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ON THE STRUCTURE OF CYCLOTOMIC NILHECKE ALGEBRAS

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In this paper we study the structure of the cyclotomic nilHecke algebras $\mathscr{H}_{\ell,n}^{(0)}$, where $\ell, n \in \mathbb{N}$. We construct a monomial basis for $\mathscr{H}_{\ell,n}^{(0)}$ which verifies a conjecture of Mathas. We show that the graded basic algebra of $\mathscr{H}_{\ell,n}^{(0)}$ is commutative and hence isomorphic to the center Z of $\mathscr{H}_{\ell,n}^{(0)}$. We further prove that $\mathscr{H}_{\ell,n}^{(0)}$ is isomorphic to the full matrix algebra over Z and construct an explicit basis for the center Z. We also construct a complete set of pairwise orthogonal primitive idempotents of $\mathscr{H}_{\ell,n}^{(0)}$. Finally, we present a new homogeneous symmetrizing form Tr on $\mathscr{H}_{\ell,n}^{(0)}$ by explicitly specifying its values on a given homogeneous basis of $\mathscr{H}_{\ell,n}^{(0)}$ and show that it coincides with Shan–Varagnolo–Vasserot's symmetrizing form $\mathrm{Tr}^{\mathrm{SVV}}$ on $\mathscr{H}_{\ell,n}^{(0)}$.

1. Introduction

Quiver Hecke algebras $\mathscr{R}_\alpha$ and their finite dimensional quotients $\mathscr{R}_\alpha^\Lambda$ (i.e., cyclotomic quiver Hecke algebras) have been hot topics in recent years. These algebras are remarkable because they can be used to categorify quantum groups and their integrable highest weight modules; see [Kang and Kashiwara 2012; Khovanov and Lauda 2009; Rouquier 2008; 2012; Varagnolo and Vasserot 2011]. These algebras can be regarded as some $\mathbb{Z}$-graded analogues of the affine Hecke algebras and their finite dimensional quotients. Many results concerning the representation theory of the affine Hecke algebras and the cyclotomic Hecke algebras of type A have their $\mathbb{Z}$-graded analogues for the quiver Hecke algebras $\mathscr{R}_\alpha$ and the cyclotomic quotients $\mathscr{R}_\alpha^\Lambda$; see [Brundan and Kleshchev 2009b; Brundan et al. 2011; Lauda and Vazirani 2011]. It is natural to expect that the structure of the affine Hecke algebras and the cyclotomic Hecke algebras of type A also have their $\mathbb{Z}$-graded analogues for the algebras $\mathscr{R}_\alpha$ and $\mathscr{R}_\alpha^\Lambda$. In fact, this is indeed the case for the quiver Hecke algebras $\mathscr{R}_\alpha$. For example, we have faithful polynomial representations, standard basis and a nice description of the center for the algebra $\mathscr{R}_\alpha$ in a similar way as in the case of the affine Hecke algebras of type A. However, the situation turns out to be much more tricky for the cyclotomic quiver Hecke algebras $\mathscr{R}_\alpha^\Lambda$. Only partial

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progress has been made for the structure of the cyclotomic quiver Hecke algebras $\mathscr{R}^{\Lambda}_{\alpha}$ so far. For example:

(1) The cyclotomic quiver Hecke algebra of type A has a $\mathbb{Z}$-graded cellular basis by [Hu and Mathas 2010].

(2) The cyclotomic quiver Hecke algebra is a $\mathbb{Z}$-graded symmetric algebra by [Shan et al. 2017].

(3) The center of the cyclotomic quiver Hecke algebra $\mathscr{R}^{\Lambda}_{\alpha}$ is the image of the center of the quiver Hecke algebra $\mathscr{R}_{\alpha}$ whenever the associated Cartan matrix is symmetric of finite type by [Webster 2015].

Apart from the type A case, one does not even know any explicit bases for arbitrary cyclotomic quiver Hecke algebras. On the other hand, for the classical cyclotomic Hecke algebra of type A, we have not only a Dipper–James–Mathas's cellular basis [Dipper et al. 1998] but also a monomial basis (or Ariki–Koike basis [Ariki and Koike 1994]). But even for the cyclotomic quiver Hecke algebra of type A we do not know any explicit monomial basis. This motivates our first question:

Question 1.1. *Can we construct an explicit monomial basis for any cyclotomic quiver Hecke algebra?*

Shan, Varagnolo and Vasserot [Shan et al. 2017] have shown that each cyclotomic quiver Hecke algebra can be endowed with a homogeneous symmetrizing form $\mathrm{Tr}^{\mathrm{SVV}}$ which makes it into a graded symmetric algebra (see Remark 4.7 and [Hu and Mathas 2010, §6.3] for the type A case). However, the SVV symmetrizing form $\mathrm{Tr}^{\mathrm{SVV}}$ is defined in an inductive manner. It is difficult to compute the explicit value of the form $\mathrm{Tr}^{\mathrm{SVV}}$ on any specified homogeneous element. On the other hand, it is well-known that the classical cyclotomic Hecke algebra of type A is symmetric [Malle and Mathas 1998; Brundan and Kleshchev 2008] and the definition of its symmetrizing form is explicit in that it specifies its value on each monomial basis element. This motivates our second question:

Question 1.2. *Can we determine the explicit values of the Shan–Varagnolo–Vasserot symmetrizing form* Tr^{SVV} *on some monomial bases (or at least a set of K-linear generators) of the cyclotomic quiver Hecke algebra?*

An explicit basis for the center of $\mathscr{R}^{\Lambda}_{\alpha}$ is unknown. Even for the classical cyclotomic Hecke algebra of type A, except in the level one case [Geck and Pfeiffer 2000] or in the degenerate case [Brundan 2008], one does not know any explicit basis for the center.

Question 1.3. *Can we give an explicit basis for the center of the cyclotomic quiver Hecke algebra?*

The starting point of this paper is to try to answer the above three questions. As a first step toward this goal, we need to consider the case of the cyclotomic quiver Hecke algebra which corresponds to a quiver with a single vertex and no edges. That is, the cyclotomic nilHecke algebra of type A. Let us recall its definition.

Definition 1.4. Let $\ell, n \in \mathbb{N}$. The nilHecke algebra $\mathscr{H}_n^{(0)}$ of type A is the unital associative K-algebra generated by $\psi_1, \dots, \psi_{n-1}, y_1, \dots, y_n$ which satisfy the following relations:

$$\begin{aligned}
\psi_r^2 &= 0, && \forall\, 1 \le r < n,\\
\psi_r\psi_k &= \psi_k\psi_r, && \forall\, 1 \le k < r-1 < n-1,\\
\psi_r\psi_{r+1}\psi_r &= \psi_{r+1}\psi_r\psi_{r+1}, && \forall\, 1 \le r < n-1,\\
y_r y_k &= y_k y_r, && \forall\, 1 \le r, k \le n,\\
\psi_r y_{r+1} &= y_r\psi_r + 1, \quad y_{r+1}\psi_r = \psi_r y_r + 1, && \forall\, 1 \le r < n,\\
\psi_r y_k &= y_k\psi_r, && \forall\, k \ne r, r+1.
\end{aligned}$$

The cyclotomic nilHecke algebra $\mathscr{H}_{\ell,n}^{(0)}$ of type A is the quotient of $\mathscr{H}_n^{(0)}$ by the two-sided ideal generated by y_1^{ℓ}.

The nilHecke algebras $\mathscr{H}_n^{(0)}$ was introduced by Kostant and Kumar [1986]. It plays an important role in the theory of Schubert calculus; see [Hiller 1982]. Mathas [2015, §2.5] has observed that the Specht module over $\mathscr{H}_{n,n}^{(0)}$ can be realized as the coinvariant algebra with standard bases of Specht modules being identified with the Schubert polynomials of the coinvariant algebras. It is clear that both $\mathscr{H}_n^{(0)}$ and $\mathscr{H}_{\ell,n}^{(0)}$ are $\mathbb{Z}$-graded K-algebras such that each ψ_r is homogeneous with $\deg \psi_r = -2$ and each y_s is homogeneous with $\deg y_s = 2$ for all $1 \le r < n$, $1 \le s \le n$. Mathas [2015, §2.5] has conjectured a monomial basis of the cyclotomic nilHecke algebra $\mathscr{H}_{n,n}^{(0)}$. In this paper, we shall construct a monomial basis of the cyclotomic nilHecke algebra $\mathscr{H}_{\ell,n}^{(0)}$ for arbitrary ℓ (Theorem 2.34) that, in particular, verifies Mathas's conjecture. As an application, we shall construct a basis for the center Z of $\mathscr{H}_{\ell,n}^{(0)}$ (Theorem 3.7). Thus we shall answer Question 1.1 and Question 1.3 for the cyclotomic nilHecke algebra $\mathscr{H}_{\ell,n}^{(0)}$. Furthermore, we shall construct a new homogeneous symmetrizing form Tr (Proposition 4.13) by specifying its values on a homogeneous basis element of $\mathscr{H}_{\ell,n}^{(0)}$. We prove that this new form Tr actually coincides with Shan–Varagnolo–Vasserot's symmetrizing form $\mathrm{Tr}^{\mathrm{SVV}}$ [Shan et al. 2017] on $\mathscr{H}_{\ell,n}^{(0)}$. Thus we also answer Question 1.2 for the cyclotomic nilHecke algebra $\mathscr{H}_{\ell,n}^{(0)}$.

The content of the paper is organized as follows. In Section 2, we shall first review some basic knowledge about the structure and representation of $\mathscr{H}_{\ell,n}^{(0)}$. Lemma 2.12 provides a useful commutator relation which will be used frequently in later discussion. In Corollary 2.18 and 2.19 we determine the graded dimensions of the graded simple modules and their graded projective covers as well as the graded decomposition numbers and the graded Cartan numbers. We construct a monomial basis of the cyclotomic nilHecke algebra $\mathscr{H}_{\ell,n}^{(0)}$ for arbitrary ℓ in Theorem 2.34. We also construct a complete set of pairwise orthogonal primitive idempotents in Corollary 2.25 and Theorem 2.31. In Section 3, we shall first present a basis for the graded basic algebra of $\mathscr{H}_{\ell,n}^{(0)}$ and show that it is isomorphic to the center Z of $\mathscr{H}_{\ell,n}^{(0)}$ in Lemma 3.2. Then we shall give a basis for the center in Theorem 3.7 which consists of certain symmetric polynomials in $y_1, \dots, y_n$. We also show in Proposition 3.8 that $\mathscr{H}_{\ell,n}^{(0)}$ is isomorphic to the full matrix algebra over Z. In Section 4, we shall first show in Lemma 4.4 that the center Z is a graded symmetric algebra by specifying an explicit homogeneous symmetrizing form on Z. Then we shall introduce two homogeneous symmetrizing forms: one is defined by using its isomorphism with the full matrix algebra over the center Z (Lemma 4.6); another is defined by specifying its values on a homogeneous basis element (Definition 4.11 and Proposition 4.13). We show in Proposition 4.14 that these two symmetrizing forms are the same. In Section 5 we show that the form Tr also coincides with Shan–Varagnolo–Vasserot's symmetrizing form $\mathrm{Tr}^{\mathrm{SVV}}$ (which was introduced in [Shan et al. 2017] for general cyclotomic quiver Hecke algebras).

After the submission of this paper, Professor Lauda emailed us that he wonders if our results have some connections with his papers [Khovanov et al. 2012; Lauda 2012]. In the latter paper he proved that the cyclotomic nilHecke algebra is isomorphic to the matrix ring of size $n!$ over the cohomology of a Grassmannian. Combining it with Proposition 3.8 in this paper this implies that the center of the cyclotomic nilHecke algebra is isomorphic to that cohomology of a Grassmannian. He also proposed an interesting question of comparing the trace form Tr in this paper with the natural form on the matrix ring over the cohomology of the Grassmannian which can be defined using integration over the volume form.

2. The structure and representation of $\mathscr{H}_{\ell,n}^{(0)}$

Let $\mathfrak{S}_n$ be the symmetric group on $\{1, 2, \dots, n\}$ and let $s_i := (i, i+1) \in \mathfrak{S}_n$, for $1 \le i < n$. Then $\{s_1, \dots, s_{n-1}\}$ is the standard set of Coxeter generators for $\mathfrak{S}_n$. If $w \in \mathfrak{S}_n$ then the length of w is

$$\ell(w) := \min\{k \in \mathbb{N} \mid w = s_{i_1} \dots s_{i_k} \text{ for some } 1 \le i_1, \dots, i_k < n\}.$$

If $w = s_{i_1} \dots s_{i_k}$ with $k = \ell(w)$ then $s_{i_1} \dots s_{i_k}$ is a reduced expression for w. In this case, we define $\psi_w := \psi_{i_1} \dots \psi_{i_k}$. The braid relation in Definition 1.4 ensures that ψ_w does not depend on the choice of the reduced expression of w. Let $w_{0,n}$ be the unique longest element in $\mathfrak{S}_n$. When n is clear from the context we shall write w_0

instead of $w_{0,n}$ for simplicity. Then $w_0 = w_0^{-1}$ and $\ell(w_0) = n(n-1)/2$. Let $*$ be the unique K-algebra antiautomorphism of $\mathscr{H}_{\ell,n}^{(0)}$ which fixes each of its ψ and y generators.

Lemma 2.1 [Manivel 2001]. *The elements in the set*

$$\{\psi_w y_1^{c_1} \dots y_n^{c_n} \mid w \in \mathfrak{S}_n, c_1, \dots, c_n \in \mathbb{N}\}$$

form a K-basis of the nilHecke algebra $\mathscr{H}_n^{(0)}$ and the center of $\mathscr{H}_n^{(0)}$ is the set of symmetric polynomials in $y_1, \dots, y_n$.

Let $\pi : \mathscr{H}_n^{(0)} \twoheadrightarrow \mathscr{H}_{\ell,n}^{(0)}$ be the canonical surjective homomorphism.

Definition 2.2. An element z in $\mathscr{H}_{\ell,n}^{(0)}$ is said to be symmetric if $z = \pi(f(y_1, \dots, y_n))$ for some symmetric polynomial $f(t_1, \dots, t_n) \in K[t_1, \dots, t_n]$, where $t_1, \dots, t_n$ are n indeterminates over K.

Corollary 2.3. *Any symmetric element in $\mathscr{H}_{\ell,n}^{(0)}$ lies in the center of $\mathscr{H}_{\ell,n}^{(0)}$.*

Proof. This follows from Lemma 2.1 and the surjective homomorphism π. □

Let Γ be a quiver without loops and I its vertex set. For any $i, j \in I$ let d_{ij} be the number of arrows $i \to j$ and set $m_{ij} := d_{ij} + d_{ji}$. This defines a *symmetric* generalized Cartan matrix $(a_{ij})_{i,j\in I}$ by putting $a_{ij} := -m_{ij}$ for $i \neq j$ and $a_{ii} := 2$ for any $i \in I$. Let u, v be two indeterminates over $\mathbb{Z}$. We define $Q_{ij} := (-1)^{d_{ij}}(u-v)^{m_{ij}}$ for any $i \neq j \in I$ and $Q_{ii}(u, v) := 0$ for any $i \in I$. Let $(\mathfrak{h}, \Pi, \Pi^\vee)$ be a realization of the generalized Cartan matrix $(a_{ij})_{i,j\in I}$. Let P be the associated weight lattice which is a finite rank free abelian group and contains $\Pi = \{\alpha_i \mid i \in I\}$, let $P^\vee$ be the associated coweight lattice which is a finite rank free abelian group too and contains $\Pi^\vee = \{\alpha_i^\vee \mid i \in I\}$. Let $Q^+ := \mathbb{N}\Pi \subset P$ be the semigroup generated by Π and $P^+ \subset P$ be the set of integral dominant weights. Let $\Lambda \in P^+$ and $\beta \in Q_n^+$. One can associate it with a quiver Hecke algebra $\mathscr{R}_\beta$ as well as its cyclotomic quotient $\mathscr{R}_\beta^\Lambda$. We refer the readers to [Khovanov and Lauda 2009; Rouquier 2012; Shan et al. 2017] for precise definitions.

Let $\{\Lambda_i \mid i \in I\}$ be the set of fundamental weights. The nilHecke algebra and its cyclotomic quotient can be regarded as a special quiver Hecke algebra and cyclotomic quiver Hecke algebra. That is, the quiver with single one vertex $\{0\}$ and no edges. More precisely, we have

$$\mathscr{H}_n^{(0)} = \mathscr{R}_{n\alpha_0}, \quad \mathscr{H}_{\ell,n}^{(0)} = \mathscr{R}_{n\alpha_0}^{\ell\Lambda_0}. \tag{2.4}$$

Throughout this paper, unless otherwise stated, we shall work in the category of $\mathbb{Z}$-graded $\mathscr{H}_{\ell,n}^{(0)}$-modules. Note that $\mathscr{H}_{\ell,n}^{(0)}$ is a special type A cyclotomic quiver Hecke algebra so that we can apply the theory of graded cellular algebras developed in [Hu and Mathas 2010]. We now recall the definition of graded cellular basis in this special situation (i.e., for $\mathscr{H}_{\ell,n}^{(0)}$).

We use $\varnothing$ to denote the empty partition and (1) to denote the unique partition of 1. Set $|\varnothing| := 0$, $|(1)| := 1$. We define

$$\mathscr{P}_0 := \Big\{ \boldsymbol{\lambda} := (\lambda^{(1)}, \dots, \lambda^{(\ell)}) \,\Big|\, \sum_{i=1}^{\ell} |\lambda^{(i)}| = n,\, \lambda^{(i)} \in \{\varnothing, (1)\},\ \forall\, 1 \le i \le \ell \Big\}.$$

Definition 2.5. If $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(\ell)}) \in \mathscr{P}_0$, then we define $\theta(\boldsymbol{\lambda})$ to be the unique n-tuple $(k_1, \dots, k_n)$ such that $1 \le k_1 < k_2 < \cdots < k_n \le \ell$ and

$$\lambda^{(j)} = \begin{cases} (1) & \text{if } j = k_i \text{ for some } 1 \le i \le n, \\ \varnothing & \text{otherwise.} \end{cases}$$

Given any two n-tuples $(k_1, \dots, k_n)$, $(k'_1, \dots, k'_n)$ of increasing positive integers, we define

$$(k_1, \dots, k_n) \ge (k'_1, \dots, k'_n) \Leftrightarrow k_i \ge k'_i,\ \forall\, 1 \le i \le n,$$

and $(k_1, \dots, k_n) > (k'_1, \dots, k'_n)$ if $(k_1, \dots, k_n) \ge (k'_1, \dots, k'_n)$ and $(k_1, \dots, k_n) \ne (k'_1, \dots, k'_n)$. For any $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathscr{P}_0$, we define

$$\boldsymbol{\lambda} > \boldsymbol{\mu} \Leftrightarrow \theta(\boldsymbol{\lambda}) < \theta(\boldsymbol{\mu}).$$

Then ">" is a partial order on $\mathscr{P}_0$.

The following is a special case of [Hu and Mathas 2010, Definition 4.15].

Definition 2.6. Let $\boldsymbol{\lambda} \in \mathscr{P}_0$ with $\theta(\boldsymbol{\lambda}) = (k_1, \dots, k_n)$. We define

$$y_{\boldsymbol{\lambda}} := y_1^{\ell-k_1} \dots y_n^{\ell-k_n}, \quad \deg y_{\boldsymbol{\lambda}} := 2\ell n - 2\sum_{i=1}^{n} k_i.$$

By the main results in [Hu and Mathas 2010], the elements in the set

$$\{\psi^{\boldsymbol{\lambda}}_{w,u} := \psi^*_w y_{\boldsymbol{\lambda}} \psi_u \mid \boldsymbol{\lambda} \in \mathscr{P}_0,\, w, u \in \mathfrak{S}_n\} \tag{2.7}$$

form a graded cellular K-basis of $\mathscr{H}^{(0)}_{\ell,n}$. Each basis element $\psi^{\boldsymbol{\lambda}}_{w,u}$ is homogeneous with degree equal to

$$\deg \psi^{\boldsymbol{\lambda}}_{w,u} := \deg y_{\boldsymbol{\lambda}} - 2\ell(w) - 2\ell(u) = 2\ell n - 2\sum_{i=1}^{n} k_i - 2\ell(w) - 2\ell(u).$$

In particular, $\dim_K \mathscr{H}^{(0)}_{\ell,n} = \ell(\ell-1)\dots(\ell-n+1)n!$. Note that $\mathscr{P}_0 \ne \varnothing$ if and only if $\ell \ge n$. Therefore, $\mathscr{H}^{(0)}_{\ell,n} = 0$ whenever $\ell < n$. Henceforth, we always assume that $\ell \ge n$.

By the general theory of (graded) cellular algebras [Graham and Lehrer 1996; Hu and Mathas 2010], for each $\boldsymbol{\lambda} \in \mathscr{P}_0$, we have a graded Specht module $S^{\boldsymbol{\lambda}}$, which is equipped with an associative homogeneous bilinear form $\langle -,- \rangle_{\boldsymbol{\lambda}}$. Let $\mathrm{rad}\langle -,- \rangle_{\boldsymbol{\lambda}}$ be the radical of that bilinear form. We define $D^{\boldsymbol{\lambda}} := S^{\boldsymbol{\lambda}} / \mathrm{rad}\langle -,- \rangle_{\boldsymbol{\lambda}}$. By [Hu and

Mathas 2010, Corollary 5.11], we know that $D^{\boldsymbol{\lambda}} \neq 0$ if and only if $\boldsymbol{\lambda}$ is a Kleshchev multipartition with respect to $(p; 0, 0, \dots, 0)$, where $p = \operatorname{char} K$.

Let $\boldsymbol{\lambda} \in \mathscr{P}_0$ with $\theta(\boldsymbol{\lambda}) = (k_1, \dots, k_n)$. A $\boldsymbol{\lambda}$-tableau is a bijection $\mathfrak{t} : \{k_1, \dots, k_n\} \to \{1, 2, \dots, n\}$. We use $\operatorname{Tab}(\boldsymbol{\lambda})$ to denote the set of $\boldsymbol{\lambda}$-tableaux. For any $\mathfrak{t} \in \operatorname{Tab}(\boldsymbol{\lambda})$, we define

$$\begin{aligned}\deg \mathfrak{t} := \sum_{i=1}^{n} \Big(\#\big\{k_i < j \le \ell \mid \text{ either } j \notin \{k_1, \dots, k_n\} \text{ or } j = k_b \text{ with } \mathfrak{t}(j) > \mathfrak{t}(k_i)\big\} \\ - \#\big\{k_i < j \le \ell \mid j \in \{k_1, \dots, k_n\} \text{ and } \mathfrak{t}(j) < \mathfrak{t}(k_i)\big\}\Big).\end{aligned}$$

It is clear that in our special case (i.e., for $\mathscr{P}_0$) the above definition of $\deg \mathfrak{t}$ coincides with that in [Brundan et al. 2011; Hu and Mathas 2010].

Definition 2.8. We define

$$\boldsymbol{\lambda}_{\max} := \big(\underbrace{(1), \dots, (1)}_{n \text{ copies}}, \underbrace{\varnothing, \dots, \varnothing}_{\ell - n \text{ copies}}\big), \quad \boldsymbol{\lambda}_{\min} := \big(\underbrace{\varnothing, \dots, \varnothing}_{\ell - n \text{ copies}}, \underbrace{(1), \dots, (1)}_{n \text{ copies}}\big).$$

It is clear that for any $\boldsymbol{\mu} \in \mathscr{P}_0 \setminus \{\boldsymbol{\lambda}_{\max}, \boldsymbol{\lambda}_{\min}\}$, we have that

$$\boldsymbol{\lambda}_{\min} < \boldsymbol{\mu} < \boldsymbol{\lambda}_{\max}, \quad \deg y_{\boldsymbol{\lambda}_{\min}} < \deg y_{\boldsymbol{\mu}} < \deg y_{\boldsymbol{\lambda}_{\max}}. \tag{2.9}$$

Using [Brundan and Kleshchev 2009a] and the definition of the Kleshchev multipartition in [Ariki and Mathas 2000], it is clear that $\boldsymbol{\lambda}_{\min}$ is the unique Kleshchev multipartition in $\mathscr{P}_0$. Therefore, for any $\boldsymbol{\lambda} \in \mathscr{P}_0$, $D^{\boldsymbol{\lambda}} \neq 0$ if and only if $\boldsymbol{\lambda} = \boldsymbol{\lambda}_{\min}$. Furthermore, $D^{\boldsymbol{\lambda}_{\min}}$ is the unique (self-dual) graded simple module for $\mathscr{H}_{\ell,n}^{(0)}$. Let $P^{\boldsymbol{\lambda}_{\min}}$ be its graded projective cover.

Definition 2.10. We define

$$D_0 := D^{\boldsymbol{\lambda}_{\min}}, \quad P_0 := P^{\boldsymbol{\lambda}_{\min}}.$$

For each $\boldsymbol{\mu} \in \mathscr{P}_0$, we use $(\mathscr{H}_{\ell,n}^{(0)})^{>\boldsymbol{\mu}}$ to denote the K-subspace of $\mathscr{H}_{\ell,n}^{(0)}$ spanned by all the elements of the form $\psi_w^* y_{\boldsymbol{\lambda}} \psi_u$, where $\boldsymbol{\lambda} > \boldsymbol{\mu}$, $w, u \in \mathfrak{S}_n$. Then $(\mathscr{H}_{\ell,n}^{(0)})^{>\boldsymbol{\mu}}$ is a two-sided ideal of $\mathscr{H}_{\ell,n}^{(0)}$. By [Hu and Mathas 2012, Corollary 3.11], for any $1 \le r \le n$, if $\theta(\boldsymbol{\mu}) = (k_1, \dots, k_n)$ then

$$y_{\boldsymbol{\mu}} y_r = y_1^{\ell - k_1} \dots y_n^{\ell - k_n} y_r \in (\mathscr{H}_{\ell,n}^{(0)})^{>\boldsymbol{\mu}}. \tag{2.11}$$

Lemma 2.12. *For any $1 \le i \le n$, $1 \le j < n$, there exists elements $h_{i,j}, h'_{i,j} \in \mathscr{H}_{\ell,n}^{(0)}$ such that*

$$\psi_{w_0} y_1^{n-1} y_2^{n-2} \dots y_{n-1} = (-1)^{n(n-1)/2} + \sum_{\substack{1 \le i \le n \\ 1 \le j < n}} y_i h_{i,j} \psi_j. \tag{2.13}$$

Similarly, we have

$$y_1^{n-1}y_2^{n-2}\dots y_{n-1}\psi_{w_0}=(-1)^{n(n-1)/2}+\sum_{\substack{1\le i\le n\\1\le j<n}}\psi_j h_{i,j}^{*}y_i. \tag{2.14}$$

Proof. We only prove the first equality as the second one follows from the first one by applying the anti-involution $*$. We use induction on n. If $n=1$, it is clear that (2.13) holds. Suppose that the lemma holds for the nilHecke algebra $\mathscr{H}_{\ell,n-1}^{(0)}$. We are going to prove (2.13) for $\mathscr{H}_{\ell,n}^{(0)}$.

Recall that the unique longest element $w_0:=w_{0,n}$ of $\mathfrak{S}_n$ has a reduced expression

$$w_0=s_1(s_2s_1)\dots(s_{n-2}s_{n-3}\dots s_1)(s_{n-1}s_{n-2}\dots s_1).$$

Recall that $w_{0,n-1}$ denotes the unique longest element in $\mathfrak{S}_{n-1}$ and

$$w_0=w_{0,n-1}(s_{n-1}s_{n-2}\dots s_1)$$

and $s_1(s_2s_1)\dots(s_{n-2}s_{n-3}\dots s_1)$ is a reduced expression for $w_{0,n-1}$.

We define

$$J_n:=\sum_{i=1}^{n}y_i\mathscr{H}_{\ell,n}^{(0)}.$$

Then we have, with all congruences modulo J_n,

$$\begin{aligned}
&\psi_{w_0}y_1^{n-1}y_2^{n-2}\dots y_{n-1}\\
&\quad=\psi_{w_0}(y_1y_2\dots y_{n-1})y_1^{n-2}y_2^{n-3}\dots y_{n-2}\\
&\quad=\psi_{w_{0,n-1}}(\psi_{n-1}\psi_{n-2}\dots\psi_1y_1y_2\dots y_{n-1})y_1^{n-2}y_2^{n-3}\dots y_{n-2}\\
&\quad=\psi_{w_{0,n-1}}(\psi_{n-1}y_1y_2\dots y_{n-1}\psi_{n-2}\dots\psi_1)y_1^{n-2}y_2^{n-3}\dots y_{n-2}\quad\text{(by Corollary 2.3)}\\
&\quad=\psi_{w_{0,n-1}}(y_1y_2\dots y_{n-2}\psi_{n-1}y_{n-1}\psi_{n-2}\dots\psi_1)y_1^{n-2}y_2^{n-3}\dots y_{n-2}\\
&\quad=\psi_{w_{0,n-1}}(y_1y_2\dots y_{n-2}(y_n\psi_{n-1}-1)\psi_{n-2}\dots\psi_1)y_1^{n-2}y_2^{n-3}\dots y_{n-2}\\
&\quad\equiv-\psi_{w_{0,n-1}}(y_1y_2\dots y_{n-2}\psi_{n-2}\dots\psi_1)y_1^{n-2}y_2^{n-3}\dots y_{n-2}\quad\text{(by (2.11))}\\
&\quad\equiv-\psi_{w_{0,n-2}}(\psi_{n-2}\psi_{n-3}\dots\psi_1y_1y_2\dots y_{n-2})(\psi_{n-2}\dots\psi_1)y_1^{n-2}y_2^{n-3}\dots y_{n-2}\\
&\quad\equiv-\psi_{w_{0,n-2}}(\psi_{n-2}y_1y_2\dots y_{n-2}\psi_{n-3}\dots\psi_1)(\psi_{n-2}\dots\psi_1)y_1^{n-2}y_2^{n-3}\dots y_{n-2}\\
&\quad\equiv-\psi_{w_{0,n-2}}(y_1y_2\dots y_{n-3}(\psi_{n-2}y_{n-2})\psi_{n-3}\dots\psi_1)(\psi_{n-2}\dots\psi_1)\\
&\qquad\qquad\times y_1^{n-2}y_2^{n-3}\dots y_{n-2}\\
&\quad\equiv-\psi_{w_{0,n-2}}(y_1y_2\dots y_{n-3}(y_{n-1}\psi_{n-2}-1)\psi_{n-3}\dots\psi_1)(\psi_{n-2}\dots\psi_1)\\
&\qquad\qquad\times y_1^{n-2}y_2^{n-3}\dots y_{n-2}\\
&\quad\equiv(-1)^2\psi_{w_{0,n-2}}(y_1y_2\dots y_{n-3}\psi_{n-3}\dots\psi_1)(\psi_{n-2}\dots\psi_1)y_1^{n-2}y_2^{n-3}\dots y_{n-2}\\
&\quad\equiv(-1)^2\psi_{w_{0,n-2}}(y_1y_2\dots y_{n-3})\big((\psi_{n-3}\dots\psi_1)(\psi_{n-2}\dots\psi_1)\big)y_1^{n-2}y_2^{n-3}\dots y_{n-2}
\end{aligned}$$

$$
\begin{aligned}
&\equiv (-1)^2\psi_{w_{0,n-3}}(\psi_{n-3}\psi_{n-4}\dots\psi_1 y_1 y_2\dots y_{n-3})\\
&\qquad\qquad\qquad\qquad \times \big((\psi_{n-3}\dots\psi_1)(\psi_{n-2}\dots\psi_1)(y_1^{n-2}y_2^{n-3}\dots y_{n-2})\big)\\
&\qquad\qquad\qquad\qquad\vdots\\
&\equiv (-1)^{n-1}\big(\psi_1(\psi_2\psi_1)\dots(\psi_{n-3}\dots\psi_1)(\psi_{n-2}\dots\psi_1)(y_1^{n-2}y_2^{n-3}\dots y_{n-2})\big)\\
&\equiv (-1)^{n-1}\psi_{w_{0,n-1}}(y_1^{n-2}y_2^{n-3}\dots y_{n-2})\\
&\equiv (-1)^{n-1}(-1)^{(n-1)(n-2)/2}\equiv(-1)^{n(n-1)/2},
\end{aligned}
$$

as required, where we have used induction in the second-to-last congruence.

Therefore, we have proved that

$$\psi_{w_0}y_1^{n-1}y_2^{n-2}\dots y_{n-1}=(-1)^{n(n-1)/2}+\sum_{\substack{1\le i\le n\\ 1\le j<n}} y_i h_i,$$

where $h_i\in\mathscr{H}_{\ell,n}^{(0)}$. Comparing the degree on both sides, we can assume that each h_i is homogeneous with $h_i\neq 0$ only if $\deg(h_i)=-2<0$. On the other hand, we can express each nonzero h_i as a K-linear combination of some monomials of the form $y_1^{c_1}\dots y_n^{c_n}\psi_w$, where $c_1,\dots,c_n\in\mathbb{N}$, $w\in\mathfrak{S}_n$. Since each y_j has degree 2, we can thus deduce that each nonzero h_i must be equal to a K-linear combination of some monomials of the form $y_1^{c_1}\dots y_n^{c_n}\psi_w$ with $c_1,\dots,c_n\in\mathbb{N}$ and $1\neq w\in\mathfrak{S}_n$. This completes the proof of the lemma. □

Lemma 2.15. (1) *For any $u,w\in\mathfrak{S}_n$, if $\ell(u)+\ell(w)>\ell(uw)$, then $\psi_u\psi_w=0$.*

(2) *For any $1\le r<n$, $\psi_r\psi_{w_0}=0=\psi_{w_0}\psi_r$.*

Proof. (1) follows from the defining relations for $\mathscr{H}_{\ell,n}^{(0)}$, while (2) follows from the defining relations for $\mathscr{H}_{\ell,n}^{(0)}$ and the fact that w_0 has both a reduced expression which starts with s_r as well as a reduced expression which ends with s_r for any $1\le r<n$. □

Let $s\in\mathbb{Z}$. For any $\mathbb{Z}$-graded $\mathscr{H}_{\ell,n}^{(0)}$-module M, we define $M\langle s\rangle$ to be a new $\mathbb{Z}$-graded $\mathscr{H}_{\ell,n}^{(0)}$-module as follows:

- $M\langle s\rangle=M$ as an ungraded $\mathscr{H}_{\ell,n}^{(0)}$-module.
- As a $\mathbb{Z}$-graded module, $M\langle s\rangle$ is obtained by shifting the grading on M up by s. That is, $M\langle s\rangle_d=M_{d-s}$, for $d\in\mathbb{Z}$.

Lemma 2.16. *Let $\boldsymbol{\mu}\in\mathscr{P}_0$ with $\theta(\boldsymbol{\mu})=(k_1,\dots,k_n)$. Then*

$$\dim D_0=n!,\quad \dim P_0=\binom{\ell}{n}n!,\quad S^{\boldsymbol{\mu}}\cong D_0\Big\langle n\ell-\frac{n(n-1)}{2}-\sum_{i=1}^n k_i\Big\rangle.$$

Proof. By the definitions of $\mathscr{P}_0$ and Specht modules over $\mathscr{H}_{\ell,n}^{(0)}$, it is clear that $S^{\mu} \cong S^{\boldsymbol{\lambda}_{\min}}\langle n\ell - n(n-1)/2 - \sum_{i=1}^{n} k_i\rangle$. Thus it suffices to show that $S^{\boldsymbol{\lambda}_{\min}} = D^{\boldsymbol{\lambda}_{\min}}$. To this end, we need to compute the bilinear form between standard bases of the Specht module $S^{\boldsymbol{\lambda}_{\min}}$.

By definition, $S^{\boldsymbol{\lambda}_{\min}}$ has a standard basis

$$\{y_1^{n-1}y_2^{n-2}\dots y_{n-1}\psi_w + (\mathscr{H}_{\ell,n}^{(0)})^{>\boldsymbol{\lambda}_{\min}} \mid w \in \mathfrak{S}_n\}.$$

For any $w, u \in \mathfrak{S}_n$, by Lemma 2.15, we see that

$$\begin{aligned} y_1^{n-1}y_2^{n-2}&\dots y_{n-1}\psi_w\psi_u^* y_1^{n-1}y_2^{n-2}\dots y_{n-1} \\ &= y_1^{n-1}y_2^{n-2}\dots y_{n-1}(\psi_w\psi_{u^{-1}})y_1^{n-1}y_2^{n-2}\dots y_{n-1} = 0 \end{aligned}$$

unless $\ell(wu^{-1}) = \ell(w) + \ell(u^{-1})$.

Now we assume that $\ell(wu^{-1}) = \ell(w) + \ell(u^{-1})$. By the commutator relations between y and ψ generators, (2.11) and the fact that $\ell(w_0) = n(n-1)2$, we can deduce that

$$\begin{aligned} y_1^{n-1}y_2^{n-2}&\dots y_{n-1}(\psi_w\psi_{u^{-1}})y_1^{n-1}y_2^{n-2}\dots y_{n-1} \\ &= y_1^{n-1}y_2^{n-2}\dots y_{n-1}\psi_{wu^{-1}}y_1^{n-1}y_2^{n-2}\dots y_{n-1} \in (\mathscr{H}_{\ell,n}^{(0)})^{>\boldsymbol{\lambda}_{\min}} \end{aligned}$$

unless $wu^{-1} = w_0$. In that case, by Lemma 2.12, we have that

$$\begin{aligned} y_1^{n-1}y_2^{n-2}&\dots y_{n-1}\psi_w\psi_u^* y_1^{n-1}y_2^{n-2}\dots y_{n-1} \\ &= y_1^{n-1}y_2^{n-2}\dots y_{n-1}\psi_{w_0}y_1^{n-1}y_2^{n-2}\dots y_{n-1} \\ &= (-1)^{n(n-1)/2}y_1^{n-1}y_2^{n-2}\dots y_{n-1} \pmod{(\mathscr{H}_{\ell,n}^{(0)})^{>\boldsymbol{\lambda}_{\min}}}. \end{aligned}$$

Thus we have proved that if $\ell(wu^{-1}) = \ell(w) + \ell(u^{-1})$ and $wu^{-1} = w_0$, then

$$\begin{aligned} \big\langle y_1^{n-1}y_2^{n-2}\dots y_{n-1}\psi_w + (\mathscr{H}_{\ell,n}^{(0)})^{>\boldsymbol{\lambda}_{\min}}, y_1^{n-1}y_2^{n-2}\dots y_{n-1}\psi_u + (\mathscr{H}_{\ell,n}^{(0)})^{>\boldsymbol{\lambda}_{\min}}\big\rangle_{\boldsymbol{\lambda}_{\min}} \\ = (-1)^{n(n-1)/2}; \end{aligned}$$

otherwise it is equal to 0. This means the Gram matrix of $S^{\boldsymbol{\lambda}_{\min}}$ is invertible and hence the bilinear form $\langle -,-\rangle_{\boldsymbol{\lambda}_{\min}}$ on $S^{\boldsymbol{\lambda}_{\min}}$ is nondegenerate. It follows that $S^{\boldsymbol{\lambda}_{\min}} = D^{\boldsymbol{\lambda}_{\min}} = D_0$ as required. Therefore, $\dim D_0 = \dim S^{\boldsymbol{\lambda}_{\min}} = n!$. Finally, since $\mathscr{H}_{\ell,n}^{(0)} \cong P_0^{\oplus \dim D_0}$, we can deduce that $\dim P_0 = \dim \mathscr{H}_{\ell,n}^{(0)} / \dim D_0 = \binom{\ell}{n}(n!)^2/n! = \binom{\ell}{n}n!$. □

Let q be an indeterminate. The *graded dimension* of M is the Laurent polynomial

$$\dim_q M = \sum_{d\in\mathbb{Z}} (\dim_K M_d)\, q^d \in \mathbb{N}[q, q^{-1}], \tag{2.17}$$

where M_d is the homogeneous component of M which has degree d. In particular,

$\dim_K M = (\dim_q M)|_{q=1}$. As a consequence, we can determine the graded dimension of the unique self-dual graded simple module D_0 and its projective cover P_0, and compute the graded decomposition number $d_{\boldsymbol{\mu},\boldsymbol{\lambda}_{\min}}(q) := [S^{\boldsymbol{\mu}} : D^{\boldsymbol{\lambda}_{\min}}]_q$ and graded Cartan number $c_{\boldsymbol{\lambda}_{\min},\boldsymbol{\lambda}_{\min}}(q) := [P^{\boldsymbol{\lambda}_{\min}} : D^{\boldsymbol{\lambda}_{\min}}]_q$.

Corollary 2.18. *We have*

$$\dim_q D_0 = \sum_{\mathfrak{t} \in \mathrm{Tab}(\boldsymbol{\lambda}_{\min})} q^{\deg \mathfrak{t}},$$

$$\dim_q P_0 = \sum_{\substack{k_1,\dots,k_n \in \mathbb{N} \\ 1 \le k_1 < k_2 < \cdots < k_n \le \ell}} \sum_{\mathfrak{t} \in \mathrm{Tab}(\boldsymbol{\lambda}_{\min})} q^{\deg \mathfrak{t} + 2n\ell - n(n-1) - \sum_{i=1}^n 2k_i}.$$

Corollary 2.19. *Let $\boldsymbol{\mu} \in \mathscr{P}_0$ with $\theta(\boldsymbol{\mu}) = (k_1, \dots, k_n)$. We have*

$$d_{\boldsymbol{\mu},\boldsymbol{\lambda}_{\min}}(q) = q^{n\ell - n(n-1)/2 - \sum_{i=1}^n k_i} \in \delta_{\boldsymbol{\mu},\boldsymbol{\lambda}_{\min}} + q\mathbb{N}[q],$$

$$c_{\boldsymbol{\lambda}_{\min},\boldsymbol{\lambda}_{\min}}(q) = \sum_{\substack{l_1,\dots,l_n \in \mathbb{N} \\ 1 \le l_1 < l_2 < \cdots < l_n \le \ell}} q^{2n\ell - n(n-1) - \sum_{i=1}^n 2l_i} \in 1 + q\mathbb{N}[q].$$

Lemma 2.20 [Hoffnung and Lauda 2010, Proposition 7]. *For any $1 \le s \le n$, we have*

$$\sum_{\substack{l_1,\dots,l_s \in \mathbb{N} \\ l_1 + \cdots + l_s = \ell - s + 1}} y_1^{l_1} y_2^{l_2} \dots y_s^{l_s} = 0.$$

Remark 2.21. Note that one should identify our generator y_r with the generator $-x_{r,\boldsymbol{i}}$ in [Hoffnung and Lauda 2010] so that the relation $\psi_r y_{r+1} = y_r \psi_r + 1$ in Definition 1.4 matches up with the relation $x_{r,\boldsymbol{i}} \delta_{r,i} - \delta_{r,\boldsymbol{i}} x_{r+1,i} = e(\boldsymbol{i})$ when $i_r = i_{r+1}$.

Lemma 2.22 [Hoffnung and Lauda 2010, Proposition 8]. *Let $1 \le m < n$ and $b \in \mathbb{N}$. If $y_{m-1}^b = 0$ then $y_m^b = 0$.*

Lemma 2.23. *For any $2 \le m \le n$ and $\omega_m > \ell - m$, we have*

$$y_1^{\ell-1} y_2^{\ell-2} \dots y_{m-1}^{\ell-m+1} y_m^{\omega_m} = 0. \tag{2.24}$$

Proof. We use induction on m. If $m = 1$, then (2.24) reduces to $y_1^{\omega_1} = 0$ for $\omega_1 > \ell - 1$, which certainly holds by the fact that $y_1^\ell = 0$.

If $m = 2$, then we need to show that $y_1^{\ell-1} y_2^{\omega_2} = 0$ whenever $\omega_2 > \ell - 2$. By Lemma 2.22, we can deduce that $y_2^\ell = 0$ from the equality $y_1^\ell = 0$. Therefore, it remains to show that $y_1^{\ell-1} y_2^{\ell-1} = 0$. In this case, applying Lemma 2.20, we get that

$$y_2^{\ell-1} = \sum_{\substack{l_1, l_2 \in \mathbb{N}, l_1 \neq 0 \\ l_1 + l_2 = \ell - 1}} y_1^{l_1} y_2^{l_2}.$$

It follows that

$$y_1^{\ell-1}y_2^{\ell-1}=-\sum_{\substack{l_1,l_2\in\mathbb{N},l_1\neq 0\\ l_1+l_2=\ell-1}} y_1^{\ell-1+l_1}y_2^{l_2}=0,$$

as required.

Now assume that (2.24) holds for $2\le k\le m$. Hence $y_1^{\ell-1}y_2^{\ell-2}\dots y_{k-1}^{\ell-k+1}y_k^{\omega_k}=0$ whenever $\omega_k>\ell-k$.

Applying Lemma 2.20 for $s=m+1$, we get that

$$y_{m+1}^{\ell-m}=\sum_{\substack{l_1,\dots,l_{m+1}\in\mathbb{N}\\ l_{m+1}\neq\ell-m,l_1+\dots+l_{m+1}=\ell-m}} y_1^{l_1}y_2^{l_2}\dots y_{m+1}^{l_{m+1}}.$$

It follows that for any $\omega_{m+1}>\ell-(m+1)$,

$$\begin{aligned}
&y_1^{\ell-1}y_2^{\ell-2}\dots y_{m-1}^{\ell-m+1}y_{m+1}^{\omega_{m+1}}\\
&\quad=y_1^{\ell-1}y_2^{\ell-2}\dots y_{m-1}^{\ell-m+1}y_{m+1}^{\omega_{m+1}-(\ell-m)}y_{m+1}^{\ell-m}\\
&\quad=-\sum_{\substack{l_2,\dots,l_{m+1}\in\mathbb{N}\\ l_{m+1}\neq\ell-m,l_2+\dots+l_{m+1}=\ell-m}} y_1^{\ell-1}y_2^{\ell-2+l_2}\dots y_{m+1}^{\omega_{m+1}-(\ell-m)+l_{m+1}}\\
&\quad=-\sum_{\substack{l_m,l_{m+1}\in\mathbb{N}\\ l_{m+1}\neq\ell-m,l_m+l_{m+1}=\ell-m}} y_1^{\ell-1}y_2^{\ell-2}\dots y_{m-1}^{\ell-m+1}y_m^{\ell-m+l_m}y_{m+1}^{\omega_{m+1}-(\ell-m)+l_{m+1}}\\
&\quad=0,
\end{aligned}$$

where we have used the induction hypothesis in the third and fourth equalities. This completes the proof of the lemma. □

Corollary 2.25. *For any $z_1,z_2\in\mathfrak{S}_n$, we define $F_{z_1,z_2}:=(-1)^{n(n-1)/2}\psi^{\boldsymbol{\lambda}_{\min}}_{w_0z_1,z_2}$. Then $F_{z_1,z_2}\neq 0$ is a homogeneous element of degree $2\ell(z_1)-2\ell(z_2)$. Suppose that $\ell=n$. Then $\sum_{w\in\mathfrak{S}_n}F_{w,w}=1$ and*

$$F_{z_1,z_2}F_{u_1,u_2}=\delta_{z_2,u_1}F_{z_1,u_2},\quad \forall\, u_1,u_2\in\mathfrak{S}_n.$$

In particular, $\mathscr{H}^{(0)}_{n,n}$ is isomorphic to the full matrix algebra $M_{n!\times n!}(K)$ over K with $\{F_{u,w}\}_{u,w\in\mathfrak{S}_n}$ being a complete set of matrix units.

Proof. As a cellular basis element, we know that $\psi^{\boldsymbol{\lambda}_{\min}}_{w_0z_1,z_2}\neq 0$ and hence $F_{z_1,z_2}\neq 0$. By definition, F_{z_1,z_2} is a homogeneous element of degree $2\ell(z_1)-2\ell(z_2)$.

Suppose that $\ell=n$. By Lemma 2.23, for any $1\le r\le n$, we have

$$\begin{aligned}
(2.26)\quad & y_1^{n-1}y_2^{n-2}\dots y_{n-1}y_r\\
&=(y_1^{n-1}y_2^{n-2}\dots y_{r+1}^{n-r-1}y_r^{n-r+1})y_{r-1}^{n-r+1}y_{r-2}^{n-r+2}\dots y_{n-1}=0.
\end{aligned}$$

For any $u_1, u_2 \in \mathfrak{S}_n$,

$$F_{z_1,z_2}F_{u_1,u_2} = \psi^*_{w_0z_1}y_1^{n-1}y_2^{n-2}\dots y_{n-1}\psi_{z_2}\psi^*_{w_0u_1}y_1^{n-1}y_2^{n-2}\dots y_{n-1}\psi_{u_2}.$$

By Lemma 2.15, this quantity is zero unless $\ell(z_2(w_0u_1)^{-1}) = \ell(z_2)+\ell((w_0u_1)^{-1})$. So we can assume that $\ell(z_2(w_0u_1)^{-1}) = \ell(z_2)+\ell((w_0u_1)^{-1})$. Then we get

$$F_{z_1,z_2}F_{u_1,u_2} = \psi^*_{w_0z_1}y_1^{n-1}y_2^{n-2}\dots y_{n-1}\psi_{z_2u_1^{-1}w_0^{-1}}y_1^{n-1}y_2^{n-2}\dots y_{n-1}\psi_{u_2}.$$

Note that w_0 is the unique longest element in $\mathfrak{S}_n$ with length $(n-1)n/2$. If $z_2u_1^{-1}w_0^{-1} \neq w_0$ then we must have

$$\psi_{z_2u_1^{-1}w_0^{-1}}y_1^{n-1}y_2^{n-2}\dots y_{n-1} \in \sum_{j=1}^{n} y_j\mathscr{H}^{(0)}_{n,n}.$$

In that case, $F_{z_1,z_2}F_{u_1,u_2} = 0$ by (2.26). Therefore, we can further assume that $z_2u_1^{-1}w_0^{-1} = w_0$ and hence $z_2 = u_1$. In the latter case, $F_{z_1,z_2}F_{u_1,u_2} = F_{z_1,u_2}$ by Lemma 2.12 and (2.26). This proves the first part of the corollary.

The second part of the corollary follows from Corollary 2.25 and the fact that $\dim \mathscr{H}^{(0)}_{n,n} = (n!)^2$ and $\{F_{z_1,z_2} \mid z_1, z_2 \in \mathfrak{S}_n\}$ is a basis of $\mathscr{H}^{(0)}_{n,n}$. □

Recall that the weak Bruhat order "$\succeq$" on $\mathfrak{S}_n$ is defined as follows (see [Dipper and James 1986]): For $u, w \in \mathfrak{S}_n$, let $u \succeq w$ if there is a reduced expression $w = s_{j_1}\dots s_{j_k}$ for w and $u = s_{j_1}\dots s_{j_l}$ for some $l \le k$. We write $u \succ w$ if $u \succeq w$ and $u \neq w$.

Corollary 2.27. *Let $\ell, n \in \mathbb{N}$. For any $z_1, z_2 \in \mathfrak{S}_n$, we define*

$$F'_{z_1,z_2} := \psi^*_{w_0z_1}y_1^{n-1}y_2^{n-2}\dots y_{n-1}\psi_{w_0}y_1^{n-1}y_2^{n-2}\dots y_{n-1}\psi_{z_2}.$$

Then $F'_{z_1,z_2} \neq 0$ is a homogeneous element of degree $2\ell(z_1)-2\ell(z_2)$, and

$$(F'_{z_1,z_1})^2 = F'_{z_1,z_1}, \quad F'_{z_1,z_2} = F'_{z_1,z_1}F'_{z_1,z_2} = F'_{z_1,z_2}F'_{z_2,z_2},$$
$$F'_{z_1,z_2}F'_{z_2,u_2} = F'_{z_1,u_2}, \quad F'_{z_1,z_2}F'_{u_1,u_2} = 0, \quad \forall\, u_1, u_2 \in \mathfrak{S}_n \text{ with } z_2^{-1} \not\succeq u_1^{-1}.$$

Proof. By Lemma 2.12 and (2.11), we have

$$F'_{z_1,z_2} \equiv (-1)^{(n-1)n/2}\psi^{\boldsymbol{\lambda}_{\min}}_{w_0z_1,z_2} \pmod{(\mathscr{H}^{(0)}_{\ell,n})^{>\boldsymbol{\lambda}_{\min}}}. \tag{2.28}$$

In particular, this implies that $F'_{z_1,z_2} \neq 0$ by the cellular structure of $\mathscr{H}^{(0)}_{\ell,n}$. By definition, it is clear that F'_{z_1,z_2} is a homogeneous element of degree $2\ell(z_1)-2\ell(z_2)$.

Again by Lemma 2.12 and Lemma 2.15, we have

$$\begin{aligned}(F'_{z_1,z_1})^2 &= \psi^*_{w_0z_1}y_1^{n-1}y_2^{n-2}\dots y_{n-1}\psi_{w_0}y_1^{n-1}y_2^{n-2}\dots y_{n-1}(\psi_{z_1}\psi^*_{w_0z_1})\\ &\qquad\times y_1^{n-1}y_2^{n-2}\dots y_{n-1}\psi_{w_0}y_1^{n-1}y_2^{n-2}\dots y_{n-1}\psi_{z_1}\\ &= \psi^*_{w_0z_1}y_1^{n-1}y_2^{n-2}\dots y_{n-1}(\psi_{w_0}y_1^{n-1}y_2^{n-2}\dots y_{n-1}\psi_{w_0})\\ &\qquad\times y_1^{n-1}y_2^{n-2}\dots y_{n-1}\psi_{w_0}y_1^{n-1}y_2^{n-2}\dots y_{n-1}\psi_{z_1}\\ &= (-1)^{(n-1)n/2}\psi^*_{w_0z_1}y_1^{n-1}y_2^{n-2}\dots y_{n-1}(\psi_{w_0}y_1^{n-1}y_2^{n-2}\dots y_{n-1}\psi_{w_0})\\ &\qquad\times y_1^{n-1}y_2^{n-2}\dots y_{n-1}\psi_{z_1}\\ &= F'_{z_1,z_1}.\end{aligned}$$

A similar argument shows that $F'_{z_1,z_2} = F'_{z_1,z_1}F'_{z_1,z_2} = F'_{z_1,z_2}F'_{z_2,z_2}$ and $F'_{z_1,z_2}F'_{z_2,u_2} = F'_{z_1,u_2}$.

Finally, let $u_1, u_2 \in \mathfrak{S}_n$ such that $z_2^{-1} \not\succeq u_1^{-1}$. We have

$$\begin{aligned}F'_{z_1,z_2}F'_{u_1,u_2} &= \psi^*_{w_0z_1}y_1^{n-1}y_2^{n-2}\dots y_{n-1}\psi_{w_0}y_1^{n-1}y_2^{n-2}\dots y_{n-1}(\psi_{z_2}\psi^*_{w_0u_1})\\ &\qquad\times y_1^{n-1}y_2^{n-2}\dots y_{n-1}\psi_{w_0}y_1^{n-1}y_2^{n-2}\dots y_{n-1}\psi_{u_2}.\end{aligned}$$

Note that the assumption $z_2^{-1} \not\succeq u_1^{-1}$ implies that $\ell(z_2u_1^{-1}w_0^{-1}) \neq \ell(z_2)+\ell(u_1^{-1}w_0^{-1})$ because otherwise we would have some $x \in \mathfrak{S}_n$ such that $xz_2 = u_1$ and

$$\begin{aligned}\ell(x) &= \ell(w_0) - \ell(z_2u_1^{-1}w_0^{-1}) = \ell(w_0) - (\ell(z_2) + \ell(u_1^{-1}w_0^{-1}))\\ &= \ell(w_0) - \ell(z_2) - (\ell(w_0) - \ell(u_1^{-1})) = \ell(u_1) - \ell(z_2).\end{aligned}$$

By Lemma 2.15, $\ell(z_2u_1^{-1}w_0^{-1}) \neq \ell(z_2) + \ell(u^{-1}w_0^{-1})$ implies that $\psi_{z_2}\psi^*_{w_0u_1} = 0$. We thus proved that $F'_{z_1,z_2}F'_{u_1,u_2} = 0$ as required. □

Definition 2.29. We fix a total order on $\mathfrak{S}_n$ and list the elements in $\mathfrak{S}_n$ as $1 = w_1, w_2, \dots, w_{n!}$ such that

$$w_i^{-1} \succ w_j^{-1} \implies i < j.$$

We define a set of elements $\{\tilde{F}_{w_i,w_j} \mid 1 \le i, j \le n!\}$ in $\mathscr{H}^{(0)}_{\ell,n}$ inductively as follows:

$$\tilde{F}_{w_1,w_j} = \tilde{F}_{1,w_j} := F'_{1,w_j}, \quad \forall\, 1 \le j \le n!.$$

Suppose that $\tilde{F}_{w_k,w_j}$ was already defined for any $1 \le k < i$ and $1 \le j \le n!$. Then we define

$$\tilde{F}_{w_i,w_j} := F'_{w_i,w_j} - \sum_{1\le k<i} \tilde{F}_{w_k,w_k}F'_{w_i,w_j}, \quad \forall\, 1 \le j \le n!.$$

By construction and Corollary 2.27, we see that

$$\tilde{F}_{w_i,w_j}F'_{w_j,w_a} = \tilde{F}_{w_i,w_a}, \quad \forall\, 1 \le a \le n!. \tag{2.30}$$

Theorem 2.31. *For any $1 \le i, j \le n!$, we have that $\tilde{F}_{w_i,w_j} \neq 0$ is a homogeneous element of degree $2\ell(w_i) - 2\ell(w_j)$ and*

$$\tilde{F}_{w_i,w_j}\tilde{F}_{w_k,w_l} = \delta_{j,k}\tilde{F}_{w_i,w_l}, \quad \forall\, 1 \le k, l \le n!. \tag{2.32}$$

Moreover, for each $1 \le i \le n!$, $\tilde{F}_{w_i,w_i}\mathscr{H}_{\ell,n}^{(0)} \cong P_0$ is an ungraded right $\mathscr{H}_{\ell,n}^{(0)}$-module, $1 = \sum_{i=1}^{n!}\tilde{F}_{w_i,w_i}$, and $\{\tilde{F}_{w_i,w_i} \mid 1 \le i \le n!\}$ is a complete set of pairwise orthogonal primitive idempotents of $\mathscr{H}_{\ell,n}^{(0)}$.

Proof. By (2.28), for any $u \in \mathfrak{S}_n$ with $u^{-1} \succ w_1^{-1}$, we have the following relations modulo $\mathscr{H}_{\ell,n}^{(0)})^{>\boldsymbol{\lambda}_{\min}}$:

$$\begin{aligned}
F'_{u,u}F'_{w_1,w_2} &\equiv \psi_{w_0u,u}^{\boldsymbol{\lambda}_{\min}}\psi_{w_0w_1,w_2}^{\boldsymbol{\lambda}_{\min}} \\
&\equiv \psi_{w_0u}^{*}y_1^{n-1}y_2^{n-2}\dots y_{n-1}(\psi_u(\psi_{w_0w_1})^*)y_1^{n-1}y_2^{n-2}\dots y_{n-1}\psi_{w_2} \\
&\equiv \psi_{w_0u}^{*}y_1^{n-1}y_2^{n-2}\dots y_{n-1}\psi_{uw_1^{-1}w_0}y_1^{n-1}y_2^{n-2}\dots y_{n-1}\psi_{w_2} \\
&\equiv \sum_{j=1}^{n} r_j\psi_{w_0u}^{*}y_1^{n-1}y_2^{n-2}\dots y_{n-1}y_jh_j\psi_{w_2} \\
&\equiv 0,
\end{aligned}$$

where $r_j \in K$, $h_j \in \mathscr{H}_{\ell,n}^{(0)}$ for any z, j. Combining this with Corollary 2.27 and (2.28) we can deduce that

$$\tilde{F}_{w_i,w_j} \equiv (-1)^{(n-1)n/2}\psi_{w_0w_i,w_j}^{\boldsymbol{\lambda}_{\min}} \pmod{(\mathscr{H}_{\ell,n}^{(0)})^{>\boldsymbol{\lambda}_{\min}}}. \tag{2.33}$$

In particular, $\tilde{F}_{w_i,w_j} \neq 0$. By definition, Corollary 2.27, and an easy induction, we see that $\tilde{F}_{w_i,w_j}$ is a homogeneous element of degree $2\ell(w_i) - 2\ell(w_j)$.

We are going to prove (2.32). We use induction on k. Suppose that $k = 1$. If $j \neq 1$, then $j > 1$. By construction,

$$\tilde{F}_{w_i,w_j} \in \sum_{w\in\mathfrak{S}_n}\mathscr{H}_{\ell,n}^{(0)}F'_{w,w_j}, \quad \tilde{F}_{1,w_l} = F'_{1,w_l}.$$

By Corollary 2.27, we have $F'_{w,w_j}F'_{1,u} = 0$. It follows that $\tilde{F}_{w_i,w_j}\tilde{F}_{w_1,w_l} = 0$. If $j = 1$, then by (2.30) we have

$$\tilde{F}_{w_i,w_1}\tilde{F}_{w_1,w_l} = \tilde{F}_{w_i,1}F'_{1,w_l} = \tilde{F}_{w_i,w_l},$$

as required.

In general, suppose that (2.32) holds for any $k < m$. Let us consider the case when $k = m$. By construction, we have

$$\tilde{F}_{w_i,w_j} \in \sum_{w\in\mathfrak{S}_n}\mathscr{H}_{\ell,n}^{(0)}F'_{w,w_j}, \quad \tilde{F}_{w_m,w_l} \in \sum_{\substack{u\in\mathfrak{S}_n\\ 1\le a\le m}}F'_{w_a,u}\mathscr{H}_{\ell,n}^{(0)}.$$

Therefore, if $j > m$ then $\tilde{F}_{w_i,w_j}\tilde{F}_{w_m,w_l} = 0$ by Corollary 2.27.

Suppose that $j < m$. Then

$$\begin{aligned}\tilde{F}_{w_i,w_j}\tilde{F}_{w_m,w_l} &= \tilde{F}_{w_i,w_j}\Big(F'_{w_m,w_l} - \sum_{1\le k<m}\tilde{F}_{w_k,w_k}F'_{w_m,w_l}\Big)\\ &= \tilde{F}_{w_i,w_j}\Big(F'_{w_m,w_l} - \sum_{1\le k<m}\delta_{k,j}\tilde{F}_{w_k,w_k}F'_{w_m,w_l}\Big)\\ &= \tilde{F}_{w_i,w_j}F'_{w_m,w_l} - \tilde{F}_{w_i,w_j}F'_{w_m,w_l}\\ &= 0,\end{aligned}$$

as required, where we have used induction hypothesis in the second and the third equalities.

Suppose that $j = m$. Then

$$\begin{aligned}\tilde{F}_{w_i,w_m}\tilde{F}_{w_m,w_l} &= \tilde{F}_{w_i,w_m}\Big(F'_{w_m,w_l} - \sum_{1\le k<m}\tilde{F}_{w_k,w_k}F'_{w_m,w_l}\Big)\\ &= \tilde{F}_{w_i,w_m}F'_{w_m,w_l} - \sum_{1\le k<m}\tilde{F}_{w_i,w_m}\tilde{F}_{w_k,w_k}F'_{w_m,w_l}\\ &= \tilde{F}_{w_i,w_m}F'_{w_m,w_l} - 0 = \tilde{F}_{w_i,w_l},\end{aligned}$$

as required, where we used (2.30) in the last equality, and used the induction hypothesis in the second last equality.

Since

$$P_0^{\oplus \dim D_0} = P_0^{\oplus n!} \cong \mathscr{H}_{\ell,n}^{(0)} \cong \Big(1 - \sum_{w\in\mathfrak{S}_n}\tilde{F}_{w,w}\Big)\mathscr{H}_{\ell,n}^{(0)} \oplus \Big(\bigoplus_{w\in\mathfrak{S}_n}\tilde{F}_{w,w}\mathscr{H}_{\ell,n}^{(0)}\Big),$$

and $\tilde{F}_{w,w}\mathscr{H}_{\ell,n}^{(0)} \neq 0$ for each $w \in \mathfrak{S}_n$. By the Krull–Schmidt theorem we can deduce that for each $w \in \mathfrak{S}_n$, $F_{w,w}\mathscr{H}_{\ell,n}^{(0)} \cong P_0$ is an ungraded right $\mathscr{H}_{\ell,n}^{(0)}$-module and $1 = \sum_{w\in\mathfrak{S}_n}\tilde{F}_{w,w}$. In other words, $\{\tilde{F}_{w_i,w_i} \mid 1 \le i \le n!\}$ is a complete set of pairwise orthogonal primitive idempotents of $\mathscr{H}_{\ell,n}^{(0)}$. □

The following result was first conjectured by A. Mathas [2015, §2.5, before Corollary 2.5.2] in the special case when $\ell = n$.

Theorem 2.34. *The elements in the set*

(2.35) $$\{\psi_w y_1^{a_1}\dots y_n^{a_n} \mid 0 \le a_i \le \ell - i, \quad \forall\, 1 \le i \le n,\ w \in \mathfrak{S}_n\}$$

form a K-basis of $\mathscr{H}_{\ell,n}^{(0)}$.

Proof. We first claim that for any $b_1,\dots,b_{m-1},\omega_m \in \mathbb{N}$ with $0 \le b_j \le l - j$, $\forall\, 1 \le j \le m$,

(2.36) $$y_1^{b_1}y_2^{b_2}\dots y_{m-1}^{b_{m-1}}y_m^{\omega_m} = \sum_{\substack{c_1,\dots,c_m\in\mathbb{N}\\ 0\le c_i\le \ell-i,\forall\, 1\le i\le m}} r_{c_1,\dots,c_m}y_1^{c_1}y_2^{c_2}\dots y_{m-1}^{c_{m-1}}y_m^{c_m},$$

where $r_{c_1,\dots,c_m} \in K$ for each m-tuple $(c_1, \dots, c_m)$.

We use induction on m. If $m = 1$, there is nothing to prove as $y_1^{\omega_1} = 0$ whenever $\omega_1 > \ell - 1$. Suppose that (2.36) holds for any $1 \le k \le m$.

We now consider the case where $k = m+1$. Applying Lemma 2.20 for $s = m+1$, we get that

$$y_{m+1}^{\ell-m} = - \sum_{\substack{l_1,\dots,l_{m+1}\in\mathbb{N}\\ l_{m+1}\neq\ell-m, l_1+\cdots+l_{m+1}=\ell-m}} y_1^{l_1} y_2^{l_2} \dots y_{m+1}^{l_{m+1}}.$$

It follows that

$$\begin{aligned}
&y_1^{b_1} y_2^{b_2} \dots y_{m-1}^{b_{m-1}} y_m^{b_m} y_{m+1}^{\omega_{m+1}}\\
&\qquad = y_1^{b_1} y_2^{b_2} \dots y_{m-1}^{b_{m-1}} y_m^{b_m} y_{m+1}^{\omega_{m+1}-(\ell-m)} y_{m+1}^{\ell-m}\\
&\qquad = - \sum_{\substack{l_1,\dots,l_{m+1}\in\mathbb{N}\\ l_{m+1}\neq\ell-m, l_1+\cdots+l_{m+1}=\ell-m}} y_1^{b_1+l_1} y_2^{b_2+l_2} \dots y_{m-1}^{b_{m-1}+l_{m-1}} y_m^{b_m+l_m} y_{m+1}^{b'_{m+1}},
\end{aligned}$$

where $b'_{m+1} := \omega_{m+1} - (l-m) + l_{m+1}$.

Our purpose is to show that

$$\begin{aligned}
(2.37)\quad & y_1^{b_1} y_2^{b_2} \dots y_m^{b_m} y_{m+1}^{\omega_{m+1}}\\
&\in K\text{-Span}\{y_1^{c_1} y_2^{c_2} \dots y_m^{c_m} y_{m+1}^{c_{m+1}} \mid c_i \in \mathbb{N}, 0 \le c_i \le \ell - i, \forall\, 1 \le i \le m+1\}.
\end{aligned}$$

We use induction on ω_{m+1}. Suppose that for any $b_1, \dots, b_m \in \mathbb{N}$ and any $0 \le b < \omega_{m+1}$, we have

$$\begin{aligned}
&y_1^{b_1} y_2^{b_2} \dots y_m^{b_m} y_{m+1}^{b}\\
&\qquad \in K\text{-Span}\{y_1^{c_1} y_2^{c_2} \dots y_m^{c_m} y_{m+1}^{c_{m+1}} \mid c_i \in \mathbb{N}, 0 \le c_i \le \ell - i, \forall\, 1 \le i \le m+1\}.
\end{aligned}$$

We are now going to prove (2.37). If $b'_{m+1} \le l-m$, then by our induction hypothesis we have

$$\begin{aligned}
&y_1^{b_1+l_1} y_2^{b_2+l_2} \dots y_{m-1}^{b_{m-1}+l_{m-1}} y_m^{b_m+l_m}\\
&\qquad \in K\text{-Span}\{y_1^{c_1} y_2^{c_2} \dots y_{m-1}^{c_{m-1}} y_m^{c_m} \mid c_1, \dots, c_m \in \mathbb{N}, 0 \le c_i \le \ell - i, \forall\, 1 \le i \le m\},
\end{aligned}$$

hence

$$\begin{aligned}
&y_1^{b_1+l_1} y_2^{b_2+l_2} \dots y_{m-1}^{b_{m-1}+l_{m-1}} y^{b_m+l_m} y_{m+1}^{b'_{m+1}}\\
&\in K\text{-Span}\{y_1^{c_1} y_2^{c_2} \dots y_m^{c_m} y_{m+1}^{c_{m+1}} \mid c_1, \dots, c_{m+1} \in \mathbb{N}, 0 \le c_i \le \ell - i, \forall\, 1 \le i \le m+1\}.
\end{aligned}$$

Therefore, it remains to consider those terms which satisfy $b'_{m+1} > \ell - m$. Since $l_1 + \cdots + l_{m+1} = \ell - m$ and $l_{m+1} \neq \ell - m$, we have $0 \le l_{m+1} \le \ell - m - 1$; furthermore, we have $b'_{m+1} \le \omega_{m+1} - 1$. By our induction hypothesis on ω_{m+1}, we have

$$y_1^{b_1+l_1} y_2^{b_2+l_2} \dots y_{m-1}^{b_{m-1}+l_{m-1}} y_m^{b_m+l_m} y_{m+1}^{b'_{m+1}} \\ \in K\text{-Span}\{y_1^{c_1} y_2^{c_2} \dots y_m^{c_m} y_{m+1}^{c_{m+1}} \mid c_i \in \mathbb{N}, 0 \le c_i \le \ell - i, \forall\, 1 \le i \le m+1\}.$$

Therefore, we can conclude that (2.37) always holds. This completes the proof of (2.36).

Now we have proved that the elements in (2.35) form a K-linear generator of $\mathscr{H}^{(0)}_{\ell,n}$. Since the set (2.35) has cardinality equal to $\binom{\ell}{n}(n!)^2$, which is equal to the dimension of $\mathscr{H}^{(0)}_{\ell,n}$, the elements in (2.35) must form a K-basis of $\mathscr{H}^{(0)}_{\ell,n}$. $\square$

Remark 2.38. We shall call the basis (2.35) a *monomial basis* of $\mathscr{H}^{(0)}_{\ell,n}$. It bears much resemblance to the Ariki–Koike basis of the cyclotomic Hecke algebra of type $G(\ell, 1, n)$. For arbitrary cyclotomic quiver Hecke algebras, Question 1.1 (on how to construct a monomial basis) remains open. Anyhow, we regard Theorem 2.34 as a first step in our effort of answering that open question.

3. A basis of the center

The purpose of this section is to give an explicit basis of the center of $\mathscr{H}^{(0)}_{\ell,n}$. Let $Z := Z(\mathscr{H}^{(0)}_{\ell,n})$ be the center of $\mathscr{H}^{(0)}_{\ell,n}$.

Definition 3.1. For each $\boldsymbol{\mu} \in \mathscr{P}_0$, we define

$$b_{\boldsymbol{\mu}} := \psi_{w_0} y_{\boldsymbol{\mu}} \psi_{w_0} y_1^{n-1} y_2^{n-2} \dots y_{n-1}.$$

By Definition 2.29, Corollary 2.27, Lemma 2.12, and Lemma 2.15, we have

$$\begin{aligned}\tilde{F}_{1,1} = F'_{1,1} &= \psi^*_{w_0} y_1^{n-1} y_2^{n-2} \dots y_{n-1} \psi_{w_0} y_1^{n-1} y_2^{n-2} \dots y_{n-1} \\ &= (\psi_{w_0} y_1^{n-1} y_2^{n-2} \dots y_{n-1}) \psi_{w_0} y_1^{n-1} y_2^{n-2} \dots y_{n-1} \\ &= (-1)^{n(n-1)/2} \psi_{w_0} y_1^{n-1} y_2^{n-2} \dots y_{n-1} = F_{1,1}.\end{aligned}$$

Note that each $y_{\boldsymbol{\mu}}$ has a left factor $y_1^{n-1} y_2^{n-2} \dots y_{n-1}$. It follows that

$$b_{\boldsymbol{\mu}} \in \tilde{F}_{1,1} \mathscr{H}^{(0)}_{\ell,n} \tilde{F}_{1,1} \cong \mathrm{End}_{\mathscr{H}^{(0)}_{\ell,n}}(\tilde{F}_{1,1} \mathscr{H}^{(0)}_{\ell,n}) \cong \mathrm{End}_{\mathscr{H}^{(0)}_{\ell,n}}(P_0).$$

Suppose further that $\theta(\boldsymbol{\mu}) = (k_1, \dots, k_n)$, where $1 \le k_1 < k_2 < \dots < k_n \le \ell$. Then by (2.11),

$$\begin{aligned} b_{\boldsymbol{\mu}} &= \psi_{w_0} y_1^{\ell-k_1} y_2^{\ell-k_2} \dots y_n^{\ell-k_n} \psi_{w_0} y_1^{n-1} y_2^{n-2} \dots y_{n-1} \\ &\equiv (-1)^{n(n-1)/2} \psi_{w_0} y_1^{\ell-k_1} y_2^{\ell-k_2} \dots y_n^{\ell-k_n} \pmod{(\mathscr{H}^{(0)}_{\ell,n})^{>\boldsymbol{\mu}}} \\ &\equiv (-1)^{n(n-1)/2} \psi^{\boldsymbol{\mu}}_{w_0,1} \pmod{(\mathscr{H}^{(0)}_{\ell,n})^{>\boldsymbol{\mu}}}. \end{aligned}$$

It follows that $\{b_{\boldsymbol{\mu}} \mid \boldsymbol{\mu} \in \mathscr{P}_0\}$ are K-linearly independent elements in $\tilde{F}_{1,1} \mathscr{H}^{(0)}_{\ell,n} \tilde{F}_{1,1}$.

Lemma 3.2. *The elements in* $\{b_{\boldsymbol{\mu}} \mid \boldsymbol{\mu} \in \mathscr{P}_0\}$ *form a K-basis of* $\tilde{F}_{1,1}\mathscr{H}^{(0)}_{\ell,n}\tilde{F}_{1,1}$. *Moreover, the basic algebra* $\mathrm{End}_{\mathscr{H}^{(0)}_{\ell,n}}(P_0)$ *of* $\mathscr{H}^{(0)}_{\ell,n}$ *is commutative and is isomorphic to the center Z of* $\mathscr{H}^{(0)}_{\ell,n}$. *In particular*, $\dim_K Z = \binom{\ell}{n}$.

Proof. Since $\#\mathscr{P}_0 = \binom{\ell}{n}$ and $\tilde{F}_{1,1}\mathscr{H}^{(0)}_{\ell,n}\tilde{F}_{1,1} \cong \mathrm{End}_{\mathscr{H}^{(0)}_{\ell,n}}(P_0)$, it suffices to show that

$$\dim_K \mathrm{End}_{\mathscr{H}^{(0)}_{\ell,n}}(P_0) = \binom{\ell}{n}.$$

By Lemma 2.16 and Corollary 2.18, we know that $[P_0 : D_0] = \binom{\ell}{n}$ and hence $\dim_K \mathrm{End}_{\mathscr{H}^{(0)}_{\ell,n}}(P_0) = \binom{\ell}{n}$, as required. Thus, the first part of the lemma follows from this together with the discussion in the paragraph above this lemma.

It remains to show that the endomorphism algebra $\mathrm{End}_{\mathscr{H}^{(0)}_{\ell,n}}(P_0)$ is commutative. Once this is proved, and since $\mathscr{H}^{(0)}_{\ell,n}$ is Morita equivalent to $\mathrm{End}_{\mathscr{H}^{(0)}_{\ell,n}}(P_0)$, it will follow from [Curtis and Reiner 1981, (3.54)(iv)] that

$$Z = Z(\mathscr{H}^{(0)}_{\ell,n}) \cong Z(\mathrm{End}_{\mathscr{H}^{(0)}_{\ell,n}}(P_0)) = \mathrm{End}_{\mathscr{H}^{(0)}_{\ell,n}}(P_0),$$

as required.

To show that $\mathrm{End}_{\mathscr{H}^{(0)}_{\ell,n}}(P_0)$ of $\mathscr{H}^{(0)}_{\ell,n}$ is commutative, it suffices to show that $\tilde{F}_{1,1}\mathscr{H}^{(0)}_{\ell,n}\tilde{F}_{1,1}$ is commutative. Furthermore, it is enough to show that $b_{\boldsymbol{\mu}}b_{\boldsymbol{\nu}} = b_{\boldsymbol{\nu}}b_{\boldsymbol{\mu}}$ for any $\boldsymbol{\mu}, \boldsymbol{\nu} \in \mathscr{P}_0$.

By definition,

$$\begin{aligned} b_{\boldsymbol{\mu}}b_{\boldsymbol{\nu}} &= \psi_{w_0}y_{\boldsymbol{\mu}}\psi_{w_0}y_1^{n-1}y_2^{n-2}\dots y_{n-1}\psi_{w_0}y_{\boldsymbol{\nu}}\psi_{w_0}y_1^{n-1}y_2^{n-2}\dots y_{n-1} \\ &= (-1)^{n(n-1)/2}\psi_{w_0}(y_{\boldsymbol{\mu}}\psi_{w_0}y_{\boldsymbol{\nu}})\psi_{w_0}y_1^{n-1}y_2^{n-2}\dots y_{n-1}. \end{aligned}$$

We set

$$J_{1,1} := \sum_{j=1}^{n-1}\psi_j\mathscr{H}^{(0)}_{\ell,n} + \sum_{j=1}^{n-1}\mathscr{H}^{(0)}_{\ell,n}\psi_j.$$

Using the graded cellular basis $\{\psi^{\boldsymbol{\mu}}_{w,u} \mid \boldsymbol{\mu} \in \mathscr{P}_0\}$ of $\mathscr{H}^{(0)}_{\ell,n}$, we can write

$$y_{\boldsymbol{\mu}}\psi_{w_0}y_{\boldsymbol{\nu}} \equiv \sum_{\boldsymbol{\rho}\in\mathscr{P}_0} c_{\boldsymbol{\rho}}y_{\boldsymbol{\rho}} \pmod{J_{1,1}},$$

where $c_\alpha \in K$ for each $\alpha \in \mathscr{P}_0$. Applying the anti-involution "$*$" on both sides of the above equality, we get that

$$y_{\boldsymbol{\nu}}\psi_{w_0}y_{\boldsymbol{\mu}} \equiv \sum_{\boldsymbol{\rho}\in\mathscr{P}_0} c_{\boldsymbol{\rho}}y_{\boldsymbol{\rho}} \pmod{J_{1,1}}.$$

Now using Lemma 2.15 we can deduce that

$$b_{\boldsymbol{\mu}}b_{\boldsymbol{\nu}} = (-1)^{n(n-1)/2}\sum_{\boldsymbol{\rho}\in\mathscr{P}_0} c_{\boldsymbol{\rho}}\psi_{w_0}y_{\boldsymbol{\rho}}\psi_{w_0}y_1^{n-1}y_2^{n-2}\dots y_{n-1} = b_{\boldsymbol{\nu}}b_{\boldsymbol{\mu}},$$

as required. □

Definition 3.3. Let $\boldsymbol{\mu} \in \mathscr{P}_0$ with $\theta(\boldsymbol{\mu}) = (k_1, \dots, k_n)$, where $1 \le k_1 < k_2 < \cdots < k_n \le \ell$. Inside the quiver Hecke algebra $\mathscr{H}_n^{(0)}$, we define $z(\boldsymbol{\mu}) \in K[y_1, \dots, y_n]$ such that

$$y_1^{\ell-k_1} \dots y_n^{\ell-k_n} \psi_{w_0} = z(\boldsymbol{\mu}) + \sum_{r=1}^{n-1} \psi_r h_r,$$

where $h_r \in \mathscr{H}_n^{(0)}$ for each $1 \le r < n$. We define

$$z_{\boldsymbol{\mu}} := \pi(z(\boldsymbol{\mu})) \in \mathscr{H}_{\ell,n}^{(0)}.$$

It is clear that $z_{\boldsymbol{\mu}}$ is a homogeneous element with degree $2\ell n - n(n-1) - 2\sum_{i=1}^n k_i$.

Lemma 3.4. *Let $\boldsymbol{\mu} \in \mathscr{P}_0$. Then $z(\boldsymbol{\mu})$ is a symmetric polynomial in $y_1, \dots, y_n$. In particular, $z(\boldsymbol{\mu})$ lives inside the center of $\mathscr{H}_n^{(0)}$ and hence $z_{\boldsymbol{\mu}}$ lives inside the center of $\mathscr{H}_{\ell,n}^{(0)}$. Moreover, $z(\boldsymbol{\lambda}_{\max}) = (-1)^{n(n-1)/2}(y_1 \dots y_n)^{\ell-n}$ and $z(\boldsymbol{\lambda}_{\min}) = (-1)^{n(n-1)/2}$.*

Proof. It suffices to show that $z(\boldsymbol{\mu})$ is symmetric in y_r, y_{r+1} for each $1 \le r < n-1$. In fact, for any $1 \le r < n-1$ and $a, b \in \mathbb{N}$, if $a > b$ then

$$\begin{aligned} y_r^a y_{r+1}^b \psi_r &= y_r^{a-b}(y_r y_{r+1})^b \psi_r = y_r^{a-b} \psi_r (y_r y_{r+1})^b \\ &\equiv -\left(\sum_{k=0}^{a-b-1} y_r^k y_{r+1}^{a+b-1-k}\right)(y_r y_{r+1})^b \left(\operatorname{mod} \sum_{r=1}^{n-1} \psi_r \mathscr{H}_n^{(0)}\right); \end{aligned}$$

if $a < b$, then

$$\begin{aligned} y_r^a y_{r+1}^b \psi_r &= y_{r+1}^{b-a}(y_r y_{r+1})^a \psi_r = y_{r+1}^{b-a} \psi_r (y_r y_{r+1})^a \\ &\equiv \left(\sum_{k=0}^{b-a-1} y_r^k y_{r+1}^{b-a-1-k}\right)(y_r y_{r+1})^a \ \left(\operatorname{mod} \sum_{r=1}^{n-1} \psi_r \mathscr{H}_n^{(0)}\right); \end{aligned}$$

if $a = b$, then $y_r^a y_{r+1}^b \psi_r = (y_r y_{r+1})^a \psi_r = \psi_r (y_r y_{r+1})^a \in \sum_{r=1}^{n-1} \psi_r \mathscr{H}_n^{(0)}$. This implies that for any monomial $y_1^{c_1} \dots y_n^{c_n} \in \mathscr{H}_n^{(0)}$,

$$y_1^{c_1} \dots y_n^{c_n} \psi_r \equiv f_r(y_1, \dots, y_n) \ \left(\operatorname{mod} \sum_{r=1}^{n-1} \psi_r \mathscr{H}_n^{(0)}\right),$$

where $f_r(y_1, \dots, y_n) \in K[y_1, \dots, y_n]$ is symmetric in y_r, y_{r+1}.

Since for each $1 \le r < n$, w_0 has a reduced expression which ends with s_r and the element $z(\boldsymbol{\mu})$ is uniquely determined by $\boldsymbol{\mu}$ by Lemma 2.1, it follows that $z(\boldsymbol{\mu})$ is symmetric in y_r, y_{r+1} for any $1 \le r < n-1$. Hence $z(\boldsymbol{\mu})$ is symmetric in $y_1, \dots, y_n$. This completes the proof of the first part of the lemma. The second part of the lemma follows from Lemma 2.12 and direct calculation. □

Lemma 3.5. (1) *For each $\boldsymbol{\mu} \in \mathscr{P}_0$, we have that*

$$\psi_{w_0} y_{\boldsymbol{\mu}} \psi_{w_0} y_1^{n-1} \dots y_{n-1} = \psi_{w_0} y_1^{n-1} \dots y_{n-1} z_{\boldsymbol{\mu}}.$$

In particular,

$$\psi_{w_0} y_{\boldsymbol{\mu}} \equiv (-1)^{n(n-1)/2} \psi_{w_0} y_1^{n-1} y_2^{n-2} \dots y_{n-1} z_{\boldsymbol{\mu}} \ (\mathrm{mod}\ (\mathscr{H}_{\ell,n}^{(0)})^{>\boldsymbol{\mu}}).$$

(2) *As a left Z-module, $P_0 \cong Z^{\oplus n!}$. In particular, P_0 is a free Z-module of rank $n!$.*

Proof. First, since $\mathscr{H}_{\ell,n}^{(0)} \cong P_0^{\oplus n!}$, it follows that the center Z must act faithfully on P_0. In other words, the left multiplication defines an injective homomorphism $\iota : Z \hookrightarrow \mathrm{End}_{\mathscr{H}_{\ell,n}^{(0)}}(P_0)$. Comparing the dimensions of both sides, we can deduce that ι is an isomorphism. On the other hand, by Lemma 3.2,

$$0 \neq b_{\boldsymbol{\mu}} \in \tilde{F}_{1,1} \mathscr{H}_{\ell,n}^{(0)} \tilde{F}_{1,1} \cong \mathrm{End}_{\mathscr{H}_{\ell,n}^{(0)}}(P_0).$$

It follows that there exists a unique nonzero homogeneous element $z'_{\boldsymbol{\mu}}$ with degree $2(\ell - k_1 + \dots + \ell - k_n) - (n-1)n$ such that

$$\begin{aligned} \psi_{w_0} y_{\boldsymbol{\mu}} \psi_{w_0} y_1^{n-1} \dots y_{n-1} &= z'_{\boldsymbol{\mu}} \psi_{w_0} y_1^{n-1} \dots y_{n-1} \\ &= \psi_{w_0} z'_{\boldsymbol{\mu}} y_1^{n-1} \dots y_{n-1} = \psi_{w_0} y_1^{n-1} \dots y_{n-1} z'_{\boldsymbol{\mu}}. \end{aligned} \tag{3.6}$$

By Lemma 3.4 and Lemma 2.15, we can see that $z'_{\boldsymbol{\mu}} = z_{\boldsymbol{\mu}}$. In particular, $z_{\boldsymbol{\mu}} \neq 0$.

Since

$$\begin{aligned} \psi_{w_0} y_{\boldsymbol{\mu}} \psi_{w_0} y_1^{n-1} \dots y_{n-1} &\equiv (-1)^{n(n-1)/2} \psi_{w_0} y_{\boldsymbol{\mu}} \\ &\equiv (-1)^{n(n-1)/2} \psi_{w_0,1}^{\boldsymbol{\mu}} \ (\mathrm{mod}\ (\mathscr{H}_{\ell,n}^{(0)})^{>\boldsymbol{\mu}}), \end{aligned}$$

we see that $\psi_{w_0} y_{\boldsymbol{\mu}} \equiv (-1)^{n(n-1)/2} \psi_{w_0} y_1^{n-1} y_2^{n-2} \dots y_{n-1} z_{\boldsymbol{\mu}}$ (mod $(\mathscr{H}_{\ell,n}^{(0)})^{>\boldsymbol{\mu}}$). This proves (1).

Recall that $\tilde{F}_{1,1} = F'_{1,1} = (-1)^{n(n-1)/2} \psi_{w_0} y_1^{n-1} y_2^{n-2} \dots y_{n-1}$. It follows from (1) that for any $\boldsymbol{\mu} \in \mathscr{P}_0$ and $w \in \mathfrak{S}_n$,

$$\tilde{F}_{1,1} z_{\boldsymbol{\mu}} \psi_w \equiv \psi_{w_0,w}^{\boldsymbol{\mu}} \ (\mathrm{mod}\ (\mathscr{H}_{\ell,n}^{(0)})^{>\boldsymbol{\mu}}).$$

In particular, the elements in the set $\{\tilde{F}_{1,1} z_{\boldsymbol{\mu}} \psi_w \mid \boldsymbol{\mu} \in \mathscr{P}_0, w \in \mathfrak{S}_n\}$ must be K-linearly independent. Since it has the cardinality $\binom{\ell}{n} n!$, we can deduce that it is a K-basis of the right $\mathscr{H}_{\ell,n}^{(0)}$-module $P_0 \cong \tilde{F}_{1,1} \mathscr{H}_{\ell,n}^{(0)}$. Since P_0 is a faithful Z-module, it follows that for any $z \in Z$, $\tilde{F}_{1,1} z = 0$ if and only if $z = 0$. For each $w \in \mathfrak{S}_n$, the subspace spanned by the basis elements in $\{\tilde{F}_{1,1} z_{\boldsymbol{\mu}} \psi_w \mid \boldsymbol{\mu} \in \mathscr{P}_0\}$ is a Z-submodule of P_0 which is isomorphic to Z. This proves that P is a free Z-module with rank $n!$. $\square$

Theorem 3.7. *The elements in the set* $\{z_{\boldsymbol{\mu}} \mid \boldsymbol{\mu} \in \mathscr{P}_0\}$ *form a* K*-basis of the center* $Z := Z(\mathscr{H}_{\ell,n}^{(0)})$ *of* $\mathscr{H}_{\ell,n}^{(0)}$. *In particular, the center of* $\mathscr{H}_{\ell,n}^{(0)}$ *is the set of symmetric polynomials in* $y_1, \dots, y_n$.

Proof. Since the elements in $\{b_{\boldsymbol{\mu}} \mid \boldsymbol{\mu} \in \mathscr{P}_0\}$ are K-linearly independent, it follows that the elements in $\{z_{\boldsymbol{\mu}} \mid \boldsymbol{\mu} \in \mathscr{P}_0\}$ are K-linearly independent and hence form a K-basis of the center $Z := Z(\mathscr{H}_{\ell,n}^{(0)})$ by dimension consideration. By Lemma 3.4, each $z_{\boldsymbol{\mu}}$ is a symmetric polynomial in $y_1, \dots, y_n$, hence the center of $\mathscr{H}_{\ell,n}^{(0)}$ is the set of symmetric polynomials in $y_1, \dots, y_n$. □

The following proposition gives a generalization of Corollary 2.25. It can be regarded as a cyclotomic analogue of the results in [Lauda 2010, Proposition 3.5] and [Kleshchev et al. 2013, Theorem 4.5].

Proposition 3.8. *Let* $\{E_{i,j} \mid 1 \le i, j \le n!\}$ *be the matrix units of the full matrix algebra* $M_{n!\times n!}(K)$. *Then the map*

$$E_{i,j} \otimes z \mapsto \tilde{F}_{w_i,w_j} z, \quad \forall\, 1 \le i, j \le n!, z \in Z,$$

extends linearly to a well-defined K*-algebra isomorphism* η *from* $M_{n!\times n!}(K) \otimes_K Z$ *onto* $\mathscr{H}_{\ell,n}^{(0)}$. *In particular,* $\mathscr{H}_{\ell,n}^{(0)} \cong M_{n!\times n!}(Z)$.

Proof. In view of Theorem 2.31, it is clear that η is a well-defined K-algebra homomorphism. By Lemma 3.2, it suffices to show that η is an injective map.

Suppose that $\eta(x) = 0$, where $x = \sum_{1\le i,j\le n!} E_{i,j} z_{i,j}$, where $z_{i,j} \in Z$ for each pair (i, j). Then

$$\sum_{1\le i,j\le n!} \tilde{F}_{w_i,w_j} z_{i,j} = \eta(x) = 0.$$

For any pair (i, j) with $1 \le i, j \le n!$, left multiplying with $\tilde{F}_{w_j,w_i}$ and right multiplying with $\tilde{F}_{w_j,w_j}$ we get (by Theorem 2.31) that

$$\begin{aligned} \tilde{F}_{w_j,w_j} z_{i,j} &= \sum_{1\le k,l\le n!} (\tilde{F}_{w_j,w_i} \tilde{F}_{w_k,w_l} \tilde{F}_{w_j,w_j}) z_{k,l} \\ &= \tilde{F}_{w_j,w_i} \Bigg(\sum_{1\le k,l\le n!} \tilde{F}_{w_k,w_l} z_{k,l} \Bigg) \tilde{F}_{w_i,w_j} = 0. \end{aligned}$$

Since $\tilde{F}_{w_j,w_j} \mathscr{H}_{\ell,n}^{(0)} \cong P_0$ is ungraded right $\mathscr{H}_{\ell,n}^{(0)}$-module and Z acts faithfully on P_0, it follows that $z_{i,j} = 0$. This proves that $x = 0$ and hence η is injective. Finally, comparing the dimensions of both sides, we see that η is an isomorphism. □

4. A homogeneous symmetrizing form on $\mathscr{H}_{\boldsymbol{\ell},\boldsymbol{n}}^{(\mathbf{0})}$

By the work of Shan, Varagnolo and Vasserot [Shan et al. 2017], each cyclotomic quiver Hecke algebra can be endowed with a homogeneous symmetrizing form which makes it into a graded symmetric algebra (see Remark 4.7 and [Hu and

Mathas 2010, §6.3] for the type A case). In particular, the nilHecke algebra $\mathscr{H}_{\ell,n}^{(0)}$ is a graded symmetric algebra. However, the SVV symmetrizing form $\mathrm{Tr}^{\mathrm{SVV}}$ is defined in an inductive manner which relies on some deep results about certain decompositions of the cyclotomic quiver Hecke algebras which come from the biadjointness of the i-induction functors and i-restriction functors in the work of Kang and Kashiwara [2012] and of Kashiwara [2012]. It is rather difficult to compute the explicit value of the form $\mathrm{Tr}^{\mathrm{SVV}}$ on any specified homogeneous element in the cyclotomic quiver Hecke algebra because its inductive definition involves some mysterious correspondence (i.e., $z \mapsto \tilde{z}$, $\ell \mapsto \tilde{\pi}_\ell$ in [Shan et al. 2017, Theorem 3.8]) whose explicit descriptions are not available. In this section, we shall introduce a new homogeneous symmetrizing form Tr such that the value of the form Tr on each graded cellular basis element of $\mathscr{H}_{\ell,n}^{(0)}$ is explicitly given. We will prove in the next section that this form Tr actually coincides with Shan–Varagnolo–Vasserot's symmetrizing form $\mathrm{Tr}^{\mathrm{SVV}}$ on $\mathscr{H}_{\ell,n}^{(0)}$.

The following result seems to be well-known. We add a proof as we can not find a suitable reference.

Lemma 4.1. *Let A, B be two finite dimensional (ungraded) K-algebras. Suppose that B is Morita equivalent to A. Then there exists a K-linear map $\rho : A^* \to B^*$ such that for any symmetrizing form $\tau \in A^*$ on A, $\rho(\tau) \in B^*$ is a symmetrizing form on B. In particular, if A is a symmetric algebra over K, then B is a symmetric algebra over K too.*

Proof. By assumption, $B^{\mathrm{op}} \cong \mathrm{End}_A(P)$ for a finite dimensional (ungraded) projective left A-module P. Moreover, there exists a natural number k such that $A^{\oplus k} \cong P \oplus P'$ as left A-modules. Let e be the idempotent of $M_{k\times k}(A)$ which corresponds to the map $A^{\oplus k} \overset{\mathrm{pr}}{\twoheadrightarrow} P \overset{\iota}{\hookrightarrow} A^{\oplus k}$. Then $B^{\mathrm{op}} \cong \mathrm{End}_A(P) \cong eM_{k\times k}(A)e$.

We define $\rho_0 : A^* \to (M_{k\times k}(A))^*$ as follows: for any $f \in A^*$ and $(a_{i,j})_{k\times k} \in M_{k\times k}(A)$,

$$\rho_0(f)((a_{i,j})_{k\times k}) := f\bigg(\sum_{i=1}^{k} a_{ii}\bigg).$$

We define $\mathrm{res} : (M_{k\times k}(A))^* \to (eM_{k\times k}(A)e)^*$ as follows: for any $f \in (M_{k\times k}(A))^*$ and $(a_{i,j})_{k\times k} \in M_{k\times k}(A)$,

$$\mathrm{res}(f)(e(a_{i,j})_{k\times k}e) := f(e(a_{i,j})_{k\times k}e).$$

It is easy to check that $\rho := \mathrm{res} \circ \rho_0$ has the property that for any symmetrizing form $\tau \in A^*$ on A, $\rho(\tau) \in B^*$ is a symmetrizing form on $\mathrm{End}_A(P) \cong eM_{k\times k}(A)e \cong B^{\mathrm{op}}$. It is clear that $\rho(\tau)$ is a symmetrizing form on B too. □

The following lemma is clear.

Lemma 4.2. *Let $A = \oplus_{k=0}^{m} A_k$ be a finite dimensional positively $\mathbb{Z}$-graded K-algebra. Let τ be a (not necessarily homogeneous) symmetrizing form on A. We define $\tilde{\tau} : A^* \to K$ as follows: for any homogeneous element $y \in A$,*

$$\tilde{\tau}(y) := \begin{cases} \tau(x) & \textit{if } \deg x = m, \\ 0 & \textit{otherwise.} \end{cases}$$

Then $\tilde{\tau}$ can be linearly extended to a well-defined homogeneous symmetrizing form on A.

The following definition comes from [Shan et al. 2017, 3.1.5].

Definition 4.3. We define

$$d_\Lambda := 2\ell n - 2n^2.$$

Recall that by Theorem 3.7, the center Z is a positively $\mathbb{Z}$-graded K-algebra with each homogeneous component being one dimensional. In particular, $\deg z \leq d_\Lambda$ for all $z \in Z$, and $\deg z_{\boldsymbol{\lambda}_{\max}} = d_\Lambda$.

Lemma 4.4. *The center Z can be endowed with a homogeneous symmetrizing form of degree $-d_\Lambda$ as follows: for any homogeneous element $z \in Z$,*

$$\operatorname{tr}(z) := \begin{cases} 1 & \textit{if } z = z_{\boldsymbol{\lambda}_{\max}}, \\ 0 & \textit{if } \deg z < d_\Lambda. \end{cases}$$

In particular, Z is a graded symmetric algebra over K.

Proof. By Lemma 3.2, we know that Z is Morita equivalent to $\mathscr{H}_{\ell,n}^{(0)}$. Since $\mathscr{H}_{\ell,n}^{(0)}$ is a symmetric algebra by [Shan et al. 2017], we can deduce from Lemma 4.1 and Lemma 4.2 that Z is a graded symmetric algebra too.

On the other hand, by Lemma 3.2 and Corollary 2.19, we know that the center Z is a positively graded K-algebra with each homogeneous component being one dimensional. Therefore, we are in a position to apply [Hu and Lam 2017, Proposition 3.9] or Lemma 4.1 and Lemma 4.2 to show that tr is a well-defined homogeneous symmetrizing form on Z. □

Since tr is a homogeneous symmetrizing form on Z, for each nonzero homogeneous element $0 \neq z \in Z$, there exists a homogeneous element $\hat{z} \in Z$ with degree $d_\Lambda - \deg z$ such that $\operatorname{tr}(z\hat{z}) \neq 0$. This motivates the following definition.

Definition 4.5. For each $\boldsymbol{\lambda} \in \mathscr{P}_0$, we fix a nonzero homogeneous element $\hat{z}_{\boldsymbol{\lambda}} \in Z$ with degree $d_\Lambda - \deg z_{\boldsymbol{\lambda}}$ such that $\operatorname{tr}(z_{\boldsymbol{\lambda}} \hat{z}_{\boldsymbol{\lambda}}) \neq 0$.

Now we are using Proposition 3.8 and Lemma 4.4 to define a homogeneous symmetrizing form $\widehat{\operatorname{Tr}}$ on $\mathscr{H}_{\ell,n}^{(0)}$ as follows: for any $1 \leq i, j \leq n!$ and any homogeneous element $z \in Z$,

$$\widehat{\operatorname{Tr}}(\tilde{F}_{w_i,w_j} z) := \begin{cases} c & \text{if } i = j \text{ and } z = c z_{\boldsymbol{\lambda}_{\max}} \text{ for some } c \in K, \\ 0 & \text{if } i \neq j \text{ or } \deg z < d_\Lambda. \end{cases}$$

Lemma 4.6. *The map $\widehat{\mathrm{Tr}}$ extends linearly to a well-defined homogeneous symmetrizing form of degree $-d_\Lambda$ on $\mathscr{H}_{\ell,n}^{(0)}$.*

Proof. This follows directly from Lemma 4.4 and Proposition 3.8. □

Remark 4.7. Shan, Varagnolo, and Vasserot [Shan et al. 2017] show that each cyclotomic quiver Hecke algebra $\mathscr{R}_\beta^\Lambda$ can be endowed with a homogeneous symmetrizing form $\mathrm{Tr}^{\mathrm{SVV}}$ of degree $d_{\Lambda,\beta}$ which makes it into a graded symmetric algebra, where

$$\beta \in Q_n^+, \quad \Lambda \in P^+, \quad d_{\Lambda,\beta} := 2(\Lambda, \beta) - (\beta, \beta).$$

In the type A case we consider the cyclic quiver or linear quiver with vertices labeled by $\mathbb{Z}/e\mathbb{Z}$, where $e \neq 1$ is a nonnegative integer. In this case, $\mathscr{R}_\beta^\Lambda$ can be identified with the block of the cyclotomic Hecke algebra of type A which corresponds to β by Brundan–Kleshchev's isomorphism [Brundan and Kleshchev 2009a] when the ground field K contains a primitive e-th root of unity or e is equal to the characteristic of the ground field K. There is another homogeneous symmetrizing form $\mathrm{Tr}^{\mathrm{HM}}$ which can be defined (see [Hu and Mathas 2010, §6.3]) as follows: let τ be the ungraded symmetrizing form on $\mathscr{R}_\beta^\Lambda$ defined in [Malle and Mathas 1998] (nondegenerate case) and [Brundan and Kleshchev 2008] (degenerate case). Following [Hu and Mathas 2010, Definition 6.15], for any homogeneous element $x \in \mathscr{R}_\beta^\Lambda$, we define

$$\mathrm{Tr}^{\mathrm{HM}}(x) := \begin{cases} \tau(x) & \text{if } \deg(x) = d_{\Lambda,\beta}, \\ 0 & \text{otherwise.} \end{cases}$$

By the proof of [Hu and Mathas 2010, Theorem 6.17], $\mathrm{Tr}^{\mathrm{HM}}$ is a homogenous symmetrizing form on $\mathscr{R}_\beta^\Lambda$ of degree $-d_{\Lambda,\beta}$. The associated homogenous bilinear form $\langle -,- \rangle$ on $\mathscr{R}_\beta^\Lambda$ of degree $-d_{\Lambda,\beta}$ can be defined as follows: $\langle x, y \rangle := \mathrm{Tr}^{\mathrm{HM}}(xy)$. We take this chance to remark that the bilinear form $\langle -,- \rangle_\beta$ in the paragraph above [Hu and Mathas 2010, Theorem 6.17] should be replaced with the bilinear form $\langle -,- \rangle$ we defined here.

Conjecture 4.8. *The two symmetrizing forms $\mathrm{Tr}^{\mathrm{SVV}}$ and $\mathrm{Tr}^{\mathrm{HM}}$ on $\mathscr{R}_\beta^\Lambda$ differ by a nonzero scalar in K.*

Definition 4.9. For each $\boldsymbol{\mu} \in \mathscr{P}_0$ and $z_1, z_2 \in \mathfrak{S}_n$, we define

$$\phi_{z_1,z_2}^{\boldsymbol{\mu}} := \psi_{z_1}^* y_1^{n-1} y_2^{n-2} \dots y_{n-1} \psi_{w_0} y_{\boldsymbol{\mu}} \psi_{w_0} y_1^{n-1} y_2^{n-2} \dots y_{n-1} \psi_{z_2}.$$

Lemma 4.10. (1) *For each $\boldsymbol{\mu} \in \mathscr{P}_0$ and $z_1, z_2 \in \mathfrak{S}_n$, we have*

$$\phi_{w_0 z_1,z_2}^{\boldsymbol{\mu}} = F'_{z_1,z_2} z_{\boldsymbol{\mu}} = \psi_{w_0 z_1}^* y_1^{n-1} y_2^{n-2} \dots y_{n-1} \psi_{w_0} y_1^{n-1} y_2^{n-2} \dots y_{n-1} z_{\boldsymbol{\mu}} \psi_{z_2}$$

and

$$\phi_{z_1,z_2}^{\boldsymbol{\mu}} \equiv \psi_{z_1,z_2}^{\boldsymbol{\mu}} \pmod{(\mathscr{H}_{\ell,n}^{(0)})^{>\boldsymbol{\mu}}}.$$

(2) *The elements in the set* $\{\phi^{\boldsymbol{\mu}}_{z_1,z_2} \mid \boldsymbol{\mu} \in \mathscr{P}_0, z_1, z_2 \in \mathfrak{S}_n\}$ *form a homogeneous K-basis of* $\mathscr{H}^{(0)}_{\ell,n}$.

Proof. The first part of (1) follows from Lemma 3.5, while the second part of (1) follows from Lemma 2.12. Finally, (2) follows from (1) and (2.7). □

We are going to define another homogeneous symmetrizing form "Tr" on $\mathscr{H}^{(0)}_{\ell,n}$. Let $\boldsymbol{\lambda} \in \mathscr{P}_0$ and $w, u \in \mathfrak{S}_n$. By the same argument used in the proof of Lemma 3.4, there is an element $z_{w,u}$ in the center $Z(\mathscr{H}^{(0)}_{\ell,n})$ of $\mathscr{H}^{(0)}_{\ell,n}$ such that

$$\psi_{w_0} y_1^{n-1} y_2^{n-2} \dots y_{n-1} \psi_u \psi_{w^{-1}w_0} y_1^{n-1} y_2^{n-2} \dots y_{n-1} \psi_{w_0} = \psi_{w_0} z_{w,u}.$$

If $\deg z_{\boldsymbol{\lambda}} + \deg z_{w,u} = d_\Lambda$, then we denote $c_{w,u} \in K$ the unique scalar which satisfies that $z_{w,u} z_{\boldsymbol{\lambda}} = c_{w,u} z_{\boldsymbol{\lambda}_{\max}}$. Note that $\deg z_{\boldsymbol{\lambda}} + \deg z_{w,u} = d_\Lambda$ if and only if $\deg \phi^{\boldsymbol{\lambda}}_{w_0 w,u} = d_\Lambda$.

Definition 4.11. For any $\boldsymbol{\mu} \in \mathscr{P}_0$ and $w, u \in \mathfrak{S}_n$, we define

$$\operatorname{Tr}(F'_{w,u} z_{\boldsymbol{\mu}}) = \operatorname{Tr}(\phi^{\boldsymbol{\mu}}_{w_0 w,u}) := \begin{cases} c_{w,u} & \text{if } \deg F'_{w,u} z_{\boldsymbol{\mu}} = d_\Lambda, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, if $w = u$ and $\boldsymbol{\mu} = \lambda_{\max}$ then $\operatorname{Tr}(\phi^{\boldsymbol{\mu}}_{w,u}) = 1$. Note that

$$\begin{aligned} 1 &= \operatorname{Tr}(\phi^{\boldsymbol{\lambda}_{\max}}_{w_0,1}) = \operatorname{Tr}(F'_{1,1} z_{\boldsymbol{\lambda}_{\max}}) \\ &= \operatorname{Tr}(\psi^*_{w_0} y_1^{n-1} y_2^{n-2} \dots y_{n-1} \psi_{w_0} y_1^{n-1} y_2^{n-2} \dots y_{n-1} z_{\boldsymbol{\lambda}_{\max}}) \\ &= (-1)^{n(n-1)/2} \operatorname{Tr}(\psi^*_{w_0} y_1^{n-1} y_2^{n-2} \dots y_{n-1} z_{\boldsymbol{\lambda}_{\max}}) \\ &= \operatorname{Tr}(\psi^*_{w_0} y_{\boldsymbol{\lambda}_{\max}}), \end{aligned}$$

which implies that

$$\operatorname{Tr}(\psi^*_{w_0} y_{\boldsymbol{\lambda}_{\max}}) = 1. \tag{4.12}$$

Proposition 4.13. *The map* Tr *can be linearly extended to a well-defined homogeneous symmetrizing form of degree $-d_\Lambda$ on* $\mathscr{H}^{(0)}_{\ell,n}$.

Proof. By construction, it is clear that the map Tr can be linearly extended to a well-defined homogeneous linear map of degree $-d_\Lambda$ on $\mathscr{H}^{(0)}_{\ell,n}$.

We want to show that $\widehat{\operatorname{Tr}} = \operatorname{Tr}$. Once this is proved, it is automatically proved that Tr is symmetric and nondegenerate. To this end, by Lemma 4.10, it suffices to show that $\widehat{\operatorname{Tr}}(F'_{z_1,z_2} z_{\boldsymbol{\mu}}) = \operatorname{Tr}(F'_{z_1,z_2} z_{\boldsymbol{\mu}})$ for any $\boldsymbol{\mu} \in \mathscr{P}_0$ and $z_1, z_2 \in \mathfrak{S}_n$.

Without loss of generality we can assume that $\deg(F'_{z_1,z_2}z_{\boldsymbol{\mu}}) = d_\Lambda$. Since $\widehat{\mathrm{Tr}}$ is a trace form and $z_{\boldsymbol{\mu}}$ is central, we have

$$\begin{aligned}
&\widehat{\mathrm{Tr}}(F'_{z_1,z_2}z_{\boldsymbol{\mu}})\\
&\quad= \widehat{\mathrm{Tr}}(F'_{z_1,z_1}F'_{z_1,z_2}z_{\boldsymbol{\mu}})\\
&\quad= \widehat{\mathrm{Tr}}\big(\psi^*_{w_0z_1}y_1^{n-1}y_2^{n-2}\dots y_{n-1}\psi_{w_0}y_1^{n-1}y_2^{n-2}\dots\\
&\qquad\qquad \times y_{n-1}\psi_{z_1}\psi^*_{w_0z_1}y_1^{n-1}y_2^{n-2}\dots y_{n-1}\psi_{w_0}y_1^{n-1}y_2^{n-2}\dots y_{n-1}\psi_{z_2}z_{\boldsymbol{\mu}}\big)\\
&\quad= \widehat{\mathrm{Tr}}\big(\psi^*_{w_0z_1}y_1^{n-1}y_2^{n-2}\dots y_{n-1}\psi_{w_0}y_1^{n-1}y_2^{n-2}\dots\\
&\qquad\qquad \times y_{n-1}\psi_{w_0}y_1^{n-1}y_2^{n-2}\dots y_{n-1}\psi_{w_0}y_1^{n-1}y_2^{n-2}\dots y_{n-1}\psi_{z_2}z_{\boldsymbol{\mu}}\big)\\
&\quad= \widehat{\mathrm{Tr}}\big(y_1^{n-1}y_2^{n-2}\dots y_{n-1}\psi_{w_0}y_1^{n-1}y_2^{n-2}\dots\\
&\qquad\qquad \times y_{n-1}\psi_{w_0}y_1^{n-1}y_2^{n-2}\dots y_{n-1}\psi_{z_2}\psi_{z_1^{-1}w_0}y_1^{n-1}y_2^{n-2}\dots y_{n-1}\psi_{w_0}z_{\boldsymbol{\mu}}\big)\\
&\quad= \widehat{\mathrm{Tr}}(y_1^{n-1}y_2^{n-2}\dots y_{n-1}\psi_{w_0}y_1^{n-1}y_2^{n-2}\dots y_{n-1}\psi_{w_0}z_{z_1,z_2}z_{\boldsymbol{\mu}})\\
&\quad= (-1)^{n(n-1)/2}\widehat{\mathrm{Tr}}(y_1^{n-1}y_2^{n-2}\dots y_{n-1}\psi_{w_0}c_{z_1,z_2}z_{\boldsymbol{\lambda}_{\max}})\\
&\quad= (-1)^{n(n-1)/2}c_{z_1,z_2}\widehat{\mathrm{Tr}}(\psi_{w_0}y_1^{n-1}y_2^{n-2}\dots y_{n-1}z_{\boldsymbol{\lambda}_{\max}})\\
&\quad= c_{z_1,z_2}\widehat{\mathrm{Tr}}(\tilde{F}_{1,1}z_{\boldsymbol{\lambda}_{\max}}) = c_{z_1,z_2}\\
&\quad= \mathrm{Tr}(F'_{z_1,z_2}z_{\boldsymbol{\mu}}).
\end{aligned}$$

This completes the proof of $\widehat{\mathrm{Tr}} = \mathrm{Tr}$. In particular, this implies that Tr is symmetric and nondegenerate. That says, Tr can be linearly extended to a well-defined homogeneous symmetrizing form of degree $-d_\Lambda$ on $\mathscr{H}^{(0)}_{\ell,n}$. □

Proposition 4.14. $$\widehat{\mathrm{Tr}} = \mathrm{Tr}.$$

Proof. This follows from the proof of Proposition 4.13. □

5. Comparing Tr with the Shan–Varagnolo–Vasserot symmetrizing form $\mathrm{Tr}^{\mathrm{SVV}}$

In this section, we compare the symmetrizing form Tr with the Shan–Varagnolo–Vasserot symmetrizing form $\mathrm{Tr}^{\mathrm{SVV}}$ introduced in [Shan et al. 2017] and show that they are actually the same.

Let A, B be two K-algebras and $i : B \to A$ is a K-algebra homomorphism. Let $A^B := \{x \in A \mid xb = bx, \forall\, b \in B\}$ be the centralizer of B in A. For any $f \in A^B$, we set

$$\mu_f : A \otimes_B A \to A, \quad a \otimes a' \mapsto afa'.$$

Recall that $\mathscr{H}^{(0)}_{\ell,n} = \mathscr{R}^{\ell\Lambda_0}_{n\alpha_0}$. In the notations of [Shan et al. 2017, §3.1.4], we set

$$\lambda_0 := \langle \ell\Lambda_0 - (n-1)\alpha_0, \alpha_0^\vee\rangle = \ell - 2(n-1). \tag{5.1}$$

We first recall the definition of $\mathrm{Tr}^{\mathrm{SVV}}$ in the case of nilHecke algebra $\mathscr{R}^{\ell\Lambda_0}_{n\alpha_0}$.

Definition 5.2 [Kang and Kashiwara 2012; Shan et al. 2017, Theorem 3.6, (6), (8)]. If $\lambda_0 \geq 0$ then for any $z \in \mathscr{R}^{\ell\Lambda_0}_{n\alpha_0}$ there are unique elements $p_k(z) \in \mathscr{R}^{\ell\Lambda_0}_{(n-1)\alpha_0}$ and $\pi(z) \in \mathscr{R}^{\ell\Lambda_0}_{(n-1)\alpha_0} \otimes_{R^{\ell\Lambda_0}_{(n-2)\alpha_0}} \mathscr{R}^{\ell\Lambda_0}_{(n-1)\alpha_0}$ such that

$$z = \mu_{\psi_{n-1}}(\pi(z)) + \sum_{k=0}^{\lambda_0-1} p_k(z) y_n^k,$$

where the above summation is understood as 0 when $\lambda_0 = 0$.

If $\lambda_0 \leq 0$ then for any $z \in \mathscr{R}^{\ell\Lambda_0}_{n\alpha_0}$, there is a unique element $\tilde{z} \in \mathscr{R}^{\ell\Lambda_0}_{(n-1)\alpha_0} \otimes_{\mathscr{R}^{\ell\Lambda_0}_{(n-2)\alpha_0}} \mathscr{R}^{\ell\Lambda_0}_{(n-1)\alpha_0}$ such that

$$\mu_{\psi_{n-1}}(\tilde{z}) = z \quad \text{and} \quad \mu_{y^k_{n-1}}(\tilde{z}) = 0, \forall\, k \in \{0, 1, \dots, -\lambda_0 - 1\},$$

where the range of k is understood as $\varnothing$ when $\lambda_0 = 0$.

Definition 5.3 [Shan et al. 2017, Theorem 3.8]. For each $n \in \mathbb{N}$, we define $\hat{\varepsilon}_n : \mathscr{R}^{\ell\Lambda_0}_{n\alpha_0} \to \mathscr{R}^{\ell\Lambda_0}_{(n-1)\alpha_0}$ as follows: for any $z \in \mathscr{R}^{\ell\Lambda_0}_{n\alpha_0}$, if $\lambda_0 := \ell - 2(n-1) > 0$ then $\hat{\varepsilon}_n(z) := p_{\ell-2(n-1)-1}(z)$; if $\lambda_0 := \ell - 2(n-1) \leq 0$ then $\hat{\varepsilon}_n(z) := \mu_{y_{n-1}^{-\ell+2(n-1)}}(\tilde{z})$.

Definition 5.4 [Shan et al. 2017, A.3.]. For any $z \in \mathscr{R}^{\ell\Lambda_0}_{n\alpha_0}$,

$$\mathrm{Tr}^{\mathrm{SVV}}(z) := \hat{\varepsilon}_1 \circ \hat{\varepsilon}_2 \circ \cdots \circ \hat{\varepsilon}_n : \mathscr{R}^{\ell\Lambda_0}_{n\alpha_0} \to \mathscr{R}^{\ell\Lambda_0}_{0\alpha_0} = K.$$

Definition 5.5. For each $n \in \mathbb{N}$, we define

$$Z_{0,n} := \psi_{w_{0,n}} y_1^{\ell-1} y_2^{\ell-2} \dots y_n^{\ell-n} \in \mathscr{H}^{(0)}_{\ell,n}.$$

We want to compute the value $\mathrm{Tr}^{\mathrm{SVV}}(Z_{0,n})$. According to Definition 5.2, we need to understand the value $p_{\ell-2(n-1)-1}(Z_{0,n})$ when $\ell > 2(n-1)$ and the value $\mu_{y_{n-1}^{-\ell+2(n-1)}}(\widetilde{Z}_{0,n})$ when $\ell \leq 2(n-1)$.

Lemma 5.6. *Suppose that $\lambda_0 := \ell - 2(n-1) \geq 0$. Then*

$$\begin{aligned}\pi(Z_{0,n}) = ((\psi_1 \dots \psi_{n-2}) y_{n-1}^{\ell-n}) \\ \otimes (\psi_1 \dots \psi_{n-3}\psi_{n-2}) \dots (\psi_1\psi_2)\psi_1 y_1^{\ell-1} y_2^{\ell-2} \dots y_{n-1}^{\ell-n+1} \\ \in \mathscr{R}^{\ell\Lambda_0}_{(n-1)\alpha_0} \otimes_{\mathscr{R}^{\ell\Lambda_0}_{(n-2)\alpha_0}} \mathscr{R}^{\ell\Lambda_0}_{(n-1)\alpha_0},\end{aligned}$$

and for any $k \in \{0, 1, \dots, \lambda_0 - 1\}$,

$$\begin{aligned}p_k(Z_{0,n}) \\ = (\psi_1 \dots \psi_{n-2})(\psi_1 \dots \psi_{n-3}) \dots (\psi_1\psi_2)\psi_1 y_1^{\ell-1} y_2^{\ell-2} \dots y_{n-2}^{\ell-n+2} y_{n-1}^{\ell-n+\lambda_0-k}.\end{aligned}$$

In particular, $p_{\lambda_0-1}(Z_{0,n}) = Z_{0,n-1}$.

Proof. By definition, we have

$$\begin{aligned}
Z_{0,n} &= \psi_{w_{0,n}} y_1^{\ell-1} y_2^{\ell-2} \dots y_n^{\ell-n} \\
&= (\psi_1 \dots \psi_{n-2}\psi_{n-1})(\psi_1 \dots \psi_{n-3}\psi_{n-2}) \dots (\psi_1\psi_2)\psi_1 y_1^{\ell-1} y_2^{\ell-2} \dots y_n^{\ell-n} \\
&= (\psi_1 \dots \psi_{n-2})(\psi_{n-1} y_n^{\ell-n}) \psi_{w_{0,n-1}} y_1^{\ell-1} y_2^{\ell-2} \dots y_{n-1}^{\ell-n+1} \\
&= (\psi_1 \dots \psi_{n-2}) \Bigg(y_{n-1}^{\ell-n}\psi_{n-1} + \sum_{\substack{a_1+a_2=\ell-n-1 \\ a_1,a_2\geq 0}} y_{n-1}^{a_1} y_n^{a_2} \Bigg) \\
&\qquad\qquad \times \psi_{w_{0,n-1}} y_1^{\ell-1} y_2^{\ell-2} \dots y_{n-1}^{\ell-n+1} \\
&= (\psi_1 \dots \psi_{n-2})(y_{n-1}^{\ell-n}\psi_{n-1}) \psi_{w_{0,n-1}} y_1^{\ell-1} y_2^{\ell-2} \dots y_{n-1}^{\ell-n+1} \\
&\qquad + \sum_{\substack{a_1+a_2=\ell-n-1 \\ a_1,a_2\geq 0}} (\psi_1 \dots \psi_{n-2} y_{n-1}^{a_1} \psi_{w_{0,n-1}} y_1^{\ell-1} y_2^{\ell-2} \dots y_{n-1}^{\ell-n+1} y_n^{a_2}) \\
&= (\psi_1 \dots \psi_{n-2})(y_{n-1}^{\ell-n}\psi_{n-1}) \psi_{w_{0,n-1}} y_1^{\ell-1} y_2^{\ell-2} \dots y_{n-1}^{\ell-n+1} \\
&\quad + \sum_{\substack{a_1+a_2=\ell-n-1 \\ a_1,a_2\geq 0}} \big(\psi_1 \dots \psi_{n-2} y_{n-1}^{a_1} (\psi_1 \dots \psi_{n-3}\psi_{n-2})(\psi_1 \dots \psi_{n-4}\psi_{n-3}) \dots \\
&\qquad\qquad \times (\psi_1\psi_2)\psi_1 y_1^{\ell-1} y_2^{\ell-2} \dots y_{n-1}^{\ell-n+1} y_n^{a_2}\big) \\
&= (\psi_1 \dots \psi_{n-2})(y_{n-1}^{\ell-n}\psi_{n-1}) \psi_{w_{0,n-1}} y_1^{\ell-1} y_2^{\ell-2} \dots y_{n-1}^{\ell-n+1} \\
&\quad + \sum_{\substack{a_1+a_2=\ell-n-1 \\ a_1,a_2\geq 0}} \big((\psi_1 \dots \psi_{n-2})(\psi_1 \dots \psi_{n-3}) \\
&\qquad\qquad \times (\psi_1 \dots \psi_{n-4}) \dots \psi_1 y_{n-1}^{a_1} (\psi_{n-2}\psi_{n-3} \dots \psi_2\psi_1) \\
&\qquad\qquad\qquad \times y_1^{\ell-1} y_2^{\ell-2} \dots y_{n-1}^{\ell-n+1} y_n^{a_2}\big) \\
&= \mu_{\psi_{n-1}}\big((\psi_1 \dots \psi_{n-2} y_{n-1}^{\ell-n}) \otimes (\psi_{w_{0,n-1}} y_1^{\ell-1} y_2^{\ell-2} \dots y_{n-1}^{\ell-n+1})\big) \\
&\qquad + \sum_{\substack{a_1+a_2=\ell-n-1 \\ a_1,a_2\geq 0}} \psi_{w_{0,n-1}} (y_{n-1}^{a_1}\psi_{n-2} \dots \psi_2\psi_1) y_1^{\ell-1} y_2^{\ell-2} \dots y_{n-1}^{\ell-n+1} y_n^{a_2}.
\end{aligned}$$

Using the uniqueness in Definition 5.2, we see that to prove the lemma, it suffices to show that

$$\begin{aligned}
\sum_{\substack{a_1+a_2=\ell-n-1 \\ a_1,a_2\geq 0}} & \psi_{w_{0,n-1}} (y_{n-1}^{a_1}\psi_{n-2} \dots \psi_2\psi_1) y_1^{\ell-1} y_2^{\ell-2} \dots y_{n-1}^{\ell-n+1} y_n^{a_2} \\
&= \sum_{k=0}^{\lambda_0-1} \psi_{w_{0,n-1}} y_1^{\ell-1} y_2^{\ell-2} \dots y_{n-2}^{\ell-n+2} y_{n-1}^{\ell-n+\lambda_0-k} y_n^{k}.
\end{aligned}$$

In fact,

$$\begin{aligned}
&\sum_{\substack{a_1+a_2=\ell-n-1\\ a_1,a_2\geq 0}} \psi_{w_{0,n-1}}(y_{n-1}^{a_1}\psi_{n-2}\dots\psi_2\psi_1)y_1^{\ell-1}y_2^{\ell-2}\dots y_{n-1}^{\ell-n+1}y_n^{a_2}\\
&\quad=\sum_{\substack{a_1+a_2=\ell-n-1\\ a_1,a_2\geq 0}} \psi_{w_{0,n-1}}(y_{n-1}^{a_1}\psi_{n-2}\dots\psi_2\psi_1)y_1^{\ell-1}y_2^{\ell-2}\dots y_{n-1}^{\ell-n+1}y_n^{a_2}\\
&\quad=\sum_{\substack{a_1+a_2=\ell-n-1\\ a_1\geq n-2,a_2\geq 0}} \psi_{w_{0,n-1}}(y_{n-1}^{a_1}\psi_{n-2}\dots\psi_2\psi_1)y_1^{\ell-1}y_2^{\ell-2}\dots y_{n-1}^{\ell-n+1}y_n^{a_2}\\
&\quad=\psi_{w_{0,n-1}}\psi_{n-2}\dots\psi_2\psi_1 y_1^{\ell-1}y_2^{\ell-2}\dots y_{n-2}^{\ell-n+2}y_{n-1}^{\ell-n+1}y_n^{\ell-2n+1}\\
&\qquad\qquad+\psi_{w_{0,n-1}}y_1^{\ell-1}y_2^{\ell-2}\dots y_{n-2}^{\ell-n+2}y_{n-1}^{\ell-n+2}y_n^{\ell-2n}\\
&\qquad\qquad+\psi_{w_{0,n-1}}y_1^{\ell-1}y_2^{\ell-2}\dots y_{n-2}^{\ell-n+2}y_{n-1}^{\ell-n+3}y_n^{\ell-2n-1}\\
&\qquad\qquad\qquad\qquad\vdots\\
&\qquad\qquad+\psi_{w_{0,n-1}}y_1^{\ell-1}y_2^{\ell-2}\dots y_{n-2}^{\ell-n+2}y_{n-1}^{2\ell-3n+1}y_n\\
&\qquad\qquad+\psi_{w_{0,n-1}}y_1^{\ell-1}y_2^{\ell-2}\dots y_{n-2}^{\ell-n+2}y_{n-1}^{2\ell-3n+2}\\
&\quad=\sum_{k=0}^{\lambda_0-1}\psi_{w_{0,n-1}}y_1^{\ell-1}y_2^{\ell-2}\dots y_{n-2}^{\ell-n+2}y_{n-1}^{\ell-n+\lambda_0-k}y_n^{k},
\end{aligned}$$

where we have used the commutator relations for the ψ and y generators of $\mathscr{H}_{\ell,n}^{(0)}$ and the fact that

$$\psi_{w_{0,n-1}}\psi_r=0 \quad \text{for any } 1\leq r<n-1$$

in the second and the last equalities. This completes the proof of the lemma. □

Lemma 5.7. *Suppose that* $\lambda_0:=\ell-2(n-1)\leq 0$. *Then*

$$\begin{aligned}
&\widetilde{Z}_{0,n}=\\
&((\psi_1\psi_2\dots\psi_{n-2})y_{n-1}^{\ell-n})\otimes((\psi_1\dots\psi_{n-3}\psi_{n-2})\dots(\psi_1\psi_2)\psi_1 y_1^{\ell-1}y_2^{\ell-2}\dots y_{n-1}^{\ell-n+1})\\
&\qquad\qquad\qquad\qquad\in\mathscr{R}^{\ell\Lambda_0}_{(n-1)\alpha_0}\otimes_{R^{\ell\Lambda_0}_{(n-2)\alpha_0}}\mathscr{R}^{\ell\Lambda_0}_{(n-1)\alpha_0}
\end{aligned}$$

and

$$\begin{aligned}
\mu_{y_{n-1}^{-\lambda_0}}(\widetilde{Z}_{0,n})&=Z_{0,n-1}\\
&=(\psi_1\dots\psi_{n-2})(\psi_1\dots\psi_{n-3})\dots(\psi_1\psi_2)\psi_1 y_1^{\ell-1}y_2^{\ell-2}\dots y_{n-1}^{\ell-n+1}.
\end{aligned}$$

Proof. By definition, we have

$$\begin{aligned}
Z_{0,n} &= \psi_{w_{0,n}} y_1^{\ell-1} y_2^{\ell-2} \dots y_n^{\ell-n} \\
&= (\psi_1 \dots \psi_{n-2}\psi_{n-1})(\psi_1 \dots \psi_{n-3}\psi_{n-2}) \dots (\psi_1\psi_2)\psi_1 y_1^{\ell-1} y_2^{\ell-2} \dots y_n^{\ell-n} \\
&= (\psi_1 \dots \psi_{n-2})(\psi_{n-1} y_n^{\ell-n})(\psi_1 \dots \psi_{n-3}\psi_{n-2}) \dots \\
&\qquad\qquad\qquad \times (\psi_1\psi_2)\psi_1 y_1^{\ell-1} y_2^{\ell-2} \dots y_{n-1}^{\ell-n+1} \\
&= (\psi_1 \dots \psi_{n-2}) \Bigg(y_{n-1}^{\ell-n}\psi_{n-1} + \sum_{\substack{a_1+a_2=\ell-n-1 \\ a_1,a_2\geq 0}} y_{n-1}^{a_1} y_n^{a_2} \Bigg) \\
&\qquad\qquad\qquad \times \psi_{w_{0,n-1}} y_1^{\ell-1} y_2^{\ell-2} \dots y_{n-1}^{\ell-n+1} \\
&= (\psi_1 \dots \psi_{n-2})(y_{n-1}^{\ell-n}\psi_{n-1})\psi_{w_{0,n-1}} y_1^{\ell-1} y_2^{\ell-2} \dots y_{n-1}^{\ell-n+1} \\
&\qquad + \sum_{\substack{a_1+a_2=\ell-n-1 \\ a_1,a_2\geq 0}} \psi_1 \dots \psi_{n-2}(y_{n-1}^{a_1})\psi_{w_{0,n-1}} y_1^{\ell-1} y_2^{\ell-2} \dots y_{n-1}^{\ell-n+1} y_n^{a_2}.
\end{aligned}$$

We now claim that

$$\sum_{\substack{a_1+a_2=\ell-n-1 \\ a_1,a_2\geq 0}} \psi_1 \dots \psi_{n-2}(y_{n-1}^{a_1})\psi_{w_{0,n-1}} y_1^{\ell-1} y_2^{\ell-2} \dots y_{n-1}^{\ell-n+1} y_n^{a_2} = 0. \tag{5.8}$$

In fact, we have

$$\begin{aligned}
&\sum_{\substack{a_1+a_2=\ell-n-1 \\ a_1,a_2\geq 0}} \psi_1 \dots \psi_{n-2}(y_{n-1}^{a_1})\psi_{w_{0,n-1}} y_1^{\ell-1} y_2^{\ell-2} \dots y_{n-1}^{\ell-n+1} y_n^{a_2} \\
&\quad = \sum_{\substack{a_1+a_2=\ell-n-1 \\ a_1,a_2\geq 0}} \psi_1 \dots \psi_{n-2}(y_{n-1}^{a_1})(\psi_1 \dots \psi_{n-3}\psi_{n-2})(\psi_1 \dots \psi_{n-4}\psi_{n-3}) \dots \\
&\qquad\qquad\qquad\qquad \times (\psi_1\psi_2)(\psi_1) y_1^{\ell-1} y_2^{\ell-2} \dots y_{n-1}^{\ell-n+1} y_n^{a_2} \\
&\quad = \sum_{\substack{a_1+a_2=\ell-n-1 \\ a_1,a_2\geq 0}} \psi_{w_{0,n-1}}(y_{n-1}^{a_1}\psi_{n-2} \dots \psi_2\psi_1) y_1^{\ell-1} y_2^{\ell-2} \dots y_{n-1}^{\ell-n+1} y_n^{a_2} \\
&\quad = \sum_{\substack{a_1+a_2=\ell-n-1 \\ a_1>0,a_2\geq 0}} \psi_{w_{0,n-1}}(y_{n-1}^{a_1}\psi_{n-2} \dots \psi_2\psi_1) y_1^{\ell-1} y_2^{\ell-2} \dots y_{n-1}^{\ell-n+1} y_n^{a_2},
\end{aligned}$$

where the last equality follows from the fact that $\psi_{w_{0,n-1}}\psi_{n-2} = 0$. Now by assumption, $a_1 \leq \ell - n - 1 \leq 2(n-1) - n - 1 = n - 3 < n - 2$. It follows that $y_{n-1}^{a_1}\psi_{n-2} \dots \psi_2\psi_1$ is a sum of some elements which have a left factor of the form ψ_r for some $1 \leq r < n-1$. Therefore, using the fact that $\psi_{w_{0,n-1}}\psi_r = 0$ for any $1 \leq r < n-1$ again, we can deduce that the above sum is 0. This completes the proof of the claim (5.8).

By Definition 5.2, to complete the proof of the lemma, it remains to show that for any $0 \le k \le -\lambda_0 - 1$,

$$\mu_{y_{n-1}^k}\big((\psi_1\psi_2\dots\psi_{n-2}y_{n-1}^{\ell-n}) \otimes (\psi_{w_{0,n-1}}y_1^{\ell-1}y_2^{\ell-2}\dots y_{n-1}^{\ell-n+1})\big) = 0. \tag{5.9}$$

In fact, we have

$$\begin{aligned}
&\mu_{y_{n-1}^k}\big((\psi_1\psi_2\dots\psi_{n-2}y_{n-1}^{\ell-n}) \otimes (\psi_{w_{0,n-1}}y_1^{\ell-1}y_2^{\ell-2}\dots y_{n-1}^{\ell-n+1})\big)\\
&\quad= \mu_{y_{n-1}^k}\big((\psi_1\dots\psi_{n-2}y_{n-1}^{\ell-n})\\
&\qquad\qquad \otimes (\psi_1\dots\psi_{n-3}\psi_{n-2})\dots(\psi_1\psi_2)\psi_1 y_1^{\ell-1}y_2^{\ell-2}\dots y_{n-1}^{\ell-n+1}\big)\\
&\quad= (\psi_1\dots\psi_{n-2})(y_{n-1}^{\ell-n+k})(\psi_1\dots\psi_{n-3}\psi_{n-2})(\psi_1\dots\psi_{n-4}\psi_{n-3})\dots\\
&\qquad\qquad \times (\psi_1\psi_2)\psi_1 y_1^{\ell-1}y_2^{\ell-2}\dots y_{n-1}^{\ell-n+1}\\
&\quad= (\psi_1\dots\psi_{n-2})(\psi_1\dots\psi_{n-3})\dots\\
&\qquad\qquad \times (\psi_1\psi_2)\psi_1(y_{n-1}^{\ell-n+k}\psi_{n-2}\psi_{n-3}\dots\psi_2\psi_1)y_1^{\ell-1}y_2^{\ell-2}\dots y_{n-1}^{\ell-n+1}\\
&\quad= \psi_{w_{0,n-1}}(y_{n-1}^{\ell-n+k}\psi_{n-2}\psi_{n-3}\dots\psi_1)y_1^{\ell-1}y_2^{\ell-2}\dots y_{n-1}^{\ell-n+1} = 0,
\end{aligned}$$

where the last equality follows from the fact that $\psi_{w_{0,n-1}}\psi_r = 0$ for any $1 \le r < n-1$ and the assumption that

$$\ell - n + k \le \ell - n - \lambda_0 - 1 = \ell - n - (\ell - 2(n-1)) - 1 = n - 3 < n - 2$$

so that $y_{n-1}^{\ell-n+k}\psi_{n-2}\psi_{n-3}\dots\psi_1$ is a sum of some elements which have a left factor of the form ψ_r for some $1 \le r < n-1$. This completes the proof of (5.9) and hence the proof of the lemma. $\square$

Corollary 5.10. $\mathrm{Tr}^{\mathrm{SVV}}(Z_{0,n}) = 1.$

Proof. This follows from Definition 5.3, Definition 5.4, Lemma 5.6, Lemma 5.7, and an induction on n. $\square$

Theorem 5.11. *The two symmetrizing forms* $\mathrm{Tr}^{\mathrm{SVV}}$ *and* Tr *on the cyclotomic nil-Hecke algebra* $\mathscr{H}_{\ell,n}^{(0)}$ *coincide with each other.*

Proof. Let $1 \le i,\ j \le n!$, and $z \in Z$. Suppose that $i \ne j$. Then as $\mathrm{Tr}^{\mathrm{SVV}}$ is a symmetrizing form and z is central, we have

$$\begin{aligned}
\mathrm{Tr}^{\mathrm{SVV}}(\tilde F_{w_i,w_j}z) &= \mathrm{Tr}^{\mathrm{SVV}}(\tilde F_{w_i,w_i}\tilde F_{w_i,w_j}z) = \mathrm{Tr}^{\mathrm{SVV}}(\tilde F_{w_i,w_j}z\tilde F_{w_i,w_i})\\
&= \mathrm{Tr}^{\mathrm{SVV}}(\tilde F_{w_i,w_j}\tilde F_{w_i,w_i}z) = \mathrm{Tr}^{\mathrm{SVV}}(0z) = 0.
\end{aligned}$$

It remains to consider the case when $i = j$.

If $\deg z < d_\Lambda$, then as $\mathrm{Tr}^{\mathrm{SVV}}$ is homogeneous of degree $-d_\Lambda$ and $\deg \tilde F_{w_i,w_i} = 0$, we have $\mathrm{Tr}^{\mathrm{SVV}}(\tilde F_{w_i,w_i}z) = 0$. Therefore, without loss of generality, we can assume that $z = z_{\boldsymbol\lambda_{\max}}$. Our purpose is to compare $\mathrm{Tr}^{\mathrm{SVV}}(\tilde F_{w_i,w_i}z_{\boldsymbol\lambda_{\max}})$ and $\mathrm{Tr}(\tilde F_{w_i,w_i}z_{\boldsymbol\lambda_{\max}})$.

Note that for any $\boldsymbol{\mu} \in \mathscr{P}_0$ with $\boldsymbol{\mu} > \boldsymbol{\lambda}_{\min}$, we have that

$$\deg(y_{\boldsymbol{\mu}} z_{\boldsymbol{\lambda}_{\max}}) > n(n-1) + 2n(\ell - n) = 2\ell n - n(n+1) = \deg(y_1^{\ell-1} y_2^{\ell-2} \dots y_n^{\ell-n}),$$

which implies that $y_{\boldsymbol{\mu}} z_{\boldsymbol{\lambda}_{\max}} = 0$ by Theorem 2.34. By (2.33) and Lemma 3.5, we have

$$\begin{aligned}
\mathrm{Tr}^{\mathrm{SVV}}(\tilde{F}_{w_i,w_i} z_{\boldsymbol{\lambda}\max}) &= (-1)^{n(n-1)/2}\, \mathrm{Tr}^{\mathrm{SVV}}(\psi^{\boldsymbol{\lambda}_{\min}}_{w_0 w_i, w_i} z_{\boldsymbol{\lambda}\max}) = \mathrm{Tr}^{\mathrm{SVV}}(\psi^{\boldsymbol{\lambda}\max}_{w_0 w_i, w_i}) \\
&= \mathrm{Tr}^{\mathrm{SVV}}(\psi_{w_i} \psi^*_{w_0 w_i} y_1^{\ell-1} y_2^{\ell-2} \dots y_n^{\ell-n}) \\
&= \mathrm{Tr}^{\mathrm{SVV}}(\psi_{w_0} y_1^{\ell-1} y_2^{\ell-2} \dots y_n^{\ell-n}) \\
&= \mathrm{Tr}^{\mathrm{SVV}}(Z_{0,n}) = 1, \quad \text{(by Corollary 5.10)} \\
\mathrm{Tr}(\tilde{F}_{w_i,w_i} z_{\boldsymbol{\lambda}\max}) &= (-1)^{n(n-1)/2}\, \mathrm{Tr}(\psi^{\boldsymbol{\lambda}_{\min}}_{w_0 w_i, w_i} z_{\boldsymbol{\lambda}\max}) = \mathrm{Tr}(\psi^{\boldsymbol{\lambda}\max}_{w_0 w_i, w_i}) \\
&= \mathrm{Tr}(\psi_{w_i} \psi^*_{w_0 w_i} y_1^{\ell-1} y_2^{\ell-2} \dots y_n^{\ell-n}) \\
&= \mathrm{Tr}(\psi_{w_0} y_1^{\ell-1} y_2^{\ell-2} \dots y_n^{\ell-n}) = 1. \quad \text{(by (4.12))}
\end{aligned}$$

This shows that $\mathrm{Tr}^{\mathrm{SVV}}(\tilde{F}_{w_i,w_i} z_{\boldsymbol{\lambda}\max}) = \mathrm{Tr}(\tilde{F}_{w_i,w_i} z_{\boldsymbol{\lambda}\max})$.

As a result, we have shown that $\mathrm{Tr}^{\mathrm{SVV}}(\tilde{F}_{w_i,w_j} z) = \mathrm{Tr}(\tilde{F}_{w_i,w_j} z)$ for any $1 \le i, j \le n!$, and $z \in Z$. It follows that $\mathrm{Tr}^{\mathrm{SVV}} = \mathrm{Tr}$, as required. □

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