to this new representation is related to E_ρ by a global change of coordinates using the action of ξ on the fibres of E_ρ. Therefore the two flat bundles are complex gauge-equivalent, and so conjugate representations give $\mathcal{G}^{\mathbb{C}}(E_c)$-equivalent flat connections, which gives us a continuous map $\zeta : \mathcal{M}_{\text{char}} \to \mathcal{A}^{\mathbb{C},\text{irr}}_{\text{flat}}(E_c)/\mathcal{G}^{\mathbb{C}}(E_c)$. Lemma 2.21 shows that $\tau \circ \zeta = \text{id}$. Since τ is injective then this implies that $\zeta \circ \tau = \text{id}$ and so τ is a homeomorphism $\mathcal{A}^{\mathbb{C},\text{irr}}_{\text{flat}}(E_c)/\mathcal{G}^{\mathbb{C}}(E_c) \cong \mathcal{M}^c_{\text{char}}$. □

2B. *Relationship to Higgs bundles.* Given a complex X with universal cover $\widetilde{X}$, fix an irreducible representation $\rho : \pi_1(X) \to \mathrm{SL}(r, \mathbb{C})$, and let $E = \widetilde{X} \times_\rho \mathbb{C}^r \to X$ be as before. We also fix a ρ-equivariant map $u : \widetilde{X} \to \mathrm{SL}(r, \mathbb{C})/\mathrm{SU}(r)$, locally Lipschitz away from the 0-skeleton X^0 of X. We now recall the basic construction from [Corlette 1988; Donaldson 1987]:

(1) The complexified tangent space $T^{\mathbb{C}}_h(\mathrm{SL}(r, \mathbb{C})/\mathrm{SU}(r))$ can be identified (independent of h) with the space of traceless matrices and this gives a trivialization of the complexified tangent bundle $T^{\mathbb{C}}(\mathrm{SL}(r, \mathbb{C})/\mathrm{SU}(r)) \cong \mathrm{SL}(r, \mathbb{C})/\mathrm{SU}(r) \times \mathfrak{sl}(r, \mathbb{C})$.

(2) In the trivialization given in (1) the Levi-Civita connection at a point $h \in \mathrm{SL}(r, \mathbb{C})/\mathrm{SU}(r)$ has the form

$$\nabla_X Y = dY(X) - \tfrac{1}{2}\big(dh(X)h^{-1}Y + Yh^{-1}dh(X)\big),$$

where we use the notation h to indicate left translation by h.

(3) The identification $h^{-1}(T_h^{\mathbb{C}}(\mathrm{SL}(r,\mathbb{C})/\mathrm{SU}(r))) \cong T_{\mathrm{id}}^{\mathbb{C}}(\mathrm{SL}(r,\mathbb{C})/\mathrm{SU}(r)) \cong \mathfrak{sl}(r,\mathbb{C})$ gives another isomorphism $\theta : T^{\mathbb{C}}(\mathrm{SL}(r,\mathbb{C})/\mathrm{SU}(r)) \to \mathrm{SL}(r,\mathbb{C})/\mathrm{SU}(r) \times \mathfrak{sl}(r,\mathbb{C})$. It follows immediately from (2) that in the coordinates given by θ, the Levi-Civita connection is given by

$$\nabla_X s = h^{-1}\nabla_X(hs) = ds(X) + \tfrac{1}{2}[h^{-1}dh(X), s].$$

We thus conclude that in the above coordinates

$$\nabla = d + \tfrac{1}{2}[h^{-1}dh, \cdot\,]. \tag{2-8}$$

(4) The isomorphism θ is equivariant with respect to the $\mathrm{PSL}(r,\mathbb{C})$ action on the complexified tangent bundle $T^{\mathbb{C}}(\mathrm{SL}(r,\mathbb{C})/\mathrm{SU}(r))$ and the adjoint representation on $T_{\mathrm{id}}^{\mathbb{C}}(\mathrm{SL}(r,\mathbb{C})/\mathrm{SU}(r)) \cong \mathfrak{s}l(r,\mathbb{C})$.

(5) Given u as above, consider the pullbacks $\mathcal{D} = u^*d$ and $d_A = u^*\nabla$ on the trivial bundle $\widetilde{X} \times T_{\mathrm{id}}^{\mathbb{C}}(\mathrm{SL}(r,\mathbb{C})/\mathrm{SU}(r)) \cong \widetilde{X} \times \mathfrak{s}l(r,\mathbb{C})$. First notice, that since u^*d is trivial and u is ρ-equivariant, $\mathcal{D}$ descends to a flat connection of holonomy ρ on E_ρ. Again, by the ρ-equivariance of u and (4), the connection d_A descends to a connection on $\mathrm{ad}(E_\rho)$ over X. Moreover, since its connection form acts by the adjoint representation, it defines an $\mathrm{SL}(r,\mathbb{C})$ connection on the bundle E_ρ over X and (2-8) implies

$$\mathcal{D} = d_A + \psi, \quad \psi = -\tfrac{1}{2}u^{-1}\,du. \tag{2-9}$$

Since $\mathcal{D}$ is a flat connection,

$$F_A + \psi \wedge \psi = 0, \tag{2-10}$$

$$d_A\psi = 0. \tag{2-11}$$

2C. *The balancing condition.*

Definition 2.27. A smooth 1-form $\omega = \{\omega_\sigma\}_{\sigma\in\mathcal{F}} \in \Omega^1(X)$ satisfies the *balancing condition* if for every $(n-1)$-cell τ, we have

$$\sum_{\sigma>\tau} \omega_\sigma(e_\sigma) = 0, \tag{2-12}$$

where $\sigma > \tau$ implies that τ is a face of σ, and e_σ is an inward-pointing normal vector field along τ in σ. The set $\Omega^1_{\mathrm{bal}}(X)$ is the subset of $\Omega^1(X)$ consisting of forms satisfying the balancing condition.

Definition 2.28. Let E be smooth vector bundle on X of rank r and let $p : \widetilde{X} \to X$ be the universal cover. We assume that the pullback bundle $p^*(E)$ over $\widetilde{X}$ is trivial *with a fixed trivialization* $p^*(E) \cong \widetilde{X} \times \mathbb{C}^r$ (if the connection is flat then this is always valid by Corollary 2.22). A connection $D \in \mathcal{A}^{\mathbb{C},\mathrm{irr}}(E)$ is called *balanced* if its pullback $p^*(D)$ to $p^*(E)$ can be written (in the given trivialization) as $p^*(D) = d + A$ where all the components satisfy $A_{ij} \in \Omega^1_{\mathrm{bal}}(\widetilde{X})$. Let $\mathcal{A}^{\mathbb{C},\mathrm{irr}}_{\mathrm{bal}}(E)$ be the space of irreducible, smooth, balanced $\mathrm{GL}(r, \mathbb{C})$ connections, and let $\mathcal{A}^{\mathrm{irr}}_{\mathrm{bal}}(E)$ denote the space of irreducible, smooth, balanced connections compatible with the Hermitian metric h on E. In what follows, if the meaning is clear then the notation for the metric is suppressed.

Definition 2.29. Let E be as in the previous definition. Given $g \in \mathcal{G}^{\mathbb{C}}(E)$, let $\tilde{g}$ denote the induced gauge transformation of $p^*(E)$. We define $\mathcal{G}^{\mathbb{C}}_{\mathrm{bal}}(E)$ (resp. $\mathcal{G}_{\mathrm{bal}}(E)$) to be the group of complex (resp. unitary) gauge transformations such that $g \in \mathcal{G}^{\mathbb{C}}_{\mathrm{bal}}(E)$ (resp. $g \in \mathcal{G}_{\mathrm{bal}}(E)$) implies that $d\tilde{g}_{ij} \in \Omega^1_{\mathrm{bal}}(\widetilde{X})$.

Remark 2.30. Via (2-4), the group $\mathcal{G}^{\mathbb{C}}_{\mathrm{bal}}(E)$ acts on the space $\mathcal{A}^{\mathbb{C},\mathrm{irr}}_{\mathrm{bal}}(E)$, and $\mathcal{G}_{\mathrm{bal}}(E)$ acts on $\mathcal{A}^{\mathrm{irr}}_{\mathrm{bal}}(E)$.

Remark 2.31. In this paper we are interested in flat bundles. Corollary 2.22 implies that the pullback of a flat bundle to the universal cover is trivial. By choosing a trivialization it thus makes sense to talk about balanced connections and gauge transformations.

3. Harmonic maps and Higgs bundles

In this section we describe the relationship between Higgs bundles and harmonic maps from a complex X into the space $\mathrm{SL}(n, \mathbb{C})/\mathrm{SU}(n)$, a generalisation of the construction of [Donaldson 1987; Corlette 1988]. *From now on X will denote an admissible 2-dimensional simplicial complex without boundary. We will further assume that X is equipped with a Riemannian metric g such that for any 2-simplex σ, (σ, g_σ) is isometric to an interior of an equilateral triangle in $\mathbb{R}^2$ and for any 1-simplex τ, (τ, g_τ) is isometric to the open unit interval in $\mathbb{R}$.* It is not hard to extend the results of this section to general Riemannian metrics and also general 2-dimensional complexes. We endow $\mathrm{SL}(n, \mathbb{C})/\mathrm{SU}(n)$ with a Riemannian metric of nonpositive sectional curvature such that $\mathrm{SL}(n, \mathbb{C})$ acts by isometries.

3A. *Estimates of harmonic maps.*

Theorem 3.1. *Let X be a 2-complex as before with universal cover $\widetilde{X}$ and $\rho : \pi_1(X) \to \mathrm{SL}(n, \mathbb{C})$ be an irreducible representation. Then there exists a unique ρ-equivariant harmonic map $u : \widetilde{X} \to Y := \mathrm{SL}(n, \mathbb{C})/\mathrm{SU}(n)$.*

Proof. The existence is a special case of Theorem 4.5 of [Daskalopoulos and Mese 2006]. Uniqueness follows from [Mese 2002]. □

Let p be a vertex (i.e., 0-cell) of X. Given a 1-cell τ of X, define $\mathcal{S}_2(\tau)$ be the set of 2-cells of X containing τ in its closure.

Theorem 3.2. *If $u : X \to Y$ is a harmonic map, then for any 1-simplex τ and 2-simplex $\sigma \in \mathcal{S}_2(\tau)$ we have $u \in C^\infty(\sigma \cup \tau)$. (In other words, the restriction of u to σ is C^∞ up to τ in the extended plane of σ). Moreover, for every 1-simplex τ and $p \in \tau$ assume that u is given in a neighbourhood of $u(p)$ in local coordinates by $u = (u^1, \dots, u^M)$.*

Then,

$$\sum_{\sigma > \tau} \frac{\partial u_j^m}{\partial e_\sigma} = 0, \tag{3-1}$$

where $\sigma > \tau$ implies that τ is a face of σ, and e_σ is an inward-pointing normal vector field along τ in σ.

Proof. The fact that $u^m \in C^\infty(\sigma \cup \tau)$ follows from Theorem 4 and Corollary 6 of [Daskalopoulos and Mese 2008]. Equation (3-1), follows from Corollary 5 of the same paper. □

For an edge τ and $\sigma \in \mathcal{S}_2(\tau)$, we define polar coordinates (r, θ) of $\sigma \cup \tau$ centred at p by setting r to be the distance from p to a point $q \in \sigma \cup \tau$ and θ to be the angle between τ and the line $\overline{pq}$ connecting p and q. The next theorem is one of the main technical results of the paper and describes the singular behaviour of harmonic maps near the lower-dimensional strata.

Theorem 3.3. *Let $u : X \to Y$ be a harmonic map. If (r, θ) are the polar coordinates of $\sigma \cup \tau$ centred at a 0-cell p and u is given in local coordinates $(u^1, \dots, u^M)$ in a neighbourhood of $u(p)$, we have the following derivative bounds for u^m in a neighbourhood of p:*

$$\left|\frac{\partial u^m}{\partial r}\right| \le Cr^{\alpha-1}, \quad \left|\frac{\partial u^m}{\partial \theta}\right| \le Cr^{\alpha},$$

$$\left|\frac{\partial^2 u^m}{\partial r^2}\right| \le Cr^{\alpha-2}, \quad \left|\frac{\partial^2 u^m}{\partial r\, \partial \theta}\right| \le Cr^{\alpha-1}, \quad \left|\frac{\partial^2 u^m}{\partial \theta^2}\right| \le Cr^{\alpha},$$

$$\left|\frac{\partial^3 u^m}{\partial r^3}\right| \le Cr^{\alpha-3}, \quad \left|\frac{\partial^3 u^m}{\partial^2 r \partial \theta}\right| \le Cr^{\alpha-2}, \quad \left|\frac{\partial^3 u^m}{\partial r\, \partial^2 \theta}\right| \le Cr^{\alpha-1}, \quad \left|\frac{\partial^3 u^m}{\partial \theta^3}\right| \le Cr^{\alpha}$$

for some constants $C > 0$ and $\alpha > 0$ depending on the total energy of u and the geometry of the complex X. Furthermore, α can be chosen independently of the choice of the 0-cell p of X.

Proof. Let $\sigma = \sigma_1, \dots, \sigma_J$ be the 2-cells in $\mathcal{S}_2(\tau)$. For each $j = 1, \dots, J$, we let (x, y) be the Euclidean coordinates of $\overline{\sigma_j \cup \tau}$ so that (i) p is given as $(x, y) = (0, 0)$, (ii) if $(x, y) \in \tau$ then $x > 0$ and $y = 0$ and (iii) if $(x, y) \in \sigma_j$ then $x, y > 0$. Let $u_j^m = u^m|_{\sigma_j}$.

We will now compute the first-derivative bounds with respect to the polar coordinates r and θ. By Theorem 6.2 of [Daskalopoulos and Mese 2006], we have the inequality

$$|\nabla u|^2(r,\theta) \le Cr^{2\alpha-2}$$

for some $\alpha > 0$. More specifically, α can be chosen to be the order of u at p; i.e.,

$$\alpha = \lim_{r\to 0} \frac{r\int_{B_r(p)} |\nabla u|^2\, d\mu}{\int_{\partial B_r(p)} d^2(u,u(p))\, ds}.$$

Hence,

$$\left|\frac{\partial u_j^m}{\partial x}\right| \le Cr^{\alpha-1} \quad \text{and} \quad \left|\frac{\partial u_j^m}{\partial y}\right| \le Cr^{\alpha-1}. \tag{3-2}$$

Using the fact that $x = r\cos\theta$ and $y = r\sin\theta$, we get

$$\frac{\partial u_j^m}{\partial r} = \frac{\partial u_j^m}{\partial x}\cos\theta + \frac{\partial u_j^m}{\partial y}\sin\theta \quad \text{and} \quad \frac{\partial u_j^m}{\partial \theta} = -\frac{\partial u_j^m}{\partial x} r\sin\theta + \frac{\partial u_j^m}{\partial y} r\cos\theta.$$

This immediately implies

$$\left|\frac{\partial u_j^m}{\partial r}\right| \le Cr^{\alpha-1} \quad \text{and} \quad \left|\frac{\partial u_j^m}{\partial \theta}\right| \le Cr^{\alpha}.$$

We will now establish the second derivative estimates of u_j^m for a points (r,θ) on $\sigma_j \cup \tau$ with θ sufficiently small. We will need the following notation: for a function φ and a domain $\Omega \subset \mathbb{R}^2$, we set

$$\begin{aligned}
|\varphi|_{0;\Omega} &= \sup_{p\in\Omega} |\varphi(p)|,\\
|D\varphi|_{0;\Omega} &= \sup_{p\in\Omega} \max\left\{\left|\frac{\partial\varphi}{\partial x}(p)\right|, \left|\frac{\partial\varphi}{\partial y}(p)\right|\right\},\\
|D^2\varphi|_{0;\Omega} &= \sup_{p\in\Omega} \max\left\{\left|\frac{\partial^2\varphi}{\partial x^2}(p)\right|, \left|\frac{\partial^2\varphi}{\partial x\,\partial y}(p)\right|, \left|\frac{\partial^2\varphi}{\partial y^2}(p)\right|\right\},\\
[\varphi]_{\beta;\Omega} &= \sup_{\substack{p,q\in\Omega\\ p\neq q}} \frac{|\varphi(p)-\varphi(q)|}{|p-q|^\beta},\\
[D\varphi]_{\beta;\Omega} &= \sup_{\substack{p,q\in\Omega\\ p\neq q}} \frac{1}{|p-q|^\beta} \max\left\{\left|\frac{\partial\varphi}{\partial x}(p) - \frac{\partial\varphi}{\partial x}(q)\right|, \left|\frac{\partial\varphi}{\partial y}(p) - \frac{\partial\varphi}{\partial y}(q)\right|\right\},\\
[D^2\varphi]_{\beta;\Omega} &= \sup_{\substack{p,q\in\Omega\\ p\neq q}} \frac{1}{|p-q|^\beta} \max\left\{\left|\frac{\partial^2\varphi}{\partial x^2}(p) - \frac{\partial^2\varphi}{\partial x^2}(q)\right|,\right.\\
&\qquad\left.\left|\frac{\partial^2\varphi}{\partial x\,\partial y}(p) - \frac{\partial^2\varphi}{\partial x\,\partial y}(q)\right|, \left|\frac{\partial^2\varphi}{\partial y^2}(p) - \frac{\partial^2\varphi}{\partial y^2}(q)\right|\right\}.
\end{aligned}$$

Let

$$\begin{aligned} T &:= \{(x, y) \in \mathbb{R}^2 : y \geq 0,\, y < \sqrt{3}x,\, y < -\sqrt{3}x + \sqrt{3}\}, \\ T^- &= \{(x, -y) \in \mathbb{R}^2 : (x, y) \in T\}, \\ \widehat{T} &= T \cup T^-. \end{aligned}$$

Fix m and j and define $U : \widehat{T} \to \mathbb{R}$ by setting

$$U(x, y) = \begin{cases} u_j^m(x, y) & \text{if } y \geq 0, \\ -u_j^m(x, -y) + (2/J) \sum_{j'=1}^{J} u_{j'}^m(x, -y) & \text{if } y < 0. \end{cases}$$

Let

$$\Gamma_j^m = \sum_{p,q=1}^{M} \Gamma_{pq}^m(u_j) \left(\frac{\partial u_j^p}{\partial x} \frac{\partial u_j^q}{\partial x} + \frac{\partial u_j^p}{\partial y} \frac{\partial u_j^q}{\partial y} \right), \tag{3-3}$$

where Γ_{pq}^m are the Christoffel symbols of Y with respect to the local coordinates $(u^1, \dots, u^M)$. Since the harmonic map equation

$$\triangle u_j^m = \Gamma_j^m$$

is satisfied in T, if we set

$$f(x, y) = \begin{cases} \Gamma_j^m(x, y) & \text{if } y \geq 0, \\ -\Gamma_j^m(x, -y) + (2/J) \sum_{j'=1}^{J} \Gamma_{j'}^m(x, -y) & \text{if } y < 0, \end{cases}$$

then U satisfies the Poisson equation

$$\triangle U = f \tag{3-4}$$

weakly in $\widehat{T}$. Indeed, let ξ be a test function supported in a neighbourhood $B_R(q)$ of a point $q = (x_0, 0) \in \widehat{T}$. Since U is a C^1 function we have by the divergence theorem,

$$\begin{aligned} &\int_{\widehat{T}} \operatorname{div}(\xi \nabla U)\, dx\, dy \\ &= \int_T \operatorname{div}(\xi \nabla U)\, dx\, dy + \int_{T^-} \operatorname{div}(\xi \nabla U)\, dx\, dy \\ &= -\int_{x_0-R}^{x_0+R} \xi \frac{\partial u_j^m}{\partial y}(x, 0)\, dx + \int_{x_0-R}^{x_0+R} \left(\xi \frac{\partial u_j^m}{\partial y}(x, 0) - \frac{2}{J} \sum_{j'=1}^{J} \xi \frac{\partial u_j^m}{\partial y}(x, 0) \right) dx = 0, \end{aligned}$$

where the last equality is because of (3-1). On the other hand,

$$\int_{\widehat{T}} \operatorname{div}(\xi \nabla U)\, dx\, dy = \int_{\widehat{T}} \nabla \xi \cdot \nabla U + \int_{\widehat{T}} \xi f\, dx\, dy,$$

which along with the previous equation implies (3-4). If $B_{2R}(q) \subset \widehat{T}$, then elliptic regularity theory, see [Gilbarg and Trudinger 1983; Simon 1996, Lemma 3, p. 13], implies

$$R^{1+\beta}[DU]_{\beta;B_{3R/2}(q)} \le C(|U|_{0;B_{2R}(q)} + R^2|f|_{0;B_{2R}(q)}).$$

If we choose R to be the largest number so that $B_{2R}(q) \subset \widehat{T}$, then R is proportional to r, where r is the distance of q to the vertex p. Furthermore, the distance from p to any point of $B_{2R}(q)$ is bounded uniformly by some constant multiple of r. Hence, assuming $U(0,0)=0$ without a loss of generality, we have

$$\begin{aligned}[DU]_{\beta;B_{3R/2}(p)} &\le C(r^{-1-\beta}|U|_{0;B_{2R}(p)} + r^{1-\beta}|f|_{0;B_{2R}(p)})\\ &\le C(r^{-1-\beta+\alpha} + r^{-\beta+2\alpha-1}) \le Cr^{-\beta+\alpha-1}.\end{aligned}$$

Here, we have used the Hölder continuity of u_j^m (hence of U) near p with Hölder exponent α, see Theorem 3.7 of [Daskalopoulos and Mese 2006], and the inequalities of (3-2) along with the fact that f is quadratic in Du_j^m from (3-3). Thus, with $B^+_{3R/2}(q) = B_{3R/2}(q) \cap \{y \ge 0\}$, we obtain

$$[Du_j^m]_{\beta;B^+_{3R/2}(q)} \le Cr^{-\beta+\alpha-1}.$$

This equation along with (3-2) and (3-3) implies that

$$[\Gamma_j^m]_{\beta;B^+_{3R/2}(q)} \le C|Du_j^k|_{0;B^+_{3R/2}(q)}[Du_j^\ell]_{\beta;B^+_{3R/2}(q)} \le Cr^{-\beta+2\alpha-2}. \tag{3-5}$$

We are now ready to prove the second-derivative bounds of u_j^m. Note that we have the set of partial differential equations

$$\triangle u_j^m = \Gamma_j^m, \quad j=1,\dots,J, \quad m=1,\dots M, \tag{3-6}$$

in T, along with boundary conditions

$$u_j^m - u_1^m = 0, \quad j=2,\dots,J, \quad m=1,\dots,M, \tag{3-7}$$

$$\sum_{j=1}^{J} \frac{\partial u_j^m}{\partial y} = 0, \quad m=1,\dots,M, \tag{3-8}$$

in $B=\{(x,y)\in\mathbb{R}^2 : y=0,\ 0<x<1\}$. This is a system of JM equations containing JM unknowns (i.e., u_j^m) along with JM boundary conditions. If we assign weights $s_j^m = 0$ to the equations, weights $t_j^m = 2$ to the unknowns, weights $r_j^m = -2$ for $j=2,\dots,M$ and $r_1^m = -1$ to the boundary conditions, then this system is said to be elliptic with complementing boundary condition according to the elliptic regularity theory of [Agmon et al. 1964] (or elliptic and coercive in [Kinderlehrer et al. 1978]). Hence, we have the Schauder estimates, see Theorem 9.1 of [Agmon et al. 1964],

$$\begin{aligned}&R^2|D^2u_j^m|_{0;B_R^+(q)} + R^{2+\beta}[D^2u_j^m]_{\beta;B_R^+(q)}\\ &\qquad\le C\big(|\Gamma_j^m|_{0;B^+_{3R/2}(q)} + R^{2+\beta}[\Gamma_j^m]_{\beta;B^+_{3R/2}(q)} + |u_j^m|_{0;B^+_{3R/2}(q)}\big).\end{aligned}$$

With the same choice of q and R as above, we obtain

$$|D^2u_j^m|_{0;B_R^+(q)} \le C\big(|\Gamma_j^m|_{0;B_{3R/2}^+(q)} + r^\beta[\Gamma_j^m]_{\beta;B_{3R/2}^+(q)} + r^{-2}|u_j^m|_{0;B_{3R/2}^+(q)}\big).$$

The above inequality, along with (3-5), implies

$$|D^2u_j^m|_{0;B_R^+(q)} \le C(r^{2\alpha-2}+r^{2\alpha-2}+r^{\alpha-2}) \le Cr^{\alpha-2}.$$

Since

$$\frac{\partial^2 u_j^m}{\partial r^2} = \frac{\partial^2 u_j^m}{\partial x^2}\cos^2\theta + 2\frac{\partial^2 u_j^m}{\partial x\,\partial y}\sin\theta\cos\theta + \frac{\partial^2 u_j^m}{\partial y^2}\sin^2\theta,$$

$$\begin{aligned}\frac{\partial^2 u_j^m}{\partial r\,\partial\theta} = -\frac{\partial^2 u_j^m}{\partial x^2} r\sin\theta\cos\theta + \frac{\partial^2 u_j^m}{\partial x\,\partial y} r\cos^2\theta - \frac{\partial u_j^m}{\partial x}\sin\theta - \frac{\partial^2 u_j^m}{\partial x\,\partial y} r\sin^2\theta \\ + \frac{\partial^2 u_j^m}{\partial y^2} r\sin\theta\cos\theta + \frac{\partial u_j^m}{\partial y}\cos\theta,\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 u_j^m}{\partial\theta^2} &= \frac{\partial^2 u_j^m}{\partial x^2} r^2\sin^2\theta + 2\frac{\partial^2 u_j^m}{\partial x\,\partial y} r^2\sin^2\theta + \frac{\partial^2 u_j^m}{\partial y^2} r^2\cos\theta \\ &= -\frac{\partial u_j^m}{\partial x} r\cos\theta - \frac{\partial u_j^m}{\partial y} r\sin\theta,\end{aligned}$$

we immediately obtain

$$\left|\frac{\partial^2 u_j^m}{\partial r^2}\right| \le Cr^{\alpha-2}, \quad \left|\frac{\partial^2 u_j^m}{\partial r\,\partial\theta}\right| \le Cr^{\alpha-1} \quad \text{and} \quad \left|\frac{\partial^2 u_j^m}{\partial\theta^2}\right| \le Cr^{\alpha}$$

at (r,θ) for θ sufficiently small. This restriction on θ is due to the choice of R and q. For (r,θ) with θ sufficiently large, we can use a similar argument using standard elliptic regularity theory, see, e.g., [Gilbarg and Trudinger 1983; Simon 1996, Lemma 3, p. 13], in the interior of σ. The third-derivative estimates follow the same way as the first two by bootstrapping the elliptic equations (3-6) with boundary conditions (3-7) and (3-8).

Section 4 of [Daskalopoulos and Mese 2008] shows the that order of u at p can be bounded from below by $2\lambda_v^{\text{comb}}$ where λ_v^{comb} is the combinatorial eigenvalue of the link of v, which is always a positive quantity. Hence choosing α to be the minimum of $2\lambda_v^{\text{comb}}$ over all 0-cells of X, we have established the last assertion of the Theorem. □

3B. *Weighted Sobolev spaces.* In this subsection we recall the important features of the weighted Sobolev spaces used in this paper. The main references are [Adams 1975; Daskalopoulos and Wentworth 1997; Lockhart and McOwen 1985]. In the following we fix a smooth vector bundle E of rank r over a 2-complex X with a Hermitian metric, and a fixed Riemannian metric on the base space X. Define the space $C_0^\infty(E)$ to be the space of smooth sections $s \in \Omega^0(X, E)$ that satisfy $s(p) = 0$ whenever p is a vertex of X. In the local model $\widetilde{B}(r)$ around each vertex p, we define local coordinates $(t, \theta) = (-\log r, \theta)$, where (r, θ) are the standard polar coordinates in a neighbourhood of the vertex p. To define a norm on $C_0^\infty(E)$, let $\{x_i\}_{i=1,\dots,V}$ denote the vertices of X and choose disjoint open neighbourhoods U_{x_i} for each vertex x_i. Then cover the rest of X with open sets $\{V_\alpha\}_{\alpha=1,\dots,K}$ that do not contain any of the vertices. For $\delta \in \mathbb{R}$, the space L_δ^p is the completion of $C_0^\infty(E)$ in the norm

$$\|s\|_{L_\delta^p} = \left(\sum_{i=1}^{V} \int_{U_{x_i}} e^{t\delta} |s|^p + \sum_{\alpha=1}^{K} \int_{V_\alpha} |s|^p \right)^{1/p}, \tag{3-9}$$

where we use $e^{t\delta}$ to denote the coordinates in a neighbourhood of a vertex. Away from all of the vertices, $e^{t\delta}$ is bounded and s is continuous, and so the question of whether the norm $\|\cdot\|_{L_\delta^p}$ is finite only depends on the choice of coordinates near each vertex. Different choices of V_α will lead to equivalent norms.

Given a vertex p and a trivialization of E near p, we say that a connection is *trivial* in a neighbourhood of p if with respect to the above trivialization $\nabla = d$. Given a fixed connection ∇_0 trivial near the vertices, and a positive integer k, we define the *weighted Sobolev space* $L_{k,\delta}^q(E)$ as the completion of $C_0^\infty(E)$ in the norm

$$\|s\|_{L_{k,\delta}^q} = \sum_{\ell=0}^{k} \|\nabla_0^\ell s\|_{L_\delta^q}. \tag{3-10}$$

Note that in this paper we are considering bundles with a fixed trivialization on the universal cover (see Remark 2.31). Since the star of a vertex p in X is simply connected it follows that we have a fixed trivialization of E in a neighbourhood of p. It thus makes sense to talk about connections on E trivial near the vertices.

It is a standard fact that the spaces $L_{k,\delta}^q$ do not change if we either (a) change the connection ∇_0 outside a neighbourhood of the vertices of X, or (b) change the coordinates outside a neighbourhood of the vertices. The usual multiplication theorems for Sobolev spaces on compact manifolds carry over to the weighted Sobolev spaces studied here. To be more precise, we have that the multiplication map $L_{s_1,\delta_1}^2 \times L_{s_2,\delta_2}^2 \to L_{s,\delta}^2$ is continuous if $s_1, s_2 \geq s$, $s < s_1 + s_2 - \frac{n}{2}$ and $\delta < \delta_1 + \delta_2 + \frac{n}{2}$, where n is the dimension of the complex X.

Following Section 3.1 of [Daskalopoulos and Wentworth 1997] we define the space of weighted connections $\mathcal{A}_\delta^{\mathbb{C}}(E)$ to be the space of all connections whose connection form is an element of $L_{1,\delta}^2$, and the space $\mathcal{A}_\delta(E) \subset \mathcal{A}_\delta^{\mathbb{C}}(E)$ to be the subset of all unitary connections. The weighted gauge group $\mathcal{G}_\delta(E)$ is defined as follows. Let ∇_0 be a connection as above and define

$$\mathcal{R} = \{v \in L_{2,\mathrm{loc}}^2(\mathrm{End}(E)) : \|\nabla_0 v\|_{L_{1,\delta}^2} < \infty\}. \tag{3-11}$$

Then the *weighted gauge group* is defined as

$$\mathcal{G}_\delta(E) = \{v \in \mathcal{R} : vv^* = \mathrm{id},\ \det v = 1\} \tag{3-12}$$

and the complexified gauge group is

$$\mathcal{G}_\delta^{\mathbb{C}}(E) = \{v \in \mathcal{R} : \det v = 1\}. \tag{3-13}$$

The multiplication theorem for weighted Sobolev spaces shows that both $\mathcal{G}_\delta(E)$ and $\mathcal{G}_\delta^{\mathbb{C}}(E)$ have a group structure, and that there are well-defined actions of $\mathcal{G}_\delta(E)$ on $\mathcal{A}_\delta$ and $\mathcal{G}_\delta^{\mathbb{C}}(E)$ on $\mathcal{A}_\delta^{\mathbb{C}}(E)$ respectively.

Similarly we have balanced versions of these spaces $\mathcal{G}_{\mathrm{bal},\delta}(E)$, $\mathcal{A}_{\mathrm{bal},\delta}(E)$ and $\Omega^1_{\mathrm{bal},\delta}(\mathrm{ad}(E))$. When a smooth pair $(d_A, \psi) \in \mathcal{A}_{\mathrm{bal},\delta}(E) \times \Omega^1_{\mathrm{bal},\delta}(\mathrm{ad}(E))$ solves (2-10) and (2-11), then the *holonomy of the pair* (d_A, ψ) refers to the holonomy of the flat connection $d_A + \psi \in \mathcal{A}^{\mathbb{C}}_{\mathrm{bal,flat},\delta}(E)$.

Proposition 3.4. *If $D_i \in \mathcal{A}^{\mathbb{C}}_{\mathrm{bal,flat},\delta}(E)$, $i = 1, 2$, are smooth and $\mathcal{G}^{\mathbb{C}}_{\mathrm{bal},\delta}(E)$-gauge-equivalent then they are $\mathcal{G}^{\mathbb{C}}_{\mathrm{bal}}(E)$-gauge-equivalent.*

Proof. Since the result is local, it follows by elliptic regularity. □

Proposition 3.5. *Let $D \in \mathcal{A}^{\mathbb{C}}_{\mathrm{bal,flat},\delta}(E)$ be smooth. Then D has trivial holonomy around the vertices of X.*

Proof. For $D = d + A$ write $A(t,\theta) = B(t,\theta)\,dt + C(t,\theta)\,d\theta$. Consider the family of loops $c_t : [0, 2\pi] \to X$ given by $c_t(\theta) = (t, \theta)$ and consider the holonomy equation from Definition 2.18 along $c_t(\theta)$

$$\frac{ds_t(\theta)}{d\theta} + C(t,\theta)s_t(\theta) = 0 \quad \text{with } s_t(0) = \mathrm{id}. \tag{3-14}$$

Lemma IV.4.1 on p. 54 of [Hartman 1964] implies

$$|s_t(\theta)| \le |s_t(0)| \exp\left\{\int_0^\theta |C(t,\theta)|\,d\theta\right\} \le K \exp\left\{\int_0^{2\pi} |C(t,\theta)|\,d\theta\right\}, \tag{3-15}$$

where K is a dimensional constant. Since

$$\int_0^\infty e^{t\delta} \int_0^{2\pi} |C(t,\theta)|^2\,d\theta\,dt < \infty,$$

there exists a sequence $t_i \to \infty$ such that $\int_0^{2\pi} |C(t_i,\theta)|^2\,d\theta \to 0$. By Cauchy–Schwarz we also have

$$\int_0^{2\pi} |C(t_i,\theta)|\,d\theta \to 0. \tag{3-16}$$

Combined with (3-15) this implies that $|s_{t_i}(\theta)|$ is uniformly bounded. By integrating (3-14) with respect to θ, we obtain from (3-16)

$$|s_{t_i}(2\pi) - s_{t_i}(0)| \le \int_0^{2\pi} |s_{t_i}(\theta)||C(t_i, \theta)|\, d\theta \to 0. \tag{3-17}$$

Since the holonomy is independent of t we obtain that $s_{t_i}(2\pi) = s_{t_i}(0)$ and thus it must be trivial. $\square$

Propositions 3.4 and 3.5 allow us to define the notion of conjugacy class of holonomy for a smooth flat connection $D \in \mathcal{A}^{\mathbb{C},\mathrm{irr}}_{\mathrm{bal,flat},\delta}(E)$ as follows.

Definition 3.6. Let $D \in \mathcal{A}^{\mathbb{C},\mathrm{irr}}_{\mathrm{bal,flat},\delta}(E)$ be a smooth flat connection and let $\rho_* : \pi_1(X_*) \to \mathrm{SL}(r, \mathbb{C})$ be the holonomy of D, where $X_* = X \backslash X^0$ and X^0 denotes the 0-skeleton of X. Since the star of a vertex is contractible, Van Kampen's theorem implies that $\pi_1(X) = \pi_1(X_*)/\pi$, where π denotes the subgroup of $\pi_1(X_*)$ generated by $\bigcup_{p \in X^0} \pi_1(Lk(p))$. By Proposition 3.5, the restriction of ρ_* to π is trivial; hence it induces a homomorphism $\rho : \pi_1(X) \to \mathrm{SL}(r, \mathbb{C})$. We say that the conjugacy class of holonomy of D is $[\rho]$. Notice that the map is well-defined since gauge-equivalent pairs yield conjugate holonomies. Furthermore, ρ is irreducible because D is irreducible.

4. Equivalence of moduli spaces

4A. *Higgs moduli space.* We fix a vector bundle $E_c = E$ of rank r over a 2-complex X with a Hermitian metric, and a fixed Riemannian metric on the base space X.

Definition 4.1. The *Higgs moduli space* is the space $\mathcal{M}_{\mathrm{Higgs}}(E)$ of $\mathcal{G}_{\mathrm{bal},\delta}(E)$-equivalence classes of pairs $(d_A, \psi) \in \mathcal{A}_{\mathrm{bal},\delta}(E) \times \Omega^1_{\mathrm{bal},\delta}(\sqrt{-1}\,\mathrm{ad}(E))$ that are *smooth, irreducible* and solve the equations

$$F_A + \psi \wedge \psi = 0, \tag{4-1}$$

$$d_A \psi = 0, \tag{4-2}$$

$$d_A^* \psi = 0. \tag{4-3}$$

We endow $\mathcal{M}_{\mathrm{Higgs}}(E)$ with the $L^2_{1,\delta}$-topology.

Given $[(d_A, \psi)] \in \mathcal{M}_{\mathrm{Higgs}}(E)$, we can assign by Definition 3.6 the holonomy $[\rho]$ of the flat connection $d_A + \psi$ and set $\alpha[(d_A, \psi)] := [\rho]$. The map α is well-defined. The next proposition follows from continuous dependence of solutions of ODE upon the initial condition.

Proposition 4.2. *The map* $\alpha : \mathcal{M}_{\mathrm{Higgs}}(E) \to \mathcal{M}^c_{\mathrm{char}}$, *where* $\alpha[(d_A, \psi)] = [\rho]$, *is well-defined and continuous.*

The following is the main theorem of this paper.

Theorem 4.3. *The map* $\alpha : \mathcal{M}_{\mathrm{Higgs}}(E) \to \mathcal{M}^c_{\mathrm{char}}$ *is a homeomorphism.*

In the next section we will construct the inverse map. We end this section with a proposition that will be used later.

Proposition 4.4. *Let* (d_{A_1}, ψ_1) *and* (d_{A_2}, ψ_2) *be solutions to* (4-1)–(4-3) *and assume that they are* $\mathcal{G}^{\mathbb{C}}_{\mathrm{bal},\delta}(E)$*-gauge-equivalent. Then they are* $\mathcal{G}_{\mathrm{bal},\delta}(E)$*-gauge-equivalent.*

Proof. Assume that there exists $g \in \mathcal{G}^{\mathbb{C}}_{\mathrm{bal},\delta}(E)$ such that $(d_{A_1}, \psi_1) = g \cdot (d_{A_2}, \psi_2)$, and we have to show that g is unitary. Let $h = g^* g$ and we will show that h is constant. By [Simpson 1988, Lemma 3.1(d)] we have the following pointwise estimate away from the vertices (notice that the sign of our Laplacian is the opposite from Simpson's):

$$\Delta \operatorname{tr}(h) \leq 0. \tag{4-4}$$

Now since g is balanced, so is $\operatorname{tr} h$, and therefore an application of Stokes' theorem on each face of X shows that

$$\begin{aligned} \int_X \Delta \operatorname{tr} h \, dx &= \lim_{r\to 0} \int_{X \setminus \bigcup_{\text{0-cells } v} B_r(v)} \Delta \operatorname{tr} h \, dx \\ &= \lim_{r\to 0} \sum_{\text{2-cells } \sigma} \int_{F \setminus \bigcup_{\text{0-cells } v} B_r(v)} \Delta \operatorname{tr} h \, dx \\ &= \lim_{r\to 0} \sum_{\text{2-cells } \sigma} \int_{\partial\left(F \setminus \bigcup_{\text{0-cells } v} B_r(v)\right)} \frac{\partial \operatorname{tr} h}{\partial \nu} \, ds, \end{aligned} \tag{4-5}$$

where ν is the outward-pointing normal vector on $\partial\big(\sigma \setminus \bigcup_{\text{0-cells } v} B_r(v)\big)$. The boundary $\partial\big(\sigma \setminus \bigcup_{\text{vertices } v} B_\sigma(v)\big)$ consists of points on the 1-cells of σ, and points on $\partial B_r(v) \cap \sigma$. Breaking the integral into these two parts, we obtain

$$\begin{aligned} &\sum_{\text{2-cells } \sigma} \int_{\partial\left(\sigma \setminus \bigcup_{\text{0-cells } v} B_r(v)\right)} \frac{\partial \operatorname{tr} h}{\partial \nu} \, ds \\ &= \sum_{\text{2-cells } \sigma} \Bigg(\sum_{\text{1-cells } \tau : \tau \cap \bar{\sigma} \neq \varnothing} \int_{\tau \setminus \bigcup_v B_r(v) \cap \tau} \frac{\partial \operatorname{tr} h}{\partial \nu} \, ds \Bigg) + \sum_{\text{2-cells } \sigma} \int_{\bigcup_v \partial B_r(v) \cap \sigma} \frac{\partial \operatorname{tr} h}{\partial \nu} \, ds. \end{aligned} \tag{4-6}$$

The balancing condition shows that the first term is zero. Therefore we are left with

$$\int_X \Delta \operatorname{tr} h \, dx = \lim_{r \to 0} \sum_{\text{2-cells } \sigma} \int_{\bigcup_v \partial B_r(v) \cap \sigma} \frac{\partial \operatorname{tr} h}{\partial \nu} \, ds. \tag{4-7}$$

In polar coordinates, each component of this integral becomes

$$\int_{\partial B_r(v) \cap F} \frac{\partial \operatorname{tr} h}{\partial \nu} \, ds = r \int_0^{\frac{\pi}{3}} \frac{\partial \operatorname{tr} h}{\partial r} \, d\theta. \tag{4-8}$$

Since $h \in \mathcal{G}(E)^{\mathbb{C}}_{\mathrm{bal},\delta}$ (and in particular, the integral of $\partial^2 h/\partial r^2$ is bounded), we have

$$\lim_{r\to 0}\left(\sigma \int_0^{\frac{\pi}{3}} \operatorname{tr}\left(\frac{\partial h}{\partial r}\right) d\theta\right) = 0 \tag{4-9}$$

and so (4-7) becomes

$$\int_X \Delta \operatorname{tr} h \, dx = 0. \tag{4-10}$$

Combined with $\Delta \operatorname{tr} h \leq 0$ from (4-4), we see that $\Delta \operatorname{tr} h = 0$. The second-to-the-last formula in [Simpson 1988, p. 876] implies that $D(h) = 0$ pointwise away from the vertices. This implies that the connection D splits according to the eigenspaces of h, and since the connection D is indecomposable, h must be a constant multiple of the identity matrix, which concludes the proof. □

4B. *The inverse map.* For an irreducible representation $\rho : \pi_1(X) \to \mathrm{SL}(r, \mathbb{C})$, with $[\rho] \in \mathcal{M}^c_{\mathrm{char}}$ and $E = E_c$, Theorem 3.3 then shows that there exists a unique ρ-equivariant harmonic map $u : \widetilde{X} \to \mathrm{SL}(r, \mathbb{C})/\,\mathrm{SU}(r)$. As in Section 2B, let d_A and ψ be the associated unitary connection and Higgs field. Since u is harmonic, d_A is the pullback of the Levi-Civita connection on $\mathrm{SL}(r, \mathbb{C})/\,\mathrm{SU}(r)$, and ψ is the derivative of u, we also have the equation

$$d_A^* \psi = 0 \tag{4-11}$$

almost everywhere (in fact by Theorem 3.3 everywhere away from the 0-skeleton).

Proposition 4.5. *If u is harmonic, α is as in Theorem 3.3 and $\delta < \alpha$, then $\mathcal{D} \in \mathcal{A}^{\mathbb{C}}_{\mathrm{bal,flat},\delta}(E)$. The metric on the bundle E induces a decomposition of $\mathcal{D}$ into skew-adjoint and self-adjoint parts, $\mathcal{D} = d_A + \psi$, where $d_A \in \mathcal{A}_{\mathrm{bal},\delta}(E)$ and $\psi \in \Omega^1_{\mathrm{bal},\delta}(i\,\mathrm{ad}(E))$. Furthermore, $\mathcal{D}$, d_A and ψ are smooth (over X_*).*

Proof. The construction in Section 2B shows that the connection $\mathcal{D}$ is induced from the trivial connection on the universal cover; hence it is clearly balanced, flat and $L^2_{1,\delta}$. Furthermore, since $d_A = u^*\nabla$ and $\psi = u^{-1}du$, Theorem 3.3 and (2-9) imply that d_A and ψ are balanced. Therefore, since $u : X \to \mathrm{SL}(r, \mathbb{C})/\,\mathrm{SU}(r)$ is a Lipschitz map over the compact space X, in order to show $d_A \in \mathcal{A}_{\mathrm{bal},\delta}(E)$ and $\psi \in \Omega^1_{\mathrm{bal}}(i\,\mathrm{ad}(E))_\delta$, it suffices to show $du \in L^2_{1,\delta}$.

First we show $du \in L^2_\delta$. By Theorem 3.3, $|\partial u/\partial r| \leq C r^{\alpha-1}$ and $|\partial u/\partial\theta| \leq C r^\alpha$ for some positive α. Using the coordinate transformation $r = e^{-t}$ we see that $|\partial u/\partial\theta| \leq C e^{-\alpha t}$ and

$$\left|\frac{\partial u}{\partial t}\right| = \left|\frac{\partial u}{\partial r}\frac{dr}{dt}\right| \leq C r^{\alpha-1} r = C e^{-\alpha t}.$$

Therefore, $du \in L^2_\delta$ if $\delta < \alpha$. Similarly, we use the estimates on the second derivatives of u to show that $du \in L^2_{1,\delta}$. We have $|\partial^2 u/\partial\theta^2| \leq C e^{-\alpha t}$, and we can

compute
$$\left|\frac{\partial^2 u}{\partial t\,\partial\theta}\right| = \left|\frac{\partial^2 u}{\partial r\,\partial\theta}\frac{dr}{dt}\right| \le Cr^{\alpha-1}r = Ce^{-\alpha t}$$
and similarly
$$\left|\frac{\partial^2 u}{\partial t^2}\right| = \left|\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial r}\frac{dr}{dt}\right)\right| = \left|\frac{\partial^2 u}{\partial r^2}\left(\frac{dr}{dt}\right)^2 + \frac{\partial u}{\partial t}\left(\frac{d}{dr}\frac{dr}{dt}\right)\frac{dr}{dt}\right|$$
$$\le Cr^{\alpha-2}r^2 + Cr^{\alpha-1}r^2 \le Ce^{-\alpha t} + Ce^{-(\alpha+1)t} \le Ce^{-\alpha t},$$
where in the last step we use the fact that $t \ge 0$ near a vertex. Therefore, $du \in L^2_{1,\delta}$ if $\delta < \alpha$. □

Theorem 4.6. *The map $\beta : \mathcal{M}^c_{\text{char}} \to \mathcal{M}_{\text{Higgs}}(E)$ defined by $\beta([\rho]) = [(d_A, \psi)]$ is a continuous inverse of α.*

Proof. The first step is to show the map β is well-defined. Given ρ, Proposition 4.5 implies that $d_A \in \mathcal{A}_{\text{bal},\delta}(E)$ and $\psi \in \Omega_{\text{bal}^1,\delta}(i\,\text{ad}(E))$. Moreover, we claim that the pair (d_A, ψ) is irreducible. If $\rho_* : \pi_1(X_*) \to \mathrm{SL}(r, \mathbb{C})$ denotes the holonomy of the flat connection $d_A + \psi$ then, as pointed out in Definition 3.6, $\rho_* = \rho \circ p$, where $p : \pi_1(X_*) \to \pi_1(X) = \pi_1(X_*)/\pi$ is the natural quotient map. Since by assumption ρ is irreducible, it follows that ρ_* is also irreducible, proving our claim.

Now, let ρ and $\rho' = \gamma\rho\gamma^{-1}$ be two representatives of $[\rho]$ and let u and u' be the two corresponding equivariant harmonic maps. It follows that $u' = \gamma \cdot u$, where $\cdot$ denotes the action of $\mathrm{SL}(r, \mathbb{C})$ on $\mathrm{SL}(r, \mathbb{C})/\mathrm{SU}(r)$. It follows that the induced decompositions $\mathcal{D} = d_A + \psi$ on the universal cover agree; hence after taking the quotients by ρ and $\rho' = \gamma\rho\gamma^{-1}$ respectively, the corresponding pairs are complex gauge-equivalent by γ. Proposition 4.4 then shows that they are $\mathcal{G}_{\text{bal},\delta}$-gauge-equivalent, which completes the proof that β is well-defined.

Next we will show that $\alpha(\beta([\rho])) = [\rho]$. Let $\beta([\rho]) = [(d_A, \psi)]$. According to (2-9), we have $d_A + \psi = \mathcal{D}$, where $\mathcal{D}$ is the connection on $\text{ad}(E_\rho)$ induced by the trivial connection on the universal cover which has holonomy ρ. Hence, $\alpha(\beta([\rho])) = [\rho]$.

Conversely, $\beta(\alpha([(d_A, \psi)])) = [(d_A, \psi)]$. Indeed, let (d_B, ϕ) be a smooth representative of $\beta(\alpha([(d_A, \psi)]))$. By applying α on both sides and what we just proved, $\alpha([(d_A, \psi)]) = \alpha([(d_B, \phi)])$. In other words, (d_A, ψ) and (d_B, ϕ) have conjugate holonomies. Since the holonomies of these pairs near the vertices are trivial by Proposition 3.5, Proposition 2.26 implies that the corresponding flat connections (and hence also the pairs) are complex gauge-equivalent. Thus Proposition 4.4 implies that (d_A, ψ) and (d_B, ϕ) are $\mathcal{G}_{\text{bal},\delta}$-gauge-equivalent; hence $\beta(\alpha([(d_A, \psi)])) = [(d_A, \psi)]$.

In order to prove continuity, let $\rho_i \to \rho \in \mathcal{M}^c_{\text{char}}$ and let u_i, u be the associated equivariant harmonic maps. Fix a compact fundamental domain $F \subset \widetilde{X}$ for the action of Γ and define ρ_i-equivariant maps $\tilde{u}_i$ by setting $\tilde{u}_i = u$ on F and extending ρ_i equivariantly on $\widetilde{X}$. Since the u_i are harmonic, the energy E^{u_i} satisfies

$$E^{u_i} \leq E^{\tilde{u}_i} = E^u.$$

The global Hölder bound, see [Daskalopoulos and Mese 2006, Theorem 3.12], implies that there is a subsequence (we call it again by $\{i\}$ by a slight abuse of notation) such that $u_i \to u_\infty$ uniformly on F. Furthermore, the convergence of the representations $\rho_i \to \rho$ implies that u_∞ is ρ-equivariant and Theorem 5.1 of [Daskalopoulos and Mese 2006] implies that u_∞ is harmonic. Finally, the uniqueness theorem, Theorem 4.6 of the same paper, implies that $u_\infty = u$. We have thus shown so far

$$u_i \to u \quad \text{locally uniformly.}$$

Let (d_{A_i}, ψ_i) denote the unitary connection and Higgs field associated with the harmonic map u_i. By Theorem 3.3 together with the proof of Proposition 4.5 (in this we use the third-derivative estimates) we obtain that the $L^2_{2,\delta}$-norm of (A_i, ψ_i) is uniformly bounded, and thus there exists a subsequence (we call it again by $\{i\}$ by a slight abuse of notation) such that $(d_{A_i}, \psi_i) \to (d_A, \psi)$ weakly in $L^2_{2,\delta}$ and hence strongly in $L^2_{1,\delta}$. □

References

[Adams 1975] R. A. Adams, *Sobolev spaces*, Pure and Applied Mathematics **65**, Academic Press, New York, 1975. MR Zbl

[Agmon et al. 1964] S. Agmon, A. Douglis, and L. Nirenberg, "Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions, II", *Comm. Pure Appl. Math.* **17** (1964), 35–92. MR Zbl

[Arapura et al. 2016] D. Arapura, A. Dimca, and R. Hain, "On the fundamental groups of normal varieties", *Commun. Contemp. Math.* **18**:4 (2016), art. id. 1550065. MR Zbl

[Armstrong 1983] M. A. Armstrong, *Basic topology*, Springer, 1983. MR Zbl

[Balaji et al. 2013] V. Balaji, P. Barik, and D. S. Nagaraj, "On degenerations of moduli of Hitchin pairs", *Electron. Res. Announc. Math. Sci.* **20** (2013), 103–108. MR Zbl

[Carlson and Toledo 1989] J. A. Carlson and D. Toledo, "Harmonic mappings of Kähler manifolds to locally symmetric spaces", *Inst. Hautes Études Sci. Publ. Math.* **69** (1989), 173–201. MR Zbl

[Chen 1995] J. Chen, "On energy minimizing mappings between and into singular spaces", *Duke Math. J.* **79**:1 (1995), 77–99. MR Zbl

[Corlette 1988] K. Corlette, "Flat G-bundles with canonical metrics", *J. Differential Geom.* **28**:3 (1988), 361–382. MR Zbl

[Corlette 1992] K. Corlette, "Archimedean superrigidity and hyperbolic geometry", *Ann. of Math.* (2) **135**:1 (1992), 165–182. MR Zbl

[Corlette and Simpson 2008] K. Corlette and C. Simpson, "On the classification of rank-two representations of quasiprojective fundamental groups", *Compos. Math.* **144**:5 (2008), 1271–1331. MR Zbl

[Daskalopoulos and Mese 2006] G. Daskalopoulos and C. Mese, "Harmonic maps from 2-complexes", *Comm. Anal. Geom.* **14**:3 (2006), 497–549. MR Zbl

[Daskalopoulos and Mese 2008] G. Daskalopoulos and C. Mese, "Harmonic maps from a simplicial complex and geometric rigidity", *J. Differential Geom.* **78**:2 (2008), 269–293. MR Zbl

[Daskalopoulos and Mese 2009] G. Daskalopoulos and C. Mese, "Fixed point and rigidity theorems for harmonic maps into NPC spaces", *Geom. Dedicata* **141** (2009), 33–57. MR Zbl

[Daskalopoulos and Wentworth 1997] G. D. Daskalopoulos and R. A. Wentworth, "Geometric quantization for the moduli space of vector bundles with parabolic structure", pp. 119–155 in *Geometry, topology and physics* (Campinas, 1996), edited by B. N. Apanasov et al., de Gruyter, Berlin, 1997. MR Zbl

[Donaldson 1987] S. K. Donaldson, "Twisted harmonic maps and the self-duality equations", *Proc. London Math. Soc.* (3) **55**:1 (1987), 127–131. MR Zbl

[Eells and Fuglede 2001] J. Eells and B. Fuglede, *Harmonic maps between Riemannian polyhedra*, Cambridge Tracts in Mathematics **142**, Cambridge University Press, 2001. MR Zbl

[Gilbarg and Trudinger 1983] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, 2nd ed., Grundlehren der Mathematischen Wissenschaften **224**, Springer, 1983. MR Zbl

[Griffiths and Morgan 1981] P. A. Griffiths and J. W. Morgan, *Rational homotopy theory and differential forms*, Progress in Mathematics **16**, Birkhäuser, Boston, 1981. MR Zbl

[Gromov and Schoen 1992] M. Gromov and R. Schoen, "Harmonic maps into singular spaces and p-adic superrigidity for lattices in groups of rank one", *Inst. Hautes Études Sci. Publ. Math.* **76** (1992), 165–246. MR Zbl

[Hartman 1964] P. Hartman, *Ordinary differential equations*, John Wiley & Sons, New York, 1964. MR Zbl

[Hitchin 1987] N. J. Hitchin, "The self-duality equations on a Riemann surface", *Proc. London Math. Soc.* (3) **55**:1 (1987), 59–126. MR Zbl

[Jost 1997] J. Jost, *Nonpositive curvature: geometric and analytic aspects*, Birkhäuser, Basel, 1997. MR Zbl

[Jost and Zuo 1996] J. Jost and K. Zuo, "Harmonic maps and $\mathrm{Sl}(r, \mathbb{C})$-representations of fundamental groups of quasiprojective manifolds", *J. Algebraic Geom.* **5**:1 (1996), 77–106. MR Zbl

[Jost et al. 2007] J. Jost, Y.-H. Yang, and K. Zuo, "The cohomology of a variation of polarized Hodge structures over a quasi-compact Kähler manifold", *J. Algebraic Geom.* **16**:3 (2007), 401–434. MR Zbl

[Kapovich and Kollár 2014] M. Kapovich and J. Kollár, "Fundamental groups of links of isolated singularities", *J. Amer. Math. Soc.* **27**:4 (2014), 929–952. MR Zbl

[Kinderlehrer et al. 1978] D. Kinderlehrer, L. Nirenberg, and J. Spruck, "Regularity in elliptic free boundary problems", *J. Analyse Math.* **34** (1978), 86–119. MR Zbl

[Lockhart and McOwen 1985] R. B. Lockhart and R. C. McOwen, "Elliptic differential operators on noncompact manifolds", *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* (4) **12**:3 (1985), 409–447. MR Zbl

[Lojasiewicz 1964] S. Lojasiewicz, "Triangulation of semi-analytic sets", *Ann. Scuola Norm. Sup. Pisa* (3) **18** (1964), 449–474. MR Zbl

[Mese 2002] C. Mese, "Uniqueness theorems for harmonic maps into metric spaces", *Commun. Contemp. Math.* **4**:4 (2002), 725–750. MR Zbl

[Simon 1996] L. Simon, *Theorems on regularity and singularity of energy minimizing maps*, Birkhäuser, Basel, 1996. MR Zbl

[Simpson 1988] C. T. Simpson, “Constructing variations of Hodge structure using Yang–Mills theory and applications to uniformization”, *J. Amer. Math. Soc.* **1**:4 (1988), 867–918. MR Zbl

[Simpson 1990] C. T. Simpson, “Harmonic bundles on noncompact curves”, *J. Amer. Math. Soc.* **3**:3 (1990), 713–770. MR Zbl

[Simpson 1992] C. T. Simpson, “Higgs bundles and local systems”, *Inst. Hautes Études Sci. Publ. Math.* **75** (1992), 5–95. MR Zbl

[Siu 1980] Y. T. Siu, “The complex-analyticity of harmonic maps and the strong rigidity of compact Kähler manifolds”, *Ann. of Math.* (2) **112**:1 (1980), 73–111. MR Zbl

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GEORGIOS DASKALOPOULOS
DEPARTMENT OF MATHEMATICS
BROWN UNIVERSITY
PROVIDENCE, RI
UNITED STATES

daskal@math.brown.edu

CHIKAKO MESE
DEPARTMENT OF MATHEMATICS
JOHNS HOPKINS UNIVERSITY
BALTIMORE, MD
UNITED STATES

cmese@math.jhu.edu

GRAEME WILKIN
DEPARTMENT OF MATHEMATICS
NATIONAL UNIVERSITY OF SINGAPORE
SINGAPORE

graeme@nus.edu.sg

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