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MINIMAL REGULARITY SOLUTIONS OF SEMILINEAR GENERALIZED TRICOMI EQUATIONS

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We prove the local existence and uniqueness of minimal regularity solutions u of the semilinear generalized Tricomi equation $\partial_t^2 u - t^m \Delta u = F(u)$ with initial data $(u(0,\cdot), \partial_t u(0,\cdot)) \in \dot{H}^\gamma(\mathbb{R}^n) \times \dot{H}^{\gamma-2/(m+2)}(\mathbb{R}^n)$ under the assumptions that $|F(u)| \lesssim |u|^\kappa$ and $|F'(u)| \lesssim |u|^{\kappa-1}$ for some $\kappa > 1$. Our results improve previous results of M. Beals and ourselves. We establish Strichartz-type estimates for the linear generalized Tricomi operator $\partial_t^2 - t^m \Delta$ from which the semilinear results are derived.

1. Introduction

In this paper, we are concerned with the local well-posedness problem for minimal regularity solutions u of the semilinear generalized Tricomi equation

$$\begin{aligned} &\partial_t^2 u - t^m \Delta u = F(u) \quad \text{in } [0,T] \times \mathbb{R}^n, \\ &u(0,\cdot) = \varphi \in \dot{H}^\gamma(\mathbb{R}^n), \quad \partial_t u(0,\cdot) = \psi \in \dot{H}^{\gamma-2/(m+2)}(\mathbb{R}^n), \end{aligned} \tag{1-1}$$

where $n \geq 2$, $m \in \mathbb{N}$, $\gamma \in \mathbb{R}$, $\Delta = \sum_{i=1}^n \partial_i^2$, and $T > 0$. The nonlinearity $F \in C^1(\mathbb{R})$ obeys the estimates

$$|F(u)| \lesssim |u|^\kappa, \quad |F'(u)| \lesssim |u|^{\kappa-1} \tag{1-2}$$

for some $\kappa > 1$. For $n \geq 3$ and $\kappa > \kappa_3$ (see below) we further assume that $\kappa \in \mathbb{N}$ and $F(u) = \pm u^\kappa$.

The main objective of this paper is to find the minimal number γ for which (1-1) under assumption (1-2) possesses a unique local solution

$$u \in C([0,T], \dot{H}^\gamma(\mathbb{R}^n)) \cap L^s((0,T); L^q(\mathbb{R}^n))$$

for certain s, q with $\min\{s,q\} \geq \kappa$. Then $F(u) \in L^{s/\kappa}((0,T); L^{q/\kappa}(\mathbb{R}^n)) \subseteq L^1_{\text{loc}}((0,T) \times \mathbb{R}^n)$ holds, and (1-1) is understood in distributions.

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We first introduce notation used throughout this paper. Set

$$\mu_* = \frac{(m+2)n+2}{2}, \qquad \kappa_* = \frac{\mu_*+2}{\mu_*-2} = \frac{(m+2)n+6}{(m+2)n-2},$$

$$\kappa_0 = 1 + \frac{6\mu_* + m}{\mu_*(m+2)n} \quad \text{if } n \ge 3 \text{ or } n = 2,\ m \ge 3,$$

$$\kappa_1 = \begin{cases} 2 & \text{if } n = 2,\ m = 1; \\ \dfrac{(\mu_*+2)(m+2)(n-1)+8}{(\mu_*-2)(m+2)(n-1)+8} & \text{if } n \ge 3 \text{ or } n = 2,\ m \ge 2; \end{cases}$$

$$\kappa_2 = \frac{\mu_*(\mu_*+2)(n-1)-2(n+1)}{\mu_*(\mu_*-2)(n-1)-2(n+1)},$$

$$\kappa_3 = \frac{\mu_*-m}{\mu_*-m-4} \quad \text{if } n \ge 3.$$

Note that μ_* is the homogeneous dimension of the degenerate differential operator $\partial_t^2 - t^m\Delta$ and κ_* is the power κ for which the equation $\partial_t^2 u - t^m \Delta u = \pm |u|^{\kappa-1}u$ is conformally invariant.

Note further that $1 < \kappa_0 < \kappa_1 < \kappa_* < \kappa_2 < \kappa_3$ whenever it applies.

Next we state the main results of this paper.

Theorem 1.1. *Let $n \ge 2$ and F be as above. Suppose further $\kappa > \kappa_1$ and $(\varphi, \psi) \in \dot H^\gamma(\mathbb{R}^n) \times \dot H^{\gamma-2/(m+2)}(\mathbb{R}^n)$, where*

$$\text{(1-3)} \quad \gamma = \gamma(\kappa, m, n) = \begin{cases} \frac14(n+1) - \dfrac{n+1}{\mu_*(\kappa-1)} - \dfrac{m}{2\mu_*(m+2)} & \text{if } \kappa_1 < \kappa \le \kappa_*, \\ \frac12 n - \dfrac{4}{(m+2)(\kappa-1)} & \text{if } \kappa \ge \kappa_*. \end{cases}$$

Then problem (1-1) *possesses a unique solution*

$$u \in C([0,T]; \dot H^\gamma(\mathbb{R}^n)) \cap L^s((0,T); L^q(\mathbb{R}^n))$$

for some $T > 0$, where

$$\text{(1-4)} \quad \|u\|_{C([0,T];\dot H^\gamma(\mathbb{R}^n))} + \|u\|_{L^s((0,T);L^q(\mathbb{R}^n))} \lesssim \|\varphi\|_{\dot H^\gamma(\mathbb{R}^n)} + \|\psi\|_{\dot H^{\gamma-2/(m+2)}(\mathbb{R}^n)}$$

and $q = \mu_*(\kappa-1)/2$,

$$\frac1s = \begin{cases} \frac14(m+2)(n-1)\left(\frac12 - \frac1q\right) + \dfrac{m}{4\mu_*} & \text{if } \kappa_1 < \kappa \le \kappa_*, \\ \frac1q & \text{if } \kappa \ge \kappa_*. \end{cases}$$

Remark 1.2. As a byproduct of the proof of Theorem 1.1, we see that problem (1-1) admits a unique global solution $u \in C([0,\infty); \dot H^\gamma(\mathbb{R}^n)) \cap L^\infty((0,\infty); \dot H^\gamma(\mathbb{R}^n)) \cap L^{\mu_*(\kappa-1)/2}(\mathbb{R}_+ \times \mathbb{R}^n)$ in case $n \ge 2$, $\kappa \ge \kappa_*$ if $(\varphi, \psi) = \varepsilon(u_0, u_1)$, $(u_0, u_1) \in$

$\dot{H}^{\gamma}(\mathbb{R}^n)\times\dot{H}^{\gamma-2/(m+2)}(\mathbb{R}^n)$, and $\varepsilon>0$ is small (see Section 5.1.2 and Section 5.1.3 in the proof of Theorem 1.1 below). With a different argument, the global result $u\in L^{\mu_*(\kappa-1)/2}(\mathbb{R}_+\times\mathbb{R}^n)$ for problem (1-1) was obtained in [He et al. 2017].

Remark 1.3. For $\gamma<n/2-4/((m+2)(\kappa-1))$, one obtains ill-posedness for problem (1-1) by scaling. More specifically, if $u=u(t,x)$ solves the Cauchy problem (1-1), where $F(u)=\pm|u|^{\kappa-1}u$, then

$$u_\varepsilon(t,x)=\varepsilon^{-2/(\kappa-1)}u(\varepsilon^{-1}t,\varepsilon^{-(m+2)/2}x),\quad \varepsilon>0,$$

also solves (1-1), with $u_\varepsilon(0,x)=\varphi_\varepsilon(x)$, $\partial_t u_\varepsilon(0,x)=\psi_\varepsilon(x)$ for some resulting φ_ε, ψ_ε. Observe that

$$\frac{\|\varphi_\varepsilon\|_{\dot{H}^{\gamma}(\mathbb{R}^n)}}{\|\varphi\|_{\dot{H}^{\gamma}(\mathbb{R}^n)}}=\frac{\|\psi_\varepsilon\|_{\dot{H}^{\gamma}(\mathbb{R}^n)}}{\|\psi\|_{\dot{H}^{\gamma}(\mathbb{R}^n)}}=\varepsilon^{\frac{1}{2}(m+2)\left(\frac{1}{2}n-\gamma\right)-\frac{2}{\kappa-1}},$$

and $\frac{1}{2}(m+2)(\frac{n}{2}-\gamma)-2/(\kappa-1)>0$ for $\gamma<n/2-4/((m+2)(\kappa-1))$. Hence, $\gamma<n/2-4/((m+2)(\kappa-1))$ implies that both the norm of the data $(\varphi_\varepsilon,\psi_\varepsilon)$ and the lifespan $T_\varepsilon=\varepsilon T$ of the solution u_ε go to zero as $\varepsilon\to 0$, where T is the lifespan of the solution u.

In case $\kappa_*\le\kappa<\kappa_2$, as a supplement to Theorem 1.1, we consider the local existence and uniqueness of solutions u of problem (1-1) in the space $C([0,T];\dot{H}^{\gamma}(\mathbb{R}^n))\cap L^s((0,T);L^q(\mathbb{R}^n))$ for certain $s\ne q$.

Theorem 1.4. *Let $n\ge 2$, F be as above, $\gamma=\gamma(\kappa,m,n)$ be as in Theorem 1.1, and suppose that $\kappa_*\le\kappa<\kappa_2$. Then the unique solution u of problem* (1-1) *also belongs to the space $L^s((0,T);L^q(\mathbb{R}^n))$, where*

$$\frac{1}{q}=\frac{1}{(m+2)(n-1)}\Big(\frac{8}{\kappa-1}-\frac{m}{\mu_*}\Big)-\frac{n-1}{2(n+1)}$$

and

$$\frac{1}{s}=\frac{(m+2)(n-1)}{4}\Big(\frac{1}{2}-\frac{1}{q}\Big)+\frac{m}{4\mu_*}.$$

Moreover, estimate (1-4) *is satisfied.*

If $n\ge 3$ or $n=2$, $m\ge 3$, then we find a number $\gamma(\kappa,m,n)$ also for certain κ in the range $\kappa_0\le\kappa<\kappa_1$.

Theorem 1.5. *Let $n\ge 3$ or $n=2$ with $m\ge 3$. Let F be as above and $\kappa_0\le\kappa<\kappa_1$. In addition, let the exponent $\gamma=\gamma(\kappa,m,n)$ in* (1-1) *be given by*

(1-5)
$$\gamma(\kappa,m,n)=\frac{n+1}{4}-\frac{n+1}{4\mu_*(m+2)}\cdot\frac{\mu_*(m+2)(n-1)+12\mu_*+2m}{2n\kappa-(n+1)}-\frac{m}{2\mu_*(m+2)}.$$

Then problem (1-1) *possesses a unique solution* $u \in C([0,T]; \dot H^{\gamma}(\mathbb{R}^n)) \cap L^s((0,T); L^q(\mathbb{R}^n))$ *for some* $T > 0$, *where*

$$\frac{1}{q} = \frac{1}{2n\kappa-(n+1)}\Bigl(\tfrac{1}{2}(n-1) + \frac{6}{m+2} + \frac{m}{\mu_*(m+2)}\Bigr)$$

and

$$\frac{1}{s} = \tfrac{1}{4}(m+2)(n-1)\Bigl(\frac{1}{2}-\frac{1}{q}\Bigr) + \frac{m}{4\mu_*}.$$

Moreover, *estimate* (1-4) *is satisfied.*

Remark 1.6. Other than for the wave equation when $m = 0$ (see also Remark 1.8 below), here γ can be negative in certain situations. In fact, $\gamma(\kappa, m, n) < 0$ holds in the following cases:

(i) $\kappa_1 < \kappa < \frac{35}{17}$ $(< \kappa_*)$ if $n = 2$, $m = 1$ and $\kappa_1 < \kappa < \frac{13}{7}$ $(< \kappa_*)$ if $n = 2$, $m = 2$ (see Theorem 1.1);

(ii) $$\kappa_0 < \kappa < \frac{\mu_*(\mu_*+2)(n+1)}{\mu_*(\mu_*-1)(n+1)-mn} \quad (\le \kappa_1)$$

if $n \ge 3$ or $n = 2$, $m \ge 3$ (see Theorem 1.5).

Remark 1.7. For initial data (φ, ψ) belonging to $H^{\gamma}(\mathbb{R}^n) \times H^{\gamma-2/(m+2)}(\mathbb{R}^n)$, where $\gamma \ge \gamma(\kappa, m, n)$, Theorems 1.1, 1.4, and 1.5 remain valid.

Remark 1.8. For $m = 0$, (1-1) becomes

$$\partial_t^2 u - \Delta u = F(u) \quad \text{in } (0,T)\times\mathbb{R}^n,$$
$$u(0,\cdot) = \varphi \in \dot H^{\gamma}(\mathbb{R}^n), \quad \partial_t u(0,\cdot) = \psi \in \dot H^{\gamma-1}(\mathbb{R}^n),$$

while the exponents $\kappa_*, \kappa_0, \kappa_1, \kappa_2$, and κ_3 are

$$\kappa_* = \frac{n+3}{n-1}, \quad \kappa_2 = \frac{(n+1)^2-6}{(n-1)^2-2}, \quad \kappa_1 = \frac{(n+1)^2}{(n-1)^2+4} \quad \text{if } n \ge 3,$$
$$\kappa_0 = \frac{n+3}{n}, \quad \kappa_3 = \frac{n+1}{n-3} \quad \text{if } n \ge 4.$$

For $n \ge 3$, γ defined in (1-3) equals

$$\gamma(\kappa,0,n) = \begin{cases} \frac{1}{4}(n+1) - 1/(\kappa-1) & \text{if } \kappa_1 < \kappa \le \kappa_*, \\ \frac{1}{2}n - 2/(\kappa-1) & \text{if } \kappa \ge \kappa_*, \end{cases} \tag{1-6}$$

whereas, for $n \ge 4$, γ defined in (1-5) equals

$$\gamma(\kappa,0,n) = \tfrac{1}{4}(n+1) - \tfrac{1}{4}(n+1)(n+5)\,\frac{1}{2n\kappa-(n+1)}. \tag{1-7}$$

Note that the numbers in (1-6) and (1-7) are exactly those in [Lindblad and Sogge 1995, (2.1) and (2.5)]. In that paper, the local existence problem for minimal regularity solutions of the semilinear wave equation was systematically studied.

The results were achieved by establishing Strichartz-type estimates for the linear wave operator $\partial_t^2 - \Delta$. Under certain restrictions on the nonlinearity $F(u, \nabla u)$, for the more general semilinear wave equation

$$\partial_t^2 u - \Delta u = F(u, \nabla u), \quad u(0, x) = \varphi(x), \quad \partial_t u(0, x) = \psi(x),$$

many remarkable results on the ill-posedness or well-posedness problem on the local existence of low regularity solutions have been obtained; see [Kapitanski 1994; Lindblad 1998; Lindblad and Sogge 1995; Ponce and Sideris 1993; Smith and Tataru 2005; Struwe 1992].

Remark 1.9. There are some essential differences between degenerate hyperbolic equations and strictly hyperbolic equations. Amongst others, the symmetry group is smaller (see [Lupo and Payne 2005]) and there is a loss of regularity for the linear Cauchy problem (see, e.g., [Dreher and Witt 2005; Taniguchi and Tozaki 1980]). Therefore, when compared to the semilinear wave equation, a more delicate analysis is required when one studies minimal regularity results for the semilinear generalized Tricomi equation in the degenerate hyperbolic region.

The Tricomi equation (i.e., (1-1) for $n = 1$, $m = 1$) was first studied by Tricomi [1923], who initiated work on boundary value problems for linear partial differential operators of mixed elliptic-hyperbolic type. So far, these equations have been extensively studied in bounded domains under suitable boundary conditions and several applications to transonic flow problems were given (see [Bers 1958; Germain 1954; Tricomi 1923; Morawetz 2004]). Conservation laws for equations of mixed type were derived by Lupo and Payne [2003; 2005]. In [Ruan et al. 2015b], we established the local solvability for low regularity solutions of the semilinear equation $\partial_t^2 u - t^m \Delta u = F(u)$, where $n \geq 2$, $m \in \mathbb{N}$ is odd, in the domain $(-T, T) \times \mathbb{R}^n$ for some $T > 0$. In [Barros-Neto and Gelfand 1999; 2002; Yagdjian 2004; 2015], fundamental solutions for the linear Tricomi operator and the linear generalized Tricomi operator have been explicitly computed. In the case $n = 2$ and $m = 1$, Beals [1992] obtained the local existence of the solution u of the equation $\partial_t^2 u - t\Delta u = F(u)$ with initial data of H^s-regularity, where $s > \frac{1}{2}n$. For the equation $\partial_t^2 u - t^m \Delta u = a(t)F(u)$, where $n \geq 2$, $m \in \mathbb{N}$ is even, and both a and F are of power type, Yadgjian [2006] obtained global existence and uniqueness for small data solutions provided the solution v of the linear problem $\partial_t^2 v - t^m \Delta v = 0$ fulfills $t^\beta v \in C([0, \infty); L^q(\mathbb{R}^n))$ for certain β, q depending on n, m, and the powers occurring in a and F.

In [Ruan et al. 2014; 2015a], for the semilinear generalized Tricomi equation $\partial_t^2 u - t^m \Delta u = F(u)$ with initial data of a special structure, i.e., homogeneous of degree 0 or piecewise smooth along a hyperplane, we obtained local existence and uniqueness via establishing L^∞ estimates on the solutions v of the linear equation $\partial_t^2 v - t^m \Delta v = g$. Note that when the nonlinear term $F(u)$ is of power type, for higher and higher powers of κ, these L^∞ estimates are basically required to guarantee existence. In this paper, where the initial data in $\dot{H}^\gamma(\mathbb{R}^n)$ is of no special structure and γ is minimal to guarantee local well-posedness of problem (1-1), the arguments of [Ruan et al. 2014; 2015a] fail. Inspired by the methods in [Lindblad and Sogge 1995], however, we are able to overcome the technical difficulties related to degeneracy and low regularity and eventually obtain the local well-posedness of problem (1-1).

We first study the linear problem

$$\begin{aligned} &\partial_t^2 u - t^m \Delta u = f(t,x) \quad \text{in } (0,T)\times\mathbb{R}^n, \\ &u(0,\cdot) = \varphi(x), \quad \partial_t u(0,\cdot) = \psi(x), \end{aligned} \tag{1-8}$$

and establish Strichartz-type estimates of the form

$$\begin{aligned} \|u\|_{C_t^0 \dot{H}_x^\gamma(S_T)} + \|u\|_{L_t^s L_x^q(S_T)} \\ \le C\big(\|\varphi\|_{\dot{H}^\gamma(\mathbb{R}^n)} + \|\psi\|_{\dot{H}^{\gamma-2/(m+2)}(\mathbb{R}^n)} + \|f\|_{L_t^r L_x^p(S_T)}\big) \end{aligned} \tag{1-9}$$

for certain s, q, r, p (see below) and some constant $C = C(T,\gamma,s,q,r,p) > 0$, where $S_T = (0,T)\times\mathbb{R}^n$. Note that, by scaling, a necessary condition for this estimate in case $T=\infty$ to hold is

$$\tfrac{1}{2}(m+2)n\Big(\frac{1}{p} - \frac{1}{q}\Big) + \frac{1}{r} - \frac{1}{s} = 2. \tag{1-10}$$

In doing so, in Section 2, we introduce certain Fourier integral operators W $(= W^0)$ and W^α for $\alpha\in\mathbb{C}$. These operators depend on a parameter $\mu\ge 2$, introduced in (2-15), which plays an auxiliary role for the linear problems and agrees with the homogeneous dimension μ_* when applied to the semilinear problems. Along with the operators W and W^α we also consider their dyadic parts W_j and W_j^α, respectively, resulting from a dyadic decomposition of frequency space. Continuity of the operators W_j and W_j^α between function spaces which holds uniformly in j ultimately provides linear estimates on the solutions u of (1-8).

In Section 3, we prove boundedness of the operators W_j^α from $L_t^r L_x^p(\mathbb{R}_+^{1+n})$ to $L_t^{r'} L_x^{p'}(\mathbb{R}_+^{1+n})$ (see Theorem 3.1) and from $L_t^r L_x^p(\mathbb{R}_+^{1+n})$ to $L_t^\infty L_x^2(\mathbb{R}_+^{1+n})$ (see Theorem 3.4), where μ has to satisfy the lower bound $\mu\ge\max\{2, m/2\}$. Combining Theorem 3.1 and Stein's analytic interpolation theorem, we show boundedness of the operators W_j^α from $L^q(\mathbb{R}_+^{1+n})$ to $L^{p_0}(\mathbb{R}_+^{1+n})$, where $q_0 \le q \le \infty$ (see Theorem 3.6). Through an additional dyadic decomposition now with respect to the time variable t, using Theorems 3.1 and 3.6 together with interpolation, we prove boundedness of the operators W_j from $L_t^r L_x^p((0,T)\times\mathbb{R}^n)$ to $L_t^s L_x^q((0,T)\times\mathbb{R}^n)$ for any $T>0$ (see Theorems 3.7 and 3.8), where μ has to satisfy the new lower bounds $\mu\ge\mu_*$ (Theorem 3.7) and $\mu\ge\max\{2, mn/2\}$ (Theorem 3.8), respectively.

In the sequel, we shall use the following notation:

$$\frac{1}{p_0}=\frac{1}{2}+\frac{2\mu-m}{\mu(2\mu_*-m)},\quad \frac{1}{p_1}=\frac{1}{2}+\frac{2\mu-m}{\mu(m+2)(n-1)},\quad \frac{1}{p_2}=\frac{2}{p_0}-\frac{1}{p_1}.$$

Note that

$$1<p_1\le p_0\le p_2\le 2 \quad \text{if } n\ge 3 \text{ or } n=2,\, m\ge 2,$$

while $1\le p_1$ in case of $n=2$ and $m=1$ requires $\mu=2$ (and then $p_1=1$). For $1\le p\le 2$, p' denotes the conjugate exponent of p defined by $1/p+1/p'=1$. Further, q_ℓ denotes p'_ℓ for $\ell=0,1,2$, while q_0^* equals q_0 when $\mu=\mu_*$ (see Remark 4.2). We often abbreviate function spaces $C_t^0\dot{H}_x^\gamma(S_T)=C([0,T];\dot{H}^\gamma(\mathbb{R}^n))$ and $L_t^rL_x^p(S_T)=L^r((0,T);L^p(\mathbb{R}^n))$, and $A\lesssim B$ means that $A\le CB$ holds for some generic constant $C>0$.

The paper is organized as follows: In Section 2, we define a class of Fourier integral operators associated with the linear generalized Tricomi operator $\partial_t^2-t^m\Delta$ in $\mathbb{R}_+\times\mathbb{R}^n$. Then, in Section 3, we establish a series of mixed-norm space-time estimates for those Fourier integral operators. These estimates are applied, in Section 4, to obtain Strichartz-type estimates for the solutions of the linear generalized Tricomi equation which in turn, in Section 5, allow us to prove the local existence and uniqueness results for problem (1-1).

2. Some preliminaries

In this section, we first recall an explicit formula for the solution of the linear generalized Tricomi equation obtained in [Taniguchi and Tozaki 1980] and then apply it to define a class of Fourier integral operators which will play a key role in proving our main results.

Consider the Cauchy problem of the linear generalized Tricomi equation

$$\partial_t^2u-t^m\Delta u=f(t,x)\quad \text{in } \mathbb{R}_+\times\mathbb{R}^n,\quad u(0,\cdot)=\varphi,\quad \partial_tu(0,\cdot)=\psi. \tag{2-1}$$

Its solution u can be written as $u=v+w$, where v solves the Cauchy problem of the homogeneous equation

$$\partial_t^2v-t^m\Delta v=0\quad \text{in } \mathbb{R}_+\times\mathbb{R}^n,\quad v(0,\cdot)=\varphi,\quad \partial_tv(0,\cdot)=\psi, \tag{2-2}$$

and w solves the inhomogeneous equation with zero initial data:

$$\partial_t^2w-t^m\Delta w=f(t,x)\quad \text{in } \mathbb{R}_+\times\mathbb{R}^n,\quad w(0,\cdot)=\partial_tw(0,\cdot)=0. \tag{2-3}$$

Recall that (see [Taniguchi and Tozaki 1980] or [Yagdjian 2006]) the solutions v and w of problems (2-2) and (2-3) can be expressed as

$$v(t,x)=V_0(t,D_x)\varphi(x)+V_1(t,D_x)\psi(x)$$

and

$$w(t,x)=\int_0^t \big(V_1(t,D_x)V_0(\tau,D_x)-V_0(t,D_x)V_1(\tau,D_x)\big)f(\tau,x)\,d\tau, \tag{2-4}$$

where the symbols $V_j(t,\xi)$ ($j=0,1$) of the Fourier integral operators $V_j(t,D_x)$ are

$$\begin{aligned} V_0(t,\xi)&=e^{-z/2}\,\Phi\Big(\frac{m}{2(m+2)},\frac{m}{m+2};z\Big),\\ V_1(t,\xi)&=te^{-z/2}\,\Phi\Big(\frac{m+4}{2(m+2)},\frac{m+4}{m+2};z\Big),\end{aligned} \tag{2-5}$$

with $z=2i\phi(t)|\xi|$ and $\phi(t)=(2/(m+2))t^{(m+2)/2}$. Here, $\Phi(a,c;z)$ is the confluent hypergeometric function which is an analytic function of z. Recall (see [Erdélyi et al. 1953, p. 254]) that

$$\frac{d^n}{dz^n}\Phi(a,c;z)=\frac{(a)_n}{(c)_n}\Phi(a+n,c+n;z), \tag{2-6}$$

where $(a)_0=1$, $(a)_n=a(a+1)\dots(a+n-1)$. In addition, for $0<\arg(z)<\pi$, one has that (see [Yagdjian 2006, (3.5)–(3.7)])

$$e^{-z/2}\,\Phi(a,c;z)=\frac{\Gamma(c)}{\Gamma(a)}\,e^{z/2}H_+(a,c;z)+\frac{\Gamma(c)}{\Gamma(c-a)}\,e^{-z/2}H_-(a,c;z), \tag{2-7}$$

where

$$\begin{aligned} H_+(a,c;z)&\\ &=\frac{e^{-i\pi(c-a)}}{e^{i\pi(c-a)}-e^{-i\pi(c-a)}}\frac{1}{\Gamma(c-a)}\,z^{a-c}\int_\infty^{(0+)}e^{-\theta}\theta^{c-a-1}\Big(1-\frac{\theta}{z}\Big)^{a-1}d\theta,\\ H_-(a,c;z)&=\frac{1}{e^{i\pi a}-e^{-i\pi a}}\frac{1}{\Gamma(a)}\,z^{-a}\int_\infty^{(0+)}e^{-\theta}\theta^{a-1}\Big(1+\frac{\theta}{z}\Big)^{c-a-1}d\theta.\end{aligned}$$

Moreover, it holds that

$$\begin{aligned} \big|\partial_\xi^\beta\big(H_+(a,c;2i\phi(t)|\xi|)\big)\big|&\lesssim(\phi(t)|\xi|)^{a-c}(1+|\xi|)^{-|\beta|}\quad\text{if }\phi(t)|\xi|\ge 1,\\ \big|\partial_\xi^\beta\big(H_-(a,c;2i\phi(t)|\xi|)\big)\big|&\lesssim(\phi(t)|\xi|)^{-a}(1+|\xi|)^{-|\beta|}\quad\text{if }\phi(t)|\xi|\ge 1.\end{aligned} \tag{2-8}$$

Choose $\eta\in C_c^\infty(\mathbb{R}_+)$ such that $0\le\eta\le 1$ with $\eta(r)=1$ if $r\le 1$ and $\eta(r)=0$ if $r\ge 2$. Then from (2-5) and (2-7), we can write

$$\begin{aligned} &V_0(t,D_x)\varphi(x)\\ &\quad=\int_{\mathbb{R}^n}e^{i(x\cdot\xi-\phi(t)|\xi|)}b_1(t,\xi)\hat{\varphi}(\xi)\,đ\xi+\int_{\mathbb{R}^n}e^{i(x\cdot\xi+\phi(t)|\xi|)}b_2(t,\xi)\hat{\varphi}(\xi)\,đ\xi\end{aligned} \tag{2-9}$$

and

$$V_1(t, D_x)\psi(x) = \int_{\mathbb{R}^n} e^{i(x\cdot\xi-\phi(t)|\xi|)} b_3(t,\xi)\hat{\psi}(\xi)\, đ\xi + \int_{\mathbb{R}^n} e^{i(x\cdot\xi+\phi(t)|\xi|)} b_4(t,\xi)\hat{\psi}(\xi)\, đ\xi, \tag{2-10}$$

where

$$b_1(t,\xi) = \eta(\phi(t)|\xi|)\Phi\Big(\frac{m}{2(m+2)}, \frac{m}{m+2}; z\Big) + \big(1-\eta(\phi(t)|\xi|)\big)H_-\Big(\frac{m}{2(m+2)}, \frac{m}{m+2}; z\Big),$$

$$b_2(t,\xi) = \big(1-\eta(\phi(t)|\xi|)\big)H_+\Big(\frac{m}{2(m+2)}, \frac{m}{m+2}; z\Big),$$

$$b_3(t,\xi) = t\eta(\phi(t)|\xi|)\Phi\Big(\frac{m+4}{2(m+2)}, \frac{m+4}{m+2}; z\Big) + t\big(1-\eta(\phi(t)|\xi|)\big)H_-\Big(\frac{m+4}{2(m+2)}, \frac{m+4}{m+2}; z\Big),$$

$$b_4(t,\xi) = t\big(1-\eta(\phi(t)|\xi|)\big)H_+\Big(\frac{m+4}{2(m+2)}, \frac{m+4}{m+2}; z\Big),$$

and $đ\xi = (2\pi)^{-n}\, d\xi$. We can also write

$$\begin{aligned}\int_0^t V_0(t, D_x)V_1(\tau, D_x) f(\tau, x)\, d\tau &= \int_0^t \int_{\mathbb{R}^n} e^{i(x\cdot\xi+(\phi(t)+\phi(\tau))|\xi|)} b_2(t,\xi)b_4(\tau,\xi)\hat{f}(\tau,\xi)\, đ\xi\, d\tau \\ &\quad + \int_0^t \int_{\mathbb{R}^n} e^{i(x\cdot\xi+(\phi(t)-\phi(\tau))|\xi|)} b_2(t,\xi)b_3(\tau,\xi)\hat{f}(\tau,\xi)\, đ\xi\, d\tau \\ &\quad + \int_0^t \int_{\mathbb{R}^n} e^{i(x\cdot\xi-(\phi(t)+\phi(\tau))|\xi|)} b_1(t,\xi)b_3(\tau,\xi)\hat{f}(\tau,\xi)\, đ\xi\, d\tau \\ &\quad + \int_0^t \int_{\mathbb{R}^n} e^{i(x\cdot\xi-(\phi(t)-\phi(\tau))|\xi|)} b_1(t,\xi)b_4(\tau,\xi)\hat{f}(\tau,\xi)\, đ\xi\, d\tau\end{aligned} \tag{2-11}$$

and

$$\begin{aligned}\int_0^t V_1(t, D_x)V_0(\tau, D_x) f(\tau, x)\, d\tau &= \int_0^t \int_{\mathbb{R}^n} e^{i(x\cdot\xi+(\phi(t)+\phi(\tau))|\xi|)} b_4(t,\xi)b_2(\tau,\xi)\hat{f}(\tau,\xi)\, đ\xi\, d\tau \\ &\quad + \int_0^t \int_{\mathbb{R}^n} e^{i(x\cdot\xi-(\phi(t)-\phi(\tau))|\xi|)} b_3(t,\xi)b_2(\tau,\xi)\hat{f}(\tau,\xi)\, đ\xi\, d\tau \\ &\quad + \int_0^t \int_{\mathbb{R}^n} e^{i(x\cdot\xi-(\phi(t)+\phi(\tau))|\xi|)} b_3(t,\xi)b_1(\tau,\xi)\hat{f}(\tau,\xi)\, đ\xi\, d\tau\end{aligned} \tag{2-12}$$

$$+\int_0^t\int_{\mathbb{R}^n} e^{i(x\cdot\xi+(\phi(t)-\phi(\tau))|\xi|)}b_4(t,\xi)b_1(\tau,\xi)\hat{f}(\tau,\xi)\,đ\xi\,d\tau,$$

where $\hat{f}(\tau,\xi)$ is the Fourier transform of $f(\tau,x)$ with respect to the variable x.

In view of the analyticity of $\Phi(a,c;z)$ with respect to the variable z, identity (2-6), and estimates (2-8), we have that, for $(t,\xi)\in\mathbb{R}_+^{1+n}$,

$$|\partial_\xi^\beta b_\ell(t,\xi)|\lesssim(1+\phi(t)|\xi|)^{-\frac{m}{2(m+2)}}|\xi|^{-|\beta|},\quad \ell=1,2, \tag{2-13}$$

and

$$|\partial_\xi^\beta b_\ell(t,\xi)|\lesssim t(1+\phi(t)|\xi|)^{-\frac{m+4}{2(m+2)}}|\xi|^{-|\beta|},\quad \ell=3,4. \tag{2-14}$$

Thus, for $\ell=1,2$, $k=3,4$, $\mu\geq 2$, $t,\tau>0$, and $\xi\in\mathbb{R}^n$, one has from (2-13) and (2-14) that

$$\begin{aligned}\big|\partial_\xi^\beta\big(b_k(t,\xi)b_\ell(\tau,\xi)\big)\big| &\lesssim t(1+\phi(t)|\xi|)^{-\frac{m+4}{2(m+2)}}(1+\phi(\tau)|\xi|)^{-\frac{m}{2(m+2)}}|\xi|^{-|\beta|}\\ &\lesssim(1+\phi(t)|\xi|)^{-\frac{m}{2(m+2)}}(1+\phi(\tau)|\xi|)^{-\frac{m}{2(m+2)}}|\xi|^{-\frac{2}{m+2}-|\beta|}\\ &\lesssim(1+|\phi(t)-\phi(\tau)||\xi|)^{-\frac{m}{\mu(m+2)}}|\xi|^{-\frac{2}{m+2}-|\beta|}.\end{aligned} \tag{2-15}$$

Furthermore, estimates (2-13)–(2-15) yield that, for $\ell=1,2$, $k=3,4$, or $\ell=3,4$, $k=1,2$ and for $\mu\geq 2$, $t,s>0$, and $\xi\in\mathbb{R}^n$, one has

$$\left|\partial_\xi^\beta\left(\int_t^\infty\overline{b_\ell(\tau,\xi)b_k(t,\xi)}\,\partial_\tau(b_\ell(\tau,\xi)b_k(s,\xi))\,d\tau\right)\right| \lesssim(1+|\phi(t)-\phi(s)||\xi|)^{-\frac{m}{\mu(m+2)}}|\xi|^{-\frac{4}{m+2}-|\beta|} \tag{2-16}$$

and

$$\left|\partial_\xi^\beta\left(\int_s^\infty\overline{b_\ell(\tau,\xi)b_k(t,\xi)}\,\partial_\tau(b_\ell(\tau,\xi)b_k(s,\xi))\,d\tau\right)\right| \lesssim(1+|\phi(t)-\phi(s)||\xi|)^{-\frac{m}{\mu(m+2)}}|\xi|^{-\frac{4}{m+2}-|\beta|}. \tag{2-17}$$

In order to study the function w in (2-4), in view of (2-11), (2-12), and (2-15)–(2-17), it suffices to consider, for a given $\mu\geq 2$, the Fourier integral operator W:

$$Wf(t,x)=\int_0^t\int_{\mathbb{R}^n} e^{i(x\cdot\xi+(\phi(t)-\phi(s))|\xi|)}b(t,s,\xi)\hat{f}(s,\xi)\,đ\xi\,ds, \tag{2-18}$$

where $b\in C^\infty(\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}^n)$ satisfies the following:

(i) for $t,s>0$ and $\xi\in\mathbb{R}^n$,

$$|\partial_\xi^\beta b(t,s,\xi)|\lesssim(1+|\phi(t)-\phi(s)||\xi|)^{-\frac{m}{\mu(m+2)}}|\xi|^{-\frac{2}{m+2}-|\beta|}; \tag{2-19}$$

(ii) for $t, s > 0$ and $\xi \in \mathbb{R}^n$,

$$\left|\partial_\xi^\beta\left(\int_t^\infty \overline{b(\tau,t,\xi)}\,\partial_\tau b(\tau,s,\xi)\,d\tau\right)\right| \lesssim (1+|\phi(t)-\phi(s)||\xi|)^{-\frac{m}{\mu(m+2)}}|\xi|^{-\frac{4}{m+2}-|\beta|} \tag{2-20}$$

and

$$\left|\partial_\xi^\beta\left(\int_s^\infty \overline{b(\tau,t,\xi)}\,\partial_\tau b(\tau,s,\xi)\,d\tau\right)\right| \lesssim (1+|\phi(t)-\phi(s)||\xi|)^{-\frac{m}{\mu(m+2)}}|\xi|^{-\frac{4}{m+2}-|\beta|}. \tag{2-21}$$

Let $\Theta \in C_c^\infty(\mathbb{R}_+)$ satisfy $\operatorname{supp}\Theta \subseteq \left[\frac{1}{2}, 2\right]$ and

$$\sum_{j=-\infty}^{\infty} \Theta(t/2^j) = 1 \quad \text{for } t > 0.$$

Then, as in [Lindblad and Sogge 1995], for $j \in \mathbb{Z}$ and $\alpha \in \mathbb{C}$, we define dyadic operators W_j and W_j^α as

$$W_j f(t,x) = \int_0^t \int_{\mathbb{R}^n} e^{i(x\cdot\xi+(\phi(t)-\phi(s))|\xi|)} b_j(t,s,\xi)\hat{f}(s,\xi)\,đ\xi\,ds$$

and

$$W_j^\alpha f(t,x) = \int_0^t \int_{\mathbb{R}^n} e^{i(x\cdot\xi+(\phi(t)-\phi(s))|\xi|)} b_j(t,s,\xi)\hat{f}(s,\xi)\,\frac{đ\xi}{|\xi|^\alpha}\,ds, \tag{2-22}$$

where $b_j(t,s,\xi) = \Theta(|\xi|/2^j)b(t,s,\xi)$. Here, $b \in C^\infty(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^n)$ satisfies estimates (2-19)–(2-21).

Littlewood–Paley theory gives us a relationship between Wf and $W_j f$ $(= W_j^0 f)$, which will play an important role in our arguments in Section 4.

Proposition 2.1. *Let* $n \geq 2$. *For* $1 < p \leq 2$, $1 \leq r \leq 2$, $2 \leq q < \infty$, *and* $2 \leq s \leq \infty$, *let*

$$\|W_j f\|_{L_t^s L_x^q} \lesssim \|f\|_{L_t^r L_x^p} \tag{2-23}$$

hold uniformly in j. *Then*

$$\|Wf\|_{L_t^s L_x^q} \lesssim \|f\|_{L_t^r L_x^p}.$$

Proof. This is actually an application of [Lindblad and Sogge 1995, Lemma 3.8]. For the sake of completeness, we give the proof here. By Littlewood–Paley theory (see, e.g., [Stein 1970]), for any $1 < \rho < \infty$,

$$\|Wf(t,\cdot)\|_{L^\rho(\mathbb{R}^n)} \lesssim \left\|\left(\sum_{j=-\infty}^{\infty} |W_j f(t,\cdot)|^2\right)^{1/2}\right\|_{L^\rho(\mathbb{R}^n)} \lesssim \|Wf(t,\cdot)\|_{L^\rho(\mathbb{R}^n)}.$$

Together with the Minkowski inequality, this yields

$$\|Wf\|_{L_t^s L_x^q} \lesssim \Big(\sum_{j=-\infty}^{\infty} \|W_j f\|^2_{L_t^s L_x^q} \Big)^{1/2} \tag{2-24}$$

and

$$\Big(\sum_{j=-\infty}^{\infty} \|W_j f\|^2_{L_t^r L_x^p} \Big)^{1/2} \lesssim \|Wf\|_{L_t^r L_x^p}. \tag{2-25}$$

Notice that

$$f = \sum_{k=-\infty}^{\infty} f_k,$$

where $f_k(\tau, x) = \Theta(\tau/2^k) f(\tau, x)$. Therefore, for some $M_0 \in \mathbb{N}$,

$$\begin{aligned}
&\|Wf\|^2_{L_t^s L_x^q} \\
&\quad\lesssim \sum_{j=-\infty}^{\infty} \|W_j f\|^2_{L_t^s L_x^q} && \text{(by (2-24))} \\
&\quad= \sum_{j=-\infty}^{\infty} \Big\| W_j \Big(\sum_{|j-k|\le M_0} f_k \Big) \Big\|^2_{L_t^s L_x^q} && \text{(due to the compact support of } \Theta) \\
&\quad\lesssim \sum_{j=-\infty}^{\infty} \Big(\sum_{|j-k|\le M_0} \|W_j f_k\|_{L_t^s L_x^q} \Big)^2 && \text{(by Minkowski inequality)} \\
&\quad\lesssim \sum_{j=-\infty}^{\infty} \sum_{|j-k|\le M_0} \|f_k\|^2_{L_t^r L_x^p} && \text{(by (2-23))} \\
&\quad\lesssim \sum_{j=-\infty}^{\infty} \|f_j\|^2_{L_t^r L_x^p} \lesssim \|f\|^2_{L_t^r L_x^p} && \text{(by (2-25)),}
\end{aligned}$$

which completes the proof of Proposition 2.1. □

3. Mixed-norm estimates for a class of Fourier integral operators

In this section, for $j \in \mathbb{Z}$, $\alpha \in \mathbb{C}$, and $\mu \ge 2$, we shall study mixed norm estimates for the class of Fourier integral operators W_j^α defined in (2-22).

We start by considering the boundedness of the operator W_j^α from $L_t^r L_x^p$ to $L_t^{r'} L_x^{p'}$, where $1 < r,\ p \le 2$. We denote $\lambda_j = 2^j$. *All the following estimates hold uniformly in* j.

Theorem 3.1. *Let* $n \ge 2$ *and* $\mu \ge \max\{2, m/2\}$. *Then*:

(i) *For* $\max\{p_1, 1\} < p \le 2$ *and*

$$\frac{1}{r} = 1 - \frac{m}{4\mu} - \tfrac{1}{4}(m+2)(n-1)\Big(\frac{1}{p} - \frac{1}{2}\Big), \tag{3-1}$$

we have that

$$\|W_j^\alpha f\|_{L_t^{r'} L_x^{p'}(\mathbb{R}_+^{1+n})} \lesssim \lambda_j^{\left(\frac{1}{p}-\frac{1}{2}\right)(n+1)-\frac{m}{\mu(m+2)}-\frac{2}{m+2}-\operatorname{Re}\alpha} \|f\|_{L_t^r L_x^p(\mathbb{R}_+^{1+n})}. \tag{3-2}$$

Consequently,

$$\|W_j^\alpha f\|_{L_t^{r'} L_x^{p'}(\mathbb{R}_+^{1+n})} \lesssim \|f\|_{L_t^r L_x^p(\mathbb{R}_+^{1+n})} \quad \text{if } \operatorname{Re}\alpha = \Big(\frac{1}{p} - \frac{1}{2}\Big)(n+1) - \frac{m}{\mu(m+2)} - \frac{2}{m+2}. \tag{3-3}$$

(ii) *For* $p_1 > 1$ *and* $1 < p < p_1$, *we have that*

$$\|W_j^\alpha f\|_{L_t^2 L_x^{p'}(\mathbb{R}_+^{1+n})} \lesssim \lambda_j^{n\left(\frac{2}{p}-1\right)-\frac{4}{m+2}-\operatorname{Re}\alpha} \|f\|_{L_t^2 L_x^p(\mathbb{R}_+^{1+n})}. \tag{3-4}$$

In particular,

$$\|W_j^\alpha f\|_{L_t^2 L_x^{p'}(\mathbb{R}_+^{1+n})} \lesssim \|f\|_{L_t^2 L_x^p(\mathbb{R}_+^{1+n})} \quad \text{if } \operatorname{Re}\alpha = n\Big(\frac{2}{p} - 1\Big) - \frac{4}{m+2}. \tag{3-5}$$

To prove Theorem 3.1, for fixed $t,\ \tau > 0$, we first consider the operator B_j^α:

$$B_j^\alpha f(t,\tau,x) = \int_{\mathbb{R}^n} e^{i(x\cdot\xi + (\phi(t)-\phi(\tau))|\xi|)}\, b_j(t,\tau,\xi)\, \hat{f}(\tau,\xi)\, \frac{đ\xi}{|\xi|^\alpha}.$$

Lemma 3.2. *Let* $n \ge 2$ *and* $1 \le p \le 2$. *Then*, *for* $t, \tau > 0$,

$$\begin{aligned} &\|B_j^\alpha f(t,\tau,\cdot)\|_{L^{p'}(\mathbb{R}^n)} \\ &\quad \lesssim \lambda_j^{\left(\frac{1}{p}-\frac{1}{2}\right)(n+1)-\frac{m}{\mu(m+2)}-\frac{2}{m+2}-\operatorname{Re}\alpha} \\ &\qquad \times \big(\lambda_j^{-\frac{2}{m+2}} + |t-\tau|\big)^{-(m+2)\left(\frac{1}{p}-\frac{1}{2}\right)\frac{n-1}{2}-\frac{m}{2\mu}} \|f(\tau,\cdot)\|_{L^p(\mathbb{R}^n)}. \end{aligned} \tag{3-6}$$

Proof. Denote

$$K_j^\alpha(t,\tau,x,y) = \int_{\mathbb{R}^n} e^{i((x-y)\cdot\xi + (\phi(t)-\phi(\tau))|\xi|)}\, b_j(t,\tau,\xi)\, \frac{đ\xi}{|\xi|^\alpha}. \tag{3-7}$$

Then $B_j^\alpha f$ can be written as

$$B_j^\alpha f(t,\tau,x) = \int_{\mathbb{R}^n} K_j^\alpha(t,\tau,x,y) f(\tau,y)\, dy.$$

Since $\operatorname{supp}_\xi b_j \subseteq \{\xi \in \mathbb{R}^n \mid \lambda_j/2 \le |\xi| \le 2\lambda_j\}$, we have from (2-19) that

$$|\partial_\xi^\beta b_j(t,\tau,\xi)| \lesssim \lambda_j^{-\frac{m}{\mu(m+2)}-\frac{2}{m+2}-|\beta|}(\lambda_j^{-\frac{2}{m+2}}+|t-\tau|)^{-\frac{m}{2\mu}}. \tag{3-8}$$

We now apply (3-8) to derive estimate (3-6) by Plancherel's theorem when $p=2$ and by the stationary phase method when $p=1$. By interpolation, we then obtain (3-6) for $1<p<2$.

Indeed, it follows from Plancherel's theorem that

$$\begin{aligned} \|B_j^\alpha f(t,\tau,\cdot)\|_{L_x^2(\mathbb{R}^n)} &= \|e^{i(\phi(t)-\phi(\tau))|\xi|}b_j(t,\tau,\xi)\hat f(\tau,\xi)|\xi|^{-\alpha}\|_{L_\xi^2(\mathbb{R}^n)} \\ &\lesssim \lambda_j^{-\frac{m}{\mu(m+2)}-\frac{2}{m+2}-\operatorname{Re}\alpha}(\lambda_j^{-\frac{2}{m+2}}+|t-\tau|)^{-\frac{m}{2\mu}}\|f(\tau,\cdot)\|_{L^2(\mathbb{R}^n)}. \end{aligned} \tag{3-9}$$

On the other hand, by the stationary phase method (see, e.g., [Sogge 1993, Lemma 7.2.4]), we have that, for any $N\ge 0$,

$$\begin{aligned} |K_j^\alpha(t,\tau,x,y)| &\lesssim \lambda_j^n(1+|\phi(t)-\phi(\tau)|\lambda_j)^{-\frac{n-1}{2}}(\lambda_j^{-\frac{2}{m+2}}+|t-\tau|)^{-\frac{m}{2\mu}} \\ &\quad\times \lambda_j^{-\frac{m}{\mu(m+2)}-\frac{2}{m+2}-\operatorname{Re}\alpha}\big(1+\lambda_j\big||x-y|-|\phi(t)-\phi(\tau)|\big|\big)^{-N} \\ &\lesssim \lambda_j^{\frac{n+1}{2}-\frac{m}{\mu(m+2)}-\frac{2}{m+2}-\operatorname{Re}\alpha}(\lambda_j^{-\frac{2}{m+2}}+|t-\tau|)^{-\frac{(m+2)(n-1)}{4}-\frac{m}{2\mu}} \\ &\quad\times\big(1+\lambda_j\big||x-y|-|\phi(t)-\phi(\tau)|\big|\big)^{-N}. \end{aligned} \tag{3-10}$$

Choosing $N=0$ in (3-10) gives

$$\begin{aligned} &\|(B_j^\alpha f)(t,\tau,\cdot)\|_{L^\infty(\mathbb{R}^n)} \\ &\quad\le \|K_j^\alpha(t,\tau,\cdot,\cdot)\|_{L^\infty_{x,y}}\|f(\tau,\cdot)\|_{L^1(\mathbb{R}^n)} \\ &\quad\lesssim \lambda_j^{\frac{n+1}{2}-\frac{m}{\mu(m+2)}-\frac{2}{m+2}-\operatorname{Re}\alpha}(\lambda_j^{-\frac{2}{m+2}}+|t-\tau|)^{-\frac14(m+2)(n-1)-\frac{m}{2\mu}}\|f(\tau,\cdot)\|_{L^1(\mathbb{R}^n)}. \end{aligned}$$

Interpolation between (3-9) and this last estimate yields (3-6) in case $1\le p\le 2$, which completes the proof of estimate (3-6). □

Proof of Theorem 3.1. Now we return to the proof of Theorem 3.1. From (3-7), we have

$$W_j^\alpha f(t,x)=\int_0^t (B_j^\alpha f)(t,\tau,x)\,d\tau. \tag{3-11}$$

Using Minkowski's inequality and estimate (3-6), we thus have that

$$\begin{aligned}(3\text{-}12)\quad &\|W_j^\alpha f(t,\cdot)\|_{L^{p'}(\mathbb{R}^n)}\\ &\lesssim \lambda_j^{\left(\frac{1}{p}-\frac{1}{2}\right)(n+1)-\frac{m}{\mu(m+2)}-\frac{2}{m+2}-\operatorname{Re}\alpha}\\ &\quad\times\int_0^\infty \left(\lambda_j^{-\frac{2}{m+2}}+|t-\tau|\right)^{-(m+2)\left(\frac{1}{p}-\frac{1}{2}\right)\frac{n-1}{2}-\frac{m}{2\mu}}\|f(\tau,\cdot)\|_{L^p(\mathbb{R}^n)}\,d\tau.\end{aligned}$$

Case 1: $\max\{p_1,1\}<p\le 2$. In this case, we have $1<r<2$. Note that

$$\frac{1}{r}-\frac{1}{r'}=-(m+2)\left(\frac{1}{p}-\frac{1}{2}\right)\frac{n-1}{2}-\frac{m}{2\mu}+1.$$

Then it follows from the Hardy–Littlewood–Sobolev theorem and (3-12) that estimate (3-2) holds.

Case 2: $p_1>1$ *and* $1<p<p_1$. In this case,

$$(m+2)\left(\frac{1}{p}-\frac{1}{2}\right)\frac{n-1}{2}+\frac{m}{2\mu}>1.$$

Thus,

$$\sup_{t>0}\int_0^\infty \left(\lambda_j^{-\frac{2}{m+2}}+|t-\tau|\right)^{-(m+2)\left(\frac{1}{p}-\frac{1}{2}\right)\frac{n-1}{2}-\frac{m}{2\mu}}\,d\tau<\infty,$$

which together with Schur's lemma and (3-12) yields (3-4). □

We would like to stress that in the proof of Theorem 3.1 only condition (2-19) on the function $b\in C^\infty(\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}^n)$ was used, whereas the conditions (2-20) and (2-21) were not required,

Remark 3.3. Note that the adjoint operator $(W_j^\alpha)^*$ of W_j^α is of the form

$$(3\text{-}13)\quad (W_j^\alpha)^*f(t,x)=\int_t^\infty\int_{\mathbb{R}^n} e^{i(x\cdot\xi+(\phi(t)-\phi(\tau))|\xi|)}\,\overline{b_j(\tau,t,\xi)}\,\hat f(\tau,\xi)\frac{đ\xi}{|\xi|^\alpha}\,d\tau.$$

By duality, we infer from Theorem 3.1 that

$$\begin{aligned}(3\text{-}14)\quad &\|(W_j^\alpha)^*f\|_{L_t^{r'}L_x^{p'}(\mathbb{R}_+^{1+n})}\\ &\qquad\lesssim \lambda_j^{\left(\frac{1}{p}-\frac{1}{2}\right)(n+1)-\frac{m}{\mu(m+2)}-\frac{2}{m+2}-\operatorname{Re}\alpha}\|f\|_{L_t^rL_x^p(\mathbb{R}_+^{1+n})}\end{aligned}$$

if $\max\{p_1,1\}<p\le 2$ and

$$(3\text{-}15)\qquad \|(W_j^\alpha f)^*\|_{L_t^2L_x^{p'}(\mathbb{R}_+^{1+n})}\lesssim \lambda_j^{n\left(\frac{2}{p}-1\right)-\frac{4}{m+2}-\operatorname{Re}\alpha}\|f\|_{L_t^2L_x^p(\mathbb{R}_+^{1+n})}$$

if $p_1>1$ and $1<p<p_1$. Here, r is given in (3-1).

As an application of Theorem 3.1, we obtain the boundedness of the operator W_j^α from $L_t^rL_x^p$ to $L_t^\infty L_x^2$, where $1<r,p\le 2$.

Theorem 3.4. *Let $n \geq 2$ and $\mu \geq \max\{2, m/2\}$. Then*:

(i) *For* $\max\{p_1, 1\} < p \leq 2$ *and r as in* (3-1), *we have that*

$$\|W_j^\alpha f\|_{L_t^\infty L_x^2(\mathbb{R}_+^{1+n})} \lesssim \lambda_j^{(\frac{1}{p}-\frac{1}{2})\frac{n+1}{2}-\frac{m}{2\mu(m+2)}-\frac{2}{m+2}-\operatorname{Re}\alpha} \|f\|_{L_t^r L_x^p(\mathbb{R}_+^{1+n})}. \tag{3-16}$$

Consequently,

$$\|W_j^\alpha f\|_{L_t^\infty L_x^2(\mathbb{R}_+^{1+n})} \lesssim \|f\|_{L_t^r L_x^p(\mathbb{R}_+^{1+n})} \quad \text{if } \operatorname{Re}\alpha = \Big(\frac{1}{p}-\frac{1}{2}\Big)\frac{n+1}{2} - \frac{m}{2\mu(m+2)} - \frac{2}{m+2}. \tag{3-17}$$

(ii) *For $p_1 > 1$ and $1 < p < p_1$, we have that*

$$\|W_j^\alpha f\|_{L_t^\infty L_x^2(\mathbb{R}_+^{1+n})} \lesssim \lambda_j^{n\left(\frac{1}{p}-\frac{1}{2}\right)-\frac{3}{m+2}-\operatorname{Re}\alpha} \|f\|_{L_t^2 L_x^p(\mathbb{R}_+^{1+n})}. \tag{3-18}$$

In particular,

$$\|W_j^\alpha f\|_{L_t^\infty L_x^2(\mathbb{R}_+^{1+n})} \lesssim \|f\|_{L_t^2 L_x^p(\mathbb{R}_+^{1+n})} \quad \text{if } \operatorname{Re}\alpha = n\Big(\frac{1}{p}-\frac{1}{2}\Big) - \frac{3}{m+2}. \tag{3-19}$$

Proof. For given $j \in \mathbb{Z}$ and $\alpha \in \mathbb{C}$, denote $U = W_j^\alpha f$. Then from (2-22) we have

$$U(t) = \int_0^t e^{i(\phi(t)-\phi(\tau))\sqrt{-\Delta}} b_j(t,\tau,D_x)(-\Delta)^{-\alpha/2} f(\tau)\, d\tau,$$

where $b_j(t,\tau,D_x)$ is the pseudodifferential operator with full symbol $b_j(t,\tau,\xi)$. Then $U(t)$ solves the Cauchy problem

$$\begin{aligned}
i\partial_t U(t) &= -t^{m/2}\sqrt{-\Delta}U(t) + i b_j(t,t,D_x)(-\Delta)^{-\alpha/2} f(t) \\
&\qquad + i\int_0^t e^{i(\phi(t)-\phi(\tau))\sqrt{-\Delta}} \partial_t b_j(t,\tau,D_x)(-\Delta)^{-\alpha/2} f(\tau)\, d\tau, \\
U(0) &= 0.
\end{aligned}$$

Multiplying by $\overline{U(t)}$ and then integrating over $\mathbb{R}^n$ yields

$$\begin{aligned}
&i\langle \partial_t U(t), U(t)\rangle \\
&\quad = -t^{m/2}\langle \sqrt{-\Delta}U(t), U(t)\rangle + i\langle b_j(t,t,D_x)(-\Delta)^{-\alpha/2} f(t), U(t)\rangle \\
&\qquad + i\Big\langle \int_0^t e^{i(\phi(t)-\phi(\tau))\sqrt{-\Delta}} \partial_t b_j(t,\tau,D_x)(-\Delta)^{-\alpha/2} f(\tau)\, d\tau, U(t)\Big\rangle,
\end{aligned}$$

and, therefore,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|U(t)\|^2 &= \operatorname{Re}\left\langle \int_0^t e^{i(\phi(t)-\phi(\tau))\sqrt{-\Delta}}\partial_t b_j(t,\tau,D_x)(-\Delta)^{-\alpha/2} f(\tau)\,d\tau, U(t)\right\rangle \\ &\qquad + \operatorname{Re}\langle b_j^*(t,t,D_x)(-\Delta)^{-\alpha/2}U(t), f(t)\rangle.\end{aligned}$$

Consequently,

$$\begin{aligned}\|U(s)\|^2 &= 2\operatorname{Re}\int_0^s \left\langle \int_0^t e^{i(\phi(t)-\phi(\tau))\sqrt{-\Delta}}\partial_t b_j(t,\tau,D_x)(-\Delta)^{-\alpha/2} f(\tau)\,d\tau, U(t)\right\rangle dt \\ &\qquad + 2\operatorname{Re}\int_0^s \langle b_j^*(t,t,D_x)(-\Delta)^{-\alpha/2}U(t), f(t)\rangle\,dt \\ &\lesssim \left|\int_0^s\int_{\mathbb{R}^n} L_j^\alpha f(t,x)\overline{W_j^\alpha f(t,x)}\,dx\,dt\right| \\ &\qquad + \left|\int_0^s\int_{\mathbb{R}^n} b_j^*(t,t,D_x)W_j^{2\alpha} f(t,x)\overline{f(t,x)}\,dx\,dt\right| \\ &= \mathrm{I}+\mathrm{II},\end{aligned}$$

where

$$\begin{aligned}\mathrm{I} &= \left|\int_0^s\int_{\mathbb{R}^n} L_j^\alpha f(t,x)\overline{W_j^\alpha f(t,x)}\,dx\,dt\right|, \\ \mathrm{II} &= \left|\int_0^s\int_{\mathbb{R}^n} b_j^*(t,t,D_x)W_j^{2\alpha} f(t,x)\overline{f(t,x)}\,dx\,dt\right|,\end{aligned}$$

and

$$L_j^\alpha f(t,x) = \int_0^t\int_{\mathbb{R}^n} e^{i(x\cdot\xi+(\phi(t)-\phi(\tau))|\xi|)}\partial_t b_j(t,\tau,\xi)\hat{f}(\tau,\xi)\,\frac{đ\xi}{|\xi|^\alpha}\,d\tau.$$

From (2-19), one has that, for any fixed $t>0$, $b_j(t,t,D_x)\in\Psi^{-2/(m+2)}(\mathbb{R}^n)$, and then $b_j^*(t,t,D_x)\in\Psi^{-2/(m+2)}(\mathbb{R}^n)$, which yields that the term II is essentially

$$\left|\int_0^s\int_{\mathbb{R}^n}(W_j^{2\alpha+2/(m+2)} f)(t,x)\overline{f(t,x)}\,dx\,dt\right|,$$

and thus by application of Theorem 3.1 it follows that

$$\mathrm{II} \lesssim \begin{cases} \lambda_j^{(n+1)\left(\frac{1}{p}-\frac{1}{2}\right)-\frac{m}{\mu(m+2)}-\frac{4}{m+2}-2\operatorname{Re}\alpha}\|f\|^2_{L_t^r L_x^p(\mathbb{R}_+^{1+n})} & \text{if } \max\{p_1,1\}<p\le 2, \\ \lambda_j^{n\left(\frac{2}{p}-1\right)-\frac{6}{m+2}-2\operatorname{Re}\alpha}\|f\|^2_{L_t^2 L_x^p(\mathbb{R}_+^{1+n})} & \text{if } 1<p<p_1. \end{cases} \tag{3-20}$$

As for the term I, note that

$$\mathrm{I} = \left| \int_0^s \int_{\mathbb{R}^n} (W_j^\alpha)^* L_j^\alpha f(t,x) \overline{f(t,x)}\, dx\, dt \right| \\ \le \|(W_j^\alpha)^* L_j^\alpha f\|_{L_t^{\rho'} L_x^{p'}(\mathbb{R}_+^{1+n})} \|f\|_{L_t^{\rho} L_x^{p}(\mathbb{R}_+^{1+n})}.$$

For any $t > 0$, we have from (3-13) that

$$\begin{aligned} (W_j^\alpha)^* L_j^\alpha f(t,x) &= \int_t^\infty \int_0^\tau \int_{\mathbb{R}^n} e^{i(x\cdot\xi + (\phi(t)-\phi(s))|\xi|)} \\ &\qquad \times \overline{b_j(\tau,t,\xi)} \partial_\tau b_j(\tau,s,\xi) \hat{f}(s,\xi) \frac{đ\xi}{|\xi|^{2\alpha}}\, ds\, d\tau \\ &= \int_0^t \int_{\mathbb{R}^n} e^{i(x\cdot\xi + (\phi(t)-\phi(s))|\xi|)} \\ &\qquad \times \left(\int_t^\infty \overline{b_j(\tau,t,\xi)} \partial_\tau b_j(\tau,s,\xi)\, d\tau \right) \hat{f}(s,\xi) \frac{đ\xi}{|\xi|^{2\alpha}}\, ds \\ &\quad + \int_t^\infty \int_{\mathbb{R}^n} e^{i(x\cdot\xi + (\phi(t)-\phi(s))|\xi|)} \\ &\qquad \times \left(\int_s^\infty \overline{b_j(\tau,t,\xi)} \partial_\tau b_j(\tau,s,\xi)\, d\tau \right) \hat{f}(s,\xi) \frac{đ\xi}{|\xi|^{2\alpha}}\, ds. \end{aligned} \tag{3-21}$$

Due to conditions (2-19)–(2-21), one has that the first and second term in (3-21) are essentially $W_j^{2\alpha+2/(m+2)} f$ and $(W_j^{2\alpha+2/(m+2)})^* f$, respectively, where $b \in C^\infty(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^n)$ satisfies condition (2-19). Then, by applying Theorem 3.1 and estimates (3-14) and (3-15), we have that

$$\mathrm{I} \lesssim \begin{cases} \lambda_j^{(n+1)\left(\frac{1}{p}-\frac{1}{2}\right) - \frac{m}{\mu(m+2)} - \frac{4}{m+2} - 2\operatorname{Re}\alpha} \|f\|^2_{L_t^r L_x^p(\mathbb{R}_+^{1+n})} & \text{if } \max\{p_1, 1\} < p \le 2, \\ \lambda_j^{n\left(\frac{2}{p}-1\right) - \frac{6}{m+2} - 2\operatorname{Re}\alpha} \|f\|^2_{L_t^2 L_x^p(\mathbb{R}_+^{1+n})} & \text{if } p_1 > 1 \text{ and } 1 < p < p_1, \end{cases}$$

which together with (3-20) yields that

$$\|U(t)\|^2 \lesssim \begin{cases} \lambda_j^{(n+1)\left(\frac{1}{p}-\frac{1}{2}\right) - \frac{m}{\mu(m+2)} - \frac{4}{m+2} - 2\operatorname{Re}\alpha} \|f\|^2_{L_t^r L_x^p(\mathbb{R}_+^{1+n})} & \text{if } \max\{p_1, 1\} < p \le 2, \\ \lambda_j^{n\left(\frac{2}{p}-1\right) - \frac{6}{m+2} - 2\operatorname{Re}\alpha} \|f\|^2_{L_t^2 L_x^p(\mathbb{R}_+^{1+n})} & \text{if } p_1 > 1 \text{ and } 1 < p < p_1. \end{cases}$$

Note that $\|W_j^\alpha f(t,\cdot)\|_{L^2(\mathbb{R}^n)} = \|U(t)\|$. Therefore, we have obtained estimates (3-16)–(3-19), which completes the proof of Theorem 3.4. □

Remark 3.5. With similar arguments as in the proof of Theorem 3.4, we have from Theorem 3.1 and estimates (3-14) and (3-15) that the operator $(W_j^\alpha)^*$ also satisfies the estimates (3-16)–(3-19).

Note that if $r = p$ for r defined in (3-1), then $r = p = p_0$. Combining Theorem 3.1 and the kernel estimate (3-10), we obtain boundedness of the operator W_j^α from $L^{p_0}(\mathbb{R}_+^{1+n})$ to $L^q(\mathbb{R}_+^{1+n})$ for certain $\alpha \in \mathbb{C}$ when $q_0 \le q \le \infty$.

Theorem 3.6. *Let $\mu \ge \max\{2, m/2\}$ and $q_0 \le q \le \infty$. Then*

$$\|W_j^\alpha f\|_{L^q(\mathbb{R}_+^{1+n})} \lesssim \|f\|_{L^{p_0}(\mathbb{R}_+^{1+n})}, \tag{3-22}$$

where

$$\operatorname{Re}\alpha = n - \frac{2}{m+2} - \Big(n + \frac{2}{m+2}\Big)\Big(\frac{1}{q} + \frac{1}{q_0}\Big).$$

Proof. Case (i): $q = q_0$. Note that

$$n - \frac{2}{q_0}\Big(n + \frac{2}{m+2}\Big) = \Big(\frac{1}{p_0} - \frac{1}{2}\Big)(n+1) - \frac{m}{\mu(m+2)}.$$

An application of (3-3) with $r = p$ yields that

$$\|W_j^\alpha f\|_{L^{q_0}(\mathbb{R}_+^{1+n})} \lesssim \|f\|_{L^{p_0}(\mathbb{R}_+^{1+n})}, \quad \operatorname{Re}\alpha = n - \frac{2}{m+2} - \frac{2}{q_0}\Big(n + \frac{2}{m+2}\Big). \tag{3-23}$$

Case (ii): $q = \infty$. In order to derive (3-22), it suffices to show that the integral kernel K_j^α defined in (3-7) satisfies

$$\begin{gathered}\sup_{(t,x)\in\mathbb{R}_+^{1+n}} \int_{\mathbb{R}_+^{1+n}} |K_j^\alpha(t,\tau,x,y)|^{q_0}\, d\tau\, dy < \infty,\\ \operatorname{Re}\alpha = n - \frac{2}{m+2} - \frac{1}{q_0}\Big(n + \frac{2}{m+2}\Big).\end{gathered} \tag{3-24}$$

In fact, from (3-7) we have

$$W_j^\alpha f(t,x) = \int_0^t \int_{\mathbb{R}^n} K_j^\alpha(t,\tau,x,y) f(\tau,y)\, dy\, d\tau.$$

By Hölder's inequality, then

$$\|W_j^\alpha f\|_{L^\infty(\mathbb{R}_+^{1+n})} \lesssim \|f\|_{L^{p_0}(\mathbb{R}_+^{1+n})}, \quad \operatorname{Re}\alpha = n - \frac{2}{m+2} - \frac{1}{q_0}\Big(n + \frac{2}{m+2}\Big). \tag{3-25}$$

Now it remains to derive estimate (3-24). In fact, due to the kernel estimate (3-10), for any $N > n$ and $\alpha \in \mathbb{C}$ with $\operatorname{Re}\alpha = n - 2/(m+2) - 1/q_0(n + 2/(m+2))$, we have by (3-10)

$$
\begin{aligned}
\int_{\mathbb{R}^{1+n}_+} |K_j^\alpha(t,\tau,x,y)|^{q_0}\, d\tau\, dy
&\lesssim \lambda_j^{\left(\frac{n+1}{2}-\operatorname{Re}\alpha-\frac{m}{\mu(m+2)}-\frac{2}{m+2}\right)q_0} \\
&\qquad \times \int_0^\infty (\lambda_j^{-\frac{2}{m+2}}+|t-\tau|)^{-\left(\frac{(m+2)(n-1)}{4}+\frac{m}{2\mu}\right)q_0}\, d\tau \\
&\qquad \times \int_{\mathbb{R}^n} (1+\lambda_j\big||x-y|-|\phi(t)-\phi(\tau)|\big|)^{-N}\, dy \\
&\lesssim \lambda_j^{\left(\frac{n+1}{2}-\operatorname{Re}\alpha-\frac{m}{\mu(m+2)}-\frac{2}{m+2}\right)q_0} \\
&\qquad \times \int_0^\infty (\lambda_j^{-\frac{2}{m+2}}+|t-\tau|)^{-\left(\frac{(m+2)(n-1)}{4}+\frac{m}{2\mu}\right)q_0}\, d\tau \\
&\qquad \times \lambda_j^{-1}\int_0^\infty (1+r)^{-N}(\lambda_j^{-1}r+|\phi(t)-\phi(\tau)|)^{n-1}\, dr \\
&= \lambda_j^{\left(\frac{n+1}{2}-\operatorname{Re}\alpha-\frac{m}{\mu(m+2)}-\frac{2}{m+2}\right)q_0-1} \\
&\qquad \times \int_0^\infty (\lambda_j^{-\frac{2}{m+2}}+|t-\tau|)^{-\left(\frac{(m+2)(n-1)}{4}+\frac{m}{2\mu}\right)q_0} \\
&\qquad\qquad (\lambda_j^{-1}+|\phi(t)-\phi(\tau)|)^{n-1}\, d\tau \\
&\qquad \times \int_0^\infty (1+r)^{-N}\Big(\frac{r+\lambda_j|\phi(t)-\phi(\tau)|}{1+\lambda_j|\phi(t)-\phi(\tau)|}\Big)^{n-1}\, dr \\
&\lesssim \lambda_j^{\left(\frac{n+1}{2}-\operatorname{Re}\alpha-\frac{m}{\mu(m+2)}-\frac{2}{m+2}\right)q_0-1} \\
&\qquad \times \int_0^\infty (\lambda_j^{-\frac{2}{m+2}}+|t-\tau|)^{-\left(\frac{(m+2)(n-1)}{4}+\frac{m}{2\mu}\right)q_0+\frac{(m+2)(n-1)}{2}}\, d\tau \\
&\lesssim \lambda_j^{\left(n-\operatorname{Re}\alpha-\frac{2}{m+2}\right)q_0-n-\frac{2}{m+2}} = 1,
\end{aligned}
$$

and hence (3-24) holds.

Case (iii): $q_0 < q < \infty$. Applying Stein's interpolation theorem, one obtains that estimate (3-22) holds by interpolating between estimates (3-23) and (3-25). □

Now we consider boundedness of the operator W_j from $L_t^r L_x^p(S_T)$ to $L_t^s L_x^q(S_T)$, where $1/p$ is symmetric around $1/p_0$.

Theorem 3.7. *Let $n \ge 2$. Further let $p_1 < p < p_2$ if $n = 2$, $m \ge 2$, or if $n \ge 3$, and $1 < p < 7\mu/(4\mu - 2)$ if $n = 2$, $m = 1$. Then, for any $\mu \ge \mu_*$ and $T > 0$,*

$$
\|W_j f\|_{L_t^s L_x^q(S_T)} \lesssim \|f\|_{L_t^r L_x^p(S_T)}, \tag{3-26}
$$

where r is defined as in (3-1) *and*

$$\frac{1}{q}=\frac{1}{p}-\frac{4}{(m+2)(n+1)}\Big(1+\frac{m}{2\mu}\Big),\qquad \frac{1}{s}=\frac{(m+2)(n-1)}{4}\Big(\frac{1}{2}-\frac{1}{q}\Big)+\frac{m}{4\mu}. \tag{3-27}$$

Proof. Since $1/p$ is symmetric around $1/p_0$, by duality it suffices to consider the case $\max\{p_1,1\}<p\le p_0$.

In order to derive (3-26), we now need a further dyadic decomposition with respect to the time variable t. Choose a function $\eta\in C_c^\infty(\mathbb{R}_+)$ such that $0\le\eta\le 1$, $\operatorname{supp}\eta\subseteq\big[\frac{1}{2},2\big]$, and

$$\sum_{\ell=-\infty}^{\infty}\eta(2^{-\ell}t)=1.$$

Let us fix $\lambda=2^j$ and set

$$\eta_0(t)=\sum_{k\le 0}\eta(\lambda 2^{-k}t),\quad \eta_\ell(t)=\eta(\lambda 2^{-\ell}t)\quad\text{for } \ell\in\mathbb{N}.$$

Then,

$$W_j f(t,x)=\sum_{k=0}^{\infty}G_k f(t,x),$$

where

$$G_k f(t,x)=\int_0^t\int_{\mathbb{R}^n}e^{i(x\cdot\xi+(\phi(t)-\phi(\tau))|\xi|)}\eta_k(t-\tau)\,b_j(t,\tau,\xi)\,\hat{f}(\tau,\xi)\,đ\xi\,d\tau. \tag{3-28}$$

Hence, to derive (3-26), it suffices to show that, for any $k\in\mathbb{N}_0$,

$$\|G_k f\|_{L_t^s L_x^q(S_T)}\lesssim 2^{-\varepsilon_p k}\|f\|_{L_t^r L_x^p(S_T)} \tag{3-29}$$

for some $\varepsilon_p>0$. From (3-1) and (3-27), we know that

$$\frac{(m+2)n}{2}\Big(\frac{1}{p}-\frac{1}{q}\Big)+\frac{1}{r}-\frac{1}{s}=2.$$

Due to scaling invariance, we need to consider only the case $\lambda=1$ (by a change of variable if $\lambda\ne 1$). Repeating the arguments which are used to prove (3-2), we get that, for any $k\in\mathbb{N}_0$,

$$\|G_k f\|_{L_t^{r'}L_x^{p'}(S_T)}\lesssim 2^{-k((m+2)(1/p-1/2)(n-1)/2+m/(2\mu))}\|f\|_{L_t^r L_x^p(S_T)}. \tag{3-30}$$

Note that $(m+2)\big(1/p-\frac{1}{2}\big)\frac{1}{2}(n-1)+m/(2\mu)>\frac{1}{3}$, since $p\le p_0$.

Furthermore, an immediate consequence of (3-16) for $\alpha = 0$ is

$$\|G_k f\|_{L_t^\infty L_x^2(S_T)} \lesssim \|f\|_{L_t^r L_x^p(S_T)},$$

and thus, for any $1 < \rho < \infty$,

$$\|G_k f\|_{L_t^\rho L_x^2(S_T)} \lesssim \|f\|_{L_t^r L_x^p(S_T)}. \tag{3-31}$$

Choose

$$\theta = \frac{4p(2\mu+m)}{\mu(m+2)(n+1)(2-p)} - 1. \tag{3-32}$$

Then $0 \le \theta \le 1$ and, for the number q from (3-27),

$$\frac{1}{q} = \frac{\theta}{p'} + \frac{1-\theta}{2}.$$

For s from (3-27) and θ from (3-32), we define s_0 by

$$2\Big(\frac{1}{s} - \frac{1}{s_0}\Big) = \theta\Big((m+2)\Big(\frac{1}{p} - \frac{1}{2}\Big)\frac{n-1}{2} + \frac{m}{2\mu}\Big)$$

and then set $\rho = \rho_*$ such that

$$\frac{1}{s_0} = \frac{\theta}{r'} + \frac{1-\theta}{\rho_*}.$$

Since $2 < s < s_0$, by interpolating between (3-30) and (3-31) when $\rho = \rho_*$, we obtain that

$$\|G_k f\|_{L_t^{s_0} L_x^q(S_T)} \lesssim 2^{-2k(1/s-1/s_0)}\|f\|_{L_t^r L_x^p(S_T)}. \tag{3-33}$$

Let $\{I_\ell\}$ be nonoverlapping intervals of side length 2^k and $\bigcup_\ell I_\ell = \mathbb{R}_+$, and denote by χ_I the characteristic function of I. In view of (3-28) and the compact support of η_k, we have that if $f(t,x) = 0$ for $t \notin I_\ell$, then $G_k f(t,x) = 0$ for $t \notin I_\ell^*$, where I_ℓ^* is the interval with the same center as I_ℓ but of side length $C_0 2^k$ with some constant $C_0 = C_0(\eta) > 0$. Thus, from Minkowski's inequality,

$$\|G_k f(t,\cdot)\|_{L^q(\mathbb{R}^n)}^s \le \Big(\sum_\ell \|G_k(\chi_{I_\ell} f)(t,\cdot)\|_{L^q(\mathbb{R}^n)}\Big)^s \lesssim \sum_\ell \|G_k(\chi_{I_\ell} f)(t,\cdot)\|_{L^q(\mathbb{R}^n)}^s. \tag{3-34}$$

Denote $\overline{I_\ell^*} = I_\ell^* \cap (0,T)$. Estimate (3-34) together with Hölder's inequality and (3-33) yields that, for any $k \in \mathbb{N}_0$,

$$\begin{aligned}\|G_k f\|^s_{L^s_t L^q_x(S_T)} &\lesssim \sum_\ell \|G_k(\chi_{I_\ell} f)\|^s_{L^s_t L^q_x(\overline{I_\ell^*}\times\mathbb{R}^n)} \\ &\lesssim \sum_\ell |\overline{I_\ell^*}|^{1-s/s_0} \|G_k(\chi_{I_\ell} f)\|^s_{L^{s_0}_t L^q_x(\overline{I_\ell^*}\times\mathbb{R}^n)} \\ &\lesssim 2^{k(1-s/s_0)} 2^{-2ks(1/s-1/s_0)} \sum_\ell \|\chi_{I_\ell} f\|^s_{L^r_t L^p_x(S_T)} \\ &\lesssim 2^{-k(1-s/s_0)} \|f\|_{L^r_t L^p_x(S_T)}.\end{aligned}$$

Therefore, we get estimate (3-29) with $\varepsilon_p = 1 - s/s_0$ and, hence, (3-26) holds. □

By a similar argument as in the proof of Theorem 3.7, we obtain the boundedness of operator W_j from $L^2_t L^p_x(S_T)$ to $L^s_t L^q_x(S_T)$ when $p_1 > 1$ and $1 < p < p_1$.

Theorem 3.8. *Let $n \geq 3$ or $n = 2$, $m \geq 2$. Suppose $1 < p < p_1$. Then, for $\mu \geq \max\{2, mn/2\}$ and $T > 0$, we have that*

$$\|W_j f\|_{L^s_t L^q_x(S_T)} \lesssim \|f\|_{L^2_t L^p_x(S_T)}, \tag{3-35}$$

where

$$\begin{aligned}\frac{1}{q} &= \frac{2n}{p(n+1)} - \frac{n-1}{2(n+1)} - \frac{m+6\mu}{\mu(m+2)(n+1)}, \\ \frac{1}{s} &= (m+2)\Big(\frac{1}{2} - \frac{1}{q}\Big)\Big(\frac{n-1}{4}\Big) + \frac{m}{4\mu}.\end{aligned} \tag{3-36}$$

Proof. Note that when $1 < p < p_1$, we have

$$(m+2)\Big(\frac{1}{p} - \frac{1}{2}\Big)\Big(\frac{n-1}{2}\Big) + \frac{m}{2\mu} > 1.$$

Then we can apply similar arguments as in the proof of Theorem 3.7 to obtain (3-35). We omit the details. □

Remark 3.9. By similar arguments as above one can show that under assumptions (3-27) and (3-36), adjoints $(W_j)^*$ of W_j also satisfy estimates (3-26) and (3-35), respectively.

4. Mixed-norm estimates for the linear generalized Tricomi equation

In this section, based on the mixed-norm space-time estimates of the Fourier integral operators W_j^α obtained in Section 3, we shall establish Strichartz-type estimates for the linear generalized Tricomi equation.

First we consider the inhomogeneous equation with zero initial data, i.e., problem (2-3).

Theorem 4.1. *Let $n \geq 2$. Suppose w is a solution of* (2-3) *in S_T for some $T > 0$. Then*:

(i) *For $\mu \geq \mu_*$,*

$$\|w\|_{L_t^s L_x^q(S_T)} \lesssim \|f\|_{L_t^r L_x^p(S_T)}, \tag{4-1}$$

provided that $p_1 < p < p_2$ if $n \geq 3$ or $n = 2$, $m \geq 2$; and $1 < p < 7\mu/(4\mu - 2)$ if $n = 2$ and $m = 1$. Here $r = r(p, \mu)$ is as in (3-1) *and q and s are taken from* (3-27).

(ii) *For $\mu \geq \max\{2, m/2\}$,*

$$\|w\|_{L^q(S_T)} \lesssim \||D_x|^{\gamma-\gamma_0} f\|_{L^{p_0}(S_T)}, \quad q_0 \leq q < \infty, \tag{4-2}$$

where

$$\begin{aligned} \gamma &= \gamma(m, n, q) = \frac{n}{2} - \frac{1}{q}\Big(n + \frac{2}{m+2}\Big), \\ \gamma_0 &= \gamma_0(m, n, \mu) = \frac{1}{q_0}\Big(n + \frac{2}{m+2}\Big) + \frac{2}{m+2} - \frac{n}{2}. \end{aligned} \tag{4-3}$$

(iii) *For $\mu \geq \max\{2, m/2\}$, $\max\{p_1, 1\} < p \leq 2$, and $0 < t \leq T$,*

$$\|w(t, \cdot)\|_{\dot{H}^\gamma(\mathbb{R}^n)} \lesssim \|f\|_{L_t^r L_x^p(S_T)}, \tag{4-4}$$

where $r = r(m, n, p, \mu)$ is defined in (3-1) *and*

$$\gamma = \gamma(m, n, \mu, p) = \frac{2}{m+2} + \frac{m}{2\mu(m+2)} - \Big(\frac{1}{p} - \frac{1}{2}\Big)\frac{n+1}{2}.$$

(iv) *For $\mu \geq \max\{2, m/2\}$, $\gamma \in \mathbb{R}$, and $0 \leq t \leq T$,*

$$\|w(t, \cdot)\|_{\dot{H}^\gamma(\mathbb{R}^n)} \lesssim \big\||D_x|^{\gamma-\gamma_0} f\big\|_{L^{p_0}(S_T)}, \tag{4-5}$$

where γ_0 is from (4-3).

Remark 4.2. If we choose $\mu = \mu_*$, then

$$p_0 = p_0^* = \frac{2\mu_*}{\mu_* + 2}, \quad q_0 = q_0^* = \frac{2\mu_*}{\mu_* - 2},$$

and for γ and γ_0 defined in (4-3),

$$\gamma(m, n, q_0^*) = \gamma_0(m, n, \mu_*) = \frac{1}{m+2}.$$

Thus, we have from (4-2) that

$$\|w\|_{L^{q_0^*}(S_T)} \lesssim \|f\|_{L^{p_0^*}(S_T)},$$

which, for any $\rho \in \mathbb{R}$, together with $[|D_x|^\rho, \partial_t^2 - t^m\Delta] = 0$ implies that

$$\big\| |D_x|^\rho w \big\|_{L^{q_0^*}(S_T)} \lesssim \big\| |D_x|^\rho f \big\|_{L^{p_0^*}(S_T)}.$$

Proof of Theorem 4.1. (i): One obtains (4-1) by applying Proposition 2.1 and Theorem 3.7 directly.

(ii): For $\alpha \in \mathbb{C}$, the Fourier transform of $|D_x|^\alpha f(t,x)$ with respect to the variable x is $|\xi|^\alpha \hat{f}(t,\xi)$. Thus, we can write $W_j f$ as

$$W_j f(t,x) = \int_0^t \int_{\mathbb{R}^n} e^{i(x\cdot\xi + (\phi(t)-\phi(\tau))|\xi|)} \Theta(|\xi|/2^j)\, b(t,\tau,\xi) (\widehat{|D_x|^\alpha f})(\tau,\xi) |\xi|^{-\alpha}\, đ\xi\, d\tau$$

and $W_j(f) = W_j^\alpha(|D_x|^\alpha f)$.

Therefore, applying Theorem 3.6, we get that

$$\|W_j f\|_{L^q(S_T)} = \big\| W_j^{\gamma-\gamma_0}(|D_x|^{\gamma-\gamma_0} f) \big\|_{L^q(S_T)} \lesssim \big\| |D_x|^{\gamma-\gamma_0} f \big\|_{L^{p_0}(S_T)},$$

which together with Proposition 2.1 yields (4-2).

(iii): Note that $[|D_x|^\gamma, \partial_t^2 - t^m\Delta] = 0$ and then

$$(\partial_t^2 - t^m\Delta)(|D_x|^\gamma w) = |D_x|^\gamma f. \tag{4-6}$$

From (ii) we know that $W_j(|D_x|^\gamma f) = W_j^{-\gamma}(f)$. Thus, for $\gamma = 2/(m+2) + m/(2\mu(m+2)) - (1/p - 1/2)(n+1)/2$, we have from estimate (3-17) that

$$\big\| W_j(|D_x|^\gamma f)(t,\cdot) \big\|_{L^2(\mathbb{R}^n)} = \| W_j^{-\gamma} f(t,\cdot) \|_{L^2(\mathbb{R}^n)} \lesssim \|f\|_{L_t^r L_x^p}.$$

Thus, by (4-6) and Proposition 2.1 it follows that

$$\big\| (|D_x|^\gamma w)(t,\cdot) \big\|_{L^2(\mathbb{R}^n)} \lesssim \|f\|_{L_t^r L_x^p},$$

which together with Plancherel's theorem implies that

$$\|w(t,\cdot)\|_{\dot{H}^\gamma(\mathbb{R}^n)} = \big\| |\xi|^\gamma \hat{w}(t,\xi) \big\|_{L_\xi^2(\mathbb{R}^n)} = \big\| (|D_x|^\gamma w)(t,\cdot) \big\|_{L_x^2(\mathbb{R}^n)} \lesssim \|f\|_{L_t^r L_x^p},$$

and estimate (4-4) holds.

(iv): From (ii) we also know that

$$W_j(g) = W_j^{-\gamma_0}(|D_x|^{-\gamma_0} g).$$

In (3-1), we have $r = p = p_0$ when $r = p$. The estimate (3-17) for

$$\alpha = -\gamma_0 = \Big(\frac{1}{p_0} - \frac{1}{2}\Big)\frac{n+1}{2} - \frac{m}{2\mu(m+2)} - \frac{2}{m+2}$$

with $p = p_0$ yields that

$$\|W_j(g)(t,\cdot)\|_{L^2(\mathbb{R}^n)} = \big\|W_j^{-\gamma_0}(|D_x|^{-\gamma_0} g)(t,\cdot)\big\|_{L^2(\mathbb{R}^n)} \lesssim \Big\| |D_x|^{-\gamma_0} g\Big\|_{L^{p_0}(S_T)},$$

and then, for $g = |D_x|^\gamma f$, where $\gamma \in \mathbb{R}$,

$$\big\|W_j(|D_x|^\gamma f)(t,\cdot)\big\|_{L^2(\mathbb{R}^n)} \lesssim \big\||D_x|^{\gamma-\gamma_0} f\big\|_{L^{p_0}(S_T)}. \tag{4-7}$$

Therefore, one has from Plancherel's theorem, Proposition 2.1, (4-6), and (4-7) that

$$\|w(t,\cdot)\|_{\dot H^\gamma(\mathbb{R}^n)} = \big\|(|D_x|^\gamma w)(t,\cdot)\big\|_{L^2(\mathbb{R}^n)} \lesssim \big\||D_x|^{\gamma-\gamma_0} f\big\|_{L^{p_0}(S_T)}$$

Hence, estimate (4-5) holds. □

In case $n \ge 2$ and $m \ge 2$ if $n = 2$, we have a more complete set of inequalities for the solution of the linear generalized Tricomi equation.

Theorem 4.3. *Let $n \ge 3$ or $n = 2$ with $m \ge 2$. Suppose w solves* (2-3) *in S_T. Then*:

(i) *For $\mu \ge \max\{2, mn/2\}$ and $1/p_1 < 1/p \le \frac{1}{2} + (m+6\mu)/(2\mu n(m+2))$,*

$$\|w\|_{L^s_t L^q_x(S_T)} \lesssim \|f\|_{L^2_t L^p_x(S_T)}, \tag{4-8}$$

where q and s are defined in (3-36).

(ii) *For $\mu \ge \max\{2, mn/2\}$ and $\frac{1}{2} \le 1/p < \frac{1}{2} + (2\mu(n-3)+m(3n-1))/(\mu(m+2)(n^2-1))$,*

$$\|w\|_{L^2_t L^q_x(S_T)} \lesssim \|f\|_{L^r_t L^p_x(S_T)}, \tag{4-9}$$

where r is defined in (3-1) *and*

$$\frac{1}{q} = \frac{n+1}{2np} + \frac{n-1}{4n} - \frac{m+6\mu}{2\mu(m+2)n}. \tag{4-10}$$

(iii) *For $\mu \ge \max\{2, m/2\}$ and $1 < p < p_1$ and $\gamma = 3/(m+2) - n\big(1/p - \frac{1}{2}\big)$,*

$$\|w(t,\cdot)\|_{\dot H^\gamma(\mathbb{R}^n)} \lesssim \|f\|_{L^2_t L^p_x(S_T)}. \tag{4-11}$$

Proof. (i) Note that, under these assumptions,

$$1 < \frac{2\mu n(m+2)}{\mu n(m+2)+6\mu+m} \le p < p_1, \quad 2 \le q < \infty, \quad 2 \le s < \infty.$$

Thus, we get estimate (4-8) by applying Proposition 2.1 and Theorem 3.8.

(ii): This will follow from the dual version of Theorem 3.8. Indeed, when

$$\frac{1}{2} \le \frac{1}{p} < \frac{1}{2} + \frac{2\mu(n-3)+m(3n-1)}{\mu(m+2)(n^2-1)},$$

then, for q defined in (4-10),

$$1 < \frac{2\mu(m+2)n}{\mu(m+2)n+6\mu+m} \le q' < p_1$$

and

$$\frac{1}{p'} = \frac{2n}{q'(n+1)} - \frac{n-1}{2(n+1)} - \frac{m+6\mu}{\mu(m+2)(n+1)}.$$

For r defined by (3-1), the conjugate exponent r' can be expressed by

$$r' = \frac{8\mu p'}{\mu(m+2)(n-1)(p'-2)+2mp'}.$$

Thus, from Remark 3.9, we have that

$$\|W_j^* f\|_{L_t^{r'} L_x^{p'}(S_T)} \lesssim \|f\|_{L_t^2 L_x^{q'}(S_T)},$$

and then, by duality,

$$\|W_j f\|_{L_t^2 L_x^q(S_T)} \lesssim \|f\|_{L_t^r L_x^p(S_T)}.$$

Therefore, from Proposition 2.1 we have that estimate (4-9) holds.

(iii): Note again that $W_j(|D_x|^\gamma f) = W_j^{-\gamma}(f)$. Then, in view of (4-6) and estimate (3-19) for $\alpha = -\gamma = n\big(1/p - \frac{1}{2}\big) - 3/(m+2)$, one has that estimate (4-11) holds. □

Now we consider the Cauchy problem (2-2).

Theorem 4.4. *Let $n \ge 2$ and $\mu \ge \max\{2, m/2\}$. Suppose v solves the Cauchy problem* (2-2). *Then*:

(i) *For $q_0 \le q < \infty$,*

$$\|v\|_{L^q(\mathbb{R}_+^{1+n})} \lesssim \|\varphi\|_{\dot H^\gamma(\mathbb{R}^n)} + \|\psi\|_{\dot H^{\gamma-2/(m+2)}(\mathbb{R}^n)}, \tag{4-12}$$

where $\gamma = n/2 - ((m+2)n+2)/(q(m+2))$.

(ii) *For $2 \le q < \infty$ when $n = 2$ and $m = 1$, and $2 \le q < q_1$ when $n \ge 2$ and $m \ge 2$ if $n = 2$,*

$$\|v\|_{L_t^s L_x^q(\mathbb{R}_+^{1+n})} \lesssim \|\varphi\|_{\dot H^\gamma(\mathbb{R}^n)} + \|\psi\|_{\dot H^{\gamma-2/(m+2)}(\mathbb{R}^n)}, \tag{4-13}$$

where

$$\frac{1}{s} = \frac{(m+2)(n-1)}{4}\Big(\frac{1}{2} - \frac{1}{q}\Big) + \frac{m}{4\mu}, \quad \gamma = \frac{n+1}{2}\Big(\frac{1}{2} - \frac{1}{q}\Big) - \frac{m}{2\mu(m+2)}.$$

(iii) *For $q_1 < q < \infty$ as well as $n \ge 2$ and $m \ge 2$ if $n = 2$,*

$$\|v\|_{L_t^2 L_x^q(\mathbb{R}_+^{1+n})} \lesssim \|\varphi\|_{\dot H^\gamma(\mathbb{R}^n)} + \|\psi\|_{\dot H^{\gamma-2/(m+2)}(\mathbb{R}^n)}, \tag{4-14}$$

where $\gamma = n\big(\frac{1}{2} - 1/q\big) - 1/(m+2)$.

Proof. The goal is to prove that

$$\|v\|_{L^\sigma_t L^\rho_x(\mathbb{R}^{1+n}_+)} \lesssim \|\varphi\|_{\dot H^\gamma(\mathbb{R}^n)} + \|\psi\|_{\dot H^{\gamma-2/(m+2)}(\mathbb{R}^n)} \tag{4-15}$$

for certain $2 \le \sigma \le \infty$ and $2 \le \rho < \infty$.

Note that

$$t(1+\phi(t)|\xi|)^{-\frac{m+4}{2(m+2)}} \le (1+\phi(t)|\xi|)^{-\frac{m}{2(m+2)}}|\xi|^{-\frac{2}{m+2}}$$
$$\le (1+\phi(t)|\xi|)^{-\frac{m}{\mu(m+2)}}|\xi|^{-\frac{2}{m+2}}.$$

In order to establish (4-15), from the expression of the function v in (4-22) together with (2-9) and (2-10) and the estimates of $b_\ell(t,\xi)(1 \le \ell \le 4)$ in (2-13) and (2-14), it suffices to show that

$$\|P\varphi\|_{L^\sigma_t L^\rho_x(\mathbb{R}^{1+n}_+)} \lesssim \|\varphi\|_{\dot H^\gamma(\mathbb{R}^n)}, \tag{4-16}$$

where the operator P is of the form

$$(P\varphi)(t,x) = \int_{\mathbb{R}^n} e^{i(x\cdot\xi+\phi(t)|\xi|)} a(t,\xi)\hat\varphi(\xi)\, đ\xi$$

with $a \in C^\infty(\mathbb{R}_+ \times \mathbb{R}^n)$ and, for any $(t,\xi) \in \mathbb{R}^{1+n}_+$,

$$|\partial^\beta_\xi a(t,\xi)| \lesssim (1+\phi(t)|\xi|)^{-m/(\mu(m+2))}|\xi|^{-|\beta|}. \tag{4-17}$$

Note that $P\varphi$ can be written as

$$(P\varphi)(t,x) = \int_{\mathbb{R}^n} e^{i(x\cdot\xi+\phi(t)|\xi|)} a(t,\xi)\widehat{|D_x|^\gamma\varphi}(\xi)\, \frac{đ\xi}{|\xi|^\gamma},$$

and, for $h = |D_x|^\gamma\varphi$, by Plancherel's theorem,

$$\|h\|_{L^2(\mathbb{R}^n)} = \||\xi|^\gamma\hat\varphi\|_{L^2(\mathbb{R}^n)} = \|\varphi\|_{\dot H^\gamma(\mathbb{R}^n)}.$$

Therefore, in order to prove (4-16), it suffices to show that the operator Q, where

$$(Qh)(t,x) = \int_{\mathbb{R}^n} e^{i(x\cdot\xi+\phi(t)|\xi|)} a(t,\xi)\hat h(\xi)\, \frac{đ\xi}{|\xi|^\gamma}, \tag{4-18}$$

is bounded from $L^2(\mathbb{R}^n)$ to $L^\sigma_t L^\rho_x(\mathbb{R}^{1+n}_+)$. By duality, it suffices to show that the adjoint Q^* of Q,

$$(Q^* f)(x) = \int_0^\infty \int_{\mathbb{R}^n} e^{i(x\cdot\xi-\phi(\tau)|\xi|)} \overline{a(\tau,\xi)}|\xi|^{-\gamma} \hat f(\tau,\xi)\, đ\xi\, d\tau, \tag{4-19}$$

satisfies

$$\|Q^* f\|_{L^2(\mathbb{R}^n)} \lesssim \|f\|_{L^{\sigma'}_t L^{\rho'}_x(\mathbb{R}^{1+n}_+)}. \tag{4-20}$$

Note that

$$\begin{aligned}\|Q^* f\|^2_{L^2(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} (Q^* f)(x)\overline{(Q^* f)(x)}\,dx \\ &= \int_{\mathbb{R}^{1+n}_+} Q Q^* f(t,x)\overline{f(t,x)}\,dt\,dx \le \|QQ^* f\|_{L^\sigma_t L^\rho_x}\|f\|_{L^{\sigma'}_t L^{\rho'}_x}.\end{aligned}$$

Thus, in order to get (4-20), we only need to show that

$$\|QQ^* f\|_{L^\sigma_t L^\rho_x} \lesssim \|f\|_{L^{\sigma'}_t L^{\rho'}_x}. \tag{4-21}$$

From (4-18) and (4-19), we have that

$$QQ^* f(t,x) = \int_0^\infty \int_{\mathbb{R}^n} e^{i(x\cdot\xi+(\phi(t)-\phi(\tau))|\xi|)} a(t,\xi)\overline{a(\tau,\xi)}\hat{f}(\tau,\xi)\,\frac{d\!\!\!{}^-\xi}{|\xi|^{2\gamma}}\,d\tau.$$

By (4-17), we further have that

$$\left|\partial_\xi^\beta (a(t,\xi)\overline{a(\tau,\xi)})\right| \lesssim (1+|\phi(t)-\phi(\tau)||\xi|)^{-\frac{m}{\mu(m+2)}}|\xi|^{-|\beta|}.$$

Thus, by Proposition 2.1, in order to get (4-21), it suffices to show that

$$\|G_j f\|_{L^\sigma_t L^\rho_x} \lesssim \|f\|_{L^{\sigma'}_t L^{\rho'}_x},$$

where the operator G_j is defined as

$$G_j f(t,x) = \int_0^\infty \int_{\mathbb{R}^n} e^{i(x\cdot\xi+(\phi(t)-\phi(\tau))|\xi|)} \Theta(|\xi|/2^j) a(t,\xi)\overline{a(\tau,\xi)}\hat{f}(\tau,\xi)\,\frac{d\!\!\!{}^-\xi}{|\xi|^{2\gamma}}\,d\tau.$$

Note that $G_j f$ is essentially $W_j^{2\gamma-2/(m+2)} f$. Therefore, in order to get (4-14), it suffices to show that

$$\|W_j^{2\gamma-2/(m+2)} f\|_{L^\sigma_t L^\rho_x} \lesssim \|f\|_{L^{\sigma'}_t L^{\rho'}_x}. \tag{4-22}$$

We first show (4-12): For $\gamma = n/2 - (n(m+2)+2)/(q(m+2))$ and $q = q_0$, we have that

$$\left(2\gamma - \frac{2}{m+2}\right) = \left(\frac{1}{p_0} - \frac{1}{2}\right)(n+1) - \frac{m}{\mu(m+2)} - \frac{2}{m+2}.$$

Thus, we have from estimate (3-3) when $r = p = p_0$ that

$$\left\| W_j^{2\gamma-2/(m+2)}\right\|_{L^{q_0}(\mathbb{R}^{1+n}_+)} \lesssim \|f\|_{L^{p_0}(\mathbb{R}^{1+n}_+)}. \tag{4-23}$$

On the other hand, from (2-22) and the compact support of Θ,

$$\|W_j^{2\gamma-2/(m+2)} f\|_{L^\infty(\mathbb{R}^{1+n}_+)} \lesssim \|f\|_{L^1(\mathbb{R}^{1+n}_+)}. \tag{4-24}$$

By interpolation between (4-23) and (4-24), we obtain that

$$\|W_j^{2\gamma-2/(m+2)} f\|_{L^q(\mathbb{R}^{1+n}_+)} \lesssim \|f\|_{L^{q'}(\mathbb{R}^{1+n}_+)}, \quad q_0 \le q \le \infty,$$

where q' is the conjugate exponent q. Therefore, we get estimate (4-12).

Next we derive (4-13). Since

$$\frac{1}{s} = \frac{(m+2)(n-1)}{4}\Big(\frac{1}{2}-\frac{1}{q}\Big) + \frac{m}{4\mu},$$

we can write

$$\frac{1}{s'} = 1 - \frac{(m+2)(n-1)}{4}\Big(\frac{1}{q'}-\frac{1}{2}\Big) - \frac{m}{4\mu}.$$

Thus, when $\gamma = (n+1)/2\left(\frac{1}{2}-1/q\right) - m/(2\mu(m+2))$, applying estimate (3-3) for $\max\{p_1, 1\} < q' \le 2$, we have

$$\|W_j^{2\gamma-2/(m+2)} f\|_{L^s_t L^q_x(\mathbb{R}^{1+n}_+)} \lesssim \|f\|_{L^{s'}_t L^{q'}_x(\mathbb{R}^{1+n}_+)},$$

and, therefore, estimate (4-13) holds.

Finally we prove (4-14). When $\gamma = n\left(\frac{1}{2}-1/q\right) - 1/(m+2)$, we have from (3-5) that, for $p_1 > 1$ and $1 < q' < p_1$,

$$\|W_j^{2\gamma-2/(m+2)} f\|_{L^2_t L^q_x(\mathbb{R}^{1+n}_+)} \lesssim \|f\|_{L^2_t L^{q'}_x(\mathbb{R}^{1+n}_+)}.$$

Thus, estimate (4-14) holds. □

Combining Theorems 4.1, 4.3, and 4.4, we obtain the following results:

Theorem 4.5. *Let u solve the Cauchy problem* (2-1) *in the strip* S_T. *Then*

$$\|u\|_{C^0_t \dot{H}^\gamma_x(S_T)} + \|u\|_{L^s_t L^q_x(S_T)} \lesssim \|\varphi\|_{\dot{H}^\gamma(\mathbb{R}^n)} + \|\psi\|_{\dot{H}^{\gamma-2/(m+2)}(\mathbb{R}^n)} + \|f\|_{L^r_t L^p_x(S_T)}, \tag{4-25}$$

provided that the exponents p, q, r, *and* s *satisfy scaling invariance condition* (1-10) *and one of the following sets of conditions*:

(i)
$$\frac{1}{p} - \frac{1}{q} = \frac{4}{(m+2)(n+1)}\Big(1+\frac{m}{2\mu}\Big),$$
$$\frac{1}{s} = \frac{(m+2)(n-1)}{4}\Big(\frac{1}{2}-\frac{1}{q}\Big) + \frac{m}{4\mu},$$
$$\gamma = \frac{n+1}{2}\Big(\frac{1}{2}-\frac{1}{q}\Big) - \frac{m}{2\mu(m+2)},$$

where $\mu \ge \mu_*$,

$$-\frac{1}{6\mu} < \gamma < \frac{47}{84} + \frac{25}{42\mu} \qquad \textit{if } n = 2, m = 1,$$

$$|\gamma - \gamma_*| < \gamma_d = \frac{2(2\mu-m)(n+1)}{\mu(m+2)(n-1)(2\mu_*-m)} \qquad \textit{if } n \ge 3 \textit{ or } n = 2, m \ge 2,$$

and

$$\gamma_* = \frac{2}{m+2} + \frac{m}{2\mu(m+2)} - \frac{(2\mu-m)(n+1)}{2\mu(2\mu_*-m)}.$$

(ii) *$n \geq 3$ or $n = 2$, $m \geq 2$ and $r = 2$,*

$$\frac{1}{s} = \frac{(m+2)(n-1)}{4}\Big(\frac{1}{2} - \frac{1}{q}\Big) + \frac{m}{4\mu}, \quad \gamma = \frac{n+1}{2}\Big(\frac{1}{2} - \frac{1}{q}\Big) - \frac{m}{2\mu(m+2)},$$

where $\mu \geq \max\{2, mn/2\}$ and

$$-\frac{m}{2\mu(m+2)} \leq \gamma < \frac{3}{m+2} - \frac{n(2\mu-m)}{\mu(m+2)(n-1)}.$$

(iii) *$n \geq 3$ or $n = 2$, $m \geq 2$ and $s = 2$,*

$$\frac{1}{r} = 1 - \frac{m}{4\mu} - \frac{(m+2)(n-1)}{4}\Big(\frac{1}{p} - \frac{1}{2}\Big), \quad \gamma = n\Big(\frac{1}{2} - \frac{1}{q}\Big) - \frac{1}{m+2},$$

where $\mu \geq \max\{2, mn/2\}$ and

$$\frac{\mu(n+1)-mn}{\mu(m+2)(n-1)} < \gamma < \frac{2}{m+2} + \frac{m}{2\mu(m+2)}.$$

Remark 4.6. We can rewrite the conditions of (4-5) in terms of q.

(i) For $\mu \geq \mu_*$,

$$\text{(4-26)} \quad \begin{cases} \frac{8}{63}\Big(1 - \frac{4}{\mu}\Big) < \frac{1}{q} \leq \frac{1}{2} & \text{if } n = 2, m = 1, \\ \frac{1}{p_2} < \frac{1}{q} + \frac{4}{(m+2)(n+1)}\Big(1 + \frac{m}{2\mu}\Big) < \frac{1}{p_1} & \text{if } n \geq 3 \text{ or } n = 2, m \geq 2. \end{cases}$$

(ii) For $\mu \geq \max\{2, mn/2\}$,

$$\text{(4-27)} \quad \frac{2n}{(n+1)p_1} - \frac{n-1}{2(n+1)} - \frac{1}{(m+2)(n+1)}\Big(6 + \frac{m}{\mu}\Big) < \frac{1}{q} \leq \frac{1}{2}.$$

(iii) For $\mu \geq \max\{2, mn/2\}$,

$$\text{(4-28)} \quad \frac{1}{2} - \frac{1}{2(m+2)n}\Big(6 + \frac{m}{\mu}\Big) < \frac{1}{q} < \frac{1}{q_1}.$$

Theorem 4.7. *Let u solve the Cauchy problem* (2-1) *in the strip S_T. Then*

$$\text{(4-29)} \quad \|u\|_{C_t^0 \dot{H}_x^\gamma(S_T)} + \|u\|_{L^q(S_T)} \lesssim \|\varphi\|_{\dot{H}^\gamma(\mathbb{R}^n)} + \|\psi\|_{\dot{H}^{\gamma-2/(m+2)}(\mathbb{R}^n)} + \||D_x|^{\gamma-\gamma_0} f\|_{L^{p_0}(S_T)}$$

provided that the exponents p, q, r, and s satisfy (1-10) *and $\mu \geq \max\{2, m/2\}$, $q_0 \leq q < \infty$, where*

$$\gamma = \tfrac{1}{2}n - \frac{n(m+2)+2}{q(m+2)}, \quad \gamma_0 = \frac{2}{m+2} + \frac{m}{2\mu(m+2)} - \frac{n+1}{2}\Big(\frac{1}{p_0} - \frac{1}{2}\Big).$$

Corollary 4.8. *Under the conditions of Theorem 4.7, one has*

$$\|u\|_{C_t^0\dot H_x^\gamma(S_T)}+\|u\|_{L^q(S_T)}+\||D_x|^{\gamma-1/(m+2)}u\|_{L^{q_0^*}(S_T)} \lesssim \|\varphi\|_{\dot H^\gamma(\mathbb{R}^n)}+\|\psi\|_{\dot H^{\gamma-2/(m+2)}(\mathbb{R}^n)}+\||D_x|^{\gamma-1/(m+2)}f\|_{L^{p_0^*}(S_T)}, \tag{4-30}$$

where $\gamma=n/2-((m+2)n+2)/(q(m+2))$ *and* $q_0^*\le q<\infty$.

Proof. This follows by combining estimate (4-29) and Remark 4.2 when $\mu=\mu_*$. □

An application of Theorem 4.5 yields the following:

Corollary 4.9. *Let u solve the Cauchy problem*

$$\partial_t^2u-t^m\Delta u=f(t,x)g(t,x)\quad \text{in } S_T,$$
$$u(0,\cdot)=\partial_tu(0,\cdot)=0.$$

Then, for any $\mu\ge\mu_*$ *and* $0<R\le\infty$,

$$\|u\|_{C_t^0\dot H_x^\gamma(S_T\cap\Lambda_R)}+\|u\|_{L_t^sL_x^q(S_T\cap\Lambda_R)}+\|u\|_{L_t^\infty L_x^\delta(S_T\cap\Lambda_R)} \lesssim \|f\|_{L_t^\sigma L_x^\rho(S_T\cap\Lambda_R)}\|g\|_{L_t^sL_x^q(S_T\cap\Lambda_R)}, \tag{4-31}$$

where q is as in (4-26),

$$\rho=\frac{\mu(m+2)(n+1)}{2(2\mu+m)},\qquad \sigma=\frac{\mu(n+1)}{2\mu-mn}, \tag{4-32}$$

$$\frac1s=\frac{(m+2)(n-1)}{4}\Big(\frac12-\frac1q\Big)+\frac{m}{4\mu},\qquad \frac n\delta=\frac nq+\frac{2}{m+2}\Big(\frac1s-\frac{m}{4\mu}\Big), \tag{4-33}$$

and

$$\Lambda_R=\{(t,x)\in\mathbb{R}_+\times\mathbb{R}^n \mid |x|+\phi(t)<R\}.$$

Proof. First we study the case $R=\infty$. Note that (4-33) gives that

$$n\Big(\frac12-\frac1\delta\Big)=\frac{n+1}{2}\Big(\frac12-\frac1q\Big)-\frac{m}{2\mu(m+2)}.$$

Applying estimate (4-25) in case (i) together with the Sobolev embedding

$$\dot H^{n(1/2-1/\delta)}(\mathbb{R}^n)\hookrightarrow L^\delta(\mathbb{R}^n),$$

we have

$$\|u\|_{C_t^0\dot H_x^\gamma(S_T)}+\|u\|_{L_t^sL_x^q(S_T)}+\|u\|_{L_t^\infty L_x^\delta(S_T)}\lesssim\|fg\|_{L_t^rL_x^p(S_T)},$$

where $1/p=1/q+1/\rho$ and $1/r=1/s+1/\sigma$. In addition, from Hölder's inequality,

$$\|fg\|_{L_t^rL_x^p(S_T)}\le\|f\|_{L_t^\sigma L_x^\rho(S_T)}\|g\|_{L_t^sL_x^q(S_T)}. \tag{4-34}$$

Thus, estimate (4-31) holds for $R=\infty$.

Now let $R<\infty$. Let χ denote the characteristic function of $S_T\cap\Lambda_R$. If u solves $\partial_t^2u-t^m\Delta u=fg$ with vanishing initial data and u_χ solves $\partial_t^2u_\chi-t^m\Delta u_\chi=\chi fg$ with vanishing initial data, then $u=u_\chi$ in $S_T\cap\Lambda_R$ due to finite propagation speed (see [Taniguchi and Tozaki 1980]). Therefore,

$$\begin{aligned}\|u\|_{C_t^0\dot H_x^\gamma(S_T\cap\Lambda_R)}+\|u\|_{L_t^sL_x^q(S_T\cap\Lambda_R)}+\|u\|_{L_t^\infty L_x^\delta(S_T\cap\Lambda_R)}&\\ =\|u_\chi\|_{C_t^0\dot H_x^\gamma(S_T)}+\|u_\chi\|_{L_t^sL_x^q(S_T)}+\|u_\chi\|_{L_t^\infty L_x^\delta(S_T)}&\\ \le\|\chi f\|_{L_t^\sigma L_x^\rho(S_T)}\|\chi g\|_{L_t^sL_x^q(S_T)}.&\end{aligned}$$

Consequently, estimate (4-31) holds. □

As another application of Theorem 4.5 we have the following:

Corollary 4.10. *Let u be a solution of*

$$\begin{aligned}\partial_t^2u-t^m\Delta u&=F(v)\quad\text{in } S_T,\\ u(0,\cdot)&=\partial_tu(0,\cdot)=0.\end{aligned}$$

If $q<\infty$ and $1/(m+2)\le\gamma=n/2-(n(m+2)+2)/(q(m+2))\le(m+3)/(m+2)$, then

$$\|u\|_{C_t^0\dot H_x^\gamma(S_T)}+\|u\|_{L^q(S_T)}+\||D_x|^{\gamma-1/(m+2)}u\|_{L^{q_0^*}(S_T)}\lesssim\|F'(v)\|_{L^{\mu_*/2}(S_T)}\||D_x|^{\gamma-1/(m+2)}v\|_{L^{q_0^*}(S_T)}.\tag{4-35}$$

Proof. This follows from estimate (4-30) by taking fractional derivatives. Indeed, for $0\le\gamma-1/(m+2)\le 1$, one has

$$\begin{aligned}\|u\|_{C_t^0\dot H_x^\gamma(S_T)}+\|u\|_{L^q(S_T)}+\||D_x|^{\gamma-1/(m+2)}u\|_{L^{q_0^*}(S_T)}&\\ \lesssim\big\||D_x|^{\gamma-1/(m+2)}(F(v))\big\|_{L^{p_0^*}(S_T)}&\\ \lesssim\|F'(v)\|_{L^{\mu_*/2}(S_T)}\big\||D_x|^{\gamma-1/(m+2)}v\big\|_{L^{q_0^*}(S_T)}.&\end{aligned}$$ □

5. Solvability of the semilinear generalized Tricomi equation

In this section, we will apply Theorems 4.5 and 4.7 and Corollaries 4.8–4.10 with $\mu=\mu_*$ to establish the existence and uniqueness of the solution u of problem (1-1). Thereby, we will use the following iteration scheme: For $j\in\mathbb{N}_0$, let u_j be the solution of

$$\begin{aligned}\partial_t^2u_j-t^m\Delta u_j&=F(u_{j-1})\quad\text{in }\mathbb{R}_+\times\mathbb{R}^n,\\ u_j(0,\cdot)=\varphi,&\quad\partial_tu_j(0,\cdot)=\psi,\end{aligned}\tag{5-1}$$

where $u_{-1}=0$.

Notice that, for $\mu = \mu_*$, the exponents from (4-25) in case (i) are

$$\gamma_* = \frac{1}{m+2}, \quad \gamma_d = \frac{2(n+1)}{\mu_*(m+2)(n-1)}.$$

In order to get the existence of solutions of the Cauchy problem (1-1) as stated in Theorems 1.1, 1.4, and 1.5, we need to show that, for the sequences $\{u_j\}_{j=0}^{\infty}$ and $\{F(u_j)\}_{j=0}^{\infty}$ defined by (5-1), there exist a $T > 0$ and a function u such that

$$u_j \to u \quad \text{in } L^1_{\mathrm{loc}}(S_T) \quad \text{as } j \to \infty, \tag{5-2}$$

$$F(u_j) \to F(u) \quad \text{in } L^1_{\mathrm{loc}}(S_T) \quad \text{as } j \to \infty. \tag{5-3}$$

From (5-2) and (5-3), one obviously has that the limit function u solves problem (1-1) in S_T.

Furthermore, let u, $\tilde{u}$ both solve the Cauchy problem (1-1) in S_T. Then $v = u - \tilde{u}$ satisfies

$$\begin{aligned} \partial_t^2 v - t^m \Delta v &= G(u, \tilde{u}) v \quad \text{in } S_T, \\ v(0,\cdot) &= \partial_t v(0,\cdot) = 0, \end{aligned} \tag{5-4}$$

where $G(u, \tilde{u}) = (F(u) - F(\tilde{u}))/(u - \tilde{u})$ if $u \neq \tilde{u}$ and $G(u, u) = F'(u)$. For certain s, $q \geq 2$, we will show that $v \in L^s_t L^q_x(S_T)$ and

$$\|v\|_{L^s_t L^q_x(S_T)} \leq \tfrac{1}{2} \|v\|_{L^s_t L^q_x(S_T)}. \tag{5-5}$$

Uniqueness of the solution of the Cauchy problem (1-1) in S_T follows.

5.1. *Proof of Theorem 1.1.*

5.1.1. *Case $\kappa_1 < \kappa < \kappa_*$.* From the assumptions of Theorem 1.1, we have

$$\gamma = \frac{n+1}{4} - \frac{n+1}{\mu_*(\kappa-1)} - \frac{m}{2\mu_*(m+2)}$$

and

$$q = \frac{\mu_*(\kappa-1)}{2}, \quad \frac{1}{s} = \frac{(m+2)(n-1)}{4}\Big(\frac{1}{2} - \frac{1}{q}\Big) + \frac{m}{4\mu_*}. \tag{5-6}$$

Thus,

$$\gamma = \frac{n+1}{2}\Big(\frac{1}{2} - \frac{1}{q}\Big) - \frac{m}{2\mu_*(m+2)}, \quad \frac{1}{m+2} - \frac{2(n+1)}{\mu_*(m+2)(n-1)} < \gamma < \frac{1}{m+2}.$$

Existence. In order to show (5-2), set

$$\begin{aligned} H_j(T) &= \|u_j\|_{C^0_t \dot{H}^{\gamma}_x(S_T)} + \|u_j\|_{L^s_t L^q_x(S_T)}, \\ N_j(T) &= \|u_j - u_{j-1}\|_{L^s_t L^q_x(S_T)}. \end{aligned} \tag{5-7}$$

We claim that there exists a constant $\varepsilon_0 > 0$ small such that

$$2T^{1/q-1/s} H_0(T) \le \varepsilon_0 \tag{5-8}$$

and

$$H_j(T) \le 2H_0(T), \quad N_j(T) \le \tfrac{1}{2} N_{j-1}(T). \tag{5-9}$$

Indeed, from the iteration scheme (5-1), we have

$$(\partial_t^2 - t^m \Delta)(u_{j+1} - u_{k+1}) = G(u_j, u_k)(u_j - u_k). \tag{5-10}$$

Note that in (4-32),

$$\rho = \sigma = \tfrac{1}{2}\mu_*$$

when $\mu = \mu_*$. Thus, from (4-31) and condition (1-2),

$$\begin{aligned} \|u_{j+1} - u_{k+1}\|_{C_t^0 \dot H_x^\gamma(S_T)} + \|u_{j+1} - u_{k+1}\|_{L_t^s L_x^q(S_T)} &\lesssim \|G(u_j, u_k)\|_{L^{\mu_*/2}(S_T)} \|u_j - u_k\|_{L_t^s L_x^q(S_T)} \\ &\lesssim \big(\|u_j\|_{L^q(S_T)}^{\kappa-1} + \|u_k\|_{L^q(S_T)}^{\kappa-1}\big)\|u_j - u_k\|_{L_t^s L_x^q(S_T)}. \end{aligned} \tag{5-11}$$

Note that $s > q$ for $\kappa < \kappa_*$. By Hölder's inequality, we arrive at

$$\|u_j\|_{L^q(S_T)} \le T^{1/q-1/s} \|u_j\|_{L_t^s L_x^q(S_T)}. \tag{5-12}$$

Since $u_{-1} = 0$, (5-11) together with (5-12) implies that

$$\|u_{j+1} - u_0\|_{L_t^s L_x^q(S_T)} + \|u_{j+1} - u_0\|_{C_t^0 \dot H_x^\gamma(S_T)} \lesssim T^{(\kappa-1)(1/q-1/s)} \|u_j\|_{L_t^s L_x^q(S_T)}^\kappa.$$

From the Minkowski inequality, we have that there exists an ε_0 with $0 < \varepsilon_0 \le 2^{-2/(\kappa-1)}$ such that

$$H_{j+1}(T) \le H_0(T) + \tfrac{1}{2} H_j(T) \quad \text{if } T^{1/q-1/s} H_j(T) \le \varepsilon_0.$$

Therefore, by induction on j,

$$H_j(T) \le 2H_0(T) \quad \text{if } 2T^{1/q-1/s} H_0(T) \le \varepsilon_0. \tag{5-13}$$

Taking $k = j-1$ in (5-10), estimates (5-11)–(5-13) yield that

$$N_{j+1}(T) \le \tfrac{1}{2} N_j(T) \quad \text{if } 2H_0(T) T^{1/q-1/s} \le \varepsilon_0,$$

which together with (5-13) implies that (5-9) holds as long as (5-8) holds.

Since $u_{-1} \equiv 0$ and u_0 is a solution of problem (2-2), we have from (4-13) that, for $\varphi \in \dot H^\gamma(\mathbb{R}^n)$ and $\psi \in \dot H^{\gamma - 2/(m+2)}(\mathbb{R}^n)$,

$$N_0(T) \le H_0(T) \lesssim \|\varphi\|_{\dot H^\gamma(\mathbb{R}^n)} + \|\psi\|_{\dot H^{\gamma-2/(m+2)}(\mathbb{R}^n)}.$$

Thus, by choosing $T > 0$ small, (5-8) holds. Consequently, there is a function $u \in C_t^0 \dot{H}_x^\gamma(S_T) \cap L_t^s L_x^q(S_T)$ such that

$$u_j \to u \quad \text{in } L_t^s L_x^q(S_T) \text{ as } j \to \infty, \tag{5-14}$$

and, therefore, (5-2) holds. It also follows that u_j converges to u almost everywhere. By Fatou's lemma, it follows that

$$\|u\|_{C_t^0 \dot{H}_x^\gamma(S_T)} + \|u\|_{L_t^s L_x^q(S_T)} \le \liminf_{j\to\infty}\big(\|u_j\|_{C_t^0 \dot{H}_x^\gamma(S_T)} + \|u_j\|_{L_t^s L_x^q(S_T)}\big) \le 2H_0(T), \tag{5-15}$$

which shows that estimate (1-4) holds.

Now we prove (5-3). It suffices to show that $F(u)$ is bounded in $L_t^r L_x^p(S_T)$ and $F(u_j)$ converges to $F(u)$ in $L_t^r L_x^p(S_T)$ as $j \to \infty$, where $p = q/\kappa$ and $1/r = 1 - m/(4\mu_*) - (m+2)(n-1)/4\,\big(1/p - \frac{1}{2}\big)$. In fact, $r\kappa < s$ if $\kappa < \kappa_*$, thus, for $q = p\kappa$, by condition (1-2) and Hölder's inequality, we have

$$\|F(u)\|_{L_t^r L_x^p(S_T)} \lesssim \|u\|^\kappa_{L_t^{r\kappa} L_x^{p\kappa}(S_T)} \lesssim T^{1/r-\kappa/s}\|u\|^\kappa_{L_t^s L_x^q(S_T)}.$$

Moreover, in view of $1/p - 1/q = 1/r - 1/s = 2/\mu_*$, by Hölder's inequality and estimates (5-11)–(5-13) and (5-15), we have

$$\begin{aligned}\|F(u_j) - F(u)\|_{L_t^r L_x^p(S_T)} &\le \|G(u_j, u)\|_{L^{\mu_*/2}(S_T)}\|u_j - u\|_{L_t^s L_x^q(S_T)}\\ &\lesssim T^{(\kappa-1)(1/q-1/s)} H_0(T)^{\kappa-1}\|u_j - u\|_{L_t^s L_x^q(S_T)}\\ &\lesssim \|u_j - u\|_{L_t^s L_x^q(S_T)}.\end{aligned}$$

Applying (5-14), we have that $F(u_j)$ converges to $F(u)$ in $L_t^r L_x^p(S_T)$ and, therefore, (5-3) holds.

From (5-2) and (5-3), we have that the limit function $u \in C_t^0 \dot{H}_x^\gamma(S_T)) \cap L_t^s L_x^q(S_T)$ solves the Cauchy problem (1-1) in S_T.

Uniqueness. Suppose $u, \tilde{u} \in C([0,T], \dot{H}^\gamma(\mathbb{R}^n)) \cap L_t^s L_x^q(S_T)$ solve the Cauchy problem (1-1) in S_T. Then $v = u - \tilde{u} \in C([0,T], \dot{H}^\gamma(\mathbb{R}^n)) \cap L_t^s L_x^q(S_T)$ is a solution of problem (5-4). From Corollary 4.9, we have that

$$\begin{aligned}&\|v\|_{L_t^s L_x^q(S_T)}\\ &\le C(\|u\|^{\kappa-1}_{L^q(S_T)} + \|\tilde{u}\|^{\kappa-1}_{L^q(S_T)})\|v\|_{L_t^s L_x^q(S_T)} && \text{(by (4-31) and (1-2))}\\ &\le C T^{(\kappa-1)(1/q-1/s)} \times (\|u\|^{\kappa-1}_{L_t^s L_x^q(S_T)} + \|\tilde{u}\|^{\kappa-1}_{L_t^s L_x^q(S_T)})\|v\|_{L_t^s L_x^q(S_T)} && \text{(by Hölder's inequality)}\\ &\le C 2^\kappa (T^{1/q-1/s} H_0(T))^{\kappa-1}\|v\|_{L_t^s L_x^q(S_T)} && \text{(by (5-15))}\\ &\le \tfrac{1}{2}\|v\|_{L_t^s L_x^q(S_T)} && \text{(by (5-8)).}\end{aligned}$$

Thus (5-5) holds and $u=\tilde{u}$ in S_T.

5.1.2. *Case $\kappa_*\le\kappa$ if $n=2$ or $\kappa_*\le\kappa\le\kappa_3$ if $n\ge 3$.*

Existence. From the assumptions of Theorem 1.1, we have

$$\gamma=\tfrac{1}{2}n-\frac{4}{(m+2)(\kappa-1)},\qquad s=q=\frac{\mu_*(\kappa-1)}{2}.$$

Thus,

$$\frac{1}{m+2}\le\gamma=\tfrac{1}{2}n-\frac{(m+2)n+2}{q(m+2)}\le\frac{m+3}{m+2}.$$

To show (5-2), we set

$$H_j(T)=\|u_j\|_{C_t^0\dot{H}_x^\gamma(S_T)}+\|u_j\|_{L^q(S_T)}+\||D_x|^{\gamma-1/(m+2)}u_j\|_{L^{q_0^*}(S_T)},$$

and

$$N_j(T)=\|u_j-u_{j-1}\|_{L^{q_0^*}(S_T\cap\Lambda_R)}.\tag{5-16}$$

We claim that there exists a constant $\varepsilon_0>0$ such that

$$H_0(T)\le\varepsilon_0,\tag{5-17}$$

and

$$H_j(T)\le 2H_0(T),\quad N_j(T)\le\tfrac{1}{2}N_{j-1}(T).\tag{5-18}$$

Indeed, since $u_{-1}=0$, from the iteration scheme (5-1), we have

$$(\partial_t^2-t^m\Delta)(u_{j+1}-u_0)=F(u_j).\tag{5-19}$$

Thus, estimate (4-35) together with condition (1-2) yields, for $0\le\gamma-1/(m+2)\le 1$,

$$\begin{aligned}H_{j+1}(T)&\le H_0(T)+C\|F'(u_j)\|_{L^{\mu_*/2}(S_T)}\||D_x|^{\gamma-1/(m+2)}u_j\|_{L^{q_0^*}(S_T)}\\&\le H_0(T)+C\|u_j\|_{L^q(S_T)}^{\kappa-1}\||D_x|^{\gamma-1/(m+2)}u_j\|_{L^{q_0^*}(S_T)}\\&\le H_0(T)+CH_j(T)^\kappa.\end{aligned}$$

Therefore, by induction, we have that

$$H_j(T)\le 2H_0(T)\quad\text{if } C2^\kappa H_0(T)^{\kappa-1}<1.$$

Consequently,

$$H_j(T)\le 2H_0(T)\quad\text{if } H_0(T)\le\varepsilon_0\tag{5-20}$$

for some $\varepsilon_0>0$ small. Notice that, for q and s from (5-6), when $q=s$, so $q=s=q_0^*$. Hence, by using estimates (5-11)–(5-13) together with (5-20), we get that for N_j defined in (5-16),

$$N_j(T)\le\tfrac{1}{2}N_{j-1}(T)\quad\text{if } H_0(T)\le\varepsilon_0.\tag{5-21}$$

Estimates (5-20) and (5-21) tell us that (5-18) holds as long as (5-17) holds. To get (5-17), from estimate (4-30) (with $f = 0$) we have that, for $\varphi \in \dot{H}^{\gamma}(\mathbb{R}^n)$ and $\psi \in \dot{H}^{\gamma-2/(m+2)}(\mathbb{R}^n)$,

$$H_0(T) \lesssim \|\varphi\|_{\dot{H}^{\gamma}(\mathbb{R}^n)} + \|\psi\|_{\dot{H}^{\gamma-2/(m+2)}(\mathbb{R}^n)}. \tag{5-22}$$

Due to the continuity of the norm in $L^q(S_T)$, (5-17) holds for some $T > 0$ small. (If $\|\varphi\|_{\dot{H}^{\gamma}(\mathbb{R}^n)} + \|\psi\|_{\dot{H}^{\gamma-2/(m+2)}(\mathbb{R}^n)}$ is small, then (5-17) holds for any $T > 0$, consequently, we get global existence.)

Note that $q = \mu_*(\kappa-1)/2 \geq q_0^*$ when $\kappa \geq \kappa_*$. Thus, from Hölder's inequality and (5-22),

$$N_0(T) = \|u_0\|_{L^{q_0^*}(S_T\cap\Lambda_R)} \lesssim \|u_0\|_{L^q(S_T)} \lesssim H_0(T). \tag{5-23}$$

From estimates (5-17), (5-18), and (5-23), we get that there exists a function $u \in C_t^0\dot{H}_x^{\gamma}(S_T) \cap L^q(S_T)$ with $|D_x|^{\gamma-1/(m+2)}u \in L^{q_0^*}(S_T)$ such that

$$u_j \to u \quad \text{in } L^{q_0^*}(S_T\cap\Lambda_R) \text{ as } j \to \infty, \tag{5-24}$$

and (5-2) holds. Thus, from Fatou's lemma and (5-18), it follows that

$$\|u\|_{C_t^0\dot{H}_x^{\gamma}(S_T)} + \|u\|_{L^q(S_T)} + \||D_x|^{\gamma-1/(m+2)}u\|_{L^{q_0^*}(S_T)} \leq 2H_0(T) \tag{5-25}$$

and u satisfies estimate (1-4).

Since $q = \mu_*(\kappa-1)/2 \geq \kappa$ when $\kappa \geq \kappa_*$, we have from condition (1-2) that $F(u)$ is locally integrable for $u \in L^q(S_T)$. By Hölder's inequality,

$$\begin{aligned}\int_{S_T\cap\Lambda_R} |F(u_j)-F(u)|\,dt\,dx &= \int_{S_T\cap\Lambda_R} |G(u_j,u)||u_j-u|\,dt\,dx\\ &\leq \|G(u_j,u)\|_{L^{p_0^*}(S_T\cap\Lambda_R)}\|u_j-u\|_{L^{q_0^*}(S_T\cap\Lambda_R)}.\end{aligned}$$

Note that $p_0^* < \mu_*/2$. Thus, from condition (1-2) we have that

$$\begin{aligned}\|G(u_j,u)\|_{L^{p_0^*}(S_T\cap\Lambda_R)} &\lesssim \|u_j\|^{\kappa-1}_{L^{p_0^*(\kappa-1)}(S_T\cap\Lambda_R)} + \|u\|^{\kappa-1}_{L^{p_0^*(\kappa-1)}(S_T\cap\Lambda_R)}\\ &\lesssim \|u_j\|^{\kappa-1}_{L^q(S_T\cap\Lambda_R)} + \|u\|^{\kappa-1}_{L^q(S_T\cap\Lambda_R)} \lesssim H_0(T)^{\kappa-1},\end{aligned}$$

which together with (5-24) implies that $F(u_j) \to F(u)$ in $L^1_{\text{loc}}(S_T)$. Hence, (5-3) holds.

From (5-2) and (5-3), we have that the limit function $u \in C_t^0\dot{H}_x^{\gamma}(S_T) \cap L^q(S_T)$ with $|D_x|^{\gamma-1/(m+2)}u \in L^{q_0^*}(S_T)$ is a weak solution of the Cauchy problem (1-1) in S_T.

Uniqueness. Suppose $u, \tilde{u} \in C_t^0 \dot{H}_x^\gamma(S_T) \cap L^q(S_T)$ with $|D_x|^{\gamma-1/(m+2)} u$ and $|D_x|^{\gamma-1/(m+2)} \tilde{u} \in L^{q_0^*}(S_T)$ solving the Cauchy problem (1-1) in S_T. Then $v = u - \tilde{u} \in C_t^0 \dot{H}_x^\gamma(S_T) \cap L^q(S_T)$ is a weak solution of problem (5-4). Thus, it follows from Corollary 4.9 that

$$\begin{aligned} \|v\|_{L^q(S_T)} &\le C\big(\|u\|_{L^q(S_T)}^{\kappa-1} + \|\tilde{u}\|_{L^q(S_T)}^{\kappa-1}\big)\|v\|_{L^q(S_T)} && \text{(by (4-31) and (1-2))} \\ &\le C 2^\kappa H_0(T)^{\kappa-1} \|v\|_{L^q(S_T)} && \text{(by (5-25))} \\ &\le \tfrac{1}{2}\|v\|_{L_t^s L_x^q(S_T)} && \text{(by (5-17)).} \end{aligned}$$

Thus (5-5) holds and $u = \tilde{u}$ in S_T.

5.1.3. *Case $n \ge 3$ and $\kappa > \kappa_3$, $\kappa \in \mathbb{N}$.*

Existence. From the assumptions of Theorem 1.1, we have

$$\gamma = \tfrac{1}{2}n - \frac{4}{(m+2)(\kappa-1)}, \qquad s = q = \frac{\mu_*(\kappa-1)}{2}, \qquad F(u) = \pm u^\kappa,$$

and

$$\gamma = \tfrac{1}{2}n - \frac{(m+2)n+2}{q(m+2)} > 1 + \frac{1}{m+2}.$$

To verify (5-2), we set

$$H_j(T) = \|u_j\|_{C_t^0 \dot{H}_x^\gamma(S_T)} + \sup_{q_0^* \le \tau \le \frac{1}{2}\mu_*(\kappa-1)} \Big\| |D_x|^{\frac{(m+2)n+2}{\tau(m+2)} - \frac{4}{(m+2)(\kappa-1)}} u_j \Big\|_{L^\tau(S_T)}$$

and

$$N_j(T) = \|u_j - u_{j-1}\|_{L^{q_0^*}(S_T \cap \Lambda_R)}.$$

We claim that there exists a constant $\varepsilon_0 > 0$ such that

$$H_0(T) \le \varepsilon_0 \tag{5-26}$$

and

$$H_j(T) \le 2H_0(T), \quad N_j(T) \le \tfrac{1}{2}N_{j-1}(T). \tag{5-27}$$

In fact, applying Minkowski's inequality and estimate (4-30) (with $\varphi = \psi = 0$),

$$\begin{aligned} H_{j+1}(T) \le{}& H_0(T) \\ &+ C \sup_{q_0^* \le \tau \le \mu_*(\kappa-1)/2} \big\| |D_x|^{\frac{1}{2}n - \frac{1}{m+2} - \frac{4}{(m+2)(\kappa-1)}} (u_j^\kappa) \big\|_{L^{p_0^*}(S_T)}. \end{aligned} \tag{5-28}$$

Note that $\alpha = n/2 - 1/(m+2) - 4/((m+2)(\kappa-1)) > 1$ when $\kappa > \kappa_3$. Thus, $|D_x|^\alpha (u_j^\kappa)$ can be expressed as a finite linear combination of $\prod_{\ell=1}^\kappa |D_x|^{\alpha_\ell} u_j$, where $0 \le \alpha_\ell \le \alpha$ $(1 \le \ell \le \kappa)$ and $\sum_{\ell=1}^{\kappa} \alpha_\ell = \alpha$. By Hölder's inequality, $\| |D_x|^\alpha (u_j^\kappa) \|_{L^{p_0^*}(S_T)}$ is dominated by a finite sum of terms of the form

$$\prod_{\ell=1}^{\kappa} \| |D_x|^{\alpha_\ell} u_j \|_{L^{\tau_\ell}(S_T)},$$

where $\sum_{\ell=1}^{\kappa} 1/\tau_\ell = 1/p_0^*$. We choose τ_ℓ so that

$$\alpha_\ell = \frac{n(m+2)+2}{\tau_\ell(m+2)} - \frac{4}{(m+2)(\kappa-1)}.$$

Then

$$q_0^* \le \tau_\ell \le \frac{\mu_*(\kappa-1)}{2}, \quad \sum_{\ell=1}^{\kappa} \frac{1}{\tau_\ell} = \frac{1}{p_0^*},$$

and, therefore,

$$\| |D_x|^{\alpha_\ell} u_j \|_{L^{\tau_\ell}(S_T)} \le H_j(T),$$

which together with (5-28) yields that

$$H_{j+1}(T) \le H_0(T) + C_\kappa H_j(T)^\kappa.$$

By induction, we have that

$$H_j(T) \le 2H_0(T) \quad \text{if } H_0(T) \le \varepsilon_0. \tag{5-29}$$

For q and s from (5-6), when $q = s$, then $q = s = q_0^*$. Hence, by estimates (5-11)–(5-13) and together with (5-29), we get that

$$N_j(T) \le \tfrac{1}{2} N_{j-1}(T) \quad \text{if } H_0(T) \le \varepsilon_0. \tag{5-30}$$

From (5-29) and (5-30), we get that (5-27) holds as long as (5-26) holds.

Note that

$$\frac{n(m+2)+2}{\tau(m+2)} - \frac{4}{(m+2)(\kappa-1)} = 0, \tag{5-31}$$

for $\tau = \mu_*(\kappa-1)/2$ and

$$\frac{n(m+2)+2}{\tau(m+2)} - \frac{4}{(m+2)(\kappa-1)} = \gamma - \frac{1}{m+2}. \tag{5-32}$$

for $\tau = q_0^*$. On the other hand, we have from (4-30) (with $f = 0$) that, for $\varphi \in \dot{H}^\gamma(\mathbb{R}^n)$ and $\psi \in \dot{H}^{\gamma-2/(m+2)}(\mathbb{R}^n)$,

$$\|u_0\|_{C_t^0 \dot{H}_x^\gamma(S_T)} + \|u_0\|_{L^{\mu_*(\kappa-1)/2}(S_T)} + \| |D_x|^{\gamma-1/(m+2)} u_0 \|_{L^{p_0^*}(S_T)} \lesssim \|\varphi\|_{\dot{H}^\gamma(\mathbb{R}^n)} + \|\psi\|_{\dot{H}^{\gamma-2/(m+2)}(\mathbb{R}^n)}. \tag{5-33}$$

By interpolation together with (5-31)–(5-33), we conclude that

$$H_0(T) \lesssim \|\varphi\|_{\dot{H}^\gamma(\mathbb{R}^n)} + \|\psi\|_{\dot{H}^{\gamma-2/(m+2)}(\mathbb{R}^n)}.$$

It follows that (5-26) holds by choosing $T > 0$ small. (We can take $T = \infty$ if $\|\varphi\|_{\dot H^{\gamma}(\mathbb{R}^n)} + \|\psi\|_{\dot H^{\gamma-2/(m+2)}(\mathbb{R}^n)}$ is small which then yields global existence.)

From Hölder's inequality and (5-31),

$$N_0(T) = \|u_0\|_{L^{q_0^*}(S_T\cap\Lambda_R)} \le C_R\|u_0\|_{L^{\mu_*(\kappa-1)/2}(S_T)} \le C_R H_0(T) < \infty. \tag{5-34}$$

Therefore, we have from (5-27), (5-26), and (5-34) that there exists a function $u \in C_t^0\dot H_x^{\gamma}(S_T)\cap L^q(S_T)$ with $|D_x|^{\gamma-1/(m+2)}u \in L^{q_0^*}(S_T)$ such that

$$u_j \to u \quad \text{in } L^{q_0^*}(S_T\cap\Lambda_R) \text{ as } j\to\infty,$$

and, therefore, (5-2) holds. Thus, from Fatou's lemma and (5-27),

$$\|u\|_{C_t^0\dot H_x^{\gamma}(S_T)} + \|u\|_{L^q(S_T)} + \big\||D_x|^{\gamma-1/(m+2)}u\big\|_{L^{q_0^*}(S_T)} \le 2H_0(T) \tag{5-35}$$

and u satisfies estimate (1-4).

Note that $q = \mu_*(\kappa-1)/2 \ge \kappa$ when $\kappa > \kappa_3$. Thus, for $u \in L^q(S_T)$, by Hölder's inequality and condition (1-2), we get that $F(u)$ is locally integrable and $F(u_j)$ converges to $F(u)$ in $L^1_{\mathrm{loc}}(S_T)$, and hence (5-3) holds.

Applying (5-2) and (5-3), it follows that the limit function $u \in C_t^0\dot H_x^{\gamma}(S_T)\cap L^q(S_T)$ with $|D_x|^{\gamma-1/(m+2)}u \in L^{q_0^*}(S_T)$ is a weak solution of the Cauchy problem (1-1) in S_T.

Uniqueness. This follows from the same arguments as in 5.1.2. □

5.2. ***Proof of Theorem 1.4.*** From the assumption of Theorem 1.4, we have

$$\gamma = \frac{n}{2} - \frac{4}{(m+2)(\kappa-1)},$$
$$\frac{1}{q} = \frac{1}{(m+2)(n+1)}\Big(\frac{8}{\kappa-1} - \frac{m}{\mu_*}\Big) - \frac{n-1}{2(n+1)},$$

and

$$\frac{1}{s} = \frac{(m+2)(n-1)}{4}\Big(\frac{1}{2} - \frac{1}{q}\Big) + \frac{m}{4\mu_*}.$$

Thus,

$$\gamma = \Big(\frac{n+1}{2}\Big)\Big(\frac{1}{2} - \frac{1}{q}\Big) - \frac{m}{2\mu_*(m+2)}$$

and

$$\frac{1}{m+2} \le \gamma < \frac{1}{m+2} + \frac{2(n+1)}{\mu_*(m+2)(n-1)},$$

where $\kappa_* \le \kappa < \kappa_2$.

To show (5-2), we set

$$H_j(T) = \|u_j\|_{C_t^0\dot H_x^{\gamma}(S_T)} + \|u_j\|_{L_t^sL_x^q(S_T)} + \|u_j - u_0\|_{L_t^\infty L_x^{\tilde s}(S_T)}$$

and

$$N_j(T) = \|u_j - u_{j-1}\|_{L_t^s L_x^q(S_T)},$$

where

$$\frac{1}{s} + \frac{(m+2)n}{2q} = \frac{(m+2)n}{2\delta} = \frac{m+2}{2}\Big(\frac{n}{2} - \gamma\Big). \tag{5-36}$$

We claim that there exist a constant $\varepsilon_0 > 0$ and a $\theta \in [0, 1]$ such that

$$2H_0(T)^\theta (2H_0(T) + \|u_0\|_{L_t^\infty L_x^\delta(S_T)})^{1-\theta} \le \varepsilon_0 \tag{5-37}$$

and

$$H_j(T) \le 2H_0(T), \qquad N_j(T) \le \tfrac{1}{2} N_{j-1}(T). \tag{5-38}$$

Indeed, due to (5-36), from Sobolev's embedding theorem we have that

$$\|u(t,\cdot)\|_{L^\delta(\mathbb{R}^n)} \lesssim \|u(t,\cdot)\|_{\dot{H}^\gamma(\mathbb{R}^n)}.$$

Applying Hölder's inequality, we get that

$$\|u_j\|_{L^{\mu_*(\kappa-1)/2}(S_T)} \le \|u_j\|^{\theta}_{L_t^s L_x^q(S_T)} \|u_j\|^{1-\theta}_{L_t^\infty L_x^\delta(S_T)},$$

where $\theta = 2/(n(m+2)+2) + 4n(m+2)/(\mu_*(m+2)(n-1)(q-2)+2mq)$. Note that $0 \le \theta \le 1$ for $\gamma \ge 1/(m+2)$.

By the same arguments as in the proof of Theorem 1.1, we get that (5-37) and (5-38) hold. Consequently, (5-2) and (5-3) also hold. Hence, the limit $u \in C_t^0 \dot{H}_x^\gamma(S_T) \cap L_t^s L_x^q(S_T)$ of the sequence $\{u_j\}$ is a solution of the Cauchy problem (1-1) in S_T. Moreover, by Fatou's lemma and (5-38), we have that

$$\|u\|_{C_t^0 \dot{H}_x^\gamma(S_T)} + \|u\|_{L_t^s L_x^q(S_T)} \le 2H_0(T),$$

which together with (5-37) yields that u satisfies estimate (1-4).

Further, by the same arguments as in the proof of Theorem 1.1, it follows that if both $u, \tilde{u}$ solve the Cauchy problem (1-1) in S_T, then $u = \tilde{u}$ in S_T. □

5.3. ***Proof of Theorem 1.5.*** From the assumptions of Theorem 1.5, we have

$$\gamma = \frac{n+1}{2}\Big(\frac{1}{2} - \frac{1}{q}\Big) - \frac{m}{2\mu_*(m+2)}$$

and

$$-\frac{m}{2\mu_*(m+2)} \le \gamma < \frac{1}{m+2} - \frac{2(n+1)}{\mu_*(m+2)(n-1)} = \frac{3}{m+2} - \frac{n(2\mu_* - m)}{\mu_*(m+2)(n-1)}.$$

To verify (5-2), we set

$$H_j(T) = \|u_j\|_{C_t^0 \dot{H}_x^\gamma(S_T)} + \|u_j\|_{L_t^s L_x^q(S_T)}, \quad N_j(T) = \|u_j - u_{j-1}\|_{L_t^s L_x^q(S_T)}.$$

Let $p = q/\kappa$. Then

$$\frac{2n}{(n+1)p} = \frac{1}{q} + \frac{6\mu+m}{\mu(m+2)(n+1)} - \frac{n-1}{2(n+1)}.$$

Thus we can apply Theorem 4.5 in case (ii) together with Hölder's inequality to find that

$$\begin{aligned}\|u_{j+1}-u_{k+1}\|_{C_t^0\dot H_x^\gamma(S_T)} + \|u_{j+1}-u_{k+1}\|_{L_t^sL_x^q(S_T)} \\ &\lesssim \|F(u_j)-F(u_k)\|_{L_t^2L_x^p(S_T)} \\ &\lesssim \|G(u_j,u_k)\|_{L_t^\rho L_x^\sigma(S_T)}\|u_j-u_k\|_{L_t^sL_x^q(S_T)},\end{aligned}$$

where $1/\rho = \frac{1}{2} - 1/s$, and $1/\sigma = 1/p - 1/q = (\kappa-1)/q$.

Note that $s > (\kappa-1)\rho$ when $\gamma < 1/(m+2) - 2(n+1)/(\mu_*(m+2)(n-1))$. Due to condition (1-2) and Hölder's inequality,

$$\begin{aligned}\|G(u_j,u_k)\|_{L_t^\rho L_x^\sigma(S_T)} &\lesssim \|u_j\|^{\kappa-1}_{L_t^{\rho(\kappa-1)}L_x^q(S_T)} + \|u_k\|^{\kappa-1}_{L_t^{\rho(\kappa-1)}L_x^q(S_T)} \\ &\lesssim T^{1/2-1/s}(\|u_j\|^{\kappa-1}_{L_t^sL_x^q(S_T)} + \|u_k\|^{\kappa-1}_{L_t^sL_x^q(S_T)}).\end{aligned}$$

As in the proof of Theorem 1.1, we get that

(5-39) $$H_j(T) \le 2H_0(T), \quad N_j(T) \le \tfrac{1}{2}N_{j-1}(T),$$

and

(5-40) $$N_0(T) \le H_0(T)T^{1/2-\kappa/s} \le \varepsilon_0,$$

for $\varepsilon_0 > 0$ small by choosing $T > 0$ small. Therefore, there is a function $u \in C_t^0\dot H_x^\gamma(S_T) \cap L_t^sL_x^q(S_T)$ such that

$$u_j \to u \quad \text{in } L_t^sL_x^q(S_T) \text{ as } j \to \infty$$

and (5-2) holds. Combining Fatou's lemma and (5-39), we see that

$$\|u\|_{C_t^0\dot H_x^\gamma(S_T)} + \|u\|_{L_t^sL_x^q(S_T)} \le 2H_0(T).$$

Together with (5-40) we get that u satisfies estimate (1-4).

Moreover, since $2\kappa > s$, by condition (1-2) and Hölder's inequality, we have that, for $p = q/\kappa$,

$$\begin{aligned}\|F(u)\|_{L_t^2L_x^p(S_T)} &\lesssim \|u\|^\kappa_{L_t^{2\kappa}L_x^q(S_T)} \\ &\lesssim T^{1/2-\kappa/s}\|u\|^\kappa_{L_t^sL_x^q(S_T)}\end{aligned}$$

and

$$\begin{aligned}\|F(u_j)-F(u)\|_{L_t^2L_x^p(S_T)}&\\ &\lesssim T^{1/2-1/s}(\|u_j\|^{\kappa-1}_{L_t^sL_x^q(S_T)}+\|u\|^{\kappa-1}_{L_t^sL_x^q(S_T)})\|u_j-u\|_{L_t^sL_x^q(S_T)}\\ &\lesssim T^{1/2-1/s}H_0(T)^{\kappa-1}\|u_j-u\|_{L_t^sL_x^q(S_T)}.\end{aligned}$$

Therefore, $F(u)\in L_t^2L_x^{q/\kappa}(S_T)$ and $F(u_j)\to F(u)$ in $L_t^2L_x^{q/\kappa}(S_T)$ as $j\to\infty$, hence (5-3) holds. Consequently, the limit function $u\in C_t^0\dot{H}_x^\gamma(S_T)\cap L_t^sL_x^q(S_T)$ solves the Cauchy problem (1-1) in S_T.

Now suppose $u,\tilde{u}\in C_t^0\dot{H}_x^\gamma(S_T)\cap L_t^sL_x^q(S_T)$ both solve the Cauchy problem (1-1) in S_T. Then $v=u-\tilde{u}\in C_t^0\dot{H}_x^\gamma(S_T)\cap L_t^sL_x^q(S_T)$ is a solution of (5-4). Applying Theorem 4.5 in case (ii) and Hölder's inequality, it follows that

$$\begin{aligned}\|v\|_{L_t^sL_x^q(S_T)}&\le C\|G(u,\tilde{u})v\|_{L_t^2L_x^p(S_T)}\\ &\le CT^{1/2-1/s}(\|u\|^{\kappa-1}_{L_t^sL_x^q(S_T)}+\|\tilde{u}\|^{\kappa-1}_{L_t^sL_x^q(S_T)})\|v\|_{L_t^sL_x^q(S_T)}\\ &\le CT^{1/2-1/s}H_0(T)^{\kappa-1}\|v\|_{L_t^sL_x^q(S_T)}\le\tfrac{1}{2}\|v\|_{L_t^sL_x^q(S_T)}.\end{aligned}$$

Thus (5-5) holds and $u=\tilde{u}$ in S_T. □

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