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DAVID TAYLOR, V. S. VARADARAJAN,
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TEMPEREDNESS OF MEASURES DEFINED BY POLYNOMIAL EQUATIONS OVER LOCAL FIELDS

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Dedicated to the memory of Professor Jun-Ichi Igusa

We investigate the asymptotic growth of the canonical measures on the fibers of morphisms between vector spaces over local fields of arbitrary characteristic. For a single polynomial over $\mathbb{R}$, this is due to Igusa and Raghavan. For nonarchimedean local fields we use a version of the Łojasiewicz inequality which follows from work of Greenberg, together with the theory of the Brauer group of local fields to construct definite forms of arbitrarily high degree, and to transfer questions at infinity to questions near the origin. We then use these to generalize results of Hörmander on estimating the growth of polynomials at infinity in terms of the distance to their zero loci. Specifically, when a fiber corresponds to a noncritical value which is stable, i.e., remains noncritical under small perturbations, we show that the canonical measure on the fiber is tempered, which generalizes results of Igusa and Raghavan, and Virtanen and Weisbart.

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1. Introduction

Let V be a finite dimensional vector space over $\mathbb{R}$ and $f\colon V \to \mathbb{R}$ a smooth nonconstant function. In the physics and mathematics literature the measure denoted by $\delta(f - c)$ figures prominently; it is a measure living on the smooth part of the zero locus $Z(f-c)$ of $f-c$, $c \in \mathbb{R}$ [Gel'fand and Shilov 1964]. Given f and choices of Haar measures on V and $\mathbb{R}$, $\delta(f-c)$ is uniquely defined for all c. Similarly if $\boldsymbol{f} = (f_1, f_2, \ldots, f_r)\colon V \to \mathbb{R}^r$ is a smooth map with $df_1 \wedge \cdots \wedge df_r \neq 0$, for given Haar measures on V and $\mathbb{R}^r$, there is a canonical measure on the smooth part of the common zero locus $Z(\boldsymbol{f} - \boldsymbol{c}) = Z(f_1 - c_1, f_2 - c_2, \ldots, f_r - c_r)$ of the $f_i - c_i$ for all $\boldsymbol{c} = (c_1, c_2, \ldots, c_r)$. We denote this measure by $\mu_{\boldsymbol{f},\boldsymbol{c}}$. In this context, the finiteness of $\mu_{\boldsymbol{f},\boldsymbol{c}}$ around the singular points of $Z(f_1 - c_1, f_2 - c_2, \ldots, f_r - c_r)$, as well as the behavior at infinity of the extended measure, viewed as a Borel measure on V, are interesting questions. If the f_i are polynomials and $Z(\boldsymbol{f} - \boldsymbol{c})$ is smooth, then it is natural to expect that $\mu_{\boldsymbol{f},\boldsymbol{c}}$ is tempered. That is,

Definition 1.1 (tempered measure). Let V be any finite dimensional k-vector space, k a *local field*. A Borel measure μ on V is *tempered* if

$$\int_V (1 + \|x\|^2)^{-\alpha}\, d\mu(x) < \infty$$

for some integer α (in any norm).

This is equivalent to saying that there are constants $A > 0$, $b \geq 0$ such that

$$\text{(G)} \qquad \mu(B_R) \leq A R^b$$

for all $R \geq 1$, B_R being the closed ball in V of radius R and center $\mathbf{0}$ (in any norm).

In [Igusa 1978] Igusa and Raghavan proved that if $k = \mathbb{R}$ and f is a nonconstant polynomial on V and $c \in \mathbb{R}$ is a noncritical value of f, i.e., the locus $Z(f-c)$ is smooth, then $\mu_{f,c}$ is tempered, and further that the growth estimate G for the measure is uniform in a neighborhood of c; here we must remember that by the algebraic Sard's theorem (Proposition 2.4), f has only finitely many critical values, so that every noncritical value c has neighborhoods consisting only of noncritical values.

The measures $\mu_{f,c}$, $\mu_{\boldsymbol{f},\boldsymbol{c}}$ can be defined over any local field. Throughout this paper by local field we mean a locally compact nondiscrete field of any characteristic, other than $\mathbb{C}$; measure theoretic questions over $\mathbb{C}$ usually reduce to $\mathbb{R}$, and so we do not treat the case of $\mathbb{C}$ separately. In [Igusa 1978] Igusa and Raghavan define the measures $\mu_{f,c}$ for any local field but do not consider their behavior at infinity, the reason being that over a nonarchimedean field they were concerned only with integrating Schwartz–Bruhat functions (i.e., compactly supported complex-valued locally constant functions). However the work of Harish-Chandra [1973] shows the necessity as well as utility of working with locally constant functions that do not vanish outside a compact set. The question of extending the results of [Igusa 1978] to the nonarchimedean case and for $r > 1$ is certainly a natural one. In [Virtanen and Weisbart 2014] the measures $\mu_{f,c}$ were shown to be tempered when f is a nondegenerate quadratic form and $c \neq 0$; moreover for the case $c = 0$ the locus $Z(f)$ has 0 as its only singularity, and it was shown that the measure $\mu_{f,0}$ is finite in the neighborhood of 0 if $\dim V \geq 3$, and the extended measure is tempered in V. The work of [Virtanen and Weisbart 2014] was motivated by physical questions arising in the theory of elementary particles over p-adic spacetimes. In this paper we generalize the results of [Igusa 1978] and [Virtanen and Weisbart 2014] to the measures $\mu_{\boldsymbol{f},\boldsymbol{c}}$ where the f_i $(1 \leq i \leq r)$ are polynomials on a vector space V over a local field k, with $\dim(V) = m$ and $df_1 \wedge df_2 \wedge \cdots \wedge df_r \not\equiv 0$, so that $m \geq r$. Note that for $r > 1$ and $k = \mathbb{R}$ this question is already more general than the one treated in [Igusa 1978].

We now describe our main result using the above notation. Let $\boldsymbol{f} : V \to k^r$ be the polynomial map whose components are the f_i, with $df_1 \wedge \cdots \wedge df_r \not\equiv 0$. A point $x \in V$ is called a critical point (CP) of $\boldsymbol{f}$ if the differentials $df_{i,x}$ are linearly dependent. We write $C(\boldsymbol{f})$ for the set of critical points of $\boldsymbol{f}$; the image $\boldsymbol{f}(C(\boldsymbol{f}))$ in k^r is called the set of critical values of $\boldsymbol{f}$, and is denoted by $CV(\boldsymbol{f})$. By the algebraic Sard's theorem (Proposition 2.4) one knows that *in characteristic zero* the Zariski closure in k^r of $CV(\boldsymbol{f})$ is a *proper* algebraic subset of k^r. A point $c \in k^r$ is called *stably noncritical* if it has an open neighborhood (in the k-topology) consisting only of noncritical values. This is the same as saying that the fibers above points sufficiently close to c are smooth. If k has characteristic zero, stably noncritical values exist and form a nonempty open set in k^r whose complement in the image of $\boldsymbol{f}$ has measure 0. Then the following is our main result. For $r = 1$ and $k = \mathbb{R}$ it was proved in [Igusa 1978]. Note that in this case the characteristic is 0 and there are only finitely many critical values and so every noncritical value is stably noncritical.

Theorem 1.2. *Fix $\boldsymbol{f}$ and write $\mu_{\boldsymbol{c}} = \mu_{\boldsymbol{f},\boldsymbol{c}}$. Suppose $\boldsymbol{c}$ is stably noncritical. Then $\mu_{\boldsymbol{c}}$ is tempered and there are constants $A > 0$, $\gamma \geq 0$ such that for all $\boldsymbol{d}$ in an open neighborhood of $\boldsymbol{c}$*

$$\mu_{\boldsymbol{d}}(B_R) \leq AR^{m-r+\gamma} \qquad (R \geq 1, \boldsymbol{d} \in U).$$

Suppose k has characteristic 0; *then stably noncritical values form a nonempty dense open set whose complement in the image of $\boldsymbol{f}$ has measure* 0; *for $r = 1$, the critical set is finite and all noncritical values are stably noncritical.*

Remark 1.3. In view of the failure of Sard's theorem over characteristic $p > 0$ (see page 233), we do not know if stably noncritical values of $\boldsymbol{c}$ always exist when k is a local field of positive characteristic.

Remark 1.4. The results and ideas in the paper lie at the interface of analysis of geometry over local fields and are motivated by the themes from quantum theory over p-adic spacetimes. We do not know what, if any, are the arithmetic consequences of our results.

As an application of our theory we prove that if k has characteristic 0, the orbits of regular semisimple elements of a semisimple Lie algebra over k are closed, and the invariant measures on them are tempered. For $k = \mathbb{R}$ this is a result of Harish-Chandra [1957].

2. Canonical measures on level sets of polynomial maps

Canonical measures on the fibers of submersive maps. The construction below is well known and our treatment is a very mild variant of Harish-Chandra's [1964] for the case $k = \mathbb{R}$ (see also [Varadarajan 1977]). Serre's book [2006] is a good reference for the theory of analytic manifolds and maps over a local field of arbitrary characteristic. (All of our manifolds are second countable.)

Lemma 2.1. *Let V, W be vector spaces of finite dimension m, r respectively, and $L\colon V \to W$ be a surjective linear map. Let $U = \ker L$. Let σ, τ be exterior forms on V, W of degrees m, r respectively, with $\tau \neq 0$. Then there exists a unique exterior $(m-r)$-form ρ on U such that if $\{u_1, u_2, \ldots, u_{m-r}\}$ is a basis for U, then*

$$\rho(u_1, u_2, \ldots, u_{m-r}) = \frac{\sigma(u_1, \ldots, u_{m-r}, v_1, \ldots, v_r)}{\tau(Lv_1, \ldots, Lv_r)},$$

where $v_i \in V$ are such that $\{u_1, \ldots, u_{m-r}, v_1, \ldots, v_r\}$ is a basis for V.

Proof. For fixed v_i it is obvious that this defines an exterior $(m-r)$-form on U. Its independence of the choice of the v_i is easy to check. □

We write $\rho = \sigma/\tau$. Note that this definition is relative to L.

Theorem 2.2. *Let k be a local field of arbitrary characteristic and M, N be analytic manifolds over k of dimensions m, r respectively, and $\pi : M \to N$ be an analytic map, surjective, and submersive everywhere. Let σ_M (resp. τ_N) be an analytic exterior m-form (resp. r-form) on M (resp. N), with $\tau_N \neq 0$ everywhere on N. Then there is a unique analytic exterior form $\rho := \rho_{M/N}$ on M such that for any $y \in N$, the pull back of ρ to the fiber $\pi^{-1}(y)$ is the exterior $(m-r)$-form $x \mapsto \sigma_x/\tau_y$ relative to $d\pi_x : T_x(M) \to T_y(N)$.*

Proof. The pointwise definition of ρ is clear after the preceding lemma. For analyticity we use local coordinates around x and $y = \pi(x)$, say $x_1, \ldots, x_m$, such that π is the projection $(x_1, \ldots, x_m) \mapsto (x_1, \ldots, x_r)$. Then

$$\sigma_M = s(x_1, \ldots, x_m)\, dx_1 \cdots dx_m, \qquad \tau = t(x_1, \ldots, x_r)\, dx_1 \cdots dx_r,$$

and

$$\rho = \bigl(s(x_1, \ldots, x_m)/t(x_1, \ldots, x_r)\bigr)\, dx_{n+1} \cdots dx_m. \qquad \square$$

Remark 2.3. Let s_M (resp. t_N) be the measures defined on M (resp. N) by $|\sigma_M|$ (resp. $|\tau_N|$). We denote by $r_{M/N,y}$ the measures defined on $\pi^{-1}(y)$ by $|\rho|$. The smooth functions in the nonarchimedean case are the locally constant functions. Then we have [Harish-Chandra 1964]

$$\int_M \alpha\, ds_M = \int_N f_\alpha\, dt_N, \qquad f_\alpha(y) = \int_{\pi^{-1}(y)} \alpha\, dr_{M/N,y}$$

for all smooth compactly supported complex-valued functions α on M.

It is easy to show, using partitions of unity that the map $\alpha \mapsto f_\alpha$ is surjective, and continuous when $k = \mathbb{R}$. This gives rise to an injection of the space of distributions on N into the space of distributions on M, say $T \mapsto T^*$. Then $r_{M/N,y} = \delta(y)^*$, $\delta(y)$ being the Dirac distribution at $y \in N$. Replacing $\delta(y)$ by its derivatives, we get distributions on M, supported by $\pi^{-1}(y)$. If F is a locally integrable function on N, it defines a distribution on N, say T_F, and T_F^* is $T_{F\circ\pi}$ where $F \circ \pi$ is a locally integrable function on M. Thus the map $T \mapsto T^*$ is the natural extension of the map $F \mapsto F \circ \pi$ from the space of locally integrable functions on N to the corresponding space on M. The map $T \mapsto T^*$ plays a fundamental role in Harish-Chandra's theory [1964] of characters on real semisimple Lie groups. Finally, in algebrogeometric terminology, ρ above is the top *relative* exterior form.

We shall now apply this result to polynomial maps $\boldsymbol{f} : V \to k^r$ where V is a vector space of finite dimension m over a local field k of arbitrary characteristic such that $df_1 \wedge \cdots \wedge df_r \not\equiv 0$ on V, the f_i being the components of $\boldsymbol{f}$; let $V^\times$ be the set of points where this exterior form is nonzero in V, so that $V^\times$ is nonempty Zariski open in V; let $N(\boldsymbol{f}) = \boldsymbol{f}(V^\times)$. Clearly $m \geq r$ and $N(\boldsymbol{f})$ is nonempty open (in the k-topology) in k^r. Then, by Theorem 2.2 with $M = V^\times$, $N = N(\boldsymbol{f})$, we have a measure $\mu_{\boldsymbol{c}}$ for $\boldsymbol{c} \in N$ on $L'_{\boldsymbol{c}} := L_{\boldsymbol{c}} \cap V^\times$ where $L_{\boldsymbol{c}}$ is the level set

$$L_{\boldsymbol{c}} = Z(f_1 - c_1, \ldots, f_r - c_r) = \{x \in V \mid f_1(x) = c_1, \ldots, f_r(x) = c_r\}. \tag{2-1}$$

Exactly as before, we may view the $\mu_{\boldsymbol{f},\boldsymbol{c}}$ as distributions living on $L'_{\boldsymbol{c}}$ which is all of $L_{\boldsymbol{c}}$ if $\boldsymbol{c}$ is a noncritical value. The derivatives of $\mu_{\boldsymbol{f},\boldsymbol{c}}$ with respect to the differential operators of k^m (when $k = \mathbb{R}$) then yield distributions supported by $L_{\boldsymbol{c}}$. Examples of such distributions have important applications ([Gel'fand and Shilov 1964],[Kolk and Varadarajan 1992]) in analysis and physics.

Fix a noncritical value $\boldsymbol{c}$ of $\boldsymbol{f}$. Let $J = \{i_1 < i_2, \ldots, < i_r\}$ be an ordered subset of r elements in $\{1, 2, \ldots, m\}$. Let

$$\partial_J := \frac{\partial(f_1, \ldots, f_r)}{\partial(x_{i_1}, \ldots, x_{i_r})}. \tag{2-2}$$

Then $L_{\boldsymbol{c}}$ is smooth and $L_{\boldsymbol{c}} = \bigcup_J L_{\boldsymbol{c},J}$ where the sum is over all sets J as above and

$$L_{\boldsymbol{c},J} := \{x \in L_{\boldsymbol{c}} \mid \partial_J(x) \neq 0\}. \tag{2-3}$$

Locally on $L_{c,J}$, $(f_1, \ldots, f_r, y_1, \ldots, y_{m-r})$ is a new coordinate system, the y_j being some enumeration of the $x_i (i \neq i_\nu)$. Obviously $dy_1 \cdots dy_m = \varepsilon \partial_J(x)\, dx_1 \cdots dx_m$, where ε is locally constant and equal to ± 1. Another way of interpreting this formula is the following: if π_J is the projection map from $L_{c,J}$ that takes x to $(y_1, \ldots, y_{m-r})$, then π_J is a local analytic isomorphism and

$$\rho_c = \varepsilon \frac{1}{\partial_J(x)} \pi_J^*(dy_1 \cdots dy_{m-r}), \tag{2-4}$$

where ε is locally constant and ± 1-valued. Hence to control the growth of the measure defined by $|\rho|$ at infinity, we must find *lower bounds* of the $\|\partial_J(x)\|$ on $L_{c,J}$ for $\|x\| \geq 1$. Let

$$\nabla_r(x) = (\partial_J(x)).$$

We call ∇_r the generalized gradient of $(f_1, \ldots, f_r)$. Then we must find lower bounds for $\|\nabla_r(x)\| := \max_J \|\partial_J(x)\|$ for $\|x\| \geq 1$ on $L_{c,J}$. In this quest we follow [Igusa 1978], and our techniques force us to assume c to be stably noncritical. For $r = 1$, ∇_1 is just the gradient ∇, and that work reduces the issue of the lower bounds for the gradient field by replacing ∇f (for $k = \mathbb{R}$) by $\sum_{1 \leq j \leq m} |\partial_j f|^2$, where $\partial_j f = \partial f / \partial x_j$. For nonarchimedean local fields and for $r > 1$ we have to replace the sum of squares by suitable *definite* forms whose degrees will grow with m. Igusa and Raghavan find lower bounds for $|\nabla|$ using Hörmander's inequalities [1958] over $\mathbb{R}$. We generalize Hörmander's inequalities to any local field and use them with the existence of definite forms of sufficiently high degree to get lower bounds for $\|\nabla_r\|$ on the level sets $L_{c,J}$.

The Hörmander inequalities over $\mathbb{R}$ are of two types: H1 and H2. H1 is local and is essentially the Łojasiewicz inequality [1959]; Hörmander derives H2 from H1 by inversion. Over nonarchimedean k, H1 turns out to be a consequence of a Henselization lemma of Greenberg [1966], as observed in [Bollaerts 1990]. The reduction of H2 to H1 is more subtle in the nonarchimedean case. We prove it by embedding V in a division algebra D, central over k, prove H2 for D, and then deduce H2 for V. The descent from D to V is elementary. To prove H2 in D we use the map $x \mapsto x^{-1}$ on $D \setminus \{0\}$ to reduce H2 to H1. The existence of central division algebras over k of arbitrarily high dimension is nontrivial and follows from the theory of the Brauer group of k. The lower bounds of $\nabla_r f$ obtained from these arguments allow us to prove that when c is a *stably noncritical* value of f, $\mu_{f,c}(B_r) = O(R^{m-r+\gamma})$ for some $\gamma \geq 0$, uniformly near c. We do not know if we can take $\gamma = 0$ always. If $\|\nabla_r f\|$ is bounded away from zero at infinity on L_c, then it is obvious that we may take $\gamma = 0$; but $\inf \|\nabla_r f\|$ may be zero on L_c. (See page 252).

Algebraic Sard's theorem in characteristic* 0 *for polynomial maps. Let V be a vector space over k of finite dimension m. Recall the definitions of $C(f)$ and $CV(f)$.

Proposition 2.4. *Let k be of characteristic* 0*. The Zariski closure, $Cl(CV(\boldsymbol{f}))$ is a proper subset of k^r; in particular, if $r = 1$, then $CV(\boldsymbol{f})$ is finite.*

Proof. Fix a basis of V so that $V \simeq k^m$. The field generated by the coefficients of the f_j, say $k_1 \supset k$, can be embedded in $\mathbb{C}$. It is thus enough to prove Proposition 2.4 over $\mathbb{C}$ itself, where it is just the statement that the fibers of $\boldsymbol{f}$ are generically smooth. Over $\mathbb{C}$ this is essentially Sard's lemma for affine algebraic varieties treated by Mumford [1995]. $\square$

Analytic Sard's theorem in characteristic $p > 0$. In characteristic $p > 0$, the algebraic Sard's lemma fails abysmally [Mumford and Oda 2015, p. 179] over algebraically closed fields. Indeed, let f be a polynomial in two variables X, Y giving rise to a map $K^2 \to K$ where K is algebraically closed and of characteristic $p > 0$, for example,

$$f = X^{p+1} + X^p Y + Y^p.$$

Then the gradient of f vanishes precisely on the Y-axis, and f on the Y-axis is the map $y \mapsto y^p$ which is surjective. So the image of the singular set is all of K, and every fiber has a singular point. But if we replace K by a local field, then $y \mapsto y^p$ is *not* surjective, and in fact the image under f of the singular set is k^p which is a closed proper subset of k (in the k-topology), and is of measure zero in k. Thus the generic fiber (in the k-topology) is smooth in k.

We shall now consider the situation over local fields of characteristic $p > 0$. From Sard [1942] we know that when $k = \mathbb{R}$ and the map is of class $C^{(a)}$ $(a > 0)$, $f(C)$ has measure zero when $a > m - r$. Now, when k has characteristic $p > 0$, the derivatives of f are not enough to determine the coefficients of the power series expansion of f whose order is greater than $p - 1$. So there is an analogy with the case of $C^{(p-1)}$ over $\mathbb{R}$, suggesting that over k the condition $p > m - r + 1$ would be sufficient to guarantee that $f(C)$ is a null set. This suggestion, which leads to Theorem 2.5, is due to Professor Pierre Deligne (personal communication, 2016), which we gratefully acknowledge.

Theorem 2.5. *Let X, Y be analytic manifolds over a local field k of characteristic $p > 0$, of dimensions m, r respectively. Let $f : X \to Y$ be an analytic. Let C be the critical set for f. Then $f(C)$ has measure zero in Y if $p > m - n + 1$.*

Proof. The proof that $f(C)$ has measure zero in Y when $p > m - r + 1$ is a minor adaptation of [Guillemin and Pollack 1974], needed because we have an additional restriction on p.

The result is local and so we may take X to be a compact open set $U \in k^m$. We use induction on m. We define the filtration $C = C_0 \supset C_1 \supset \cdots \supset C_{p-1}$, where C_s $(1 \le s \le p - 1)$ is the set where all derivatives of the components of f of order $\le s$ vanish. The sets C, C_s are compact while $C_s \setminus C_{s+1}$ is locally compact and second countable, hence a countable union of compact sets. So $f(C)$, $f(C_s)$ are compact, and $f(C_s \setminus C_{s+1})$ is a countable union of compact sets.

The inductive proof that $f(C \setminus C_1)$ is a null set reduces to the case when (m, r) becomes $(m-1, r-1)$. Since $m - r = (m-1) - (r-1)$, the condition on p remains the same and induction applies.

The inductive proof that $f(C_s \setminus C_{s+1})$ is a null set reduces to the case when (m, r) becomes $(m-1, r)$. Since $p > m - r + 1 > (m-1) - r + 1$, induction applies again.

It remains to show that $f(C_{p-1})$ is a null set when $p > m - r + 1$. We shall show actually that $f(C_{p-1})$ is a null set when $p > m/r$. This is enough since $m/r \le m - r + 1$. This is a local result and so we may work around a point of C_{p-1} which can be taken to be the origin. We use the max norm on k^m and k^r so that the norms take values in $q^{\mathbb{Z}}$, where $q > 1$ is the cardinality of the residue field of k. By scaling, if necessary, we may assume that all components of f are given by power series expansions, absolutely convergent on the ball $B(q) := \{x \in k^m \mid \|x\| \le q\}$. Note that $B(1) = R^m$, where R is the ring of integers of k. In order to estimate the growth of these series we need a lemma:

Lemma 2.6. *Let g be an analytic function on $B(q)$ given by an absolutely convergent power series expansion about* 0 *on $B(q)$. Let D be the set in $B(1)$ where $\partial^\beta f = 0$ for all β with $|\beta| \le p - 1$. Then we have*

$$|g(x+h) - g(x)| \le A\|h\|^p$$

uniformly for $x \in D$, $\|h\| \le 1 \le q - 1$, the constant $A > 0$ depending only on g.

Proof. We use [Serre 2006, pp. 67–75]. We have

$$g(x) = \sum_\alpha c_\alpha X^\alpha, \quad \sum_\alpha |c_\alpha| = A < \infty.$$

For $x \in B(1)$ we have $g(x+h) = \sum_\beta \Delta^\beta g(x) h^\beta$, where

$$\Delta^\beta g(x) = \sum_{\alpha \ge \beta} c_\alpha \binom{\alpha}{\beta} x^{\alpha - \beta}, \quad \beta! \Delta^\beta g(x) = \partial^\beta g(x).$$

Then $|\Delta^\beta g(x)| \le A$ on $B(1)$. If $x \in D$, $\|h\| \le 1 \le q - 1$, then $x + h \in B(1)$. Moreover, for $|\beta| \le p - 1$, $\beta! \Delta^\beta g(x) = 0$ so that $\Delta^\beta g(x) = 0$. Hence,

$$g(x+h) = g(x) + \sum_{|\beta| \ge p} (\Delta^\beta g)(x) h^\beta.$$

But, for $y \in B(1)$,

$$|\Delta^\beta g(y)| \le \sum |c_\alpha| = A.$$

So,

$$|g(x+h) - g(x)| \le A\|h\|^p \qquad (x \in D, \|h\| \le 1),$$

proving the lemma. □

We now divide $B(1)^m$ into very small "cells". Let P be the maximal ideal in R. Let N be any integer ≥ 1. Then $B(1)$ is the disjoint union of q^N cosets of P^N each of which is a compact open set that has diameter $\leq q^{-N}$ and volume q^{-N}. This gives a partition of $B(1)^m$ into q^{mN} compact open sets ("cells") of diameter $\leq q^{-N}$ and volume q^{-Nm}. By the above lemma, if $x, x+h \in D$ and are in one of these cells, say γ, then

$$\|f(x+h) - f(x)\| \leq A\|h\|^p \leq Aq^{-Np}, \tag{2-5}$$

where A is a constant independent of x. Hence, $f(\gamma)$ is contained in a set of diameter $\leq q^{-Np}$ and hence volume $\leq q^{-Npr}$. Thus $f(D \cap C_{p-1})$ is enclosed in a set of volume $\leq q^{mN-Npr} = q^{-N(pr-m)}$. If $p > m/r$ this expression goes to 0 as $N \to \infty$, and we are done. $\square$

Remark 2.7. If $\boldsymbol{f} = (f_1, \dots, f_r)$ is a polynomial map of k^m into k^r such that $df_1 \wedge \cdots \wedge df_r \neq 0$, then $\boldsymbol{f}(k^m)$ is open and Sard's theorem shows that almost every fiber of f is smooth in k. So there are always noncritical values. Whether some of them are stable is not known to us.

Remark 2.8. When $r = 1$, the above condition reduces to $p > m$. Both this condition and the fact that when $m \geq p+1$ it is possible that the image of the critical set can be all of k were communicated to us by Professor Pierre Deligne (2016). We are grateful for his generosity and for giving us permission to discuss his example.

Example 2.9 (Deligne). We take $m = p+1$ with coordinates $y, x_1, \dots, x_p$. The field $k := \mathbb{F}[\![t]\!][1/t]$, where $\mathbb{F}$ is a finite field of characteristic p, is a local field of characteristic p. Then k is a vector space of dimension p over $k^{(p)} := \{x^p \mid x \in k\}$. Let $(a_i)_{1 \leq i \leq p}$ be a basis for $k/k^{(p)}$, for instance $a_i = t^{i-1}$, $(1 \leq i \leq p)$. Consider, for an integer $n > 1$, prime to p,

$$f = y^n + a_1 x_1^p + \cdots + a_p x_p^p.$$

Then the critical locus is given by $y = 0$. Its image under f is obviously all of k. If we do not insist that $df \not\equiv 0$, we can omit y so that f maps the critical set k^p onto k.

This example is easily modified for the case $r > 1$. We consider k^{p+r} with coordinates $y_1, \dots, y_{r-1}, y, x_1 \dots, x_p$ and take the map $f : k^{p+r} \to k^r$ defined by

$$f : (y_1, \dots, y_{r-1}, y, x_1 \dots, x_p) \mapsto \left(y_1, \dots, y_{r-1}, y^n + \sum_{i=1}^{p} a_i x_i^p\right),$$

where the notation is as before. The critical set is again given by $y = 0$, and the map restricted to this set is

$$f : (y_1, \dots, y_{r-1}, 0, x_1 \dots, x_p) \mapsto \left(y_1, \dots, y_{r-1}, \sum_{i=1}^{p} a_i x_i^p\right),$$

whose range is k^r. Exactly as before, if we omit y, we get a map where $df_1\wedge\cdots\wedge df_r$ is zero but f maps the critical set k^{p+r-1} onto k^r.

3. Construction of definite forms and their associated norms

As mentioned in Remark 2.3 we begin by discussing the construction of definite forms in an arbitrary number of variables over k.

Proposition 3.1. *Let V be a finite dimensional vector space over a local field. If $k=\mathbb{R}$, and $\nu(x)$ is a positive definite quadratic form on V, then $|\nu(x)|^{1/2}$ is a norm on V. If k is nonarchimedean, and r is an integer such that $r^2\geq m$, then there is a homogeneous polynomial $\nu\colon V\to k$ of degree r such that*

(a) *ν is definite, i.e., for $x\in V$, $\nu(x)=0$ if and only if $x=0$;*

(b) *$|\nu(x)|^{1/r}$ is a nonarchimedean norm on V.*

Proof. We deal only with the case of nonarchimedean k. By the theory of the Brauer group of k [Weil 1967, chapter XII, theorem 1] and its corollary we can find a division algebra D over k which is central over k and $\dim_k(D)=r^2$. Since $V\hookrightarrow D$, it is enough to prove the proposition for $V=D$. The advantage is that we can use the algebraic structure of D.

Let ν be the reduced norm [Weil 1967, chapter IX, proposition 6] of D. Then, $\nu\colon D\to k$ is a homogeneous polynomial function on D of degree r, and $\nu(x)^r=\det(\lambda(x))$ where $\lambda(x)$ is the endomorphism $y\mapsto xy$ of D. Note that $\det(\lambda)$ is a polynomial function on D with values in k, homogeneous of degree r^2. As $\lambda(x)$ is invertible for any $x\neq 0$ in D, $\det(\lambda(x))$ and hence $\nu(x)$, is nonzero for $x\neq 0$ in D. Hence, ν is a definite form of degree r on D. It remains to prove that $N(x):=|\nu(x)|^{1/r}$ is a nonarchimedean norm on D. This reduces to showing that $N(1+u)\leq 1$ if $u\in D$ and $N(u)\leq 1$, or equivalently, that $|\lambda(1+u)|\leq 1$ if $u\in D$ and $|\lambda(u)|\leq 1$, which follows from [Weil 1967, chapter I, section 4]. $\square$

Remark 3.2. Actually, $\nu(x)^r=\det\lambda(x)$ will serve our purposes as well and is obviously a homogeneous polynomial of degree r^2, Then $|\nu(x)|^{1/r}=|\det\lambda(x)|^{1/r^2}$. We introduced ν because it is of smaller degree and this may be of use in other contexts.

4. Hörmander's inequalities over nonarchimedean local fields

Let V be a finite dimensional vector space over a local, nonarchimedean field k, with its canonical norm $|\cdot|$. Let $\|\cdot\|$ be a nonarchimedean norm on V. We may assume that the norms on k and V take values in the set $\{0,q^{\pm1},q^{\pm2},\dots\}$, where q is the cardinality of the residue field of k. Also, let $f\colon V\to k$ be a polynomial function, and let $Z(f)$ denote its zero locus. For $x\in V$ and nonempty $E\subset V$ let $\operatorname{dist}(x,E):=\inf_{y\in E}\|x-y\|$.

Theorem 4.1 (H1). *Let $f : V \to k$ be a polynomial function on V. Suppose that $Z(f) \neq \varnothing$. Then there exist constants $C > 0$, $\alpha \geq 0$ such that*

$$|f(x)| \geq C \cdot \operatorname{dist}(x, Z(f))^{\alpha} \tag{4-1}$$

for all $x \in V$ with $\|x\| \leq 1$.

Theorem 4.2 (H2). *Let $f : V \to k$ be a polynomial function, $Z(f)$ as above. Then*

(a) *if $Z(f) = \varnothing$, then there exist constants $C > 0$ and $\beta \geq 0$ such that*

$$|f(x)| \geq C \cdot \frac{1}{\|x\|^{\beta}} \quad (x \in V,\ \|x\| \geq 1); \tag{4-2}$$

(b) *if $Z(f) \neq \varnothing$, then there exist constants $C > 0$ and $\alpha, \beta \geq 0$ such that*

$$\|f(x)\| \geq C \cdot \frac{\operatorname{dist}(x, Z(f))^{\alpha}}{\|x\|^{\beta}} \quad (x \in V,\ \|x\| \geq 1); \tag{4-3}$$

Remark 4.3. Theorem 4.1 and 4.2 were proved by Hörmander [1958] when $k = \mathbb{R}$. Also, H1 is a special case of the Łojasiewicz inequality for f a real analytic function [Łojasiewicz 1959].

In proving H1 we may assume that $V = k^m$ and $f \in R[x_1, \dots, x_m]$, R being the ring of integers in k. Let $P \subset R$ be the maximal ideal of R. Suppose that $Z(f) \neq \varnothing$ but $Z(f) \cap R^m = \varnothing$. Then there exists a constant $b > 0$ such that $|f(x)| \geq b > 0$ for $x \in R^m$. On the other hand, as R^m is compact, there exists $b_1 > 0$ such that $\operatorname{dist}(x, Z(f)) \leq b_1$ for all $x \in R^m$. Hence $|f(x)| \geq bb_1^{-1}b_1 \geq bb_1^{-1} \operatorname{dist}(x, Z(f))$ for all $x \in R^m$. Hence we may assume in addition that $Z(f) \cap R^m \neq \varnothing$ in the proof of H1.

Proof of H1: k nonarchimedean. We follow [Greenberg 1966], specialized to the case of a single polynomial.

Proof. By theorem 1 there, applied to the single polynomial f, we can find integers, $N, c \geq 1$ and $s \geq 0$ such that if $\nu \geq N$ and $f(x) \equiv 0 \pmod{P^{\nu}}$, and $x \in R^m$, then there exists $y \in R^m$ such that $f(y) = 0$ and $x_i - y_i \equiv 0 \pmod{P^{[\nu/c]-s}}$ for all i.

Assume $|f(x)| = q^{-(N+\ell)}$, $\ell \geq 0$. Then there exists $y \in Z(f) \cap R^m$ such that

$$\|x - y\| \leq q^{-[(N+\ell)/c]+s} \leq q^{-[(N+\ell)/c-1]+s} \leq q^{s+1}|f(x)|^{1/c},$$

which implies that

$$\operatorname{dist}(x, Z(f) \cap R^m) \leq q^{s+1}|f(x)|^{1/c},$$

so that

$$|f(x)| \geq \frac{\operatorname{dist}(x, Z(f) \cap R^m)^c}{q^{c(s+1)}} \geq \frac{\operatorname{dist}(x, Z(f))^c}{q^{c(s+1)}}.$$

Thus, H1 is proved for $x \in R^m$ with $|f(x)| \le q^{-N}$. For x in R^m with $|f(x)| > q^{-N}$, we have $q^{-N} < |f(x)| \le 1$, while $\operatorname{dist}(x, Z(f) \cap R^m) \le 1$ since $\|x - y\| \le 1$ for $x, y \in R^m$. Hence

$$|f(x)| \ge q^{-N} \operatorname{dist}(x, Z(f) \cap R^m) \ge q^{-N} \operatorname{dist}(x, Z(f) \cap R^m)^c \ge q^{-N} \operatorname{dist}(x, Z(f))^c.$$

If $C = \min(q^{-N}, q^{-(s+1)c})$, then we have H1 with $\alpha = c$. □

Remark 4.4. That the local version of the Łojasiewicz inequality comes out of [Greenberg 1966] has been observed in [Bollaerts 1990]; we give this proof since it includes the case when k has characteristic > 0. Greenberg's result is applicable here because R is then complete ($k^* = k$ in his notation).

Proof of H2.

Lemma 4.5. *If* H2 *is true for a k-vector space V, then it is also true for any subspace W of V. In particular, for a central division algebra, D_r over k, of dimension $r^2 \ge \dim_k V$, it is enough to prove* H2 *for D_r.*

Proof. Let $W \subseteq V$ be a subspace, and $U \subseteq V$ such that $V = W \oplus U \simeq W \times U$. Let f be a polynomial on W. Define the polynomial g on V by $g(w+u) := f(w)$. For $w \in W, u \in U$, we take $\|w+u\| = \max(\|u\|, \|w\|)$; because U and W are complementary, this is nonarchimedean. Clearly $Z(g) = Z(f) \times U$.

Suppose $Z(f) = \varnothing$. Then $Z(g) = \varnothing$. Since H2 is true for V and $W \subset V$, there exist constants $C > 0$, $\beta \ge 0$ such that $|f(w)| \ge C\|w\|^{-\beta}$ for $w \in W$, $\|w\| \ge 1$. We may therefore assume that $Z(f) \ne \varnothing$, so $Z(g) \ne \varnothing$.

Then, $|g(x)| \ge C \operatorname{dist}(x, Z(g))^\alpha \|x\|^{-\beta}$ for $x \in V$, $\|x\| \ge 1$ where $C > 0$, $\alpha, \beta \ge 0$ are constants. If $x = w \in W$, $\operatorname{dist}(w, Z(g)) = \operatorname{dist}(w, Z(f))$. □

Now we prove H2 for D_r. Our proof is inspired by Hörmander's [1958]. It replaces the inversion in his proof by the involution $x \mapsto x^{-1}$ of $D_r^\times := D_r \setminus \{0\}$.

For a division algebra D_r of dimension r^2, central over k, let us recall $\nu := \nu_r : D_r \to k$ of Proposition 3.1, and note that it has the following property: if k' is any field containing k such that there exists an isomorphism $F : k' \otimes_k D_r \xrightarrow{\sim} M_r(k') = M_r$ where M_r is the algebra of $r \times r$ matrices over k', then $\nu(a) = \det F(a)$ for $a \in D_r$ [Weil 1967, Proposition 6, p. 168],

Lemma 4.6. *For any polynomial function $f : D \to k$ of degree d, f not necessarily homogeneous, let $f^*(x) := f(x^{-1})\nu(x)^d$ for $x \ne 0$; then $f^*(x)$ extends uniquely to a polynomial function $D_r \to k$. Moreover, for nonzero x, $x \in Z(f)$ if and only if $x^{-1} \in Z(f^*)$.*

Proof. Uniqueness is obvious. To prove that f^* has a polynomial extension it suffices to prove it for $k' \otimes_k D_r$, where k' is a separable extension of k such that $k' \otimes_k D_r \simeq M_r(k')$. The required result is compatible with addition and multiplication of the f so that it is enough to verify it for $f = 1$ (obvious) and $f = a_{ij}$, a matrix entry; then $f^* = a^{ij} \det = A_{ij}$, the corresponding cofactor. The last statement of the lemma is obvious □

Remark 4.7. From now on we use the norm $\|x\| = |\nu(x)|^{1/r}$ for D_r, $r \geq 2$.

Lemma 4.8. *If x, y, $x - y$ are all nonzero, then* $\|x - y\| = \|x^{-1} - y^{-1}\| \|x\| \|y\|$

Proof. Use $y - x = x(x^{-1} - y^{-1})y$ and the multiplicativity of $\|\cdot\|$. □

The next two lemmas are auxiliary before we prove H2 for D_r.

Lemma 4.9. *If $Z(f)$ is nonempty, there exists a constant $A \geq 1$ such that*

$$\operatorname{dist}(x, Z(f)) \leq A\|x\| \quad \text{for all } x \text{ with } \|x\| \geq 1.$$

Proof. Choose $z_0 \in Z(f)$. Then $\operatorname{dist}(x, Z(f)) \leq \|x - z_0\| \leq \max(\|x\|, \|z_0\|)$. If $\|x\| \geq \|z_0\|$, then $\operatorname{dist}(x, Z(f)) \leq \|x\|$ and we can take $A = 1$. If $\|x\| < \|z_0\|$ then $\|x - z_0\| = \|z_0\| \leq \|z_0\| \|x\|$ for $\|x\| \geq 1$; and as $\|z_0\| \geq 1$, the lemma is proved if we take $A = 1 + \|z_0\|$. □

Lemma 4.10. *Suppose $Z(f)$ contains a nonzero element. Then there exists a constant $C > 0$ such that*

$$\operatorname{dist}(x^{-1}, Z(f^*)) \geq C \frac{\operatorname{dist}(x, Z(f))}{\|x\|^2} \qquad (\|x\| \geq 1). \tag{4-4}$$

Proof. First assume $0 \notin Z(f^*)$. Then $Z(f^*) = Z(f^*) \setminus \{0\} \neq \varnothing$. Then, with $\|x\| \geq 1$,

$$\operatorname{dist}(x^{-1}, Z(f^*) \setminus \{0\}) = \inf_{0 \neq z \in Z(f^*)} \|x^{-1} - z\| = \inf_{0 \neq y \in Z(f)} \|x^{-1} - y^{-1}\| = \inf_{0 \neq y \in Z(f)} E,$$

where $E := \|x - y\| \|x\|^{-1} \|y\|^{-1}$.

We consider cases: (a) $\|y\| > \|x\|$ and (b) $\|y\| \leq \|x\|$. In case (a) $\|x - y\| = \|y\|$ so that $E = \|x\|^{-1} = \|x\| \|x\|^{-2} \geq A^{-1} \operatorname{dist}(x, Z(f)) \|x\|^{-2}$, where $A \geq 1$ is as in Lemma 4.9. In case (b) $E \geq \|x - y\| \|x\|^{-2}$ so that $\inf E \geq \operatorname{dist}(x, Z(f)) \|x\|^{-2}$. These give (4-4) with $C = 1/A$.

If $0 \in Z(f^*)$, then $\operatorname{dist}(x^{-1}, Z(f^*)) = \min(\operatorname{dist}(x^{-1}, Z(f^*) \setminus \{0\}), \|x^{-1}\|)$. Now $\|x\|^{-1} = \|x\| \|x\|^{-2} \geq C \|x\|^{-2} \operatorname{dist}(x, Z(f))$ by Lemma 4.9 where $C = 1/A$, while $\operatorname{dist}(x^{-1}, Z(f^*) \setminus \{0\}) \geq C \|x\|^{-2} \operatorname{dist}(x, Z(f))$, by above. □

Proof of H2 *for D_r.* We consider two cases: (a) $Z(f) = \varnothing$, (b) $Z(f) \neq \varnothing$.

Case (a): Then $Z(f^*) = \varnothing$ or $\{0\}$. If $Z(f^*) = \varnothing$, then there exists a constant $C > 0$ such that $|f^*(x)| \geq C > 0$ with $\|x\| \leq 1$. So, $|f^*(y)| = |f(y^{-1})| \|y\|^{rd} \geq C > 0$ for $0 < \|y\| \leq 1$, which becomes $|f(x)| \geq C \|x\|^{rd} \geq C > 0$ for $\|x\| \geq 1$.

If $Z(f^*) = \{0\}$, then $\operatorname{dist}(z, Z(f^*)) = \|z\|$, and $|f^*(y)| \geq C \|y\|^{\beta}$ with $0 < \|y\| \leq 1$ for constants $C > 0$, $\beta \geq 0$ by Theorem 4.1. Then $|f(y^{-1})| \|y\|^{rd} \geq C \|y\|^{\beta}$ with $\|y\| \leq 1$ or $|f(x)| \geq C \|x\|^{rd} \|x\|^{-\beta} \geq C \|x\|^{-\beta}$ with $\|x\| \geq 1$.

Case (b): $Z(f)$ is now nonempty, and hence either $Z(f) = \{0\}$ or $Z(f)$ contains a nonzero element. If $Z(f) = \{0\}$, then $Z(f^*) = \varnothing$ or $\{0\}$. This comes under case (a), above, and we have $|f(x)| \geq C\|x\|^{-\beta}$ with $\|x\| \geq 1$ which gives (a).

Suppose $Z(f)$ contains a nonzero element. By H1, there exists constants $C_1 > 0$, $\alpha \geq 0$ such that $|f^*(x^{-1})| \geq C_1 \operatorname{dist}(x^{-1}, Z(f^*))^\alpha$ with $\|x\| \geq 1$. So by Lemma 4.10, for $C_2 = C_1 C^\alpha$, $|f(x)| \geq C_2 \operatorname{dist}(x, Z(f))^\alpha \|x\|^{-2\alpha}$ for $\|x\| \geq 1$, proving (b). ☐

Criterion for a polynomial not to be rapidly decreasing on a set S. In [Igusa 1978] Igusa and Raghavan develop what is essentially a criterion for a polynomial on an real vector space *not to be rapidly decreasing* on a set of vectors of norm ≥ 1. In this section we generalize that method to all local fields, introducing several polynomials in the criterion.

Lemma 4.11. *Let* $\boldsymbol{f} : V \to k^r$ *be a polynomial map and* d *the maximum of the degrees of its components. Then there exists a constant* $C > 0$ *such that for all* $x, y \in V$ *with* $\|x\| \geq 1$,

$$\|\boldsymbol{f}(x) - \boldsymbol{f}(y)\| \leq C\|x\|^{d-1} \max_{0\leq r\leq d} (\|x-y\|^r).$$

Proof. It is enough to prove this for $r = 1$, $\boldsymbol{f} = f$. The estimate is compatible with addition in f and so we may assume f to be a monomial of degree d in some coordinate system on V. Assume the result for all monomials of degree $d-1$. Then $f = x_i g$, where g is a monomial of degree $d-1$. We have

$$x_i g(x) - y_i g(y) = x_i(g(x) - g(y)) + (x_i - y_i)(g(y) - g(x)) + (x_i - y_i)g(x),$$

and the estimate is obvious for each of the three terms. ☐

Proposition 4.12. *Let* $S \subseteq V$ *be a set with* $\|x\| \geq 1$ *for all* $x \in S$. *Let* g *be polynomial on* V. *If* $Z(g) = \varnothing$, *we have*

$$|g(x)| \geq \frac{C}{\|x\|^\gamma} \qquad (\|x\| \geq 1)$$

for some $C > 0$, $\gamma \geq 0$. *Suppose* $Z(g) \neq \varnothing$ *and suppose that there exist polynomials* $f_i : V \to k$, $i = 1, \dots r$, *and a constant* $b > 0$ *such that* $\max |f_i(x) - f_i(y)| \geq b > 0$ *for all* $x \in S$, $y \in Z(g)$. *Then there exist constants* $C > 0$ *and* $\gamma \geq 0$ *such that*

$$|g(x)| \geq \frac{C}{\|x\|^\gamma} \qquad (x \in S). \tag{4-5}$$

Proof. The first statement is (a) of H2. We now assume $Z(g) \neq \varnothing$. We identify $V \simeq k^m$ and work in coordinates. Set $d := \max_i(\deg(f_i))$. In what follows, $C_1, C_2, \dots,$ are constants > 0.

For all $x \in S$ and $y \in Z(g)$, by Lemma 4.11 for some constant $C > 0$, we have $0 < b \leq \max_{1\leq i\leq r} |f_i(x) - f_i(y)| \leq C\|x\|^{d-1} \max_{1\leq r\leq r} \|x-y\|^r$ for all $x \in S$, $y \in Z(g)$.

Choose $y \in Z(g)$ such that $\|x - y\| = \mathrm{dist}(x, Z(g))$. Then for all $x \in S$, we have

$$0 < b \le C_1 \|x\|^{d-1} \max_{1 \le r \le d} (\mathrm{dist}(x, Z(g))^r).$$

We consider the two cases (a) $\mathrm{dist}(x, Z(g)) \le 1$, so the maximum above is $\mathrm{dist}(x, Z(g))$, and (b) $\mathrm{dist}(x, Z(g)) > 1$, so the maximum is $\mathrm{dist}(x, Z(g))^d$.

By H2, there exist constants $C_2 > 0,\ \alpha, \beta \ge 0$ such that

$$|g(x)| \ge C_2 \mathrm{dist}(x, Z(g))^\alpha \|x\|^{-\beta},$$

so $\mathrm{dist}(x, Z(g)) \le C_3 |g(x)|^{1/\alpha} \|x\|^{\beta/\alpha}$. In case (a), $0 < b \le C_3 |g(x)|^{1/\alpha} \|x\|^{\beta/\alpha + (d-1)}$, and in case (b), $0 < b \le C_4 |g(x)|^{d/\alpha} \|x\|^{d\beta/\alpha + (d-1)}$. So in both cases, with $\delta = d\beta/\alpha + (d-1)$, one has

$$0 < b \le C_5 \|x\|^\delta \max(|g(x)|^{1/\alpha}, |g(x)|^{d/\alpha}).$$

Hence, $\max(|g(x)|, |g(x)|^d) \ge C_6 \|x\|^{-\delta\alpha}$, giving in all cases $|g(x)| \ge C_7 \|x\|^{-\delta\alpha}$ with $x \in S$. □

Lower bounds of $\|\nabla_r f\|$ on stably noncritical level sets. Let V and $\boldsymbol{f} =: V \to k^r$ ($\boldsymbol{f} = (f_1, \dots f_r)$, $r \le m = \dim_k V$) be as usual. Let $C(\boldsymbol{f})$ be the critical set of $\boldsymbol{f}$, and $CV(\boldsymbol{f}) = \boldsymbol{f}(C(\boldsymbol{f}))$ have their usual meanings. Write $W = CV(\boldsymbol{f})$. We assume that the closure $\overline{W}$, in the k-topology of k^r, of W is a proper subset of k^r. Our assumption is equivalent to assuming that stably noncritical values of $\boldsymbol{f}$ exist, which is true in characteristic zero (see page 232). Let $L_{\boldsymbol{c}}$, $\nabla_r \boldsymbol{f}$, and $\partial_J \boldsymbol{f}$ be defined as in Section 2.

If $\omega \subset k^r \setminus \overline{W}$ is a compact set, then there exists $b > 0$ such that $\|u - v\| \ge b > 0$ for $u \in \omega$, $v \in \overline{W}$. This means $\max_i |f_i(x) - f_i(y)| \ge b > 0$, with $\boldsymbol{c} \in \omega$, $x \in L_{\boldsymbol{c}}$, $y \in C(\boldsymbol{f})$.

Proposition 4.13. *Let $\omega \subset k^r$ be an open set whose closure consists entirely of noncritical values of $\boldsymbol{f} = (f_1, \dots, f_r)$. For $\boldsymbol{c} \in \omega$, let $L_{\boldsymbol{c}}$ be defined as above. Then there exist constants, $C, \gamma > 0$ such that*

$$\|\nabla_r \boldsymbol{f}(x)\| \ge \frac{C}{\|x\|^\gamma} \qquad (x \in L_{\boldsymbol{c}}, \boldsymbol{c} \in \omega, \|x\| \ge 1) \tag{4-6}$$

Proof. We write (y_J) for the coordinates on $k^{\binom{m}{r}}$ and select a definite homogeneous form ν, which is positive definite of degree 2 if k archimedean, and of degree R on $k^{\binom{m}{r}}$, where R is any integer ≥ 2 such that $R^2 \ge \binom{m}{r}$, with the property that $|\nu(y)|^{1/R}$ is a norm on $k^{\binom{m}{r}}$, if k is nonarchimedean. Then $\nu(\nabla_r \boldsymbol{f}(x)) = 0$ if and only if $\nabla_r \boldsymbol{f}(x) = 0$, i.e., if and only if x is a critical point of $\boldsymbol{f}$. Let $g(x) = \nu(\nabla_r \boldsymbol{f}(x))$. Then $Z(g)$ is the set of critical points of $\boldsymbol{f}$. Suppose first that $Z(g) \ne \varnothing$. Now there exists $b > 0$ such that

$$\|u - v\| = \max_{1 \le i \le r} |u_i - v_i| \ge b > 0 \quad (u \in \omega, v \in \overline{W})$$

Hence, as $\boldsymbol{f}(x) \in \omega$ for $x \in L_{\boldsymbol{c}}$ $(c \in \omega)$ and $\boldsymbol{f}(y) \in \overline{W}$ for $y \in Z(g)$, $\|\boldsymbol{f}(x) - \boldsymbol{f}(y)\| \ge b > 0$. So by Proposition 4.12 there exist constants $C > 0$, $\delta \ge 0$ such that

$$|\nu(\nabla_r \boldsymbol{f}(x))| = |g(x)| \ge \frac{C}{\|x\|^\delta} \quad (x \in L_{\boldsymbol{c}},\ \boldsymbol{c} \in \omega,\ \|x\| \ge 1).$$

But ν is homogeneous of degree d ($d = 2$ for archimedean and R for nonarchimedean k) and definite. So there exist constants $C_1, C_2 > 0$ such that

$$C_1 \|\nabla_r \boldsymbol{f}(x)\|^d \le |\nu(\nabla_r \boldsymbol{f}(x))| = |g(x)| \le C_2 \|\nabla_r \boldsymbol{f}(x)\|^d.$$

So for suitable $C > 0$, $\gamma \ge 0$, we have $\|\nabla_r \boldsymbol{f}(x)\| \ge C \|x\|^{-\gamma}$. The case $Z(g) = \varnothing$ is taken care of by the first statement of Proposition 4.12. □

Remark 4.14. We cannot make $\gamma = 0$ in all cases. For instance, let $\operatorname{char} k = 0$ and $r = 1$, $f(x, y, z) = x^2 z^2 + y^3 z$ and $c = -1$. Consider $x_n = n$, $z_n = 1/n$, $y_n = -(2n)^{1/3}$. Then $F(x_n, y_n, z_n) = 1 - 2 = -1$, $\partial F/\partial X(x_n, y_n, z_n) = 2x_n z_n^2 \to 0$, and $\partial F/\partial Y(x_n, y_n, z_n) = 3y_n^2 z_n \to 0$, $\partial F/\partial Z(x_n, y_n, z_n) = 2x_n^2 z_n + y_n^3 = 2n - 2n = 0$. But $\|(x_n, y_n, z_n)\| = n$, $\|\nabla f(x_n, y_n, z_m)\| \sim \mathrm{Const} \cdot 1/n^{1/3}$. So $\gamma \ge 1/3$. We do not know the minimal value of γ.

5. Proof of temperedness of canonical measures on stably noncritical level sets

Consequences of Krasner's lemma. The well-known lemma of Krasner [Artin 1967] has an important consequence (Corollary 5.3). Let k be a local field of arbitrary characteristic and K its algebraic closure. The following lemma must be well known, but we prove it in this form.

Lemma 5.1. *We can find a countable family $\{k_n\}$ of finite extensions of k with the property that any finite extension of k is contained in one of the k_n. In particular $K = \bigcup_n k_n$.*

Proof. We first work with separable extensions of fixed degree n over k. Let S_n be the set of monic, irreducible and separable elements of $k[X]$ of degree n. Then it follows from Krasner's lemma that if $f \in S_n$, there is an $\varepsilon = \varepsilon(f) > 0$ with the following property: if g is monic and $\|f - g\| < \varepsilon$, then $g \in S_n$ and $K(f) = K(g)$ in K, where $K(h)$ denotes the splitting field of h. Since S_n is a separable metric space, it follows that there are at most a countable number of these splitting fields, and any separable extension of degree n over k is contained in one of these. Let us enumerate these splitting fields as $\{k_{nj}\}$ $(j = 1, 2, \ldots)$. If k has characteristic 0 we are already finished. Suppose k has characteristic $p > 0$. Let $F(x \mapsto x^p)$ be the Frobenius automorphism of K. Define the extension $k_{njr} = F^{-r}(k_{nj})$ for $r = 1, 2, \ldots,$ which are clearly finite over k. Clearly, any finite extension of k of finite degree is contained in one of the k_{njr}. $\square$

Remark 5.2. If k has characteristic 0, then there are only a finite number of extensions of fixed degree n. But in prime characteristic this is not true: the field $k = F_2[\![X]\!][X^{-1}]$ of Laurent series in X with F_2 a finite field of characteristic 2 has a countably infinite number of separable quadratic extensions. Indeed, the extensions defined by $T^2 - T - c = 0$ are distinct for infinitely many values of c.

Corollary 5.3. *If M is an affine subvariety of some A^n_K and $M(k')$ is countable for all finite extensions k' of k, then M is finite.*

Proof. By Lemma 5.1, $M(K) = \bigcup_{k'} M(k')$ is countable, hence finite. $\square$

A consequence of the refined Bézout's theorem. The refinement of Bézout's theorem due to Fulton [1998, Example 8.4.7, p. 148, and Section 12.3] (see also [Vogel 1984, Corollary 2.26, p. 85]), is the statement that if Z_i $(1 \le i \le r)$ are r $(r \ge 2)$ pure dimensional varieties in $\mathbb{P}^m_K$, then the number of irreducible components of $\bigcap_i Z_i$ is bounded by the Bézout number $\prod_i \deg(Z_i)$. It has the following simple consequence.

Lemma 5.4. *Let U be a nonempty Zariski open subset of $\mathbb{A}^r_K$ so that $U \subset \mathbb{A}^r_K \subset \mathbb{P}^r_K$. Let h_i $(i = 1, 2, \ldots, r)$ be polynomials on A^r_K with $\deg h_i =: d_i$, and let Z_i be the zero locus of h_i. Let $Z_i^\times = Z_i \cap U$ and $\bar{Z}_i$ the closure of Z_i in $\mathbb{P}^r_K$. If $\bigcap_i Z_i^\times = F$ is nonempty and finite, then F has at most $D := \prod_i d_i$ elements.*

Proof. Since $\mathbb{A}^r_K$ is Zariski dense in $\mathbb{P}_K$ we have $\bar{Z}_i \cap A^r_K = Z_i$; moreover, $\bar{Z}_i$ is of pure degree d_i. Let W_0 be an irreducible component of $W := \bigcap \bar{Z}_i$ that meets U. Since W_0 is irreducible and $W_0 \cap U$ is nonempty open in W_0, it is dense in W_0. Let $w \in W_0 \cap U$. Then w is in each of the $\bar{Z}_i \cap U$ and so $w \in F$. So $W_0 \cap U$ is finite and contained in F. Since $W_0 \cap U$ is dense in W_0, it follows that $W_0 \cap U$ must consist of a single element of F and W_0 itself consists of that point. Moreover all points of F are accounted for in this manner as F is contained in the union of irreducible components of W which meet U. Hence the cardinality of F is at most the number of irreducible components of W, which is at most D. $\square$

The maps π_J and a universal bound for the cardinality of their fibers. Let $V \simeq k^m$ so that $\boldsymbol{f} = (f_1, \ldots, f_r)$ with $f_j \in k[x_1, \ldots, x_m]$. Assume that $\boldsymbol{c}$ is a noncritical value of $\boldsymbol{f}$ so that $L_{\boldsymbol{c}}$ has no singularities. Fix $J \subset \underline{m} := \{1, \ldots, m\}$, and let $\pi_J : k^m \to k^{m-r}$ map $(x_1, \ldots, x_m)$ to $(y_1, \ldots, y_{m-r})$, where $\{y_j\}_{j=1}^{m-r} = \{x_i \mid i \in \underline{m} \setminus J\}$. We wish to prove that the map π_J restricted to $L_{\boldsymbol{c}}$ has fibers of cardinality $\le D := d_1 \cdots d_r$, where $d_i := \deg(f_i)$. Without loss of generality assume $J = \{1, \ldots, r\}$, so that

$\pi_J : (x_1, \dots, x_m) \mapsto (x_{r+1}, \dots, x_m)$. Write $x = (x_1, \dots, x_m)$ and $y = (x_{r+1}, \dots, x_m)$. Define z so that $x = (z, y)$.

We regard $L_{\boldsymbol{c}}$ as an affine variety and $L_{\boldsymbol{c},J}$ as an affine open subvariety. For any k' with $k \subset k' \subset K$ we have the respective sets of k'-points, $L_{\boldsymbol{c}}(k')$ and $L_{\boldsymbol{c},J}(k')$. Denote the restriction of π_J to $L_{\boldsymbol{c},J}$ by $\bar{\pi}_J$.

Proposition 5.5. *Let $D = \prod_{1 \le i \le r} d_i$. Then the fibers of $\bar{\pi}_J$ are all of cardinality $\le D$.*

Proof. Note that $d\bar{\pi}_J$ is an isomorphism on $L_{\boldsymbol{c},J}(k)$. Hence $U_J(k) := \bar{\pi}_J(L_{\boldsymbol{c},J}(k))$ is open in k^{m-r} and $\bar{\pi}_J$ is a local analytic isomorphism of $L_{\boldsymbol{c},J}(k)$ onto $U_J(k)$. For any field k' between k and K, we write again $\bar{\pi}_J$ for the map $L_{\boldsymbol{c},J}(k') \to k'^{m-r}$, and $U_J(k')$ for its image. If k' is a *finite* extension of k, then k' is again a local field; exactly as for k, we have $d\bar{\pi}_J : L_{\boldsymbol{c},J}(k') \to U_J(k')$ is an analytic isomorphism. For any k', $k \subset k' \subset K$ with k'/k finite, $U_J(k')$ is open in k'^{m-r} and the fibers of $\bar{\pi}_J$ on $L_{\boldsymbol{c},j}(k')$ are discrete and at most countable. If we then fix $y \in U_J(k)$, and write W_y for the affine variety $\bar{\pi}_J^{-1}(y)$, then $W_y(k')$ is at most countable for all finite extensions k'/k. Hence, by Corollary 5.3, $W_y(K)$ is finite. Let $F := W_y(K)$.

On the other hand, $\pi_J^{-1}(y)(K) = K^r \times \{y\} \simeq K^r$. Let $h_i(z) := f_i(z, y) - c_i$. Then h_i is a polynomial on K^r of degree $\le d_i$. Moreover, since $\bar{\pi}_J^{-1}(y)(k)$ is nonempty, $\partial(h_1, \dots, h_r)/\partial(x_1, \dots, x_r) = \partial_J(z, y)$ is not identically zero on K^r. Thus, $\{z | \partial_J(z, y) \neq 0\}$ is a nonempty affine open U_1 in K^r. Moreover, $F = \bigcap_{1 \le i \le r} Z(h_i)^\times$ where $Z(h_i)^\times := Z(h_i) \cap U_1$. So Lemma 5.4 applies and proves that $\#F \le D$. □

Lemma 5.6. *Let ∂_J be as on page 232. Then if ω_{m-r} is the exterior form corresponding to the Haar measure on k^{m-r}, the exterior form*

$$\rho_{\boldsymbol{c}} := \frac{1}{\partial_J(x)} \bar{\pi}_J^*(\omega_{m-r})$$

on $L_{\boldsymbol{c},J}$ has the property that $|\rho_{\boldsymbol{c}}|$ generates the measure $\mu_{\boldsymbol{c}} := \mu_{\boldsymbol{f},\boldsymbol{c}}$. In particular, if λ is the Haar measure on k^{m-r} and ν is the measure generated by $|\bar{\pi}_J^*(\omega_{m-r})|$, then $\bar{\pi}_J$ takes ν to λ in small open neighborhoods of each point of $L_{\boldsymbol{c},J}(k)$, and $d\mu_{\boldsymbol{c}} = |\partial_J(x)|^{-1}\, d\nu$.

Proof. This is clear from (2-4). □

Proof of Theorem 1.2. This follows from three things: the lower bounds on $\|\nabla_r\|$ when $\boldsymbol{c}$ is a stably noncritical value of $\boldsymbol{f}$, the relationship between λ, ν, $\mu_{\boldsymbol{f},\boldsymbol{c}}$, and the temperedness of λ. The simple measure-theoretic lemma below explains this. Let R, S be locally compact metric spaces which are second countable, with Borel measures r, s respectively on them, and $\pi : R \to S$ a continuous surjective map which is a local homeomorphism, and takes r to s in a small neighborhood of each point of R: this means that for each $x \in R$ there are open sets M_x, $N_{\pi(x)}$ containing x and $\pi(x)$ respectively, such that π is a homeomorphism of M_x with $N_{\pi(x)}$ and takes r to s.

Lemma 5.7. *If there is a natural number d such that all fibers of π have cardinality at most d, then for each Borel set $E \subset R$, $\pi(E)$ is a Borel set in S, and we have*

$$r(E) \leq d \cdot s(\pi(E)).$$

Moreover if $f \geq 0$ is a continuous function on R and t is the Borel measure on R defined by $dt = f\,dr$, then for any Borel set $E \subset R$ we have

$$t(E) \leq \sup_E |f| \cdot d \cdot s(\pi(E)).$$

Proof. The second inequality follows trivially from the first, so that we need only prove the first. We use induction on d. For $d = 1$, π is a continuous bijection of R with S; being a local homeomorphism, it is then a global homeomorphism. It is easy to see that it takes r to s globally, and so the results are trivial. Let $d > 1$, assume the results for $d - 1$, and suppose that there are points of S the fibers over which have cardinality exactly d. Let S_d be the set of such points in S. Now, if the fiber above a point has e elements, the fibers of neighboring points have cardinality $\geq e$, and so S_d is open in S. Let $R_d = \pi^{-1}(S_d)$. Then $\pi : R_d \to S_d$ is a d-sheeted covering map. If $x \in R_d$, we can find an open set M containing $\pi(x)$ such that $N := \pi^{-1}(M) = \bigsqcup_{1 \leq j \leq d} N_j$ where $\pi : N_j \to M$ is a homeomorphism taking r to s. If $E \subset N$ is a Borel set, then $E = \bigsqcup_j E \cap N_j$, so that $\pi(E) = \bigcup_j \pi(E \cap N_j)$ is Borel as π is a homeomorphism on each N_j. Moreover,

$$r(E) = \sum_j r(E \cap N_j) = \sum_j s(\pi(E \cap N_j) \leq d \cdot s(\pi(E)).$$

These two properties are true with any Borel $M' \subset M$ and $N' = \pi'(M')$ replacing M, N respectively. Write now $S_d = \bigcup_n M_n$ where the M_n are open and have the properties described above for M. Then $S_d = \bigsqcup_n M'_n$ where $M'_n \subset M_n$, so that $R_d = \bigsqcup_n \pi^{-1}(M'_n)$. The two properties above are valid for any Borel set contained in any $\pi^{-1}(M'_n)$, hence they follow for any Borel set $E \subset R_d$. Write $S' = S \setminus S_d$, $R' = \pi^{-1}(S') = R \setminus R_d$. Then (R', S', π) inherit the properties of (R, S, π) with $d - 1$ instead of d. The result is valid for (R', S', π) and hence for (R, S, π), as is easily seen.

We are now ready to prove Theorem 1.2. Assume that $\boldsymbol{c}$ is a stably noncritical value of $\boldsymbol{f}$. For simplicity of notation we will suppress mentioning $\boldsymbol{c}$, because all of our estimates are locally uniform in $\boldsymbol{c}$. On $L_{\boldsymbol{c}} = L$ we have the estimate

$$\|\nabla_r(x)\| = \max_J |\partial_J(x)| > \frac{C}{\|x\|^\gamma} \qquad (\|x\| \geq 1),$$

where $C > 0$, $\gamma \geq 0$ are constants that remain the same when $\boldsymbol{c}$ is varied in a small neighborhood of $\boldsymbol{c}$. Let us write L^+ for the subset of L where $\|x\| > 1$. Now, at each point $x \in L^+$ some $|\partial_J(x)|$ equals $\|\nabla_r(x)\|$. Hence if we write

$$M_J = \{x \in L^+ \mid |\partial_J(x)| > C\|x\|^{-\gamma}\},$$

then

$$L^+ = \bigcup_J M_J.$$

The map $\overline{\pi}_J$ is open on M_J onto its image W_J and is a local analytic isomorphism. Moreover, if λ, ν, $\mu = \mu_{\boldsymbol{c}}$ have the same meaning as before, we have, on M_J,

$$d\mu = |\partial_J(x)|^{-1}\, d\nu$$

and hence, for any Borel set $E \subset M_J$, with D as in Lemma 5.4,

$$\mu(E) \leq D\cdot \sup_E |\partial_J(x)^{-1}|\cdot\lambda(\pi_J(E)).$$

Remembering that $|\partial_J(x)|^{-1} < C^{-1}\|x\|^\gamma$, we get from this that

$$\mu(E) \leq DC^{-1}\cdot \sup_E \|x\|^\gamma\cdot\lambda(\pi_J(E)).$$

If we take $E = B_R \cap M_J$ where $B_R = \{x \in k^m \mid \|x\| < R\}$, we see that $\pi_J(E)$ is a subset of the open ball of k^{m-r} of radius R, and hence $\lambda(\pi_J(E)) \leq AR^{m-r}$ where A is a universal constant. Hence

$$\mu(B_R \cap M_J) \leq ADC^{-1}\cdot R^{m-r+\gamma}.$$

Since this is true for all J, the temperedness of μ together with the growth estimate is proved, as well as the assertion that the last estimate remains unchanged if $\boldsymbol{c}$ varies in a small neighborhood of its original value. This finishes the proof of Theorem 1.2. □

6. Invariant measures on regular adjoint orbits of a semisimple Lie algebra

As an application of our Theorem 1.2 we shall prove that the invariant measures on *regular* semisimple orbits of a semisimple Lie algebra $\mathfrak{g} := \mathfrak{g}_K$ over a local field k of *characteristic* 0 are tempered.

The restriction to regular orbits is a consequence of the methods we use; the result is expected to be true without any condition on the orbit of the adjoint action.

For the moment let k be any field of characteristic 0 and K the algebraic closure of k. We write $\mathfrak{g}_K = K \otimes_k \mathfrak{g}_k$. Let $P(K)$ be the K-algebra of polynomial functions on $\mathfrak{g}_K$ with values in K. Since such a polynomial is determined by its restriction to $\mathfrak{g}_k$, the restriction to k defines an isomorphism of $P(K)$ with the K-algebra $P_k(K)$ of K-valued polynomial functions on $\mathfrak{g}_k$.

Let G be the connected adjoint group of $\mathfrak{g}_k$. It is a linear algebraic group defined over k and we write $G(k')$ for the group of its points over k', $k \subset k' \subset K$. We regard $G(k')$ as a subset of $G = G(K)$. From [Borel 1991] we know that $G(k)$ is Zariski-dense in $G(K)$. Now $G(K)$ acts on $P(K)$ and we denote by $J(K)$ the K-algebra of invariants of this action, which is a graded algebra in the obvious way. By a theorem of Chevalley, $J(K)$ is freely generated by homogeneous elements $p_1, \ldots, p_r$ of degrees $d_1, \ldots, d_r$ respectively, where r is the rank of $\mathfrak{g}_k$. In view of our remarks above, $J(K)$ is isomorphic to the graded K-algebra of invariants of $G(k)$ in $P_k(K)$. The action by $G(k)$ leaves $P_k(k)$ invariant, and we write $J(k)$ for the graded k-subalgebra of $G(k)$- invariants in $P_k(k)$. It is clear that

$$J(k) \simeq J(K)^{\mathrm{Gal}(K/k)}$$

as graded k-algebras.

The following lemma is surely known but we include it for the sake of completeness.

Lemma 6.1. *The graded k-algebra $J(k)$ is freely generated by homogeneous elements $q_1, \ldots, q_r$ of degrees $d_1, \ldots, d_r$ respectively.*

Proof. There is a finite extension k' of k with $k \subset k' \subset K$ such that the free homogeneous generators p_i of $J(K)$ have their coefficients in k'. Hence we may come down from K to k'. Let (e_α) be a k-basis for k'. Then we can write each p_i as

$$p_i = \sum_\alpha p_{i,\alpha} e_\alpha \quad (p_{i,\alpha} \in P_k(k)).$$

Since the $p_{i,\alpha}$ are k-valued, the $G(k)$-invariance of the p_i implies that the $p_{i,\alpha}$ are in $J(k)$. Now the p_i are algebraically independent, and so, $\omega := dp_1 \wedge \cdots \wedge dp_r \not\equiv 0$. Let

$$\omega_{\alpha_1,\ldots,\alpha_r} = dp_{1,\alpha_1} \wedge \cdots \wedge dp_{r,\alpha_r}.$$

Then

$$\omega = \sum_{\alpha_1,\ldots,\alpha_r} \omega_{\alpha_1,\ldots,\alpha_r} e_{\alpha_1} \wedge e_{\alpha_2} \cdots \wedge e_{\alpha_r} \not\equiv 0.$$

Hence we can choose $\alpha_1, \ldots, \alpha_r$ such that $\omega_{\alpha_1,\ldots,\alpha_r} \not\equiv 0$. With this choice, let

$$q_i = p_{i,\alpha_i} \qquad (1 \le i \le r).$$

Then the q_i are homogeneous elements of $J(k)$ and $\deg(q_i) = d_i$, and they are algebraically independent.

Now $J(k')$ is freely generated by the p_i of degree d_i. Hence, its Poincaré series is $\prod_i (1 - T^{d_i})^{-1}$. For any integer $m \ge 1$ let D_m be the dimension of $J(k')_m$, the subspace of degree m in $J(k')$. So $\dim(J(k)_m) \le D_m$. On the other hand, let $J_1(k)$ be the subalgebra of $J(k)$ generated by the q_i. Since the q_i are homogeneous, this is a graded subalgebra of $J(k)$, and it has the same Poincaré series as $J(k')$. Now $J_1(k)_m \subset J(k)_m$ for all m, and so

$$D_m = \dim J_1(k)_m \leq \dim J(k)_m \leq \dim J(k')_m = D_m.$$

This proves that $J_1(k)_m = J(k)m$ for all m, so that $J_1(k) = J(k)$. This finishes the proof of the lemma. □

Let $r = \operatorname{rank}(\mathfrak{g})$. Then by assumption we can choose $g_1, \ldots, g_r \in J(k)$ freely generating $J(k)$, hence also $J(K)$ (over K). An element $H \in \mathfrak{g}_K$ is semisimple (resp. nilpotent) if $\operatorname{ad} X$ is semisimple (resp. nilpotent). A semisimple element H is called *regular* if its centralizer is a Cartan subalgebra (CSA) of $\mathfrak{g}_K$. There is an invariant polynomial $D \in J(k)$, called the discriminant of $\mathfrak{g}$, such that if $X \in \mathfrak{g}_k$, X is semisimple and regular if and only if $D(X) \neq 0$. If $Y \in \mathfrak{g}$ is any element, we can write $Y = H + X$ where H is semisimple and X is a nilpotent in the derived algebra of the centralizer of H in $\mathfrak{g}_K$ (which is semisimple). It is known [Kostant 1963] that the orbit of $H + X$ has H in its closure, and so, for any $g \in J(K)$, we have $g(H) = g(H + X)$. If $\mathfrak{h}_K$ is a CSA of $\mathfrak{g}_K$, it is further known that the restriction map from $\mathfrak{g}_K$ to $\mathfrak{h}_K$ is an isomorphism of $J(K)$ with the algebra $J(\mathfrak{h}_K)$ of polynomials on $\mathfrak{h}_K$ invariant under the Weyl group W_K of $\mathfrak{h}_K$. It is known that the differentials $dg_1, \ldots, dg_r$ are linearly independent at an element Y of $\mathfrak{g}_K$ if and only if Y lies in an adjoint orbit of maximal dimension, which is $\dim(\mathfrak{g}_K) - \operatorname{rank}(\mathfrak{g}_K) = n - r$, where $n = \dim(\mathfrak{g}_K)$ [Kostant 1963]. If Y is semisimple, this happens if and only if Y is regular. Let $\mathfrak{g}'_K$ be the invariant open set of regular semisimple elements. We write

$$\boldsymbol{F} = (g_1, \ldots, g_r) \colon \mathfrak{g}_K \mapsto K^r$$

and view it as a polynomial map of $\mathfrak{g}_K$ into K^r commuting with the action of the adjoint group. Before we apply Theorem 1.2 to this set up, we need some preliminary discussion. Let $\mathcal{R} = \boldsymbol{F}(\mathfrak{g}'_K)$. The next lemma deals with the situation over K.

Lemma 6.2. *We have $\mathfrak{g}'_K = \boldsymbol{F}^{-1}(\mathcal{R})$. Moreover $\mathcal{R}$ is Zariski open in K^r, and is precisely the set of noncritical values of $\boldsymbol{F}$, so that all the noncritical values are also stably noncritical. Moreover, for any $\boldsymbol{c} \in \mathcal{R}$, the preimage $\boldsymbol{F}^{-1}(\boldsymbol{c})$ is an orbit under the adjoint group, consisting entirely of regular semisimple elements, hence smooth.*

Proof. Since $dg_1 \wedge \cdots \wedge dg_r \neq 0$ everywhere on $\mathfrak{g}'_K$, the map $\boldsymbol{F}$ is smooth on $\mathfrak{g}'_K$. Hence it is an open map [Görtz and Wedhorn 2010, Corollary 14.34], showing that $\boldsymbol{F}(\mathfrak{g}'_K) = \mathcal{R}$ is open in K^r.

We shall prove that if $Y \in \mathfrak{g}_K$ and $X \in \mathfrak{g}'_K$ are such that $\boldsymbol{F}(Y) = \boldsymbol{F}(X)$, then Y is regular semisimple, and is conjugate to X under the adjoint group. Suppose Y is not regular semisimple. Write $Y = Z + N$, where Z is semisimple and N is a nilpotent in the derived algebra of the centralizer of Z. The $\boldsymbol{F}(Y) = \boldsymbol{F}(Z) = \boldsymbol{F}(X)$.

Using the action of the adjoint group separately on X and Z we may assume that $X, Z \in \mathfrak{h}_K$ where $\mathfrak{h}_K$ is a CSA, and $\boldsymbol{F}(X) = \boldsymbol{F}(Z)$. Then all Weyl group invariant polynomials take the same value at Z and X and so Z and X are conjugate under the Weyl group. But as X is regular, so is Z, hence $N = 0$ or Y itself is regular semisimple. So, $\mathfrak{g}'_K = \boldsymbol{F}^{-1}(\mathcal{R})$. But then the above argument already shows that Y and X are conjugate under the adjoint group. Since the fibers of $\boldsymbol{F}$ above points of $\mathcal{R}$ are smooth, all points of $\mathcal{R}$ are stably noncritical. It remains to show that there are no other noncritical values. Suppose $Y \in \mathfrak{g}_K$ is such that $\boldsymbol{d} = \boldsymbol{F}(Y)$ is a noncritical value where $\boldsymbol{d} \notin \mathcal{R}$. Then $Y \notin \mathfrak{g}'_K$. Now $Y = Z + N$ as before, where Z is no longer regular (it is semisimple still). Then $\boldsymbol{F}(Z) = \boldsymbol{F}(Y)$ and so $Z \in \boldsymbol{F}^{-1}(\boldsymbol{d})$. But as Z is semisimple but not regular, $dg_1 \wedge \cdots \wedge dg_r$ is zero at Z [Kostant 1963]. Thus Z is a singular point of $\boldsymbol{F}^{-1}(\boldsymbol{d})$, contradicting the fact that $\boldsymbol{d}$ is noncritical. The lemma is thus completely proved. □

We now come to the case where the ground field is k, a local field of characteristic 0. We assume that the g_i have coefficients in k. Fix a regular semisimple element H_0 in $\mathfrak{g}_k$. Let

$$W(k) := W_{H_0}(k) = \{X \in \mathfrak{g}(k) \mid g_i(X) = g_i(H_0)(1 \le i \le r)\}.$$

Theorem 6.3. *Then the canonical measure on $W(k)$ is tempered, and the growth estimate* G *(see Section 1) is uniform when H varies in a neighborhood of H_0.*

Proof. For the map $\boldsymbol{F}$ on $\mathfrak{g}_k$ we know that $(g_1(H_0), \dots, g_r(H_0))$ is a stably noncritical value and so the theorem follows at once from Theorem 1.2. □

Although $W(K)$ is a single orbit under $G(K)$, this may no longer true over k. $W(k)$ is a k-analytic manifold of dimension $n - r$. On the other hand, the stabilizer in $G(k)$ of any point of $W(k)$ has dimension r and so its orbit under $G(k)$ is an open submanifold of $W(k)$. If we do this at every point of $W(k)$ we obtain a decomposition of $W(k)$ into a disjoint union of $G(k)$-orbits which are open submanifolds of dimension $n - r$ and so all these submanifolds are closed also. Thus the orbit $G(k).H_0$ is an open and closed submanifold of $W(k)$ of dimension $n - r$. Now the canonical measure on $W(k)$ is invariant under $G(k)$ and so on the orbit $G(k).H_0$ it is a multiple of the invariant measure on the orbit. Note that the orbit being closed, the invariant measure on it is a Borel measure on $\mathfrak{g}_k$. Since the canonical measure is tempered on $W(k)$ by Theorem 6.3, it is immediate that the invariant measure on the orbit $G(k).H_0$ is also tempered. Hence we have proved the following theorem:

Theorem 6.4. *The orbits of regular semisimple elements of $\mathfrak{g}_k$ are closed, and the invariant measures on them are tempered.*

For temperedness of invariant measures on semisimple symmetric spaces at the Lie algebra level over $\mathbb{R}$; see [Heckman 1982].

Remark 6.5. Ranga Rao [1972] and Deligne have independently shown that for any $X \in \mathfrak{g}_k$, there is an invariant measure on the adjoint orbit of X, and this measure extends to a Borel measure on the k-closure of the adjoint orbit of X. It is natural to ask if these are tempered in our sense when k is nonarchimedean. We shall consider this question in another paper since it does not follow from the results proved here.

7. Examples

In this section we give some examples. We consider only single polynomials ($r = 1$) of degree $d \geq 3$, defined over a local field k of characteristic 0. Let $f \in k[x_1, \ldots, x_m]$.

Elementary methods when r = 1 and f is homogeneous. For f homogeneous we have Euler's theorem on homogeneous functions, which asserts that $\sum_i x_i \partial f/\partial x_i = d \cdot f$. Let $L_c = \{x \in k^m \mid f(x) = c\}$ for $c \in k$. Then, for any critical point x of f, we have $f(x) = 0$, i.e., L_0 contains all the critical points. So every $c \in k \setminus \{0\}$ is a noncritical value and so is also stably noncritical. Moreover, Euler's identity for $x \in L_c$, $c \neq 0$, gives $\sum_i x_i \partial f/\partial x_i = dc$, so that we have

$$|d||c| = \left|\sum_i x_i \frac{\partial f}{\partial x_i}\right| \leq C\|x\|\|\nabla f(x)\| \quad (C > 0),$$

giving the estimate, with A a constant > 0,

$$\|\nabla f(x)\| \geq A\|x\|^{-1}, \quad \|x\| \geq 1,\ x \in L_c.$$

Moreover the projection $(x_1, \ldots, x_m) \mapsto (x_1, \ldots, \hat{x}_i, \ldots x_m)$ has the property that all fibers have cardinality $\leq d$. We thus have Theorem 1.2 with

$$\mu_{f,c} = O(R^m) \qquad (R \to \infty),$$

where O is uniform locally around c. We can actually say more.

Proposition 7.1. *Suppose* $\mathbf{0}$ *is the only singularity in* L_0, *i.e., the projective locus of* L_0 *is smooth. Then for any compact set* $W \subset k \setminus \{\mathbf{0}\}$, *we have*

$$\inf_{c \in W, x \in L_c, \|x\| \geq 1} \|\nabla f(x)\| > 0. \tag{7-1}$$

Moreover, the measure $\mu_{f,0}$ *defined on* $L_0 \setminus \{0\}$ *is finite in open neighborhoods of* $\mathbf{0}$ *if* $m > d$, *so that it extends to a Borel measure on* L_0. *Finally, for all* $c \in k$,

$$\mu_{f,c}(B_r) = O(R^{m-1}).$$

If $m \leq d$, *there are examples where* $\mu_{f,0}$ *is not finite in neighborhoods of* $\mathbf{0}$.

Proof. To prove (7-1) assume (7-1) is not true. Then we can find sequences $c_n \in W$, $x_n \in L_{c_n}$ such that $c_n \to c \in W$, $\nabla f(x_n) \to 0$ as $n \to \infty$. By passing to a subsequence and permuting the coordinates we may assume that $x_n = (x_{n1}, \ldots, x_{nm})$ where $|x_{n1}| \geq |x_{nj}|$ $(j \geq 2)$ and $|x_{n1}| \to \infty$. Now,

$$f(x_{n1}, \ldots, x_{nm}) = x_{n1}^d f(1, x_{n1}^{-1} x_{n2}, \ldots, x_{n1}^{-1} x_{nm}) = c_n \to c$$

and

$$(\nabla f)(x_{n1}, \ldots, x_{nm}) = x_{n1}^{d-1} (\nabla f)(1, x_{n1}^{-1} x_{n2}, \ldots, x_{n1}^{-1} x_{nm}) \to 0.$$

Now $|x_{n1}^{-1} x_{nj}| \leq 1$ for $2 \leq j \leq m$ and so, passing to a subsequence, we may assume that $x_{n1}^{-1} x_{nj} \to v_j$ for $j \geq 2$. Hence,

$$f(1, v_2, \ldots, v_m) = 0 \quad \text{and} \quad (\nabla f)(1, v_2, \ldots, v_m) = 0,$$

showing that $(1, v_2, \ldots, v_m) \neq (0, \ldots, 0)$ is a singularity of L_0. Then (7-1) leads to the conclusion

$$\mu_{f,c}(B_R) = O(R^{m-1}) \quad (R \to \infty)$$

locally uniformly at each $c \neq 0$.

For $\mu_{f,0}$ defined on $L_0 \setminus \{0\}$, one must first show that it is finite on small neighborhoods of 0, i.e., it extends to a Borel measure on L_0, if $m > d$. Let $S = \{u \in L_0 \mid \|u\| = 1\}$. Then there exist constants $a, b > 0$ such that $a \leq \|\nabla f(x)\| \leq b$ for all $x \in S$. Hence, by homogeneity,

$$a\|x\|^{d-1} \leq \|\nabla f(x)\| \leq b\|x\|^{d-1} \qquad (x \in L_0 \setminus \{0\}).$$

Hence

$$\|\nabla f(x)\| \geq a > 0 \qquad (x \in L_0, \|x\| \geq 1).$$

As before, this leads to $\mu_{f,0}(B_R \setminus B_1) = O(R^{m-1})$ as $R \to \infty$. Around **0** we obtain the finiteness of $\mu_{f,0}$ from the estimate $b^{-1}\|x\|^{-(d-1)} \leq \|\nabla f(x)\|^{-1} \leq a^{-1}\|x\|^{-(d-1)}$ and the fact that

$$\int_{x \in k^{m-1}, 0 < \|x\| < 1} \|x\|^{-(d-1)} \, d^{m-1}x < \infty$$

if $m > d$ for both $k = \mathbb{R}$ and k nonarchimedean. We shall now suppose that $f = X^4 + Y^4 - Z^4$. Then **0** is the only critical point. The map $(x, y, z) \mapsto (x, y)$ on $L_0 \cap \{(x, y, z) \mid x > 0\}$ is a diffeomorphism and the measure $\mu_{f,0}$ is

$$\frac{1}{|\partial f / \partial z|} \, dx \, dy = \frac{1}{4} \frac{dx \, dy}{(x^4 + y^4)^{3/4}}$$

and it is easy to verify that

$$\iint_N \frac{dx \, dy}{(x^4 + y^4)^{3/4}} = \infty$$

for any neighborhood N of $(0, 0)$. $\square$

Remark 7.2. It follows from Proposition 7.1 that to have

$$\inf_{x\in L_c,\|x\|\geq 1} \|\nabla f(x)\| = 0 \quad (c\neq 0) \tag{7-2}$$

we must look for f such that L_0 has singular points $\neq \mathbf{0}$. In the next section we describe some of these examples.

Some hypersurfaces in $\mathbb{P}_k^{m-1}$ with $[1:0:\cdots:0]$ as an isolated singularity. We do not try to give a "normal form" for such hypersurfaces; nevertheless large families of these can be described. We work in k^m, k a local field of characteristic 0. Since the first coordinate axis in k^m is chosen to be an isolated critical line (ICL), the first variable will be distinguished in what follows. Let us write $X, Y_1, \ldots, Y_{m-1}$ as the variables. Write $\boldsymbol{Y} = (Y_1, \ldots, Y_{m-1})$. Let $C(\varepsilon) = \{(X, \boldsymbol{Y}) \mid \|\boldsymbol{Y}\| \leq \varepsilon|X|\}$

Lemma 7.3. *Suppose $(X_n, \boldsymbol{Y}_n)$ is a sequence of points in L_c $(c\neq 0)$ such that they are in $C(\varepsilon)$ for some $\varepsilon < 1$. Let $F(X_n, \boldsymbol{Y}_n) = c \neq 0$ and $\nabla F(X_n, \boldsymbol{Y}_n) \to 0$. Then if the X-axis is an (ICL) for F, we must have $X_n \to \infty$, $1/X_n\boldsymbol{Y}_n \to \mathbf{0}$ as $n\to\infty$.*

Proof. By Euler's theorem, there is no singularity on L_c $(c\neq 0)$. Hence $\|\nabla F\|$ is bounded away from 0 on each compact subset of L_c. Hence, item 2 above implies $\|(X_n, \boldsymbol{Y}_n)\| = |X_n| \to \infty$. Then $\|{X_n}^{-1}\boldsymbol{Y}_n\| \leq 1$ and has a limit point $\boldsymbol{\eta}$. Passing to a subsequence, if necessary, we have ${X_n}^{-1}\boldsymbol{Y}_n \to \boldsymbol{\eta}$ as $n\to\infty$. If $d = \deg(F)$ we have $X_n^d F(1, {X_n}^{-1}\boldsymbol{Y}_n) = c$, $X_n^{d-1}\partial_X F(1, {X_n}^{-1}\boldsymbol{Y}_n) \to 0$, and $X_n^{d-1}\partial_{Y_i} F(1, {X_n}^{-1}\boldsymbol{Y}_n) \to 0$. So $F(1, \boldsymbol{\eta}) = 0$ and $\nabla F(1, \eta) = 0$, while $\boldsymbol{\eta} \in C(\varepsilon)$. Hence $\boldsymbol{\eta} = \mathbf{0}$ since ε can be arbitrarily small. □

Lemma 7.4. *If $(1, \mathbf{0})$ is a critical point of F, then F has the form*

$$F = X^{d-2}p_2 + X^{d-3}p_3 + \cdots + p_d$$

where p_r is a homogeneous polynomial in $\boldsymbol{Y}$ of degree r.

Proof. Write $F = X^{d-2}p_2 + X^{d-3}p_3 + \cdots + p_d$. Then p_0 is a constant, and $F(1, \mathbf{0}) = \mathbf{0}$ gives $p_0 = 0$. Then, $\partial F/\partial Y_i(1, \mathbf{0}) = 0$ gives $p_1 = 0$. □

From now on we let $d \geq 3$ and write

$$F = X^{d-2}p_2 + \cdots + p_d, \quad G = p_2 + \cdots + p_d.$$

Note that G is a polynomial in $\boldsymbol{Y}$, but *not necessarily* homogeneous.

Lemma 7.5. *If $\mathbf{0}$ is an isolated critical point (ICP) of G, then the X-axis is an ICL of F. In particular, this is so if the quadratic form p_2 is nondegenerate.*

Proof. We must prove that if $(1, \boldsymbol{Y}_n)$ is a CP for F with $\boldsymbol{Y}_n \to \mathbf{0}$, then $\boldsymbol{Y}_n = 0$ for $n \geq 1$. The conditions for $(1, \boldsymbol{Y}_n)$ to be a CP of F are

$$F(1, \boldsymbol{Y}_n) = 0, \quad \frac{\partial}{\partial X}F(1, \boldsymbol{Y}_n) = 0, \quad \frac{\partial}{\partial \boldsymbol{Y}_i}F(1, \boldsymbol{Y}_n) = 0 \text{ for all } i.$$

Consequently $G(\boldsymbol{Y}_n) = 0$ and $\partial G/\partial \boldsymbol{Y}_i(\boldsymbol{Y}_n) = 0$ for all i. Since $\boldsymbol{Y}_n \to \mathbf{0}$ and $\mathbf{0}$ is an ICP for G, $\boldsymbol{Y}_n = 0$ for all $n \gg 1$.

For the second statement, suppose p_2 is nondegenerate. By Morse's lemma [Duistermaat 1973] for local fields k, ch. $= 0$, there is a local diffeomorphism of k^{m-1} fixing $\mathbf{0}$ taking G to p_2. But $\mathbf{0}$ is an isolated CP for p_2, which makes it isolated for G. □

We remark that Duistermaat's proof [1973] of Morse's lemma is over $\mathbb{R}$, but its proof applies to the nonarchimedean case without any change, so we omit it.

Lemma 7.6. *The converse to the first statement of Lemma 7.5 is true if*

$$F = X^{d-r} p_r + p_d \quad (r \geq 2).$$

Proof. We must show that $G = p_r + p_d$ has $\mathbf{0}$ as an ICP if $(1, \mathbf{0})$ is an ICP for F. Suppose $\boldsymbol{w}_n$ are CPs for $G = p_r + p_d$ with $\boldsymbol{w}_n \to \mathbf{0}$. Then $G(w_n) = F(1, \boldsymbol{w}_n) = 0$ for all n, and $G_i(\boldsymbol{w}_n) = \partial F/\partial Y_i(1, \boldsymbol{w}_n) = 0$ for all n. Hence, $p_{r,i}(\boldsymbol{w}_n) + p_{d,i}(\boldsymbol{w}_n) = 0$ for all n. By Euler's theorem, $rp_r(\boldsymbol{w}_n) + dp_d(\boldsymbol{w}_n) = 0$ for all n. But, $p_r(\boldsymbol{w}_n) + p_d(\boldsymbol{w}_n) = 0$ for all n as well. So, $p_r(\boldsymbol{w}_n) = p_d(\boldsymbol{w}_n) = 0$ for all n. Hence, $\partial F/\partial X(1, \boldsymbol{w}_n) = (d-r)p_r(\boldsymbol{w}_n) = 0$ for all n. So $(1, \boldsymbol{w}_n)$ is a CP of F for all n. As $(1, \mathbf{0})$ is assumed to be an ICP for F, $\boldsymbol{w}_n = \mathbf{0}$ for $n \gg 1$. So $\mathbf{0}$ is an ICP for F. □

Study of condition* (7-2) *for $F = X^{d-2}p_2 + p_d$ where $G = p_2 + p_d$ has* 0 *as an ICP. Let us consider $F = X^2 + P_4(Y)$ where P_4 is a homogeneous quartic polynomial in Y, Z. For this to have $(t, 0, 0)$ as and ICL we must have $(0, 0)$ as an ICP for $G = Z^2 + P_4(Y, Z)$.

Lemma 7.7. *$G = Z^2 + P_4(Y, Z)$ has* $\mathbf{0}$ *as an ICP if and only if $Z^2 \nmid P_4(Y, Z)$, i.e.,*

$$P_4(Y, Z) = a_0Y^4 + a_1Y^3Z + a_2Y^2Z^2 + a_3YZ^3 + a_4Z^4$$

where at least one of a_0, a_1 is nonzero. In this case $\mathbf{0}$ *is its only CP.*

Proof. The equations which determine whether (y, z) is a CP of G are

$$z^2 + P_4(y, z) = 0, \quad \frac{\partial P_4}{\partial Y}(y, z) = 0 \quad \text{and} \quad 2Z + \frac{\partial P_4}{\partial Z}(y, z) = 0.$$

From the second and third equations just defined, using Euler's theorem, we have $2z^2 + 4P_4(y, z) = 0$, which implies $z^2 = 0$ and $P_4(yz) = 0$.

So the only critical points are of the form $(y, 0)$. Then $(0, 0)$ is certainly a CP. If $(y, 0)$ is a critical point for some $y \neq 0$, then $4a_0y^3 = 0$, $a_1y^3 = 0$ which implies a_0, a_1 both vanish. The entire Y-axis consists of critical points, and so for $(0, 0)$ to be an ICP, at least one of $a_0, a_1 \neq 0$. in which case $(0, 0)$ is the only CP.

We consider the cases (I) $a_0 \neq 0$ and (II) $a_0 = 0$, $a_1 \neq 0$. We consider case (I). We shall now verify that $\inf_{\|\boldsymbol{u}\|>1} \|\nabla F(\boldsymbol{u})\| > 0$ if $\boldsymbol{u} \in L_c$, $\|\boldsymbol{u}\| \geq 1$. Assume $F = X^2Z^2 + P_4(Y, Z)$, and in view of Lemma 7.3, choose a sequence (x_n, y_n, z_n) such that $x_n \to \infty$, $y_n/x_n \to 0$, $z_n/x_n \to 0$ and:

(i) $x_n^2 y_n^2 + P_4(y_n, z_n) = c$,

(ii) $\partial F/\partial X = 2x_n z_n^2 \to 0$,

(iii) $\partial P_4/\partial Y(y_n, z_n) \to 0$,

(iv) $2x_n^2 z_n + \partial P_4/\partial Z(y_n, z_n) \to 0$.

From (ii) we get $z_n \to 0$. Assuming we are in case (I), y_n is bounded. Otherwise, by passing to a subsequence we may assume $y_n \to \infty$ giving $\partial P_4/\partial Y(y_n, z_n) = 4a_0 y_n^3 + 3a_1 y_n^2 z_n + \cdots \to 0$. If $a_0 \neq 0$, then $\partial P_4/\partial Y(y_n, z_n) = 4a_0 y_n^3(1+o(z_n/y_n)) \to \infty$, which is a contradiction. But if $\eta \neq 0$ is a limit point of y_n, then

$$\frac{\partial P_4}{\partial Y}(z_n, y_n) \to 4a_0\eta^3 \neq 0$$

which is a contradiction. So, $y_n \to 0$ necessarily. Then, $P_4(y_n, z_n) \to 0$ and $\partial P_4/\partial Z(y_n, z_n) \to 0$. Hence by (iv), $x_n^2 z_n \to 0$, by (i) $x_n^2 z_n^2 \to c \neq 0$, a contradiction. This finishes case (I).

Assuming we are in case (II), $a_0 = 0,\ a_1 \neq 0$, we claim $y_n \to \infty$. Otherwise, by passing to a subsequence, we may assume $y_n \to \eta$. Then $P_4(y_n, z_n) = a_1 y_n^3 z_n + \cdots$ so that $P_4(y_n, z_n) \to 0$. Hence, $x_n^2 z_n^2 \to c$. But $\partial P_4/\partial Z(y_n, z_n) = a_1 y_n^3 + \cdots \to a_1\eta^3$. Hence, by (iv), $x_n^2 z_n = o(1)$. So, as $z_n \to 0$, we have $x_n^2 y_n^2 \to 0$. Hence, $c = 0$ is a contradiction.

We are left with the case $x_n \to \infty,\ y_n \to \infty,\ z_n \to 0,\ (y_n/x_n)(z_n/x_n) \to 0$, and $P_4(Y, Z) = a_1 Y^3 Z + \cdots$, for $a_1 \neq 0$. But $\partial P_4/\partial Y(y_n, z_n) = 3a_1 y_n^2 z_n(1+o(z_n/y_n)) \to 0$ if and only if $y_n^2 z_n \to 0$. In this case may we have a counterexample to statement (7-2). Remark 4.14 gives an example of this kind. Note that case (I) is generic among the families we consider. □

References

[Artin 1967] E. Artin, *Algebraic numbers and algebraic functions*, Gordon and Breach Science Publishers, New York-London-Paris, 1967. MR Zbl

[Bollaerts 1990] D. Bollaerts, "An estimate of approximation constants for p-adic and real varieties", *Manuscripta Math.* **69**:4 (1990), 411–442. MR Zbl

[Borel 1991] A. Borel, *Linear algebraic groups*, 2nd ed., Graduate Texts in Mathematics **126**, Springer, 1991. MR Zbl

[Duistermaat 1973] J. J. Duistermaat, *Fourier integral operators*, Courant Institute of Mathematical Sciences, New York University, New York, 1973. Translated from Dutch notes of a course given at Nijmegen University, February 1970 to December 1971. MR Zbl

[Fulton 1998] W. Fulton, *Intersection theory*, Springer, Berlin, 1998.

[Gel'fand and Shilov 1964] I. M. Gel'fand and G. E. Shilov, *Generalized functions. Vol. I: Properties and operations*, Translated by Eugene Saletan, Academic Press, New York-London, 1964. MR

[Görtz and Wedhorn 2010] U. Görtz and T. Wedhorn, *Algebraic geometry I*, Advanced Lectures in Mathematics, Vieweg + Teubner, Wiesbaden, 2010. Schemes with examples and exercises. MR Zbl

[Greenberg 1966] M. J. Greenberg, "Rational points in Henselian discrete valuation rings", *Inst. Hautes Études Sci. Publ. Math.* 31 (1966), 59–64. MR Zbl

[Guillemin and Pollack 1974] V. Guillemin and A. Pollack, *Differential topology*, Prentice-Hall, Englewood Cliffs, N.J., 1974. MR Zbl

[Harish-Chandra 1957] Harish-Chandra, "Fourier transforms on a semisimple Lie algebra. I", *Amer. J. Math.* **79** (1957), 193–257. MR

[Harish-Chandra 1964] Harish-Chandra, "Invariant distributions on Lie algebras", *Amer. J. Math.* **86** (1964), 271–309. MR

[Harish-Chandra 1973] Harish-Chandra, "Harmonic analysis on reductive p-adic groups", *Proc. Sympos. Pure Math., Vol. XXVI, Williams Coll., Williamstown, Mass., 1972* (1973), 167–192. MR

[Heckman 1982] G. J. Heckman, "Projections of orbits and asymptotic behavior of multiplicities for compact connected Lie groups", *Invent. Math.* **67**:2 (1982), 333–356. MR Zbl

[Hörmander 1958] L. Hörmander, "On the division of distributions by polynomials", *Ark. Mat.* **3** (1958), 555–568. MR

[Igusa 1978] J.-i. Igusa, *Forms of higher degree*, Tata Institute of Fundamental Research Lectures on Mathematics and Physics **59**, Tata Institute of Fundamental Research, Bombay; by the Narosa Publishing House, New Delhi, 1978. MR Zbl

[Kolk and Varadarajan 1992] J. A. C. Kolk and V. S. Varadarajan, "Lorentz invariant distributions supported on the forward light cone", *Compositio Math.* **81**:1 (1992), 61–106. MR Zbl

[Kostant 1963] B. Kostant, "Lie group representations on polynomial rings", *Amer. J. Math.* **85** (1963), 327–404. MR Zbl

[Łojasiewicz 1959] S. Łojasiewicz, "Sur le problème de la division", *Studia Math.* **18** (1959), 87–136. MR

[Mumford 1995] D. Mumford, *Algebraic geometry. I*, Classics in Mathematics, Springer, 1995. Complex projective varieties, Reprint of the 1976 edition. MR Zbl

[Mumford and Oda 2015] D. Mumford and T. Oda, *Algebraic geometry. II*, Texts and Readings in Mathematics **73**, Hindustan Book Agency, New Delhi, 2015. MR Zbl

[Ranga Rao 1972] R. Ranga Rao, "Orbital integrals in reductive groups", *Ann. of Math.* (2) **96** (1972), 505–510. MR Zbl

[Sard 1942] A. Sard, "The measure of the critical values of differentiable maps", *Bull. Amer. Math. Soc.* **48** (1942), 883–890. MR Zbl

[Serre 2006] J.-P. Serre, *Lie algebras and Lie groups*, Lecture Notes in Mathematics **1500**, Springer, 2006. 1964 lectures given at Harvard University, Corrected fifth printing of the second (1992) edition. MR

[Varadarajan 1977] V. S. Varadarajan, *Harmonic analysis on real reductive groups*, Lecture Notes in Mathematics, Vol. 576, Springer, 1977. MR Zbl

[Virtanen and Weisbart 2014] J. Virtanen and D. Weisbart, "Elementary particles on p-adic spacetime and temperedness of invariant measures", *p-Adic Numbers Ultrametric Anal. Appl.* **6**:4 (2014), 318–332. MR Zbl

[Vogel 1984] W. Vogel, *Lectures on results on Bezout's theorem*, Tata Institute of Fundamental Research Lectures on Mathematics and Physics **74**, Springer, 1984. Notes by D. P. Patil. MR Zbl

[Weil 1967] A. Weil, *Basic number theory*, Die Grundlehren der mathematischen Wissenschaften, Band 144, Springer, 1967. MR Zbl

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DAVID TAYLOR
DEPARTMENT OF MATHEMATICS
UCLA
LOS ANGELES, CA
UNITED STATES
dwtaylor@math.ucla.edu

V. S. VARADARAJAN
DEPARTMENT OF MATHEMATICS
UCLA
LOS ANGELES, CA
UNITED STATES
vsv@math.ucla.edu

JUKKA VIRTANEN
DEPARTMENT OF MATHEMATICS
UCLA
LOS ANGELES, CA
UNITED STATES
semisimplemath@gmail.com

DAVID WEISBART
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA, RIVERSIDE
RIVERSIDE, CA
UNITED STATES
dweisbart@gmail.com

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