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FOUR-MANIFOLDS WITH POSITIVE YAMABE CONSTANT

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We refine a theorem due to Gursky (2000). As applications, we give some rigidity theorems on four-manifolds with positive Yamabe constant. We recover some of Gursky's results (1998, 2000). We prove some classification theorems of four-manifolds according to some conformal invariants, which reprove and generalize the conformally invariant sphere theorem of Chang, Gursky and Yang (2003).

1. Introduction and main results

In [Fu 2017], the author proved that an n-manifold with harmonic curvature is isometric to a quotient of the standard sphere or Einstein manifold, if the upper bound of some curvature functional is given by Yamabe constant. By this we mean that we can precisely characterize the case of equality. The aim of this paper is to present some rigidity results in the subject of curvature pinching on four-manifolds with positive Yamabe constant.

Let (M^n, g) be an n-dimensional Riemannian manifold. The decomposition of the Riemannian curvature tensor Rm into irreducible components yields

$$Rm = W + \frac{1}{n-2}\mathring{\mathrm{R}}\mathrm{ic} \owedge g + \frac{R}{2n(n-1)} g \owedge g,$$

where W, Ric, $\mathring{\mathrm{R}}\mathrm{ic} = \mathrm{Ric} - (R/n)g$ and R denote the Weyl curvature tensor, Ricci tensor, trace-free Ricci tensor and scalar curvature, respectively. When the divergence of the Weyl curvature tensor W is vanishing, i.e., $\delta W = 0$, (M^n, g) is said to be a manifold with harmonic Weyl tensor. The norm of a (k, l)- tensor $T = T^{j_1\cdots j_l}_{i_1\cdots i_k}$ deduced by the Riemannian metric g is defined as

$$|T|^2 = g^{i_1 m_1} \cdots g^{i_k m_k} g_{j_1 n_1} \cdots g_{j_l n_l} T^{j_1\cdots j_l}_{i_1\cdots i_k} T^{n_1\cdots n_l}_{m_1\cdots m_k}.$$

The sphere theorem for $\frac{1}{4}$-pinched Riemannian manifolds, conjectured by Rauch in 1951, is a good example of the deep connections between the topology and the

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geometry of Riemannian manifolds. Now we know that the answer is positive, due to the fundamental work of Klingenberg, Berger and Rauch for the topological statement and the recent proof of the original conjecture by Brendle and Schoen [2008], based on the results of Böhm and Wilking [2008].

From the work of Huisken [1985] and Margerin [1998], we know that there exists a positive-dimensional constant $C(n)$ such that if

$$|W + 1/(n-2)\mathring{\mathrm{R}}\mathrm{ic} \owedge g|^2 < C(n)R^2,$$

then M^n is diffeomorphic to a quotient of the standard unit sphere. In particular, Margerin improved the constant in dimension four, and obtained the optimal theorem in [Margerin 1998].

The common feature of all the above results is to give topological information on a manifold that carries a metric whose curvature satisfies a certain pinching at each point. The question one raises here is whether one can characterize the topology and the geometry of Riemannian manifolds by means of integral pinching conditions instead of pointwise ones. Some results in this direction on four manifolds were obtained in [Bour and Carron 2015; Chang et al. 2003; Chen and Zhu 2014; Gursky 1998; Gursky 2000; Hebey and Vaugon 1996].

In four-manifolds, the Weyl functional $\int |W_g|^2$ has long been an object of interest to physicists. Suppose M^4 is a 4-dimensional manifold. Then the Hodge $*$-operator induces a splitting of the space of two-forms $\wedge^2 = \wedge^2_+ + \wedge^2_-$ into the subspace of self-dual forms $\wedge^2_+$ and anti-self-dual forms $\wedge^2_-$. This splitting in turn induces a decomposition of the Weyl curvature into its self-dual and anti-self-dual components $W^{\pm}$. A four-manifold is said to be self-dual (resp., anti-self-dual) if $W^- = 0$ (resp., $W^+ = 0$). It is said to be a manifold with half harmonic Weyl tensor if $\delta W^{\pm} = 0$. By the Hirzebruch signature formula (see [Besse 1987]),

$$\int_M |W^+|^2 - \int_M |W^-|^2 = 48\pi^2 \sigma(M), \tag{1-1}$$

where $\sigma(M)$ denotes the signature of M. A consequence of (1-1) is that the study of the Weyl functional is completely equivalent to the study of the self-dual Weyl functional $\int_M |W^+|^2$. M. J. Gursky [1998; 2000] has obtained some good and interesting results by studying $\int_M |W^+|^2$ (see Theorems A, B and C). For background material on this condition we recommend [Besse 1987, Chapter 16] and [Derdziński 1983].

Our formulation of some results will be given in terms of the Yamabe invariant. Now we introduce the definition of the Yamabe constant. Given a compact Riemannian n-manifold M, we consider the Yamabe functional

$$Q_g : C^\infty_+(M) \to \mathbb{R} : f \mapsto Q_g(f) = \frac{\frac{4(n-1)}{n-2} \int_M |\nabla f|^2 \, \mathrm{d}v_g + \int_M Rf^2 \, \mathrm{d}v_g}{\left(\int_M f^{\frac{2n}{n-2}} \, \mathrm{d}v_g\right)^{\frac{n-2}{n}}},$$

where R denotes the scalar curvature of M. It follows that Q_g is bounded below by the Hölder inequality. We set

$$\mu([g]) = \inf\{Q_g(f) \mid f \in C_+^\infty(M)\}.$$

This constant $\mu([g])$ is an invariant of the conformal class of (M, g), called the Yamabe constant. The important works of Aubin, Schoen, Trudinger and Yamabe showed that the infimum in the above is always achieved; see [Aubin 1998; Lee and Parker 1987]. The Yamabe constant of a given compact manifold is determined by the sign of scalar curvature [Aubin 1998]. The scalar curvature $R_{\tilde{g}}$ of a conformal metric $\tilde{g} = f^{4/(n-2)}g$ is equal to $\mu([g])/\mathrm{Vol}(g)^{2/n}$. We call such a metric $\tilde{g}$ a Yamabe minimizer.

Gursky [1998; 2000] proved the three striking Theorems A, B and C, and as byproducts obtained these integral pinching results, which are generalizations of the Bochner theorem in dimension 4 (see Propositions E, F and G).

Theorem A [Gursky 1998]. *Let (M^4, g) be a 4-dimensional compact Riemannian manifold with positive Yamabe constant and the space of self-dual harmonic two-forms $H_+^2(M^4) \neq 0$. Then*

$$\int_M |W^+|^2 \geq \tfrac{16}{3}\pi^2(2\chi(M^4) + 3\sigma(M^4)),$$

where $\chi(M)$ is the Euler–Poincaré characteristic of M. Furthermore, equality holds in the above inequality if and only if g is conformal to a positive Kähler–Einstein metric.

Theorem B [Gursky 2000]. *Let (M^4, g) be a 4-dimensional compact Riemannian manifold with positive Yamabe constant and $\delta W^+ = 0$. Then either (M^4, g) is anti-self-dual, or*

$$\int_M |W^+|^2 \geq \tfrac{16}{3}\pi^2(2\chi(M^4) + 3\sigma(M^4)).$$

Furthermore, equality holds in the above inequality if and only if g is a positive Einstein metric which is either Kähler, or the quotient of a Kähler manifold by a free, isometric, antiholomorphic involution.

Theorem C [Gursky 1998]. *Let (M^4, g) be a 4-dimensional compact Riemannian manifold with positive Yamabe constant and the space of harmonic 1-forms $H^1(M^4) \neq 0$. Then*

$$\int_M |W^+|^2 \geq 8\pi^2(2\chi(M^4) + 3\sigma(M^4)).$$

Furthermore, equality holds in the above inequality if and only if (M^4, g) is conformal to a quotient of $\mathbb{R}^1 \times \mathbb{S}^3$ with the product metric.

Chang, Gursky and Yang [Chang et al. 2003] proved that a four manifold with positive Yamabe constant which satisfies the strict inequality for the Weyl functional $\int |W|^2$ is actually diffeomorphic to a quotient of the sphere and precisely characterizes the case of equality. We state this result of Chang, Gursky and Yang as follows:

Theorem D [Chang et al. 2003]. *Let (M^4, g) be a 4-dimensional compact Riemannian manifold with positive Yamabe constant. If*

$$\int_M |W|^2 \leq 16\pi^2 \chi(M),$$

then one of the following must be true:

(1) *M^4 is diffeomorphic to the round sphere $\mathbb{S}^4$ or the real projective space $\mathbb{RP}^4$.*

(2) *M^4 is conformal to a manifold which is isometrically covered by $\mathbb{S}^1 \times \mathbb{S}^3$ with the product metric.*

(3) *M^4 is conformal to the complex projective plane $\mathbb{CP}^2$ with the Fubini–Study metric.*

Bour and Carron [2015] reprove and extend to higher degrees and higher dimensions Propositions F and G obtained by Gursky. Bour [2010] gives a new proof of Theorem D under a stronger pinching assumption, which is entirely based on the study of a geometric flow, and doesn't rely on the pointwise version of the theorem, due to Margerin. Chen and Zhu [2014] proved a classification theorem of 4-manifolds according to some conformal invariants, which generalizes the conformally invariant sphere theorem in [Chang et al. 2003], i.e., Theorem D under the strict inequality assumption, and relies on Chen, Tang and Zhu's classification on four-manifolds with positive isotropic curvature [Chen et al. 2012].

In this note, we refine Theorems B and Proposition E due to Gursky, and obtain Theorem 1.1 which can not be deduced from the Gursky's proof [2000] of Theorem B as follows:

Theorem 1.1. *Let (M^4, g) be a 4-dimensional compact Riemannian manifold with $\delta W^{\pm} = 0$ and positive Yamabe constant $\mu([g])$. If*

$$\int_M |W^{\pm}|^2 = \tfrac{1}{6}\mu^2([g]), \tag{1-2}$$

then $\nabla W^{\pm} = 0$ and $W^{\pm}$ has exactly two distinct eigenvalues at each point. Hence (M^4, g) is a Kähler manifold of positive constant scalar curvature.

Theorem 1.2. *Let (M^4, g) be a 4-dimensional compact Riemannian manifold with harmonic Weyl tensor and positive Yamabe constant. If*

$$\int_M |W|^2 = \tfrac{1}{6}\mu^2([g]),$$

then (M^4, g) *is* $\mathbb{CP}^2$ *with the Fubini–Study metric.*

Combing some results due to Gursky, and Chen, Tang and Zhu's classification on four-manifolds with positive isotropic curvature with Theorem 1.1, we give the following Theorem 1.3 which generalizes the conformally invariant sphere theorem of [Chang et al. 2003], i.e., Theorem D.

Theorem 1.3. *Let* (M^4, g) *be a* 4*-dimensional compact Riemannian manifold with positive Yamabe constant. If*

$$\int_M |W|^2 \leq \tfrac{1}{6}\mu^2([g]),$$

then one of the following must be true:

(1) $\tilde{g}$ *is a Yamabe minimizer and* $(M^4, \tilde{g})$ *is* $\mathbb{CP}^2$ *with the Fubini–Study metric.*

(2) (M^4, g) *is diffeomorphic to* $\mathbb{S}^4$, $\mathbb{RP}^4$, $\mathbb{S}^3 \times \mathbb{R}/G$ *or a connected sum of them. Here* G *is a cocompact fixed point free discrete subgroup of the isometry group of the standard metric on* $\mathbb{S}^3 \times \mathbb{R}$.

Theorem 1.4. *Let* (M^4, g) *be a* 4*-dimensional compact Riemannian manifold with harmonic Weyl tensor and positive Yamabe constant. If*

$$\int_M |W|^2 < \tfrac{64}{3}\pi^2\chi(M),$$

then one of the following must be true:

(1) $\tilde{g}$ *is a Yamabe minimizer and* $(M^4, \tilde{g})$ *is the round sphere* $\mathbb{S}^4$ *or the real projective space* $\mathbb{RP}^4$.

(2) (M^4, g) *is* $\mathbb{CP}^2$ *with the Fubini–Study metric.*

Theorem 1.5. *Let* (M^4, g) *be a* 4*-dimensional compact Riemannian manifold with harmonic Weyl tensor and positive Yamabe constant. If*

$$\int_M |W|^2 = \tfrac{64}{3}\pi^2\chi(M),$$

then one of the following must be true:

(1) $\tilde{g}$ *is a Yamabe minimizer and* $(M^4, \tilde{g})$ *is the manifold which is isometrically covered by* $\mathbb{S}^1 \times \mathbb{S}^3$ *with the product metric or* $\mathbb{S}^1 \times \mathbb{S}^3$ *with a rotationally symmetric Derdziński metric* (*see* [Catino 2016b; Derdziński 1982]).

(2) (M^4, g) *is isometric to a quotient of* $\mathbb{S}^2 \times \mathbb{S}^2$ *with the product metric.*

Theorem 1.6. *Let* (M^4, g) *be a* 4*-dimensional compact Riemannian manifold which is not homeomorphic to* $\mathbb{S}^4$ *or* $\mathbb{RP}^4$ *with positive Yamabe constant. If*

$$16\pi^2\chi(M) < \int_M |W|^2 \leq \tfrac{64}{3}\pi^2\chi(M),$$

then one of the following must be true:

(1) $\tilde{g}$ *is a Yamabe minimizer and* $(M^4, \tilde{g})$ *is isometric to a quotient of* $\mathbb{S}^2 \times \mathbb{S}^2$ *with the product metric.*

(2) (M, g) *has* $\chi(M) = 3,\ b_1 = 0$ *and* $b_2 = 1$, *where* b_i *denotes the* i*-th Betti number of* M, *and does not have a harmonic Weyl tensor.*

Theorem 1.7. *Let* (M^4, g) *be a* 4*-dimensional compact Riemannian manifold with harmonic curvature and positive scalar curvature. If*

$$\int_M |W|^2 \leq \tfrac{1}{3}\mu^2([g]),$$

then one of the following must be true:

(1) (M^4, g) *is conformally flat with positive constant scalar curvature.*

(2) (M^4, g) *is* $\mathbb{CP}^2$ *with the Fubini–Study metric.*

(3) (M^4, g) *is isometric to a quotient of* $\mathbb{S}^2 \times \mathbb{S}^2$ *with the product metric.*

Corollary 1.8. Let (M^4, g) be a 4-dimensional complete Einstein manifold with positive scalar curvature. If

$$\int_M |W|^2 \leq \tfrac{1}{3}\mu^2([g]),$$

then one of the following must be true:

(1) (M^4, g) is isometric to either $\mathbb{S}^4$ or $\mathbb{RP}^4$.

(2) (M^4, g) is isometric to $\mathbb{CP}^2$ with the Fubini–Study metric.

(3) (M^4, g) is isometric to a quotient of $\mathbb{S}^2 \times \mathbb{S}^2$ with the product metric.

Remark 1.9. For Riemannian manifolds with harmonic curvature and dimensions $n \geq 4$, the author proved some similar results in [Fu 2017].

2. Four manifolds with half harmonic Weyl tensor

In order to prove some results in this article, we need the following Weyl estimate proved by Gursky [2000].

Proposition E [Gursky 2000]. *Let* (M^4, g) *be a* 4*-dimensional compact Riemannian manifold with* $\delta W^{\pm} = 0$ *and positive Yamabe constant* $\mu([g])$. *If*

$$\int_M |W^{\pm}|^2 < \tfrac{1}{6}\mu^2([g]),$$

then (M^4, g) *is anti-self-dual* (*resp.*, *self-dual*).

Remark 2.1. Gursky [2000] obtained an improved Kato inequality $|\nabla W^+|^2 \geq \frac{5}{3}\big|\nabla|W^+|\big|^2$. Thus using the Bochner technique, Gursky proved Theorem B and Propositions E, F and G by introducing the corresponding functional and conformal invariant with the modified scalar curvature $R - \sqrt{6}|W^\pm|$ in [Gursky 2000]. Based on Gursky's improved Kato inequality, we can reprove Theorem B and Propositions E, F and G only by using the modified Bochner technique (see [Bour and Carron 2015; Fu 2017; Fu and Li 2010]). In order to prove Theorem 1.1 which can not be deduced from Gursky's proofs of Theorem B and Proposition E, we also need the following different proof of Proposition E.

Proof. First, we recall the following Weitzenböck formula (see [Besse 1987] and [Bour 2010])

$$\triangle|W^\pm|^2 = 2|\nabla W^\pm|^2 + R|W^\pm|^2 - 144\det\nolimits_{\wedge^2_\pm} W^\pm. \tag{2-1}$$

From (2-1), by the Kato inequality $|\nabla W^+|^2 \geq \frac{5}{3}\big|\nabla|W^+|\big|^2$ ([Gursky 2000]), we obtain

$$|W^\pm|\triangle|W^\pm| \geq \tfrac{2}{3}\big|\nabla|W^\pm|\big|^2 + \tfrac{1}{2}R|W^\pm|^2 - 72\det\nolimits_{\wedge^2_\pm} W^\pm. \tag{2-2}$$

By a simple Lagrange multiplier argument it is easily verified that

$$-144\det\nolimits_{\wedge^2_\pm} W^\pm \geq -\sqrt{6}|W^\pm|^3 \tag{2-3}$$

and equality is attained at a point where $W^\pm \neq 0$ if and only if $W^\pm$ has precisely two eigenvalues. By (2-2) and (2-3), we get

$$|W^\pm|\triangle|W^\pm| \geq \tfrac{2}{3}\nabla|W^\pm|\big|^2 + \tfrac{1}{2}R|W^\pm|^2 - \tfrac{\sqrt{6}}{2}|W^\pm|^3. \tag{2-4}$$

Let $u_\epsilon = \sqrt{|W^\pm|^2 + \epsilon^2}$ and $u = |W^\pm|$. Thus we have

$$|\nabla u_\epsilon|^2 = \frac{u^2|\nabla u|^2}{u^2+\epsilon^2} \leq |\nabla u|^2.$$

By (2-4), we compute

$$\begin{aligned} u_\epsilon^\alpha \triangle u_\epsilon^\alpha &= u_\epsilon^\alpha(\alpha(\alpha-1)u_\epsilon^{\alpha-2}|\nabla u|^2 + \alpha u_\epsilon^{\alpha-1}\triangle u_\epsilon) \\ &= \frac{\alpha-1}{\alpha}|\nabla u_\epsilon^\alpha|^2 + \alpha u_\epsilon^{2\alpha-2}u_\epsilon\triangle u_\epsilon \\ &= \frac{\alpha-1}{\alpha}|\nabla u_\epsilon^\alpha|^2 + \alpha u_\epsilon^{2\alpha-2}\big(\tfrac{1}{2}\triangle u_\epsilon^2 - |\nabla u_\epsilon|^2\big) \\ &= \frac{\alpha-1}{\alpha}|\nabla u_\epsilon^\alpha|^2 + \alpha u_\epsilon^{2\alpha-2}(u\triangle u + |\nabla u|^2 - |\nabla u_\epsilon|^2) \\ &\geq \Big(1 - \frac{1}{3\alpha}\Big)|\nabla u_\epsilon^\alpha|^2 - \frac{\sqrt{6}}{2}\alpha u_\epsilon^{2(\alpha-1)}u^3 + \frac{R\alpha}{2}u_\epsilon^{2(\alpha-1)}u^2, \end{aligned} \tag{2-5}$$

where α is a positive constant. Integrating (2-5) by parts, choosing $\alpha = \frac{1}{3}$, and letting ϵ go to zero, we get

$$(2\text{-}6)\qquad \Big(2-\frac{1}{3\alpha}\Big)\int_M |\nabla u^\alpha|^2 - \frac{\sqrt{6}}{2}\alpha\int_M u^{2\alpha+1} + \frac{\alpha}{2}\int_M Ru^{2\alpha} \le 0.$$

By the Hölder inequality and (2-6), we have

$$(2\text{-}7)\qquad \Big(2-\frac{1}{3\alpha}\Big)\int_M |\nabla u^\alpha|^2 - \frac{\sqrt{6}}{2}\alpha\Big(\int_M u^{4\alpha}\Big)^{\frac{1}{2}}\Big(\int_M u^2\Big)^{\frac{1}{2}} + \frac{\alpha}{2}\int_M Ru^{2\alpha} \le 0.$$

By the definition of Yamabe constant and (2-7), we get

$$(2\text{-}8)\qquad 0 \ge \Big(2-\frac{1}{3\alpha}\Big)\frac{1}{6}\mu([g])\Big(\int_M u^{4\alpha}\Big)^{\frac{1}{2}} - \frac{\sqrt{6}}{2}\alpha\Big(\int_M u^{4\alpha}\Big)^{\frac{1}{2}}\Big(\int_M u^2\Big)^{\frac{1}{2}} + \frac{9\alpha^2-6\alpha+1}{18\alpha}\int_M Ru^{2\alpha},$$

that is,

$$(2\text{-}9)\qquad 0 \ge \Big[\frac{1}{\sqrt{6}}\mu([g]) - \Big(\int_M u^2\Big)^{\frac{1}{2}}\Big]\Big(\int_M u^{\frac{4}{3}}\Big)^{\frac{1}{2}}.$$

We choose $\big(\int_M |W^\pm|^2\big) < \frac{1}{6}\mu^2([g])$ such that the above inequality imply $\int_M u^{4/3} = 0$, that is, $W^\pm = 0$, i.e., (M^4, g) is anti-self-dual, or self-dual. □

Remark 2.2. For $0 \le k \le \frac{n}{2}$, by the Kato inequality for harmonic k-form ω (see [Bourguignon 1990]) $(n+1-k)/(n-k)\big|\nabla|\omega|\big|^2 \le |\nabla\omega|^2$ and the two Weitzenböck formulas in [Gursky 1998], one has

$$\tfrac{1}{2}\triangle|\omega|^2 \ge |\nabla\omega|^2 - \tfrac{\sqrt{6}}{3}|W^\pm||\omega|^2 + \tfrac{1}{3}R|\omega|^2 \ge \tfrac{3}{2}\big|\nabla|\omega|\big|^2 - \tfrac{\sqrt{6}}{3}|W^\pm||\omega|^2 + \tfrac{1}{3}R|\omega|^2,$$

for all $\omega \in H^2_\pm(M^4)$ and

$$\tfrac{1}{2}\triangle|\omega|^2 \ge \tfrac{4}{3}|\nabla|\omega||^2 - \tfrac{\sqrt{3}}{2}|\mathring{\mathrm{R}}\mathrm{ic}||\omega|^2 + \tfrac{1}{4}R|\omega|^2, \qquad \text{for all } \omega \in H^1(M^4).$$

Based on the above two Weitzenböck formulas, using the same argument as in the proof of Proposition E, we can obtain two results of Gursky as follows:

Proposition F [Gursky 1998; 2000]. *Let (M^4, g) be a 4-dimensional compact Riemannian manifold with positive Yamabe constant $\mu([g])$.*

(i) *If*

$$\int_M |W^\pm|^2 < \tfrac{1}{6}\mu^2([g]),$$

then $H^2_\pm(M^4) = 0$ and $b_2^\pm(M) = 0$.

(ii) *If*

$$\int_M |\mathring{\mathrm{R}}\mathrm{ic}|^2 < \tfrac{1}{12}\mu^2([g]),$$

then $H^1(M^4) = 0$ *and* $b_1(M) = 0$.

Proof of Theorem 1.1. Equation (1-2) implies that the equality holds in (2-9). When the equality holds in (2-9), all the inequalities leading to (2-7) become equalities. From (2-8), the function u^α attains the infimum in the Yamabe functional. From (2-7), the equality for the Hölder inequality implies that u is constant, i.e., $|W^\pm|$ is constant. Hence at every point, it has an eigenvalue of multiplicity 2 and another of multiplicity 1, i.e., $W^\pm$ has eigenvalues $\{-\frac{R}{12}, -\frac{R}{12}, \frac{R}{6}\}$, and R is constant. From (2-1), we get $\nabla W^\pm = 0$. By Proposition 5 in [Derdziński 1983], (M^4, g) is a Kähler manifold of positive constant scalar curvature. □

Remark 2.3. Since $\int_M |W^\pm|^2 \geq 16 \int_M \sigma_2(A)$, we have

$$\int_M |W^\pm|^2 \geq \tfrac{16}{3}\pi^2(2\chi(M^4) \pm 3\sigma(M^4)). \tag{2-10}$$

In fact, we recall the following lower bound for the Yamabe invariant on compact four-manifolds which was proved by Gursky [1994]:

$$96 \int_M \sigma_2(A) = \int_M R^2 - 12 \int_M |\mathring{\mathrm{R}}\mathrm{ic}|^2 \leq \mu^2([g]), \tag{2-11}$$

where $\sigma_2(A)$ denotes the second-elementary function of the eigenvalues of the Schouten tensor A; the inequality is strict unless (M^4, g) is conformally Einstein. By the Chern–Gauss–Bonnet formula (see Equation 6.31 of [Besse 1987])

$$\int_M |W|^2 - 2\int_M |\mathring{\mathrm{R}}\mathrm{ic}|^2 + \frac{1}{6}\int_M R^2 = 32\pi^2\chi(M),$$

we obtain

$$\int_M |W^\pm|^2 \geq -2\int_M |\mathring{\mathrm{R}}\mathrm{ic}|^2 + \frac{1}{6}\int_M R^2 = 32\pi^2\chi(M) - \int_M |W|^2. \tag{2-12}$$

Combining (1-1) with (2-12), we can prove (2-10).

Since $\int_M |W^\pm|^2 = 16 \int_M \sigma_2(A)$, we have

$$\int_M |W^\pm|^2 = \tfrac{16}{3}\pi^2(2\chi(M^4) \pm 3\sigma(M^4)). \tag{2-13}$$

In fact, by Proposition E and (2-11), we have $\int_M |W^\pm|^2 = \frac{1}{6}\mu^2([g]) = 16\int_M \sigma_2(A)$. Hence from (2-12), (2-13) holds.

For four-manifolds M^4 with harmonic Weyl tensor and positive Yamabe constant $\mu([g])$ which is not locally conformally flat, the lower bound for $\mu([g])$ is given by $\mu^2([g]) \le 6\int_M |W^-|^2$ if M is anti-self-dual; $\mu^2([g]) \le 6\int_M |W^+|^2$ if M is self-dual; and $\mu^2([g]) \le 6\min\{\int_M |W^-|^2, \int_M |W^+|^2\}$ if M is neither anti-self-dual nor self-dual. The Yamabe constant $\mu^2([g])$ of a compact positive Kähler–Einstein manifold (M^4, g) is equal to $32\pi^2(2\chi(M^4)+3\sigma(M^4))$.

By Remark 2.3, we can rewrite Theorem B as follows:

Theorem B*. *Let (M^4, g) be a 4-dimensional compact Riemannian manifold with $\delta W^+ = 0$ and positive Yamabe constant $\mu([g])$. Then either (M^4, g) is anti-self-dual, or*

$$\int_M |W^+|^2 \ge 16\int_M \sigma_2(A). \tag{2-14}$$

Furthermore, equality holds in (2-14) if and only if (M^4, g) is a positive Einstein manifold which is either Kähler, or the quotient of a Kähler manifold by a free, isometric, antiholomorphic involution.

Proof. By Proposition E and (2-11), we get

$$\int_M |W^+|^2 \ge \tfrac{1}{6}\mu^2([g]) \ge 16\int_M \sigma_2(A).$$

Since the equality holds in (2-14), we have

$$\int_M |W^+|^2 = \tfrac{1}{6}\mu^2([g]) = 16\int_M \sigma_2(A).$$

So g is conformal to an Einstein metric $\tilde{g}$. By Theorem 1.1, we get that (M^4, g) is a Kähler manifold of positive constant scalar curvature.

Assume that $\tilde{g} = \lambda^2 g$. We now claim that λ is constant, i.e., g is an Einstein metric. To see this, first notice that $\tilde{g}$ being an Einstein metric implies that $\delta W^+_{\tilde{g}} = 0$. We recall this transformation law about W^+, i.e.,

$$\delta_{\tilde{g}} W^+_{\tilde{g}} = \delta_g W^+_g - W^+_g\Big(\frac{\nabla\lambda}{\lambda}, \dots\Big). \tag{2-15}$$

It is easy to see from (2-15) that

$$W^+_g\Big(\frac{\nabla\lambda}{\lambda}, \dots\Big) = 0. \tag{2-16}$$

Now any oriented four-manifold W^+ satisfies (see [Derdziński 1983])

$$(W^+)^{ikpq}(W^+)_{jkpq} = |W^+|^2\delta^i_j. \tag{2-17}$$

Pairing both sides of (2-17) with $(d\lambda\otimes d\lambda)^j_i$ and using (2-16) we get $|W^+_g|^2|\nabla\lambda|^2 = 0$. Since $|W^+_g|$ is constant, W^+ never vanishes, so $\nabla\lambda = 0$ and λ is constant.

We conclude that (M^4, g) is an Einstein manifold which is either Kähler, or the quotient of a Kähler manifold by a free, isometric, antiholomorphic involution. □

3. Four manifolds with harmonic Weyl tensor

Proof of Theorem 1.2. By Proposition E, we have that $W^+ = 0$ and $\int_M |W^-| = \frac{1}{6}\mu^2([g])$, or $W^- = 0$ and $\int_M |W^+| = \frac{1}{6}\mu^2([g])$. By Theorem 1.1, (M^4, g) is a Kähler manifold of positive constant scalar curvature.

When $W^+ = 0$, by Corollary 1 in [Derdziński 1983], the scalar curvature of (M^4, g) is 0, and $\mu([g]) = 0$. This is a contradiction.

When $W^- = 0$, by Lemma 7 in [Derdziński 1983], (M^4, g) is locally symmetric. By the result of Bourguignon [1981], (M^4, g) is Einstein. Then g is both Einstein and half conformally flat. By the classification theorem of Hitchin (see [Besse 1987]), (M^4, g) is isometric to either a quotient of $\mathbb{S}^4$ with the round metric or $\mathbb{CP}^2$ with the Fubini–Study metric. Since we are assuming that is not locally conformal flat, (M^4, g) is $\mathbb{CP}^2$ with the Fubini–Study metric. □

Corollary 3.1. Let (M^4, g) be a 4-dimensional complete Einstein manifold with positive scalar curvature. If

$$\int_M |W|^2 = \tfrac{1}{6}\mu^2([g]), \tag{3-1}$$

then M^4 is $\mathbb{CP}^2$ with the Fubini–Study metric.

Remark 3.2. If the equality in (3-1) is replaced by a strict inequality, we have proved in [Fu and Xiao 2017a; 2017b] that M^4 is a quotient of the round $\mathbb{S}^4$, which is proved by Proposition E. For dimensions $n > 4$, under some $L^{\frac{n}{2}}$ pinching condition, we proved in [Fu and Xiao 2017a; Fu and Xiao 2017b], as did G. Catino in [Catino 2016a], that M^n is a quotient of the round $\mathbb{S}^n$.

Proposition 3.3. Let (M^4, g) be a 4-dimensional compact Riemannian manifold with harmonic Weyl tensor and positive Yamabe constant. If

$$\int_M |W|^2 + 2\int_M |\mathring{\mathrm{Ric}}|^2 \le \frac{1}{6}\int_M R^2, \quad \text{i.e.,} \quad \int_M |W|^2 \le 16\pi^2\chi(M),$$

then one of the following must be true:

(1) M^4 is a locally conformally flat manifold. In particular, $\tilde{g}$ is a Yamabe minimizer and $(M^4, \tilde{g})$ is the round sphere $\mathbb{S}^4$, the real projective space $\mathbb{RP}^4$, or the manifold which is isometrically covered by $\mathbb{S}^1 \times \mathbb{S}^3$ with the product metric, or $\mathbb{S}^1 \times \mathbb{S}^3$ with a rotationally symmetric Derdziński metric.

(2) (M^4, g) is $\mathbb{CP}^2$ with the Fubini–Study metric.

Proof. By the Chern–Gauss–Bonnet formula, we get

$$\int_M |W|^2 + 2\int_M |\mathring{\mathrm{Ric}}|^2 - \frac{1}{6}\int_M R^2 = 2\int_M |W|^2 - 32\pi^2\chi(M) \le 0, \quad \text{i.e.,} \quad \int_M |W|^2 \le 16\pi^2\chi(M). \tag{3-2}$$

From (2-11), we get

$$\int_M |W|^2 - \frac{1}{6}\mu^2([g]) \le \int_M |W|^2 + 2\int_M |\mathring{\mathrm{Ric}}|^2 - \frac{1}{6}\int_M R^2. \tag{3-3}$$

Moreover, the above inequality is strict unless (M^4, g) is conformally Einstein.

In the case of strict inequality, we have

$$\int_M |W|^2 < \tfrac{1}{6}\mu^2([g]), \quad \text{i.e.,} \quad \int_M |W^{\pm}|^2 < \tfrac{1}{6}\mu^2([g]).$$

By Proposition E, we get that M^4 is conformally flat. Since $\int_M |W|^2$, $\mu^2([g])$ and $\int_M \sigma_2(A)$ are conformally invariant, there exists a conformal metric $\tilde{g}$ of g such that $\mu^2([g]) = \int_M R_{\tilde{g}}^2$, and

$$\int_M |W_{\tilde{g}}|^2 + 2\int_M |\mathring{\mathrm{Ric}}_{\tilde{g}}|^2 - \frac{1}{6}\int_M R_{\tilde{g}}^2 = \int_M |W|^2 + 2\int_M |\mathring{\mathrm{Ric}}|^2 - \frac{1}{6}\int_M R^2 \le 0, \tag{3-4}$$

i.e.,

$$2\int_M |\mathring{\mathrm{Ric}}_{\tilde{g}}|^2 - \tfrac{1}{6}\mu^2([g]) \le 0.$$

By Theorems 1.5 and 1.6 in [Fu and Xiao 2018], $(M^4, \tilde{g})$ is isometric to the round $\mathbb{S}^4$, the real projective space $\mathbb{RP}^4$, or a manifold which is isometrically covered by $\mathbb{S}^1 \times \mathbb{S}^3$ with the product metric, or $\mathbb{S}^1 \times \mathbb{S}^3$ with a rotationally symmetric Derdziński metric.

In the case of equality, we have

$$\int_M |W|^2 = \tfrac{1}{6}\mu^2([g]).$$

Here g is conformally Einstein. By Theorem 1.2, (M^4, g) is $\mathbb{CP}^2$ with the Fubini–Study metric. □

Remark 3.4. Any compact conformally flat 4-manifold with $\mu([g]) > 0$ and $\chi(M) \ge 0$ has been classified [Gursky 1994; 1998]. Gursky proved that M^4 is conformal to the round $\mathbb{S}^4$, the real projective space $\mathbb{RP}^4$, or a quotient of $\mathbb{R}^1 \times \mathbb{S}^3$ with the product metric in [Gursky 1994; 1998]. Comparing with Theorem D, it is easy to see that the condition and conclusion in Proposition 3.3 are both strong.

Proof of Theorem 1.4. By the Chern–Gauss–Bonnet formula, we get

$$\int_M |W|^2 + 4\int_M |\mathring{\mathrm{Ric}}|^2 - \frac{1}{3}\int_M R^2 = 3\int_M |W|^2 - 64\pi^2\chi(M) < 0, \quad \text{i.e.,} \quad \int_M |W|^2 < \tfrac{64}{3}\pi^2\chi(M). \tag{3-5}$$

From (2-11), we get

$$\int_M |W|^2 - \frac{1}{3}\mu^2([g]) \le \int_M |W|^2 + 4\int_M |\mathring{\mathrm{R}}\mathrm{ic}|^2 - \frac{1}{3}\int_M R^2.$$

Moreover, the above inequality is strict unless (M^4, g) is conformally Einstein. Then we have

$$\int_M |W|^2 < \tfrac{1}{3}\mu^2([g]).$$

Since $\int_M |W|^2$, $\mu^2([g])$ and $\int_M \sigma_2(A)$ are conformally invariant, there exists a conformally metric $\tilde{g}$ of g such that $\mu^2([g]) = \int_M R^2_{\tilde{g}}$, and from (3-5) we have

$$4\int_M |\mathring{\mathrm{R}}\mathrm{ic}_{\tilde{g}}|^2 - \tfrac{1}{3}\mu^2([g]) < 0.$$

(a) $W = 0$. By Theorem 1.5 in [Fu and Xiao 2018], $(M^4, \tilde{g})$ is the round $\mathbb{S}^4$ or the real projective space $\mathbb{RP}^4$.

(b) $W \neq 0$. By Proposition F, $b_1(M) = 0$. Hence $\chi(M) = 2 + b_2$. By Proposition 3.3, we assume $16\pi^2\chi(M) < \int_M |W|^2$. Since $\mu^2([g]) \le \mu^2(\mathbb{S}^4) = 384\pi^2$ and the inequality is strict unless (M^4, g) is conformal to $\mathbb{S}^4$, $\int_M |W|^2 < \frac{1}{3}\mu^2([g])$ implies that $\chi(M) \le 7$. By Proposition E, we have that $W^+ = 0$ and $\int_M |W^-|^2 \ge \frac{1}{6}\mu^2([g])$, or $W^- = 0$ and $\int_M |W^+|^2 \ge \frac{1}{6}\mu^2([g])$. By Proposition F and the Hirzebruch signature formula, $b_2(M) = b_2^-(M) \neq 0$ or $b_2(M) = b_2^+(M) \neq 0$. Hence $3 \le \chi(M) = 2 + b_2 \le 7$.

When $W^- = 0$ and $\int_M |W^+|^2 \ge \frac{1}{6}\mu^2([g])$, $3 \le \chi(M) = 2 + b_2^+(M) \le 7$. By the Hirzebruch signature formula

$$\frac{4\chi(M)}{9} > \frac{1}{48\pi^2}\int_M |W_g^+|^2 = b_2^+,$$

only the case $b_2^+ = 1$ occurs. Thus we have $\chi(M) = 3$, $\sigma(M) = 1$, and

$$\int_M |W_g^+|^2 = 48\pi^2 = \frac{16\pi^2}{3}(2\chi(M) + 3\sigma(M)).$$

By Remark 2.3,

$$\int_M |W|^2 = \int_M |W_g^+|^2 = \frac{\mu^2([g])}{6}.$$

Hence by Theorem 1.2, (M^4, g) is $\mathbb{CP}^2$ with the Fubini–Study metric.

When $W^+ = 0$ and $\int_M |W^-|^2 \ge \frac{1}{6}\mu^2([g])$. Similarly, we obtain

$$\int_M |W|^2 = \int_M |W_g^-|^2 = \frac{\mu^2([g])}{6}.$$

From the proof of Theorem 1.2, this can't happen. □

Proof of Theorem 1.5. (i) When $\chi(M)=0$. This pinching condition implies $W=0$. From (3-4), there exists a conformally metric $\tilde{g}$ of g such that $\mu^2([g])=\int_M R_{\tilde{g}}^2$ and

$$2\int_M |\mathring{\mathrm{R}}\mathrm{ic}_{\tilde{g}}|^2-\tfrac{1}{6}\mu^2([g])=0.$$

By Theorem 1.6 in [Fu and Xiao 2018], $(M^4,\tilde{g})$ is a manifold which is isometrically covered by $\mathbb{S}^1\times\mathbb{S}^3$ with the product metric, or a manifold which is isometrically covered by $\mathbb{S}^1\times\mathbb{S}^3$ with a rotationally symmetric Derdziński metric.

(ii) When $\chi(M)\neq 0$. Since $\int_M|W|^2\le\frac{1}{3}\mu^2([g])$, $\int_M|W|^2=\frac{64}{3}\pi^2\chi(M)$ implies that $\chi(M)\le 5$. Since $\int_M|W|^2=\frac{64}{3}\pi^2\chi(M)$, by (3-3) and Proposition F, $b_1(M)=0$. Hence $\chi(M)=2+b_2$.

Case 1: In the case of strict inequality, we have

$$\int_M|W|^2<\tfrac{1}{3}\mu^2([g]).$$

From the proof of Theorem 1.4, we have $W^{\mp}=0$ and $\int_M|W^{\pm}|^2\ge\frac{1}{6}\mu^2([g])$, and $3\le\chi(M)=2+b_2(M)=2+b_2^{+}(M)\le 5$. By the Hirzebruch signature formula

$$\frac{4\chi(M)}{9}=\frac{1}{48\pi^2}\int_M|W_g^{\pm}|^2=b_2^{\pm},$$

we get that $b_2^{\pm}$ is not an integer. Hence there exists no such manifold.

Case 2: In the case of strict equality, we have

$$\int_M|W|^2=\tfrac{1}{3}\mu^2([g]).$$

Here g is conformal to an Einstein metric. Since (M^4,g) has harmonic Weyl tensor, from the proof of Theorem B*, we get that (M^4,g) is also Einstein. By Corollary 1.8, (M^4,g) is a quotient of $\mathbb{S}^2\times\mathbb{S}^2$ with the product metric. □

Proposition 3.5. Let (M^4,g) be a 4-dimensional compact Riemannian manifold with harmonic Weyl tensor and positive Yamabe constant. If

$$\tfrac{1}{6}\mu^2([g])\le\int_M|W|^2\le\tfrac{1}{3}\mu^2([g]),$$

then one of the following must be true:

(1) (M^4,g) is self-dual, but is not anti-self-dual, which has either even $\chi(M^4)\le 4$ and $b_2^{+}=2$ or odd $\chi(M^4)\le 1$ and $b_2^{+}=1$.

(2) (M^4,g) is anti-self-dual, but is not self-dual, which has either even $\chi(M^4)\le 4$ and $b_2^{-}=2$ or odd $\chi(M^4)\le 1$ and $b_2^{-}=1$.

(3) (M^4,g) is a $\mathbb{CP}^2$ with the Fubini–Study metric.

(4) (M^4, g) is a quotient of a quotient of $\mathbb{S}^2 \times \mathbb{S}^2$ with the product metric.

Proof. By Proposition E, we get that $W^- = 0$, $W^+ = 0$ or $\int_M |W^\pm|^2 = \frac{1}{6}\mu^2([g])$.

When $W^\mp = 0$, $\int_M |W^\pm|^2 \geq \frac{1}{6}\mu^2([g])$. By Proposition F, we have $b_2^\mp = 0$. By the Hirzebruch signature formula

$$\pm\frac{1}{48\pi^2}\int_M |W_g^\pm|^2 = \frac{1}{48\pi^2}\int_M (|W_g^+|^2 - |W_g^-|^2) = b_2^+ - b_2^- = \pm b_2^\pm = \sigma(M), \tag{3-6}$$

we get $\pm\sigma(M) = b_2^\pm \geq 1$. Since $\int_M |W|^2 \leq \frac{1}{3}\mu^2([g])$, by the fact that $\mu^2([g]) \leq \mu^2(\mathbb{S}^4) = 384\pi^2$ and the inequality is strict unless (M^4, g) is conformal to $\mathbb{S}^4$, we get $b_2^\pm \leq 2$. Then we get $\chi(M) \leq 4$.

If $\chi(M) = 3$, then $b_2^\pm = 1$ and $b_1 = 0$. By Remark 2.3, we have

$$\int_M |W^\pm|^2 \geq \tfrac{16}{3}\pi^2(2\chi(M^4) \pm 3\sigma(M^4)).$$

Combining with (3-6), we have

$$48\pi^2 = \pm 48\pi^2\sigma(M^4) = \int_M |W^\pm|^2 \geq \tfrac{16}{3}\pi^2\big(2\chi(M^4) \pm 3\sigma(M^4)\big) = 48\pi^2.$$

By Remark 2.3, $\int_M |W|^2 = \frac{1}{6}\mu^2([g])$. By Theorem 1.2, (M^4, g) is $\mathbb{CP}^2$ with the Fubini–Study metric.

When $\int_M |W^\pm|^2 = \frac{1}{6}\mu^2([g])$, by Theorem 1.1, (M^4, g) is a Kähler manifold of positive constant scalar curvature, and the Weyl tensor is parallel. Since (M^4, g) is a Kähler manifold with harmonic Weyl tensor, by Proposition 1 in [Derdziński 1983], the Ricci tensor is parallel. Hence $\nabla Rm = 0$, i.e., M is locally symmetric. From (2-4), by the maximum principle we get $|W^\pm|^2 = R^2/6$, and $W^\pm$ has eigenvalues $\{-\frac{R}{12}, -\frac{R}{12}, \frac{R}{6}\}$. Thus Rm has eigenvalues $\{0, 0, 1, 0, 0, 1\}$. By the classification of 4-dimensional symmetric spaces, it is isometric to a quotient of $\mathbb{S}^2 \times \mathbb{S}^2$ with the product metric. □

4. Four manifolds with harmonic curvature

Proposition 4.1. Let (M^4, g) be a 4-dimensional compact Riemannian manifold with harmonic curvature and positive scalar curvature. If

$$\tfrac{1}{6}\mu^2([g]) \leq \int_M |W|^2 \leq \tfrac{1}{3}\mu^2([g]),$$

then one of the following must be true:

(1) (M^4, g) is $\mathbb{CP}^2$ with the Fubini–Study metric.

(2) (M^4, g) is isometric to a quotient of $\mathbb{S}^2 \times \mathbb{S}^2$ with the product metric.

Proof. By Proposition 3.5, We just need to consider whether (M^4, g) is self-dual or anti-self-dual.

When (M^4, g) is self-dual, since it has harmonic curvature, it is analytic [DeTurck and Goldschmidt 1989]. By Proposition 7 in [Derdziński 1983], we get that (M^4, g) is Einstein. By the classification theorem of Hitchin, (M^4, g) is isometric to $\mathbb{CP}^2$ with the Fubini–Study metric g.

When (M^4, g) is anti-self-dual, $\frac{R}{6}I - W^+ = \frac{R}{6}I > 0$. Since (M^4, g) is not self-dual, by Theorem 4.3 of [Micallef and Wang 1993], only (c) and (d) therein occur, i.e., (M^4, g) is a Kähler manifold of positive constant scalar curvature. By Corollary 1 in [Derdziński 1983], the scalar curvature of (M^4, g) is 0. This is a contradiction. □

By Theorem 1.1 and Propositions 4.1 and E, we have Theorem 1.7.

Proposition 4.2. Let (M^4, g) be a 4-dimensional compact Riemannian manifold with harmonic curvature and positive scalar curvature. If

$$\int_M |W|^2 + 4\int_M |\mathring{\mathrm{Ric}}|^2 = \tfrac{1}{3}\mu^2([g]), \tag{4-1}$$

then one of the following must be true:

(1) M^4 is a quotient of $\mathbb{S}^2 \times \mathbb{S}^2$ with the product metric.

(2) M^4 is covered isometrically by $\mathbb{S}^1 \times \mathbb{S}^3$ with the product metric.

(3) M^4 is covered isometrically by $\mathbb{S}^1 \times \mathbb{S}^3$ with a rotationally symmetric Derdziński metric.

Proof. **Case 1:** $\mathring{\mathrm{Ric}} = 0$, i.e., M is Einstein. By Corollary 1.8, (M^4, g) falls under (1).

Case 2: $\mathring{\mathrm{Ric}} \neq 0$. It is easy to see from (4-1) that $\int_M |W|^2 < \frac{1}{3}\mu^2([g])$. By Theorem 1.7, we have $W = 0$, i.e., M is locally conformally flat and $\int_M |\mathring{\mathrm{Ric}}|^2 = \frac{1}{12}\mu^2([g])$. By Theorem 1.6 in [Fu and Xiao 2018], (M^4, g) falls under (2) or (3). □

Proposition 4.3. Let (M^4, g) be a 4-dimensional compact Riemannian manifold with harmonic curvature and positive scalar curvature. If

$$\int_M |W|^2 + 4\int_M |\mathring{\mathrm{Ric}}|^2 < \tfrac{1}{3}\mu^2([g]), \tag{4-2}$$

then one of the following must be true:

(1) M^4 is a quotient of the round $\mathbb{S}^4$.

(2) M^4 is $\mathbb{CP}^2$ with the Fubini–Study metric.

Proof. Suppose $\mathring{\mathrm{Ric}} \neq 0$. It is easy to see from (4-2) that $\int_M |W|^2 < \frac{1}{3}\mu^2([g])$. By Theorem 1.7, we have $W = 0$, i.e., M is locally conformally flat and $\int_M |\mathring{\mathrm{Ric}}|^2 <$

$\frac{1}{12}\mu^2([g])$. By Theorem 1.5 in [Fu and Xiao 2018], that (M^4, g) is a quotient of the round $\mathbb{S}^4$. This is a contradiction.

Thus (4-2) implies that $\mathring{\mathrm{R}}\mathrm{ic} = 0$, i.e., M is Einstein, and $\int_M |W|^2 < \frac{1}{3}\mu^2([g])$. By Theorem 1.7, M is $\mathbb{CP}^2$ with the Fubini–Study metric, or locally conformally flat. Hence M^4 is a constant curvature space. Since the Yamabe constant is positive, M^4 is a quotient of the round $\mathbb{S}^4$. □

Corollary 4.4. Let (M^4, g) be a 4-dimensional compact Riemannian manifold with harmonic curvature and positive scalar curvature. If

$$\int_M |W|^2 + 8\int_M |\mathring{\mathrm{R}}\mathrm{ic}|^2 \le \frac{1}{3}\int_M R^2, \tag{4-3}$$

then one of the following must be true:

(1) M^4 is isometric to a quotient of the round $\mathbb{S}^4$.

(2) M^4 is a quotient of $\mathbb{S}^2 \times \mathbb{S}^2$ with the product metric.

(3) M^4 is $\mathbb{CP}^2$ with the Fubini–Study metric.

Remark 4.5. The pinching condition (4-3) in Corollary 4.4 is equivalent to

$$\int_M |W|^2 + \frac{1}{15}\int_M R^2 \le \frac{128}{5}\pi^2\chi(M).$$

Proofs of Corollary 4.4 and Remark 4.5. From (2-11), we get

$$\int_M |W|^2 + 4\int_M |\mathring{\mathrm{R}}\mathrm{ic}|^2 - \frac{1}{3}\mu^2([g]) \le \int_M |W|^2 + 8\int_M |\mathring{\mathrm{R}}\mathrm{ic}|^2 - \frac{1}{3}\int_M R^2.$$

Moreover, the inequality is strict unless (M^4, g) is conformally Einstein.

In the case of strict inequality, Proposition 4.3 immediately implies Corollary 4.4.

In the case of equality, we have that g is conformally Einstein and

$$\int_M |W|^2 + 4\int_M |\mathring{\mathrm{R}}\mathrm{ic}|^2 = \frac{1}{3}\mu^2([g]).$$

Since g has constant scalar curvature, g is Einstein from the proof of Obata's theorem. By Proposition 4.2, we complete the proof of this corollary.

By the Chern–Gauss–Bonnet formula, the right-hand sides of the above can be written as

$$\int_M |W|^2 + 8\int_M |\mathring{\mathrm{R}}\mathrm{ic}|^2 - \frac{1}{3}\int_M R^2 = 5\int_M |W|^2 + \frac{1}{3}\int_M R^2 - 128\pi^2\chi(M).$$

This proves Remark 4.5. □

5. Four manifolds with positive Yamabe constant

Chang, Gursky and Yang's proof of Theorem D is based on establishing the existence of a solution of a fourth order fully nonlinear equation. Avoiding the requirement for the existence of a fourth order fully nonlinear equation, we can reprove Theorem D which is rewritten as follows:

Theorem D*. *Let (M^4, g) be a 4-dimensional compact Riemannian manifold with positive Yamabe constant. If*

$$\int_M |W|^2 \leq 16\pi^2\chi(M),$$

then one of the following must be true:

(1) *$\tilde{g}$ is a Yamabe minimizer and $(M^4, \tilde{g})$ is the manifold which is isometrically covered by $\mathbb{S}^1 \times \mathbb{S}^3$ with the product metric*, *or $\mathbb{S}^1 \times \mathbb{S}^3$ with a rotationally symmetric Derdziński metric.*

(2) *M^4 is diffeomorphic to the round sphere $\mathbb{S}^4$ or the real projective space $\mathbb{RP}^4$.*

(3) *$\tilde{g}$ is a Yamabe minimizer and $(M^4, \tilde{g})$ is $\mathbb{CP}^2$ with the Fubini–Study metric.*

Proof. (i) When $\chi(M) = 0$. This pinching condition implies $W = 0$. Since $\int_M |W|^2$, $\mu^2([g])$ and $\int_M \sigma_2(A)$ are conformally invariant, there exists a conformally metric $\tilde{g}$ of g such that $\mu^2([g]) = \int_M R_{\tilde{g}}^2$, and from (3-2) we have

$$2\int_M |\mathring{\mathrm{Ric}}_{\tilde{g}}|^2 - \tfrac{1}{6}\mu^2([g]) = 0.$$

By Theorem 1.6 in [Fu and Xiao 2018], $(M^4, \tilde{g})$ is a manifold which is isometrically covered by $\mathbb{S}^1 \times \mathbb{S}^3$ with the product metric, or $\mathbb{S}^1 \times \mathbb{S}^3$ with a rotationally symmetric Derdziński metric.

(ii) When $\chi(M) \neq 0$. **Case 1:** If $\int_M |W|^2 < 16\pi^2\chi(M)$ or $\int_M |W|^2 = 16\pi^2\chi(M)$ and $-2\int_M |\mathring{\mathrm{Ric}}|^2 + \frac{1}{6}\int_M R^2 < \frac{1}{6}\mu^2([g])$, then from (3-2) we have

$$\int_M |W|^2 < \tfrac{1}{6}\mu^2([g]).$$

By Proposition F, $b_2(M) = 0$.

(a) $W = 0$. Since $\int_M |W|^2$, $\mu^2([g])$ and $\int_M \sigma_2(A)$ are conformally invariant, there exists a conformally metric $\tilde{g}$ of g such that $\mu^2([g]) = \int_M R_{\tilde{g}}^2$, and from (3-2) we have

$$2\int_M |\mathring{\mathrm{Ric}}_{\tilde{g}}|^2 - \tfrac{1}{6}\mu^2([g]) \leq 0.$$

Since $\chi(M) \neq 0$, by Theorem 1.5 in [Fu and Xiao 2018], $(M^4, \tilde{g})$ is the round $\mathbb{S}^4$, the real projective space $\mathbb{RP}^4$.

(b) $W \neq 0$. Since $\int_M |W|^2$, $\mu^2([g])$ and $\int_M \sigma_2(A)$ are conformally invariant, and $W \neq 0$, there exists a conformally metric $\tilde{g}$ of g such that $\mu^2([g]) = \int_M R_{\tilde{g}}^2$, and

$$2\int_M |\mathring{\mathrm{Ric}}_{\tilde{g}}|^2 - \tfrac{1}{6}\mu^2([g]) < 0.$$

By Proposition F, $b_1(M) = 0$. By Freedman's result [1982], M^4 is covered by a homeomorphism sphere. For any metric g' of unit volume in the conformal class of g, we have

$$\text{(5-1)}\quad \int_M R_{g'} - \sqrt{6}\int_M |W_{g'}| \geq \mu([g]) - \sqrt{6}\Big(\int_M |W_{g'}|^2\Big)^{\frac{1}{2}} = \mu([g]) - \sqrt{6}\Big(\int_M |W|^2\Big)^{\frac{1}{2}} > 0.$$

Thus by [Chen and Zhu 2014, Section 2] and [Gursky 2000, Section 3], from (5-1) there is a metric $\tilde{g}$ of unit volume in the conformal class of g such that

$$\sqrt{6}|W_{\tilde{g}}| < R_{\tilde{g}}.$$

Let $\lambda_1^{\pm} \geq \lambda_2^{\pm} \geq \lambda_3^{\pm}$ be the eigenvalues of $W^{\pm}$. Since $W^{\pm}$ is trace free, we have $\lambda_1^{\pm} + \lambda_2^{\pm} + \lambda_3^{\pm} = 0$, and

$$\begin{aligned}\tfrac{3}{2}\lambda_1^{+2} + \tfrac{3}{2}\lambda_1^{-2} &\leq [\lambda_1^{+2} + \tfrac{1}{2}(\lambda_2^{+} + \lambda_3^{+})^2] + [\lambda_1^{-2} + \tfrac{1}{2}(\lambda_2^{-} + \lambda_3^{-})^2] \\ &= (\lambda_1^{+2} + \lambda_2^{+2} + \lambda_3^{+2}) + (\lambda_1^{-2} + \lambda_2^{-2} + \lambda_3^{-2}) \\ &= \tfrac{1}{4}|W|^2 < \tfrac{1}{24}R^2,\end{aligned}$$

i.e., $\lambda_1^{\pm} < \frac{1}{6}R$. Hence $\lambda_2^{\pm} + \lambda_3^{\pm} > -\frac{1}{6}R$, i.e., $\lambda_2^{\pm} + \lambda_3^{\pm} + \frac{1}{6}R > 0$. This implies the sum of the least two eigenvalues of $\frac{1}{12}R + W^{\pm}$ is positive. So $(M^4, \tilde{g})$ has positive isotropic curvature. Since M^4 is covered by a homeomorphism sphere, by the main theorem of [Chen et al. 2012], M^4 is diffeomorphic to the standard sphere $\mathbb{S}^4$ or the real projective space $\mathbb{RP}^4$;

Case 2: If $\int_M |W|^2 = 16\pi^2\chi(M)$ and $-2\int_M |\mathring{\mathrm{Ric}}|^2 + \frac{1}{6}\int_M R^2 = \frac{1}{6}\mu^2([g])$, then from (3-2) and (3-3) we have

$$\int_M |W|^2 = \tfrac{1}{6}\mu^2([g])$$

and g is conformal to a Einstein metric $\tilde{g}$. Thus we have

$$\int_M |W_{\tilde{g}}|^2 = \tfrac{1}{6}\mu^2([g]).$$

By Corollary 3.1, $(M^4, \tilde{g})$ is $\mathbb{CP}^2$ with the Fubini–Study metric. Hence $(M^4, \tilde{g})$ is conformal to $\mathbb{CP}^2$ with the Fubini–Study metric. □

Remark 5.1. The proof of Chang, Gursky and Yang consists of two steps. First, they prove the case for strict inequality, and second, based on the first step, they prove the case for equality. We unify the two cases. Chen and Zhu [2014] prove a classification theorem of 4-manifolds which generalizes Theorem C under the strict inequality assumption.

Based on the first Weitzenböck formulas in Remark 2.2, using the same argument as in the proof of Theorem 1.1, we can obtain the following result of Gursky [2000].

Proposition G [Gursky 2000]. *Let (M^4, g) be a 4-dimensional compact Riemannian manifold with positive Yamabe constant $\mu([g])$. If $b_2^{\pm} \neq 0$ and*

$$\int_M |W^{\pm}|^2 = \tfrac{1}{6}\mu^2([g]), \tag{5-2}$$

then (M^4, g) is conformal to a Kähler manifold of positive constant scalar curvature.

Proof. Since $b_2^{\pm} \neq 0$, there exists a nonzero $\omega^{\pm} \in H_{\pm}^2(M)$. Setting $u = |\omega^{\pm}|$. Based on the first Weitzenböck formulas in Remark 2.2, using the same argument as in the proof of Theorem 1.1, we get

$$\begin{aligned} 0 \geq \Big(2 - \frac{1}{2\alpha}\Big)\frac{1}{6}\mu([g])\Big(\int_M u^{4\alpha}\Big)^{\frac{1}{2}} \\ - \frac{\sqrt{6}}{3}\alpha\Big(\int_M u^{4\alpha}\Big)^{\frac{1}{2}}\Big(\int_M |W^{\pm}|^2\Big)^{\frac{1}{2}} + \frac{(2\alpha-1)^2}{12\alpha}\int_M Ru^{2\alpha}. \end{aligned} \tag{5-3}$$

We choose $\alpha = \frac{1}{2}$, from (5-3) we get

$$0 \geq \Big[\frac{1}{\sqrt{6}}\mu([g]) - \Big(\int_M |W^{\pm}|^2\Big)^{\frac{1}{2}}\Big]\Big(\int_M u^2\Big)^{\frac{1}{2}}. \tag{5-4}$$

Equation (5-2) implies that the equality holds in (5-4). When the equality holds in (5-4), all inequalities leading to (5-3) become equalities. From (5-3), the function u^{α} attains the infimum in the Yamabe functional. Hence the metric $\tilde{g} = u^{2\alpha}g$ is a Yamabe minimizer. Then we get $|\omega^{\pm}|_{\tilde{g}} = 1$. Since $\int_M |W^{\pm}|^2$ is conformally invariant, the equality for the Hölder inequality implies that $|W^{\pm}|_{\tilde{g}}$ is constant. From (5-2), we get $|W^{\pm}|_{\tilde{g}}^2 = \frac{1}{6}R_{\tilde{g}}^2$. By the first Weitzenböck formula and the maximum principle, we get that $|\omega|$ is constant, thus $\nabla\omega = 0$, i.e., $(M^4, \tilde{g})$ is a Kähler manifold of positive constant scalar curvature. Hence (M^4, g) is conformal to a Kähler manifold of positive constant scalar curvature. □

Based on Propositions F and G, using the same arguments as in the proof of Theorem B*, we can reprove Theorems A and C proved by Gursky by using some results on functional determinant and the Bochner technique which are rewritten as follows:

Theorem A*. *Let* (M^4, g) *be a* 4*-dimensional compact Riemannian manifold with positive Yamabe constant* $\mu([g])$ *and* $H_+^2(M) \neq 0$*. Then*

$$\int_M |W_g^+|^2 \geq 16 \int_M \sigma_2(A).$$

Furthermore, equality holds in the above inequality if and only if g is conformal to a positive Kähler–Einstein metric.

Theorem C*. *Let* (M^4, g) *be a* 4*-dimensional compact Riemannian manifold with positive Yamabe constant and the space of harmonic* 1*-forms* $H^1(M^4) \neq 0$*. Then*

$$\int_M |W^+|^2 = 8\pi^2(2\chi(M^4) + 3\sigma(M^4)) - 8\int_M \sigma_2(A) \geq 8\pi^2(2\chi(M^4) + 3\sigma(M^4)).$$

Furthermore, the equality holds in the above inequalities if and only if (M^4, g) *is conformal to a quotient of* $\mathbb{R}^1 \times \mathbb{S}^3$ *with the product metric.*

Proof. By Proposition F, $\int_M |\mathring{\mathrm{R}}\mathrm{ic}|^2 \geq \frac{1}{12}\mu^2([g])$ for $H^1(M^4) \neq 0$. Since M^4 is compact, there exists a conformally metric $\tilde{g}$ of g such that $\mu^2([g]) = \int_M R_{\tilde{g}}^2$. Hence we get

$$-2\int_M |\mathring{\mathrm{R}}\mathrm{ic}_{\tilde{g}}|^2 + \frac{1}{6}\mu^2([g]) = -2\int_M |\mathring{\mathrm{R}}\mathrm{ic}_{\tilde{g}}|^2 + \frac{1}{6}\int_M R_{\tilde{g}}^2 = -2\int_M |\mathring{\mathrm{R}}\mathrm{ic}|^2 + \frac{1}{6}\int_M R^2 \leq 0,$$

i.e.,

$$16\int_M \sigma_2(A) \leq 0.$$

By the Chern–Gauss–Bonnet formula,

$$\int_M |W^+|^2 = 8\pi^2(2\chi(M^4) + 3\sigma(M^4)) - 8\int_M \sigma_2(A).$$

Hence

$$\int_M |W^+|^2 \geq 8\pi^2(2\chi(M^4) + 3\sigma(M^4)).$$

From the proof of Proposition F and the above, the equality holding in the above inequalities implies that $|\nabla|\omega||^2 = \frac{3}{4}|\nabla\omega|^2$ and $\int_M |\mathring{\mathrm{R}}\mathrm{ic}|^2 = \frac{1}{12}\mu^2([g]) = \frac{1}{12}\int_M R^2$. By [Bour and Carron 2015, Proposition 5.1 and Section 7.2], (M^4, g) is conformal to a quotient of $\mathbb{R}^1 \times \mathbb{S}^3$ with the product metric. □

It is easy to see from the proof of Theorem A* that the assumption that $H_+^2(M) \neq 0$ in Theorem 1 of [Gursky 1998] can be dropped for metrics with zero Yamabe constant.

Proposition 5.2. Let (M^4, g) be a 4-dimensional compact Riemannian manifold with zero Yamabe constant $\mu([g])$. Then

$$\int_M |W^+|^2 \geq \tfrac{16}{3}\pi^2(2\chi(M^4) + 3\sigma(M^4)).$$

Furthermore, equality holds in the above inequality if and only if g is conformal to a Ricci-flat and anti-self-dual metric, if and only if the universal cover of M is conformal to either $\mathbb{R}^4$ or a $K3$ surface.

Proof of Theorem 1.3. **Case 1:** $\int_M |W^{\pm}|^2 \leq \int_M |W|^2 < \frac{1}{6}\mu^2([g])$. By Proposition F, we get $b_2^{\pm} = 0$. From the proof of Theorem D*, we get that (M^4, g) has positive isotropic curvature. According to the main theorem in [Chen et al. 2012], it is diffeomorphic to $\mathbb{S}^4$, $\mathbb{RP}^4$, $\mathbb{S}^3 \times \mathbb{R}/G$ or a connected sum of them. Here G is a cocompact fixed point free discrete subgroup of the isometry group of the standard metric on $\mathbb{S}^3 \times \mathbb{R}$.

Case 2: $W^+ = 0$, $\int_M |W^-|^2 = \frac{1}{6}\mu^2([g])$, or $W^- = 0$, $\int_M |W^+|^2 = \frac{1}{6}\mu^2([g])$. By the Hirzebruch signature formula

$$\frac{1}{48\pi^2}\int_M (|W_g^+|^2 - |W_g^-|^2) = b_2^+ - b_2^- = \sigma(M),$$

we get $b_2^- \neq 0$ or $b_2^+ \neq 0$. By Proposition G, (M^4, g) is conformal to a Kähler manifold of positive constant scalar curvature.

When $W^+ = 0$, by Corollary 1 in [Derdziński 1983], the scalar curvature of $(M^4, \tilde{g})$ is 0, and $\mu([g]) = 0$. This is a contradiction.

When $W^- = 0$, by Lemma 7 in [Derdziński 1983], $(M^4, \tilde{g})$ is locally symmetric. By the result of Bourguignon, $(M^4, \tilde{g})$ is Einstein. Then $\tilde{g}$ is both Einstein and half conformally flat. By the classification theorem of Hitchin, $(M^4, \tilde{g})$ is isometric to either a quotient of $\mathbb{S}^4$ with the round metric or $\mathbb{CP}^2$ with the Fubini–Study metric. Since we are assuming that is not locally conformal flat, $(M^4, \tilde{g})$ is $\mathbb{CP}^2$ with the Fubini–Study metric. □

Remark 5.3. We do not know whether Theorem 1.3 can be deduced directly from [Chen and Zhu 2014, Theorem 1.6], which has weaker conditions. For their proof, Chen and Zhu used Micallef and Wang's result [1993], which we do not use in the proof of Theorem 1.3. Theorem D can be deduced from the proof of Theorem D* and Theorem 1.3.

Proof of Theorem 1.6. From the proof of Theorem 1.4, we have $b_1 = 0$ and $2 \leq \chi(M) \leq 7$.

Case 1: In the case of strict inequality, we have

$$\int_M |W|^2 < \tfrac{1}{3}\mu^2([g]).$$

When $\int_M |W^{\pm}|^2 < \frac{1}{6}\mu^2([g])$, by Proposition F, $b_2^+(M) = b_2^-(M) = 0$. Hence M^4 is covered by a homeomorphism sphere, i.e., M^4 is homeomorphic to the standard sphere $\mathbb{S}^4$ or the real projective space $\mathbb{RP}^4$.

When $\int_M |W^+|^2 < \frac{1}{6}\mu^2([g])$ and $\int_M |W^-|^2 \geq \frac{1}{6}\mu^2([g])$, or $\int_M |W^-|^2 < \frac{1}{6}\mu^2([g])$ and $\int_M |W^+|^2 \geq \frac{1}{6}\mu^2([g])$. By Proposition E and the Hirzebruch signature formula, $b_2(M) = b_2^-(M) \neq 0$ or $b_2(M) = b_2^+(M) \neq 0$. Hence $3 \leq \chi(M) = 2 + b_2 \leq 7$. From the proof of Theorem 1.4, we have $b_2^{\pm} = 1$ and $\chi(M) = 3$. If (M^4, g) has harmonic Weyl tensor, by Theorem 1.4 we have $\int_M |W|^2 = \frac{1}{6}\mu^2([g])$, which contradicts $\int_M |W|^2 > \frac{1}{6}\mu^2([g])$.

Case 2: In the case of equality, we have

$$\int_M |W|^2 = \tfrac{1}{3}\mu^2([g]) = \tfrac{64}{3}\pi^2\chi(M).$$

Hence g is conformal to an Einstein metric $\tilde{g}$. By Corollary 1.8, (M^4, g) is conformal to a quotient of $\mathbb{S}^2 \times \mathbb{S}^2$ with the product metric. □

Proposition 5.4. Let (M^4, g) be a 4-dimensional compact Riemannian manifold with positive Yamabe constant. If

$$\tfrac{1}{6}\mu^2([g]) \leq \int_M |W|^2 \leq \tfrac{1}{3}\mu^2([g]),$$

and the universal cover of (M^4, g) is not homeomorphic to $\mathbb{S}^4$, then one of the following must be true:

(1) (M^4, g) has $\chi(M^4) \leq 4$ and $1 \leq b_2 = b_2^+ \leq 2$.

(2) (M^4, g) has $\chi(M^4) \leq 4$ and $1 \leq b_2 = b_2^- \leq 2$.

(3) The universal cover of (M^4, g) is conformal to a Kähler manifold of positive constant scalar curvature. In particular, (M^4, g) is a quotient of $(\Sigma_1, g_1) \times (\Sigma_2, g_2)$, where the surface (Σ_i, g_i) has constant Gaussian curvature k_i, and $k_1 + k_2 > 0$.

Proof. When $\int_M |W^+|^2 < \frac{1}{6}\mu^2([g])$ and $\int_M |W^-|^2 < \frac{1}{6}\mu^2([g])$. By Proposition F, we have $b_2 = 0$. Hence the universal cover of (M^4, g) is homeomorphic to $\mathbb{S}^4$.

When $\int_M |W^{\mp}|^2 < \frac{1}{6}\mu^2([g])$ and $\int_M |W^{\pm}|^2 \geq \frac{1}{6}\mu^2([g])$. By Proposition E, we have $b_2^{\mp} = 0$. By the Hirzebruch signature formula

$$\frac{1}{48\pi^2}\int_M (|W_g^+|^2 - |W_g^-|^2) = b_2^+ - b_2^- = \pm b_2^{\pm} = \sigma(M), \tag{5-5}$$

we get $\pm\sigma(M) = b_2^{\pm} \geq 1$. Since $\int_M |W|^2 \leq \frac{1}{3}\mu^2([g])$, by the fact that $\mu^2([g]) \leq \mu^2(\mathbb{S}^4) = 384\pi^2$ and the inequality is strict unless (M^4, g) is conformal to $\mathbb{S}^4$, we get $b_2^{\pm} \leq 2$. Then we get $\chi(M) \leq 4$.

When $\int_M |W^+|^2 = \int_M |W^-|^2 = \frac{1}{6}\mu^2([g])$. We have that $\sigma(M) = 0$ and $\chi(M)$ is even. For any metric g' of unit volume in the conformal class of g, we have

$$\int_M R_{g'} - \sqrt{6}\int_M |W^{\pm}_{g'}| \geq \mu([g]) - \sqrt{6}\Big(\int_M |W^{\pm}_{g'}|^2\Big)^{\frac{1}{2}} = \mu([g]) - \sqrt{6}\Big(\int_M |W^{\pm}|^2\Big)^{\frac{1}{2}} = 0.$$

Case 1: $\int_M R_{g'} - \sqrt{6}\int_M |W^{\pm}_{g'}| > 0$. By [Chen and Zhu 2014, Section 2] and [Gursky 2000, Section 3], there is a metric $\tilde{g}$ of unit volume in the conformal class of g such that

$$\sqrt{6}|W^{\pm}_{\tilde{g}}| < R_{\tilde{g}}.$$

From the proof of Proposition G, we have $b_2 = 0$ for $\sigma(M) = 0$. Hence the universal cover of (M^4, g) is homeomorphic to $\mathbb{S}^4$.

Case 2: $\int_M R_{g'} - \sqrt{6}\int_M |W^{\pm}_{g'}| = 0$. Thus there are two metrics $\tilde{g_1}$ and $\tilde{g_2}$ of unit volume in the conformal class of g such that

$$\sqrt{6}|W^+_{\tilde{g_1}}| = R_{\tilde{g_1}}, \qquad \sqrt{6}|W^-_{\tilde{g_2}}| = R_{\tilde{g_2}}.$$

We have

$$\int_M |W^+_{\tilde{g_1}}|^2 = \int_M |W^-_{\tilde{g_2}}|^2 = \frac{1}{6}\int_M R^2_{\tilde{g_1}} = \frac{1}{6}\int_M R^2_{\tilde{g_2}} = \frac{1}{6}\mu^2([g]).$$

Hence $\tilde{g_1}$ and $\tilde{g_2}$ are the Yamabe minimizers of g. So $(M^4, \tilde{g_1})$ has nonnegative isotropic curvature. If $b_2 = 0$, by Theorem 4.10 of [Micallef and Wang 1993], $(M^4, \tilde{g_1})$ becomes positive isotropic curvature. By the proof of Theorem 1.3(1), the universal cover of (M^4, g) is diffeomorphic to $\mathbb{S}^4$. If $b_2 > 0$, from the proof of Proposition G, the universal cover of $(M^4, \tilde{g_1})$ is a Kähler manifold of positive constant scalar curvature. Since the scalar curvature is positive, the universal cover of $(M^4, \tilde{g_1})$ is diffeomorphic to $(\Sigma_1, g_1) \times (\Sigma_2, g_2)$, where (Σ_i, g_i) is a 2-dimensional manifold, and the Gaussian curvature k_i of g_i must be a constant and satisfies $k_1 + k_2 > 0$. □

By Theorems 1.6 and D* and Corollary 1.8, we obtain the following theorem:

Theorem 5.5. *Let (M^4, g) be a 4-dimensional compact Riemannian manifold which is not homeomorphic to $\mathbb{S}^4$ or $\mathbb{RP}^4$ with positive Yamabe constant. If*

$$\int_M |W|^2 + 4\int_M |\mathring{\mathrm{Ric}}|^2 \leq \tfrac{1}{3}\mu^2([g]),$$

then one of the following must be true:

(1) *(M^4, g) is a quotient of $\mathbb{S}^2 \times \mathbb{S}^2$ with the product metric.*

(2) *(M^4, g) is $\mathbb{CP}^2$ with the Fubini–Study metric.*

(3) (M^4, g) *is conformal to a quotient of* $\mathbb{R}^1 \times \mathbb{S}^3$ *with the product metric.*

(4) (M^4, g) *has* $\chi(M) = 3,\ b_1 = 0$ *and* $b_2 = 1$, *and does not have harmonic Weyl tensor.*

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