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BESOV-WEAK-HERZ SPACES AND GLOBAL SOLUTIONS FOR NAVIER–STOKES EQUATIONS

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We consider the incompressible Navier-Stokes equations (NS) in $\mathbb{R}^n$ for $n \geq 2$. Global well-posedness is proved in critical Besov-weak-Herz spaces (BWH-spaces) that consist in Besov spaces based on weak-Herz spaces. These spaces are larger than some critical spaces considered in previous works for NS. For our purposes, we need to develop a basic theory for BWH-spaces containing properties and estimates such as heat semigroup estimates, embedding theorems, interpolation properties, among others. In particular, we prove a characterization of Besov-weak-Herz spaces as interpolation of Sobolev-weak-Herz ones, which is key in our arguments. Self-similarity and asymptotic behavior of solutions are also discussed. Our class of spaces and its properties developed here could also be employed to study other PDEs of elliptic, parabolic and conservation-law type.

1. Introduction

This paper is concerned with the incompressible Navier–Stokes equations

$$\text{(1-1)} \qquad \begin{cases} \frac{\partial u}{\partial t} - \Delta u + u \cdot \nabla u + \nabla \rho = 0 & \text{in } \mathbb{R}^n \times (0, \infty), \\ \nabla \cdot u = 0 & \text{in } \mathbb{R}^n \times (0, \infty), \\ u(0) = u_0 & \text{in } \mathbb{R}^n, \end{cases}$$

where $n \geq 2$, ρ is the pressure, $u = (u_j)_{j=1}^n$ is the velocity field and u_0 is a given initial velocity satisfying $\nabla \cdot u_0 = 0$.

After applying the Leray–Hopf projector $\mathbb{P}$ and using Duhamel's principle, the Cauchy problem (1-1) can be reduced to the integral formulation

$$\text{(1-2)} \quad u(t) = G(t)u_0 - \int_0^t G(t-\tau)\mathbb{P}\,\mathrm{div}(u \otimes u)(\tau)\,d\tau := G(t)u_0 + B(u,u)(t),$$

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where $u \otimes v := (u_i v_j)_{1\le i,j\le n}$ is a matrix-valued function and $G(t) = e^{t\Delta}$ is the heat semigroup. The operator $\mathbb{P}$ can be expressed as $\mathbb{P} = (\mathbb{P}_{i,j})_{n\times n}$ where $\mathbb{P}_{i,j} := \delta_{i,j} + \mathcal{R}_i\mathcal{R}_j$, $\delta_{i,j}$ is the Kronecker delta and $\mathcal{R}_i = (-\Delta)^{-1/2}\partial_i$ is the i-th Riesz transform. Divergence-free solutions for (1-2) are called mild solutions for (1-1). Note that if u is a smooth solution for (1-1) (or (1-2)), then

$$u_\lambda(x,t) := \lambda u(\lambda x, \lambda^2 t) \tag{1-3}$$

is also a solution with initial data

$$(u_0)_\lambda(x) = \lambda u_0(\lambda x). \tag{1-4}$$

Recall that given a Banach space Y we say that it has scaling degree equal to $k \in \mathbb{R}$ if $\|f(\lambda x)\|_Y \approx \lambda^k \|f\|_Y$ for all $\lambda > 0$ and $f \in Y$. Motivated by (1-4), a Banach space Y is called critical for (1-1) if it has scaling degree equal to -1, that is, if $\|f\|_Y \approx \|\lambda f(\lambda x)\|_Y$ for all $\lambda > 0$ and $f \in Y$. In turn, a solution of (1-1) which is invariant by the scaling (1-3), i.e., $u = u_\lambda$, is called a self-similar solution of (1-1). Note that in order to obtain self-similar solutions, the initial data should be homogeneous of degree -1.

Over the years, global-in-time well-posedness of small solutions for (1-1) in critical spaces has attracted the interest of a number of authors. Without making a complete list, we mention works in the following spaces: homogeneous Sobolev $\dot{H}^{1/2}(\mathbb{R}^3)$ [Fujita and Kato 1964], Lebesgue $L^n(\mathbb{R}^n)$ [Kato 1984], Marcinkiewicz $L^{n,\infty}(\mathbb{R}^n)$ [Barraza 1996; Yamazaki 2000], Morrey $\mathcal{M}^n_q(\mathbb{R}^n)$ [Giga and Miyakawa 1989; Kato 1992; Taylor 1992], weak-Morrey $\mathcal{M}^n_{q,\infty}(\mathbb{R}^n)$ [Miao and Yuan 2007; Lemarié-Rieusset 2015; Ferreira 2016], PM^{n-1}-spaces [Cannone and Karch 2004], Besov $\dot{B}^{n/p-1}_{p,\infty}(\mathbb{R}^n)$ for $p > n$ [Cannone 1997], Fourier–Besov $F\dot{B}^{n-1-n/p}_{p,\infty}$ [Iwabuchi and Takada 2014; Konieczny and Yoneda 2011], homogeneous weak-Herz spaces $W\dot{K}^0_{n,\infty}(\mathbb{R}^n)$ [Tsutsui 2011], Fourier–Herz $\mathcal{B}^{-1}_r = F\dot{B}^{-1}_{1,r}$ with $r \in [1,2]$ [Cannone and Wu 2012; Iwabuchi and Takada 2014; Lei and Lin 2011], homogeneous Besov–Morrey $\mathcal{N}^{n/r-1}_{r,q,\infty}$ with $r > n$ [Kozono and Yamazaki 1994; Mazzucato 2003], and BMO^{-1} [Koch and Tataru 2001]. The reader can find other examples in the nice review [Lemarié-Rieusset 2002]. Up until now, to the best of our knowledge, BMO^{-1} and $\mathcal{N}^{n/r-1}_{r,1,\infty}$ are maximal critical spaces for (1-1) in the sense that a larger critical space in which small solutions of (1-1) are globally well-posed is not known.

The purpose of this paper is to provide a new critical Besov type class for global well-posedness of solutions for (1-1) by assuming a smallness condition on initial data norms. Here we consider homogeneous Besov-weak-Herz spaces $\dot{B}W\dot{K}^{\alpha,s}_{p,q,r}$, which are a type of Besov space based on homogeneous weak-Herz spaces $W\dot{K}^\alpha_{p,q}$. They are a natural extension of the spaces $BK^{\alpha,s}_{p,q,r}$ introduced in [Xu 2005] (see Definition 2.5 in Section 2B). The Herz space $K^\alpha_{p,q}$ was introduced by Herz [1968] but his definition is not appropriate for our purposes. Later, Johnson [1974] obtained

a characterization of the $K^{\alpha}_{p,q}$-norm in terms of L^p-norms over annuli which is the base for the definition of the spaces $W\dot{K}^{\alpha}_{p,q}$ in [Tsutsui 2011] and is the same one that we use. In order to achieve our aims, we need to develop properties for $W\dot{K}^{\alpha}_{p,q,r}$- and $\dot{B}W\dot{K}^{\alpha,s}_{p,q,r}$-spaces such as the Hölder inequality, estimates for convolution operators, embedding theorems, interpolation properties, among others (see Section 2). In particular, a characterization of Besov-weak-Herz spaces in terms of interpolation of Sobolev-weak-Herz ones is proved, which is key in our arguments (see Lemma 2.14). Moreover, we prove estimates for the heat semigroup, as well as for the bilinear term $B(u, v)$ in (1-2), in the context of $\dot{B}W\dot{K}^{\alpha,s}_{p,q,r}$-spaces. We also point out that these spaces and their basic theory developed here could be employed to study other PDEs of elliptic, parabolic and conservation-law type. It is worth observing that some arguments in this paper are inspired by some of those in [Kozono and Yamazaki 1994] that analyzed (1-1) in Besov–Morrey spaces.

In what follows, we state our global well-posedness result.

Theorem 1.1. *Let* $1 \le q \le \infty$, $n/2 < p < \infty$ *and* $0 \le \alpha < \min\{1 - n/(2p), n/(2p)\}$. *There exist* $\epsilon > 0$ *and* $\delta > 0$ *such that if* $u_0 \in \dot{B}W\dot{K}^{\alpha,\alpha+n/p-1}_{p,q,\infty}$ *with* $\nabla \cdot u_0 = 0$ *and* $\|u_0\|_{\dot{B}W\dot{K}^{\alpha,\alpha+n/p-1}_{p,q,\infty}} \le \delta$, *then problem* (1-1) *has a unique mild solution*

$$u \in L^{\infty}\big((0,\infty);\ \dot{B}W\dot{K}^{\alpha,\alpha+n/p-1}_{p,q,\infty}\big)$$

such that

$$\|u\|_X := \|u\|_{L^{\infty}((0,\infty);\dot{B}W\dot{K}^{\alpha,\alpha+n/p-1}_{p,q,\infty})} + \sup_{t>0} t^{\frac{1}{2}-(\alpha/2+n/(4p))}\|u\|_{W\dot{K}^{\alpha}_{2p,2q}} \le 2\epsilon.$$

Moreover, $u(t) \overset{*}{\rightharpoonup} u_0$ *in* $\dot{B}^{-1}_{\infty,\infty}$, *as* $t \to 0^+$, *and solutions depend continuously on initial data.*

As a matter of fact, one can show that the solution in Theorem 1.1 is time-continuous for $t > 0$. We have the continuous inclusions $L^n \subset L^{n,\infty} \subset W\dot{K}^0_{n,\infty} \subset \dot{B}W\dot{K}^{0,0}_{n,\infty,\infty}$ (see Lemmas 2.7 and 2.12) and

$$\dot{H}^{n/2-1} \subset L^n \subset \dot{B}^{n/p-1}_{p,\infty} \subset \dot{B}W\dot{K}^{0,n/p-1}_{p,\infty,\infty}, \qquad \text{for } p \ge n \text{ (see Remark 2.6).}$$

So our initial data class extends those of some previous works; for instance, the ones in [Fujita and Kato 1964; Kato 1984; Barraza 1996; Cannone 1997; Yamazaki 2000; Tsutsui 2011].

Notice that the parameter s corresponds to the regularity index of the Besov type space $\dot{B}W\dot{K}^{\alpha,s}_{p,q,r}$. Considering the family $\{\dot{B}W\dot{K}^{0,n/p-1}_{p,\infty,\infty}\}_{p>n/2}$, in the positive regularity range $n/2 < p < n$ we are dealing with spaces smaller than those with $p > n$ (negative regularity), because of the Sobolev embedding $\dot{B}W\dot{K}^{0,n/p_2-1}_{p_2,\infty,\infty} \subset \dot{B}W\dot{K}^{0,n/p_1-1}_{p_1,\infty,\infty}$ when $p_2 < p_1$ (see Lemma 2.13). For $p > n$, it is not clear to us whether there are inclusion relations between $\dot{B}W\dot{K}^{0,n/p-1}_{p,\infty,\infty}$ and BMO^{-1} or between

$\dot{B}WK^{0,n/p-1}_{p,\infty,\infty}$ and $\mathcal{N}^{n/r-1}_{r,1,\infty}$ with $r > n$. In this sense, our result seems to give a new critical initial data class for existence of small global mild solutions for (1-1). In any case, it would be suitable to recall that well-posedness involves more properties than only existence of solutions, namely existence, uniqueness, persistence, and continuous dependence on initial data, which together characterize a good behavior of the Navier–Stokes flow in the considered space.

We finish with some comments about self-similarity and asymptotic behavior of solutions. It is not difficult to see that for $n \le p < \infty$ the function $f(x) = |x|^{-1}$ belongs to $\dot{B}WK^{0,n/p-1}_{p,\infty,\infty}$. So, the homogeneous Besov-weak-Herz spaces (at least some of them) contain homogeneous functions of degree -1. Thus, if one assumes further that the initial data u_0 is a homogeneous vector field of degree -1, then a standard procedure involving a Picard type sequence gives that the solution obtained in Theorem 1.1 is in fact self-similar. Moreover, following some estimates and arguments in the proof of Theorem 1.1, with some extra effort, it is possible to prove that if we have u_0 and v_0 satisfying $\lim_{t\to\infty} \|G(t)(u_0 - v_0)\|_{\dot{B}WK^{\alpha,\alpha+n/p-1}_{p,q,\infty}} = 0$, then

$$\lim_{t\to\infty} \|u(\cdot,t) - v(\cdot,t)\|_{\dot{B}WK^{\alpha,\alpha+n/p-1}_{p,q,\infty}} = 0,$$

where u and v are the solutions obtained in Theorem 1.1 with initial data u_0 and v_0, respectively.

The plan of this paper is as follows. Section 2 is devoted to function spaces, with Herz and Sobolev–Herz spaces considered in Section 2A, while Sobolev-weak-Herz and Besov-weak-Herz spaces are addressed in Section 2B. The proof of Theorem 1.1 is performed in the final section: In Section 3A we provide linear estimates for the heat semigroup. Section 3B is devoted to bilinear estimates for $B(\cdot,\cdot)$ in our setting. After obtaining the required estimates, the proof is concluded in Section 3C by means of a contraction argument.

2. Function spaces

In this section we recall some definitions and properties about function spaces that will be considered throughout this paper.

2A. ***Weak-Herz and Sobolev-weak-Herz spaces.*** For an integer $k \in \mathbb{Z}$, we define the set A_k as

$$A_k = \{x \in \mathbb{R}^n : 2^{k-1} \le |x| < 2^k\}, \tag{2-1}$$

and observe that $\mathbb{R}^n \setminus \{0\} = \bigcup_{k\in\mathbb{Z}} A_k$. Taking $x \in A_k$ we have that

$$y \in A_m \text{ and } m \le k \;\Rightarrow\; 2^{k-1} - 2^m \le |x-y| < 2^k + 2^m,$$
$$y \in A_m \text{ and } m \ge k \;\Rightarrow\; 2^{m-1} - 2^k \le |x-y| < 2^m + 2^k.$$

Consider also the sets

$$\begin{aligned} C_{m,k} &= \{\xi : 2^{k-1} - 2^m \le |\xi| < 2^k + 2^m\}, \\ \widetilde{C}_{m,k} &= \{\xi : 2^{m-1} - 2^k \le |\xi| < 2^m + 2^k\}. \end{aligned} \tag{2-2}$$

We now define the weak-Herz spaces:

Definition 2.1. Let $1 < p \le \infty$, $1 \le q \le \infty$ and $\alpha \in \mathbb{R}$. The homogeneous weak-Herz space $W\dot{K}^{\alpha}_{p,q} = W\dot{K}^{\alpha}_{p,q}(\mathbb{R}^n)$ is defined as the set of all measurable functions such that the following quantity is finite:

$$\|f\|_{W\dot{K}^{\alpha}_{p,q}} := \begin{cases} \left(\sum_{k\in\mathbb{Z}} 2^{k\alpha q} \|f\|^q_{L^{p,\infty}(A_k)} \right)^{1/q} & \text{if } q < \infty, \\ \sup_{k\in\mathbb{Z}} 2^{k\alpha} \|f\|_{L^{p,\infty}(A_k)} & \text{if } q = \infty. \end{cases} \tag{2-3}$$

For $\alpha \in \mathbb{R}$, $1 < p \le \infty$ and $1 \le q \le \infty$, the quantity $\|\cdot\|_{W\dot{K}^{\alpha}_{p,q}}$ defines a norm in $W\dot{K}^{\alpha}_{p,q}$ and the pair $(W\dot{K}^{\alpha}_{p,q}, \|\cdot\|_{W\dot{K}^{\alpha}_{p,q}})$ is a Banach space (see, e.g., [Hernández and Yang 1999; Tsutsui 2011]).

The Hölder inequality holds in the setting of homogeneous weak-Herz spaces (see [Tsutsui 2011]). To be more precise, if $1 < p, p_1, p_2 \le \infty$, $1 \le q, q_1, q_2 \le \infty$ and $\alpha, \alpha_1, \alpha_2 \in \mathbb{R}$ are such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$ and $\alpha = \alpha_1 + \alpha_2$, then

$$\|fg\|_{W\dot{K}^{\alpha}_{p,q}} \le C \|f\|_{W\dot{K}^{\alpha_1}_{p_1,q_1}} \|g\|_{W\dot{K}^{\alpha_2}_{p_2,q_2}}, \tag{2-4}$$

where $C > 0$ is a universal constant. In fact, for all $k \in \mathbb{Z}$, we have

$$\|fg\|_{L^{p,\infty}(A_k)} \le C \|f\|_{L^{p_1,\infty}(A_k)} \|g\|_{L^{p_2,\infty}(A_k)},$$

and therefore

$$\begin{aligned} \|fg\|_{W\dot{K}^{\alpha}_{p,q}} &= \left(\sum_{k\in\mathbb{Z}} 2^{k\alpha q} \|fg\|^q_{L^{p,\infty}(A_k)} \right)^{1/q} \\ &\le C \left(\sum_{k\in\mathbb{Z}} 2^{k\alpha_1 q} \|f\|^q_{L^{p_1,\infty}(A_k)} 2^{k\alpha_2 q} \|g\|^q_{L^{p_2,\infty}(A_k)} \right)^{1/q} \\ &\le C \|f\|_{W\dot{K}^{\alpha_1}_{p_1,q_1}} \|g\|_{W\dot{K}^{\alpha_2}_{p_2,q_2}}. \end{aligned} \tag{2-5}$$

Taking in particular $(\alpha_1, p_1, q_1) = (0, \infty, \infty)$ in (2-5), we obtain

$$\|fg\|_{W\dot{K}^{\alpha}_{p,q}} \le C \|f\|_{L^{\infty}(\mathbb{R}^n)} \|g\|_{W\dot{K}^{\alpha}_{p,q}}. \tag{2-6}$$

Later, we will need to estimate some convolution operators, particularly the heat semigroup, in weak-Herz and Besov-weak-Herz spaces. The following lemma will be useful for that purpose.

Lemma 2.2 (convolution). *Let* $1 \le p_1 < \infty$ *and* $1 < r, p_2 < \infty$ *be such that* $1 + \frac{1}{r} = \frac{1}{p_1} + \frac{1}{p_2}$. *Further, let* $1 \le q \le \infty$, $-\frac{n}{r} < \alpha < n\big(1 - \frac{1}{p_2}\big)$, *and* $\theta \in L^{p_1}(\mathbb{R}^n)$ *be such that* $\theta|\cdot|^{n/p_1} \in L^\infty(\mathbb{R}^n)$. *There exists a positive constant* C *independent of* θ *such that*

$$\|\theta * f\|_{W\dot{K}^{\alpha}_{r,q}} \le C \max\Big\{\|\theta\|_{L^{p_1}}, \||\cdot|^{n/p_1}\theta\|_{L^\infty}\Big\}\|f\|_{W\dot{K}^{\alpha}_{p_2,q}}, \tag{2-7}$$

for all $f \in W\dot{K}^{\alpha}_{p_2,q}$.

Proof. Denote $f_m = f|_{A_m}$. Recalling the decomposition (2-1), for $k \in \mathbb{Z}$ we can estimate

$$\begin{aligned} &2^{k\alpha}\|\theta * f\|_{L^{r,\infty}(A_k)} \\ &\quad\le 2^{k\alpha}\left\{\Big\|\sum_{m\le k-2}\theta * f_m\Big\|_{L^{r,\infty}(A_k)} + \Big\|\sum_{m=k-1}^{k+1}\theta * f_m\Big\|_{L^{r,\infty}(A_k)} + \Big\|\sum_{m\ge k+2}\theta * f_m\Big\|_{L^{r,\infty}(A_k)}\right\} \\ &\quad=: I_1^k + I_2^k + I_3^k. \end{aligned} \tag{2-8}$$

Using the notations in (2-2) and the change of variable $z = k - m$, we handle the term I_3^k as follows:

$$\begin{aligned} I_3^k &\le 2^{k\alpha}\Big\|\sum_{m\ge k+2}\theta * f_m\Big\|_{L^{r,\infty}(A_k)} \\ &\le 2^{k\alpha}\Big\|\sum_{m\ge k+2}\theta * f_m\Big\|_{L^{r}(A_k)} \\ &\le 2^{k\alpha}\left(\int_{A_k}\Big|\sum_{m\ge k+2}\int_{\mathbb{R}^n}\theta(x-y)f_m(y)\,dy\Big|^r dx\right)^{1/r} \\ &= 2^{k\alpha}\left(\int_{A_k}\Big|\sum_{m\ge k+2}\int_{\mathbb{R}^n}\theta(x-y)\chi_{\widetilde{C}_{m,k}}(x-y)f_m(y)dy\Big|^r dx\right)^{1/r} \\ &\le C\||\cdot|^{n/p_1}\theta\|_{L^\infty}2^{k\alpha} \\ &\quad\times\left(\int_{A_k}\left(\sum_{m\ge k+2}\int_{\mathbb{R}^n}|x-y|^{-n/p_1}\chi_{\widetilde{C}_{m,k}}(x-y)\times|f_m(y)|\,dy\right)^r dx\right)^{1/r} \\ &\le C\||\cdot|^{n/p_1}\theta\|_{L^\infty}2^{k\alpha}\left(\int_{A_k}\left(\sum_{m\ge k+2}2^{-mn/p_1}\|f\|_{L^1(A_m)}\right)^r dx\right)^{1/r}. \end{aligned} \tag{2-9}$$

Recalling the inclusion $L^{p_2,\infty}(A_m) \hookrightarrow L^1(A_m)$, we can continue to estimate the

right-hand side of the fifth inequality in (2-9) in order to obtain

$$
\begin{aligned}
(2\text{-}10)\quad I_3^k &\le C\|\,|\cdot|^{n/p_1}\theta\|_{L^\infty}2^{k\alpha}\\
&\quad\times\left(\int_{A_k}\left(\sum_{m\ge k+2}2^{-mn/p_1}2^{nm(1-1/p_2)}\|f\|_{L^{p_2,\infty}(A_m)}\right)^r dx\right)^{1/r}\\
&\le C\|\,|\cdot|^{n/p_1}\theta\|_{L^\infty}2^{k\alpha}2^{kn/r}\sum_{m\ge k+2}2^{-mn/r}\|f\|_{L^{p_2,\infty}(A_m)}\\
&\le C\|\,|\cdot|^{n/p_1}\theta\|_{L^\infty}\sum_{-2\ge z}2^{k(\alpha+n/r)}2^{(z-k)n/r}\|f\|_{L^{p_2,\infty}(A_{k-z})}\\
&\le C\|\,|\cdot|^{n/p_1}\theta\|_{L^\infty}\sum_{-2\ge z}2^{k\alpha}2^{zn/r}2^{-(k-z)\alpha}2^{(k-z)\alpha}\|f\|_{L^{p_2,\infty}(A_{k-z})}\\
&\le C\|\,|\cdot|^{n/p_1}\theta\|_{L^\infty}\sum_{-2\ge z}2^{z(n/r+\alpha)}2^{(k-z)\alpha}\|f\|_{L^{p_2,\infty}(A_{k-z})}.
\end{aligned}
$$

This estimate and the Minkowski inequality lead us to (with the usual modification in the case $q=\infty$)

$$
\left(\sum_{k\in\mathbb{Z}}(I_3^k)^q\right)^{1/q}\le CM_\theta\|f\|_{W\dot{K}^\alpha_{p_2,q}}.
$$

For the summand I_2^k, we estimate

$$
\begin{aligned}
I_2^k &\le 2^{k\alpha}\sum_{m=k-1}^{k+1}\|\theta*f_m\|_{L^{r,\infty}(A_k)}\\
&\le 2^{k\alpha}\sum_{m=k-1}^{k+1}\|\theta*f_m\|_{L^{r,\infty}(\mathbb{R}^n)}\\
&\le 2^{k\alpha}\sum_{m=k-1}^{k+1}\|\theta\|_{L^{p_1,\infty}}\|f_m\|_{L^{p_2,\infty}}\\
&\le C\|\theta\|_{L^{p_1}}\sum_{l=-1}^{1}2^{(k+l)\alpha}\|f\|_{L^{p_2,\infty}(A_{k+l})},
\end{aligned}
$$

which implies

$$
\left(\sum_{k\in\mathbb{Z}}(I_2^k)^q\right)^{1/q}\le CM_\theta\|f\|_{W\dot{K}^\alpha_{p_2,q}}.
$$

Proceeding similarly to the estimates (2-9)–(2-10) but considering $C_{m,k}$ in place of $\widetilde{C}_{m,k}$, the summand I_1^k can be estimated as

$$I_1^k \le C\|\,|\cdot|^{n/p_1}\theta\|_{L^\infty} 2^{k\alpha}\left(\int_{A_k}\left(\sum_{m\le k-2}\int_{\mathbb{R}^n}|x-y|^{-n/p_1}\chi_{C_{m,k}}(x-y)\times|f_m(y)|dy\right)^r dx\right)^{1/r} \tag{2-11}$$

$$\le C\|\,|\cdot|^{n/p_1}\theta\|_{L^\infty} 2^{k\alpha}\left(\int_{A_k}\left(\sum_{m\le k-2}2^{-kn/p_1}\|f\|_{L^1(A_m)}\right)^r dx\right)^{1/r}$$

$$\le C\|\,|\cdot|^{n/p_1}\theta\|_{L^\infty} 2^{k\alpha}2^{kn/r}\sum_{m\le k-2}2^{-kn/p_1}\|f\|_{L^1(A_m)}$$

$$\le C\|\,|\cdot|^{n/p_1}\theta\|_{L^\infty}\sum_{m\le k-2}2^{k(\alpha-n+n/p_2)}\|f\|_{L^1(A_m)}$$

$$\le C\|\,|\cdot|^{n/p_1}\theta\|_{L^\infty}\sum_{2\le z}2^{k(\alpha-n+n/p_2)}2^{n(k-z)(1-1/p_2)}\|f\|_{L^{p_2,\infty}(A_{k-z})}$$

$$\le C\|\,|\cdot|^{n/p_1}\theta\|_{L^\infty}\sum_{2\le z}2^{z(\alpha-n+n/p_2)}2^{(k-z)\alpha}\|f\|_{L^{p_2,\infty}(A_{k-z})}.$$

It follows from (2-11) that

$$\left(\sum_{k\in\mathbb{Z}}(I_1^k)^q\right)^{1/q} \le C M_\theta \|f\|_{W\dot{K}^\alpha_{p_2,q}}.$$

Finally, the desired estimate is obtained after recalling the norm (2-3) and using the above estimates for I_j^k in (2-8). □

Let $\varphi \in C_c^\infty(\mathbb{R}^n\setminus\{0\})$ be radially symmetric and such that

$$\operatorname{supp}\varphi \subset \{x : \tfrac{3}{4}\le|x|\le\tfrac{8}{3}\}$$

and

$$\sum_{j\in\mathbb{N}}\varphi_j(\xi)=1, \quad \text{for all } \xi\in\mathbb{R}^n\setminus\{0\},$$

where $\varphi_j(\xi) := \varphi(\xi 2^{-j})$. Now we can define the well-known localization operators Δ_j and S_j:

$$\Delta_j f = \varphi_j(D)f = (\mathcal{F}^{-1}\varphi_j) * f,$$
$$S_k f = \sum_{j\le k}\Delta_j f.$$

It is easy to see that we have the identities

$$\Delta_j\Delta_k f = 0 \text{ if } |j-k|\ge 2 \quad \text{and} \quad \Delta_j(S_{k-2}g\,\Delta_k f) = 0 \text{ if } |j-k|\ge 5.$$

Finally, Bony's decomposition gives (see, e.g., [Bony 1981])

$$fg = T_f g + T_g f + R(fg),$$

where

$$T_f g = \sum_{j\in Z} S_{j-2} f \Delta_j g, \quad R(fg) = \sum_{j\in Z} \Delta_j f \tilde{\Delta}_j g \quad \text{and} \quad \tilde{\Delta}_j g = \sum_{|j-j'|\le 1} \Delta_{j'} g.$$

The next lemma will be useful in order to estimate some multiplier operators in Besov-weak-Herz spaces.

Lemma 2.3. *Let* $1 < p < \infty$, $1 \le q \le \infty$, $-\frac{n}{p} < \alpha < n\big(1-\frac{1}{p}\big)$, $m \in \mathbb{R}$ *and* $D_j = \{x : \frac{3}{4}2^j \le |x| \le \frac{8}{3}2^j\}$ *for* $j \in \mathbb{Z}$. *Let* P *be a* C^n*-function on*

$$\widetilde{D}_j := D_{j-1} \cup D_j \cup D_{j+1}$$

such that $|\partial_\xi^\beta P(\xi)| \le C2^{(m-|\beta|)j}$ *for all* $\xi \in \widetilde{D}_j$ *and multi-index* β *satisfying* $|\beta| \le [n/2]+1$. *Then, we have that*

$$\|(P\hat{f})^\vee\|_{W\dot{K}^\alpha_{p,q}} \le C2^{jm}\|f\|_{W\dot{K}^\alpha_{p,q}},$$

for all $f \in W\dot{K}^\alpha_{p,q}$ *such that* $\operatorname{supp}\hat{f} \subset D_j$.

Proof. We start by defining $\tilde{\varphi}_j = \varphi_{j-1} + \varphi_j + \varphi_{j+1}$ and $K(x) = (P\tilde{\varphi}_j)^\vee$. Since $\operatorname{supp}\hat{f} \subset D_j$ we have that $P(\xi)\hat{f}(\xi) = P(\xi)\tilde{\varphi}_j(\xi)\hat{f}(\xi)$, and therefore $(P\hat{f})^\vee = (P\tilde{\varphi}_j\hat{f})^\vee = K * f$.

Using Lemma 2.2 we get

$$\|(P\hat{f})^\vee\|_{W\dot{K}^\alpha_{p,q}} \le C\max\{\|K\|_{L^1}, \||\cdot|^n K\|_{L^\infty}\}\|f\|_{W\dot{K}^\alpha_{p,q}}.$$

It remains to show that $\max\{\|K\|_{L^1}, \||\cdot|^n K\|_{L^\infty}\} \le C2^{mj}$. For that, let $N \in \mathbb{N}$ be such that $n/2 < N \le n$ and proceed as follows:

$$\begin{aligned}
\|K\|_{L^1} &= \int_{B(0,2^{-j})} K(y) + \int_{|y|\ge 2^{-j}} K(y) \\
&\le \left(\int_{B(0,2^{-j})} 1\right)^{1/2}\left(\int_{B(0,2^{-j})} |K(y)|^2\right)^{1/2} \\
&\qquad + \left(\int_{|y|\ge 2^{-j}} |y|^{-2N}\right)^{1/2}\left(\int_{|y|\ge 2^{-j}} |y|^{2N}|K(y)|^2\right)^{1/2} \\
&\le C2^{-jn/2}\|P\tilde{\varphi}_j\|_{L^2} + C2^{-j(-N+n/2)}\sum_{|\beta|=N}\|(\cdot)^\beta K\|_{L^2} \\
&\le C2^{-jn/2}\|P\tilde{\varphi}_j\|_{L^2} + C2^{-j(-N+n/2)}\sum_{|\beta|=N}\|\partial^\beta(P\tilde{\varphi}_j)\|_{L^2} \\
&\le C2^{-jn/2}C2^{mj}2^{jn/2} + C2^{-j(-N+n/2)}C2^{j(m-N)}2^{jn/2} \\
&\le C2^{mj}.
\end{aligned}$$

For the norm $\| |\cdot|^n K\|_{L^\infty}$, we have that

$$\begin{aligned}\| |\cdot|^n K\|_{L^\infty} &\le \sum_{|\beta|=n} \|(\cdot)^\beta K\|_{L^\infty} \le C \sum_{|\beta|=n} \|\partial^\beta (P\tilde{\varphi}_j)\|_{L^1} \\ &\le C \sum_{|\beta|=n} 2^{j(m-n)} 2^{jn} \le C 2^{mj},\end{aligned}$$

as required. □

2B. ***Sobolev-weak-Herz spaces and Besov-weak-Herz spaces.*** In this section we introduce the homogeneous Sobolev-weak-Herz spaces and Besov-weak-Herz spaces. We also shall prove a number of properties about these spaces that will be useful in our study of the Navier–Stokes equations. These spaces are a generalization of Sobolev–Herz and Besov–Herz spaces found in [Xu 2005].

Definition 2.4. Let $1 < p \le \infty$, $1 \le q \le \infty$ and $\alpha, s \in \mathbb{R}$. Recall the Riesz operator $\widehat{I^s f} = |\xi|^s \hat{f}$. The homogeneous Sobolev-weak-Herz spaces $W\dot{K}^{\alpha,s}_{p,q} = W\dot{K}^{\alpha,s}_{p,q}(\mathbb{R}^n)$ are defined as

$$W\dot{K}^{\alpha,s}_{p,q} = \{f \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P} : \|I^s f\|_{W\dot{K}^{\alpha}_{p,q}} < \infty\}. \tag{2-12}$$

Definition 2.5. Let $1 < p \le \infty$, $1 \le q, r \le \infty$ and $\alpha, s \in \mathbb{R}$. The homogeneous Besov-weak-Herz spaces $\dot{B}W\dot{K}^{\alpha,s}_{p,q,r} = \dot{B}W\dot{K}^{\alpha,s}_{p,q,r}(\mathbb{R}^n)$ are defined as

$$\dot{B}W\dot{K}^{\alpha,s}_{p,q,r} = \{f \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P} : \|f\|_{\dot{B}W\dot{K}^{\alpha,s}_{p,q,r}} < \infty\},$$

where

$$\|f\|_{\dot{B}W\dot{K}^{\alpha,s}_{p,q,r}} := \begin{cases} \left(\sum_{j\in\mathbb{Z}} 2^{jsr} \|\Delta_j f\|^r_{W\dot{K}^{\alpha}_{p,q}}\right)^{1/r} & \text{if } r < \infty \\ \sup_{j\in\mathbb{Z}} 2^{js} \|\Delta_j f\|_{W\dot{K}^{\alpha}_{p,q}} & \text{if } r = \infty. \end{cases} \tag{2-13}$$

Remark 2.6. (i) The spaces $W\dot{K}^{\alpha,s}_{p,q}$ and $\dot{B}W\dot{K}^{\alpha,s}_{p,q,r}$ are Banach spaces endowed with the norms $\|\cdot\|_{W\dot{K}^{\alpha}_{p,q}}$ and $\|\cdot\|_{\dot{B}W\dot{K}^{\alpha,s}_{p,q,r}}$, respectively.

(ii) The continuous inclusion $\dot{B}^s_{p,r}(\mathbb{R}^n) \subset \dot{B}W\dot{K}^{0,s}_{p,\infty,r}(\mathbb{R}^n)$ holds for all $s \in \mathbb{R}$, $1 < p \le \infty$, and $1 \le r \le \infty$, where $\dot{B}^s_{p,r}$ stands for homogeneous Besov spaces. To show that, it is sufficient to recall the definition of Besov spaces (see [Bergh and Löfström 1976, p. 146]) and (2-13) and to use the inclusion $L^p \subset W\dot{K}^0_{p,\infty}$ that is going to be shown in the lemma below.

The next lemma contains relations between weak-L^p, weak-Herz and Morrey spaces. For the definition and some properties about Morrey spaces we refer the reader to [Kozono and Yamazaki 1994] (see also [Kato 1992] for an equivalent definition and further properties).

Lemma 2.7. *For $1 < p < \infty$, we have the continuous inclusion*

$$L^p \subsetneq L^{p,\infty} \subsetneq W\dot{K}^0_{p,\infty}. \tag{2-14}$$

Moreover, let $\mathcal{M}^r_q$ stand for homogeneous Morrey spaces, $1 \le q \le r < \infty$ and $n/r \neq \alpha + n/p$ when $q < p$. Then

$$W\dot{K}^\alpha_{p,\infty} \not\subset \mathcal{M}^r_q. \tag{2-15}$$

Proof. The first inclusion in (2-14) is well known, so we only prove the second one. For that, it is sufficient to note that $\|f\|_{L^{p,\infty}(A_k)} \le \|f\|_{L^{p,\infty}(\mathbb{R}^n)}$ for all $k \in \mathbb{Z}$ and after to take the supremum over k. In order to see the strictness of the inclusion, take $x_k = \frac{3}{2}2^{k-1}\vec{e}_1$ and $h(x) := \sum_{k=1}^\infty |x - x_k|^{-n/p}\chi_{B(0,1/8)}(x - x_k)$. It is clear that h is an element of $W\dot{K}^0_{p,\infty}$ but not of $L^{p,\infty}(\mathbb{R}^n)$.

Now we turn to (2-15). For $f(x) = |x|^{-n/p}$, we have that $f \in L^{p,\infty} \subset W\dot{K}^0_{p,\infty}$. On the other hand, for any $q \ge p$ note that $\|f\|_{L^q(B(0,R))} = \infty$, and then $f \notin \mathcal{M}^r_q$ for any r. Finally, if $n/r \neq \alpha + n/p$ then $W\dot{K}^\alpha_{p,\infty} \subset \mathcal{M}^r_q$ (and the reverse) never could hold. This follows from an easy scaling analysis of the space norms; in fact, the scaling of $\mathcal{M}^r_q$ is $-n/r$ and that of $W\dot{K}^0_{p,\infty}$ is $-\alpha - n/p$. □

In the next remark, we recall some inclusion and noninclusion relations involving Herz, weak-Herz, Besov and bmo^{-1} spaces that can be found in [Tsutsui 2011].

Remark 2.8. (i) For $1 < p < \sigma < \infty$ and $0 < \alpha < n(1 - 1/p)$, we have

$$W\dot{K}^\alpha_{p,\infty} \hookrightarrow \dot{B}^{-(\alpha+n(1/p-1/\sigma))}_{\sigma,\infty}, \quad \dot{K}^\alpha_{p,\infty} \hookrightarrow \dot{B}^{-\alpha}_{p,\infty} \quad \text{and} \quad W\dot{K}^0_{p,\sigma} \hookrightarrow \dot{B}^{-n(1/p-1/\sigma)}_{\sigma,\infty}.$$

(ii) For $1 < p < \infty$ and $0 \le \alpha < n(1 - 1/p)$, we have $W\dot{K}^\alpha_{p,\infty} \hookrightarrow \dot{B}^{-(\alpha+n/p)}_{\infty,\infty}$.

(iii) For $0 \le \alpha < n$, we have $\dot{K}^\alpha_{\infty,\infty} \hookrightarrow \dot{B}^{-\alpha}_{\infty,\infty}$.

(iv) For $1 < p \le \infty$ and $0 \le \alpha \le n(1 - 1/p)$, we have $W\dot{K}^\alpha_{p,1} \hookrightarrow \dot{B}^{-(\alpha+n/p)}_{\infty,\infty}$.

(v) We have $L^1 = \dot{K}^0_{1,1} \hookrightarrow \dot{B}^{-n}_{\infty,\infty}$. For $n < p \le \infty$ and $0 \le \alpha < 1 - n/p$, the inclusion $W\dot{K}^\alpha_{p,\infty} \hookrightarrow bmo^{-1}$ holds.

(vi) For $1 < p < \sigma < \infty$ and $-n(1/p - 1/\sigma) < \alpha \le 0$, $W\dot{K}^\alpha_{p,\infty} \hookrightarrow \dot{B}^{-(\alpha+n(1/p-1/\sigma))}_{\sigma,\infty}$ does not hold.

(vii) For $1 < p < \infty$ and $-n/p < \alpha < 0$, $W\dot{K}^\alpha_{p,\infty} \hookrightarrow \dot{B}^{-(\alpha+n/p)}_{\infty,\infty}$ does not hold.

Remark 2.9. Using the interpolation properties of homogeneous Besov spaces and homogeneous Besov-weak-Herz spaces (see Lemma 2.14 below) and item (ii) of Remark 2.8, for $1 < p < \infty$ and $0 \le \alpha < n(1 - 1/p)$ we can obtain

$$\dot{B}W\dot{K}^{\alpha,s}_{p,\infty,r} \hookrightarrow \dot{B}^{s-(\alpha+n/p)}_{\infty,r}. \tag{2-16}$$

In particular, $\dot{B}W\dot{K}^{\alpha,\alpha+n/p-1}_{p,\infty,\infty} \hookrightarrow \dot{B}^{-1}_{\infty,\infty}$ and

$$\dot{B}W\dot{K}^{0,n/p}_{p,\infty,1} \hookrightarrow \dot{B}^0_{\infty,1} \hookrightarrow L^\infty. \tag{2-17}$$

Moreover, from Remark 2.8(vi) and Lemma 2.12 below, it follows that the inclusion

$$\dot{B}W\dot{K}^{0,s}_{p,\infty,\infty} \hookrightarrow \dot{B}^{s-n(1/p-1/\sigma)}_{\sigma,\infty}$$

does not hold for any $s \in \mathbb{R}$, $1 < p < \infty$ and $1 \le \sigma < \infty$.

Remark 2.10. Note that for $s-(\alpha+n/p) < 0$ and $r > 1$, or $s-(\alpha+n/p) \le 0$ and $r = 1$, the inclusion (2-16) implies that for $f \in \dot{B}W\dot{K}^{\alpha,s}_{p,\infty,r}$ the series $\sum_{j=-\infty}^{\infty} \Delta_j f$ converges in $\mathcal{S}'$ to a representative of f in $\mathcal{S}'/\mathcal{P}$ (see, e.g., [Lemarié-Rieusset 2002]). So, in these cases the space $\dot{B}W\dot{K}^{\alpha,s}_{p,\infty,r}$ can be regarded as a subspace of $\mathcal{S}'$. Hereafter, we say that $f \in \mathcal{S}'$ belongs to $\dot{B}W\dot{K}^{\alpha,s}_{p,\infty,r}$ with $s-(\alpha+n/p) < 0$ and $r > 1$, or $s-(\alpha+n/p) \le 0$ and $r = 1$, if f is the canonical representative of the class in $\mathcal{S}'/\mathcal{P}$, namely $f = \sum_{j=-\infty}^{\infty} \Delta_j f$ in $\mathcal{S}'$.

A multiplier theorem of Hörmander–Mihlin type will be needed in our setting. This is the subject of the next lemma. In fact, the main part of the proof has already been done in Lemma 2.3.

Lemma 2.11. *Let $1 < p < \infty$, $1 \le q, r \le \infty$, $-n/p < \alpha < n(1-1/p)$ and $m, s \in \mathbb{R}$. Let $P \in C^n(\mathbb{R}^n \setminus \{0\})$ be a function such that $|\partial^\beta_\xi P(\xi)| \le C|\xi|^{(m-|\beta|)}$ for all multi-index β satisfying $|\beta| \le n$. Then*

$$\|P(D)f\|_{\dot{B}W\dot{K}^{\alpha,s-m}_{p,q,r}} \le C\|f\|_{\dot{B}W\dot{K}^{\alpha,s}_{p,q,r}}.$$

Proof. Note that for each $j \in \mathbb{Z}$ we have that $|\xi|^{m-|\beta|} \le C2^{j(m-|\beta|)}$ for all $\xi \in \widetilde{D}_j$, and therefore $|\partial^\beta_\xi P(\xi)| \le C2^{j(m-|\beta|)}$. On the other hand, since $\operatorname{supp} \widehat{\Delta_j f} \subset D_j$ we can use Lemma 2.3 in order to get

$$\|\Delta_j(P(D)f)\|_{W\dot{K}^\alpha_{p,q}} = \|P(D)(\Delta_j f)\|_{W\dot{K}^\alpha_{p,q}} \le C2^{jm}\|\Delta_j f\|_{W\dot{K}^\alpha_{p,q}}. \tag{2-18}$$

The result follows by multiplying (2-18) by $2^{j(s-m)}$ and then taking the l^r-norm. □

In what follows we present some inclusions involving Sobolev-weak-Herz and Besov-weak-Herz spaces.

Lemma 2.12. *Let $s \in \mathbb{R}$, $1 < p < \infty$, $1 \le q \le \infty$ and $-n/p < \alpha < n(1-1/p)$. We have the following continuous inclusions*:

$$\dot{B}W\dot{K}^{\alpha,0}_{p,q,1} \subset W\dot{K}^\alpha_{p,q} \subset \dot{B}W\dot{K}^{\alpha,0}_{p,q,\infty} \tag{2-19}$$

$$\dot{B}W\dot{K}^{\alpha,s}_{p,q,1} \subset W\dot{K}^{\alpha,s}_{p,q} \subset \dot{B}W\dot{K}^{\alpha,s}_{p,q,\infty}. \tag{2-20}$$

Proof. For $f \in \dot{B}W\dot{K}^{\alpha,0}_{p,q,1}$, we can employ the decomposition $f = \sum_{j\in\mathbb{Z}} \Delta_j f$ in order to estimate

$$\|f\|_{L^{p,\infty}(A_k)} \le \sum_{j\in\mathbb{Z}} \|\Delta_j f\|_{L^{p,\infty}(A_k)}.$$

Thus, using the Minkowski inequality, we arrive at (with the usual modification in the case $q=\infty$)

$$\begin{aligned}\|f\|_{W\dot{K}^{\alpha}_{p,q}} &\le \Big(\sum_{k\in\mathbb{Z}} 2^{k\alpha q}\|f\|^q_{L^{p,\infty}(A_k)}\Big)^{1/q} \le \Big(\sum_{k\in\mathbb{Z}}\Big(\sum_{j\in\mathbb{Z}} 2^{k\alpha}\|\Delta_j f\|_{L^{p,\infty}(A_k)}\Big)^q\Big)^{1/q}\\ &\le \sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}} 2^{k\alpha q}\|\Delta_j f\|^q_{L^{p,\infty}(A_k)}\Big)^{1/q} = \sum_{j\in\mathbb{Z}}\|\Delta_j f\|_{W\dot{K}^{\alpha}_{p,q}}\\ &= \|f\|_{\dot{B}W\dot{K}^{\alpha,0}_{p,q,1}},\end{aligned}$$

which implies the first inclusion in (2-19). Now, let $f\in W\dot{K}^{\alpha}_{p,q}$ and note that in fact we have that $f\in\mathcal{S}'/\mathcal{P}$. Moreover, using Lemma 2.2 we get

$$\|f\|_{\dot{B}W\dot{K}^{\alpha,0}_{p,q,\infty}} = \sup_{j\in\mathbb{Z}}\|\Delta_j f\|_{W\dot{K}^{\alpha}_{p,q}} \le C\sup_{j\in\mathbb{Z}}\|f\|_{W\dot{K}^{\alpha}_{p,q}} = C\|f\|_{W\dot{K}^{\alpha}_{p,q}},$$

and then the second inclusion in (2-19) holds.

For (2-20), we can use Lemma 2.3 in order to estimate

$$\begin{aligned}\|f\|_{W\dot{K}^{\alpha,s}_{p,q}} = \|I^s f\|_{W\dot{K}^{\alpha}_{p,q}} &\le \|I^s f\|_{\dot{B}W\dot{K}^{\alpha,0}_{p,q,1}} = \sum_{j\in\mathbb{Z}}\|\Delta_j I^s f\|_{W\dot{K}^{\alpha}_{p,q}}\\ &\le C\sum_{j\in\mathbb{Z}} 2^{js}\|\Delta_j f\|_{W\dot{K}^{\alpha}_{p,q}} = C\|f\|_{\dot{B}W\dot{K}^{\alpha,s}_{p,q,1}}.\end{aligned}$$

Moreover, Lemma 2.3 also can be used to obtain

$$\begin{aligned}\|f\|_{\dot{B}W\dot{K}^{\alpha,s}_{p,q,\infty}} &= \sup_{j\in\mathbb{Z}} 2^{js}\|\Delta_j f\|_{W\dot{K}^{\alpha}_{p,q}} = \sup_{j\in\mathbb{Z}} 2^{js}\|I^{-s}\Delta_j I^s f\|_{W\dot{K}^{\alpha}_{p,q}}\\ &\le C\sup_{j\in\mathbb{Z}}\|\Delta_j I^s f\|_{W\dot{K}^{\alpha}_{p,q}} \le C\sup_{j\in\mathbb{Z}}\|I^s f\|_{W\dot{K}^{\alpha}_{p,q}}\\ &= C\|I^s f\|_{W\dot{K}^{\alpha}_{p,q}} = C\|f\|_{W\dot{K}^{\alpha,s}_{p,q}},\end{aligned}$$

for all $f\in W\dot{K}^{\alpha,s}_{p,q}$, as required. □

Now we present an embedding theorem of Sobolev type.

Lemma 2.13. *Let $s\in\mathbb{R}$, $1<p<\infty$, $1\le q,r\le\infty$, $p\le p_1<\infty$, $1<p_2\le p_1$ and $-\frac{n}{p}<\alpha<n\big(1+\frac{1}{p_1}-\frac{1}{p_2}-\frac{1}{p}\big)$. Then*

$$\|f\|_{\dot{B}W\dot{K}^{\alpha,s}_{p,q,r}} \le C\|f\|_{\dot{B}W\dot{K}^{\alpha+n(1/p-1/p_1),s+n(1/p_2-1/p_1)}_{p_2,q,r}}. \tag{2-21}$$

In particular, for $\frac{n}{2}<p<\infty$ and $0\le\alpha<\min\{1-\frac{n}{2p},\frac{n}{2p}\}$, it follows that

$$\|f\|_{\dot{B}W\dot{K}^{\alpha,s}_{2p,q,r}} \le C\|f\|_{\dot{B}W\dot{K}^{2\alpha,\alpha+s+n/(2p)}_{p,q,r}}. \tag{2-22}$$

Proof. Using the Hölder inequality, it follows that

$$\|\Delta_j f\|_{W\dot{K}^{\alpha}_{p,q,r}} \le C\|\Delta_j f\|_{W\dot{K}^{\alpha+n(1/p-1/p_1)}_{p_1,q,r}}.$$

Also, we have that $\varphi_j \hat{f} = \tilde{\varphi}_j \varphi_j \hat{f}$, that is, $\Delta_j f = (\tilde{\varphi}_j)^{\vee} * \Delta_j f$. So, using Lemma 2.2 we get

$$\begin{aligned}\|\Delta_j f\|_{W\dot{K}^{\alpha+n(1/p-1/p_1)}_{p_1,q,r}} &= \|(\tilde{\varphi}_j)^{\vee} * \Delta_j f\|_{W\dot{K}^{\alpha+n(1/p-1/p_1)}_{p_1,q,r}} \\ &\leq C \max\left\{\|(\tilde{\varphi}_j)^{\vee}\|_{L^{p*}}, \||\cdot|^{\frac{n}{p*}}(\tilde{\varphi}_j)^{\vee}\|_{L^{\infty}}\right\}\|\Delta_j f\|_{W\dot{K}^{\alpha+n(1/p-1/p_1)}_{p_2,q,r}},\end{aligned}$$

where $1+\frac{1}{p_1} = \frac{1}{p*} + \frac{1}{p_2}$. It is easy to check that

$$\max\left\{\|(\tilde{\varphi}_j)^{\vee}\|_{L^{p*}}, \||\cdot|^{n/p*}(\tilde{\varphi}_j)^{\vee}\|_{L^{\infty}}\right\} \leq C 2^{jn(1/p_2-1/p_1)},$$

and then

$$\|\Delta_j f\|_{W\dot{K}^{\alpha+n(1/p-1/p_1)}_{p_1,q,r}} \leq C 2^{jn(1/p_2-1/p_1)}\|\Delta_j f\|_{W\dot{K}^{\alpha+n(1/p-1/p_1)}_{p_2,q,r}},$$

which gives (2-21). We conclude the proof by noting that for $0 \leq \alpha < n/2p$ there exists p_1 such that $p_1 \geq 2p$ and $\alpha = n\left(\frac{1}{p} - \frac{1}{p_1}\right)$. Moreover, $\alpha < n + \frac{n}{p_1} - \frac{1}{p} - \frac{n}{2p}$ because $\alpha < 1 - \frac{n}{2p} \leq \frac{n}{2} - \frac{n}{2p}$. So, (2-22) follows from (2-21) by choosing this value of p_1. □

We finish this section with a result that provides a characterization of homogeneous Besov-weak-Herz spaces as interpolation of two homogeneous Sobolev-weak-Herz ones.

Lemma 2.14. *Let $s_0, s_1, s \in \mathbb{R}$, $1 < p < \infty$, $1 \leq q, r \leq \infty$ and $-\frac{n}{p} < \alpha < n\left(1-\frac{1}{p}\right)$. If $s_0 \neq s_1$ and $s = (1-\theta)s_0 + \theta s_1$ with $\theta \in (0,1)$, then*

$$(W\dot{K}^{\alpha,s_0}_{p,q}, W\dot{K}^{\alpha,s_1}_{p,q})_{\theta,r} = \dot{B}W\dot{K}^{\alpha,s}_{p,q,r}.$$

Proof. Let $f = f_0 + f_1$ with $f_i \in W\dot{K}^{\alpha,s_i}_{p,q}$ $i = 0, 1$. By using Lemma 2.3 we get

$$\begin{aligned}\|\Delta_j f\|_{W\dot{K}^{\alpha}_{p,q}} &\leq \|\Delta_j f_0\|_{W\dot{K}^{\alpha}_{p,q}} + \|\Delta_j f_1\|_{W\dot{K}^{\alpha}_{p,q}} \\ &\leq C(2^{-s_0 j}\|I^{s_0}\Delta_j f_0\|_{W\dot{K}^{\alpha}_{p,q}} + 2^{-s_1 j}\|I^{s_1}\Delta_j f_1\|_{W\dot{K}^{\alpha}_{p,q}}) \\ &\leq C(2^{-s_0 j}\|I^{s_0} f_0\|_{W\dot{K}^{\alpha}_{p,q}} + 2^{-s_1 j}\|I^{s_1} f_1\|_{W\dot{K}^{\alpha}_{p,q}}) \\ &\leq C 2^{-s_0 j}(\|f_0\|_{W\dot{K}^{\alpha,s_0}_{p,q}} + 2^{(s_0-s_1)j}\|f_1\|_{W\dot{K}^{\alpha,s_1}_{p,q}}).\end{aligned} \tag{2-23}$$

It follows from (2-23) that

$$\|\Delta_j f\|_{W\dot{K}^{\alpha}_{p,q}} \leq C 2^{-s_0 j} K(2^{(s_0-s_1)j}, f, W\dot{K}^{\alpha,s_0}_{p,q}, W\dot{K}^{\alpha,s_1}_{p,q}).$$

Noting that $s - s_0 = -\theta(s_0 - s_1)$ and multiplying the previous inequality by 2^{js}, we arrive at

$$2^{sj}\|\Delta_j f\|_{W\dot{K}^{\alpha}_{p,q}} \leq C(2^{(s_0-s_1)j})^{-\theta} K(2^{(s_0-s_1)j}, f, W\dot{K}^{\alpha,s_0}_{p,q}, W\dot{K}^{\alpha,s_1}_{p,q}),$$

and then (see [Bergh and Löfström 1976, Lemma 3.1.3]) we can conclude that

$$\|f\|_{\dot{B}W\dot{K}^{\alpha,s}_{p,q,r}} \leq C\|f\|_{(W\dot{K}^{\alpha,s_0}_{p,q}, W\dot{K}^{\alpha,s_1}_{p,q})_{\theta,r}}.$$

To prove the reverse inequality, note that by using Lemma 2.3 again we have

$$\begin{aligned}
2^{(s-s_0)j} J(2^{(s_0-s_1)j}, \Delta_j f, W\dot{K}^{\alpha,s_0}_{p,q}, W\dot{K}^{\alpha,s_1}_{p,q}) &= 2^{(s-s_0)j} \max\{\|\Delta_j f\|_{W\dot{K}^{\alpha,s_0}_{p,q}}, 2^{(s_0-s_1)j}\|\Delta_j f\|_{W\dot{K}^{\alpha,s_1}_{p,q}}\} \\
&\le 2^{(s-s_0)j} \max\{2^{s_0 j}\|\Delta_j f\|_{W\dot{K}^{\alpha}_{p,q}}, 2^{s_0 j}\|\Delta_j f\|_{W\dot{K}^{\alpha}_{p,q}}\} \\
&\le 2^{sj} \max\{\|\Delta_j f\|_{W\dot{K}^{\alpha}_{p,q}}, \|\Delta_j f\|_{W\dot{K}^{\alpha}_{p,q}}\} \\
&= 2^{sj}\|\Delta_j f\|_{W\dot{K}^{\alpha}_{p,q}}.
\end{aligned}$$

Now the equivalence theorem (see [Bergh and Löfström 1976, Lemma 3.2.3]) leads us to

$$\|f\|_{(W\dot{K}^{\alpha,s_0}_{p,q}, W\dot{K}^{\alpha,s_1}_{p,q})_{\theta,r}} \le C\|f\|_{\dot{B}W\dot{K}^{\alpha,s}_{p,q,r}}.$$

The remainder of the proof is to show that in fact $f \in \dot{B}W\dot{K}^{\alpha,s}_{p,q,r}$ implies that $f \in W\dot{K}^{\alpha,s_0}_{p,q} + W\dot{K}^{\alpha,s_1}_{p,q}$. Suppose that $s_0 > s_1$ (without loss of generality). Using the decomposition $f = \sum_{j<0} \Delta_j f + \sum_{j\ge 0} \Delta_j f = f_0 + f_1$ and Lemma 2.3, we obtain

$$\begin{aligned}
\|f_0\|_{W\dot{K}^{\alpha,s_0}_{p,q}} &\le \sum_{j<0} \|\Delta_j f\|_{W\dot{K}^{\alpha,s_0}_{p,q}} \le \sum_{j<0} 2^{j(s_0-s)} 2^{js}\|\Delta_j f\|_{W\dot{K}^{\alpha}_{p,q}} \\
&\le C\left(\sum_{j<0} 2^{j(s_0-s)r'}\right)^{1/r'} \left(\sum_{j<0} 2^{jsr}\|\Delta_j f\|^r_{W\dot{K}^{\alpha}_{p,q}}\right)^{1/r} \\
&\le C\|f\|_{\dot{B}W\dot{K}^{\alpha,s}_{p,q,r}}.
\end{aligned}$$

Similarly, one has

$$\begin{aligned}
\|f_1\|_{W\dot{K}^{\alpha,s_1}_{p,q}} &\le \sum_{j\ge 0} \|\Delta_j f\|_{W\dot{K}^{\alpha,s_1}_{p,q}} \le \sum_{j\ge 0} 2^{j(s_1-s)} 2^{js}\|\Delta_j f\|_{W\dot{K}^{\alpha}_{p,q}} \\
&\le C\left(\sum_{j\ge 0} 2^{j(s_1-s)r'}\right)^{1/r'} \left(\sum_{j\ge 0} 2^{jsr}\|\Delta_j f\|^r_{W\dot{K}^{\alpha}_{p,q}}\right)^{1/r} \\
&\le C\|f\|_{\dot{B}W\dot{K}^{\alpha,s}_{p,q,r}}
\end{aligned}$$

and then we are done. □

3. Proof of Theorem 1.1

In the previous sections, we have derived key properties about homogeneous Besov-weak-Herz spaces. With these results in hand, we prove Theorem 1.1 in the present section.

3A. ***Heat kernel estimates.*** We start by providing estimates for the heat semigroup $\{G(t)\}_{t\geq 0}$ in Besov-weak-Herz spaces. Recall that in the whole space $\mathbb{R}^n$ this semigroup can be defined as $G(t)f = (\exp(-t|\xi|^2)\hat{f})^{\check{}}$ for all $f \in \mathcal{S}'$ and $t \geq 0$.

Lemma 3.1. *Let $s, \sigma \in \mathbb{R}$, $s \leq \sigma$, $1 < p < \infty$, $1 \leq q, r \leq \infty$ and $-\frac{n}{p} < \alpha < n\big(1-\frac{1}{p}\big)$. Then, there is $C > 0$ (independent of f) such that*

$$\|G(t)f\|_{\dot{B}W\dot{K}^{\alpha,\sigma}_{p,q,r}} \leq Ct^{(s-\sigma)/2}\|f\|_{\dot{B}W\dot{K}^{\alpha,s}_{p,q,r}}, \tag{3-1}$$

for all $t > 0$. Moreover, if $s < \sigma$, then we have the estimate

$$\|G(t)f\|_{\dot{B}W\dot{K}^{\alpha,\sigma}_{p,q,1}} \leq Ct^{(s-\sigma)/2}\|f\|_{\dot{B}W\dot{K}^{\alpha,s}_{p,q,\infty}}, \tag{3-2}$$

for all $t > 0$.

Proof. Firstly, observe that for each multi-index β there is a polynomial $p_\beta(\,\cdot\,)$ of degree $|\beta|$ such that

$$\partial^\beta_\xi(\exp(-t|\xi|^2)) = t^{|\beta|/2}p_\beta(\sqrt{t}\xi)\exp(-t|\xi|^2).$$

Therefore, for some $C > 0$ it follows that

$$|\partial^\beta_\xi(\exp(-t|\xi|^2))| \leq Ct^{-m/2}|\xi|^{-m-|\beta|}.$$

By employing Lemma 2.11, we obtain

$$\|G(t)f\|_{\dot{B}W\dot{K}^{\alpha,s-m}_{p,q,r}} \leq Ct^{-m/2}\|f\|_{\dot{B}W\dot{K}^{\alpha,s}_{p,q,r}}.$$

Taking now $m = s - \sigma$ we arrive at the inequality (3-1).

Next we turn to (3-2) and let $s < \sigma$. From (3-1) with $r = \infty$ we get

$$\|G(t)f\|_{\dot{B}W\dot{K}^{\alpha,2\sigma-s}_{p,q,\infty}} \leq Ct^{s-\sigma}\|f\|_{\dot{B}W\dot{K}^{\alpha,s}_{p,q,\infty}}$$

and

$$\|G(t)f\|_{\dot{B}W\dot{K}^{\alpha,s}_{p,q,\infty}} \leq C\|f\|_{\dot{B}W\dot{K}^{\alpha,s}_{p,q,\infty}}.$$

By using Lemma 2.14 and the reiteration theorem (see [Bergh and Löfström 1976, Theorem 3.5.3 and its remark]) we conclude that

$$G(t) : \dot{B}W\dot{K}^{\alpha,s}_{p,q,\infty} \to (\dot{B}W\dot{K}^{\alpha,2\sigma-s}_{p,q,\infty}, \dot{B}W\dot{K}^{\alpha,s}_{p,q,\infty})_{\frac{1}{2},1} = \dot{B}W\dot{K}^{\alpha,\sigma}_{p,q,1},$$

with $\|G(t)\|_{\dot{B}W\dot{K}^{\alpha,s}_{p,q,\infty}\to\dot{B}W\dot{K}^{\alpha,\sigma}_{p,q,1}} \leq Ct^{(s-\sigma)/2}$, which gives (3-2). □

3B. ***Bilinear estimate.*** Let us define the space X as

$$X=\Big\{u:(0,\infty)\to \dot{B}W\dot{K}^{\alpha,\alpha+n/p-1}_{p,q,\infty}\cap W\dot{K}^{\alpha}_{2p,2q} \text{ with } \nabla\cdot u=0 \text{ such that } \|u\|_X<\infty\Big\},$$

where

$$\|u\|_X := \|u\|_{L^\infty((0,\infty);\dot{B}W\dot{K}^{\alpha,\alpha+n/p-1}_{p,q,\infty})} + \sup_{t>0} t^{\frac{1}{2}-(\frac{\alpha}{2}+\frac{n}{4p})}\|u\|_{W\dot{K}^{\alpha}_{2p,2q}}. \tag{3-3}$$

We are going to prove the bilinear estimate

$$\|B(u,v)\|_X \le K\|u\|_X\|v\|_X. \tag{3-4}$$

We start by estimating the second part of the norm (3-3). For that, we use (2-19), (2-22), (3-2) and Lemma 2.11 in order to get

$$\begin{aligned}
\|B(u,v)(t)\|_{W\dot{K}^{\alpha}_{2p,2q}} &\le \|B(u,v)(t)\|_{\dot{B}W\dot{K}^{\alpha,0}_{2p,2q,1}}\\
&\le \|B(u,v)(t)\|_{\dot{B}W\dot{K}^{2\alpha,\alpha+n/(2p)}_{p,2q,1}}\\
&\le C\int_0^t \|G(t-\tau)\mathbb{P}\operatorname{div}(u\otimes v)\|_{\dot{B}W\dot{K}^{2\alpha,\alpha+n/(2p)}_{p,2q,1}}\,d\tau\\
&\le C\int_0^t (t-\tau)^{-\frac{1}{2}-(\frac{\alpha}{2}+\frac{n}{4p})}\|\mathbb{P}\operatorname{div}(u\otimes v)\|_{\dot{B}W\dot{K}^{2\alpha,-1}_{p,2q,\infty}}\,d\tau\\
&\le C\int_0^t (t-\tau)^{-\frac{1}{2}-(\frac{\alpha}{2}+\frac{n}{4p})}\|u\otimes v\|_{\dot{B}W\dot{K}^{2\alpha,0}_{p,2q,\infty}}\,d\tau\\
&\le C\int_0^t (t-\tau)^{-\frac{1}{2}-(\frac{\alpha}{2}+\frac{n}{4p})}\|u\otimes v\|_{W\dot{K}^{2\alpha}_{p,q}}\,d\tau\\
&\le C\int_0^t (t-\tau)^{-\frac{1}{2}-(\frac{\alpha}{2}+\frac{n}{4p})}\|u\|_{W\dot{K}^{\alpha}_{2p,2q}}\|v\|_{W\dot{K}^{\alpha}_{2p,2q}}\,d\tau\\
&\le C\int_0^t (t-\tau)^{-\frac{1}{2}-(\frac{\alpha}{2}+\frac{n}{4p})}\tau^{-2(\frac{1}{2}-(\frac{\alpha}{2}+\frac{n}{4p}))}\,d\tau\|u\|_X\|v\|_X\\
&\le Ct^{-\frac{1}{2}+(\frac{\alpha}{2}+\frac{n}{4p})}\mathcal{B}\Big(\alpha+\frac{n}{2p},\frac{1}{2}-\Big(\frac{\alpha}{2}+\frac{n}{4p}\Big)\Big)\|u\|_X\|v\|_X,
\end{aligned}$$

where $\mathcal{B}(\cdot,\cdot)$ denotes the beta function. The previous estimate leads us to

$$\sup_{t>0} t^{\frac{1}{2}-(\frac{\alpha}{2}+\frac{n}{4p})}\|B(u,v)(t)\|_{W\dot{K}^{\alpha}_{2p,2q}} \le C\|u\|_X\|v\|_X. \tag{3-5}$$

Moreover, for the first part of the norm (3-3), we have

$$
\begin{aligned}
\|B(u,v)(t)\|_{\dot{B}W\dot{K}^{\alpha,\alpha+n/p-1}_{p,q,\infty}} &\le \int_0^t \|G(t-\tau)\mathbb{P}\operatorname{div}[u\otimes v]\|_{\dot{B}W\dot{K}^{\alpha,\alpha+n/p-1}_{p,q,\infty}}\,d\tau\\
&\le C\int_0^t \|G(t-\tau)\mathbb{P}\operatorname{div}[u\otimes v]\|_{\dot{B}W\dot{K}^{2\alpha,2\alpha+n/p-1}_{p,q,\infty}}\,d\tau\\
&\le C\int_0^t (t-\tau)^{-(\alpha+\frac{n}{2p})}\|\mathbb{P}\operatorname{div}[u\otimes v]\|_{\dot{B}W\dot{K}^{2\alpha,-1}_{p,q,\infty}}\,d\tau\\
&\le C\int_0^t (t-\tau)^{-(\alpha+\frac{n}{2p})}\|u\otimes v\|_{\dot{B}W\dot{K}^{2\alpha,0}_{p,q,\infty}}\,d\tau\\
&\le C\int_0^t (t-\tau)^{-(\alpha+\frac{n}{2p})}\|u\otimes v\|_{W\dot{K}^{2\alpha}_{p,q}}\,d\tau\\
&\le C\|u\|_X\|v\|_X\int_0^t (t-\tau)^{-(\alpha+\frac{n}{2p})}\tau^{-2(\frac{1}{2}-(\frac{\alpha}{2}+\frac{n}{4p}))}d\tau\\
&\le C\mathcal{B}\Big(\alpha+\frac{n}{2p},1-\Big(\alpha+\frac{n}{2p}\Big)\Big)\|u\|_X\|v\|_X.
\end{aligned}
$$

In other words, we have obtained the estimate

$$
\|B(u,v)\|_{L^\infty((0,\infty);\dot{B}W\dot{K}^{\alpha,\alpha+n/p-1}_{p,q,\infty})}\le C\|u\|_X\|v\|_X. \tag{3-6}
$$

Finally, notice that the estimates (3-5) and (3-6) together give (3-4).

3C. ***Proof of Theorem 1.1.*** Existence and uniqueness. For $\epsilon>0$ (to be chosen later) let $\bar{B}(0,\epsilon)$ denote the closed ball in X and define the operator $\Psi:\bar{B}(0,2\epsilon)\to\bar{B}(0,2\epsilon)$ as

$$
\Psi(u)=G(t)u_0+B(u,u).
$$

First, note that by using (2-19), (3-2) , $\alpha+\frac{n}{2p}-1<0$ and (2-21) it follows that

$$
\begin{aligned}
\sup_{t>0} t^{\frac{1}{2}-(\frac{\alpha}{2}+\frac{n}{4p})}\|G(t)u_0\|_{W\dot{K}^{\alpha}_{2p,2q}} &\le C\sup_{t>0} t^{\frac{1}{2}-(\frac{\alpha}{2}+\frac{n}{4p})}\|G(t)u_0\|_{\dot{B}W\dot{K}^{\alpha,0}_{2p,2q,1}}\\
&\le C\|u_0\|_{\dot{B}W\dot{K}^{\alpha,\alpha+n/(2p)-1}_{2p,2q,\infty}}\le C\|u_0\|_{\dot{B}W\dot{K}^{\alpha,\alpha+n/p-1}_{p,q,\infty}}.
\end{aligned} \tag{3-7}
$$

Moreover, using (3-1) we obtain

$$
\|G(t)u_0\|_{\dot{B}W\dot{K}^{\alpha,\alpha+n/p-1}_{p,q,\infty}}\le C\|u_0\|_{\dot{B}W\dot{K}^{\alpha,\alpha+n/p-1}_{p,q,\infty}}.
$$

From the last two estimates, we get

$$
\|G(t)u_0\|_X\le C\|u_0\|_{\dot{B}W\dot{K}^{\alpha,\alpha+n/p-1}_{p,q,\infty}}. \tag{3-8}
$$

Take $0 < \epsilon < 1/4K$ and $0 < \delta < \epsilon/C$ where C is as in (3-8). It follows from (3-8) and (3-4) that

$$\begin{aligned}\|\Psi(u)\|_X &\le \|G(t)u_0\|_X + \|B(u,u)\|_X \\ &\le C\|u_0\|_{\dot{B}W\dot{K}^{\alpha,\alpha+n/p-1}_{p,q,\infty}} + K\|u\|_X\|v\|_X \le 2\epsilon.\end{aligned}$$

So, Ψ is well defined; moreover for $u, v \in \bar{B}(0, 2\epsilon)$ we have that

$$\begin{aligned}\|\Psi(u) - \Psi(v)\|_X &= \|B(u-v,u) + B(v,u-v)\|_X \\ &\le K\|u-v\|_X\|u\|_X + K\|v\|_X\|u-v\|_X \\ &\le 4K\epsilon\|u-v\|_X.\end{aligned} \tag{3-9}$$

Since $4K\epsilon < 1$, we get that Ψ is a contraction and then this part is concluded by the Banach fixed-point theorem. Notice that the continuous dependence with respect to the initial data u_0 follows from estimates (3-8) and (3-9).

Time-weak continuity at $t = 0$. The proof of the weak-$*$ convergence follows from the two following lemmas.

The first one is due to Kozono and Yamazaki [1994, p. 989.].

Lemma 3.2. *For every real number s and $u_0 \in \dot{B}^s_{\infty,\infty}$, we have $G(t)u_0 \overset{*}{\rightharpoonup} u_0$ in $\dot{B}^s_{\infty,\infty}$ as $t \to 0^+$.*

The second one is concerned with the weak-convergence of the bilinear term $B(u,u)$ and it concludes the proof.

Lemma 3.3. *Let $v \in X$. We have that $B(v,v)(t)$ converges to 0 in the weak-$*$ topology of $\dot{B}^{-1}_{\infty,\infty}$ as $t \to 0^+$.*

Proof. Let $\phi \in \dot{B}^1_{1,1}$ and $\epsilon > 0$ be an arbitrary number. We can choose $\tilde{\phi} \in \mathcal{S}$ such that $\|\phi - \tilde{\phi}\|_{\dot{B}^1_{1,1}} < \epsilon$. Then we have that

$$\begin{aligned}&|\langle B(v,v)(t), \phi - \tilde{\phi}\rangle| \\ &\quad\le \|B(v,v)(t)\|_{\dot{B}^{-1}_{\infty,\infty}}\|\phi - \tilde{\phi}\|_{\dot{B}^1_{1,1}} \\ &\quad\le C\|B(v,v)(t)\|_{\dot{B}W\dot{K}^{\alpha,\alpha+n/p-1}_{p,q,r}}\|\phi - \tilde{\phi}\|_{\dot{B}^1_{1,1}} \le K\|v\|_X^2\epsilon \le C\epsilon.\end{aligned} \tag{3-10}$$

On the other hand,

$$\begin{aligned}|\langle B(v,v)(t), \tilde{\phi}\rangle| &\le \int_0^t \left|\langle G(t-\tau)\mathbb{P}\operatorname{div}[v\otimes v](\tau), \tilde{\phi}\rangle\right| d\tau \\ &\le \int_0^t \left|\langle \mathbb{P}\operatorname{div}[v\otimes v](\tau), G(t-\tau)\tilde{\phi}\rangle\right| d\tau \\ &\le \int_0^t \|\operatorname{div}[v\otimes v](\tau)\|_{\dot{B}^{-1-2\alpha-n/p}_{\infty,\infty}}\|G(t-\tau)\tilde{\phi}\|_{\dot{B}^{1+2\alpha+n/p}_{1,1}} d\tau\end{aligned} \tag{3-11}$$

$$
\begin{aligned}
&\le C_{\tilde{\phi}} \int_0^t \|[v\otimes v](\tau)\|_{\dot{B}^{-2\alpha-n/p}_{\infty,\infty}} d\tau \\
&\le C_{\tilde{\phi}} \int_0^t \|[v\otimes v](\tau)\|_{W\dot{K}^{2\alpha}_{p,q}} d\tau \\
&\le C_{\tilde{\phi}} \int_0^t \tau^{[-\frac{1}{2}+(\frac{\alpha}{2}+\frac{n}{4p})]\cdot 2} \tau^{[\frac{1}{2}-(\frac{\alpha}{2}+\frac{n}{4p})]\cdot 2} \|v(\tau)\|^2_{W\dot{K}^{\alpha}_{2p,2q}} d\tau \\
&\le C_{\tilde{\phi}} \|v\|_X^2 \int_0^t \tau^{-1+\alpha+\frac{n}{2p}} d\tau \le C_{\tilde{\phi}} \|v\|_X^2 t^{\alpha+\frac{n}{2p}}.
\end{aligned}
$$

From (3-10) and (3-11), we obtain

$$
\begin{aligned}
0 &\le \limsup_{t\to 0^+} |\langle B(v,v)(t), \phi\rangle| \\
&\le \limsup_{t\to 0^+} |\langle B(v,v)(t), \phi-\tilde{\phi}\rangle| + \limsup_{t\to 0^+} |\langle B(v,v)(t), \tilde{\phi}\rangle| \le C\epsilon + 0.
\end{aligned}
$$

Since $\epsilon > 0$ is arbitrary, we conclude that $\lim_{t\to 0^+} |\langle B(v,v)(t),\phi\rangle| = 0$. Now, using that $\phi \in \dot{B}^1_{1,1}$ is arbitrary, we get the desired convergence. □

References

[Barraza 1996] O. A. Barraza, "Self-similar solutions in weak L^p-spaces of the Navier–Stokes equations", *Rev. Mat. Iberoamericana* **12**:2 (1996), 411–439. MR Zbl

[Bergh and Löfström 1976] J. Bergh and J. Löfström, *Interpolation spaces: an introduction*, Grundlehren der Math. Wissenschaften **223**, Springer, 1976. MR

[Bony 1981] J.-M. Bony, "Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires", *Ann. Sci. École Norm. Sup.* (4) **14**:2 (1981), 209–246. MR Zbl

[Cannone 1997] M. Cannone, "A generalization of a theorem by Kato on Navier–Stokes equations", *Rev. Mat. Iberoamericana* **13**:3 (1997), 515–541. MR Zbl

[Cannone and Karch 2004] M. Cannone and G. Karch, "Smooth or singular solutions to the Navier–Stokes system?", *J. Differential Equations* **197**:2 (2004), 247–274. MR Zbl

[Cannone and Wu 2012] M. Cannone and G. Wu, "Global well-posedness for Navier–Stokes equations in critical Fourier–Herz spaces", *Nonlinear Anal.* **75**:9 (2012), 3754–3760. MR Zbl

[Ferreira 2016] L. C. F. Ferreira, "On a bilinear estimate in weak-Morrey spaces and uniqueness for Navier–Stokes equations", *J. Math. Pures Appl.* (9) **105**:2 (2016), 228–247. MR Zbl

[Fujita and Kato 1964] H. Fujita and T. Kato, "On the Navier–Stokes initial value problem, I", *Arch. Rational Mech. Anal.* **16** (1964), 269–315. MR Zbl

[Giga and Miyakawa 1989] Y. Giga and T. Miyakawa, "Navier–Stokes flow in $\mathbb{R}^3$ with measures as initial vorticity and Morrey spaces", *Comm. Partial Differential Equations* **14**:5 (1989), 577–618. MR Zbl

[Hernández and Yang 1999] E. Hernández and D. Yang, "Interpolation of Herz spaces and applications", *Math. Nachr.* **205** (1999), 69–87. MR Zbl

[Herz 1968] C. S. Herz, "Lipschitz spaces and Bernstein's theorem on absolutely convergent Fourier transforms", *J. Math. Mech.* **18**:4 (1968), 283–323. MR Zbl

[Iwabuchi and Takada 2014] T. Iwabuchi and R. Takada, "Global well-posedness and ill-posedness for the Navier–Stokes equations with the Coriolis force in function spaces of Besov type", *J. Funct. Anal.* **267**:5 (2014), 1321–1337. MR Zbl

[Johnson 1974] R. Johnson, "Lipschitz spaces, Littlewood–Paley spaces, and convoluteurs", *Proc. London Math. Soc.* (3) **29** (1974), 127–141. MR Zbl

[Kato 1984] T. Kato, "Strong L^p-solutions of the Navier–Stokes equation in $\mathbb{R}^m$, with applications to weak solutions", *Math. Z.* **187**:4 (1984), 471–480. MR Zbl

[Kato 1992] T. Kato, "Strong solutions of the Navier–Stokes equation in Morrey spaces", *Bol. Soc. Brasil. Mat.* (*N.S.*) **22**:2 (1992), 127–155. MR Zbl

[Koch and Tataru 2001] H. Koch and D. Tataru, "Well-posedness for the Navier–Stokes equations", *Adv. Math.* **157**:1 (2001), 22–35. MR Zbl

[Konieczny and Yoneda 2011] P. Konieczny and T. Yoneda, "On dispersive effect of the Coriolis force for the stationary Navier–Stokes equations", *J. Differential Equations* **250**:10 (2011), 3859–3873. MR Zbl

[Kozono and Yamazaki 1994] H. Kozono and M. Yamazaki, "Semilinear heat equations and the Navier–Stokes equation with distributions in new function spaces as initial data", *Comm. Partial Differential Equations* **19**:5-6 (1994), 959–1014. MR Zbl

[Lei and Lin 2011] Z. Lei and F. Lin, "Global mild solutions of Navier–Stokes equations", *Comm. Pure Appl. Math.* **64**:9 (2011), 1297–1304. MR

[Lemarié-Rieusset 2002] P. G. Lemarié-Rieusset, *Recent developments in the Navier–Stokes problem*, Chapman & Hall/CRC Res. Notes in Math. **431**, Chapman & Hall, Boca Raton, FL, 2002. MR Zbl

[Lemarié-Rieusset 2015] P. G. Lemarié-Rieusset, "On some classes of time-periodic solutions for the Navier–Stokes equations in the whole space", *SIAM J. Math. Anal.* **47**:2 (2015), 1022–1043. MR Zbl

[Mazzucato 2003] A. L. Mazzucato, "Besov–Morrey spaces: function space theory and applications to non-linear PDE", *Trans. Amer. Math. Soc.* **355**:4 (2003), 1297–1364. MR Zbl

[Miao and Yuan 2007] C.-x. Miao and B.-q. Yuan, "Weak Morrey spaces and strong solutions to the Navier–Stokes equations", *Sci. China Ser. A* **50**:10 (2007), 1401–1417. MR Zbl

[Taylor 1992] M. E. Taylor, "Analysis on Morrey spaces and applications to Navier–Stokes and other evolution equations", *Comm. Partial Differential Equations* **17**:9-10 (1992), 1407–1456. MR Zbl

[Tsutsui 2011] Y. Tsutsui, "The Navier–Stokes equations and weak Herz spaces", *Adv. Differential Equations* **16**:11-12 (2011), 1049–1085. MR Zbl

[Xu 2005] J. Xu, "Equivalent norms of Herz-type Besov and Triebel–Lizorkin spaces", *J. Funct. Spaces Appl.* **3**:1 (2005), 17–31. MR Zbl

[Yamazaki 2000] M. Yamazaki, "The Navier–Stokes equations in the weak-L^n space with time-dependent external force", *Math. Ann.* **317**:4 (2000), 635–675. MR Zbl

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