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MONOTONICITY OF EIGENVALUES OF GEOMETRIC OPERATORS ALONG THE RICCI–BOURGUIGNON FLOW

BIN CHEN, QUN HE AND FANQI ZENG

We study monotonicity of eigenvalues of the Schrödinger-type operator $-\Delta + cR$, where c is a constant, along the Ricci–Bourguignon flow. For $c \neq 0$, we derive monotonicity of the lowest eigenvalue of the Schrödinger-type operator $-\Delta + cR$, which generalizes some results of Cao (2008). As an application, we rule out nontrivial compact steady breathers in the Ricci–Bourguignon flow. For $c = 0$, we derive monotonicity of the first eigenvalue of the Laplacian, which generalizes some results of Ma (2006).

1. Introduction

Let (M, g) be an n-dimensional closed Riemannian manifold and $g(t)$ be a solution to the following Ricci–Bourguignon flow:

$$\frac{\partial}{\partial t} g = -2\,\mathrm{Ric} + 2\rho R g = -2(\mathrm{Ric} - \rho R g), \tag{1-1}$$

where Ric is the Ricci tensor of the manifold, R is scalar curvature and ρ is a real constant. When $\rho = \frac{1}{2}, \frac{1}{n}, \frac{1}{2(n-1)}$ or 0, the tensor $\mathrm{Ric} - \rho R g$ corresponds to the Einstein tensor, the traceless Ricci tensor, the Schouten tensor or the Ricci tensor respectively. Apart from these special values of ρ, for which we will call the associated flows by the same name as the corresponding tensor, in general we will refer to the evolution equation defined by the PDE system (1-1) as the Ricci–Bourguignon flow. Moreover, by a suitable rescaling in time, when ρ is nonpositive, they can be seen as an interpolation between the Ricci flow and the Yamabe flow, see [Brendle 2005; Ye 1994] for instance, obtained as a limit when $\rho \to -\infty$.

The study of these flows was proposed by Jean-Pierre Bourguignon [1981, Question 3.24], building on some unpublished work of Lichnerowicz in the sixties and a paper of Aubin [1970]. Fischer [2004] studied a conformal version of this problem where the scalar curvature is constrained along the flow. Lu, Qing and

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Zheng [Lu et al. 2014] also proved some results on the conformal Ricci–Bourguignon flow. Recently, for suitable values of the scalar parameter involved in these flows, Catino et al. [2017] proved short time existence and provided curvature estimates. Moreover, Catino and Mazzieri [2016] stated some results on the associated solitons.

At present, the eigenvalues of geometric operators have become a powerful tool in the study of geometry and topology of manifolds. Recently, there has been a lot of work on the eigenvalue problems under the Ricci flow. On one hand, Perelman [2002] introduced the so-called $\mathscr{F}$-entropy functional and proved that it is nondecreasing along the Ricci flow coupled to a backward heat-type equation. Since the functional $\mathscr{F}$ is nondecreasing, this implies the monotonicity of the lowest eigenvalue of $-4\Delta + R$ along the Ricci flow.

Cao [2007] extended the operator $-4\Delta + R$ to the new operator $-\Delta + \frac{R}{2}$ on closed Riemannian manifolds, and showed that the eigenvalues of this new operator are nondecreasing along the Ricci flow with nonnegative curvature operator. Shortly thereafter Li [2007] dropped the curvature assumption and also obtained the above result for the operator $-\Delta + \frac{R}{2}$.

At around the same time, Cao [2008] considered the general operator

$$-\Delta + cR, \quad \text{where } c \geq \tfrac{1}{4},$$

and derived the following exact monotonicity formula; thus he showed that the lowest eigenvalue of this operator is nondecreasing along the Ricci flow without any curvature assumption.

Theorem A [Cao 2008]. *Let $(M, g(t))_{t\in[0,T)}$ be a solution of the unnormalized Ricci flow on a closed manifold M. Assume that $\lambda_0(t)$ is the lowest eigenvalue of $-\Delta + cR$, $c \geq \frac{1}{4}$, and $f = f(x,t) > 0$ satisfies*

$$-\Delta f(x,t) + cRf(x,t) = \lambda_0(t) f(x,t)$$

with $\int_M f^2\, dv = 1$. Then, under the unnormalized Ricci flow, we have

$$\frac{d}{dt}\lambda_0(t) = \frac{1}{2}\int_M |\operatorname{Ric} + \nabla^2\varphi|^2 e^{-\varphi}\, dv + \frac{4c-1}{2}\int_M |\operatorname{Ric}|^2 e^{-\varphi}\, dv \geq 0, \tag{1-2}$$

where $e^{-\varphi} = f^2$.

On the other hand, Ma [2006] obtained the monotonicity of the first eigenvalue of the Laplacian operator on a domain with Dirichlet boundary condition along the Ricci flow. Using the differentiability of the eigenvalues and the corresponding eigenfunctions of the Laplace operator under the Ricci flow, he obtained the following result.

Theorem B [Ma 2006]. *Let $g = g(t)$ be the evolving metric along the Ricci–Hamilton flow with $g(0) = g_0$ being the initial metric in M. Let D be a smooth* *bounded domain in* (M, g_0). *Let* $\mu > 0$ *be the first eigenvalue of the Laplace operator of the metric* $g(t)$. *If there is a constant* a *such that the scalar curvature satisfies* $R \geq 2a$ *in* $D \times \{t\}$ *and the Einstein tensor satisfies*

$$E_{ij} \geq -a g_{ij} \quad \text{in } D \times \{t\},$$

where $E_{ij} := R_{ij} - \frac{R}{2} g_{ij}$, *then we have* $\frac{d}{dt}\mu \geq 0$, *that is*, μ *is nondecreasing in* t; *furthermore*, $\frac{d}{dt}\mu(t) > 0$ *when the scalar curvature* R *is not the constant* $2a$. *The same monotonicity result is also true for other eigenvalues.*

Motivated by the above work, we also consider the eigenvalue of $-\Delta + cR$ with c a constant. For $c \neq 0$, inspired by [Cao 2007; 2008; Li 2007], we can derive the following monotonicity of the lowest eigenvalue of $-\Delta + cR$ under the Ricci–Bourguignon flow (1-1). That is, we obtain:

Theorem 1.1. *Let* $(M, g(t))_{t\in[0,T)}$ *be a compact maximal solution of the nontrivial Ricci–Bourguignon flow* (1-1) *and* $\lambda_0(t)$ *be the lowest eigenvalue of the operator* $-\Delta + cR$ *corresponding to the normalized eigenfunction* f, *that is*,

$$(-\Delta + cR) f = \lambda_0 f, \quad \int_M f^2 \, dv = 1.$$

(1) *If* $\rho \leq 0$,

$$c \in \left[\frac{1}{4}, \frac{n}{4(n-1)}\right] \cup \left[\frac{(1-(n-1)\rho)^2}{4(1-2(n-1)\rho)}, +\infty\right)$$

and the scalar curvature is nonnegative at the initial time, *then the lowest eigenvalue of the operator* $-\Delta + cR$ *is nondecreasing in* $[0, T)$ *under the Ricci–Bourguignon flow* (1-1). *Furthermore*, *if* $\rho \neq 0$ *or* $c \neq \frac{1}{4}$, *then the lowest eigenvalue of the operator* $-\Delta + cR$ *is strictly monotone increasing in* $[0, T)$ *under the Ricci–Bourguignon flow* (1-1).

(2) *If* $0 < \rho < \frac{1}{2(n-1)}$,

$$c \geq \frac{3(n-1)^2\sqrt{\rho}}{2(1-2(n-1)\rho)} + \frac{1}{4}$$

and the curvature operator is nonnegative at the initial time, *then the quantity*

$$(T' - t)^{-\alpha} \lambda_0(t) \tag{1-3}$$

is strictly monotone increasing under the Ricci–Bourguignon flow (1-1) *in* $[0, T')$, *where*

$$T' = \frac{1}{2(1-\rho)\epsilon}, \quad \epsilon = \max_M R(0) \quad \text{and} \quad \alpha = \frac{\rho}{1-\rho} > 0.$$

Remark 1.1. When $\rho = 0$, where the Ricci–Bourguignon flow is the Ricci–Hamilton flow, our (1) reduces to the corresponding result of Cao [2008]. When $\rho \neq 0$, we don't know the differentiability for the lowest eigenvalue, so we show that the lowest eigenvalue is strictly monotone increasing by using the sign-preserving property.

Remark 1.2. According to the proof, it is obvious that (2) will hold whenever the Ricci curvature is nonnegative, but in general, the nonnegativity of the Ricci curvature is not preserved along the Ricci–Bourguignon flow. Nevertheless, the nonnegativity of the Ricci curvature is preserved in dimension three.

Corollary 1.2. *In dimension three, let $g(t)$ and $\lambda_0(t)$ be the same as in Theorem 1.1. But here we assume the Ricci curvature is nonnegative at the initial time. If $0<\rho<\frac{1}{4}$ and*

$$c \geq \frac{6\sqrt{\rho}}{1-4\rho}+\frac{1}{4},$$

then the quantity

$$(T'-t)^{-\alpha}\lambda_0(t) \tag{1-4}$$

is strictly monotone increasing under the Ricci–Bourguignon flow (1-1) in $[0,T')$, where

$$T'=\frac{1}{2(1-\rho)\epsilon}, \quad \epsilon=\max_M R(0) \quad \textit{and} \quad \alpha=\frac{\rho}{1-\rho}>0.$$

Next, as an application of our Theorem 1.1, we rule out nontrivial compact steady breathers. That is, we obtain:

Theorem 1.3. (1) *If $\rho=0,\ c\geq\frac{1}{4}$, there is no compact steady breather other than the one which is Ricci-flat.*

(2) *If $\rho<0$,*

$$c\in\left(\frac{1}{4},\frac{n}{4(n-1)}\right]\cup\left[\frac{(1-(n-1)\rho)^2}{4(1-2(n-1)\rho)},+\infty\right),$$

there is no compact steady breather with nonnegative scalar curvature other than the one which is Ricci-flat.

For $c=0$, we derive the following monotonicity of eigenvalues on Laplacian under the Ricci–Bourguignon flow (1-1). That is, we obtain:

Theorem 1.4. *Let $(M,g(t))_{t\in[0,T)}$ be a compact maximal solution of the nontrivial Ricci–Bourguignon flow (1-1) and $\rho<\frac{1}{2(n-1)}$. Let $\lambda_1(t)$ be the first eigenvalue of the Laplace operator of the metric $g(t)$. If there is a nonnegative constant a such that*

$$R_{ij}-\frac{1+(2-n)\rho}{2}Rg_{ij}\geq -ag_{ij} \quad \textit{in } M\times[0,T), \tag{1-5}$$

$$R\geq\frac{2a}{1-n\rho} \quad \textit{in } M\times\{0\}, \tag{1-6}$$

then $\lambda_1(t)$ is strictly monotone increasing and differentiable almost everywhere along the Ricci–Bourguignon flow in $[0,T)$.

Remark 1.3. (1) Wu et al. [2010] proved a similar result about the p-Laplace operator along the Ricci flow, where they assumed $R \geq ap$ and $R \not\equiv ap$ in $M \times \{0\}$, which are a little stronger than (1-6). The key difference is that we use Lemma 2.3.

(2) It should be pointed out that for $\rho = 0$, the above theorem is similar to the main result for the first eigenvalue of the Laplace operator in [Ma 2006]. Moreover, our assumptions are weaker than Ma's.

(3) If $a < 0$, there doesn't exist any scalar curvature which satisfies (1-5) and (1-6) at the same time.

(4) The result may be useful in the study of blow-up models of Ricci–Bourguignon flow on a complete Riemannian manifold (M, g_0).

2. Preliminaries

We begin with the definition for the first eigenvalue (the lowest eigenvalue) of the Laplace operator (the Schrödinger-type operator $-\Delta + cR$) under the Ricci–Bourguignon flow on a closed manifold. Then, we will show that the first eigenvalue of the Laplace operator is a continuous function along the Ricci–Bourguignon flow. Finally, under the Ricci–Bourguignon flow, we show that if $R(g_0) := R(0) \geq \beta$, for some $\beta \in \mathbb{R}$, then either $\max_M R(t) > \beta$ or the flow is trivial (i.e., $g(t) = g(0)$) for every $t \in (0, T)$.

Throughout, M will be taken to be a closed manifold (i.e., compact without boundary). We use moving frames in all calculations and adopt the index convention

$$1 \leq i, j, k, \cdots \leq n$$

throughout this paper.

Now we recall the definition of the first eigenvalue of the Laplace operator on a closed manifold under the Ricci–Bourguignon flow. Let $(M, g(t))$ be a solution of the Ricci–Bourguignon flow on the time interval $[0, T)$. Consider the first nonzero eigenvalue of the Laplace operator at time t, where $0 \leq t < T$,

$$\lambda_1(t) = \inf\left\{ \int_M |\nabla f|^2 \, d\upsilon : f \in W^{1,2}, \int_M f^2 \, d\upsilon = 1 \text{ and } \int_M f \, d\upsilon = 0 \right\},$$

where $d\upsilon$ denotes the volume form of the metric $g = g(t)$. Meanwhile the corresponding eigenfunction f satisfies the equation

$$-\Delta f(t) = \lambda_1(t) f(t),$$

where Δ is the Laplace operator with respect to $g(t)$, given by

$$\Delta_{g(t)} = \frac{1}{\sqrt{|g(t)|}} \partial_i \left(\sqrt{|g(t)|} g(t)^{ij} \partial_j\right),$$

and $g(t)^{ij} = g(t)^{-1}_{ij}$ is the inverse of the matrix $g(t)$ and $|g| = \det(g_{ij})$.

Note that it is not clear whether the first eigenvalue or the corresponding eigenfunction of the Laplace operator is differentiable under the Ricci–Bourguignon flow. When $\rho = 0$, where the Ricci–Bourguignon flow is the Ricci–Hamilton flow, many papers have pointed out that its differentiability under the Ricci–Hamilton flow follows from eigenvalue perturbation theory; e.g., see [Kato 1984; Kleiner and Lott 2008; Reed and Simon 1978]. But for $\rho \neq 0$, as far as we are aware, the differentiability of the first eigenvalue and eigenfunction of the Laplace operator under the Ricci–Bourguignon flow has not been known until now. So we cannot use Ma's trick to derive the monotonicity of the first eigenvalue of the Laplace operator. Although, we do not know the differentiability for $\lambda_1(t)$, following the techniques of [Wu et al. 2010], we will see that $\lambda_1(t)$ in fact is a continuous function along the Ricci–Bourguignon flow on $[0, T)$.

Lemma 2.1 [Wu et al. 2010]. *If g_1 and g_2 are two metrics on M which satisfy*

$$(1+\varepsilon)^{-1} g_1 \le g_2 \le (1+\varepsilon) g_1,$$

then, we have

$$(1+\varepsilon)^{-(n+1)} \le \frac{\lambda_1(g_1)}{\lambda_1(g_2)} \le (1+\varepsilon)^{(n+1)}. \tag{2-1}$$

In particular, $\lambda_1(g(t))$ is a continuous function in the t-variable.

Proof. This can be proved using arguments similar to those for Theorem 2.1 in [Wu et al. 2010]. □

Next we recall the definition of the lowest eigenvalue of $-\Delta + cR$. Let $\lambda_0(t)$ be the lowest eigenvalue of $-\Delta + cR$. Given a metric g on a closed manifold M, we define the functional λ_0 by

$$\lambda_0(t) = \inf\left\{ \mathscr{G}(g, f) : \int_M f^2 \, dv = 1,\ f > 0 \text{ and } f \in W^{1,2} \right\}, \tag{2-2}$$

where

$$\mathscr{G}(g, f) = \int_M (f(-\Delta f) + cRf^2)\, dv = \int_M (|\nabla f|^2 + cRf^2)\, dv.$$

We also do not know the differentiability for $\lambda_0(t)$ and the corresponding eigenfunction. But, following the techniques of [Chow et al. 2008], we will see that $\lambda_0(t)$ in fact is a continuous function along the Ricci–Bourguignon flow on $[0, T)$.

Lemma 2.2 [Chow et al. 2008]. *If g_1 and g_2 are two metrics on M which satisfy*

$$(1+\varepsilon)^{-1} g_1 \le g_2 \le (1+\varepsilon) g_1 \quad \text{and} \quad R(g_1) - \varepsilon \le R(g_2) \le R(g_1) + \varepsilon,$$

then

$$\begin{aligned} \lambda_0(g_2) - \lambda_0(g_1) \le {} & ((1+\varepsilon)^{\frac{n}{2}+1} - (1+\varepsilon)^{-\frac{n}{2}})(1+\varepsilon)^{\frac{n}{2}}(\lambda_0(g_1) - \min_M |c| R(g_1)) \\ & + |c|\big((1+\delta)\max_M |R(g_2) - R(g_1)| + 2\delta \max_M |R(g_1)|\big)(1+\varepsilon)^{\frac{n}{2}}, \end{aligned} \tag{2-3}$$

where $\delta \to 0$ as $\varepsilon \to 0$. In particular, λ_0 is a continuous function with respect to the C^2-topology.

Proof. This can be proved using arguments similar to those for Lemma 5.24 in [Chow et al. 2008]. □

At last, we present the following lemma.

Lemma 2.3. *Let $(M, g_t)_{t\in[0,T)}$ be a compact maximal solution of the Ricci–Bourguignon flow (1-1). If $\rho < \frac{1}{2(n-1)}$ and $R(0) \geq \beta$ for some $\beta \in \mathbb{R}$, then either $\max_M R(t) > \beta$ or $g(t) = g(0)$ for every $t \in (0, T)$.*

Proof. From Lemma 2.6, we know that $R(t) \geq \beta$ for every $t \in [0, T)$. If $\max_M R(t_0) = \beta$ for some $t_0 \in (0, T)$, we have $R(t_0) \equiv \beta$ and $\frac{\partial R}{\partial t}\big|_{t_0} \leq 0$. From (2-7), we have

$$\frac{1}{n}R^2(t_0) \leq |\mathrm{Ric}|^2(t_0) \leq \rho R^2(t_0) < \frac{1}{2(n-1)}R^2(t_0).$$

Obviously, we have $R(t_0) = 0$ and $\mathrm{Ric}(t_0) = 0$. Hence, $\max_M R(t_0) = \beta = 0$. Therefore, if $\beta \neq 0$, we have $\max_M R(t) > \beta$ for every $t \in [0, T)$.

When $\beta = 0$, let $I = \{t > 0 : \max_M R(t) > 0\}$. If $I = \varnothing$, then we have $R(t) \equiv 0$ and $\mathrm{Ric}(t) \equiv 0$. Hence we have $g(t) = g(0)$. When $I \neq \varnothing$ and $t_1 \in I$, for any t_0 and $0 < t_0 < t_1$, if $\max_M R(t_0) = 0$, then $R(t_0) \equiv 0$ and $\mathrm{Ric}(t_0) \equiv 0$. Hence, in $[t_0, T)$, $g(t) = g(t_0)$. So we have $\mathrm{Ric}(t_1) = \mathrm{Ric}(t_0) = 0$, which is in contradiction with $\max_M R(t_1) > 0$. Hence, $t_0 \in I$. Since $t_0 \in (0, t_1)$ is arbitrary, we have $(0, t_1] \subset I$. By the strong maximum principle, we have $(0, T) \subset I$. □

For the reader's convenience, we will recall some basic knowledge about the Ricci–Bourguignon flow.

Lemma 2.4 [Catino et al. 2017]. *Under the Ricci–Bourguignon flow (1-1), we have*

$$\frac{\partial}{\partial t}g^{ij} = 2(R^{ij} - \rho R g^{ij}), \tag{2-4}$$

$$\frac{\partial}{\partial t}(d\upsilon) = (n\rho - 1)R\, d\upsilon, \tag{2-5}$$

$$\frac{\partial}{\partial t}(\Gamma^k_{ij}) = -R_{ik,j} - R_{kj,i} + R_{ij,k} + \rho(\delta^i_k R_{,j} + \delta^k_j R_{,i} - g_{ij}R_{,k}), \tag{2-6}$$

$$\frac{\partial}{\partial t}R = [1 - 2(n-1)\rho]\Delta R + 2|\mathrm{Ric}|^2 - 2\rho R^2. \tag{2-7}$$

Lemma 2.5 (short time existence [Catino et al. 2017]). *Let $\rho < \frac{1}{2(n-1)}$. Then, the evolution equation (1-1) has a unique solution for a positive time interval on any smooth, n-dimensional, compact Riemannian manifold M (without boundary) with any initial metric g_0.*

Lemma 2.6 (preserved curvature conditions [Catino et al. 2017]). *Let $(M, g_t)_{t\in[0,T)}$ be a compact maximal solution of the Ricci–Bourguignon flow (1-1). If $\rho \le \frac{1}{2(n-1)}$, the minimum of the scalar curvature is nondecreasing along the flow. In particular, if $R(g_0) \ge \alpha$, for some $\alpha \in \mathbb{R}$, then $R(g_t) \ge \alpha$ for every $t \in [0, T)$. Moreover, if $\alpha > 0$ then $T \le n/(2(1-n\rho)\alpha)$.*

Lemma 2.7 (conditions preserved in three dimensions [Catino et al. 2017]). *Let $(M, g_t)_{t\in[0,T)}$ be a compact, 3-dimensional, solution of the Ricci–Bourguignon flow (1-1). If $\rho < \frac{1}{4}$, then*

(1) *nonnegative Ricci curvature is preserved along the flow*;

(2) *the pinching inequality* $\mathrm{Ric} \ge \varepsilon R g$ *is preserved along the flow for any* $\varepsilon \le \frac{1}{3}$.

Lemma 2.8 [Catino et al. 2017]. *Let $(M, g_t)_{t\in[0,T)}$ be a compact solution of the Ricci–Bourguignon flow (1-1) with $\rho \le \frac{1}{2(n-1)}$, and such that the initial data g_0 has nonnegative curvature operator. Then $\widetilde{R}_{g(t)} \ge 0$ for every $t \in [0, T)$, where $\widetilde{R} \in \mathrm{End}(\Lambda^2 M)$ is the Riemann curvature operator.*

Lemma 2.9 [Catino et al. 2017]. *Let $\rho < \frac{1}{2(n-1)}$. If $g(t)$ is a compact solution of the Ricci–Bourguignon flow on a maximal time interval $[0, T)$, $T < +\infty$, then*

$$\limsup_{t\to T} \max_M |\mathrm{Riem}(\cdot\,, t)| = +\infty,$$

where $\mathrm{Riem}(\cdot\,, t)$ *is Riemann tensor.*

3. Proof of Theorem 1.1

We will now prove Theorem 1.1. In order to achieve this, we first prove the following two lemmas. Our proof uses some tricks from [Cao 2007, 2008].

Let M be an n-dimensional closed Riemannian manifold, and $g(t)$ be a smooth solution of the Ricci–Bourguignon flow on the time interval $[0, T)$. Let $\lambda_0(t)$ be the lowest eigenvalue of the operator $-\Delta + cR$ corresponding to the normalized eigenfunction f, that is,

$$(-\Delta + cR) f = \lambda_0 f, \qquad \int_M f^2 \, d\upsilon = 1.$$

From Theorem 7.2 in [Guo et al. 2013], we know that, for any $t_0 \in [0, T)$, there exists a smooth function $\varphi(t) > 0$ satisfying

$$\int_M \varphi^2(t) \, d\upsilon = 1 \tag{3-1}$$

and $\varphi(t_0) = f(t_0)$. Let

$$\mu(t) = \int_M \bigl(\varphi(t)(-\Delta\varphi(t)) + cR\varphi^2(t)\bigr) \, d\upsilon, \tag{3-2}$$

then $\mu(t)$ is a smooth function by definition. And at time t_0, we conclude that

$$\mu(t_0) = \lambda_0(t_0).$$

We first give the following lemma.

Lemma 3.1. *Let $(M, g_t)_{t\in[0,T)}$ be a solution of the Ricci–Bourguignon flow on an n-dimensional closed manifold M, and let $\lambda_0(t)$ be the lowest eigenvalue of $-\Delta+cR$ under the Ricci–Bourguignon flow. Assume that $f(t_0)$ is the corresponding eigenfunction of $\lambda_0(t)$ at time $t_0 \in [0, T)$. Let $\mu(t)$ be a smooth function defined by* (3-2). *Then we have*

$$\frac{d}{dt}\mu(t)\Big|_{t=t_0} = (A-2\rho)c\int_M R^2 f^2\,d\upsilon + (A-2\rho)\int_M R|\nabla f|^2\,d\upsilon - A\lambda_0\int_M Rf^2\,d\upsilon + 2\int_M \mathrm{Ric}(\nabla f,\nabla f)\,d\upsilon + 2c\int_M |\mathrm{Ric}|^2 f^2\,d\upsilon, \tag{3-3}$$

where

$$A = -1 + n\rho + 2c[1-2(n-1)\rho].$$

Proof. The proof is by straightforward computation. Notice that

$$\frac{\partial}{\partial t}(\Delta\varphi) = 2R_{ij}\varphi_{ij} + \Delta(\varphi_t) - 2\rho R\Delta\varphi - (2-n)\rho R_{,k}\varphi_k. \tag{3-4}$$

Using Lemma 2.4, we have

$$\begin{aligned}\frac{d}{dt}\mu(t)\Big|_{t=t_0} &= \int_M \partial_t(-\Delta\varphi + cR\varphi)\varphi\,d\upsilon + \int_M(-\Delta\varphi + cR\varphi)\partial_t(\varphi\,d\upsilon)\\
&= \int_M [\partial_t(-\Delta\varphi) + c\varphi\partial_t R + cR\partial_t\varphi]\varphi\,d\upsilon\\
&\qquad + \int_M(-\Delta\varphi + cR\varphi)\partial_t(\varphi\,d\upsilon)\\
&= \int_M [-2R_{ij}\varphi_{ij} - \Delta(\varphi_t) + 2\rho R\Delta\varphi + (2-n)\rho R_{,k}\varphi_k]\varphi\,d\upsilon\\
&\quad + \int_M [c\varphi\partial_t R + cR\partial_t\varphi]\varphi\,d\upsilon + \int_M(-\Delta\varphi + cR\varphi)\partial_t(\varphi\,d\upsilon)\\
&= \int_M [-2R_{ij}\varphi_{ij}\varphi + 2\rho R\varphi\Delta\varphi + (2-n)\rho R_{,k}\varphi_k\varphi]\,d\upsilon\\
&\quad + c\int_M \{[1-2(n-1)\rho]\Delta R + 2|\mathrm{Ric}|^2 - 2\rho R^2\}\varphi^2\,d\upsilon\\
&\quad + \int_M(-\Delta\varphi + cR\varphi)[\partial_t(\varphi)d\upsilon + \partial_t(\varphi\,d\upsilon)].\end{aligned} \tag{3-5}$$

From $R_{,i} = 2R_{ij,j}$ and the Stokes formula, we have

$$\int_M \varphi^2 \Delta R \, d\upsilon = \int_M 2R(|\nabla\varphi|^2 + \varphi\Delta\varphi)\, d\upsilon, \tag{3-6}$$

$$\int_M R_{,k}\varphi_k\varphi \, d\upsilon = \int_M 2R(|\nabla\varphi|^2 + \varphi\Delta\varphi)\, d\upsilon, \tag{3-7}$$

and

$$\begin{aligned} \int_M -R_{ij}\varphi_{ij}\varphi\, d\upsilon &= \int_M (R_{ij}\varphi)_j\varphi_i\, d\upsilon \\ &= \int_M R_{ij,j}\varphi\varphi_i\, d\upsilon + \int_M R_{ij}\varphi_j\varphi_i\, d\upsilon \\ &= \frac{1}{2}\int_M R_{,i}\varphi\varphi_i\, d\upsilon + \int_M R_{ij}\varphi_j\varphi_i\, d\upsilon \\ &= -\frac{1}{2}\int_M R(\varphi\varphi_i)_i\, d\upsilon + \int R_{ij}\varphi_j\varphi_i\, d\upsilon \\ &= -\frac{1}{2}\int_M R\Delta\varphi\varphi\, d\upsilon - \frac{1}{2}\int_M R|\nabla\varphi|^2\, d\upsilon + \int_M R_{ij}\varphi_j\varphi_i\, d\upsilon. \end{aligned} \tag{3-8}$$

On the other hand, at time t_0, we know φ is the eigenfunction of $\lambda_0(t_0)$, i.e., $(-\Delta + cR)\varphi = \lambda_0\varphi$, and we have

$$\int_M (-\Delta\varphi + cR\varphi)[\partial_t(\varphi)d\upsilon + \partial_t(\varphi\, d\upsilon)] = \lambda_0 \int_M \varphi[\partial_t(\varphi)d\upsilon + \partial_t(\varphi\, d\upsilon)] = 0. \tag{3-9}$$

The last equality holds because of (3-1). Inserting (3-6)–(3-9) into (3-5), at $t = t_0$, yields

$$\begin{aligned} \frac{d}{dt}\mu(t)\Big|_{t=t_0} &= (-1+n\rho+2c[1-2(n-1)\rho])\int_M R\varphi\Delta\varphi\, d\upsilon \\ &\quad +(-1+(n-2)\rho+2c[1-2(n-1)\rho])\int_M R|\nabla\varphi|^2\, d\upsilon \\ &\quad +2\int_M R_{ij}\varphi_i\varphi_j\, d\upsilon + 2c\int_M |\mathrm{Ric}|^2\varphi^2\, d\upsilon - 2c\rho\int_M R\varphi^2\, d\upsilon. \end{aligned} \tag{3-10}$$

Inserting $\Delta\varphi = cR\varphi - \lambda_0\varphi$ into (3-10), at $t = t_0$, gives

$$\begin{aligned} \frac{d}{dt}\mu(t)\Big|_{t=t_0} &= (A-2\rho)c\int_M R^2\varphi^2\, d\upsilon + (A-2\rho)\int_M R|\nabla\varphi|^2\, d\upsilon \\ &\quad - A\mu\int_M R\varphi^2\, d\upsilon + 2\int_M R_{ij}\varphi_i\varphi_j\, d\upsilon \\ &\quad + 2c\int_M |\mathrm{Ric}|^2\varphi^2\, d\upsilon. \end{aligned} \tag{3-11}$$

Therefore we finish the proof of Lemma 3.1. □

Now we give the second lemma.

Lemma 3.2. *Let $(M, g_t)_{t\in[0,T)}$ be a solution of the Ricci–Bourguignon flow on an n-dimensional closed manifold M, and let $\lambda_0(t)$ be the lowest eigenvalue of $-\Delta+cR$ under the Ricci–Bourguignon flow. Assume that $f(t_0)$ is the corresponding eigenfunction of $\lambda_0(t)$ at time $t_0\in[0,T)$. Let $\mu(t)$ be a smooth function defined by (3-2). Then we have*

$$\begin{aligned}(3\text{-}12)\quad \frac{d}{dt}\mu(t)\Big|_{t=t_0} &= \frac{1}{2k(2-k)}\int_M |R_{ij}-2k(\log f)_{ij}|^2 f^2\,d\upsilon \\ &\quad+\left(2c-\frac{1}{2k(2-k)}\right)\int_M |\mathrm{Ric}|^2 f^2\,d\upsilon \\ &\quad-\frac{(a-bk)4c-(d-ek)}{2-k}\lambda_0\int_M Rf^2\,d\upsilon \\ &\quad+\left(\frac{(a-bk)4c-(d-ek)}{2-k}-2\rho\right)\int_M (cR^2f^2+R|\nabla f|^2)\,d\upsilon,\end{aligned}$$

where

$$a=1-2(n-1)\rho,\quad b=1-(n-1)\rho,$$
$$d=1-2n\rho,\quad e=1-n\rho\quad\text{and}\quad 0<k<2.$$

Proof. The proof is by straightforward computation:

$$\begin{aligned}(3\text{-}13)\quad \int_M |R_{ij}-2k(\log f)_{ij}|^2 f^2\,d\upsilon &= \int_M |\mathrm{Ric}|^2 f^2\,d\upsilon + 4k^2\int_M |\nabla^2(\log f)|^2 f^2\,d\upsilon \\ &\quad -4k\int_M R_{ij}(\log f)_{ij} f^2\,d\upsilon.\end{aligned}$$

From [Cao 2008], we can get

$$\begin{aligned}(3\text{-}14)\quad 4k^2\int_M |\nabla^2(\log f)|^2 f^2\,d\upsilon &= 2k^2c\int_M R\Delta f^2\,d\upsilon - 4k^2\int_M R_{ij} f_i f_j\,d\upsilon \\ &= 4k^2c\int_M R(f\Delta f+|\nabla f|^2)\,d\upsilon - 4k^2\int_M R_{ij} f_i f_j\,d\upsilon\end{aligned}$$

and

$$\begin{aligned}(3\text{-}15)\quad -4k\int_M R_{ij}(\log f)_{ij} f^2\,d\upsilon &= -2k\int_M R(f\Delta f+|\nabla f|^2)\,d\upsilon + 8k\int_M R_{ij} f_i f_j\,d\upsilon.\end{aligned}$$

Combining (3-14) and (3-15), we arrive at

$$\begin{aligned}\int_M |R_{ij}-2k(\log f)_{ij}|^2 f^2\,d\upsilon &= \int_M |\mathrm{Ric}|^2 f^2\,d\upsilon + (8k-4k^2)\int_M R_{ij} f_i f_j\,d\upsilon \\ &\quad + 2k(2kc-1)\left(c\int_M R^2 f^2\,d\upsilon - \lambda_0\int_M R f^2\,d\upsilon\right) \\ &\quad + 2k(2kc-1)\int_M R|\nabla f|^2\,d\upsilon.\end{aligned} \tag{3-16}$$

Multiplying by $\frac{1}{2k(2-k)}$ on both sides of (3-16), we conclude that

$$\begin{aligned}\frac{1}{2k(2-k)}\int_M |R_{ij}-2k(\log f)_{ij}|^2 f^2\,d\upsilon &= \frac{1}{2k(2-k)}\int_M |\mathrm{Ric}|^2 f^2\,d\upsilon + 2\int_M R_{ij} f_i f_j\,d\upsilon \\ &\quad + \frac{2kc-1}{2-k}\left(c\int_M R^2 f^2\,d\upsilon - \lambda_0\int_M R f^2\,d\upsilon\right) \\ &\quad + \frac{2kc-1}{2-k}\int_M R|\nabla f|^2\,d\upsilon.\end{aligned} \tag{3-17}$$

Subtracting (3-17) from (3-3), we see that

$$\begin{aligned}\frac{d}{dt}\mu(t)\Big|_{t=t_0} &= \frac{1}{2k(2-k)}\int_M |R_{ij}-2k(\log f)_{ij}|^2 f^2\,d\upsilon \\ &\quad + \left(2c-\frac{1}{2k(2-k)}\right)\int_M |\mathrm{Ric}|^2 f^2\,d\upsilon \\ &\quad + \left(A-2\rho-\frac{2kc-1}{2-k}\right)\int_M (cR^2f^2+R|\nabla f|^2-\lambda_0 R f^2)\,d\upsilon \\ &\quad - 2\rho\int_M \lambda_0 R f^2\,d\upsilon,\end{aligned} \tag{3-18}$$

where $A=-1+n\rho+2c[1-2(n-1)\rho]$. Note that

$$\begin{aligned}A-\frac{2kc-1}{2-k} &= \frac{4\{[1-2(n-1)\rho]-[1-(n-1)\rho]k\}c-[(1-2n\rho)-(1-n\rho)k]}{2-k} \\ &:= \frac{(a-bk)c-(d-ek)}{2-k}.\end{aligned}$$

Therefore we finish the proof of Lemma 3.2. □

Proof of Theorem 1.1. We first prove (1). If $\rho<0$, inserting

$$k=\frac{a}{b}=\frac{1-2(n-1)\rho}{1-(n-1)\rho}$$

into (3-12), we obtain

$$
\text{(3-19)}\quad \frac{d}{dt}\mu(t)\Big|_{t=t_0} = \frac{(1-(n-1)\rho)^2}{2-4(n-1)\rho}\int_M \left|R_{ij}-2\frac{1-2(n-1)\rho}{1-(n-1)\rho}(\log f)_{ij}\right|^2 f^2\,d\upsilon
+\left(2c-\frac{(1-(n-1)\rho)^2}{2(1-2(n-1)\rho)}\right)\int_M |\mathrm{Ric}|^2 f^2\,d\upsilon-\rho\lambda_0\int_M Rf^2\,d\upsilon
-\rho\left\{c\int_M R^2f^2\,d\upsilon+\int_M R|\nabla f|^2\,d\upsilon\right\}.
$$

If $R\geq 0$ in $M\times\{0\}$, from Lemmas 2.3 and 2.6, we know that either $\max_M R(t)>0$ or $g(t)=g(0)$ for every $t\in(0,T)$. Assume $\max_M R(t)>0$ (otherwise the proof is trivial). By (3-19), when

$$
c\geq\frac{(1-(n-1)\rho)^2}{4(1-2(n-1)\rho)}>\frac{1}{4},
$$

we obtain

$$
\text{(3-20)}\quad \frac{d}{dt}\mu(t)\Big|_{t=t_0}>0.
$$

Moreover, setting $k=1$ in (3-12), we obtain

$$
\text{(3-21)}\quad \frac{d}{dt}\mu(t)\Big|_{t=t_0}=\frac{1}{2}\int_M |R_{ij}-2(\log f)_{ij}|^2 f^2\,d\upsilon+\left(2c-\frac{1}{2}\right)\int_M |\mathrm{Ric}|^2 f^2\,d\upsilon
-\rho[4(n-1)c-n+2]\left\{c\int_M R^2f^2\,d\upsilon+\int_M R|\nabla f|^2\,d\upsilon\right\}
+\rho[4(n-1)c-n]\lambda_0\int_M Rf^2\,d\upsilon.
$$

Then when $\frac{1}{4}\leq c\leq n/(4(n-1))$, we also obtain (3-20).

Since the eigenfunction of the lowest eigenvalue is not equal to 0 along the Ricci–Bourguignon flow, see Lemma 5.22 in [Chow et al. 2008], and $\mu(t)$ is a smooth function with respect to the t-variable, we have

$$
\text{(3-22)}\quad \frac{d}{dt}\mu(t)>0
$$

in $(t_0-\delta, t_0+\delta)$, where $\delta>0$ is sufficiently small. So we get

$$
\text{(3-23)}\quad \mu(t_0)>\mu(t_1)
$$

for any $t_1\in(t_0-\delta, t_0+\delta)$ and $t_1<t_0$.

Notice that

$$
\mu(t_0)=\lambda_0(t_0)\quad\text{and}\quad\mu(t_1)\geq\lambda_0(t_1).
$$

This implies $\lambda_0(t_0) > \lambda_0(t_1)$ for any $t_0 > t_1$. Since $\lambda_0(t)$ is continuous and $t_0 \in [0, T)$ is arbitrary, $\lambda_0(t)$ is strictly monotone increasing in $[0, T)$. Therefore we finish the proof of (1).

Next we prove (2). If $0 < \rho < \frac{1}{2(n-1)}$, in (3-18), we pick k such that

$$A - 2\rho - \frac{2kc-1}{2-k} = 0.$$

Then

$$k = \frac{(4c-1)[1-2(n-1)\rho]-2\rho}{(4c-1)[1-(n-1)\rho]-\rho}.$$

Taking $4c-1 = B \geq 0$, we have

$$k = \frac{B[1-2(n-1)\rho]-2\rho}{B[1-(n-1)\rho]-\rho} \quad \text{and} \quad 2-k = \frac{B}{B[1-(n-1)\rho]-\rho}.$$

For $0 < k < 2$, we need

$$B > \frac{2\rho}{1-2(n-1)\rho}. \tag{3-24}$$

Now, we need $2c - \frac{1}{2k(2-k)} \geq 0$, which is

$$B(B+1)(B[1-2(n-1)\rho]-2\rho) \geq (B[1-(n-1)\rho]-\rho)^2. \tag{3-25}$$

It is true when $B \to +\infty$. Next, we will prove that given

$$4c-1 = B \geq \frac{6(n-1)^2\sqrt{\rho}}{1-2(n-1)\rho}, \tag{3-26}$$

both (3-24) and (3-25) are true. Firstly, since $0 < \rho < \frac{1}{2(n-1)} < 1$, we have

$$B \geq \frac{6(n-1)^2\sqrt{\rho}}{1-2(n-1)\rho} > \frac{6(n-1)^2\rho}{1-2(n-1)\rho} > \frac{2\rho}{1-2(n-1)\rho}.$$

Thus, (3-24) holds. Secondly, let's show (3-25):

$$\begin{aligned}
&B(B+1)(B[1-2(n-1)\rho]-2\rho)-(B[1-(n-1)\rho]-\rho)^2\\
&\quad = B^2(B[1-2(n-1)\rho]-2\rho)+B(B[1-2(n-1)\rho]-2\rho)-(B[1-(n-1)\rho]-\rho)^2\\
&\quad = B^2(B[1-2(n-1)\rho]-2\rho)-((n-1)B\rho+\rho)^2\\
&\quad \geq B^2(6(n-1)^2\sqrt{\rho}-2\rho)-((n-1)B\rho+\rho)^2\\
&\quad \geq B^2(4(n-1)^2\sqrt{\rho})-((n-1)B\rho+\rho)^2\\
&\quad = [2B(n-1)\sqrt[4]{\rho}+(n-1)B\rho+\rho][2B(n-1)\sqrt[4]{\rho}-(n-1)B\rho-\rho].
\end{aligned}$$

The first factor is clearly positive. For the second factor, note $\rho^{\frac{1}{4}} > \rho^{\frac{3}{4}} > \rho$,

$$\begin{aligned} 2B(n-1)\sqrt[4]{\rho} - (n-1)B\rho - \rho &\geq B(n-1)\sqrt[4]{\rho} - \rho \\ &\geq \frac{6(n-1)^3\sqrt{\rho}}{1-2(n-1)\rho}\sqrt[4]{\rho} - \rho \\ &\geq 6(n-1)^3\rho^{\frac{3}{4}} - \rho \geq 12(n-1)^3\rho - \rho > 0. \end{aligned}$$

Therefore, given

$$4c - 1 \geq \frac{6(n-1)^2\sqrt{\rho}}{1-2(n-1)\rho},$$

we have

$$\frac{d}{dt}\mu\Big|_{t=t_0} > -2\rho\mu\int_M Rf^2\,d\upsilon. \tag{3-27}$$

By Lemma 2.8, we know that the nonnegativity of the curvature operator is preserved by the Ricci–Bourguignon flow. This implies that the Ricci curvature is also nonnegative, and we have $|\mathrm{Ric}|^2 \leq R^2$. The evolution equation of scalar curvature satisfies

$$\begin{aligned} \frac{\partial}{\partial t}R &= [1-2(n-1)\rho]\Delta R + 2|\mathrm{Ric}|^2 - 2\rho R^2 \\ &\leq [1-2(n-1)\rho]\Delta R + 2(1-\rho)R^2. \end{aligned} \tag{3-28}$$

Let $\sigma(t)$ be the solution of the following ODE with initial value:

$$\begin{cases} \partial\sigma(t)/\partial t = 2(1-\rho)\sigma^2, \\ \sigma(0) = \max_M R(0). \end{cases} \tag{3-29}$$

By the maximum principle, letting $\epsilon = \max_M R(0)$, we can get

$$R(t) \leq \sigma(t) = \left(-2(1-\rho)t + \frac{1}{\epsilon}\right)^{-1}$$

on $[0, T'')$, where $T'' = \min\{T', T\}$ and $T' = \frac{1}{2(1-\rho)\epsilon}$. Arguing now as in [Hamilton 1982, Section 14], it follows that the metrics $g(t)$ converge to some limit metric $g(T)$ in the C^∞ topology if $T < T''$; hence, we can restart the flow with this initial metric $g(T)$, obtaining a smooth flow in some larger time interval $[0, T+\delta)$, in contradiction with the fact that T was the maximal time of smooth existence. So we have $T' \leq T$. Hence $R(t) \leq \sigma(t)$ on $[0, T')$. Since the eigenfunction of the lowest eigenvalue is not equal to 0 along the Ricci–Bourguignon flow, see Lemma 5.22 in [Chow et al. 2008], from Lemma 2.3 and (3-27), we have

$$\frac{d}{dt}\mu\Big|_{t=t_0} > -2\rho\mu\int_M Rf^2\,d\upsilon \geq -2\rho\mu\sigma, \tag{3-30}$$

which implies

$$\left(\frac{d}{dt}\mu + 2\rho\mu\sigma\right)\Big|_{t=t_0} > 0.$$

By arguments similar to those in the proof of (1), we know that

$$(T' - t)^{-\alpha}\lambda_0(t)$$

is strictly monotone increasing under the Ricci–Bourguignon flow (1-1) on $[0, T')$ and

$$T' = \frac{1}{2(1-\rho)\epsilon},$$

where

$$\epsilon = \max_M R(0) \quad \text{and} \quad \alpha = \frac{\rho}{1-\rho} > 0,$$

which shows (2) holds. Therefore we finish the proof of Theorem 1.1. □

4. Proof of Theorem 1.3

We will now prove Theorem 1.3. First, we recall the definition of breathers.

Definition 4.1. A metric $g(t)$ evolving from the Ricci–Bourguignon flow is called a breather if for some $t_1 < t_2$ and $\alpha > 0$ the metrics $\alpha g(t_1)$ and $g(t_2)$ differ only by a diffeomorphism; the cases $\alpha = 1$, $\alpha < 1$ and $\alpha > 1$ correspond to steady, shrinking and expanding breathers, respectively.

Proof of Theorem 1.3. For a steady breather, let t_1 and t_2 be the same as above; we have

$$\lambda_0(t_1) = \lambda_0(t_2).$$

When $\rho = 0$ and $c \geq \frac{1}{4}$, by (1-2) of Theorem A, we have

$$\lambda_0(t_1) \leq \lambda_0(t_2)$$

provided $t_1 < t_2$. And the equality holds if and only if for any $t_1 \leq t \leq t_2$,

$$\frac{d}{dt}\lambda_0(t) = 0.$$

Since the eigenfunction f cannot be identical to zero, from (1-2) of Theorem A we must have $\mathrm{Ric} \equiv 0$.

But when $\rho < 0$,

$$c \in \left(\frac{1}{4}, \frac{n}{4(n-1)}\right] \cup \left[\frac{(1-(n-1)\rho)^2}{4(1-2(n-1)\rho)}, +\infty\right)$$

and the scalar curvature is nonnegative at the initial time, because of Theorem 1.1(1) for a nontrivial flow we have

$$\lambda_0(t_1) < \lambda_0(t_2)$$

provided $t_1 < t_2$. When $(M, g(t))_{t\in[0,T)}$ is a compact maximal solution of the trivial Ricci–Bourguignon flow (1-1), i.e., $\mathrm{Ric} \equiv 0$, we have $\lambda_0(t_1) = \lambda_0(t_2)$. Hence we have proved Theorem 1.3. □

5. Proof of Theorem 1.4

We will now prove Theorem 1.4. In order to achieve this, we first prove Lemma 5.1. Our proof involves choosing a proper smooth function, which seems to be a delicate trick.

Let M be an n-dimensional closed Riemannian manifold, and $g(t)$ be a smooth solution of the Ricci–Bourguignon flow on the time interval $[0, T)$. Let $\lambda_1(t)$ be the first eigenvalue of the Laplace operator under the Ricci–Bourguignon flow and $f(t_0)$ be the corresponding eigenfunction of $\lambda_1(t)$ at time $t_0 \in [0, T)$, i.e.,

$$-\Delta_{g(t_0)} f(t_0) = \lambda_1(t_0) f(t_0). \tag{5-1}$$

For any $t_0 \in [0, T)$, Wu et al. [2010] pointed out that there exists a smooth function

$$\phi(t) = \frac{\psi(t)}{\left(\int_M \psi(t)^2\, d\upsilon\right)^{\frac{1}{2}}}, \quad \text{where } \psi(t) = f(t_0)\left(\frac{|g(t_0)|}{|g(t)|}\right)^{\frac{1}{2}}$$

satisfying

$$\int_M \phi(t)^2\, d\upsilon = 1, \qquad \int_M \phi(t)\, d\upsilon = 0, \tag{5-2}$$

and $\varphi(t_0) = f(t_0)$. Now we define a general smooth function

$$\mu(t) = \int_M \phi(t)(-\Delta\phi(t))\, d\upsilon. \tag{5-3}$$

In general, $\mu(t)$ is not equal to $\lambda_1(t)$. But at time t_0, we conclude that

$$\mu(t_0) = \lambda_1(t_0).$$

Lemma 5.1. *Let $(M, g_t)_{t\in[0,T)}$ be a solution of the Ricci–Bourguignon flow on an n-dimensional closed manifold M and let $\lambda_1(t)$ be the first eigenvalue of the Laplace operator under the Ricci–Bourguignon flow. Assume that $f(t_0)$ is the corresponding eigenfunction of $\lambda_1(t)$ at time $t_0 \in [0, T)$, i.e.,*

$$-\Delta_{g(t_0)} f(t_0) = \lambda_1(t_0) f(t_0).$$

Let $\mu(t)$ be a smooth function defined by (5-3). Then we have

$$\frac{d}{dt}\mu(t)\Big|_{t=t_0} = \int_M \{2R_{ij} f_i f_j + (1-n\rho)\lambda_1 R f^2 - [(2-n)\rho+1] R|\nabla f|^2\}\, d\upsilon. \tag{5-4}$$

Proof. The proof is by direct computation:

$$\begin{aligned}
\text{(5-5)}\quad \frac{d}{dt}\mu(t)\Big|_{t=t_0} &= \int_M \partial_t(-\phi\Delta\phi)\,d\upsilon + \int_M (-\phi\Delta\phi)\partial_t(d\upsilon)\\
&= \int_M [-2R_{ij}\phi_{ij} - \Delta(\partial_t\phi) + 2\rho R\Delta\phi + (2-n)\rho R_{,k}\phi_k]\phi\,d\upsilon\\
&\qquad + \int_M (-\Delta\phi)\partial_t\phi\,d\upsilon + \int_M (-\Delta\phi)\phi(n\rho-1)R\,d\upsilon\\
&= \int_M -2R_{ij}\phi_{ij}\phi\,d\upsilon + \int_M -2(\Delta\phi)\partial_t\phi\,d\upsilon\\
&\quad + (2-n)\rho\int_M R_{,k}\phi_k\phi\,d\upsilon + [1+(2-n)\rho]\int_M R(\Delta\phi)\phi\,d\upsilon.
\end{aligned}$$

From (3-7) and (3-8), we have

$$\begin{aligned}
\text{(5-6)}\quad \frac{d}{dt}\mu(t)\Big|_{t=t_0} &= -\int_M R\Delta\phi\phi\,d\upsilon - \int_M R|\nabla\phi|^2\,d\upsilon + 2\int_M R_{ij}\phi_j\phi_i\,d\upsilon\\
&\quad + \int_M -2(\Delta\phi)\partial_t\phi\,d\upsilon - (2-n)\rho\int_M R\Delta\phi\phi\,d\upsilon\\
&\quad - (2-n)\rho\int_M R|\nabla\phi|^2\,d\upsilon\\
&\quad + [1+(2-n)\rho]\int_M R(\Delta\phi)\phi\,d\upsilon\\
&= 2\int_M R_{ij}\phi_j\phi_i\,d\upsilon + 2\mu\int_M \phi\partial_t\phi\,d\upsilon\\
&\quad - [1+(2-n)\rho]\int_M R|\nabla\phi|^2\,d\upsilon.
\end{aligned}$$

Under the Ricci–Bourguignon flow, from the constraint condition (5-2), we get

$$\text{(5-7)}\qquad 2\int_M \phi\partial_t\phi\,d\upsilon = -(n\rho-1)\int_M \phi^2 R\,d\upsilon.$$

Hence, at time t_0, the desired lemma follows from substituting (5-7) into (5-6). □

Proof of Theorem 1.4. We assume that for any time $t_0 \in [0, T)$, if $f(t_0)$ is the corresponding eigenfunction of the first eigenvalue $\lambda_1(t_0)$, then we have $\lambda_1(t_0) = \mu(t_0)$. By Lemma 5.1, we have

$$\begin{aligned}
\text{(5-8)}\quad \frac{d}{dt}\mu(t)\Big|_{t=t_0} &= \int_M \{(1-n\rho)\lambda_1 Rf^2 + 2R_{ij}f_i f_j - [(2-n)\rho+1]R|\nabla f|^2\}\,d\upsilon\\
&= \int_M \{2R_{ij} - [(2-n)\rho+1]Rg_{ij}\}f_i f_j\,d\upsilon + \int_M (1-n\rho)\lambda_1 Rf^2\,d\upsilon\\
&\geq \int_M (1-n\rho)\lambda_1 Rf^2\,d\upsilon - 2a\int_M |\nabla f|^2\,d\upsilon\\
&= \int_M (1-n\rho)\lambda_1 Rf^2\,d\upsilon - 2a\lambda_1 = \lambda_1\int_M f^2\{(1-n\rho)R - 2a\}\,d\upsilon,
\end{aligned}$$

where we used the first assumption of Theorem 1.4.

From Lemma 2.3, we know that either $\max_M R(t) > 0$ or $g(t) = g(0)$ for every $t \in (0, T)$. Assume $\max_M R(t) > 0$ (otherwise the proof is trivial). Since the eigenfunction of the first eigenvalue is not equal to 0 along the Ricci–Bourguignon flow, by (5-8), we obtain

$$\frac{d}{dt}\mu(t)\Big|_{t=t_0} > 0. \tag{5-9}$$

By arguments similar to those in the proof of Theorem 1.1, we have $\lambda_1(t)$ is strictly monotone increasing in $[0, T)$.

As for the differentiability for $\lambda_1(t)$, since $\lambda_1(t)$ is increasing on the time interval $[0, T)$ under curvature conditions of the theorem, by the classical Lebesgue's theorem, see for example Chapter 4 in [Mukherjea and Pothoven 1984], it is easy to see that $\lambda_1(t)$ is differentiable almost everywhere on $[0, T)$. □

Remark 5.1. (1) In the course of proving Theorem 1.4, we do not use any differentiability of the first eigenvalue or the corresponding eigenfunction of the Laplace operator under the Ricci–Bourguignon flow.

(2) Using this method, we cannot get any monotonicity for higher-order eigenvalues of the Laplace operator under the Ricci–Bourguignon flow.

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BIN CHEN
SCHOOL OF MATHEMATICAL SCIENCES
TONGJI UNIVERSITY
SHANGHAI
CHINA
chenbin@tongji.edu.cn

QUN HE
SCHOOL OF MATHEMATICAL SCIENCES
TONGJI UNIVERSITY
SHANGHAI
CHINA
hequn@tongji.edu.cn

FANQI ZENG
SCHOOL OF MATHEMATICAL SCIENCES
TONGJI UNIVERSITY
SHANGHAI
CHINA
fanzeng10@126.com

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COMPOSITION SERIES OF A CLASS OF INDUCED REPRESENTATIONS, A CASE OF ONE HALF CUSPIDAL REDUCIBILITY

IGOR CIGANOVIĆ

We determine the composition series of the induced representation

$$\delta([\nu^{-b}\rho,\nu^{c}\rho])\times\delta([\nu^{\frac{1}{2}}\rho,\nu^{a}\rho])\rtimes\sigma,$$

where $a, b, c \in \frac{1}{2}(2\mathbb{Z}+1)$ satisfy $\frac{1}{2}\le a<b<c$, ρ is an irreducible cuspidal unitary representation of a general linear group and σ is an irreducible cuspidal representation of a classical group.

Introduction

In this paper we determine the composition series of a class of standard representations in terms of Mœglin–Tadić classification of discrete series [Mœglin 2002; Mœglin and Tadić 2002]. Interesting on its own, this result should also prove valuable for extending results about Jacquet modules of segment type representations obtained in [Matić and Tadić 2015].

To describe our results we introduce some notation. Fix a local nonarchimedean field F of characteristic different from 2. Let ρ be an irreducible cuspidal unitary representation of $GL(m_\rho, F)$ (this defines m_ρ) and $x, y\in\mathbb{R}$, such that $y-x+1\in\mathbb{Z}_{\geq 0}$. The set $[\nu^x\rho,\nu^y\rho]=\{\nu^x\rho,\dots,\nu^y\rho\}$ is called a segment. The parabolically induced representation $\nu^y\rho\times\cdots\times\nu^x\rho$ has a unique irreducible subrepresentation; it is essentially square integrable and we denote it by $\delta([\nu^x\rho,\nu^y\rho])$. Also we denote $e([\nu^x\rho,\nu^y\rho])=e(\delta([\nu^x\rho,\nu^y\rho])=\frac{1}{2}(x+y)$. If δ is an essentially square integrable representation of $GL(m_\delta, F)$, there exists a segment Δ such that $\delta=\delta(\Delta)$.

Let G_n be a symplectic or (full) orthogonal group having split rank n. Given a sequence of segments $\Delta_1,\dots,\Delta_k$, $e(\Delta_i)>0$, $i=1,\dots,k$ and an irreducible tempered representation τ of some $G_{n'}$ we denote by $\mathrm{Lang}(\delta(\Delta_1)\times\cdots\times\delta(\Delta_k)\rtimes\tau)$ the unique irreducible quotient, called the Langlands quotient, of the parabolically induced representation $\delta(\Delta_{\varphi(1)})\times\cdots\times\delta(\Delta_{\varphi(k)})\rtimes\tau$ where φ is a permutation of the

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set $\{1, \dots, k\}$ such that $e(\Delta_{\varphi(1)}) \geq \cdots \geq e(\Delta_{\varphi(k)})$. These induced representations are called standard representations and are important because by the Langlands classification every irreducible representation of G_n can be described as a Langlands quotient. Further if τ is a discrete series representation then by the Mœglin–Tadić classification of discrete series it is described by an admissible triple (Jord, $\tau_{\text{cusp}}, \epsilon$). Here Jord is a set Jordan blocks, τ_{cusp} a partial cuspidal support and ϵ a function from a subset of Jord $\cup$(Jord $\times$ Jord) into $\{\pm 1\}$. Results of Muić about reducibility of the generalized principal series $\delta([\nu^x\rho, \nu^y\rho]) \rtimes \tau$ [Muić 2004; 2005] are stated case by case depending on Jord and x and y where the case $x = \frac{1}{2}$ plays an important role. In our situation, we provide some additional information, see Proposition 2.4. These results are used to compute composition series of the induced representation

$$\delta([\nu^{-b}\rho, \nu^{c}\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \rtimes \sigma,$$

where $a, b, c \in \frac{1}{2}(2\mathbb{Z}+1)$ such that $\frac{1}{2} \leq a < b < c$, ρ is an irreducible unitary cuspidal representation of $GL(m_\rho, F)$ and σ is an irreducible cuspidal representation of G_n such that $\nu^{\frac{1}{2}}\rho \rtimes \sigma$ reduces.

1. Preliminaries

Let F be a local nonarchimedean field of characteristic different from 2. Groups that we consider are as follows. As in [Mœglin and Tadić 2002] we fix a tower of symplectic or orthogonal nondegenerate F vector spaces V_n, $n \geq 0$ where n is the Witt index. We denote by G_n the group of isometries of V_n. It has split rank n. Also we fix the set of standard parabolic subgroups in the usual way. Standard parabolic proper subgroups of G_n are in bijection with the set of ordered partitions of positive integers $m \leq n$. Given positive integers $n_1, \dots, n_k$ such that $m = n_1 + \cdots + n_k \leq n$ the corresponding standard parabolic subgroup P_s, $s = (n_1, \dots, n_k)$ has the Levi factor M_s isomorphic to

$$GL(n_1, F) \times \cdots \times GL(n_k, F) \times G_{n-m}.$$

Further, if δ_i is a smooth representation of $GL(n_i, F)$, $i = 1, \dots, k$ and τ a smooth representation of G_{n-m}, denote by $\pi = \delta_1 \otimes \cdots \otimes \delta_k \otimes \tau$ the representation of M_s and by

$$\delta_1 \times \cdots \times \delta_k \rtimes \tau = \operatorname{Ind}_{M_s}^{G_n}(\pi)$$

the representation induced from π using normalized parabolic induction. If σ is a smooth representation of G_n we denote by $\mathrm{r}_s(\sigma) = \mathrm{r}_{M_s}(\sigma) = \operatorname{Jacq}_{M_s}^{G_n}(\sigma)$ the normalized Jacquet module of σ. We have the Frobenius reciprocity

$$\operatorname{Hom}_{G_n}(\sigma, \operatorname{Ind}_{M_s}^{G_n}(\pi)) = \operatorname{Hom}_{M_s}(\operatorname{Jacq}_{M_s}^{G_n}(\sigma), \pi).$$

Let ρ be an irreducible cuspidal unitary representation of $GL(m_\rho, F)$ (this defines m_ρ) and $x, y \in \mathbb{R}$, such that $y - x + 1 \in \mathbb{Z}_{\geq 0}$. The set $[\nu^x\rho, \nu^y\rho] = \{\nu^x\rho, \dots, \nu^y\rho\}$ is called a segment. The induced representation $\nu^y\rho \times \cdots \times \nu^x\rho$ has the unique irreducible subrepresentation; it is essentially square integrable, and we denote it by $\delta([\nu^x\rho, \nu^y\rho])$. We also denote

$$e([\nu^x\rho, \nu^y\rho]) = e(\delta([\nu^x\rho, \nu^y\rho]) = \frac{x+y}{2}.$$

For $y-x+1 \in \mathbb{Z}_{<0}$ define $[\nu^x\rho, \nu^y\rho] = \varnothing$ and $\delta(\varnothing)$ is the irreducible representation of the trivial group. Let $\Delta = [\nu^x\rho, \nu^y\rho]$ and $\widetilde{\Delta} = [\nu^{-y}\tilde{\rho}, \nu^{-x}\tilde{\rho}]$ where $\tilde{\rho}$ denotes the contragredient of ρ. We have $\widetilde{\delta(\Delta)} = \delta(\widetilde{\Delta})$. By [Zelevinsky 1980] if δ is an essentially square integrable representation of $GL(m_\delta, F)$, there exists a segment Δ such that $\delta = \delta(\Delta)$. If Δ' and Δ'' are segments such that $\Delta'' \subseteq \Delta'$ then $\delta(\Delta') \times \delta(\Delta'')$ is irreducible and $\delta(\Delta') \times \delta(\Delta'') \cong \delta(\Delta'') \times \delta(\Delta')$.

Given a sequence of segments $\Delta_1, \dots, \Delta_k$, $e(\Delta_i) > 0$, $i = 1, \dots, k$ and an irreducible tempered representation τ of some $G_{n'}$, we denote by

$$\mathrm{Lang}(\delta(\Delta_1) \times \cdots \times \delta(\Delta_k) \rtimes \tau)$$

the unique irreducible quotient, called the Langlands quotient, of

$$\delta(\Delta_{\varphi(1)}) \times \cdots \times \delta(\Delta_{\varphi(k)}) \rtimes \tau,$$

where φ is a permutation of the set $\{1, \dots, k\}$ such that $e(\Delta_{\varphi(1)}) \geq \cdots \geq e(\Delta_{\varphi(k)})$. It appears with multiplicity 1 in the induced representation and is the unique irreducible subrepresentation of $\delta(\widetilde{\Delta}_{\varphi(1)}) \times \cdots \times \delta(\widetilde{\Delta}_{\varphi(k)}) \rtimes \tau$. By the Langlands classification every irreducible representation of G_n can be written as a Langlands quotient.

If σ is a discrete series representation of G_n then by the Mœglin–Tadić classification of discrete series [Mœglin 2002; Mœglin and Tadić 2002] it is described by an admissible triple $(\mathrm{Jord}, \sigma_{\mathrm{cusp}}, \epsilon)$. We note that the classification, written under a natural hypothesis, is now unconditional; see page 3160 of [Matić 2016]. Here Jord is a set of pairs (a, ρ) where ρ is an irreducible self-dual cuspidal representation of $GL(m_\rho, F)$, a is a positive integer of parity depending on ρ and $\delta([\nu^{-(a-1)/2}\rho, \nu^{(a-1)/2}\rho]) \rtimes \sigma$ is irreducible. We write $\mathrm{Jord}_\rho = \{a : (a, \rho) \in \mathrm{Jord}\}$ and for $a \in \mathrm{Jord}_\rho$ let a_- be the largest element of Jord_ρ strictly less than a, if such exists. Next, σ_{cusp} is the unique irreducible cuspidal representation of some $G_{n'}$ such that there exists an irreducible representation π of $GL(m_\pi, F)$ such that $\sigma \hookrightarrow \pi \rtimes \sigma_{\mathrm{cusp}}$. It is called the partial cuspidal support of σ. Finally, ϵ is a function from a subset of $\mathrm{Jord} \cup (\mathrm{Jord} \times \mathrm{Jord})$ into $\{\pm 1\}$. It is defined on a pair $(a, \rho), (a', \rho') \in \mathrm{Jord}$ if and only if $\rho \cong \rho'$ and $a \neq a'$. In such a case we formally denote the value on the pair by $\epsilon(a, \rho)\epsilon(a', \rho)^{-1}$ and it is equal to the product of $\epsilon(a, \rho)$ and $\epsilon(a', \rho)^{-1}$ if they are defined. Suppose that $(a, \rho) \in \mathrm{Jord}$ and a_- is defined. Then

$$\epsilon(a, \rho)\epsilon(a_-, \rho)^{-1} = 1 \iff \text{there exists a representation } \pi' \text{ of some } G_{n_{\pi'}} \text{ such that } \sigma \hookrightarrow \delta([\nu^{(a_-+1)/2}\rho, \nu^{(a-1)/2}\rho]) \rtimes \pi'.$$

If $(a, \rho) \in \mathrm{Jord}$ and a is even then $\epsilon(a, \rho)$ is defined. Additionally, if $a = \min(\mathrm{Jord}_\rho)$,

$$\epsilon(a, \rho)=1 \iff \text{there exists a representation } \pi'' \text{of some } G_{n_{\pi''}} \text{ such that } \sigma \hookrightarrow \delta([\nu^{1/2}\rho, \nu^{(a-1)/2}\rho]) \rtimes \pi''.$$

Now we recall the Tadić formula for computing Jacquet modules. Let $R(G_n)$ be the Grothendieck group of the category of smooth representations of G_n of finite length. It is the free abelian group generated by classes of irreducible representations of G_n. If σ is a smooth finite length representation of G_n denote by s.s.(σ) the semisimplification of σ, that is the sum of classes of composition series of σ. Put $R(G) = \oplus_{n\geq 0} R(G_n)$. For $\pi_1, \pi_2 \in R(G)$ we define $\pi_1 \leq \pi_2$ if $\pi_2 - \pi_1$ is a linear combination of classes of irreducible representations with nonnegative coefficients. Similarly we have $R(GL) = \oplus_{n\geq 0} R(GL(n, F))$. We have the map $\mu^* : R(G) \to R(GL) \otimes R(G)$ defined by

$$\mu^*(\sigma) = 1 \otimes \sigma + \sum_{k=1}^{n} \text{s.s.}(r_{(k)}(\sigma)), \quad \sigma \in R(G_n).$$

The following result is derived from Theorems 5.4 and 6.5 of [Tadić 1995]; see also Section 1 in [Mœglin and Tadić 2002]. They are based on Bernstein and Zelevinsky's geometrical lemma [1977, Lemma 2.11].

Theorem 1.1. *Let σ be a smooth representation of a finite length of G_n, ρ an irreducible unitary cuspidal representation of $GL(m_\rho, F)$ and $x, y \in \mathbb{R}$, such that $y - x + 1 \in \mathbb{Z}_{\geq 0}$. Then*

$$\mu^*(\delta([\nu^x\rho, \nu^y\rho]) \rtimes \sigma) = \sum_{\delta'\otimes\sigma' \leq \mu^*(\sigma)} \sum_{i=0}^{y-x+1} \sum_{j=0}^{i} \delta([\nu^{i-y}\tilde{\rho}, \nu^{-x}\tilde{\rho}]) \times \delta([\nu^{y+1-j}\rho, \nu^y\rho]) \times \delta' \otimes \delta([\nu^{y+1-i}\rho, \nu^{y-j}\rho]) \rtimes \sigma', \tag{1-1}$$

where $\delta' \otimes \sigma'$ denotes an irreducible subquotient in the appropriate Jacquet module.

We also note that in the appropriate Grothendieck group

$$\delta([\nu^x\rho, \nu^y\rho]) \rtimes \sigma = \delta([\nu^{-y}\tilde{\rho}, \nu^{-x}\tilde{\rho}]) \rtimes \sigma. \tag{1-2}$$

2. Basic reducibilities

In this section we fix the notation and prepare some reducibility results. Let ρ be an irreducible unitary cuspidal representation of $GL(m_\rho, F)$ and σ an irreducible cuspidal representation of G_n such that $\nu^{\frac{1}{2}}\rho \rtimes \sigma$ reduces. By Proposition 2.4 of [Tadić 1998] ρ is self-dual. Let $a, b, c \in \frac{1}{2}(2\mathbb{Z}+1)$ such that $\frac{1}{2} \leq a < b < c$.

The following result is Theorem 2.3 from [Muić 2004] proved using Jacquet module computation.

Theorem 2.1. (i) *The induced representation* $\delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma$ *is of length* 2. *Besides its Langlands quotient it has the unique irreducible subrepresentation, the discrete series* σ_1. *In the appropriate Grothendieck group we have*

$$\delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma = \sigma_1 + \mathrm{Lang}(\delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma).$$

Here $\mathrm{Jord}(\sigma_1) = \{(2a+1, \rho)\}$ *and* $\epsilon_{\sigma_1}(2a+1, \rho) = 1$.

(ii) *The induced representation* $\delta([\nu^{-b}\rho, \nu^c\rho]) \rtimes \sigma$ *is of length* 3. *Besides its Langlands quotient it has two nonisomorphic irreducible subrepresentations* σ_2 *and* σ_3. *In the appropriate Grothendieck group we have*

$$\delta([\nu^{-b}\rho, \nu^c\rho]) \rtimes \sigma = \sigma_2 + \sigma_3 + \mathrm{Lang}(\delta([\nu^{-b}\rho, \nu^c\rho]) \rtimes \sigma).$$

Here,

$$\begin{aligned} \mathrm{Jord}(\sigma_2) = \mathrm{Jord}(\sigma_3) &= \{(2b+1, \rho), (2c+1, \rho)\} \\ \epsilon_{\sigma_2}(2b+1, \rho) &= \epsilon_{\sigma_2}(2c+1, \rho) = 1 \\ \epsilon_{\sigma_3}(2b+1, \rho) &= \epsilon_{\sigma_3}(2c+1, \rho) = -1. \end{aligned}$$

The next proposition follows from Theorem 2.1 of [Muić 2004].

Proposition 2.2. *The induced representation* $\delta([\nu^{-b}\rho, \nu^c\rho]) \rtimes \sigma_1$ *is of length* 3. *Besides its Langlands quotient it has two nonisomorphic irreducible subrepresentations, the discrete series* σ_4 *and* σ_5. *In the appropriate Grothendieck group we have*

$$\delta([\nu^{-b}\rho, \nu^c\rho]) \rtimes \sigma_1 = \sigma_4 + \sigma_5 + \mathrm{Lang}(\delta([\nu^{-b}\rho, \nu^c\rho]) \rtimes \sigma_1).$$

Here,

$$\begin{aligned} \mathrm{Jord}(\sigma_4) = \mathrm{Jord}(\sigma_5) &= \{(2a+1, \rho), (2b+1, \rho), (2c+1, \rho)\}, \\ \epsilon_{\sigma_4}(2a+1, \rho) &= \epsilon_{\sigma_4}(2b+1, \rho) = \epsilon_{\sigma_4}(2c+1, \rho) = 1, \\ \epsilon_{\sigma_5}(2a+1, \rho) &= 1, \epsilon_{\sigma_5}(2b+1, \rho) = \epsilon_{\sigma_5}(2c+1, \rho) = -1. \end{aligned}$$

Proposition 2.3. *The representation* $\delta([\nu^{-b}\rho, \nu^c\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma$ *has two irreducible subrepresentations* σ_4 *and* σ_5 *and they appear with multiplicity* 1.

Proof. By Theorem 2.1 and Proposition 2.2 we have

$$\sigma_4 \oplus \sigma_5 \hookrightarrow \delta([\nu^{-b}\rho, \nu^c\rho]) \rtimes \sigma_1 \hookrightarrow \delta([\nu^{-b}\rho, \nu^c\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma.$$

To see that there are no other irreducible subrepresentations let

$$\pi \hookrightarrow \delta([\nu^{-b}\rho, \nu^c\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma$$

be an irreducible subrepresentation. Frobenius reciprocity implies

$$\mu^*(\pi) \geq \delta([\nu^{-b}\rho, \nu^c\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \otimes \sigma.$$

We show that $\delta([\nu^{-b}\rho, \nu^{c}\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \otimes \sigma$ appears with multiplicity 2 in $\mu^*(\delta([\nu^{-b}\rho, \nu^{c}\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \rtimes \sigma)$. Looking for possible occurrences, formula (1-1) implies that there exist $i, j, k, l \in \mathbb{Z}$ such that $0 \leq l \leq k \leq a + \frac{1}{2}$, $0 \leq j \leq i \leq b+c+1$ and

$$\begin{aligned}\delta([\nu^{-b}\rho, \nu^{c}\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \leq \delta([\nu^{k-a}\rho, \nu^{-\frac{1}{2}}\rho]) \\ \times \delta([\nu^{a+1-l}\rho, \nu^{a}\rho]) \times \delta([\nu^{i-c}\rho, \nu^{b}\rho]) \times \delta([\nu^{c+1-j}\rho, \nu^{c}\rho]),\end{aligned}$$

$$\sigma \leq \delta([\nu^{a+1-k}\rho, \nu^{a-l}\rho]) \times \delta([\nu^{c+1-i}\rho, \nu^{c-j}\rho]) \rtimes \sigma.$$

Comparing cuspidal support in the first equation we see $i-c=-b$ or $c+1-j=-b$. The second inequality implies $k=l$ and $i=j$. So we have $i=j=c-b$ or $i=j=c+b+1$. Now $k=l=a+\frac{1}{2}$. This shows that there are at most two irreducible subrepresentations in $\delta([\nu^{-b}\rho, \nu^{c}\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \otimes \sigma$, so there are no others than σ_4 and σ_5. □

Proposition 2.4. *In the appropriate Grothendieck group we have*

$$\delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \rtimes \sigma_2 = \sigma_4 + \mathrm{Lang}(\delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \rtimes \sigma_2),$$

$$\delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \rtimes \sigma_3 = \sigma_5 + \mathrm{Lang}(\delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \rtimes \sigma_3).$$

Proof. By Lemma 6.1 of [Muić 2005] the induced representations on the left side of the equations reduce. The proof of that lemma claims that all irreducible subquotients of the induced representations other than Langlands quotients are discrete series. The argument as in the proof of Theorem 2.1 of [Muić 2004] implies that they are all subrepresentations.

Let π_4 be a discrete series subrepresentation of $\delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \rtimes \sigma_2$ and π_5 a discrete series subrepresentation of $\delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \rtimes \sigma_3$. By Theorem 2.1, $\sigma_2 \oplus \sigma_3 \hookrightarrow \delta([\nu^{-b}\rho, \nu^{c}\rho]) \rtimes \sigma$ so we have

$$\begin{aligned}\pi_4 \oplus \pi_5 &\hookrightarrow \delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \rtimes \sigma_2 \oplus \delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \rtimes \sigma_3 \\ &\cong \delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \rtimes (\sigma_2 \oplus \sigma_3) \\ &\hookrightarrow \delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \times \delta([\nu^{-b}\rho, \nu^{c}\rho]) \rtimes \sigma \\ &\cong \delta([\nu^{-b}\rho, \nu^{c}\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \rtimes \sigma.\end{aligned} \tag{2-1}$$

By Proposition 2.3 π_4 and π_5 are not isomorphic and we have

$$\delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \rtimes \sigma_2 = \pi_4 + \mathrm{Lang}(\delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \rtimes \sigma_2), \tag{2-2}$$

$$\delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \rtimes \sigma_3 = \pi_5 + \mathrm{Lang}(\delta([\nu^{\frac{1}{2}}\rho, \nu^{a}\rho]) \rtimes \sigma_3), \tag{2-3}$$

where $\{\pi_4, \pi_5\} = \{\sigma_4, \sigma_5\}$.

We now prove that $\pi_4 = \sigma_4$ and $\pi_5 = \sigma_5$. It is enough to see that

$$\epsilon_{\pi_4}(2a+1,\rho)\epsilon_{\pi_4}(2b+1,\rho)^{-1} = 1.$$

Since $\epsilon_{\sigma_2}(2b+1,\rho)=1$ and $\min(\mathrm{Jord}_\rho(\sigma_2))=2b+1\in 2\mathbb{Z}$ there exists an irreducible representation τ of $G_{n+(c+\frac{1}{2})m_\rho}$ such that $\sigma_2 \hookrightarrow \delta([\nu^{\frac{1}{2}}\rho,\nu^b\rho])\rtimes\tau$. Now we have

$$\begin{aligned}\pi_4 &\hookrightarrow \delta([\nu^{\frac{1}{2}}\rho,\nu^a\rho])\rtimes\sigma_2 \hookrightarrow \delta([\nu^{\frac{1}{2}}\rho,\nu^a\rho])\times\delta([\nu^{\frac{1}{2}}\rho,\nu^b\rho])\rtimes\tau\\ &\cong \delta([\nu^{\frac{1}{2}}\rho,\nu^b\rho])\times\delta([\nu^{\frac{1}{2}}\rho,\nu^a\rho])\rtimes\tau\\ &\hookrightarrow \delta([\nu^{a+1}\rho,\nu^b\rho])\times\delta([\nu^{\frac{1}{2}}\rho,\nu^a\rho])\times\delta([\nu^{\frac{1}{2}}\rho,\nu^a\rho])\rtimes\tau.\end{aligned}$$

By Lemma 3.2 of [Mœglin and Tadić 2002] there exists an irreducible representation τ' of $G_{n+(2a+c+\frac{3}{2})m_\rho}$ such that

$$\pi_4 \hookrightarrow \delta([\nu^{a+1}\rho,\nu^b\rho])\rtimes\tau'.$$

Now $\epsilon_{\pi_4}(2a+1,\rho)\epsilon_{\pi_4}(2b+1,\rho)^{-1}=1$. As we proved that $\pi_4=\sigma_4$ and $\pi_5=\sigma_5$, (2-2) and (2-3) give the claim of the proposition. □

3. The main theorem

Theorem 3.1. *The induced representation $\delta([\nu^{-b}\rho,\nu^c\rho])\times\delta([\nu^{\frac{1}{2}}\rho,\nu^a\rho])\rtimes\sigma$ is of length* 6, *and it has two nonisomorphic irreducible subrepresentations. They are discrete series. In the appropriate Grothendieck group we have*

$$\begin{aligned}&\delta([\nu^{-b}\rho,\nu^c\rho])\times\delta([\nu^{\frac{1}{2}}\rho,\nu^a\rho])\rtimes\sigma =\\ &\quad\sigma_4+\sigma_5+\mathrm{Lang}(\delta([\nu^{\frac{1}{2}}\rho,\nu^a\rho])\rtimes\sigma_2)+\mathrm{Lang}(\delta([\nu^{\frac{1}{2}}\rho,\nu^a\rho])\rtimes\sigma_3)\\ &\qquad+\mathrm{Lang}(\delta([\nu^{-b}\rho,\nu^c\rho])\rtimes\sigma_1)\\ &\qquad\quad+\mathrm{Lang}(\delta([\nu^{-b}\rho,\nu^c\rho])\times\delta([\nu^{\frac{1}{2}}\rho,\nu^a\rho])\rtimes\sigma).\end{aligned}$$

Moreover,

$$\begin{aligned}&\mathrm{Lang}(\delta([\nu^{\frac{1}{2}}\rho,\nu^a\rho])\rtimes\sigma_2)\oplus\mathrm{Lang}(\delta([\nu^{\frac{1}{2}}\rho,\nu^a\rho])\rtimes\sigma_3)\oplus\mathrm{Lang}(\delta([\nu^{-b}\rho,\nu^c\rho])\rtimes\sigma_1)\\ &\quad\hookrightarrow (\delta([\nu^{-b}\rho,\nu^c\rho])\times\delta([\nu^{\frac{1}{2}}\rho,\nu^a\rho])\rtimes\sigma)/(\sigma_4\oplus\sigma_5).\end{aligned}$$

Proof. Suppose that $-b+c\geq\frac{1}{2}+a$. Otherwise we have a similar proof. We look at the composition of some intertwining operators:

$$\begin{aligned}\delta([\nu^{-b}\rho,\nu^c\rho])\times\delta([\nu^{\frac{1}{2}}\rho,\nu^a\rho])\rtimes\sigma &\to \delta([\nu^{\frac{1}{2}}\rho,\nu^a\rho])\times\delta([\nu^{-b}\rho,\nu^c\rho])\rtimes\sigma\\ &\to \delta([\nu^{\frac{1}{2}}\rho,\nu^a\rho])\times\delta([\nu^{-c}\rho,\nu^b\rho])\rtimes\sigma\\ &\to \delta([\nu^{-c}\rho,\nu^b\rho])\times\delta([\nu^{\frac{1}{2}}\rho,\nu^a\rho])\rtimes\sigma\\ &\to \delta([\nu^{-c}\rho,\nu^b\rho])\times\delta([\nu^{-a}\rho,\nu^{-\frac{1}{2}}\rho])\rtimes\sigma.\end{aligned}$$

Since $\frac{1}{2} \leq a < b < c$ the first and the third map are isomorphisms. By Theorem 2.1 the kernel of the second map is in the appropriate Grothendieck group

$$\delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma_2 + \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma_3.$$

By Proposition 2.4 this equals

$$\sigma_4 + \sigma_5 + \mathrm{Lang}(\delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma_2) + \mathrm{Lang}(\delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma_3).$$

By Theorem 2.1 and equation (1-2), the kernel of the last map is in the appropriate Grothendieck group

$$\delta([\nu^{-c}\rho, \nu^b\rho]) \rtimes \sigma_1 = \delta([\nu^{-b}\rho, \nu^c\rho]) \rtimes \sigma_1$$

which is, by Proposition 2.2, equal to

$$\sigma_4 + \sigma_5 + \mathrm{Lang}(\delta([\nu^{-b}\rho, \nu^c\rho]) \rtimes \sigma_1).$$

The image of the composition is

$$\mathrm{Lang}(\delta([\nu^{-b}\rho, \nu^c\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma).$$

We see that σ_4 and σ_5 appear in two kernels, but by Proposition 2.3 they appear with multiplicity 1 in the induced representation, so we have proved the first formula of the theorem.

To prove the second formula of the theorem, observe that by Theorem 2.1 and Propositions 2.2 and 2.3 we have

$$\sigma_4 \oplus \sigma_5 \hookrightarrow \delta([\nu^{-b}\rho, \nu^c\rho]) \rtimes \sigma_1 \hookrightarrow \delta([\nu^{-b}\rho, \nu^c\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma$$

and

$$\mathrm{Lang}(\delta([\nu^{-b}\rho, \nu^c\rho]) \rtimes \sigma_1) \hookrightarrow (\delta([\nu^{-b}\rho, \nu^c\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma)/(\sigma_4 \oplus \sigma_5). \tag{3-1}$$

Additionally, Proposition 2.4 and (2-1) imply

$$\begin{aligned}\sigma_4 \oplus \sigma_5 &\hookrightarrow \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma_2 \oplus \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma_3 \\ &\hookrightarrow \delta([\nu^{-b}\rho, \nu^c\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma\end{aligned}$$

and

$$\mathrm{Lang}(\delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma_2) \oplus \mathrm{Lang}(\delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma_3) \hookrightarrow (\delta([\nu^{-b}\rho, \nu^c\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma)/(\sigma_4 \oplus \sigma_5). \tag{3-2}$$

Now equations (3-1) and (3-2) prove the second formula of the theorem. □

4. Consequences

We have the following result:

Corollary 4.1. *In the appropriate Grothendieck group we have*

$$\delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \mathrm{Lang}(\delta([\nu^{-b}\rho, \nu^c\rho]) \rtimes \sigma) = \\ \mathrm{Lang}(\delta([\nu^{-b}\rho, \nu^c\rho]) \rtimes \sigma_1) + \mathrm{Lang}(\delta([\nu^{-b}\rho, \nu^c\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma),$$

$$\delta([\nu^{-b}\rho, \nu^c\rho]) \rtimes \mathrm{Lang}(\delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma) = \\ \mathrm{Lang}(\delta([\nu^{-b}\rho, \nu^c\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma) \\ + \mathrm{Lang}(\delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma_2) + \mathrm{Lang}(\delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma_3).$$

Except for $\mathrm{Lang}(\delta([\nu^{-b}\rho, \nu^c\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma)$ *all irreducible subquotients of induced representations on the left-hand side appear as subrepresentations.*

Proof. Using the exactness of the parabolic induction, Theorem 2.1, Proposition 2.4, (2-1) and Theorem 3.1 we have

$$\delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \mathrm{Lang}(\delta([\nu^{-b}\rho, \nu^c\rho]) \rtimes \sigma) \\ \cong (\delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \times \delta([\nu^{-b}\rho, \nu^c\rho]) \rtimes \sigma)/(\delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes (\sigma_2 \oplus \sigma_3)) \\ \cong (\delta([\nu^{-b}\rho, \nu^c\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma)/(\delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma_2 \oplus \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma_3).$$

Comparing this with the result of the main theorem gives the first formula of the corollary. Similarly, for the second formula use Proposition 2.2 and observe that

$$\delta([\nu^{-b}\rho, \nu^c\rho]) \rtimes \mathrm{Lang}(\delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma) \cong \\ (\delta([\nu^{-b}\rho, \nu^c\rho]) \times \delta([\nu^{\frac{1}{2}}\rho, \nu^a\rho]) \rtimes \sigma)/(\delta([\nu^{-b}\rho, \nu^c\rho]) \rtimes \sigma_1). \quad \square$$

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IGOR CIGANOVIĆ
DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
UNIVERSITY OF ZAGREB
ZAGREB
CROATIA
igor.ciganovic@math.hr

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HIGGS BUNDLES OVER CELL COMPLEXES AND REPRESENTATIONS OF FINITELY PRESENTED GROUPS

GEORGIOS DASKALOPOULOS, CHIKAKO MESE AND GRAEME WILKIN

The purpose of this paper is to extend the Donaldson–Corlette theorem to the case of vector bundles over cell complexes. We define the notions of a vector bundle and a Higgs bundle over a complex, and describe the associated Betti, de Rham and Higgs moduli spaces. The main theorem is that the $\mathrm{SL}(r, \mathbb{C})$ character variety of a finitely presented group Γ is homeomorphic to the moduli space of rank-r Higgs bundles over an admissible complex X with $\pi_1(X) = \Gamma$. A key role is played by the theory of harmonic maps defined on singular domains.

1. Introduction

Higgs bundles were first introduced by Hitchin [1987] as a PDE on a vector bundle over a Riemann surface obtained by the dimensional reduction of the anti-self-dual equations on $\mathbb{R}^4$. Since then, the field has seen a remarkable explosion in different directions, most notably the work of Simpson [1988; 1992] on variations of Hodge structures and applications to Kähler groups. The work of Donaldson [1987] and Corlette [1988] provided links with the theory of flat bundles and character varieties of groups. Higgs bundles have been generalised over noncompact manifolds [Corlette and Simpson 2008; Simpson 1990; Jost et al. 2007; Jost and Zuo 1996] and singular curves [Balaji et al. 2013]. The goal of this paper is to push this even further by considering Higgs bundles over more general singular spaces; namely, finite simplicial complexes.

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As pointed out by Hitchin, Donaldson and Corlette, a key role in the relation between character varieties and Higgs bundles is played by the theory of harmonic maps. Harmonic maps have been used in the study of representations of Kähler manifold groups starting with the work of Siu [1980], also see [Carlson and Toledo 1989], and have seen some remarkable applications in providing new proofs of the celebrated Margulis superrigidity theorem, see [Jost 1997], and the only existing proof of the rank-1 superrigidity theorem due to Corlette [1992] and Gromov and Schoen [1992]. But these directions involved showing that the representations are rigid, in contrast with Hitchin's point of view, which is to study the moduli space of such representations.

In all the above references, one studies representations of fundamental groups of smooth manifolds rather than arbitrary finitely presented groups. In other words, the domain space of the harmonic map is smooth. Chen [1995] and Eells and Fuglede [2001] developed the theory of harmonic maps from a certain class of singular domains including admissible simplicial complexes. By admissible they mean complexes that are dimensionally homogeneous and locally chainable in order to avoid certain analytic pathologies (see the next section for precise definitions). Since any finitely presented group is the fundamental group of an admissible complex, there is no real restriction in considering admissible complexes. The key property of harmonic maps shown in the above references is that they are Hölder continuous but in general they fail to be Lipschitz. In fact, the work of the first two authors [Daskalopoulos and Mese 2008; 2009] shows that Lipschitz harmonic maps often imply that the representations are rigid.

The starting point of this paper is a finitely presented group Γ and a 2-dimensional admissible complex without boundary X with fundamental group $\pi_1(X) \simeq \Gamma$. We also fix a piecewise-smooth vector bundle E over X that admits a flat $\mathrm{SL}(r, \mathbb{C})$ structure. Such bundles are parametrised topologically by the (finitely many) connected components of the $\mathrm{SL}(r, \mathbb{C})$ character variety of $\pi_1(X)$. One can write down Hitchin's equations

$$F_A + \psi \wedge \psi = 0, \tag{1-1}$$

$$d_A \psi = 0 \tag{1-2}$$

for a sufficiently regular unitary connection A and Higgs field ψ. Again, as in the smooth case, the $\mathrm{SL}(r, \mathbb{C})$ connection $d_A + \psi$ is flat and one can ask what the precise condition is so that the pair (d_A, ψ) corresponds to a representation $\rho : \pi_1(X) \to \mathrm{SL}(r, \mathbb{C})$.

Given a representation ρ as above, we can associate as in the smooth case a ρ-equivariant harmonic map from the universal cover $\widetilde{X}$ to the symmetric space $\mathrm{SL}(r, \mathbb{C}) / \mathrm{SU}(r)$. The first two authors [Daskalopoulos and Mese 2008] studied harmonic maps from simplicial complexes to smooth manifolds and discovered the following crucial properties:

(1) The harmonic map is smooth away from the codimension-2 skeleton of $\widetilde{X}$.

(2) The harmonic map satisfies a balancing condition at the codimension-1 skeleton of $\widetilde{X}$ in the sense that the sum of the normal derivatives vanishes identically.

(3) The harmonic map blows up in a controlled way at the codimension-2 skeleton of $\widetilde{X}$.

All the above properties are described precisely in Theorem 3.3. This allows us to prove that the derivative of the harmonic map belongs in an appropriate weighted Sobolev space $L^2_{1,\delta}$ (see Proposition 4.5). The definition of weighted Sobolev spaces is given in Section 3B. Finally, the main theorem describing the correspondence between equivalence classes of balanced Higgs pairs of class $L^2_{1,\delta}$ and representations is given in Section 4 (see Theorem 4.3).

We would like to end this introduction with a brief discussion of some motivation and future applications of this paper that we will explore elsewhere. Note that, with the exception of [Balaji et al. 2013], the theory of Higgs bundles on singular varieties is not very well understood. For example, one of the important questions about fundamental groups of singular projective varieties is whether fundamental groups of normal varieties behave more like the ones of smooth manifolds, or in the other extreme, if there are very few restrictions on them [Arapura et al. 2016; Kapovich and Kollár 2014]. The connection with the results of this paper is as follows: By [Eells and Fuglede 2001, Example 8.9], an n-dimensional normal projective variety X admits a bi-Lipschitz triangulation with its singular set as a subcomplex of dimension at most $n-2$. Furthermore, X is admissible in the sense of Definition 2.2. Thus, studying harmonic maps on X, or more generally constructing moduli spaces of bundles on X, could imply restrictions on fundamental groups as in [Carlson and Toledo 1989; Simpson 1992].

2. Vector bundles over complexes

2A. *Basic definitions of smooth bundles.*

Definition 2.1 [Lojasiewicz 1964]. Let $\mathbb{E}^N$ be an N-dimensional affine space. A *cell* of dimension i is a nonempty, open, convex, bounded subset in some $\mathbb{E}^i \subset \mathbb{E}^N$. We will use the notation σ^i to denote a cell of dimension i and call $\mathbb{E}^i$ the *extended plane* defined by σ^i. A *locally finite convex cell complex*, or simply a *complex* X in $\mathbb{E}^N$, is a locally a finite collection $\mathcal{F} = \{\sigma\}$ of disjoint cells in $\mathbb{E}^N$ such that for any $\sigma \in \mathcal{F}$ its closure $\bar{\sigma}$ is a union of cells in $\mathcal{F}$. *The dimension of a complex* X is the maximum dimension of a cell in X.

For example, a simplicial complex is a cell complex whose cells are all simplices.

Definition 2.2. A connected complex X of dimension n is said to be admissible [Chen 1995; Eells and Fuglede 2001] if the following two conditions hold:

(i) X is dimensionally homogeneous, i.e., every cell is contained in a closure of at least one n-cell, and

(ii) X is locally $(n-1)$-chainable, i.e., given any $(n-2)$-cell v, every two n-cells σ and σ' incident to v can be joined by a sequence $\sigma = \sigma_0, \sigma_1, \ldots, \sigma_k = \sigma'$ where σ_i and σ_{i+1} are two adjacent n-cells incident to v for $i = 0, 1, \ldots, k-1$.

The boundary ∂X of X is the union of the closures of the $(n-1)$-cells contained in the closure of exactly one n-cell. Using a regular barycentric subdivision we obtain that given any locally finite complex there is a locally finite simplicial complex such that any cell is a union of simplices.

Definition 2.3. Let U be a subset of a complex X. A function $f : U \to \mathbb{R}$ is called *smooth* if for any n-cell σ of X, the restriction $f|_{\sigma\cap U}$ can be extended to a smooth function on $\mathbb{E}^i \cap U$ in the extended plane defined by σ. A map $f : U \to Z \subset \mathbb{E}^M$ into a complex Z is called *smooth* if with respect to some affine coordinate system on $\mathbb{E}^M$ we have $f = (f^1, \ldots, f^M)$ where f^j is smooth for every $j = 1, \ldots, M$.

Definition 2.4. A Riemannian metric g_σ on a cell σ is the restriction to σ of a smooth Riemannian metric on its extended plane. A Riemannian metric g on X is a smooth Riemannian metric g_σ on each n-cell σ of X satisfying the additional property that if τ is a face of σ, then $g_\sigma|_\tau = g_\tau$, where $g_\sigma|_\tau$ denotes the restriction of the extension of g_σ to the extended plane of σ. In particular, the expressions of g_σ with respect to some affine coordinates in the extended plane are smooth functions in the sense of Definition 2.3.

Definition 2.5. A *smooth complex vector bundle of rank r* over a complex X is a topological space E and a continuous, surjective map $\pi : E \to X$ such that:

(1) for each $x \in X$ the fibre $\pi^{-1}(x)$ has the structure of a complex vector space, and

(2) there exists an open cover $\{U_\alpha\}_{\alpha\in I}$ of X such that for each $\alpha \in I$ there exists a homeomorphism $\varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}^r$ such that

 (i) φ_α restricts to a linear isomorphism $\pi^{-1}(x) \cong \{x\} \times \mathbb{C}^r$ for each $x \in U_\alpha$, and

 (ii) if $U_\alpha \cap U_\beta \neq \varnothing$, then the transition function $g_{\alpha\beta} = \varphi_\alpha \circ \varphi_\beta^{-1} : U_\alpha \cap U_\beta \times \mathbb{C}^r \to U_\alpha \cap U_\beta \times \mathbb{C}^r$ induces a smooth map $g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(r, \mathbb{C})$.

A *section* of $\pi : E \to X$ is a continuous map $s : X \to E$ satisfying $\pi \circ s = \mathrm{id}_X$. The section is *smooth* if on each local trivialisation $\pi^{-1}(U_\alpha) \cong U_\alpha \times \mathbb{C}^r$ with projection onto the second factor denoted by $p_2 : \pi^{-1}(U_\alpha) \to \mathbb{C}^r$, the composition of $U_\alpha \xrightarrow{s} \pi^{-1}(U_\alpha) \xrightarrow{p_2} \mathbb{C}^r$ is a smooth map as in Definition 2.3. Let $\Omega^0(X, E)$ denote the vector space of all smooth sections of $\pi : E \to X$. If E is a smooth vector bundle, then so is any associated bundle formed by taking the dual, tensor product, etc. In particular, if E is smooth then $\mathrm{End}(E)$ is smooth.

Definition 2.6. A smooth complex p-form on a cell σ is the restriction to σ of a smooth complex p form on the extended plane of the cell. A *smooth p-form* $\omega=\{\omega_\sigma\}_{\sigma\in\mathcal{F}}$ on a complex X with values in a smooth vector bundle E is a collection of smooth p-forms ω_σ with values in E for each cell σ of X, with the additional property that if τ is a face of σ, then $\omega_\sigma|_\tau=\omega_\tau$. In particular, the expressions of ω_σ with respect to some affine coordinates in the extended plane are smooth functions in the sense of Definition 2.3. We define $\Omega^p(X,E)$ as the space of all smooth p-forms with values in E. If E is the trivial line bundle, then we write $\Omega^p(X)=\Omega^p(X,E)$ and this is the space of smooth p-forms on X. Given a smooth p-form $\omega=\{\omega_\sigma\}_{\sigma\in\mathcal{F}}\in\Omega^p(X)$, we define $d\omega=\{d\omega_\sigma\}_{\sigma\in\mathcal{F}}$ and note that this is a well-defined smooth $(p+1)$-form. Clearly, $d^2=0$ and the complex $(\Omega^*(X),d)$ denotes the smooth de Rham complex. We denote by $H^p_{\mathrm{dR}}(X)$ the cohomology groups associated with this complex; see [Griffiths and Morgan 1981, Chapter VIII].

Definition 2.7. A *smooth connection* on a smooth vector bundle $\pi:E\to X$ is a $\mathbb{C}$-linear map $D:\Omega^0(X,E)\to\Omega^1(X,E)$ that satisfies the Leibniz rule

$$D(fs)=(df)s+f(Ds),\quad f\in\Omega^0(X),\ \ s\in\Omega^0(X,E).$$

We denote the space of all smooth connections by $\mathcal{A}^{\mathbb{C}}(E)$.

The definition of D can be extended to bundle-valued forms in the usual way. More precisely, any element in $\sigma\in\Omega^p(X,E)$ can be written as a linear combination of elements of the form $\sigma=s\omega$ with $\omega\in\Omega^p(X)$ and $s\in\Omega^0(X,E)$, and define

$$D\sigma=s(d\omega)+(Ds)\wedge\omega. \tag{2-1}$$

Remark 2.8. Implicit in the definition of $\Omega^1(X,E)$ is that 1-forms with values in E must agree on the interfaces between the cells in the complex X. Therefore, the definition above implies that a connection must map sections that agree on the interfaces to bundle-valued 1-forms that agree on the interfaces.

As for the case of a smooth vector bundle over a smooth manifold, with respect to a trivialization, $\varphi_\alpha:\pi^{-1}(U_\alpha)\to U_\alpha\times\mathbb{C}^r$, $D=d+A_\alpha$, where $(A_\alpha)_{ij}$ is a complex-valued smooth 1-form. A_α is called the *connection form* of D with respect to the trivialization φ_α. In a different trivialization φ_β and with $g_{\alpha\beta}=\varphi_\alpha\circ\varphi_\beta^{-1}$ we have,

$$A_\beta=g_{\alpha\beta}^{-1}\,dg_{\alpha\beta}+g_{\alpha\beta}^{-1}A_\alpha g_{\alpha\beta}. \tag{2-2}$$

Definition 2.9. The *curvature* of a smooth connection D is the matrix-valued 2-form F_D defined by

$$D^2s=F_Ds\quad\text{for all } s\in\Omega^0(X,E).$$

Locally, we have $(F_D)_\alpha = dA_\alpha + A_\alpha \wedge A_\alpha$, where A_α is the connection form of D. Furthermore,

$$(F_D)_\beta = g_{\alpha\beta}^{-1}(F_D)_\alpha g_{\alpha\beta}, \tag{2-3}$$

and so the curvature form F_D is an element of $\Omega^2(X, \mathrm{End}(E))$.

Definition 2.10. The *complex gauge group* is the group $\mathcal{G}^{\mathbb{C}}(E)$ of all smooth automorphisms of E. If D is a smooth connection on E and $g \in \mathcal{G}^{\mathbb{C}}(E)$, then we define $g(D) = g^{-1} \circ D \circ g$. In local coordinates, the action of $\mathcal{G}^{\mathbb{C}}(E)$ on $\mathcal{A}^{\mathbb{C}}(E)$ is

$$g(d + A_\alpha) = d + g^{-1}\, dg + g^{-1} A_\alpha g. \tag{2-4}$$

Definition 2.11. A *smooth Hermitian metric* $h = (h_\sigma)$ on a rank-r complex vector bundle $\pi : E \to X$ is a smooth section h of $\mathrm{End}(E)$ such that for each cell σ its restriction h_σ is a Hermitian metric and if τ is a face of σ, then $h_\sigma|_\tau = h_\tau$. A Hermitian metric in a trivialization $\varphi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times \mathbb{C}^r$ is given locally by a smooth map $\tilde{h}_\alpha$ from U_α into the positive definite matrices in $\mathrm{GL}(r, \mathbb{C})$, and the induced inner product on the fibres of E is

$$\langle s_1(x), s_2(x)\rangle = \overline{\varphi_\alpha(s_1(x))}^T\, \tilde{h}_\alpha(x)\, \varphi_\alpha(s_2(x)) \in \mathbb{C}.$$

Definition 2.12. A connection D on a vector bundle E with a Hermitian metric h is a *unitary connection* if the following equation is satisfied:

$$d\langle s_1, s_2\rangle = \langle Ds_1, s_2\rangle + \langle s_1, Ds_2\rangle,$$

where $\langle\cdot,\cdot\rangle$ is the pointwise inner product on the fibres of E induced by the metric h. The space of smooth unitary connections on E is denoted by $\mathcal{A}(E, h)$. If $D \in \mathcal{A}(E, h)$, then the curvature F_D is a section of $\Omega^2(\mathrm{ad}(E))$. In other words, with respect to a unitary frame field the curvature satisfies $F_D^* = -F_D$.

Definition 2.13. The *unitary gauge group* $\mathcal{G}(E)$ is the subgroup of $\mathcal{G}^{\mathbb{C}}(E)$ that preserves the Hermitian metric h on each fibre of E. The action on $\mathcal{G}(E)$ on $\mathcal{A}^{\mathbb{C}}(E)$ preserves the space $\mathcal{A}(E, h)$.

Definition 2.14. A connection D on a vector bundle E is *flat* if $F_D = 0$. Given a flat connection, we can define the twisted de Rham complex $(\Omega^*(X, E), D)$. The cohomology groups will be denoted by $H^p(X, E)$.

Definition 2.15. A *flat structure* on a vector bundle $\pi : E \to X$ is given by an open cover $\{U_\alpha\}_{\alpha\in I}$ and trivialisations $\{\varphi_\alpha\}_{\alpha\in I}$ for which the transition functions $g_{\alpha\beta} = \varphi_\alpha \circ \varphi_\beta^{-1}$ are constant. A vector bundle with a flat structure is also called a *flat bundle*.

Remark 2.16. Equation (2-2) shows that the connection $D = d$ (with zero connection form) is globally defined on a flat bundle. Thus a flat bundle clearly admits a connection of curvature zero. The converse is also true.

Theorem 2.17. *Let X be n-complex, U an open subset of X and E a smooth vector bundle with a smooth flat connection on U. Then E admits a flat structure.*

Proof. Given a flat connection D on E, fix a cell σ, a point $x_0 \in \bar{\sigma} \cap U$ and consider a contractible neighbourhood V_σ of x_0 in the extended plane of σ. Choose a local frame s_σ^0 of E on V_σ and let A^σ be the corresponding connection form. We are assuming that the local frames s_σ^0 patch together to define a piecewise smooth frame s_0 in a neighbourhood of x_0 in X. We are going to choose a different trivialisation s_σ for which the connection can be written as $D = d$. This can be done by solving the equation

$$g_\sigma^{-1} A^\sigma g_\sigma + g_\sigma^{-1}\, dg_\sigma = 0 \quad \Longleftrightarrow \quad dg_\sigma = -A^\sigma g_\sigma \tag{2-5}$$

locally for a gauge transformation g_σ. By the result in the smooth case (this is an application of the Frobenius theorem) a solution g_σ exists and by multiplying by a constant matrix we may assume without loss of generality that $g_\sigma(x_0) = \mathrm{id}$. This makes the solution unique and thus if a cell τ is a face of a cell σ then, since $A^\sigma|_\tau = A^\tau$, it must be $g_\sigma|_\tau = g_\tau$. It follows that the new frames $s_\sigma = g_\sigma \circ s_\sigma^0$ patch together to define a piecewise smooth frame s in a neighbourhood of x_0 in X. The flat structure is now defined by the local frames $\{s\}$. □

Definition 2.18. A section $s \in \Omega^0(X, E)$ is *parallel with respect to D* if $Ds = 0$. Given a smooth curve $c : [a, b] \to X$, a section s is *parallel along c with respect to D* if $D_{c'(t)} s = 0$. Given a curve $c : [a, b] \to X$ and $s_a \in \pi^{-1}(c(a))$ the *parallel transport of s along c with respect to D* is the section $s : \pi^{-1}(c([a, b])) \to E$ which is given locally by the solution to the equation

$$\frac{ds(c(t))}{dt} + A_{c(t)}(c'(t))s(c(t)) = 0.$$

Lemma 2.19. *Let $c_1, c_2 : [a, b] \to X$ be two closed smooth curves in X which are homotopy equivalent, and which satisfy $x_0 = c_1(a) = c_1(b) = c_2(a) = c_2(b)$. Let D be a smooth flat connection on a rank-r bundle $\pi : E \to X$, and let s_1 and s_2 be the parallel transport with respect to D along c_1 and c_2 respectively, with initial condition $s_0 \in \pi^{-1}(x_0)$. If $F_D = 0$ then $s_1(c_1(b)) = s_2(c_2(b))$.*

Proof. As usual, note that it suffices to show that the holonomy is trivial around a homotopically trivial loop. If there is a homotopy equivalence between two loops that is constant except on a single cell, then standard theorems for smooth manifolds show that the holonomy around the two loops is the same. Given a homotopically trivial loop γ, there is a sequence of homotopy equivalences $\gamma \sim \gamma_1,\ \gamma_1 \sim \gamma_2,\ \ldots,$

$\gamma_N \sim$ id between γ and the trivial loop (denoted id), such that each homotopy equivalence is constant except on a single n-cell. For example, one can do this by identifying the fundamental group with the edge group of a simplicial complex; see [Armstrong 1983, Section 6.4]. Therefore, the holonomy of γ is the same as the holonomy of each γ_n along this sequence of homotopy equivalences, and so the holonomy of γ is trivial. □

Definition 2.20. A flat connection D on a rank-r vector bundle $\pi : E \to X$ defines a representation $\rho : \pi_1(X) \to \mathrm{GL}(r, \mathbb{C})$ called the *holonomy representation of D*. A flat connection is called *irreducible* if its holonomy representation is irreducible. The space of irreducible, flat smooth connections is denoted by $\mathcal{A}^{\mathbb{C},\mathrm{irr}}(E)$.

Lemma 2.21. *A representation $\rho : \pi_1(X) \to \mathrm{GL}(r, \mathbb{C})$ defines a flat connection on a bundle $\pi : E_\rho \to X$ with holonomy representation ρ. Moreover, the flat connection on E_ρ depends continuously on the representation ρ.*

Proof. In the standard way, from a representation $\rho : \pi_1(X) \to \mathrm{GL}(r, \mathbb{C})$ we construct a flat vector bundle $E_\rho \to X$, with total space

$$E_\rho = \widetilde{X} \times_\rho \mathbb{C}^r, \tag{2-6}$$

where $\widetilde{X}$ is the universal cover of X, and the equivalence is by deck transformations on the left factor $\widetilde{X}$, and via the representation ρ on the right factor $\mathbb{C}^r$. On each trivialisation we have the trivial connection d, and since the transition functions of E are constant, this connection is globally defined. Since the deck transformations depend continuously on the representation ρ, the flat connection on E_ρ depends continuously on ρ. □

Corollary 2.22. *A flat connection on a vector bundle over a simply connected complex X is complex gauge-equivalent to the trivial connection d on the trivial vector bundle.*

Definition 2.23. The $\mathrm{SL}(r, \mathbb{C})$ *character variety* is the space of irreducible representations $\rho : \pi_1(X) \to \mathrm{SL}(r, \mathbb{C})$ modulo conjugation by $\mathrm{SL}(r, \mathbb{C})$:

$$\mathcal{M}_{\mathrm{char}} = \{\text{irreducible reps } \rho : \pi_1(X) \to \mathrm{SL}(r, \mathbb{C})\}/\,\mathrm{SL}(r, \mathbb{C}). \tag{2-7}$$

The next lemma is a trivial consequence of the path-lifting property and is standard.

Lemma 2.24. *If two characters defined by the representations ρ and ρ' belong to the same connected component of $\mathcal{M}_{\mathrm{char}}$ then the vector bundles E_ρ and $E_{\rho'}$ are smoothly isomorphic.*

In view of the above, let $\mathcal{C}$ denote the set of connected components of $\mathcal{M}_{\mathrm{char}}$. Then we can write

$$\mathcal{M}_{\mathrm{char}} = \bigsqcup_{c \in \mathcal{C}} \mathcal{M}^c_{\mathrm{char}}$$

and write $E_c = E_\rho$ for any representative in the isomorphism class of bundles defined by $\rho \in \mathcal{M}^c_{\mathrm{char}}$.

Remark 2.25. Since we are interested in the $\mathrm{SL}(r, \mathbb{C})$ character variety instead of the $\mathrm{GL}(r, \mathbb{C})$ character variety, we need to fix determinants in our definitions of connections and gauge transformations. Henceforth *we will impose the condition that all connection forms are traceless and all gauge transformations have determinant 1*. For the sake of notational simplicity we will keep the same notation as before for the various spaces of $\mathrm{SL}(r, \mathbb{C})$ connections and gauge groups.

Proposition 2.26. $$\mathcal{A}^{\mathbb{C},\mathrm{irr}}_{\mathrm{flat}}(E_c)/\mathcal{G}^{\mathbb{C}}(E_c) \cong \mathcal{M}^c_{\mathrm{char}}.$$

Proof. The holonomy map applied to an irreducible flat connection D gives an irreducible representation $\rho : \pi_1(X, x_0) \to \mathrm{GL}(r, \mathbb{C})$. The action of a complex gauge transformation $g \in \mathcal{G}^{\mathbb{C}}(E_c)$ on D induces the conjugate action of an element $\xi = g(x_0) \in \mathrm{GL}(r, \mathbb{C})$ on ρ. Therefore we have a continuous map $\tau : \mathcal{A}^{\mathbb{C},\mathrm{irr}}_{\mathrm{flat}}(E_c)/\mathcal{G}^{\mathbb{C}}(E_c) \to \mathcal{M}^c_{\mathrm{char}}$. Note that $\tau([D_1]) = \tau([D_2])$ implies that the flat structures associated to D_1 and D_2 by Theorem 2.17 are complex gauge-equivalent, and so D_1 and D_2 are complex gauge-equivalent. Therefore τ is injective.

Similarly, given a representation $\rho : \pi_1(X, x_0) \to \mathrm{GL}(r, \mathbb{C})$ we construct a flat connection d on the flat bundle E_ρ as in the proof of Lemma 2.21. If we conjugate the representation by an element $\xi \in \mathrm{GL}(r, \mathbb{C})$, then the flat connection associated to this new representation is related to E_ρ by a global change of coordinates using the action of ξ on the fibres of E_ρ. Therefore the two flat bundles are complex gauge-equivalent, and so conjugate representations give $\mathcal{G}^{\mathbb{C}}(E_c)$-equivalent flat connections, which gives us a continuous map $\zeta : \mathcal{M}_{\mathrm{char}} \to \mathcal{A}^{\mathbb{C},\mathrm{irr}}_{\mathrm{flat}}(E_c)/\mathcal{G}^{\mathbb{C}}(E_c)$. Lemma 2.21 shows that $\tau \circ \zeta = \mathrm{id}$. Since τ is injective then this implies that $\zeta \circ \tau = \mathrm{id}$ and so τ is a homeomorphism $\mathcal{A}^{\mathbb{C},\mathrm{irr}}_{\mathrm{flat}}(E_c)/\mathcal{G}^{\mathbb{C}}(E_c) \cong \mathcal{M}^c_{\mathrm{char}}$. □

2B. ***Relationship to Higgs bundles.*** Given a complex X with universal cover $\widetilde{X}$, fix an irreducible representation $\rho : \pi_1(X) \to \mathrm{SL}(r, \mathbb{C})$, and let $E = \widetilde{X} \times_\rho \mathbb{C}^r \to X$ be as before. We also fix a ρ-equivariant map $u : \widetilde{X} \to \mathrm{SL}(r, \mathbb{C})/\mathrm{SU}(r)$, locally Lipschitz away from the 0-skeleton X^0 of X. We now recall the basic construction from [Corlette 1988; Donaldson 1987]:

(1) The complexified tangent space $T^{\mathbb{C}}_h(\mathrm{SL}(r, \mathbb{C})/\mathrm{SU}(r))$ can be identified (independent of h) with the space of traceless matrices and this gives a trivialization of the complexified tangent bundle $T^{\mathbb{C}}(\mathrm{SL}(r, \mathbb{C})/\mathrm{SU}(r)) \cong \mathrm{SL}(r, \mathbb{C})/\mathrm{SU}(r) \times \mathfrak{sl}(r, \mathbb{C})$.

(2) In the trivialization given in (1) the Levi-Civita connection at a point $h \in \mathrm{SL}(r, \mathbb{C})/\mathrm{SU}(r)$ has the form

$$\nabla_X Y = dY(X) - \tfrac{1}{2}\big(dh(X)h^{-1}Y + Yh^{-1}dh(X)\big),$$

where we use the notation h to indicate left translation by h.

(3) The identification $h^{-1}(T_h^{\mathbb{C}}(\mathrm{SL}(r,\mathbb{C})/\mathrm{SU}(r))) \cong T_{\mathrm{id}}^{\mathbb{C}}(\mathrm{SL}(r,\mathbb{C})/\mathrm{SU}(r)) \cong \mathfrak{sl}(r,\mathbb{C})$ gives another isomorphism $\theta : T^{\mathbb{C}}(\mathrm{SL}(r,\mathbb{C})/\mathrm{SU}(r)) \to \mathrm{SL}(r,\mathbb{C})/\mathrm{SU}(r) \times \mathfrak{sl}(r,\mathbb{C})$. It follows immediately from (2) that in the coordinates given by θ, the Levi-Civita connection is given by

$$\nabla_X s = h^{-1}\nabla_X(hs) = ds(X) + \tfrac{1}{2}[h^{-1}dh(X), s].$$

We thus conclude that in the above coordinates

$$\nabla = d + \tfrac{1}{2}[h^{-1}dh, \cdot\,]. \tag{2-8}$$

(4) The isomorphism θ is equivariant with respect to the $\mathrm{PSL}(r,\mathbb{C})$ action on the complexified tangent bundle $T^{\mathbb{C}}(\mathrm{SL}(r,\mathbb{C})/\mathrm{SU}(r))$ and the adjoint representation on $T_{\mathrm{id}}^{\mathbb{C}}(\mathrm{SL}(r,\mathbb{C})/\mathrm{SU}(r)) \cong \mathfrak{s}l(r,\mathbb{C})$.

(5) Given u as above, consider the pullbacks $\mathcal{D} = u^*d$ and $d_A = u^*\nabla$ on the trivial bundle $\widetilde{X} \times T_{\mathrm{id}}^{\mathbb{C}}(\mathrm{SL}(r,\mathbb{C})/\mathrm{SU}(r)) \cong \widetilde{X} \times \mathfrak{s}l(r,\mathbb{C})$. First notice, that since u^*d is trivial and u is ρ-equivariant, $\mathcal{D}$ descends to a flat connection of holonomy ρ on E_ρ. Again, by the ρ-equivariance of u and (4), the connection d_A descends to a connection on $\mathrm{ad}(E_\rho)$ over X. Moreover, since its connection form acts by the adjoint representation, it defines an $\mathrm{SL}(r,\mathbb{C})$ connection on the bundle E_ρ over X and (2-8) implies

$$\mathcal{D} = d_A + \psi, \quad \psi = -\tfrac{1}{2}u^{-1}\,du. \tag{2-9}$$

Since $\mathcal{D}$ is a flat connection,

$$F_A + \psi \wedge \psi = 0, \tag{2-10}$$

$$d_A\psi = 0. \tag{2-11}$$

2C. *The balancing condition.*

Definition 2.27. A smooth 1-form $\omega = \{\omega_\sigma\}_{\sigma\in\mathcal{F}} \in \Omega^1(X)$ satisfies the *balancing condition* if for every $(n-1)$-cell τ, we have

$$\sum_{\sigma>\tau} \omega_\sigma(e_\sigma) = 0, \tag{2-12}$$

where $\sigma > \tau$ implies that τ is a face of σ, and e_σ is an inward-pointing normal vector field along τ in σ. The set $\Omega^1_{\mathrm{bal}}(X)$ is the subset of $\Omega^1(X)$ consisting of forms satisfying the balancing condition.

Definition 2.28. Let E be smooth vector bundle on X of rank r and let $p : \widetilde{X} \to X$ be the universal cover. We assume that the pullback bundle $p^*(E)$ over $\widetilde{X}$ is trivial *with a fixed trivialization* $p^*(E) \cong \widetilde{X} \times \mathbb{C}^r$ (if the connection is flat then this is always valid by Corollary 2.22). A connection $D \in \mathcal{A}^{\mathbb{C},\mathrm{irr}}(E)$ is called *balanced* if its pullback $p^*(D)$ to $p^*(E)$ can be written (in the given trivialization) as $p^*(D) = d + A$ where all the components satisfy $A_{ij} \in \Omega^1_{\mathrm{bal}}(\widetilde{X})$. Let $\mathcal{A}^{\mathbb{C},\mathrm{irr}}_{\mathrm{bal}}(E)$ be the space of irreducible, smooth, balanced $\mathrm{GL}(r, \mathbb{C})$ connections, and let $\mathcal{A}^{\mathrm{irr}}_{\mathrm{bal}}(E)$ denote the space of irreducible, smooth, balanced connections compatible with the Hermitian metric h on E. In what follows, if the meaning is clear then the notation for the metric is suppressed.

Definition 2.29. Let E be as in the previous definition. Given $g \in \mathcal{G}^{\mathbb{C}}(E)$, let $\tilde{g}$ denote the induced gauge transformation of $p^*(E)$. We define $\mathcal{G}^{\mathbb{C}}_{\mathrm{bal}}(E)$ (resp. $\mathcal{G}_{\mathrm{bal}}(E)$) to be the group of complex (resp. unitary) gauge transformations such that $g \in \mathcal{G}^{\mathbb{C}}_{\mathrm{bal}}(E)$ (resp. $g \in \mathcal{G}_{\mathrm{bal}}(E)$) implies that $d\tilde{g}_{ij} \in \Omega^1_{\mathrm{bal}}(\widetilde{X})$.

Remark 2.30. Via (2-4), the group $\mathcal{G}^{\mathbb{C}}_{\mathrm{bal}}(E)$ acts on the space $\mathcal{A}^{\mathbb{C},\mathrm{irr}}_{\mathrm{bal}}(E)$, and $\mathcal{G}_{\mathrm{bal}}(E)$ acts on $\mathcal{A}^{\mathrm{irr}}_{\mathrm{bal}}(E)$.

Remark 2.31. In this paper we are interested in flat bundles. Corollary 2.22 implies that the pullback of a flat bundle to the universal cover is trivial. By choosing a trivialization it thus makes sense to talk about balanced connections and gauge transformations.

3. Harmonic maps and Higgs bundles

In this section we describe the relationship between Higgs bundles and harmonic maps from a complex X into the space $\mathrm{SL}(n, \mathbb{C})/\,\mathrm{SU}(n)$, a generalisation of the construction of [Donaldson 1987; Corlette 1988]. *From now on X will denote an admissible 2-dimensional simplicial complex without boundary. We will further assume that X is equipped with a Riemannian metric g such that for any 2-simplex σ, (σ, g_σ) is isometric to an interior of an equilateral triangle in $\mathbb{R}^2$ and for any 1-simplex τ, (τ, g_τ) is isometric to the open unit interval in $\mathbb{R}$.* It is not hard to extend the results of this section to general Riemannian metrics and also general 2-dimensional complexes. We endow $\mathrm{SL}(n, \mathbb{C})/\,\mathrm{SU}(n)$ with a Riemannian metric of nonpositive sectional curvature such that $\mathrm{SL}(n, \mathbb{C})$ acts by isometries.

3A. *Estimates of harmonic maps.*

Theorem 3.1. *Let X be a 2-complex as before with universal cover $\widetilde{X}$ and $\rho : \pi_1(X) \to \mathrm{SL}(n, \mathbb{C})$ be an irreducible representation. Then there exists a unique ρ-equivariant harmonic map $u : \widetilde{X} \to Y := \mathrm{SL}(n, \mathbb{C})/\,\mathrm{SU}(n)$.*

Proof. The existence is a special case of Theorem 4.5 of [Daskalopoulos and Mese 2006]. Uniqueness follows from [Mese 2002]. □

Let p be a vertex (i.e., 0-cell) of X. Given a 1-cell τ of X, define $\mathcal{S}_2(\tau)$ be the set of 2-cells of X containing τ in its closure.

Theorem 3.2. *If $u : X \to Y$ is a harmonic map, then for any 1-simplex τ and 2-simplex $\sigma \in \mathcal{S}_2(\tau)$ we have $u \in C^\infty(\sigma \cup \tau)$. (In other words, the restriction of u to σ is C^∞ up to τ in the extended plane of σ). Moreover, for every 1-simplex τ and $p \in \tau$ assume that u is given in a neighbourhood of $u(p)$ in local coordinates by $u = (u^1, \dots, u^M)$.*

Then,

$$\sum_{\sigma > \tau} \frac{\partial u_j^m}{\partial e_\sigma} = 0, \tag{3-1}$$

where $\sigma > \tau$ implies that τ is a face of σ, and e_σ is an inward-pointing normal vector field along τ in σ.

Proof. The fact that $u^m \in C^\infty(\sigma \cup \tau)$ follows from Theorem 4 and Corollary 6 of [Daskalopoulos and Mese 2008]. Equation (3-1), follows from Corollary 5 of the same paper. □

For an edge τ and $\sigma \in \mathcal{S}_2(\tau)$, we define polar coordinates (r, θ) of $\sigma \cup \tau$ centred at p by setting r to be the distance from p to a point $q \in \sigma \cup \tau$ and θ to be the angle between τ and the line $\overline{pq}$ connecting p and q. The next theorem is one of the main technical results of the paper and describes the singular behaviour of harmonic maps near the lower-dimensional strata.

Theorem 3.3. *Let $u : X \to Y$ be a harmonic map. If (r, θ) are the polar coordinates of $\sigma \cup \tau$ centred at a 0-cell p and u is given in local coordinates $(u^1, \dots, u^M)$ in a neighbourhood of $u(p)$, we have the following derivative bounds for u^m in a neighbourhood of p:*

$$\left|\frac{\partial u^m}{\partial r}\right| \le C r^{\alpha-1}, \quad \left|\frac{\partial u^m}{\partial \theta}\right| \le C r^{\alpha},$$

$$\left|\frac{\partial^2 u^m}{\partial r^2}\right| \le C r^{\alpha-2}, \quad \left|\frac{\partial^2 u^m}{\partial r\,\partial \theta}\right| \le C r^{\alpha-1}, \quad \left|\frac{\partial^2 u^m}{\partial \theta^2}\right| \le C r^{\alpha},$$

$$\left|\frac{\partial^3 u^m}{\partial r^3}\right| \le C r^{\alpha-3}, \quad \left|\frac{\partial^3 u^m}{\partial^2 r \partial \theta}\right| \le C r^{\alpha-2}, \quad \left|\frac{\partial^3 u^m}{\partial r\,\partial^2 \theta}\right| \le C r^{\alpha-1}, \quad \left|\frac{\partial^3 u^m}{\partial \theta^3}\right| \le C r^{\alpha}$$

for some constants $C > 0$ and $\alpha > 0$ depending on the total energy of u and the geometry of the complex X. Furthermore, α can be chosen independently of the choice of the 0-cell p of X.

Proof. Let $\sigma = \sigma_1, \dots, \sigma_J$ be the 2-cells in $\mathcal{S}_2(\tau)$. For each $j = 1, \dots, J$, we let (x, y) be the Euclidean coordinates of $\overline{\sigma_j \cup \tau}$ so that (i) p is given as $(x, y) = (0, 0)$, (ii) if $(x, y) \in \tau$ then $x > 0$ and $y = 0$ and (iii) if $(x, y) \in \sigma_j$ then $x, y > 0$. Let $u_j^m = u^m|_{\sigma_j}$.

We will now compute the first-derivative bounds with respect to the polar coordinates r and θ. By Theorem 6.2 of [Daskalopoulos and Mese 2006], we have the inequality

$$|\nabla u|^2(r,\theta) \le Cr^{2\alpha-2}$$

for some $\alpha > 0$. More specifically, α can be chosen to be the order of u at p; i.e.,

$$\alpha = \lim_{r\to 0} \frac{r\int_{B_r(p)} |\nabla u|^2\,d\mu}{\int_{\partial B_r(p)} d^2(u,u(p))\,ds}.$$

Hence,

$$\left|\frac{\partial u_j^m}{\partial x}\right| \le Cr^{\alpha-1} \quad \text{and} \quad \left|\frac{\partial u_j^m}{\partial y}\right| \le Cr^{\alpha-1}. \tag{3-2}$$

Using the fact that $x = r\cos\theta$ and $y = r\sin\theta$, we get

$$\frac{\partial u_j^m}{\partial r} = \frac{\partial u_j^m}{\partial x}\cos\theta + \frac{\partial u_j^m}{\partial y}\sin\theta \quad \text{and} \quad \frac{\partial u_j^m}{\partial \theta} = -\frac{\partial u_j^m}{\partial x} r\sin\theta + \frac{\partial u_j^m}{\partial y} r\cos\theta.$$

This immediately implies

$$\left|\frac{\partial u_j^m}{\partial r}\right| \le Cr^{\alpha-1} \quad \text{and} \quad \left|\frac{\partial u_j^m}{\partial \theta}\right| \le Cr^{\alpha}.$$

We will now establish the second derivative estimates of u_j^m for a points (r,θ) on $\sigma_j \cup \tau$ with θ sufficiently small. We will need the following notation: for a function φ and a domain $\Omega \subset \mathbb{R}^2$, we set

$$|\varphi|_{0;\Omega} = \sup_{p\in\Omega} |\varphi(p)|,$$

$$|D\varphi|_{0;\Omega} = \sup_{p\in\Omega} \max\left\{\left|\frac{\partial\varphi}{\partial x}(p)\right|, \left|\frac{\partial\varphi}{\partial y}(p)\right|\right\},$$

$$|D^2\varphi|_{0;\Omega} = \sup_{p\in\Omega} \max\left\{\left|\frac{\partial^2\varphi}{\partial x^2}(p)\right|, \left|\frac{\partial^2\varphi}{\partial x\,\partial y}(p)\right|, \left|\frac{\partial^2\varphi}{\partial y^2}(p)\right|\right\},$$

$$[\varphi]_{\beta;\Omega} = \sup_{\substack{p,q\in\Omega\\ p\neq q}} \frac{|\varphi(p)-\varphi(q)|}{|p-q|^\beta},$$

$$[D\varphi]_{\beta;\Omega} = \sup_{\substack{p,q\in\Omega\\ p\neq q}} \frac{1}{|p-q|^\beta} \max\left\{\left|\frac{\partial\varphi}{\partial x}(p)-\frac{\partial\varphi}{\partial x}(q)\right|, \left|\frac{\partial\varphi}{\partial y}(p)-\frac{\partial\varphi}{\partial y}(q)\right|\right\},$$

$$[D^2\varphi]_{\beta;\Omega} = \sup_{\substack{p,q\in\Omega\\ p\neq q}} \frac{1}{|p-q|^\beta} \max\left\{\left|\frac{\partial^2\varphi}{\partial x^2}(p)-\frac{\partial^2\varphi}{\partial x^2}(q)\right|, \left|\frac{\partial^2\varphi}{\partial x\,\partial y}(p)-\frac{\partial^2\varphi}{\partial x\,\partial y}(q)\right|, \left|\frac{\partial^2\varphi}{\partial y^2}(p)-\frac{\partial^2\varphi}{\partial y^2}(q)\right|\right\}.$$

Let

$$
\begin{aligned}
T &:= \{(x,y) \in \mathbb{R}^2 : y \geq 0,\ y < \sqrt{3}x,\ y < -\sqrt{3}x + \sqrt{3}\},\\
T^- &= \{(x,-y) \in \mathbb{R}^2 : (x,y) \in T\},\\
\widehat{T} &= T \cup T^-.
\end{aligned}
$$

Fix m and j and define $U : \widehat{T} \to \mathbb{R}$ by setting

$$
U(x,y) = \begin{cases} u_j^m(x,y) & \text{if } y \geq 0,\\ -u_j^m(x,-y) + (2/J)\sum_{j'=1}^J u_{j'}^m(x,-y) & \text{if } y < 0. \end{cases}
$$

Let

$$
\Gamma_j^m = \sum_{p,q=1}^M \Gamma_{pq}^m(u_j)\left(\frac{\partial u_j^p}{\partial x}\frac{\partial u_j^q}{\partial x} + \frac{\partial u_j^p}{\partial y}\frac{\partial u_j^q}{\partial y}\right), \tag{3-3}
$$

where Γ_{pq}^m are the Christoffel symbols of Y with respect to the local coordinates $(u^1,\dots,u^M)$. Since the harmonic map equation

$$
\triangle u_j^m = \Gamma_j^m
$$

is satisfied in T, if we set

$$
f(x,y) = \begin{cases} \Gamma_j^m(x,y) & \text{if } y \geq 0,\\ -\Gamma_j^m(x,-y) + (2/J)\sum_{j'=1}^J \Gamma_{j'}^m(x,-y) & \text{if } y < 0, \end{cases}
$$

then U satisfies the Poisson equation

$$
\triangle U = f \tag{3-4}
$$

weakly in $\widehat{T}$. Indeed, let ξ be a test function supported in a neighbourhood $B_R(q)$ of a point $q = (x_0, 0) \in \widehat{T}$. Since U is a C^1 function we have by the divergence theorem,

$$
\begin{aligned}
&\int_{\widehat{T}} \operatorname{div}(\xi \nabla U)\,dx\,dy\\
&= \int_T \operatorname{div}(\xi \nabla U)\,dx\,dy + \int_{T^-} \operatorname{div}(\xi \nabla U)\,dx\,dy\\
&= -\int_{x_0-R}^{x_0+R} \xi \frac{\partial u_j^m}{\partial y}(x,0)\,dx + \int_{x_0-R}^{x_0+R}\left(\xi \frac{\partial u_j^m}{\partial y}(x,0) - \frac{2}{J}\sum_{j'=1}^J \xi \frac{\partial u_j^m}{\partial y}(x,0)\right)dx = 0,
\end{aligned}
$$

where the last equality is because of (3-1). On the other hand,

$$
\int_{\widehat{T}} \operatorname{div}(\xi \nabla U)\,dx\,dy = \int_{\widehat{T}} \nabla \xi \cdot \nabla U + \int_{\widehat{T}} \xi f\,dx\,dy,
$$

which along with the previous equation implies (3-4). If $B_{2R}(q) \subset \widehat{T}$, then elliptic regularity theory, see [Gilbarg and Trudinger 1983; Simon 1996, Lemma 3, p. 13], implies

$$R^{1+\beta}[DU]_{\beta;B_{3R/2}(q)} \le C(|U|_{0;B_{2R}(q)} + R^2|f|_{0;B_{2R}(q)}).$$

If we choose R to be the largest number so that $B_{2R}(q) \subset \widehat{T}$, then R is proportional to r, where r is the distance of q to the vertex p. Furthermore, the distance from p to any point of $B_{2R}(q)$ is bounded uniformly by some constant multiple of r. Hence, assuming $U(0,0)=0$ without a loss of generality, we have

$$\begin{aligned}[DU]_{\beta;B_{3R/2}(p)} &\le C(r^{-1-\beta}|U|_{0;B_{2R}(p)} + r^{1-\beta}|f|_{0;B_{2R}(p)})\\ &\le C(r^{-1-\beta+\alpha} + r^{-\beta+2\alpha-1}) \le Cr^{-\beta+\alpha-1}.\end{aligned}$$

Here, we have used the Hölder continuity of u_j^m (hence of U) near p with Hölder exponent α, see Theorem 3.7 of [Daskalopoulos and Mese 2006], and the inequalities of (3-2) along with the fact that f is quadratic in Du_j^m from (3-3). Thus, with $B^+_{3R/2}(q) = B_{3R/2}(q) \cap \{y \ge 0\}$, we obtain

$$[Du_j^m]_{\beta;B^+_{3R/2}(q)} \le Cr^{-\beta+\alpha-1}.$$

This equation along with (3-2) and (3-3) implies that

$$[\Gamma_j^m]_{\beta;B^+_{3R/2}(q)} \le C|Du_j^k|_{0;B^+_{3R/2}(q)}[Du_j^\ell]_{\beta;B^+_{3R/2}(q)} \le Cr^{-\beta+2\alpha-2}. \tag{3-5}$$

We are now ready to prove the second-derivative bounds of u_j^m. Note that we have the set of partial differential equations

$$\triangle u_j^m = \Gamma_j^m, \quad j=1,\dots,J, \quad m=1,\dots M, \tag{3-6}$$

in T, along with boundary conditions

$$u_j^m - u_1^m = 0, \quad j=2,\dots,J, \quad m=1,\dots,M, \tag{3-7}$$

$$\sum_{j=1}^{J} \frac{\partial u_j^m}{\partial y} = 0, \quad m=1,\dots,M, \tag{3-8}$$

in $B=\{(x,y)\in\mathbb{R}^2 : y=0,\ 0<x<1\}$. This is a system of JM equations containing JM unknowns (i.e., u_j^m) along with JM boundary conditions. If we assign weights $s_j^m = 0$ to the equations, weights $t_j^m = 2$ to the unknowns, weights $r_j^m = -2$ for $j=2,\dots,M$ and $r_1^m = -1$ to the boundary conditions, then this system is said to be elliptic with complementing boundary condition according to the elliptic regularity theory of [Agmon et al. 1964] (or elliptic and coercive in [Kinderlehrer et al. 1978]). Hence, we have the Schauder estimates, see Theorem 9.1 of [Agmon et al. 1964],

$$\begin{aligned}&R^2|D^2u_j^m|_{0;B_R^+(q)} + R^{2+\beta}[D^2u_j^m]_{\beta;B_R^+(q)}\\ &\qquad\le C\big(|\Gamma_j^m|_{0;B^+_{3R/2}(q)} + R^{2+\beta}[\Gamma_j^m]_{\beta;B^+_{3R/2}(q)} + |u_j^m|_{0;B^+_{3R/2}(q)}\big).\end{aligned}$$

With the same choice of q and R as above, we obtain

$$|D^2 u_j^m|_{0;B_R^+(q)} \le C\big(|\Gamma_j^m|_{0;B^+_{3R/2}(q)} + r^\beta[\Gamma_j^m]_{\beta;B^+_{3R/2}(q)} + r^{-2}|u_j^m|_{0;B^+_{3R/2}(q)}\big).$$

The above inequality, along with (3-5), implies

$$|D^2 u_j^m|_{0;B_R^+(q)} \le C(r^{2\alpha-2} + r^{2\alpha-2} + r^{\alpha-2}) \le Cr^{\alpha-2}.$$

Since

$$\frac{\partial^2 u_j^m}{\partial r^2} = \frac{\partial^2 u_j^m}{\partial x^2}\cos^2\theta + 2\frac{\partial^2 u_j^m}{\partial x\,\partial y}\sin\theta\cos\theta + \frac{\partial^2 u_j^m}{\partial y^2}\sin^2\theta,$$

$$\begin{aligned}\frac{\partial^2 u_j^m}{\partial r\,\partial\theta} = -\frac{\partial^2 u_j^m}{\partial x^2} r\sin\theta\cos\theta + \frac{\partial^2 u_j^m}{\partial x\,\partial y} r\cos^2\theta - \frac{\partial u_j^m}{\partial x}\sin\theta - \frac{\partial^2 u_j^m}{\partial x\,\partial y} r\sin^2\theta \\ + \frac{\partial^2 u_j^m}{\partial y^2} r\sin\theta\cos\theta + \frac{\partial u_j^m}{\partial y}\cos\theta,\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 u_j^m}{\partial \theta^2} &= \frac{\partial^2 u_j^m}{\partial x^2} r^2\sin^2\theta + 2\frac{\partial^2 u_j^m}{\partial x\,\partial y} r^2\sin^2\theta + \frac{\partial^2 u_j^m}{\partial y^2} r^2\cos\theta \\ &= -\frac{\partial u_j^m}{\partial x} r\cos\theta - \frac{\partial u_j^m}{\partial y} r\sin\theta,\end{aligned}$$

we immediately obtain

$$\left|\frac{\partial^2 u_j^m}{\partial r^2}\right| \le Cr^{\alpha-2}, \quad \left|\frac{\partial^2 u_j^m}{\partial r\,\partial\theta}\right| \le Cr^{\alpha-1} \quad\text{and}\quad \left|\frac{\partial^2 u_j^m}{\partial\theta^2}\right| \le Cr^{\alpha}$$

at (r,θ) for θ sufficiently small. This restriction on θ is due to the choice of R and q. For (r,θ) with θ sufficiently large, we can use a similar argument using standard elliptic regularity theory, see, e.g., [Gilbarg and Trudinger 1983; Simon 1996, Lemma 3, p. 13], in the interior of σ. The third-derivative estimates follow the same way as the first two by bootstrapping the elliptic equations (3-6) with boundary conditions (3-7) and (3-8).

Section 4 of [Daskalopoulos and Mese 2008] shows the that order of u at p can be bounded from below by $2\lambda_v^{\text{comb}}$ where λ_v^{comb} is the combinatorial eigenvalue of the link of v, which is always a positive quantity. Hence choosing α to be the minimum of $2\lambda_v^{\text{comb}}$ over all 0-cells of X, we have established the last assertion of the Theorem. □

3B. *Weighted Sobolev spaces.* In this subsection we recall the important features of the weighted Sobolev spaces used in this paper. The main references are [Adams 1975; Daskalopoulos and Wentworth 1997; Lockhart and McOwen 1985]. In the following we fix a smooth vector bundle E of rank r over a 2-complex X with a Hermitian metric, and a fixed Riemannian metric on the base space X. Define the space $C_0^\infty(E)$ to be the space of smooth sections $s \in \Omega^0(X, E)$ that satisfy $s(p) = 0$ whenever p is a vertex of X. In the local model $\widetilde{B}(r)$ around each vertex p, we define local coordinates $(t, \theta) = (-\log r, \theta)$, where (r, θ) are the standard polar coordinates in a neighbourhood of the vertex p. To define a norm on $C_0^\infty(E)$, let $\{x_i\}_{i=1,\dots,V}$ denote the vertices of X and choose disjoint open neighbourhoods U_{x_i} for each vertex x_i. Then cover the rest of X with open sets $\{V_\alpha\}_{\alpha=1,\dots,K}$ that do not contain any of the vertices. For $\delta \in \mathbb{R}$, the space L_δ^p is the completion of $C_0^\infty(E)$ in the norm

$$\|s\|_{L_\delta^p} = \left(\sum_{i=1}^{V} \int_{U_{x_i}} e^{t\delta} |s|^p + \sum_{\alpha=1}^{K} \int_{V_\alpha} |s|^p \right)^{1/p}, \tag{3-9}$$

where we use $e^{t\delta}$ to denote the coordinates in a neighbourhood of a vertex. Away from all of the vertices, $e^{t\delta}$ is bounded and s is continuous, and so the question of whether the norm $\|\cdot\|_{L_\delta^p}$ is finite only depends on the choice of coordinates near each vertex. Different choices of V_α will lead to equivalent norms.

Given a vertex p and a trivialization of E near p, we say that a connection is *trivial* in a neighbourhood of p if with respect to the above trivialization $\nabla = d$. Given a fixed connection ∇_0 trivial near the vertices, and a positive integer k, we define the *weighted Sobolev space* $L_{k,\delta}^q(E)$ as the completion of $C_0^\infty(E)$ in the norm

$$\|s\|_{L_{k,\delta}^q} = \sum_{\ell=0}^{k} \|\nabla_0^\ell s\|_{L_\delta^q}. \tag{3-10}$$

Note that in this paper we are considering bundles with a fixed trivialization on the universal cover (see Remark 2.31). Since the star of a vertex p in X is simply connected it follows that we have a fixed trivialization of E in a neighbourhood of p. It thus makes sense to talk about connections on E trivial near the vertices.

It is a standard fact that the spaces $L_{k,\delta}^q$ do not change if we either (a) change the connection ∇_0 outside a neighbourhood of the vertices of X, or (b) change the coordinates outside a neighbourhood of the vertices. The usual multiplication theorems for Sobolev spaces on compact manifolds carry over to the weighted Sobolev spaces studied here. To be more precise, we have that the multiplication map $L_{s_1,\delta_1}^2 \times L_{s_2,\delta_2}^2 \to L_{s,\delta}^2$ is continuous if $s_1, s_2 \geq s$, $s < s_1 + s_2 - \frac{n}{2}$ and $\delta < \delta_1 + \delta_2 + \frac{n}{2}$, where n is the dimension of the complex X.

Following Section 3.1 of [Daskalopoulos and Wentworth 1997] we define the space of weighted connections $\mathcal{A}_\delta^{\mathbb{C}}(E)$ to be the space of all connections whose connection form is an element of $L_{1,\delta}^2$, and the space $\mathcal{A}_\delta(E) \subset \mathcal{A}_\delta^{\mathbb{C}}(E)$ to be the subset of all unitary connections. The weighted gauge group $\mathcal{G}_\delta(E)$ is defined as follows. Let ∇_0 be a connection as above and define

$$\mathcal{R} = \{v \in L_{2,\mathrm{loc}}^2(\mathrm{End}(E)) : \|\nabla_0 v\|_{L_{1,\delta}^2} < \infty\}. \tag{3-11}$$

Then the *weighted gauge group* is defined as

$$\mathcal{G}_\delta(E) = \{v \in \mathcal{R} : vv^* = \mathrm{id},\ \det v = 1\} \tag{3-12}$$

and the complexified gauge group is

$$\mathcal{G}_\delta^{\mathbb{C}}(E) = \{v \in \mathcal{R} : \det v = 1\}. \tag{3-13}$$

The multiplication theorem for weighted Sobolev spaces shows that both $\mathcal{G}_\delta(E)$ and $\mathcal{G}_\delta^{\mathbb{C}}(E)$ have a group structure, and that there are well-defined actions of $\mathcal{G}_\delta(E)$ on $\mathcal{A}_\delta$ and $\mathcal{G}_\delta^{\mathbb{C}}(E)$ on $\mathcal{A}_\delta^{\mathbb{C}}(E)$ respectively.

Similarly we have balanced versions of these spaces $\mathcal{G}_{\mathrm{bal},\delta}(E)$, $\mathcal{A}_{\mathrm{bal},\delta}(E)$ and $\Omega^1_{\mathrm{bal},\delta}(\mathrm{ad}(E))$. When a smooth pair $(d_A, \psi) \in \mathcal{A}_{\mathrm{bal},\delta}(E) \times \Omega^1_{\mathrm{bal},\delta}(\mathrm{ad}(E))$ solves (2-10) and (2-11), then the *holonomy of the pair* (d_A, ψ) refers to the holonomy of the flat connection $d_A + \psi \in \mathcal{A}^{\mathbb{C}}_{\mathrm{bal,flat},\delta}(E)$.

Proposition 3.4. *If $D_i \in \mathcal{A}^{\mathbb{C}}_{\mathrm{bal,flat},\delta}(E)$, $i = 1, 2$, are smooth and $\mathcal{G}^{\mathbb{C}}_{\mathrm{bal},\delta}(E)$-gauge-equivalent then they are $\mathcal{G}^{\mathbb{C}}_{\mathrm{bal}}(E)$-gauge-equivalent.*

Proof. Since the result is local, it follows by elliptic regularity. □

Proposition 3.5. *Let $D \in \mathcal{A}^{\mathbb{C}}_{\mathrm{bal,flat},\delta}(E)$ be smooth. Then D has trivial holonomy around the vertices of X.*

Proof. For $D = d + A$ write $A(t,\theta) = B(t,\theta)\,dt + C(t,\theta)\,d\theta$. Consider the family of loops $c_t : [0, 2\pi] \to X$ given by $c_t(\theta) = (t,\theta)$ and consider the holonomy equation from Definition 2.18 along $c_t(\theta)$

$$\frac{ds_t(\theta)}{d\theta} + C(t,\theta)s_t(\theta) = 0 \quad \text{with } s_t(0) = \mathrm{id}. \tag{3-14}$$

Lemma IV.4.1 on p. 54 of [Hartman 1964] implies

$$|s_t(\theta)| \le |s_t(0)| \exp\left\{\int_0^\theta |C(t,\theta)|\,d\theta\right\} \le K \exp\left\{\int_0^{2\pi} |C(t,\theta)|\,d\theta\right\}, \tag{3-15}$$

where K is a dimensional constant. Since

$$\int_0^\infty e^{t\delta} \int_0^{2\pi} |C(t,\theta)|^2\,d\theta\,dt < \infty,$$

there exists a sequence $t_i \to \infty$ such that $\int_0^{2\pi} |C(t_i,\theta)|^2\,d\theta \to 0$. By Cauchy–Schwarz we also have

$$\int_0^{2\pi} |C(t_i,\theta)|\,d\theta \to 0. \tag{3-16}$$

Combined with (3-15) this implies that $|s_{t_i}(\theta)|$ is uniformly bounded. By integrating (3-14) with respect to θ, we obtain from (3-16)

$$(3\text{-}17)\qquad |s_{t_i}(2\pi) - s_{t_i}(0)| \le \int_0^{2\pi} |s_{t_i}(\theta)||C(t_i,\theta)|\, d\theta \to 0.$$

Since the holonomy is independent of t we obtain that $s_{t_i}(2\pi) = s_{t_i}(0)$ and thus it must be trivial. □

Propositions 3.4 and 3.5 allow us to define the notion of conjugacy class of holonomy for a smooth flat connection $D \in \mathcal{A}^{\mathbb{C},\mathrm{irr}}_{\mathrm{bal,flat},\delta}(E)$ as follows.

Definition 3.6. Let $D \in \mathcal{A}^{\mathbb{C},\mathrm{irr}}_{\mathrm{bal,flat},\delta}(E)$ be a smooth flat connection and let $\rho_* : \pi_1(X_*) \to \mathrm{SL}(r,\mathbb{C})$ be the holonomy of D, where $X_* = X\backslash X^0$ and X^0 denotes the 0-skeleton of X. Since the star of a vertex is contractible, Van Kampen's theorem implies that $\pi_1(X) = \pi_1(X_*)/\pi$, where π denotes the subgroup of $\pi_1(X_*)$ generated by $\bigcup_{p\in X^0} \pi_1(Lk(p))$. By Proposition 3.5, the restriction of ρ_* to π is trivial; hence it induces a homomorphism $\rho : \pi_1(X) \to \mathrm{SL}(r,\mathbb{C})$. We say that the conjugacy class of holonomy of D is $[\rho]$. Notice that the map is well-defined since gauge-equivalent pairs yield conjugate holonomies. Furthermore, ρ is irreducible because D is irreducible.

4. Equivalence of moduli spaces

4A. *Higgs moduli space.* We fix a vector bundle $E_c = E$ of rank r over a 2-complex X with a Hermitian metric, and a fixed Riemannian metric on the base space X.

Definition 4.1. The *Higgs moduli space* is the space $\mathcal{M}_{\mathrm{Higgs}}(E)$ of $\mathcal{G}_{\mathrm{bal},\delta}(E)$-equivalence classes of pairs $(d_A,\psi) \in \mathcal{A}_{\mathrm{bal},\delta}(E) \times \Omega^1_{\mathrm{bal},\delta}(\sqrt{-1}\,\mathrm{ad}(E))$ that are *smooth, irreducible* and solve the equations

$$(4\text{-}1)\qquad F_A + \psi\wedge\psi = 0,$$

$$(4\text{-}2)\qquad d_A\psi = 0,$$

$$(4\text{-}3)\qquad d_A^*\psi = 0.$$

We endow $\mathcal{M}_{\mathrm{Higgs}}(E)$ with the $L^2_{1,\delta}$-topology.

Given $[(d_A,\psi)] \in \mathcal{M}_{\mathrm{Higgs}}(E)$, we can assign by Definition 3.6 the holonomy $[\rho]$ of the flat connection $d_A+\psi$ and set $\alpha[(d_A,\psi)] := [\rho]$. The map α is well-defined. The next proposition follows from continuous dependence of solutions of ODE upon the initial condition.

Proposition 4.2. *The map $\alpha : \mathcal{M}_{\mathrm{Higgs}}(E) \to \mathcal{M}^c_{\mathrm{char}}$, where $\alpha[(d_A,\psi)] = [\rho]$, is well-defined and continuous.*

The following is the main theorem of this paper.

Theorem 4.3. *The map* $\alpha : \mathcal{M}_{\mathrm{Higgs}}(E) \to \mathcal{M}^c_{\mathrm{char}}$ *is a homeomorphism.*

In the next section we will construct the inverse map. We end this section with a proposition that will be used later.

Proposition 4.4. *Let* (d_{A_1}, ψ_1) *and* (d_{A_2}, ψ_2) *be solutions to* (4-1)–(4-3) *and assume that they are* $\mathcal{G}^{\mathbb{C}}_{\mathrm{bal},\delta}(E)$*-gauge-equivalent. Then they are* $\mathcal{G}_{\mathrm{bal},\delta}(E)$*-gauge-equivalent.*

Proof. Assume that there exists $g \in \mathcal{G}^{\mathbb{C}}_{\mathrm{bal},\delta}(E)$ such that $(d_{A_1}, \psi_1) = g \cdot (d_{A_2}, \psi_2)$, and we have to show that g is unitary. Let $h = g^* g$ and we will show that h is constant. By [Simpson 1988, Lemma 3.1(d)] we have the following pointwise estimate away from the vertices (notice that the sign of our Laplacian is the opposite from Simpson's):

$$\Delta \operatorname{tr}(h) \le 0. \tag{4-4}$$

Now since g is balanced, so is $\operatorname{tr} h$, and therefore an application of Stokes' theorem on each face of X shows that

$$\begin{aligned} \int_X \Delta \operatorname{tr} h \, dx &= \lim_{r\to 0} \int_{X \setminus \bigcup_{\text{0-cells } v} B_r(v)} \Delta \operatorname{tr} h \, dx \\ &= \lim_{r\to 0} \sum_{\text{2-cells } \sigma} \int_{F \setminus \bigcup_{\text{0-cells } v} B_r(v)} \Delta \operatorname{tr} h \, dx \\ &= \lim_{r\to 0} \sum_{\text{2-cells } \sigma} \int_{\partial (F \setminus \bigcup_{\text{0-cells } v} B_r(v))} \frac{\partial \operatorname{tr} h}{\partial \nu} \, ds, \end{aligned} \tag{4-5}$$

where ν is the outward-pointing normal vector on $\partial\big(\sigma \setminus \bigcup_{\text{0-cells } v} B_r(v)\big)$. The boundary $\partial\big(\sigma \setminus \bigcup_{\text{vertices } v} B_\sigma(v)\big)$ consists of points on the 1-cells of σ, and points on $\partial B_r(v) \cap \sigma$. Breaking the integral into these two parts, we obtain

$$\begin{aligned} &\sum_{\text{2-cells}\,\sigma} \int_{\partial(\sigma \setminus \bigcup_{\text{0-cells } v} B_r(v))} \frac{\partial \operatorname{tr} h}{\partial \nu} \, ds \\ &= \sum_{\text{2-cells}\,\sigma} \Bigg(\sum_{\text{1-cells } \tau : \tau \cap \bar\sigma \neq \varnothing} \int_{\tau \setminus \bigcup_v B_r(v) \cap \tau} \frac{\partial \operatorname{tr} h}{\partial \nu} \, ds \Bigg) + \sum_{\text{2-cells}\,\sigma} \int_{\bigcup_v \partial B_r(v) \cap \sigma} \frac{\partial \operatorname{tr} h}{\partial \nu} \, ds. \end{aligned} \tag{4-6}$$

The balancing condition shows that the first term is zero. Therefore we are left with

$$\int_X \Delta \operatorname{tr} h \, dx = \lim_{r\to 0} \sum_{\text{2-cells } \sigma} \int_{\bigcup_v \partial B_r(v) \cap \sigma} \frac{\partial \operatorname{tr} h}{\partial \nu} \, ds. \tag{4-7}$$

In polar coordinates, each component of this integral becomes

$$\int_{\partial B_r(v) \cap F} \frac{\partial \operatorname{tr} h}{\partial \nu} \, ds = r \int_0^{\frac{\pi}{3}} \frac{\partial \operatorname{tr} h}{\partial r} \, d\theta. \tag{4-8}$$

Since $h \in \mathcal{G}(E)^{\mathbb{C}}_{\text{bal},\delta}$ (and in particular, the integral of $\partial^2 h/\partial r^2$ is bounded), we have

$$\lim_{r\to 0}\left(\sigma \int_0^{\frac{\pi}{3}} \operatorname{tr}\left(\frac{\partial h}{\partial r}\right) d\theta\right) = 0 \tag{4-9}$$

and so (4-7) becomes

$$\int_X \Delta \operatorname{tr} h \, dx = 0. \tag{4-10}$$

Combined with $\Delta \operatorname{tr} h \le 0$ from (4-4), we see that $\Delta \operatorname{tr} h = 0$. The second-to-the-last formula in [Simpson 1988, p. 876] implies that $D(h) = 0$ pointwise away from the vertices. This implies that the connection D splits according to the eigenspaces of h, and since the connection D is indecomposable, h must be a constant multiple of the identity matrix, which concludes the proof. □

4B. *The inverse map.* For an irreducible representation $\rho : \pi_1(X) \to \mathrm{SL}(r, \mathbb{C})$, with $[\rho] \in \mathcal{M}^c_{\text{char}}$ and $E = E_c$, Theorem 3.3 then shows that there exists a unique ρ-equivariant harmonic map $u : \widetilde{X} \to \mathrm{SL}(r, \mathbb{C})/\mathrm{SU}(r)$. As in Section 2B, let d_A and ψ be the associated unitary connection and Higgs field. Since u is harmonic, d_A is the pullback of the Levi-Civita connection on $\mathrm{SL}(r, \mathbb{C})/\mathrm{SU}(r)$, and ψ is the derivative of u, we also have the equation

$$d_A^* \psi = 0 \tag{4-11}$$

almost everywhere (in fact by Theorem 3.3 everywhere away from the 0-skeleton).

Proposition 4.5. *If u is harmonic, α is as in Theorem 3.3 and $\delta < \alpha$, then $\mathcal{D} \in \mathcal{A}^{\mathbb{C}}_{\text{bal, flat},\delta}(E)$. The metric on the bundle E induces a decomposition of $\mathcal{D}$ into skew-adjoint and self-adjoint parts, $\mathcal{D} = d_A + \psi$, where $d_A \in \mathcal{A}_{\text{bal},\delta}(E)$ and $\psi \in \Omega^1_{\text{bal},\delta}(i\,\mathrm{ad}(E))$. Furthermore, $\mathcal{D}$, d_A and ψ are smooth (over X_*).*

Proof. The construction in Section 2B shows that the connection $\mathcal{D}$ is induced from the trivial connection on the universal cover; hence it is clearly balanced, flat and $L^2_{1,\delta}$. Furthermore, since $d_A = u^*\nabla$ and $\psi = u^{-1} du$, Theorem 3.3 and (2-9) imply that d_A and ψ are balanced. Therefore, since $u : X \to \mathrm{SL}(r, \mathbb{C})/\mathrm{SU}(r)$ is a Lipschitz map over the compact space X, in order to show $d_A \in \mathcal{A}_{\text{bal},\delta}(E)$ and $\psi \in \Omega^1_{\text{bal}}(i\,\mathrm{ad}(E))_\delta$, it suffices to show $du \in L^2_{1,\delta}$.

First we show $du \in L^2_\delta$. By Theorem 3.3, $|\partial u/\partial r| \le C r^{\alpha-1}$ and $|\partial u/\partial \theta| \le C r^{\alpha}$ for some positive α. Using the coordinate transformation $r = e^{-t}$ we see that $|\partial u/\partial \theta| \le C e^{-\alpha t}$ and

$$\left|\frac{\partial u}{\partial t}\right| = \left|\frac{\partial u}{\partial r}\frac{dr}{dt}\right| \le C r^{\alpha-1} r = C e^{-\alpha t}.$$

Therefore, $du \in L^2_\delta$ if $\delta < \alpha$. Similarly, we use the estimates on the second derivatives of u to show that $du \in L^2_{1,\delta}$. We have $|\partial^2 u/\partial \theta^2| \le C e^{-\alpha t}$, and we can compute

$$\left|\frac{\partial^2 u}{\partial t\,\partial\theta}\right| = \left|\frac{\partial^2 u}{\partial r\,\partial\theta}\frac{dr}{dt}\right| \leq Cr^{\alpha-1}r = Ce^{-\alpha t}$$

and similarly

$$\left|\frac{\partial^2 u}{\partial t^2}\right| = \left|\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial r}\frac{dr}{dt}\right)\right| = \left|\frac{\partial^2 u}{\partial r^2}\left(\frac{dr}{dt}\right)^2 + \frac{\partial u}{\partial t}\left(\frac{d}{dr}\frac{dr}{dt}\right)\frac{dr}{dt}\right|$$
$$\leq Cr^{\alpha-2}r^2 + Cr^{\alpha-1}r^2 \leq Ce^{-\alpha t} + Ce^{-(\alpha+1)t} \leq Ce^{-\alpha t},$$

where in the last step we use the fact that $t \geq 0$ near a vertex. Therefore, $du \in L^2_{1,\delta}$ if $\delta < \alpha$. □

Theorem 4.6. *The map $\beta : \mathcal{M}^c_{\text{char}} \to \mathcal{M}_{\text{Higgs}}(E)$ defined by $\beta([\rho]) = [(d_A, \psi)]$ is a continuous inverse of α.*

Proof. The first step is to show the map β is well-defined. Given ρ, Proposition 4.5 implies that $d_A \in \mathcal{A}_{\text{bal},\delta}(E)$ and $\psi \in \Omega_{\text{bal}^1,\delta}(i\,\text{ad}(E))$. Moreover, we claim that the pair (d_A, ψ) is irreducible. If $\rho_* : \pi_1(X_*) \to \text{SL}(r, \mathbb{C})$ denotes the holonomy of the flat connection $d_A + \psi$ then, as pointed out in Definition 3.6, $\rho_* = \rho \circ p$, where $p : \pi_1(X_*) \to \pi_1(X) = \pi_1(X_*)/\pi$ is the natural quotient map. Since by assumption ρ is irreducible, it follows that ρ_* is also irreducible, proving our claim.

Now, let ρ and $\rho' = \gamma\rho\gamma^{-1}$ be two representatives of $[\rho]$ and let u and u' be the two corresponding equivariant harmonic maps. It follows that $u' = \gamma \cdot u$, where $\cdot$ denotes the action of $\text{SL}(r, \mathbb{C})$ on $\text{SL}(r, \mathbb{C})/\text{SU}(r)$. It follows that the induced decompositions $\mathcal{D} = d_A + \psi$ on the universal cover agree; hence after taking the quotients by ρ and $\rho' = \gamma\rho\gamma^{-1}$ respectively, the corresponding pairs are complex gauge-equivalent by γ. Proposition 4.4 then shows that they are $\mathcal{G}_{\text{bal},\delta}$-gauge-equivalent, which completes the proof that β is well-defined.

Next we will show that $\alpha(\beta([\rho])) = [\rho]$. Let $\beta([\rho]) = [(d_A, \psi)]$. According to (2-9), we have $d_A + \psi = \mathcal{D}$, where $\mathcal{D}$ is the connection on $\text{ad}(E_\rho)$ induced by the trivial connection on the universal cover which has holonomy ρ. Hence, $\alpha(\beta([\rho])) = [\rho]$.

Conversely, $\beta(\alpha([(d_A, \psi)])) = [(d_A, \psi)]$. Indeed, let (d_B, ϕ) be a smooth representative of $\beta(\alpha([(d_A, \psi)]))$. By applying α on both sides and what we just proved, $\alpha([(d_A, \psi)]) = \alpha([(d_B, \phi)])$. In other words, (d_A, ψ) and (d_B, ϕ) have conjugate holonomies. Since the holonomies of these pairs near the vertices are trivial by Proposition 3.5, Proposition 2.26 implies that the corresponding flat connections (and hence also the pairs) are complex gauge-equivalent. Thus Proposition 4.4 implies that (d_A, ψ) and (d_B, ϕ) are $\mathcal{G}_{\text{bal},\delta}$-gauge-equivalent; hence $\beta(\alpha([(d_A, \psi)])) = [(d_A, \psi)]$.

In order to prove continuity, let $\rho_i \to \rho \in \mathcal{M}^c_{\text{char}}$ and let u_i, u be the associated equivariant harmonic maps. Fix a compact fundamental domain $F \subset \widetilde{X}$ for the action of Γ and define ρ_i-equivariant maps $\tilde{u}_i$ by setting $\tilde{u}_i = u$ on F and extending ρ_i equivariantly on $\widetilde{X}$. Since the u_i are harmonic, the energy E^{u_i} satisfies

$$E^{u_i} \leq E^{\tilde{u}_i} = E^u.$$

The global Hölder bound, see [Daskalopoulos and Mese 2006, Theorem 3.12], implies that there is a subsequence (we call it again by $\{i\}$ by a slight abuse of notation) such that $u_i \to u_\infty$ uniformly on F. Furthermore, the convergence of the representations $\rho_i \to \rho$ implies that u_∞ is ρ-equivariant and Theorem 5.1 of [Daskalopoulos and Mese 2006] implies that u_∞ is harmonic. Finally, the uniqueness theorem, Theorem 4.6 of the same paper, implies that $u_\infty = u$. We have thus shown so far

$$u_i \to u \quad \text{locally uniformly.}$$

Let (d_{A_i}, ψ_i) denote the unitary connection and Higgs field associated with the harmonic map u_i. By Theorem 3.3 together with the proof of Proposition 4.5 (in this we use the third-derivative estimates) we obtain that the $L^2_{2,\delta}$-norm of (A_i, ψ_i) is uniformly bounded, and thus there exists a subsequence (we call it again by $\{i\}$ by a slight abuse of notation) such that $(d_{A_i}, \psi_i) \to (d_A, \psi)$ weakly in $L^2_{2,\delta}$ and hence strongly in $L^2_{1,\delta}$. □

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GEORGIOS DASKALOPOULOS
DEPARTMENT OF MATHEMATICS
BROWN UNIVERSITY
PROVIDENCE, RI
UNITED STATES
daskal@math.brown.edu

CHIKAKO MESE
DEPARTMENT OF MATHEMATICS
JOHNS HOPKINS UNIVERSITY
BALTIMORE, MD
UNITED STATES
cmese@math.jhu.edu

GRAEME WILKIN
DEPARTMENT OF MATHEMATICS
NATIONAL UNIVERSITY OF SINGAPORE
SINGAPORE
graeme@nus.edu.sg

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BESOV-WEAK-HERZ SPACES AND GLOBAL SOLUTIONS FOR NAVIER–STOKES EQUATIONS

LUCAS C. F. FERREIRA AND JHEAN E. PÉREZ-LÓPEZ

We consider the incompressible Navier-Stokes equations (NS) in $\mathbb{R}^n$ for $n \geq 2$. Global well-posedness is proved in critical Besov-weak-Herz spaces (BWH-spaces) that consist in Besov spaces based on weak-Herz spaces. These spaces are larger than some critical spaces considered in previous works for NS. For our purposes, we need to develop a basic theory for BWH-spaces containing properties and estimates such as heat semigroup estimates, embedding theorems, interpolation properties, among others. In particular, we prove a characterization of Besov-weak-Herz spaces as interpolation of Sobolev-weak-Herz ones, which is key in our arguments. Self-similarity and asymptotic behavior of solutions are also discussed. Our class of spaces and its properties developed here could also be employed to study other PDEs of elliptic, parabolic and conservation-law type.

1. Introduction

This paper is concerned with the incompressible Navier–Stokes equations

$$\begin{cases} \frac{\partial u}{\partial t} - \Delta u + u \cdot \nabla u + \nabla \rho = 0 & \text{in } \mathbb{R}^n \times (0, \infty), \\ \nabla \cdot u = 0 & \text{in } \mathbb{R}^n \times (0, \infty), \\ u(0) = u_0 & \text{in } \mathbb{R}^n, \end{cases} \tag{1-1}$$

where $n \geq 2$, ρ is the pressure, $u = (u_j)_{j=1}^n$ is the velocity field and u_0 is a given initial velocity satisfying $\nabla \cdot u_0 = 0$.

After applying the Leray–Hopf projector $\mathbb{P}$ and using Duhamel's principle, the Cauchy problem (1-1) can be reduced to the integral formulation

$$u(t) = G(t)u_0 - \int_0^t G(t-\tau)\mathbb{P}\,\mathrm{div}(u \otimes u)(\tau)d\tau := G(t)u_0 + B(u,u)(t), \tag{1-2}$$

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where $u \otimes v := (u_i v_j)_{1\le i,j\le n}$ is a matrix-valued function and $G(t) = e^{t\Delta}$ is the heat semigroup. The operator $\mathbb{P}$ can be expressed as $\mathbb{P} = (\mathbb{P}_{i,j})_{n\times n}$ where $\mathbb{P}_{i,j} := \delta_{i,j} + \mathcal{R}_i\mathcal{R}_j$, $\delta_{i,j}$ is the Kronecker delta and $\mathcal{R}_i = (-\Delta)^{-1/2}\partial_i$ is the i-th Riesz transform. Divergence-free solutions for (1-2) are called mild solutions for (1-1). Note that if u is a smooth solution for (1-1) (or (1-2)), then

$$u_\lambda(x,t) := \lambda u(\lambda x, \lambda^2 t) \tag{1-3}$$

is also a solution with initial data

$$(u_0)_\lambda(x) = \lambda u_0(\lambda x). \tag{1-4}$$

Recall that given a Banach space Y we say that it has scaling degree equal to $k \in \mathbb{R}$ if $\|f(\lambda x)\|_Y \approx \lambda^k \|f\|_Y$ for all $\lambda > 0$ and $f \in Y$. Motivated by (1-4), a Banach space Y is called critical for (1-1) if it has scaling degree equal to -1, that is, if $\|f\|_Y \approx \|\lambda f(\lambda x)\|_Y$ for all $\lambda > 0$ and $f \in Y$. In turn, a solution of (1-1) which is invariant by the scaling (1-3), i.e., $u = u_\lambda$, is called a self-similar solution of (1-1). Note that in order to obtain self-similar solutions, the initial data should be homogeneous of degree -1.

Over the years, global-in-time well-posedness of small solutions for (1-1) in critical spaces has attracted the interest of a number of authors. Without making a complete list, we mention works in the following spaces: homogeneous Sobolev $\dot{H}^{1/2}(\mathbb{R}^3)$ [Fujita and Kato 1964], Lebesgue $L^n(\mathbb{R}^n)$ [Kato 1984], Marcinkiewicz $L^{n,\infty}(\mathbb{R}^n)$ [Barraza 1996; Yamazaki 2000], Morrey $\mathcal{M}^n_q(\mathbb{R}^n)$ [Giga and Miyakawa 1989; Kato 1992; Taylor 1992], weak-Morrey $\mathcal{M}^n_{q,\infty}(\mathbb{R}^n)$ [Miao and Yuan 2007; Lemarié-Rieusset 2015; Ferreira 2016], PM^{n-1}-spaces [Cannone and Karch 2004], Besov $\dot{B}^{n/p-1}_{p,\infty}(\mathbb{R}^n)$ for $p > n$ [Cannone 1997], Fourier–Besov $F\dot{B}^{n-1-n/p}_{p,\infty}$ [Iwabuchi and Takada 2014; Konieczny and Yoneda 2011], homogeneous weak-Herz spaces $W\dot{K}^0_{n,\infty}(\mathbb{R}^n)$ [Tsutsui 2011], Fourier–Herz $\mathcal{B}^{-1}_r = F\dot{B}^{-1}_{1,r}$ with $r \in [1,2]$ [Cannone and Wu 2012; Iwabuchi and Takada 2014; Lei and Lin 2011], homogeneous Besov–Morrey $\mathcal{N}^{n/r-1}_{r,q,\infty}$ with $r > n$ [Kozono and Yamazaki 1994; Mazzucato 2003], and BMO^{-1} [Koch and Tataru 2001]. The reader can find other examples in the nice review [Lemarié-Rieusset 2002]. Up until now, to the best of our knowledge, BMO^{-1} and $\mathcal{N}^{n/r-1}_{r,1,\infty}$ are maximal critical spaces for (1-1) in the sense that a larger critical space in which small solutions of (1-1) are globally well-posed is not known.

The purpose of this paper is to provide a new critical Besov type class for global well-posedness of solutions for (1-1) by assuming a smallness condition on initial data norms. Here we consider homogeneous Besov-weak-Herz spaces $\dot{B}W\dot{K}^{\alpha,s}_{p,q,r}$, which are a type of Besov space based on homogeneous weak-Herz spaces $W\dot{K}^{\alpha}_{p,q}$. They are a natural extension of the spaces $BK^{\alpha,s}_{p,q,r}$ introduced in [Xu 2005] (see Definition 2.5 in Section 2B). The Herz space $K^{\alpha}_{p,q}$ was introduced by Herz [1968] but his definition is not appropriate for our purposes. Later, Johnson [1974] obtained a characterization of the $K^{\alpha}_{p,q}$-norm in terms of L^p-norms over annuli which is the base for the definition of the spaces $W\dot{K}^{\alpha}_{p,q}$ in [Tsutsui 2011] and is the same one that we use. In order to achieve our aims, we need to develop properties for $W\dot{K}^{\alpha}_{p,q,r}$- and $\dot{B}W\dot{K}^{\alpha,s}_{p,q,r}$-spaces such as the Hölder inequality, estimates for convolution operators, embedding theorems, interpolation properties, among others (see Section 2). In particular, a characterization of Besov-weak-Herz spaces in terms of interpolation of Sobolev-weak-Herz ones is proved, which is key in our arguments (see Lemma 2.14). Moreover, we prove estimates for the heat semigroup, as well as for the bilinear term $B(u, v)$ in (1-2), in the context of $\dot{B}W\dot{K}^{\alpha,s}_{p,q,r}$-spaces. We also point out that these spaces and their basic theory developed here could be employed to study other PDEs of elliptic, parabolic and conservation-law type. It is worth observing that some arguments in this paper are inspired by some of those in [Kozono and Yamazaki 1994] that analyzed (1-1) in Besov–Morrey spaces.

In what follows, we state our global well-posedness result.

Theorem 1.1. *Let $1 \le q \le \infty$, $n/2 < p < \infty$ and $0 \le \alpha < \min\{1-n/(2p), n/(2p)\}$. There exist $\epsilon > 0$ and $\delta > 0$ such that if $u_0 \in \dot{B}W\dot{K}^{\alpha,\alpha+n/p-1}_{p,q,\infty}$ with $\nabla \cdot u_0 = 0$ and $\|u_0\|_{\dot{B}W\dot{K}^{\alpha,\alpha+n/p-1}_{p,q,\infty}} \le \delta$, then problem (1-1) has a unique mild solution*

$$u \in L^{\infty}\big((0,\infty);\ \dot{B}W\dot{K}^{\alpha,\alpha+n/p-1}_{p,q,\infty}\big)$$

such that

$$\|u\|_X := \|u\|_{L^{\infty}((0,\infty);\dot{B}W\dot{K}^{\alpha,\alpha+n/p-1}_{p,q,\infty})} + \sup_{t>0} t^{\frac{1}{2}-(\alpha/2+n/(4p))}\|u\|_{W\dot{K}^{\alpha}_{2p,2q}} \le 2\epsilon.$$

Moreover, $u(t) \overset{*}{\rightharpoonup} u_0$ in $\dot{B}^{-1}_{\infty,\infty}$, as $t \to 0^+$, and solutions depend continuously on initial data.

As a matter of fact, one can show that the solution in Theorem 1.1 is time-continuous for $t > 0$. We have the continuous inclusions $L^n \subset L^{n,\infty} \subset W\dot{K}^{0}_{n,\infty} \subset \dot{B}W\dot{K}^{0,0}_{n,\infty,\infty}$ (see Lemmas 2.7 and 2.12) and

$$\dot{H}^{n/2-1} \subset L^n \subset \dot{B}^{n/p-1}_{p,\infty} \subset \dot{B}W\dot{K}^{0,n/p-1}_{p,\infty,\infty}, \qquad \text{for } p \ge n \text{ (see Remark 2.6).}$$

So our initial data class extends those of some previous works; for instance, the ones in [Fujita and Kato 1964; Kato 1984; Barraza 1996; Cannone 1997; Yamazaki 2000; Tsutsui 2011].

Notice that the parameter s corresponds to the regularity index of the Besov type space $\dot{B}W\dot{K}^{\alpha,s}_{p,q,r}$. Considering the family $\{\dot{B}W\dot{K}^{0,n/p-1}_{p,\infty,\infty}\}_{p>n/2}$, in the positive regularity range $n/2 < p < n$ we are dealing with spaces smaller than those with $p > n$ (negative regularity), because of the Sobolev embedding $\dot{B}W\dot{K}^{0,n/p_2-1}_{p_2,\infty,\infty} \subset \dot{B}W\dot{K}^{0,n/p_1-1}_{p_1,\infty,\infty}$ when $p_2 < p_1$ (see Lemma 2.13). For $p > n$, it is not clear to us whether there are inclusion relations between $\dot{B}W\dot{K}^{0,n/p-1}_{p,\infty,\infty}$ and BMO^{-1} or between

$\dot{B}W\dot{K}^{0,n/p-1}_{p,\infty,\infty}$ and $\mathcal{N}^{n/r-1}_{r,1,\infty}$ with $r > n$. In this sense, our result seems to give a new critical initial data class for existence of small global mild solutions for (1-1). In any case, it would be suitable to recall that well-posedness involves more properties than only existence of solutions, namely existence, uniqueness, persistence, and continuous dependence on initial data, which together characterize a good behavior of the Navier–Stokes flow in the considered space.

We finish with some comments about self-similarity and asymptotic behavior of solutions. It is not difficult to see that for $n \le p < \infty$ the function $f(x) = |x|^{-1}$ belongs to $\dot{B}W\dot{K}^{0,n/p-1}_{p,\infty,\infty}$. So, the homogeneous Besov-weak-Herz spaces (at least some of them) contain homogeneous functions of degree -1. Thus, if one assumes further that the initial data u_0 is a homogeneous vector field of degree -1, then a standard procedure involving a Picard type sequence gives that the solution obtained in Theorem 1.1 is in fact self-similar. Moreover, following some estimates and arguments in the proof of Theorem 1.1, with some extra effort, it is possible to prove that if we have u_0 and v_0 satisfying $\lim_{t\to\infty} \|G(t)(u_0 - v_0)\|_{\dot{B}W\dot{K}^{\alpha,\alpha+n/p-1}_{p,q,\infty}} = 0$, then

$$\lim_{t\to\infty} \|u(\,\cdot\,,t) - v(\,\cdot\,,t)\|_{\dot{B}W\dot{K}^{\alpha,\alpha+n/p-1}_{p,q,\infty}} = 0,$$

where u and v are the solutions obtained in Theorem 1.1 with initial data u_0 and v_0, respectively.

The plan of this paper is as follows. Section 2 is devoted to function spaces, with Herz and Sobolev–Herz spaces considered in Section 2A, while Sobolev-weak-Herz and Besov-weak-Herz spaces are addressed in Section 2B. The proof of Theorem 1.1 is performed in the final section: In Section 3A we provide linear estimates for the heat semigroup. Section 3B is devoted to bilinear estimates for $B(\cdot,\cdot)$ in our setting. After obtaining the required estimates, the proof is concluded in Section 3C by means of a contraction argument.

2. Function spaces

In this section we recall some definitions and properties about function spaces that will be considered throughout this paper.

2A. ***Weak-Herz and Sobolev-weak-Herz spaces.*** For an integer $k \in \mathbb{Z}$, we define the set A_k as

$$A_k = \{x \in \mathbb{R}^n : 2^{k-1} \le |x| < 2^k\}, \tag{2-1}$$

and observe that $\mathbb{R}^n \setminus \{0\} = \bigcup_{k\in\mathbb{Z}} A_k$. Taking $x \in A_k$ we have that

$$y \in A_m \text{ and } m \le k \;\Rightarrow\; 2^{k-1} - 2^m \le |x-y| < 2^k + 2^m,$$
$$y \in A_m \text{ and } m \ge k \;\Rightarrow\; 2^{m-1} - 2^k \le |x-y| < 2^m + 2^k.$$

Consider also the sets

$$C_{m,k} = \{\xi : 2^{k-1} - 2^m \le |\xi| < 2^k + 2^m\},$$
$$\widetilde{C}_{m,k} = \{\xi : 2^{m-1} - 2^k \le |\xi| < 2^m + 2^k\}. \tag{2-2}$$

We now define the weak-Herz spaces:

Definition 2.1. Let $1 < p \le \infty,\ 1 \le q \le \infty$ and $\alpha \in \mathbb{R}$. The homogeneous weak-Herz space $W\dot{K}^{\alpha}_{p,q} = W\dot{K}^{\alpha}_{p,q}(\mathbb{R}^n)$ is defined as the set of all measurable functions such that the following quantity is finite:

$$\|f\|_{W\dot{K}^{\alpha}_{p,q}} := \begin{cases} \left(\sum_{k\in\mathbb{Z}} 2^{k\alpha q}\|f\|^q_{L^{p,\infty}(A_k)}\right)^{1/q} & \text{if } q < \infty, \\ \sup_{k\in\mathbb{Z}} 2^{k\alpha}\|f\|_{L^{p,\infty}(A_k)} & \text{if } q = \infty. \end{cases} \tag{2-3}$$

For $\alpha \in \mathbb{R},\ 1 < p \le \infty$ and $1 \le q \le \infty$, the quantity $\|\cdot\|_{W\dot{K}^{\alpha}_{p,q}}$ defines a norm in $W\dot{K}^{\alpha}_{p,q}$ and the pair $(W\dot{K}^{\alpha}_{p,q}, \|\cdot\|_{W\dot{K}^{\alpha}_{p,q}})$ is a Banach space (see, e.g., [Hernández and Yang 1999; Tsutsui 2011]).

The Hölder inequality holds in the setting of homogeneous weak-Herz spaces (see [Tsutsui 2011]). To be more precise, if $1 < p, p_1, p_2 \le \infty,\ 1 \le q, q_1, q_2 \le \infty$ and $\alpha, \alpha_1, \alpha_2 \in \mathbb{R}$ are such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2},\ \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$ and $\alpha = \alpha_1 + \alpha_2$, then

$$\|fg\|_{W\dot{K}^{\alpha}_{p,q}} \le C\|f\|_{W\dot{K}^{\alpha_1}_{p_1,q_1}}\|g\|_{W\dot{K}^{\alpha_2}_{p_2,q_2}}, \tag{2-4}$$

where $C > 0$ is a universal constant. In fact, for all $k \in \mathbb{Z}$, we have

$$\|fg\|_{L^{p,\infty}(A_k)} \le C\|f\|_{L^{p_1,\infty}(A_k)}\|g\|_{L^{p_2,\infty}(A_k)},$$

and therefore

$$\begin{aligned} \|fg\|_{W\dot{K}^{\alpha}_{p,q}} &= \left(\sum_{k\in\mathbb{Z}} 2^{k\alpha q}\|fg\|^q_{L^{p,\infty}(A_k)}\right)^{1/q} \\ &\le C\left(\sum_{k\in\mathbb{Z}} 2^{k\alpha_1 q}\|f\|^q_{L^{p_1,\infty}(A_k)} 2^{k\alpha_2 q}\|g\|^q_{L^{p_2,\infty}(A_k)}\right)^{1/q} \\ &\le C\|f\|_{W\dot{K}^{\alpha_1}_{p_1,q_1}}\|g\|_{W\dot{K}^{\alpha_2}_{p_2,q_2}}. \end{aligned} \tag{2-5}$$

Taking in particular $(\alpha_1, p_1, q_1) = (0, \infty, \infty)$ in (2-5), we obtain

$$\|fg\|_{W\dot{K}^{\alpha}_{p,q}} \le C\|f\|_{L^{\infty}(\mathbb{R}^n)}\|g\|_{W\dot{K}^{\alpha}_{p,q}}. \tag{2-6}$$

Later, we will need to estimate some convolution operators, particularly the heat semigroup, in weak-Herz and Besov-weak-Herz spaces. The following lemma will be useful for that purpose.

Lemma 2.2 (convolution). *Let $1 \le p_1 < \infty$ and $1 < r, p_2 < \infty$ be such that $1+\frac{1}{r}=\frac{1}{p_1}+\frac{1}{p_2}$. Further, let $1 \le q \le \infty$, $-\frac{n}{r} < \alpha < n\big(1-\frac{1}{p_2}\big)$, and $\theta \in L^{p_1}(\mathbb{R}^n)$ be such that $\theta|\cdot|^{n/p_1} \in L^\infty(\mathbb{R}^n)$. There exists a positive constant C independent of θ such that*

$$\|\theta * f\|_{W\dot{K}^{\alpha}_{r,q}} \le C \max\big\{\|\theta\|_{L^{p_1}}, \||\cdot|^{n/p_1}\theta\|_{L^\infty}\big\}\|f\|_{W\dot{K}^{\alpha}_{p_2,q}}, \tag{2-7}$$

for all $f \in W\dot{K}^{\alpha}_{p_2,q}$.

Proof. Denote $f_m = f|_{A_m}$. Recalling the decomposition (2-1), for $k \in \mathbb{Z}$ we can estimate

$$\begin{aligned} 2^{k\alpha}\|\theta * f\|_{L^{r,\infty}(A_k)} &\le 2^{k\alpha}\bigg\{\Big\|\sum_{m\le k-2}\theta*f_m\Big\|_{L^{r,\infty}(A_k)} + \Big\|\sum_{m=k-1}^{k+1}\theta*f_m\Big\|_{L^{r,\infty}(A_k)} + \Big\|\sum_{m\ge k+2}\theta*f_m\Big\|_{L^{r,\infty}(A_k)}\bigg\} \\ &=: I_1^k + I_2^k + I_3^k. \end{aligned} \tag{2-8}$$

Using the notations in (2-2) and the change of variable $z = k-m$, we handle the term I_3^k as follows:

$$\begin{aligned} I_3^k &\le 2^{k\alpha}\Big\|\sum_{m\ge k+2}\theta*f_m\Big\|_{L^{r,\infty}(A_k)} \\ &\le 2^{k\alpha}\Big\|\sum_{m\ge k+2}\theta*f_m\Big\|_{L^{r}(A_k)} \\ &\le 2^{k\alpha}\bigg(\int_{A_k}\Big|\sum_{m\ge k+2}\int_{\mathbb{R}^n}\theta(x-y)f_m(y)\,dy\Big|^r dx\bigg)^{1/r} \\ &= 2^{k\alpha}\bigg(\int_{A_k}\Big|\sum_{m\ge k+2}\int_{\mathbb{R}^n}\theta(x-y)\chi_{\widetilde{C}_{m,k}}(x-y)f_m(y)dy\Big|^r dx\bigg)^{1/r} \\ &\le C\||\cdot|^{n/p_1}\theta\|_{L^\infty}2^{k\alpha} \\ &\quad\times\bigg(\int_{A_k}\Big(\sum_{m\ge k+2}\int_{\mathbb{R}^n}|x-y|^{-n/p_1}\chi_{\widetilde{C}_{m,k}}(x-y)\times|f_m(y)|\,dy\Big)^r dx\bigg)^{1/r} \\ &\le C\||\cdot|^{n/p_1}\theta\|_{L^\infty}2^{k\alpha}\bigg(\int_{A_k}\Big(\sum_{m\ge k+2}2^{-mn/p_1}\|f\|_{L^1(A_m)}\Big)^r dx\bigg)^{1/r}. \end{aligned} \tag{2-9}$$

Recalling the inclusion $L^{p_2,\infty}(A_m) \hookrightarrow L^1(A_m)$, we can continue to estimate the right-hand side of the fifth inequality in (2-9) in order to obtain

$$
\begin{aligned}
(2\text{-}10)\quad I_3^k &\le C\||\cdot|^{n/p_1}\theta\|_{L^\infty}2^{k\alpha} \\
&\quad\times\left(\int_{A_k}\left(\sum_{m\ge k+2}2^{-mn/p_1}2^{nm(1-1/p_2)}\|f\|_{L^{p_2,\infty}(A_m)}\right)^r dx\right)^{1/r} \\
&\le C\||\cdot|^{n/p_1}\theta\|_{L^\infty}2^{k\alpha}2^{kn/r}\sum_{m\ge k+2}2^{-mn/r}\|f\|_{L^{p_2,\infty}(A_m)} \\
&\le C\||\cdot|^{n/p_1}\theta\|_{L^\infty}\sum_{-2\ge z}2^{k(\alpha+n/r)}2^{(z-k)n/r}\|f\|_{L^{p_2,\infty}(A_{k-z})} \\
&\le C\||\cdot|^{n/p_1}\theta\|_{L^\infty}\sum_{-2\ge z}2^{k\alpha}2^{zn/r}2^{-(k-z)\alpha}2^{(k-z)\alpha}\|f\|_{L^{p_2,\infty}(A_{k-z})} \\
&\le C\||\cdot|^{n/p_1}\theta\|_{L^\infty}\sum_{-2\ge z}2^{z(n/r+\alpha)}2^{(k-z)\alpha}\|f\|_{L^{p_2,\infty}(A_{k-z})}.
\end{aligned}
$$

This estimate and the Minkowski inequality lead us to (with the usual modification in the case $q=\infty$)

$$\left(\sum_{k\in\mathbb{Z}}(I_3^k)^q\right)^{1/q}\le CM_\theta\|f\|_{W\dot{K}^{\alpha}_{p_2,q}}.$$

For the summand I_2^k, we estimate

$$
\begin{aligned}
I_2^k &\le 2^{k\alpha}\sum_{m=k-1}^{k+1}\|\theta * f_m\|_{L^{r,\infty}(A_k)} \\
&\le 2^{k\alpha}\sum_{m=k-1}^{k+1}\|\theta * f_m\|_{L^{r,\infty}(\mathbb{R}^n)} \\
&\le 2^{k\alpha}\sum_{m=k-1}^{k+1}\|\theta\|_{L^{p_1,\infty}}\|f_m\|_{L^{p_2,\infty}} \\
&\le C\|\theta\|_{L^{p_1}}\sum_{l=-1}^{1}2^{(k+l)\alpha}\|f\|_{L^{p_2,\infty}(A_{k+l})},
\end{aligned}
$$

which implies

$$\left(\sum_{k\in\mathbb{Z}}(I_2^k)^q\right)^{1/q}\le CM_\theta\|f\|_{W\dot{K}^{\alpha}_{p_2,q}}.$$

Proceeding similarly to the estimates (2-9)–(2-10) but considering $C_{m,k}$ in place of $\widetilde{C}_{m,k}$, the summand I_1^k can be estimated as

$$
\begin{aligned}
I_1^k &\le C\||\cdot|^{n/p_1}\theta\|_{L^\infty}2^{k\alpha}\bigg(\int_{A_k}\bigg(\sum_{m\le k-2}\int_{\mathbb{R}^n}|x-y|^{-n/p_1}\chi_{C_{m,k}}(x-y)\\
&\qquad\qquad \times|f_m(y)|dy\bigg)^r dx\bigg)^{1/r}\\
&\le C\||\cdot|^{n/p_1}\theta\|_{L^\infty}2^{k\alpha}\bigg(\int_{A_k}\bigg(\sum_{m\le k-2}2^{-kn/p_1}\|f\|_{L^1(A_m)}\bigg)^r dx\bigg)^{1/r}\\
&\le C\||\cdot|^{n/p_1}\theta\|_{L^\infty}2^{k\alpha}2^{kn/r}\sum_{m\le k-2}2^{-kn/p_1}\|f\|_{L^1(A_m)}\\
&\le C\||\cdot|^{n/p_1}\theta\|_{L^\infty}\sum_{m\le k-2}2^{k(\alpha-n+n/p_2)}\|f\|_{L^1(A_m)}\\
&\le C\||\cdot|^{n/p_1}\theta\|_{L^\infty}\sum_{2\le z}2^{k(\alpha-n+n/p_2)}2^{n(k-z)(1-1/p_2)}\|f\|_{L^{p_2,\infty}(A_{k-z})}\\
&\le C\||\cdot|^{n/p_1}\theta\|_{L^\infty}\sum_{2\le z}2^{z(\alpha-n+n/p_2)}2^{(k-z)\alpha}\|f\|_{L^{p_2,\infty}(A_{k-z})}.
\end{aligned}
\tag{2-11}
$$

It follows from (2-11) that

$$
\bigg(\sum_{k\in\mathbb{Z}}(I_1^k)^q\bigg)^{1/q}\le CM_\theta\|f\|_{W\dot{K}^{\alpha}_{p_2,q}}.
$$

Finally, the desired estimate is obtained after recalling the norm (2-3) and using the above estimates for I_j^k in (2-8). □

Let $\varphi\in C_c^\infty(\mathbb{R}^n\setminus\{0\})$ be radially symmetric and such that

$$
\operatorname{supp}\varphi\subset\{x:\tfrac{3}{4}\le|x|\le\tfrac{8}{3}\}
$$

and

$$
\sum_{j\in\mathbb{N}}\varphi_j(\xi)=1,\quad \text{for all } \xi\in\mathbb{R}^n\setminus\{0\},
$$

where $\varphi_j(\xi):=\varphi(\xi 2^{-j})$. Now we can define the well-known localization operators Δ_j and S_j:

$$
\begin{aligned}
\Delta_j f&=\varphi_j(D)f=(\mathcal{F}^{-1}\varphi_j)*f,\\
S_k f&=\sum_{j\le k}\Delta_j f.
\end{aligned}
$$

It is easy to see that we have the identities

$$
\Delta_j\Delta_k f=0 \text{ if } |j-k|\ge 2\quad\text{and}\quad \Delta_j(S_{k-2}g\Delta_k f)=0 \text{ if } |j-k|\ge 5.
$$

Finally, Bony's decomposition gives (see, e.g., [Bony 1981])

$$
fg=T_f g+T_g f+R(fg),
$$

where

$$T_f g = \sum_{j\in Z} S_{j-2} f \Delta_j g, \quad R(fg) = \sum_{j\in Z} \Delta_j f \tilde{\Delta}_j g \quad \text{and} \quad \tilde{\Delta}_j g = \sum_{|j-j'|\le 1} \Delta_{j'} g.$$

The next lemma will be useful in order to estimate some multiplier operators in Besov-weak-Herz spaces.

Lemma 2.3. *Let* $1 < p < \infty$, $1 \le q \le \infty$, $-\frac{n}{p} < \alpha < n\big(1-\frac{1}{p}\big)$, $m \in \mathbb{R}$ *and* $D_j = \{x : \frac{3}{4}2^j \le |x| \le \frac{8}{3}2^j\}$ *for* $j \in \mathbb{Z}$. *Let* P *be a* C^n*-function on*

$$\widetilde{D}_j := D_{j-1} \cup D_j \cup D_{j+1}$$

such that $|\partial_\xi^\beta P(\xi)| \le C2^{(m-|\beta|)j}$ *for all* $\xi \in \widetilde{D}_j$ *and multi-index* β *satisfying* $|\beta| \le [n/2]+1$. *Then, we have that*

$$\|(P\hat{f})^{\vee}\|_{W\dot{K}^{\alpha}_{p,q}} \le C2^{jm}\|f\|_{W\dot{K}^{\alpha}_{p,q}},$$

for all $f \in W\dot{K}^{\alpha}_{p,q}$ *such that* $\operatorname{supp} \hat{f} \subset D_j$.

Proof. We start by defining $\tilde{\varphi}_j = \varphi_{j-1} + \varphi_j + \varphi_{j+1}$ and $K(x) = (P\tilde{\varphi}_j)^{\vee}$. Since $\operatorname{supp} \hat{f} \subset D_j$ we have that $P(\xi)\hat{f}(\xi) = P(\xi)\tilde{\varphi}_j(\xi)\hat{f}(\xi)$, and therefore $(P\hat{f})^{\vee} = (P\tilde{\varphi}_j\hat{f})^{\vee} = K * f$.

Using Lemma 2.2 we get

$$\|(P\hat{f})^{\vee}\|_{W\dot{K}^{\alpha}_{p,q}} \le C\max\{\|K\|_{L^1}, \||\cdot|^n K\|_{L^\infty}\}\|f\|_{W\dot{K}^{\alpha}_{p,q}}.$$

It remains to show that $\max\{\|K\|_{L^1}, \||\cdot|^n K\|_{L^\infty}\} \le C2^{mj}$. For that, let $N \in \mathbb{N}$ be such that $n/2 < N \le n$ and proceed as follows:

$$\begin{aligned}
\|K\|_{L^1} &= \int\limits_{B(0,2^{-j})} K(y) + \int\limits_{|y|\ge 2^{-j}} K(y) \\
&\le \left(\int_{B(0,2^{-j})} 1\right)^{1/2}\left(\int_{B(0,2^{-j})} |K(y)|^2\right)^{1/2} \\
&\qquad + \left(\int_{|y|\ge 2^{-j}} |y|^{-2N}\right)^{1/2}\left(\int_{|y|\ge 2^{-j}} |y|^{2N}|K(y)|^2\right)^{1/2} \\
&\le C2^{-jn/2}\|P\tilde{\varphi}_j\|_{L^2} + C2^{-j(-N+n/2)}\sum_{|\beta|=N}\|(\cdot)^\beta K\|_{L^2} \\
&\le C2^{-jn/2}\|P\tilde{\varphi}_j\|_{L^2} + C2^{-j(-N+n/2)}\sum_{|\beta|=N}\|\partial^\beta(P\tilde{\varphi}_j)\|_{L^2} \\
&\le C2^{-jn/2}C2^{mj}2^{jn/2} + C2^{-j(-N+n/2)}C2^{j(m-N)}2^{jn/2} \\
&\le C2^{mj}.
\end{aligned}$$

For the norm $\| |\cdot|^n K\|_{L^\infty}$, we have that

$$\begin{aligned}\| |\cdot|^n K\|_{L^\infty} &\le \sum_{|\beta|=n} \|(\cdot)^\beta K\|_{L^\infty} \le C \sum_{|\beta|=n} \|\partial^\beta (P\tilde{\varphi}_j)\|_{L^1} \\ &\le C \sum_{|\beta|=n} 2^{j(m-n)} 2^{jn} \le C 2^{mj},\end{aligned}$$

as required. □

2B. ***Sobolev-weak-Herz spaces and Besov-weak-Herz spaces.*** In this section we introduce the homogeneous Sobolev-weak-Herz spaces and Besov-weak-Herz spaces. We also shall prove a number of properties about these spaces that will be useful in our study of the Navier–Stokes equations. These spaces are a generalization of Sobolev–Herz and Besov–Herz spaces found in [Xu 2005].

Definition 2.4. Let $1 < p \le \infty$, $1 \le q \le \infty$ and $\alpha, s \in \mathbb{R}$. Recall the Riesz operator $\widehat{I^s f} = |\xi|^s \hat{f}$. The homogeneous Sobolev-weak-Herz spaces $W\dot{K}^{\alpha,s}_{p,q} = W\dot{K}^{\alpha,s}_{p,q}(\mathbb{R}^n)$ are defined as

$$W\dot{K}^{\alpha,s}_{p,q} = \{f \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P} : \|I^s f\|_{W\dot{K}^{\alpha}_{p,q}} < \infty\}. \tag{2-12}$$

Definition 2.5. Let $1 < p \le \infty$, $1 \le q, r \le \infty$ and $\alpha, s \in \mathbb{R}$. The homogeneous Besov-weak-Herz spaces $\dot{B}W\dot{K}^{\alpha,s}_{p,q,r} = \dot{B}W\dot{K}^{\alpha,s}_{p,q,r}(\mathbb{R}^n)$ are defined as

$$\dot{B}W\dot{K}^{\alpha,s}_{p,q,r} = \{f \in \mathcal{S}'(\mathbb{R}^n)/\mathcal{P} : \|f\|_{\dot{B}W\dot{K}^{\alpha,s}_{p,q,r}} < \infty\},$$

where

$$\|f\|_{\dot{B}W\dot{K}^{\alpha,s}_{p,q,r}} := \begin{cases} \left(\sum_{j\in\mathbb{Z}} 2^{jsr} \|\Delta_j f\|^r_{W\dot{K}^{\alpha}_{p,q}} \right)^{1/r} & \text{if } r < \infty \\ \sup_{j\in\mathbb{Z}} 2^{js} \|\Delta_j f\|_{W\dot{K}^{\alpha}_{p,q}} & \text{if } r = \infty. \end{cases} \tag{2-13}$$

Remark 2.6. (i) The spaces $W\dot{K}^{\alpha,s}_{p,q}$ and $\dot{B}W\dot{K}^{\alpha,s}_{p,q,r}$ are Banach spaces endowed with the norms $\|\cdot\|_{W\dot{K}^{\alpha}_{p,q}}$ and $\|\cdot\|_{\dot{B}W\dot{K}^{\alpha,s}_{p,q,r}}$, respectively.

(ii) The continuous inclusion $\dot{B}^s_{p,r}(\mathbb{R}^n) \subset \dot{B}W\dot{K}^{0,s}_{p,\infty,r}(\mathbb{R}^n)$ holds for all $s \in \mathbb{R}$, $1 < p \le \infty$, and $1 \le r \le \infty$, where $\dot{B}^s_{p,r}$ stands for homogeneous Besov spaces. To show that, it is sufficient to recall the definition of Besov spaces (see [Bergh and Löfström 1976, p. 146]) and (2-13) and to use the inclusion $L^p \subset W\dot{K}^0_{p,\infty}$ that is going to be shown in the lemma below.

The next lemma contains relations between weak-L^p, weak-Herz and Morrey spaces. For the definition and some properties about Morrey spaces we refer the reader to [Kozono and Yamazaki 1994] (see also [Kato 1992] for an equivalent definition and further properties).

Lemma 2.7. *For $1 < p < \infty$, we have the continuous inclusion*

$$L^p \subsetneq L^{p,\infty} \subsetneq W\dot{K}^0_{p,\infty}. \tag{2-14}$$

Moreover, let $\mathcal{M}^r_q$ stand for homogeneous Morrey spaces, $1 \le q \le r < \infty$ and $n/r \neq \alpha + n/p$ when $q < p$. Then

$$W\dot{K}^\alpha_{p,\infty} \not\subset \mathcal{M}^r_q. \tag{2-15}$$

Proof. The first inclusion in (2-14) is well known, so we only prove the second one. For that, it is sufficient to note that $\|f\|_{L^{p,\infty}(A_k)} \le \|f\|_{L^{p,\infty}(\mathbb{R}^n)}$ for all $k \in \mathbb{Z}$ and after to take the supremum over k. In order to see the strictness of the inclusion, take $x_k = \frac{3}{2}2^{k-1}\vec{e}_1$ and $h(x) := \sum_{k=1}^\infty |x - x_k|^{-n/p}\chi_{B(0,1/8)}(x - x_k)$. It is clear that h is an element of $W\dot{K}^0_{p,\infty}$ but not of $L^{p,\infty}(\mathbb{R}^n)$.

Now we turn to (2-15). For $f(x) = |x|^{-n/p}$, we have that $f \in L^{p,\infty} \subset W\dot{K}^0_{p,\infty}$. On the other hand, for any $q \ge p$ note that $\|f\|_{L^q(B(0,R))} = \infty$, and then $f \notin \mathcal{M}^r_q$ for any r. Finally, if $n/r \neq \alpha + n/p$ then $W\dot{K}^\alpha_{p,\infty} \subset \mathcal{M}^r_q$ (and the reverse) never could hold. This follows from an easy scaling analysis of the space norms; in fact, the scaling of $\mathcal{M}^r_q$ is $-n/r$ and that of $W\dot{K}^0_{p,\infty}$ is $-\alpha - n/p$. □

In the next remark, we recall some inclusion and noninclusion relations involving Herz, weak-Herz, Besov and bmo^{-1} spaces that can be found in [Tsutsui 2011].

Remark 2.8. (i) For $1 < p < \sigma < \infty$ and $0 < \alpha < n(1 - 1/p)$, we have

$$W\dot{K}^\alpha_{p,\infty} \hookrightarrow \dot{B}^{-(\alpha+n(1/p-1/\sigma))}_{\sigma,\infty}, \ \dot{K}^\alpha_{p,\infty} \hookrightarrow \dot{B}^{-\alpha}_{p,\infty} \quad \text{and} \quad W\dot{K}^0_{p,\sigma} \hookrightarrow \dot{B}^{-n(1/p-1/\sigma)}_{\sigma,\infty}.$$

(ii) For $1 < p < \infty$ and $0 \le \alpha < n(1 - 1/p)$, we have $W\dot{K}^\alpha_{p,\infty} \hookrightarrow \dot{B}^{-(\alpha+n/p)}_{\infty,\infty}$.

(iii) For $0 \le \alpha < n$, we have $\dot{K}^\alpha_{\infty,\infty} \hookrightarrow \dot{B}^{-\alpha}_{\infty,\infty}$.

(iv) For $1 < p \le \infty$ and $0 \le \alpha \le n(1 - 1/p)$, we have $W\dot{K}^\alpha_{p,1} \hookrightarrow \dot{B}^{-(\alpha+n/p)}_{\infty,\infty}$.

(v) We have $L^1 = \dot{K}^0_{1,1} \hookrightarrow \dot{B}^{-n}_{\infty,\infty}$. For $n < p \le \infty$ and $0 \le \alpha < 1 - n/p$, the inclusion $W\dot{K}^\alpha_{p,\infty} \hookrightarrow bmo^{-1}$ holds.

(vi) For $1 < p < \sigma < \infty$ and $-n(1/p - 1/\sigma) < \alpha \le 0$, $W\dot{K}^\alpha_{p,\infty} \hookrightarrow \dot{B}^{-(\alpha+n(1/p-1/\sigma))}_{\sigma,\infty}$ does not hold.

(vii) For $1 < p < \infty$ and $-n/p < \alpha < 0$, $W\dot{K}^\alpha_{p,\infty} \hookrightarrow \dot{B}^{-(\alpha+n/p)}_{\infty,\infty}$ does not hold.

Remark 2.9. Using the interpolation properties of homogeneous Besov spaces and homogeneous Besov-weak-Herz spaces (see Lemma 2.14 below) and item (ii) of Remark 2.8, for $1 < p < \infty$ and $0 \le \alpha < n(1 - 1/p)$ we can obtain

$$\dot{B}W\dot{K}^{\alpha,s}_{p,\infty,r} \hookrightarrow \dot{B}^{s-(\alpha+n/p)}_{\infty,r}. \tag{2-16}$$

In particular, $\dot{B}W\dot{K}^{\alpha,\alpha+n/p-1}_{p,\infty,\infty} \hookrightarrow \dot{B}^{-1}_{\infty,\infty}$ and

$$\dot{B}W\dot{K}^{0,n/p}_{p,\infty,1} \hookrightarrow \dot{B}^0_{\infty,1} \hookrightarrow L^\infty. \tag{2-17}$$

Moreover, from Remark 2.8(vi) and Lemma 2.12 below, it follows that the inclusion

$$\dot{B}W\dot{K}^{0,s}_{p,\infty,\infty} \hookrightarrow \dot{B}^{s-n(1/p-1/\sigma)}_{\sigma,\infty}$$

does not hold for any $s \in \mathbb{R}$, $1 < p < \infty$ and $1 \le \sigma < \infty$.

Remark 2.10. Note that for $s-(\alpha+n/p) < 0$ and $r > 1$, or $s-(\alpha+n/p) \le 0$ and $r = 1$, the inclusion (2-16) implies that for $f \in \dot{B}W\dot{K}^{\alpha,s}_{p,\infty,r}$ the series $\sum_{j=-\infty}^{\infty} \Delta_j f$ converges in $\mathcal{S}'$ to a representative of f in $\mathcal{S}'/\mathcal{P}$ (see, e.g., [Lemarié-Rieusset 2002]). So, in these cases the space $\dot{B}W\dot{K}^{\alpha,s}_{p,\infty,r}$ can be regarded as a subspace of $\mathcal{S}'$. Hereafter, we say that $f \in \mathcal{S}'$ belongs to $\dot{B}W\dot{K}^{\alpha,s}_{p,\infty,r}$ with $s-(\alpha+n/p) < 0$ and $r > 1$, or $s-(\alpha+n/p) \le 0$ and $r = 1$, if f is the canonical representative of the class in $\mathcal{S}'/\mathcal{P}$, namely $f = \sum_{j=-\infty}^{\infty} \Delta_j f$ in $\mathcal{S}'$.

A multiplier theorem of Hörmander–Mihlin type will be needed in our setting. This is the subject of the next lemma. In fact, the main part of the proof has already been done in Lemma 2.3.

Lemma 2.11. *Let $1 < p < \infty$, $1 \le q, r \le \infty$, $-n/p < \alpha < n(1-1/p)$ and $m, s \in \mathbb{R}$. Let $P \in C^n(\mathbb{R}^n \setminus \{0\})$ be a function such that $|\partial_\xi^\beta P(\xi)| \le C|\xi|^{(m-|\beta|)}$ for all multi-index β satisfying $|\beta| \le n$. Then*

$$\|P(D)f\|_{\dot{B}W\dot{K}^{\alpha,s-m}_{p,q,r}} \le C\|f\|_{\dot{B}W\dot{K}^{\alpha,s}_{p,q,r}}.$$

Proof. Note that for each $j \in \mathbb{Z}$ we have that $|\xi|^{m-|\beta|} \le C2^{j(m-|\beta|)}$ for all $\xi \in \widetilde{D}_j$, and therefore $|\partial_\xi^\beta P(\xi)| \le C2^{j(m-|\beta|)}$. On the other hand, since $\operatorname{supp} \widehat{\Delta_j f} \subset D_j$ we can use Lemma 2.3 in order to get

$$\|\Delta_j(P(D)f)\|_{W\dot{K}^{\alpha}_{p,q}} = \|P(D)(\Delta_j f)\|_{W\dot{K}^{\alpha}_{p,q}} \le C2^{jm}\|\Delta_j f\|_{W\dot{K}^{\alpha}_{p,q}}. \tag{2-18}$$

The result follows by multiplying (2-18) by $2^{j(s-m)}$ and then taking the l^r-norm. □

In what follows we present some inclusions involving Sobolev-weak-Herz and Besov-weak-Herz spaces.

Lemma 2.12. *Let $s \in \mathbb{R}$, $1 < p < \infty$, $1 \le q \le \infty$ and $-n/p < \alpha < n(1-1/p)$. We have the following continuous inclusions*:

$$\dot{B}W\dot{K}^{\alpha,0}_{p,q,1} \subset W\dot{K}^{\alpha}_{p,q} \subset \dot{B}W\dot{K}^{\alpha,0}_{p,q,\infty} \tag{2-19}$$

$$\dot{B}W\dot{K}^{\alpha,s}_{p,q,1} \subset W\dot{K}^{\alpha,s}_{p,q} \subset \dot{B}W\dot{K}^{\alpha,s}_{p,q,\infty}. \tag{2-20}$$

Proof. For $f \in \dot{B}W\dot{K}^{\alpha,0}_{p,q,1}$, we can employ the decomposition $f = \sum_{j\in\mathbb{Z}} \Delta_j f$ in order to estimate

$$\|f\|_{L^{p,\infty}(A_k)} \le \sum_{j\in\mathbb{Z}} \|\Delta_j f\|_{L^{p,\infty}(A_k)}.$$

Thus, using the Minkowski inequality, we arrive at (with the usual modification in the case $q=\infty$)

$$\begin{aligned}\|f\|_{W\dot{K}^{\alpha}_{p,q}} &\leq \Big(\sum_{k\in\mathbb{Z}} 2^{k\alpha q}\|f\|^q_{L^{p,\infty}(A_k)}\Big)^{1/q} \leq \Big(\sum_{k\in\mathbb{Z}}\Big(\sum_{j\in\mathbb{Z}} 2^{k\alpha}\|\Delta_j f\|_{L^{p,\infty}(A_k)}\Big)^q\Big)^{1/q}\\ &\leq \sum_{j\in\mathbb{Z}}\Big(\sum_{k\in\mathbb{Z}} 2^{k\alpha q}\|\Delta_j f\|^q_{L^{p,\infty}(A_k)}\Big)^{1/q} = \sum_{j\in\mathbb{Z}}\|\Delta_j f\|_{W\dot{K}^{\alpha}_{p,q}}\\ &= \|f\|_{\dot{B}W\dot{K}^{\alpha,0}_{p,q,1}},\end{aligned}$$

which implies the first inclusion in (2-19). Now, let $f \in W\dot{K}^{\alpha}_{p,q}$ and note that in fact we have that $f \in \mathcal{S}'/\mathcal{P}$. Moreover, using Lemma 2.2 we get

$$\|f\|_{\dot{B}W\dot{K}^{\alpha,0}_{p,q,\infty}} = \sup_{j\in\mathbb{Z}}\|\Delta_j f\|_{W\dot{K}^{\alpha}_{p,q}} \leq C\sup_{j\in\mathbb{Z}}\|f\|_{W\dot{K}^{\alpha}_{p,q}} = C\|f\|_{W\dot{K}^{\alpha}_{p,q}},$$

and then the second inclusion in (2-19) holds.

For (2-20), we can use Lemma 2.3 in order to estimate

$$\begin{aligned}\|f\|_{W\dot{K}^{\alpha,s}_{p,q}} = \|I^s f\|_{W\dot{K}^{\alpha}_{p,q}} &\leq \|I^s f\|_{\dot{B}W\dot{K}^{\alpha,0}_{p,q,1}} = \sum_{j\in\mathbb{Z}}\|\Delta_j I^s f\|_{W\dot{K}^{\alpha}_{p,q}}\\ &\leq C\sum_{j\in\mathbb{Z}} 2^{js}\|\Delta_j f\|_{W\dot{K}^{\alpha}_{p,q}} = C\|f\|_{\dot{B}W\dot{K}^{\alpha,s}_{p,q,1}}.\end{aligned}$$

Moreover, Lemma 2.3 also can be used to obtain

$$\begin{aligned}\|f\|_{\dot{B}W\dot{K}^{\alpha,s}_{p,q,\infty}} = \sup_{j\in\mathbb{Z}} 2^{js}\|\Delta_j f\|_{W\dot{K}^{\alpha}_{p,q}} &= \sup_{j\in\mathbb{Z}} 2^{js}\|I^{-s}\Delta_j I^s f\|_{W\dot{K}^{\alpha}_{p,q}}\\ &\leq C\sup_{j\in\mathbb{Z}}\|\Delta_j I^s f\|_{W\dot{K}^{\alpha}_{p,q}} \leq C\sup_{j\in\mathbb{Z}}\|I^s f\|_{W\dot{K}^{\alpha}_{p,q}}\\ &= C\|I^s f\|_{W\dot{K}^{\alpha}_{p,q}} = C\|f\|_{W\dot{K}^{\alpha,s}_{p,q}},\end{aligned}$$

for all $f \in W\dot{K}^{\alpha,s}_{p,q}$, as required. □

Now we present an embedding theorem of Sobolev type.

Lemma 2.13. *Let $s\in\mathbb{R}$, $1<p<\infty$, $1\leq q,r\leq\infty$, $p\leq p_1<\infty$, $1<p_2\leq p_1$ and $-\frac{n}{p}<\alpha<n\big(1+\frac{1}{p_1}-\frac{1}{p_2}-\frac{1}{p}\big)$. Then*

$$\|f\|_{\dot{B}W\dot{K}^{\alpha,s}_{p,q,r}} \leq C\|f\|_{\dot{B}W\dot{K}^{\alpha+n(1/p-1/p_1),s+n(1/p_2-1/p_1)}_{p_2,q,r}}. \tag{2-21}$$

In particular, for $\frac{n}{2}<p<\infty$ and $0\leq\alpha<\min\{1-\frac{n}{2p},\frac{n}{2p}\}$, it follows that

$$\|f\|_{\dot{B}W\dot{K}^{\alpha,s}_{2p,q,r}} \leq C\|f\|_{\dot{B}W\dot{K}^{2\alpha,\alpha+s+n/(2p)}_{p,q,r}}. \tag{2-22}$$

Proof. Using the Hölder inequality, it follows that

$$\|\Delta_j f\|_{W\dot{K}^{\alpha}_{p,q,r}} \leq C\|\Delta_j f\|_{W\dot{K}^{\alpha+n(1/p-1/p_1)}_{p_1,q,r}}.$$

Also, we have that $\varphi_j \hat{f} = \tilde{\varphi}_j \varphi_j \hat{f}$, that is, $\Delta_j f = (\tilde{\varphi}_j)^\vee * \Delta_j f$. So, using Lemma 2.2 we get

$$\begin{aligned}\|\Delta_j f\|_{W\dot{K}^{\alpha+n(1/p-1/p_1)}_{p_1,q,r}} &= \|(\tilde{\varphi}_j)^\vee * \Delta_j f\|_{W\dot{K}^{\alpha+n(1/p-1/p_1)}_{p_1,q,r}} \\ &\le C \max\left\{\|(\tilde{\varphi}_j)^\vee\|_{L^{p*}}, \||\cdot|^{\frac{n}{p*}}(\tilde{\varphi}_j)^\vee\|_{L^\infty}\right\} \|\Delta_j f\|_{W\dot{K}^{\alpha+n(1/p-1/p_1)}_{p_2,q,r}},\end{aligned}$$

where $1 + \frac{1}{p_1} = \frac{1}{p*} + \frac{1}{p_2}$. It is easy to check that

$$\max\left\{\|(\tilde{\varphi}_j)^\vee\|_{L^{p*}}, \||\cdot|^{n/p*}(\tilde{\varphi}_j)^\vee\|_{L^\infty}\right\} \le C 2^{jn(1/p_2-1/p_1)},$$

and then

$$\|\Delta_j f\|_{W\dot{K}^{\alpha+n(1/p-1/p_1)}_{p_1,q,r}} \le C 2^{jn(1/p_2-1/p_1)} \|\Delta_j f\|_{W\dot{K}^{\alpha+n(1/p-1/p_1)}_{p_2,q,r}},$$

which gives (2-21). We conclude the proof by noting that for $0 \le \alpha < n/2p$ there exists p_1 such that $p_1 \ge 2p$ and $\alpha = n\left(\frac{1}{p} - \frac{1}{p_1}\right)$. Moreover, $\alpha < n + \frac{n}{p_1} - \frac{1}{p} - \frac{n}{2p}$ because $\alpha < 1 - \frac{n}{2p} \le \frac{n}{2} - \frac{n}{2p}$. So, (2-22) follows from (2-21) by choosing this value of p_1. □

We finish this section with a result that provides a characterization of homogeneous Besov-weak-Herz spaces as interpolation of two homogeneous Sobolev-weak-Herz ones.

Lemma 2.14. *Let $s_0, s_1, s \in \mathbb{R}$, $1 < p < \infty$, $1 \le q, r \le \infty$ and $-\frac{n}{p} < \alpha < n\left(1 - \frac{1}{p}\right)$. If $s_0 \ne s_1$ and $s = (1-\theta)s_0 + \theta s_1$ with $\theta \in (0, 1)$, then*

$$(W\dot{K}^{\alpha,s_0}_{p,q}, W\dot{K}^{\alpha,s_1}_{p,q})_{\theta,r} = \dot{B}W\dot{K}^{\alpha,s}_{p,q,r}.$$

Proof. Let $f = f_0 + f_1$ with $f_i \in W\dot{K}^{\alpha,s_i}_{p,q}$ $i = 0, 1$. By using Lemma 2.3 we get

$$\begin{aligned}\|\Delta_j f\|_{W\dot{K}^{\alpha}_{p,q}} &\le \|\Delta_j f_0\|_{W\dot{K}^{\alpha}_{p,q}} + \|\Delta_j f_1\|_{W\dot{K}^{\alpha}_{p,q}} \\ &\le C(2^{-s_0 j}\|I^{s_0}\Delta_j f_0\|_{W\dot{K}^{\alpha}_{p,q}} + 2^{-s_1 j}\|I^{s_1}\Delta_j f_1\|_{W\dot{K}^{\alpha}_{p,q}}) \\ &\le C(2^{-s_0 j}\|I^{s_0} f_0\|_{W\dot{K}^{\alpha}_{p,q}} + 2^{-s_1 j}\|I^{s_1} f_1\|_{W\dot{K}^{\alpha}_{p,q}}) \\ &\le C 2^{-s_0 j}(\|f_0\|_{W\dot{K}^{\alpha,s_0}_{p,q}} + 2^{(s_0-s_1)j}\|f_1\|_{W\dot{K}^{\alpha,s_1}_{p,q}}).\end{aligned} \tag{2-23}$$

It follows from (2-23) that

$$\|\Delta_j f\|_{W\dot{K}^{\alpha}_{p,q}} \le C 2^{-s_0 j} K(2^{(s_0-s_1)j}, f, W\dot{K}^{\alpha,s_0}_{p,q}, W\dot{K}^{\alpha,s_1}_{p,q}).$$

Noting that $s - s_0 = -\theta(s_0 - s_1)$ and multiplying the previous inequality by 2^{js}, we arrive at

$$2^{sj}\|\Delta_j f\|_{W\dot{K}^{\alpha}_{p,q}} \le C(2^{(s_0-s_1)j})^{-\theta} K(2^{(s_0-s_1)j}, f, W\dot{K}^{\alpha,s_0}_{p,q}, W\dot{K}^{\alpha,s_1}_{p,q}),$$

and then (see [Bergh and Löfström 1976, Lemma 3.1.3]) we can conclude that

$$\|f\|_{\dot{B}W\dot{K}^{\alpha,s}_{p,q,r}} \le C\|f\|_{(W\dot{K}^{\alpha,s_0}_{p,q}, W\dot{K}^{\alpha,s_1}_{p,q})_{\theta,r}}.$$

To prove the reverse inequality, note that by using Lemma 2.3 again we have

$$\begin{aligned}
&2^{(s-s_0)j} J(2^{(s_0-s_1)j}, \Delta_j f, W\dot{K}^{\alpha,s_0}_{p,q}, W\dot{K}^{\alpha,s_1}_{p,q}) \\
&\qquad = 2^{(s-s_0)j} \max\{\|\Delta_j f\|_{W\dot{K}^{\alpha,s_0}_{p,q}}, 2^{(s_0-s_1)j}\|\Delta_j f\|_{W\dot{K}^{\alpha,s_1}_{p,q}}\} \\
&\qquad \le 2^{(s-s_0)j} \max\{2^{s_0 j}\|\Delta_j f\|_{W\dot{K}^{\alpha}_{p,q}}, 2^{s_0 j}\|\Delta_j f\|_{W\dot{K}^{\alpha}_{p,q}}\} \\
&\qquad \le 2^{sj} \max\{\|\Delta_j f\|_{W\dot{K}^{\alpha}_{p,q}}, \|\Delta_j f\|_{W\dot{K}^{\alpha}_{p,q}}\} \\
&\qquad = 2^{sj}\|\Delta_j f\|_{W\dot{K}^{\alpha}_{p,q}}.
\end{aligned}$$

Now the equivalence theorem (see [Bergh and Löfström 1976, Lemma 3.2.3]) leads us to

$$\|f\|_{(W\dot{K}^{\alpha,s_0}_{p,q}, W\dot{K}^{\alpha,s_1}_{p,q})_{\theta,r}} \le C\|f\|_{\dot{B}W\dot{K}^{\alpha,s}_{p,q,r}}.$$

The remainder of the proof is to show that in fact $f \in \dot{B}W\dot{K}^{\alpha,s}_{p,q,r}$ implies that $f \in W\dot{K}^{\alpha,s_0}_{p,q} + W\dot{K}^{\alpha,s_1}_{p,q}$. Suppose that $s_0 > s_1$ (without loss of generality). Using the decomposition $f = \sum_{j<0} \Delta_j f + \sum_{j\ge 0} \Delta_j f = f_0 + f_1$ and Lemma 2.3, we obtain

$$\begin{aligned}
\|f_0\|_{W\dot{K}^{\alpha,s_0}_{p,q}} &\le \sum_{j<0} \|\Delta_j f\|_{W\dot{K}^{\alpha,s_0}_{p,q}} \le \sum_{j<0} 2^{j(s_0-s)} 2^{js}\|\Delta_j f\|_{W\dot{K}^{\alpha}_{p,q}} \\
&\le C\Big(\sum_{j<0} 2^{j(s_0-s)r'}\Big)^{1/r'} \Big(\sum_{j<0} 2^{jsr}\|\Delta_j f\|^r_{W\dot{K}^{\alpha}_{p,q}}\Big)^{1/r} \\
&\le C\|f\|_{\dot{B}W\dot{K}^{\alpha,s}_{p,q,r}}.
\end{aligned}$$

Similarly, one has

$$\begin{aligned}
\|f_1\|_{W\dot{K}^{\alpha,s_1}_{p,q}} &\le \sum_{j\ge 0} \|\Delta_j f\|_{W\dot{K}^{\alpha,s_1}_{p,q}} \le \sum_{j\ge 0} 2^{j(s_1-s)} 2^{js}\|\Delta_j f\|_{W\dot{K}^{\alpha}_{p,q}} \\
&\le C\Big(\sum_{j\ge 0} 2^{j(s_1-s)r'}\Big)^{1/r'} \Big(\sum_{j\ge 0} 2^{jsr}\|\Delta_j f\|^r_{W\dot{K}^{\alpha}_{p,q}}\Big)^{1/r} \\
&\le C\|f\|_{\dot{B}W\dot{K}^{\alpha,s}_{p,q,r}}
\end{aligned}$$

and then we are done. □

3. Proof of Theorem 1.1

In the previous sections, we have derived key properties about homogeneous Besov-weak-Herz spaces. With these results in hand, we prove Theorem 1.1 in the present section.

3A. *Heat kernel estimates.* We start by providing estimates for the heat semigroup $\{G(t)\}_{t\geq 0}$ in Besov-weak-Herz spaces. Recall that in the whole space $\mathbb{R}^n$ this semigroup can be defined as $G(t)f = (\exp(-t|\xi|^2)\hat{f})^{\vee}$ for all $f \in \mathcal{S}'$ and $t \geq 0$.

Lemma 3.1. *Let* $s, \sigma \in \mathbb{R}$, $s \leq \sigma$, $1 < p < \infty$, $1 \leq q, r \leq \infty$ *and* $-\frac{n}{p} < \alpha < n\left(1-\frac{1}{p}\right)$. *Then, there is* $C > 0$ (*independent of* f) *such that*

$$\|G(t)f\|_{\dot{B}W\dot{K}^{\alpha,\sigma}_{p,q,r}} \leq Ct^{(s-\sigma)/2}\|f\|_{\dot{B}W\dot{K}^{\alpha,s}_{p,q,r}}, \tag{3-1}$$

for all $t > 0$. *Moreover*, *if* $s < \sigma$, *then we have the estimate*

$$\|G(t)f\|_{\dot{B}W\dot{K}^{\alpha,\sigma}_{p,q,1}} \leq Ct^{(s-\sigma)/2}\|f\|_{\dot{B}W\dot{K}^{\alpha,s}_{p,q,\infty}}, \tag{3-2}$$

for all $t > 0$.

Proof. Firstly, observe that for each multi-index β there is a polynomial $p_\beta(\cdot)$ of degree $|\beta|$ such that

$$\partial_\xi^\beta(\exp(-t|\xi|^2)) = t^{|\beta|/2}p_\beta(\sqrt{t}\xi)\exp(-t|\xi|^2).$$

Therefore, for some $C > 0$ it follows that

$$|\partial_\xi^\beta(\exp(-t|\xi|^2))| \leq Ct^{-m/2}|\xi|^{-m-|\beta|}.$$

By employing Lemma 2.11, we obtain

$$\|G(t)f\|_{\dot{B}W\dot{K}^{\alpha,s-m}_{p,q,r}} \leq Ct^{-m/2}\|f\|_{\dot{B}W\dot{K}^{\alpha,s}_{p,q,r}}.$$

Taking now $m = s - \sigma$ we arrive at the inequality (3-1).

Next we turn to (3-2) and let $s < \sigma$. From (3-1) with $r = \infty$ we get

$$\|G(t)f\|_{\dot{B}W\dot{K}^{\alpha,2\sigma-s}_{p,q,\infty}} \leq Ct^{s-\sigma}\|f\|_{\dot{B}W\dot{K}^{\alpha,s}_{p,q,\infty}}$$

and

$$\|G(t)f\|_{\dot{B}W\dot{K}^{\alpha,s}_{p,q,\infty}} \leq C\|f\|_{\dot{B}W\dot{K}^{\alpha,s}_{p,q,\infty}}.$$

By using Lemma 2.14 and the reiteration theorem (see [Bergh and Löfström 1976, Theorem 3.5.3 and its remark]) we conclude that

$$G(t) : \dot{B}W\dot{K}^{\alpha,s}_{p,q,\infty} \to (\dot{B}W\dot{K}^{\alpha,2\sigma-s}_{p,q,\infty}, \dot{B}W\dot{K}^{\alpha,s}_{p,q,\infty})_{\frac{1}{2},1} = \dot{B}W\dot{K}^{\alpha,\sigma}_{p,q,1},$$

with $\|G(t)\|_{\dot{B}W\dot{K}^{\alpha,s}_{p,q,\infty}\to\dot{B}W\dot{K}^{\alpha,\sigma}_{p,q,1}} \leq Ct^{(s-\sigma)/2}$, which gives (3-2). □

3B. ***Bilinear estimate.*** Let us define the space X as

$$X=\Big\{u:(0,\infty)\to \dot{B}W\dot{K}^{\alpha,\alpha+n/p-1}_{p,q,\infty}\cap W\dot{K}^{\alpha}_{2p,2q} \text{ with } \nabla\cdot u=0 \text{ such that } \|u\|_X<\infty\Big\},$$

where

$$\|u\|_X := \|u\|_{L^\infty((0,\infty);\dot{B}W\dot{K}^{\alpha,\alpha+n/p-1}_{p,q,\infty})} + \sup_{t>0} t^{\frac{1}{2}-(\frac{\alpha}{2}+\frac{n}{4p})}\|u\|_{W\dot{K}^{\alpha}_{2p,2q}}. \tag{3-3}$$

We are going to prove the bilinear estimate

$$\|B(u,v)\|_X \le K\|u\|_X\|v\|_X. \tag{3-4}$$

We start by estimating the second part of the norm (3-3). For that, we use (2-19), (2-22), (3-2) and Lemma 2.11 in order to get

$$\begin{aligned}
\|B(u,v)(t)\|_{W\dot{K}^{\alpha}_{2p,2q}} &\le \|B(u,v)(t)\|_{\dot{B}W\dot{K}^{\alpha,0}_{2p,2q,1}}\\
&\le \|B(u,v)(t)\|_{\dot{B}W\dot{K}^{2\alpha,\alpha+n/(2p)}_{p,2q,1}}\\
&\le C\int_0^t \|G(t-\tau)\mathbb{P}\operatorname{div}(u\otimes v)\|_{\dot{B}W\dot{K}^{2\alpha,\alpha+n/(2p)}_{p,2q,1}}\,d\tau\\
&\le C\int_0^t (t-\tau)^{-\frac{1}{2}-(\frac{\alpha}{2}+\frac{n}{4p})}\|\mathbb{P}\operatorname{div}(u\otimes v)\|_{\dot{B}W\dot{K}^{2\alpha,-1}_{p,2q,\infty}}\,d\tau\\
&\le C\int_0^t (t-\tau)^{-\frac{1}{2}-(\frac{\alpha}{2}+\frac{n}{4p})}\|u\otimes v\|_{\dot{B}W\dot{K}^{2\alpha,0}_{p,2q,\infty}}\,d\tau\\
&\le C\int_0^t (t-\tau)^{-\frac{1}{2}-(\frac{\alpha}{2}+\frac{n}{4p})}\|u\otimes v\|_{W\dot{K}^{2\alpha}_{p,q}}\,d\tau\\
&\le C\int_0^t (t-\tau)^{-\frac{1}{2}-(\frac{\alpha}{2}+\frac{n}{4p})}\|u\|_{W\dot{K}^{\alpha}_{2p,2q}}\|v\|_{W\dot{K}^{\alpha}_{2p,2q}}\,d\tau\\
&\le C\int_0^t (t-\tau)^{-\frac{1}{2}-(\frac{\alpha}{2}+\frac{n}{4p})}\tau^{-2(\frac{1}{2}-(\frac{\alpha}{2}+\frac{n}{4p}))}\,d\tau\,\|u\|_X\|v\|_X\\
&\le Ct^{-\frac{1}{2}+(\frac{\alpha}{2}+\frac{n}{4p})}\mathcal{B}\Big(\alpha+\frac{n}{2p},\frac{1}{2}-\Big(\frac{\alpha}{2}+\frac{n}{4p}\Big)\Big)\|u\|_X\|v\|_X,
\end{aligned}$$

where $\mathcal{B}(\cdot\,,\cdot)$ denotes the beta function. The previous estimate leads us to

$$\sup_{t>0} t^{\frac{1}{2}-(\frac{\alpha}{2}+\frac{n}{4p})}\|B(u,v)(t)\|_{W\dot{K}^{\alpha}_{2p,2q}} \le C\|u\|_X\|v\|_X. \tag{3-5}$$

Moreover, for the first part of the norm (3-3), we have

$$\begin{aligned}
\|B(u,v)(t)\|_{\dot{B}W\dot{K}^{\alpha,\alpha+n/p-1}_{p,q,\infty}} &\le \int_0^t \|G(t-\tau)\mathbb{P}\operatorname{div}[u\otimes v]\|_{\dot{B}W\dot{K}^{\alpha,\alpha+n/p-1}_{p,q,\infty}}\,d\tau\\
&\le C\int_0^t \|G(t-\tau)\mathbb{P}\operatorname{div}[u\otimes v]\|_{\dot{B}W\dot{K}^{2\alpha,2\alpha+n/p-1}_{p,q,\infty}}\,d\tau\\
&\le C\int_0^t (t-\tau)^{-(\alpha+\frac{n}{2p})}\|\mathbb{P}\operatorname{div}[u\otimes v]\|_{\dot{B}W\dot{K}^{2\alpha,-1}_{p,q,\infty}}\,d\tau\\
&\le C\int_0^t (t-\tau)^{-(\alpha+\frac{n}{2p})}\|u\otimes v\|_{\dot{B}W\dot{K}^{2\alpha,0}_{p,q,\infty}}\,d\tau\\
&\le C\int_0^t (t-\tau)^{-(\alpha+\frac{n}{2p})}\|u\otimes v\|_{W\dot{K}^{2\alpha}_{p,q}}\,d\tau\\
&\le C\|u\|_X\|v\|_X\int_0^t (t-\tau)^{-(\alpha+\frac{n}{2p})}\tau^{-2(\frac{1}{2}-(\frac{\alpha}{2}+\frac{n}{4p}))}d\tau\\
&\le C\mathcal{B}\Big(\alpha+\frac{n}{2p},1-\Big(\alpha+\frac{n}{2p}\Big)\Big)\|u\|_X\|v\|_X.
\end{aligned}$$

In other words, we have obtained the estimate

$$\|B(u,v)\|_{L^\infty((0,\infty);\dot{B}W\dot{K}^{\alpha,\alpha+n/p-1}_{p,q,\infty})}\le C\|u\|_X\|v\|_X. \tag{3-6}$$

Finally, notice that the estimates (3-5) and (3-6) together give (3-4).

3C. ***Proof of Theorem 1.1.*** Existence and uniqueness. For $\epsilon>0$ (to be chosen later) let $\bar{B}(0,\epsilon)$ denote the closed ball in X and define the operator $\Psi:\bar{B}(0,2\epsilon)\to\bar{B}(0,2\epsilon)$ as

$$\Psi(u)=G(t)u_0+B(u,u).$$

First, note that by using (2-19), (3-2) , $\alpha+\frac{n}{2p}-1<0$ and (2-21) it follows that

$$\begin{aligned}
\sup_{t>0}t^{\frac{1}{2}-(\frac{\alpha}{2}+\frac{n}{4p})}\|G(t)u_0\|_{W\dot{K}^{\alpha}_{2p,2q}} &\le C\sup_{t>0}t^{\frac{1}{2}-(\frac{\alpha}{2}+\frac{n}{4p})}\|G(t)u_0\|_{\dot{B}W\dot{K}^{\alpha,0}_{2p,2q,1}}\\
&\le C\|u_0\|_{\dot{B}W\dot{K}^{\alpha,\alpha+n/(2p)-1}_{2p,2q,\infty}}\le C\|u_0\|_{\dot{B}W\dot{K}^{\alpha,\alpha+n/p-1}_{p,q,\infty}}.
\end{aligned} \tag{3-7}$$

Moreover, using (3-1) we obtain

$$\|G(t)u_0\|_{\dot{B}W\dot{K}^{\alpha,\alpha+n/p-1}_{p,q,\infty}}\le C\|u_0\|_{\dot{B}W\dot{K}^{\alpha,\alpha+n/p-1}_{p,q,\infty}}.$$

From the last two estimates, we get

$$\|G(t)u_0\|_X\le C\|u_0\|_{\dot{B}W\dot{K}^{\alpha,\alpha+n/p-1}_{p,q,\infty}}. \tag{3-8}$$

Take $0 < \epsilon < 1/4K$ and $0 < \delta < \epsilon/C$ where C is as in (3-8). It follows from (3-8) and (3-4) that

$$\begin{aligned}\|\Psi(u)\|_X &\leq \|G(t)u_0\|_X + \|B(u,u)\|_X \\ &\leq C\|u_0\|_{\dot{B}W\dot{K}^{\alpha,\alpha+n/p-1}_{p,q,\infty}} + K\|u\|_X\|v\|_X \leq 2\epsilon.\end{aligned}$$

So, Ψ is well defined; moreover for $u, v \in \bar{B}(0, 2\epsilon)$ we have that

$$\begin{aligned}\|\Psi(u)-\Psi(v)\|_X &= \|B(u-v,u)+B(v,u-v)\|_X \\ &\leq K\|u-v\|_X\|u\|_X + K\|v\|_X\|u-v\|_X \\ &\leq 4K\epsilon\|u-v\|_X.\end{aligned} \tag{3-9}$$

Since $4K\epsilon < 1$, we get that Ψ is a contraction and then this part is concluded by the Banach fixed-point theorem. Notice that the continuous dependence with respect to the initial data u_0 follows from estimates (3-8) and (3-9).

Time-weak continuity at $t = 0$. The proof of the weak-$*$ convergence follows from the two following lemmas.

The first one is due to Kozono and Yamazaki [1994, p. 989.].

Lemma 3.2. *For every real number s and $u_0 \in \dot{B}^s_{\infty,\infty}$, we have $G(t)u_0 \overset{*}{\rightharpoonup} u_0$ in $\dot{B}^s_{\infty,\infty}$ as $t \to 0^+$.*

The second one is concerned with the weak-convergence of the bilinear term $B(u,u)$ and it concludes the proof.

Lemma 3.3. *Let $v \in X$. We have that $B(v,v)(t)$ converges to 0 in the weak-$*$ topology of $\dot{B}^{-1}_{\infty,\infty}$ as $t \to 0^+$.*

Proof. Let $\phi \in \dot{B}^1_{1,1}$ and $\epsilon > 0$ be an arbitrary number. We can choose $\tilde{\phi} \in \mathcal{S}$ such that $\|\phi - \tilde{\phi}\|_{\dot{B}^1_{1,1}} < \epsilon$. Then we have that

$$\begin{aligned}&|\langle B(v,v)(t), \phi-\tilde{\phi}\rangle| \\ &\quad\leq \|B(v,v)(t)\|_{\dot{B}^{-1}_{\infty,\infty}}\|\phi-\tilde{\phi}\|_{\dot{B}^1_{1,1}} \\ &\quad\leq C\|B(v,v)(t)\|_{\dot{B}W\dot{K}^{\alpha,\alpha+n/p-1}_{p,q,r}}\|\phi-\tilde{\phi}\|_{\dot{B}^1_{1,1}} \leq K\|v\|_X^2\epsilon \leq C\epsilon.\end{aligned} \tag{3-10}$$

On the other hand,

$$\begin{aligned}|\langle B(v,v)(t),\tilde{\phi}\rangle| &\leq \int_0^t \big|\langle G(t-\tau)\mathbb{P}\,\mathrm{div}[v\otimes v](\tau),\tilde{\phi}\rangle\big|\,d\tau \\ &\leq \int_0^t \big|\langle \mathbb{P}\,\mathrm{div}[v\otimes v](\tau), G(t-\tau)\tilde{\phi}\rangle\big|\,d\tau \\ &\leq \int_0^t \|\mathrm{div}[v\otimes v](\tau)\|_{\dot{B}^{-1-2\alpha-n/p}_{\infty,\infty}}\|G(t-\tau)\tilde{\phi}\|_{\dot{B}^{1+2\alpha+n/p}_{1,1}}\,d\tau\end{aligned} \tag{3-11}$$

$$
\begin{aligned}
&\le C_{\tilde{\phi}} \int_0^t \|[v\otimes v](\tau)\|_{\dot{B}^{-2\alpha-n/p}_{\infty,\infty}} d\tau \\
&\le C_{\tilde{\phi}} \int_0^t \|[v\otimes v](\tau)\|_{W\dot{K}^{2\alpha}_{p,q}} d\tau \\
&\le C_{\tilde{\phi}} \int_0^t \tau^{[-\frac{1}{2}+(\frac{\alpha}{2}+\frac{n}{4p})]\cdot 2}\tau^{[\frac{1}{2}-(\frac{\alpha}{2}+\frac{n}{4p})]\cdot 2}\|v(\tau)\|^2_{W\dot{K}^{\alpha}_{2p,2q}} d\tau \\
&\le C_{\tilde{\phi}} \|v\|_X^2 \int_0^t \tau^{-1+\alpha+\frac{n}{2p}} d\tau \le C_{\tilde{\phi}} \|v\|_X^2 t^{\alpha+\frac{n}{2p}}.
\end{aligned}
$$

From (3-10) and (3-11), we obtain

$$
\begin{aligned}
0 &\le \limsup_{t\to 0^+} |\langle B(v,v)(t),\phi\rangle| \\
&\le \limsup_{t\to 0^+} |\langle B(v,v)(t),\phi-\tilde{\phi}\rangle| + \limsup_{t\to 0^+} |\langle B(v,v)(t),\tilde{\phi}\rangle| \le C\epsilon + 0.
\end{aligned}
$$

Since $\epsilon > 0$ is arbitrary, we conclude that $\lim_{t\to 0^+}|\langle B(v,v)(t),\phi\rangle| = 0$. Now, using that $\phi \in \dot{B}^1_{1,1}$ is arbitrary, we get the desired convergence. □

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LUCAS C. F. FERREIRA
DEPARTAMENTO DE MATEMÁTICA
UNIVERSIDADE ESTADUAL DE CAMPINAS, IMECC
CAMPINAS-SP
BRAZIL
lcff@ime.unicamp.br

JHEAN E. PÉREZ-LÓPEZ
ESCUELA DE MATEMÁTICAS
UNIVERSIDAD INDUSTRIAL DE SANTANDER
BUCARAMANGA
COLOMBIA
jhean.perez@uis.edu.co

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FOUR-MANIFOLDS WITH POSITIVE YAMABE CONSTANT

HAI-PING FU

We refine a theorem due to Gursky (2000). As applications, we give some rigidity theorems on four-manifolds with positive Yamabe constant. We recover some of Gursky's results (1998, 2000). We prove some classification theorems of four-manifolds according to some conformal invariants, which reprove and generalize the conformally invariant sphere theorem of Chang, Gursky and Yang (2003).

1. Introduction and main results

In [Fu 2017], the author proved that an n-manifold with harmonic curvature is isometric to a quotient of the standard sphere or Einstein manifold, if the upper bound of some curvature functional is given by Yamabe constant. By this we mean that we can precisely characterize the case of equality. The aim of this paper is to present some rigidity results in the subject of curvature pinching on four-manifolds with positive Yamabe constant.

Let (M^n, g) be an n-dimensional Riemannian manifold. The decomposition of the Riemannian curvature tensor Rm into irreducible components yields

$$Rm = W + \frac{1}{n-2}\mathring{\mathrm{Ric}} \owedge g + \frac{R}{2n(n-1)} g \owedge g,$$

where W, Ric, $\mathring{\mathrm{Ric}} = \mathrm{Ric} - (R/n)g$ and R denote the Weyl curvature tensor, Ricci tensor, trace-free Ricci tensor and scalar curvature, respectively. When the divergence of the Weyl curvature tensor W is vanishing, i.e., $\delta W = 0$, (M^n, g) is said to be a manifold with harmonic Weyl tensor. The norm of a (k,l)- tensor $T = T_{i_1 \cdots i_k}^{j_1 \cdots j_l}$ deduced by the Riemannian metric g is defined as

$$|T|^2 = g^{i_1 m_1} \cdots g^{i_k m_k} g_{j_1 n_1} \cdots g_{j_l n_l} T_{i_1 \cdots i_k}^{j_1 \cdots j_l} T_{m_1 \cdots m_k}^{n_1 \cdots n_l}.$$

The sphere theorem for $\frac{1}{4}$-pinched Riemannian manifolds, conjectured by Rauch in 1951, is a good example of the deep connections between the topology and the

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geometry of Riemannian manifolds. Now we know that the answer is positive, due to the fundamental work of Klingenberg, Berger and Rauch for the topological statement and the recent proof of the original conjecture by Brendle and Schoen [2008], based on the results of Böhm and Wilking [2008].

From the work of Huisken [1985] and Margerin [1998], we know that there exists a positive-dimensional constant $C(n)$ such that if

$$|W + 1/(n-2)\mathring{\mathrm{R}}\mathrm{ic} \owedge g|^2 < C(n)R^2,$$

then M^n is diffeomorphic to a quotient of the standard unit sphere. In particular, Margerin improved the constant in dimension four, and obtained the optimal theorem in [Margerin 1998].

The common feature of all the above results is to give topological information on a manifold that carries a metric whose curvature satisfies a certain pinching at each point. The question one raises here is whether one can characterize the topology and the geometry of Riemannian manifolds by means of integral pinching conditions instead of pointwise ones. Some results in this direction on four manifolds were obtained in [Bour and Carron 2015; Chang et al. 2003; Chen and Zhu 2014; Gursky 1998; Gursky 2000; Hebey and Vaugon 1996].

In four-manifolds, the Weyl functional $\int |W_g|^2$ has long been an object of interest to physicists. Suppose M^4 is a 4-dimensional manifold. Then the Hodge $*$-operator induces a splitting of the space of two-forms $\wedge^2 = \wedge^2_+ + \wedge^2_-$ into the subspace of self-dual forms $\wedge^2_+$ and anti-self-dual forms $\wedge^2_-$. This splitting in turn induces a decomposition of the Weyl curvature into its self-dual and anti-self-dual components $W^\pm$. A four-manifold is said to be self-dual (resp., anti-self-dual) if $W^- = 0$ (resp., $W^+ = 0$). It is said to be a manifold with half harmonic Weyl tensor if $\delta W^\pm = 0$. By the Hirzebruch signature formula (see [Besse 1987]),

$$\int_M |W^+|^2 - \int_M |W^-|^2 = 48\pi^2\sigma(M), \tag{1-1}$$

where $\sigma(M)$ denotes the signature of M. A consequence of (1-1) is that the study of the Weyl functional is completely equivalent to the study of the self-dual Weyl functional $\int_M |W^+|^2$. M. J. Gursky [1998; 2000] has obtained some good and interesting results by studying $\int_M |W^+|^2$ (see Theorems A, B and C). For background material on this condition we recommend [Besse 1987, Chapter 16] and [Derdziński 1983].

Our formulation of some results will be given in terms of the Yamabe invariant. Now we introduce the definition of the Yamabe constant. Given a compact Riemannian n-manifold M, we consider the Yamabe functional

$$Q_g : C^\infty_+(M) \to \mathbb{R} : f \mapsto Q_g(f) = \frac{\frac{4(n-1)}{n-2}\int_M |\nabla f|^2 \,\mathrm{d}v_g + \int_M Rf^2 \,\mathrm{d}v_g}{\left(\int_M f^{\frac{2n}{n-2}} \,\mathrm{d}v_g\right)^{\frac{n-2}{n}}},$$

where R denotes the scalar curvature of M. It follows that Q_g is bounded below by the Hölder inequality. We set

$$\mu([g]) = \inf\{Q_g(f) \mid f \in C^\infty_+(M)\}.$$

This constant $\mu([g])$ is an invariant of the conformal class of (M, g), called the Yamabe constant. The important works of Aubin, Schoen, Trudinger and Yamabe showed that the infimum in the above is always achieved; see [Aubin 1998; Lee and Parker 1987]. The Yamabe constant of a given compact manifold is determined by the sign of scalar curvature [Aubin 1998]. The scalar curvature $R_{\tilde{g}}$ of a conformal metric $\tilde{g} = f^{4/(n-2)} g$ is equal to $\mu([g])/\mathrm{Vol}(g)^{2/n}$. We call such a metric $\tilde{g}$ a Yamabe minimizer.

Gursky [1998; 2000] proved the three striking Theorems A, B and C, and as byproducts obtained these integral pinching results, which are generalizations of the Bochner theorem in dimension 4 (see Propositions E, F and G).

Theorem A [Gursky 1998]. *Let (M^4, g) be a 4-dimensional compact Riemannian manifold with positive Yamabe constant and the space of self-dual harmonic two-forms $H^2_+(M^4) \neq 0$. Then*

$$\int_M |W^+|^2 \geq \tfrac{16}{3}\pi^2(2\chi(M^4) + 3\sigma(M^4)),$$

where $\chi(M)$ is the Euler–Poincaré characteristic of M. Furthermore, equality holds in the above inequality if and only if g is conformal to a positive Kähler–Einstein metric.

Theorem B [Gursky 2000]. *Let (M^4, g) be a 4-dimensional compact Riemannian manifold with positive Yamabe constant and $\delta W^+ = 0$. Then either (M^4, g) is anti-self-dual, or*

$$\int_M |W^+|^2 \geq \tfrac{16}{3}\pi^2(2\chi(M^4) + 3\sigma(M^4)).$$

Furthermore, equality holds in the above inequality if and only if g is a positive Einstein metric which is either Kähler, or the quotient of a Kähler manifold by a free, isometric, antiholomorphic involution.

Theorem C [Gursky 1998]. *Let (M^4, g) be a 4-dimensional compact Riemannian manifold with positive Yamabe constant and the space of harmonic 1-forms $H^1(M^4) \neq 0$. Then*

$$\int_M |W^+|^2 \geq 8\pi^2(2\chi(M^4) + 3\sigma(M^4)).$$

Furthermore, equality holds in the above inequality if and only if (M^4, g) is conformal to a quotient of $\mathbb{R}^1 \times \mathbb{S}^3$ with the product metric.

Chang, Gursky and Yang [Chang et al. 2003] proved that a four manifold with positive Yamabe constant which satisfies the strict inequality for the Weyl functional $\int |W|^2$ is actually diffeomorphic to a quotient of the sphere and precisely characterizes the case of equality. We state this result of Chang, Gursky and Yang as follows:

Theorem D [Chang et al. 2003]. *Let (M^4, g) be a 4-dimensional compact Riemannian manifold with positive Yamabe constant. If*

$$\int_M |W|^2 \leq 16\pi^2 \chi(M),$$

then one of the following must be true:

(1) *M^4 is diffeomorphic to the round sphere $\mathbb{S}^4$ or the real projective space $\mathbb{RP}^4$.*

(2) *M^4 is conformal to a manifold which is isometrically covered by $\mathbb{S}^1 \times \mathbb{S}^3$ with the product metric.*

(3) *M^4 is conformal to the complex projective plane $\mathbb{CP}^2$ with the Fubini–Study metric.*

Bour and Carron [2015] reprove and extend to higher degrees and higher dimensions Propositions F and G obtained by Gursky. Bour [2010] gives a new proof of Theorem D under a stronger pinching assumption, which is entirely based on the study of a geometric flow, and doesn't rely on the pointwise version of the theorem, due to Margerin. Chen and Zhu [2014] proved a classification theorem of 4-manifolds according to some conformal invariants, which generalizes the conformally invariant sphere theorem in [Chang et al. 2003], i.e., Theorem D under the strict inequality assumption, and relies on Chen, Tang and Zhu's classification on four-manifolds with positive isotropic curvature [Chen et al. 2012].

In this note, we refine Theorems B and Proposition E due to Gursky, and obtain Theorem 1.1 which can not be deduced from the Gursky's proof [2000] of Theorem B as follows:

Theorem 1.1. *Let (M^4, g) be a 4-dimensional compact Riemannian manifold with $\delta W^{\pm} = 0$ and positive Yamabe constant $\mu([g])$. If*

$$\int_M |W^{\pm}|^2 = \tfrac{1}{6}\mu^2([g]), \tag{1-2}$$

then $\nabla W^{\pm} = 0$ and $W^{\pm}$ has exactly two distinct eigenvalues at each point. Hence (M^4, g) is a Kähler manifold of positive constant scalar curvature.

Theorem 1.2. *Let (M^4, g) be a 4-dimensional compact Riemannian manifold with harmonic Weyl tensor and positive Yamabe constant. If*

$$\int_M |W|^2 = \tfrac{1}{6}\mu^2([g]),$$

then (M^4, g) *is* $\mathbb{CP}^2$ *with the Fubini–Study metric.*

Combing some results due to Gursky, and Chen, Tang and Zhu's classification on four-manifolds with positive isotropic curvature with Theorem 1.1, we give the following Theorem 1.3 which generalizes the conformally invariant sphere theorem of [Chang et al. 2003], i.e., Theorem D.

Theorem 1.3. *Let* (M^4, g) *be a* 4*-dimensional compact Riemannian manifold with positive Yamabe constant. If*

$$\int_M |W|^2 \le \tfrac{1}{6}\mu^2([g]),$$

then one of the following must be true:

(1) $\tilde{g}$ *is a Yamabe minimizer and* $(M^4, \tilde{g})$ *is* $\mathbb{CP}^2$ *with the Fubini–Study metric.*

(2) (M^4, g) *is diffeomorphic to* $\mathbb{S}^4$, $\mathbb{RP}^4$, $\mathbb{S}^3 \times \mathbb{R}/G$ *or a connected sum of them. Here* G *is a cocompact fixed point free discrete subgroup of the isometry group of the standard metric on* $\mathbb{S}^3 \times \mathbb{R}$.

Theorem 1.4. *Let* (M^4, g) *be a* 4*-dimensional compact Riemannian manifold with harmonic Weyl tensor and positive Yamabe constant. If*

$$\int_M |W|^2 < \tfrac{64}{3}\pi^2\chi(M),$$

then one of the following must be true:

(1) $\tilde{g}$ *is a Yamabe minimizer and* $(M^4, \tilde{g})$ *is the round sphere* $\mathbb{S}^4$ *or the real projective space* $\mathbb{RP}^4$.

(2) (M^4, g) *is* $\mathbb{CP}^2$ *with the Fubini–Study metric.*

Theorem 1.5. *Let* (M^4, g) *be a* 4*-dimensional compact Riemannian manifold with harmonic Weyl tensor and positive Yamabe constant. If*

$$\int_M |W|^2 = \tfrac{64}{3}\pi^2\chi(M),$$

then one of the following must be true:

(1) $\tilde{g}$ *is a Yamabe minimizer and* $(M^4, \tilde{g})$ *is the manifold which is isometrically covered by* $\mathbb{S}^1 \times \mathbb{S}^3$ *with the product metric or* $\mathbb{S}^1 \times \mathbb{S}^3$ *with a rotationally symmetric Derdziński metric* (*see* [Catino 2016b; Derdziński 1982]).

(2) (M^4, g) *is isometric to a quotient of* $\mathbb{S}^2 \times \mathbb{S}^2$ *with the product metric.*

Theorem 1.6. *Let* (M^4, g) *be a* 4*-dimensional compact Riemannian manifold which is not homeomorphic to* $\mathbb{S}^4$ *or* $\mathbb{RP}^4$ *with positive Yamabe constant. If*

$$16\pi^2\chi(M) < \int_M |W|^2 \le \tfrac{64}{3}\pi^2\chi(M),$$

then one of the following must be true:

(1) $\tilde{g}$ *is a Yamabe minimizer and* $(M^4, \tilde{g})$ *is isometric to a quotient of* $\mathbb{S}^2 \times \mathbb{S}^2$ *with the product metric.*

(2) (M, g) *has* $\chi(M) = 3,\ b_1 = 0$ *and* $b_2 = 1$, *where* b_i *denotes the* i*-th Betti number of* M, *and does not have a harmonic Weyl tensor.*

Theorem 1.7. *Let* (M^4, g) *be a* 4*-dimensional compact Riemannian manifold with harmonic curvature and positive scalar curvature. If*

$$\int_M |W|^2 \le \tfrac{1}{3}\mu^2([g]),$$

then one of the following must be true:

(1) (M^4, g) *is conformally flat with positive constant scalar curvature.*

(2) (M^4, g) *is* $\mathbb{CP}^2$ *with the Fubini–Study metric.*

(3) (M^4, g) *is isometric to a quotient of* $\mathbb{S}^2 \times \mathbb{S}^2$ *with the product metric.*

Corollary 1.8. Let (M^4, g) be a 4-dimensional complete Einstein manifold with positive scalar curvature. If

$$\int_M |W|^2 \le \tfrac{1}{3}\mu^2([g]),$$

then one of the following must be true:

(1) (M^4, g) is isometric to either $\mathbb{S}^4$ or $\mathbb{RP}^4$.

(2) (M^4, g) is isometric to $\mathbb{CP}^2$ with the Fubini–Study metric.

(3) (M^4, g) is isometric to a quotient of $\mathbb{S}^2 \times \mathbb{S}^2$ with the product metric.

Remark 1.9. For Riemannian manifolds with harmonic curvature and dimensions $n \ge 4$, the author proved some similar results in [Fu 2017].

2. Four manifolds with half harmonic Weyl tensor

In order to prove some results in this article, we need the following Weyl estimate proved by Gursky [2000].

Proposition E [Gursky 2000]. *Let* (M^4, g) *be a* 4*-dimensional compact Riemannian manifold with* $\delta W^{\pm} = 0$ *and positive Yamabe constant* $\mu([g])$. *If*

$$\int_M |W^{\pm}|^2 < \tfrac{1}{6}\mu^2([g]),$$

then (M^4, g) *is anti-self-dual* (*resp.*, *self-dual*).

Remark 2.1. Gursky [2000] obtained an improved Kato inequality $|\nabla W^+|^2 \geq \frac{5}{3}\big|\nabla|W^+|\big|^2$. Thus using the Bochner technique, Gursky proved Theorem B and Propositions E, F and G by introducing the corresponding functional and conformal invariant with the modified scalar curvature $R-\sqrt{6}|W^\pm|$ in [Gursky 2000]. Based on Gursky's improved Kato inequality, we can reprove Theorem B and Propositions E, F and G only by using the modified Bochner technique (see [Bour and Carron 2015; Fu 2017; Fu and Li 2010]). In order to prove Theorem 1.1 which can not be deduced from Gursky's proofs of Theorem B and Proposition E, we also need the following different proof of Proposition E.

Proof. First, we recall the following Weitzenböck formula (see [Besse 1987] and [Bour 2010])

$$\triangle|W^\pm|^2 = 2|\nabla W^\pm|^2 + R|W^\pm|^2 - 144\det\nolimits_{\wedge^2_\pm} W^\pm. \tag{2-1}$$

From (2-1), by the Kato inequality $|\nabla W^+|^2 \geq \frac{5}{3}\big|\nabla|W^+|\big|^2$ ([Gursky 2000]), we obtain

$$|W^\pm|\triangle|W^\pm| \geq \tfrac{2}{3}\big|\nabla|W^\pm|\big|^2 + \tfrac{1}{2}R|W^\pm|^2 - 72\det\nolimits_{\wedge^2_\pm} W^\pm. \tag{2-2}$$

By a simple Lagrange multiplier argument it is easily verified that

$$-144\det\nolimits_{\wedge^2_\pm} W^\pm \geq -\sqrt{6}|W^\pm|^3 \tag{2-3}$$

and equality is attained at a point where $W^\pm \neq 0$ if and only if $W^\pm$ has precisely two eigenvalues. By (2-2) and (2-3), we get

$$|W^\pm|\triangle|W^\pm| \geq \tfrac{2}{3}\nabla|W^\pm|^2 + \tfrac{1}{2}R|W^\pm|^2 - \tfrac{\sqrt{6}}{2}|W^\pm|^3. \tag{2-4}$$

Let $u_\epsilon = \sqrt{|W^\pm|^2+\epsilon^2}$ and $u = |W^\pm|$. Thus we have

$$|\nabla u_\epsilon|^2 = \frac{u^2|\nabla u|^2}{u^2+\epsilon^2} \leq |\nabla u|^2.$$

By (2-4), we compute

$$\begin{aligned} u_\epsilon^\alpha \triangle u_\epsilon^\alpha &= u_\epsilon^\alpha(\alpha(\alpha-1)u_\epsilon^{\alpha-2}|\nabla u|^2 + \alpha u_\epsilon^{\alpha-1}\triangle u_\epsilon) \\ &= \frac{\alpha-1}{\alpha}|\nabla u_\epsilon^\alpha|^2 + \alpha u_\epsilon^{2\alpha-2}u_\epsilon\triangle u_\epsilon \\ &= \frac{\alpha-1}{\alpha}|\nabla u_\epsilon^\alpha|^2 + \alpha u_\epsilon^{2\alpha-2}\big(\tfrac{1}{2}\triangle u_\epsilon^2 - |\nabla u_\epsilon|^2\big) \\ &= \frac{\alpha-1}{\alpha}|\nabla u_\epsilon^\alpha|^2 + \alpha u_\epsilon^{2\alpha-2}(u\triangle u + |\nabla u|^2 - |\nabla u_\epsilon|^2) \\ &\geq \Big(1-\frac{1}{3\alpha}\Big)|\nabla u_\epsilon^\alpha|^2 - \frac{\sqrt{6}}{2}\alpha u_\epsilon^{2(\alpha-1)}u^3 + \frac{R\alpha}{2}u_\epsilon^{2(\alpha-1)}u^2, \end{aligned} \tag{2-5}$$

where α is a positive constant. Integrating (2-5) by parts, choosing $\alpha=\frac{1}{3}$, and letting ϵ go to zero, we get

$$\left(2-\frac{1}{3\alpha}\right)\int_M|\nabla u^\alpha|^2-\frac{\sqrt{6}}{2}\alpha\int_M u^{2\alpha+1}+\frac{\alpha}{2}\int_M Ru^{2\alpha}\le 0. \tag{2-6}$$

By the Hölder inequality and (2-6), we have

$$\left(2-\frac{1}{3\alpha}\right)\int_M|\nabla u^\alpha|^2-\frac{\sqrt{6}}{2}\alpha\left(\int_M u^{4\alpha}\right)^{\frac{1}{2}}\left(\int_M u^2\right)^{\frac{1}{2}}+\frac{\alpha}{2}\int_M Ru^{2\alpha}\le 0. \tag{2-7}$$

By the definition of Yamabe constant and (2-7), we get

$$\begin{aligned}0\ge\left(2-\frac{1}{3\alpha}\right)\frac{1}{6}\mu([g])\left(\int_M u^{4\alpha}\right)^{\frac{1}{2}}\\ -\frac{\sqrt{6}}{2}\alpha\left(\int_M u^{4\alpha}\right)^{\frac{1}{2}}\left(\int_M u^2\right)^{\frac{1}{2}}+\frac{9\alpha^2-6\alpha+1}{18\alpha}\int_M Ru^{2\alpha},\end{aligned} \tag{2-8}$$

that is,

$$0\ge\left[\frac{1}{\sqrt{6}}\mu([g])-\left(\int_M u^2\right)^{\frac{1}{2}}\right]\left(\int_M u^{\frac{4}{3}}\right)^{\frac{1}{2}}. \tag{2-9}$$

We choose $\left(\int_M|W^\pm|^2\right)<\frac{1}{6}\mu^2([g])$ such that the above inequality imply $\int_M u^{4/3}=0$, that is, $W^\pm=0$, i.e., (M^4,g) is anti-self-dual, or self-dual. □

Remark 2.2. For $0\le k\le\frac{n}{2}$, by the Kato inequality for harmonic k-form ω (see [Bourguignon 1990]) $(n+1-k)/(n-k)\big|\nabla|\omega|\big|^2\le|\nabla\omega|^2$ and the two Weitzenböck formulas in [Gursky 1998], one has

$$\tfrac{1}{2}\triangle|\omega|^2\ge|\nabla\omega|^2-\tfrac{\sqrt{6}}{3}|W^\pm||\omega|^2+\tfrac{1}{3}R|\omega|^2\ge\tfrac{3}{2}\big|\nabla|\omega|\big|^2-\tfrac{\sqrt{6}}{3}|W^\pm||\omega|^2+\tfrac{1}{3}R|\omega|^2,$$

for all $\omega\in H^2_\pm(M^4)$ and

$$\tfrac{1}{2}\triangle|\omega|^2\ge\tfrac{4}{3}|\nabla|\omega||^2-\tfrac{\sqrt{3}}{2}|\mathring{\mathrm{Ric}}||\omega|^2+\tfrac{1}{4}R|\omega|^2,\qquad\text{for all }\ \omega\in H^1(M^4).$$

Based on the above two Weitzenböck formulas, using the same argument as in the proof of Proposition E, we can obtain two results of Gursky as follows:

Proposition F [Gursky 1998; 2000]. *Let (M^4,g) be a 4-dimensional compact Riemannian manifold with positive Yamabe constant $\mu([g])$.*

(i) *If*

$$\int_M|W^\pm|^2<\tfrac{1}{6}\mu^2([g]),$$

then $H^2_\pm(M^4)=0$ and $b_2^\pm(M)=0$.

(ii) *If*

$$\int_M |\mathring{\mathrm{Ric}}|^2 < \tfrac{1}{12}\mu^2([g]),$$

then $H^1(M^4) = 0$ *and* $b_1(M) = 0$.

Proof of Theorem 1.1. Equation (1-2) implies that the equality holds in (2-9). When the equality holds in (2-9), all the inequalities leading to (2-7) become equalities. From (2-8), the function u^α attains the infimum in the Yamabe functional. From (2-7), the equality for the Hölder inequality implies that u is constant, i.e., $|W^\pm|$ is constant. Hence at every point, it has an eigenvalue of multiplicity 2 and another of multiplicity 1, i.e., $W^\pm$ has eigenvalues $\{-\frac{R}{12}, -\frac{R}{12}, \frac{R}{6}\}$, and R is constant. From (2-1), we get $\nabla W^\pm = 0$. By Proposition 5 in [Derdziński 1983], (M^4, g) is a Kähler manifold of positive constant scalar curvature. □

Remark 2.3. Since $\int_M |W^\pm|^2 \geq 16\int_M \sigma_2(A)$, we have

$$\int_M |W^\pm|^2 \geq \tfrac{16}{3}\pi^2(2\chi(M^4) \pm 3\sigma(M^4)). \tag{2-10}$$

In fact, we recall the following lower bound for the Yamabe invariant on compact four-manifolds which was proved by Gursky [1994]:

$$96\int_M \sigma_2(A) = \int_M R^2 - 12\int_M |\mathring{\mathrm{Ric}}|^2 \leq \mu^2([g]), \tag{2-11}$$

where $\sigma_2(A)$ denotes the second-elementary function of the eigenvalues of the Schouten tensor A; the inequality is strict unless (M^4, g) is conformally Einstein. By the Chern–Gauss–Bonnet formula (see Equation 6.31 of [Besse 1987])

$$\int_M |W|^2 - 2\int_M |\mathring{\mathrm{Ric}}|^2 + \frac{1}{6}\int_M R^2 = 32\pi^2\chi(M),$$

we obtain

$$\int_M |W^\pm|^2 \geq -2\int_M |\mathring{\mathrm{Ric}}|^2 + \frac{1}{6}\int_M R^2 = 32\pi^2\chi(M) - \int_M |W|^2. \tag{2-12}$$

Combining (1-1) with (2-12), we can prove (2-10).

Since $\int_M |W^\pm|^2 = 16\int_M \sigma_2(A)$, we have

$$\int_M |W^\pm|^2 = \tfrac{16}{3}\pi^2(2\chi(M^4) \pm 3\sigma(M^4)). \tag{2-13}$$

In fact, by Proposition E and (2-11), we have $\int_M |W^\pm|^2 = \frac{1}{6}\mu^2([g]) = 16\int_M \sigma_2(A)$. Hence from (2-12), (2-13) holds.

For four-manifolds M^4 with harmonic Weyl tensor and positive Yamabe constant $\mu([g])$ which is not locally conformally flat, the lower bound for $\mu([g])$ is given by $\mu^2([g]) \le 6\int_M |W^-|^2$ if M is anti-self-dual; $\mu^2([g]) \le 6\int_M |W^+|^2$ if M is self-dual; and $\mu^2([g]) \le 6\min\{\int_M |W^-|^2, \int_M |W^+|^2\}$ if M is neither anti-self-dual nor self-dual. The Yamabe constant $\mu^2([g])$ of a compact positive Kähler–Einstein manifold (M^4, g) is equal to $32\pi^2(2\chi(M^4) + 3\sigma(M^4))$.

By Remark 2.3, we can rewrite Theorem B as follows:

Theorem B*. *Let (M^4, g) be a 4-dimensional compact Riemannian manifold with $\delta W^+ = 0$ and positive Yamabe constant $\mu([g])$. Then either (M^4, g) is anti-self-dual, or*

$$\int_M |W^+|^2 \ge 16 \int_M \sigma_2(A). \tag{2-14}$$

Furthermore, equality holds in (2-14) if and only if (M^4, g) is a positive Einstein manifold which is either Kähler, or the quotient of a Kähler manifold by a free, isometric, antiholomorphic involution.

Proof. By Proposition E and (2-11), we get

$$\int_M |W^+|^2 \ge \tfrac{1}{6}\mu^2([g]) \ge 16 \int_M \sigma_2(A).$$

Since the equality holds in (2-14), we have

$$\int_M |W^+|^2 = \tfrac{1}{6}\mu^2([g]) = 16 \int_M \sigma_2(A).$$

So g is conformal to an Einstein metric $\tilde{g}$. By Theorem 1.1, we get that (M^4, g) is a Kähler manifold of positive constant scalar curvature.

Assume that $\tilde{g} = \lambda^2 g$. We now claim that λ is constant, i.e., g is an Einstein metric. To see this, first notice that $\tilde{g}$ being an Einstein metric implies that $\delta W^+_{\tilde{g}} = 0$. We recall this transformation law about W^+, i.e.,

$$\delta_{\tilde{g}} W^+_{\tilde{g}} = \delta_g W^+_g - W^+_g\Big(\frac{\nabla\lambda}{\lambda}, \dots\Big). \tag{2-15}$$

It is easy to see from (2-15) that

$$W^+_g\Big(\frac{\nabla\lambda}{\lambda}, \dots\Big) = 0. \tag{2-16}$$

Now any oriented four-manifold W^+ satisfies (see [Derdziński 1983])

$$(W^+)^{ikpq}(W^+)_{jkpq} = |W^+|^2 \delta^i_j. \tag{2-17}$$

Pairing both sides of (2-17) with $(d\lambda \otimes d\lambda)^j_i$ and using (2-16) we get $|W^+_g|^2 |\nabla\lambda|^2 = 0$. Since $|W^+_g|$ is constant, W^+ never vanishes, so $\nabla\lambda = 0$ and λ is constant.

We conclude that (M^4, g) is an Einstein manifold which is either Kähler, or the quotient of a Kähler manifold by a free, isometric, antiholomorphic involution. □

3. Four manifolds with harmonic Weyl tensor

Proof of Theorem 1.2. By Proposition E, we have that $W^+ = 0$ and $\int_M |W^-| = \frac{1}{6}\mu^2([g])$, or $W^- = 0$ and $\int_M |W^+| = \frac{1}{6}\mu^2([g])$. By Theorem 1.1, (M^4, g) is a Kähler manifold of positive constant scalar curvature.

When $W^+ = 0$, by Corollary 1 in [Derdziński 1983], the scalar curvature of (M^4, g) is 0, and $\mu([g]) = 0$. This is a contradiction.

When $W^- = 0$, by Lemma 7 in [Derdziński 1983], (M^4, g) is locally symmetric. By the result of Bourguignon [1981], (M^4, g) is Einstein. Then g is both Einstein and half conformally flat. By the classification theorem of Hitchin (see [Besse 1987]), (M^4, g) is isometric to either a quotient of $\mathbb{S}^4$ with the round metric or $\mathbb{CP}^2$ with the Fubini–Study metric. Since we are assuming that is not locally conformal flat, (M^4, g) is $\mathbb{CP}^2$ with the Fubini–Study metric. □

Corollary 3.1. Let (M^4, g) be a 4-dimensional complete Einstein manifold with positive scalar curvature. If

$$\int_M |W|^2 = \tfrac{1}{6}\mu^2([g]), \tag{3-1}$$

then M^4 is $\mathbb{CP}^2$ with the Fubini–Study metric.

Remark 3.2. If the equality in (3-1) is replaced by a strict inequality, we have proved in [Fu and Xiao 2017a; 2017b] that M^4 is a quotient of the round $\mathbb{S}^4$, which is proved by Proposition E. For dimensions $n > 4$, under some $L^{\frac{n}{2}}$ pinching condition, we proved in [Fu and Xiao 2017a; Fu and Xiao 2017b], as did G. Catino in [Catino 2016a], that M^n is a quotient of the round $\mathbb{S}^n$.

Proposition 3.3. Let (M^4, g) be a 4-dimensional compact Riemannian manifold with harmonic Weyl tensor and positive Yamabe constant. If

$$\int_M |W|^2 + 2\int_M |\mathring{\mathrm{R}}\mathrm{ic}|^2 \le \frac{1}{6}\int_M R^2, \quad \text{i.e.,} \quad \int_M |W|^2 \le 16\pi^2\chi(M),$$

then one of the following must be true:

(1) M^4 is a locally conformally flat manifold. In particular, $\tilde{g}$ is a Yamabe minimizer and $(M^4, \tilde{g})$ is the round sphere $\mathbb{S}^4$, the real projective space $\mathbb{RP}^4$, or the manifold which is isometrically covered by $\mathbb{S}^1 \times \mathbb{S}^3$ with the product metric, or $\mathbb{S}^1 \times \mathbb{S}^3$ with a rotationally symmetric Derdziński metric.

(2) (M^4, g) is $\mathbb{CP}^2$ with the Fubini–Study metric.

Proof. By the Chern–Gauss–Bonnet formula, we get

$$\int_M |W|^2 + 2\int_M |\mathring{\mathrm{R}}\mathrm{ic}|^2 - \frac{1}{6}\int_M R^2 = 2\int_M |W|^2 - 32\pi^2\chi(M) \le 0, \quad \text{i.e.,} \quad \int_M |W|^2 \le 16\pi^2\chi(M). \tag{3-2}$$

From (2-11), we get

$$\int_M |W|^2 - \frac{1}{6}\mu^2([g]) \le \int_M |W|^2 + 2\int_M |\mathring{\mathrm{R}}\mathrm{ic}|^2 - \frac{1}{6}\int_M R^2. \tag{3-3}$$

Moreover, the above inequality is strict unless (M^4, g) is conformally Einstein.

In the case of strict inequality, we have

$$\int_M |W|^2 < \tfrac{1}{6}\mu^2([g]), \quad \text{i.e.,} \quad \int_M |W^{\pm}|^2 < \tfrac{1}{6}\mu^2([g]).$$

By Proposition E, we get that M^4 is conformally flat. Since $\int_M |W|^2$, $\mu^2([g])$ and $\int_M \sigma_2(A)$ are conformally invariant, there exists a conformal metric $\tilde{g}$ of g such that $\mu^2([g]) = \int_M R^2_{\tilde{g}}$, and

$$\int_M |W_{\tilde{g}}|^2 + 2\int_M |\mathring{\mathrm{R}}\mathrm{ic}_{\tilde{g}}|^2 - \frac{1}{6}\int_M R^2_{\tilde{g}} = \int_M |W|^2 + 2\int_M |\mathring{\mathrm{R}}\mathrm{ic}|^2 - \frac{1}{6}\int_M R^2 \le 0, \tag{3-4}$$

i.e.,

$$2\int_M |\mathring{\mathrm{R}}\mathrm{ic}_{\tilde{g}}|^2 - \tfrac{1}{6}\mu^2([g]) \le 0.$$

By Theorems 1.5 and 1.6 in [Fu and Xiao 2018], $(M^4, \tilde{g})$ is isometric to the round $\mathbb{S}^4$, the real projective space $\mathbb{RP}^4$, or a manifold which is isometrically covered by $\mathbb{S}^1 \times \mathbb{S}^3$ with the product metric, or $\mathbb{S}^1 \times \mathbb{S}^3$ with a rotationally symmetric Derdziński metric.

In the case of equality, we have

$$\int_M |W|^2 = \tfrac{1}{6}\mu^2([g]).$$

Here g is conformally Einstein. By Theorem 1.2, (M^4, g) is $\mathbb{CP}^2$ with the Fubini–Study metric. □

Remark 3.4. Any compact conformally flat 4-manifold with $\mu([g]) > 0$ and $\chi(M) \ge 0$ has been classified [Gursky 1994; 1998]. Gursky proved that M^4 is conformal to the round $\mathbb{S}^4$, the real projective space $\mathbb{RP}^4$, or a quotient of $\mathbb{R}^1 \times \mathbb{S}^3$ with the product metric in [Gursky 1994; 1998]. Comparing with Theorem D, it is easy to see that the condition and conclusion in Proposition 3.3 are both strong.

Proof of Theorem 1.4. By the Chern–Gauss–Bonnet formula, we get

$$\int_M |W|^2 + 4\int_M |\mathring{\mathrm{R}}\mathrm{ic}|^2 - \frac{1}{3}\int_M R^2 = 3\int_M |W|^2 - 64\pi^2\chi(M) < 0, \quad \text{i.e.,} \quad \int_M |W|^2 < \tfrac{64}{3}\pi^2\chi(M). \tag{3-5}$$

From (2-11), we get

$$\int_M |W|^2 - \tfrac{1}{3}\mu^2([g]) \le \int_M |W|^2 + 4\int_M |\mathring{\mathrm{R}}\mathrm{ic}|^2 - \frac{1}{3}\int_M R^2.$$

Moreover, the above inequality is strict unless (M^4, g) is conformally Einstein. Then we have

$$\int_M |W|^2 < \tfrac{1}{3}\mu^2([g]).$$

Since $\int_M |W|^2$, $\mu^2([g])$ and $\int_M \sigma_2(A)$ are conformally invariant, there exists a conformally metric $\tilde{g}$ of g such that $\mu^2([g]) = \int_M R^2_{\tilde{g}}$, and from (3-5) we have

$$4\int_M |\mathring{\mathrm{R}}\mathrm{ic}_{\tilde{g}}|^2 - \tfrac{1}{3}\mu^2([g]) < 0.$$

(a) $W = 0$. By Theorem 1.5 in [Fu and Xiao 2018], $(M^4, \tilde{g})$ is the round $\mathbb{S}^4$ or the real projective space $\mathbb{RP}^4$.

(b) $W \neq 0$. By Proposition F, $b_1(M) = 0$. Hence $\chi(M) = 2 + b_2$. By Proposition 3.3, we assume $16\pi^2\chi(M) < \int_M |W|^2$. Since $\mu^2([g]) \le \mu^2(\mathbb{S}^4) = 384\pi^2$ and the inequality is strict unless (M^4, g) is conformal to $\mathbb{S}^4$, $\int_M |W|^2 < \frac{1}{3}\mu^2([g])$ implies that $\chi(M) \le 7$. By Proposition E, we have that $W^+ = 0$ and $\int_M |W^-|^2 \ge \frac{1}{6}\mu^2([g])$, or $W^- = 0$ and $\int_M |W^+|^2 \ge \frac{1}{6}\mu^2([g])$. By Proposition F and the Hirzebruch signature formula, $b_2(M) = b_2^-(M) \neq 0$ or $b_2(M) = b_2^+(M) \neq 0$. Hence $3 \le \chi(M) = 2 + b_2 \le 7$.

When $W^- = 0$ and $\int_M |W^+|^2 \ge \frac{1}{6}\mu^2([g])$, $3 \le \chi(M) = 2 + b_2^+(M) \le 7$. By the Hirzebruch signature formula

$$\frac{4\chi(M)}{9} > \frac{1}{48\pi^2}\int_M |W_g^+|^2 = b_2^+,$$

only the case $b_2^+ = 1$ occurs. Thus we have $\chi(M) = 3$, $\sigma(M) = 1$, and

$$\int_M |W_g^+|^2 = 48\pi^2 = \frac{16\pi^2}{3}(2\chi(M) + 3\sigma(M)).$$

By Remark 2.3,

$$\int_M |W|^2 = \int_M |W_g^+|^2 = \frac{\mu^2([g])}{6}.$$

Hence by Theorem 1.2, (M^4, g) is $\mathbb{CP}^2$ with the Fubini–Study metric.

When $W^+ = 0$ and $\int_M |W^-|^2 \ge \frac{1}{6}\mu^2([g])$. Similarly, we obtain

$$\int_M |W|^2 = \int_M |W_g^-|^2 = \frac{\mu^2([g])}{6}.$$

From the proof of Theorem 1.2, this can't happen. □

Proof of Theorem 1.5. (i) When $\chi(M)=0$. This pinching condition implies $W=0$. From (3-4), there exists a conformally metric $\tilde{g}$ of g such that $\mu^2([g])=\int_M R_{\tilde{g}}^2$ and

$$2\int_M |\mathring{\mathrm{R}}\mathrm{ic}_{\tilde{g}}|^2-\tfrac{1}{6}\mu^2([g])=0.$$

By Theorem 1.6 in [Fu and Xiao 2018], $(M^4,\tilde{g})$ is a manifold which is isometrically covered by $\mathbb{S}^1\times\mathbb{S}^3$ with the product metric, or a manifold which is isometrically covered by $\mathbb{S}^1\times\mathbb{S}^3$ with a rotationally symmetric Derdziński metric.

(ii) When $\chi(M)\neq 0$. Since $\int_M|W|^2\le\frac{1}{3}\mu^2([g])$, $\int_M|W|^2=\frac{64}{3}\pi^2\chi(M)$ implies that $\chi(M)\le 5$. Since $\int_M|W|^2=\frac{64}{3}\pi^2\chi(M)$, by (3-3) and Proposition F, $b_1(M)=0$. Hence $\chi(M)=2+b_2$.

Case 1: In the case of strict inequality, we have

$$\int_M|W|^2<\tfrac{1}{3}\mu^2([g]).$$

From the proof of Theorem 1.4, we have $W^{\mp}=0$ and $\int_M|W^{\pm}|^2\ge\frac{1}{6}\mu^2([g])$, and $3\le\chi(M)=2+b_2(M)=2+b_2^{+}(M)\le 5$. By the Hirzebruch signature formula

$$\frac{4\chi(M)}{9}=\frac{1}{48\pi^2}\int_M|W_g^{\pm}|^2=b_2^{\pm},$$

we get that $b_2^{\pm}$ is not an integer. Hence there exists no such manifold.

Case 2: In the case of strict equality, we have

$$\int_M|W|^2=\tfrac{1}{3}\mu^2([g]).$$

Here g is conformal to an Einstein metric. Since (M^4,g) has harmonic Weyl tensor, from the proof of Theorem B*, we get that (M^4,g) is also Einstein. By Corollary 1.8, (M^4,g) is a quotient of $\mathbb{S}^2\times\mathbb{S}^2$ with the product metric. □

Proposition 3.5. Let (M^4,g) be a 4-dimensional compact Riemannian manifold with harmonic Weyl tensor and positive Yamabe constant. If

$$\tfrac{1}{6}\mu^2([g])\le\int_M|W|^2\le\tfrac{1}{3}\mu^2([g]),$$

then one of the following must be true:

(1) (M^4,g) is self-dual, but is not anti-self-dual, which has either even $\chi(M^4)\le 4$ and $b_2^{+}=2$ or odd $\chi(M^4)\le 1$ and $b_2^{+}=1$.

(2) (M^4,g) is anti-self-dual, but is not self-dual, which has either even $\chi(M^4)\le 4$ and $b_2^{-}=2$ or odd $\chi(M^4)\le 1$ and $b_2^{-}=1$.

(3) (M^4,g) is a $\mathbb{CP}^2$ with the Fubini–Study metric.

(4) (M^4, g) is a quotient of a quotient of $\mathbb{S}^2 \times \mathbb{S}^2$ with the product metric.

Proof. By Proposition E, we get that $W^- = 0$, $W^+ = 0$ or $\int_M |W^\pm|^2 = \frac{1}{6}\mu^2([g])$.

When $W^\mp = 0$, $\int_M |W^\pm|^2 \geq \frac{1}{6}\mu^2([g])$. By Proposition F, we have $b_2^\mp = 0$. By the Hirzebruch signature formula

$$\text{(3-6)}\quad \pm\frac{1}{48\pi^2}\int_M |W_g^\pm|^2 = \frac{1}{48\pi^2}\int_M (|W_g^+|^2 - |W_g^-|^2) = b_2^+ - b_2^- = \pm b_2^\pm = \sigma(M),$$

we get $\pm\sigma(M) = b_2^\pm \geq 1$. Since $\int_M |W|^2 \leq \frac{1}{3}\mu^2([g])$, by the fact that $\mu^2([g]) \leq \mu^2(\mathbb{S}^4) = 384\pi^2$ and the inequality is strict unless (M^4, g) is conformal to $\mathbb{S}^4$, we get $b_2^\pm \leq 2$. Then we get $\chi(M) \leq 4$.

If $\chi(M) = 3$, then $b_2^\pm = 1$ and $b_1 = 0$. By Remark 2.3, we have

$$\int_M |W^\pm|^2 \geq \tfrac{16}{3}\pi^2(2\chi(M^4) \pm 3\sigma(M^4)).$$

Combining with (3-6), we have

$$48\pi^2 = \pm 48\pi^2\sigma(M^4) = \int_M |W^\pm|^2 \geq \tfrac{16}{3}\pi^2\big(2\chi(M^4) \pm 3\sigma(M^4)\big) = 48\pi^2.$$

By Remark 2.3, $\int_M |W|^2 = \frac{1}{6}\mu^2([g])$. By Theorem 1.2, (M^4, g) is $\mathbb{CP}^2$ with the Fubini–Study metric.

When $\int_M |W^\pm|^2 = \frac{1}{6}\mu^2([g])$, by Theorem 1.1, (M^4, g) is a Kähler manifold of positive constant scalar curvature, and the Weyl tensor is parallel. Since (M^4, g) is a Kähler manifold with harmonic Weyl tensor, by Proposition 1 in [Derdziński 1983], the Ricci tensor is parallel. Hence $\nabla Rm = 0$, i.e., M is locally symmetric. From (2-4), by the maximum principle we get $|W^\pm|^2 = R^2/6$, and $W^\pm$ has eigenvalues $\{-\frac{R}{12}, -\frac{R}{12}, \frac{R}{6}\}$. Thus Rm has eigenvalues $\{0, 0, 1, 0, 0, 1\}$. By the classification of 4-dimensional symmetric spaces, it is isometric to a quotient of $\mathbb{S}^2 \times \mathbb{S}^2$ with the product metric. □

4. Four manifolds with harmonic curvature

Proposition 4.1. Let (M^4, g) be a 4-dimensional compact Riemannian manifold with harmonic curvature and positive scalar curvature. If

$$\tfrac{1}{6}\mu^2([g]) \leq \int_M |W|^2 \leq \tfrac{1}{3}\mu^2([g]),$$

then one of the following must be true:

(1) (M^4, g) is $\mathbb{CP}^2$ with the Fubini–Study metric.

(2) (M^4, g) is isometric to a quotient of $\mathbb{S}^2 \times \mathbb{S}^2$ with the product metric.

Proof. By Proposition 3.5, We just need to consider whether (M^4, g) is self-dual or anti-self-dual.

When (M^4, g) is self-dual, since it has harmonic curvature, it is analytic [DeTurck and Goldschmidt 1989]. By Proposition 7 in [Derdziński 1983], we get that (M^4, g) is Einstein. By the classification theorem of Hitchin, (M^4, g) is isometric to $\mathbb{CP}^2$ with the Fubini–Study metric g.

When (M^4, g) is anti-self-dual, $\frac{R}{6}I - W^+ = \frac{R}{6}I > 0$. Since (M^4, g) is not self-dual, by Theorem 4.3 of [Micallef and Wang 1993], only (c) and (d) therein occur, i.e., (M^4, g) is a Kähler manifold of positive constant scalar curvature. By Corollary 1 in [Derdziński 1983], the scalar curvature of (M^4, g) is 0. This is a contradiction. □

By Theorem 1.1 and Propositions 4.1 and E, we have Theorem 1.7.

Proposition 4.2. Let (M^4, g) be a 4-dimensional compact Riemannian manifold with harmonic curvature and positive scalar curvature. If

$$\int_M |W|^2 + 4\int_M |\mathring{\mathrm{Ric}}|^2 = \tfrac{1}{3}\mu^2([g]), \tag{4-1}$$

then one of the following must be true:

(1) M^4 is a quotient of $\mathbb{S}^2 \times \mathbb{S}^2$ with the product metric.

(2) M^4 is covered isometrically by $\mathbb{S}^1 \times \mathbb{S}^3$ with the product metric.

(3) M^4 is covered isometrically by $\mathbb{S}^1 \times \mathbb{S}^3$ with a rotationally symmetric Derdziński metric.

Proof. **Case 1:** $\mathring{\mathrm{Ric}} = 0$, i.e., M is Einstein. By Corollary 1.8, (M^4, g) falls under (1).

Case 2: $\mathring{\mathrm{Ric}} \neq 0$. It is easy to see from (4-1) that $\int_M |W|^2 < \frac{1}{3}\mu^2([g])$. By Theorem 1.7, we have $W = 0$, i.e., M is locally conformally flat and $\int_M |\mathring{\mathrm{Ric}}|^2 = \frac{1}{12}\mu^2([g])$. By Theorem 1.6 in [Fu and Xiao 2018], (M^4, g) falls under (2) or (3). □

Proposition 4.3. Let (M^4, g) be a 4-dimensional compact Riemannian manifold with harmonic curvature and positive scalar curvature. If

$$\int_M |W|^2 + 4\int_M |\mathring{\mathrm{Ric}}|^2 < \tfrac{1}{3}\mu^2([g]), \tag{4-2}$$

then one of the following must be true:

(1) M^4 is a quotient of the round $\mathbb{S}^4$.

(2) M^4 is $\mathbb{CP}^2$ with the Fubini–Study metric.

Proof. Suppose $\mathring{\mathrm{Ric}} \neq 0$. It is easy to see from (4-2) that $\int_M |W|^2 < \frac{1}{3}\mu^2([g])$. By Theorem 1.7, we have $W = 0$, i.e., M is locally conformally flat and $\int_M |\mathring{\mathrm{Ric}}|^2 <$

$\frac{1}{12}\mu^2([g])$. By Theorem 1.5 in [Fu and Xiao 2018], that (M^4, g) is a quotient of the round $\mathbb{S}^4$. This is a contradiction.

Thus (4-2) implies that $\mathring{\mathrm{R}}\mathrm{ic} = 0$, i.e., M is Einstein, and $\int_M |W|^2 < \frac{1}{3}\mu^2([g])$. By Theorem 1.7, M is $\mathbb{CP}^2$ with the Fubini–Study metric, or locally conformally flat. Hence M^4 is a constant curvature space. Since the Yamabe constant is positive, M^4 is a quotient of the round $\mathbb{S}^4$. □

Corollary 4.4. Let (M^4, g) be a 4-dimensional compact Riemannian manifold with harmonic curvature and positive scalar curvature. If

$$\int_M |W|^2 + 8\int_M |\mathring{\mathrm{R}}\mathrm{ic}|^2 \leq \frac{1}{3}\int_M R^2, \tag{4-3}$$

then one of the following must be true:

(1) M^4 is isometric to a quotient of the round $\mathbb{S}^4$.

(2) M^4 is a quotient of $\mathbb{S}^2 \times \mathbb{S}^2$ with the product metric.

(3) M^4 is $\mathbb{CP}^2$ with the Fubini–Study metric.

Remark 4.5. The pinching condition (4-3) in Corollary 4.4 is equivalent to

$$\int_M |W|^2 + \frac{1}{15}\int_M R^2 \leq \frac{128}{5}\pi^2\chi(M).$$

Proofs of Corollary 4.4 and Remark 4.5. From (2-11), we get

$$\int_M |W|^2 + 4\int_M |\mathring{\mathrm{R}}\mathrm{ic}|^2 - \frac{1}{3}\mu^2([g]) \leq \int_M |W|^2 + 8\int_M |\mathring{\mathrm{R}}\mathrm{ic}|^2 - \frac{1}{3}\int_M R^2.$$

Moreover, the inequality is strict unless (M^4, g) is conformally Einstein.

In the case of strict inequality, Proposition 4.3 immediately implies Corollary 4.4.

In the case of equality, we have that g is conformally Einstein and

$$\int_M |W|^2 + 4\int_M |\mathring{\mathrm{R}}\mathrm{ic}|^2 = \tfrac{1}{3}\mu^2([g]).$$

Since g has constant scalar curvature, g is Einstein from the proof of Obata's theorem. By Proposition 4.2, we complete the proof of this corollary.

By the Chern–Gauss–Bonnet formula, the right-hand sides of the above can be written as

$$\int_M |W|^2 + 8\int_M |\mathring{\mathrm{R}}\mathrm{ic}|^2 - \frac{1}{3}\int_M R^2 = 5\int_M |W|^2 + \frac{1}{3}\int_M R^2 - 128\pi^2\chi(M).$$

This proves Remark 4.5. □

5. Four manifolds with positive Yamabe constant

Chang, Gursky and Yang's proof of Theorem D is based on establishing the existence of a solution of a fourth order fully nonlinear equation. Avoiding the requirement for the existence of a fourth order fully nonlinear equation, we can reprove Theorem D which is rewritten as follows:

Theorem D*. *Let (M^4, g) be a 4-dimensional compact Riemannian manifold with positive Yamabe constant. If*

$$\int_M |W|^2 \le 16\pi^2\chi(M),$$

then one of the following must be true:

(1) *$\tilde{g}$ is a Yamabe minimizer and $(M^4, \tilde{g})$ is the manifold which is isometrically covered by $\mathbb{S}^1 \times \mathbb{S}^3$ with the product metric, or $\mathbb{S}^1 \times \mathbb{S}^3$ with a rotationally symmetric Derdziński metric.*

(2) *M^4 is diffeomorphic to the round sphere $\mathbb{S}^4$ or the real projective space $\mathbb{RP}^4$.*

(3) *$\tilde{g}$ is a Yamabe minimizer and $(M^4, \tilde{g})$ is $\mathbb{CP}^2$ with the Fubini–Study metric.*

Proof. (i) When $\chi(M) = 0$. This pinching condition implies $W = 0$. Since $\int_M |W|^2$, $\mu^2([g])$ and $\int_M \sigma_2(A)$ are conformally invariant, there exists a conformally metric $\tilde{g}$ of g such that $\mu^2([g]) = \int_M R_{\tilde{g}}^2$, and from (3-2) we have

$$2\int_M |\mathring{\mathrm{Ric}}_{\tilde{g}}|^2 - \tfrac{1}{6}\mu^2([g]) = 0.$$

By Theorem 1.6 in [Fu and Xiao 2018], $(M^4, \tilde{g})$ is a manifold which is isometrically covered by $\mathbb{S}^1 \times \mathbb{S}^3$ with the product metric, or $\mathbb{S}^1 \times \mathbb{S}^3$ with a rotationally symmetric Derdziński metric.

(ii) When $\chi(M) \neq 0$. **Case 1:** If $\int_M |W|^2 < 16\pi^2\chi(M)$ or $\int_M |W|^2 = 16\pi^2\chi(M)$ and $-2\int_M |\mathring{\mathrm{Ric}}|^2 + \frac{1}{6}\int_M R^2 < \frac{1}{6}\mu^2([g])$, then from (3-2) we have

$$\int_M |W|^2 < \tfrac{1}{6}\mu^2([g]).$$

By Proposition F, $b_2(M) = 0$.

(a) $W = 0$. Since $\int_M |W|^2$, $\mu^2([g])$ and $\int_M \sigma_2(A)$ are conformally invariant, there exists a conformally metric $\tilde{g}$ of g such that $\mu^2([g]) = \int_M R_{\tilde{g}}^2$, and from (3-2) we have

$$2\int_M |\mathring{\mathrm{Ric}}_{\tilde{g}}|^2 - \tfrac{1}{6}\mu^2([g]) \le 0.$$

Since $\chi(M) \neq 0$, by Theorem 1.5 in [Fu and Xiao 2018], $(M^4, \tilde{g})$ is the round $\mathbb{S}^4$, the real projective space $\mathbb{RP}^4$.

(b) $W \neq 0$. Since $\int_M |W|^2$, $\mu^2([g])$ and $\int_M \sigma_2(A)$ are conformally invariant, and $W \neq 0$, there exists a conformally metric $\tilde{g}$ of g such that $\mu^2([g]) = \int_M R^2_{\tilde{g}}$, and

$$2\int_M |\mathring{\mathrm{R}}\mathrm{ic}_{\tilde{g}}|^2 - \tfrac{1}{6}\mu^2([g]) < 0.$$

By Proposition F, $b_1(M) = 0$. By Freedman's result [1982], M^4 is covered by a homeomorphism sphere. For any metric g' of unit volume in the conformal class of g, we have

$$\int_M R_{g'} - \sqrt{6}\int_M |W_{g'}| \geq \mu([g]) - \sqrt{6}\Big(\int_M |W_{g'}|^2\Big)^{\frac{1}{2}} = \mu([g]) - \sqrt{6}\Big(\int_M |W|^2\Big)^{\frac{1}{2}} > 0. \tag{5-1}$$

Thus by [Chen and Zhu 2014, Section 2] and [Gursky 2000, Section 3], from (5-1) there is a metric $\tilde{g}$ of unit volume in the conformal class of g such that

$$\sqrt{6}|W_{\tilde{g}}| < R_{\tilde{g}}.$$

Let $\lambda_1^{\pm} \geq \lambda_2^{\pm} \geq \lambda_3^{\pm}$ be the eigenvalues of $W^{\pm}$. Since $W^{\pm}$ is trace free, we have $\lambda_1^{\pm} + \lambda_2^{\pm} + \lambda_3^{\pm} = 0$, and

$$\begin{aligned}\tfrac{3}{2}\lambda_1^{+2} + \tfrac{3}{2}\lambda_1^{-2} &\leq [\lambda_1^{+2} + \tfrac{1}{2}(\lambda_2^{+} + \lambda_3^{+})^2] + [\lambda_1^{-2} + \tfrac{1}{2}(\lambda_2^{-} + \lambda_3^{-})^2]\\ &= (\lambda_1^{+2} + \lambda_2^{+2} + \lambda_3^{+2}) + (\lambda_1^{-2} + \lambda_2^{-2} + \lambda_3^{-2})\\ &= \tfrac{1}{4}|W|^2 < \tfrac{1}{24}R^2,\end{aligned}$$

i.e., $\lambda_1^{\pm} < \frac{1}{6}R$. Hence $\lambda_2^{\pm} + \lambda_3^{\pm} > -\frac{1}{6}R$, i.e., $\lambda_2^{\pm} + \lambda_3^{\pm} + \frac{1}{6}R > 0$. This implies the sum of the least two eigenvalues of $\frac{1}{12}R + W^{\pm}$ is positive. So $(M^4, \tilde{g})$ has positive isotropic curvature. Since M^4 is covered by a homeomorphism sphere, by the main theorem of [Chen et al. 2012], M^4 is diffeomorphic to the standard sphere $\mathbb{S}^4$ or the real projective space $\mathbb{RP}^4$;

Case 2: If $\int_M |W|^2 = 16\pi^2\chi(M)$ and $-2\int_M |\mathring{\mathrm{R}}\mathrm{ic}|^2 + \frac{1}{6}\int_M R^2 = \frac{1}{6}\mu^2([g])$, then from (3-2) and (3-3) we have

$$\int_M |W|^2 = \tfrac{1}{6}\mu^2([g])$$

and g is conformal to a Einstein metric $\tilde{g}$. Thus we have

$$\int_M |W_{\tilde{g}}|^2 = \tfrac{1}{6}\mu^2([g]).$$

By Corollary 3.1, $(M^4, \tilde{g})$ is $\mathbb{CP}^2$ with the Fubini–Study metric. Hence $(M^4, \tilde{g})$ is conformal to $\mathbb{CP}^2$ with the Fubini–Study metric. □

Remark 5.1. The proof of Chang, Gursky and Yang consists of two steps. First, they prove the case for strict inequality, and second, based on the first step, they prove the case for equality. We unify the two cases. Chen and Zhu [2014] prove a classification theorem of 4-manifolds which generalizes Theorem C under the strict inequality assumption.

Based on the first Weitzenböck formulas in Remark 2.2, using the same argument as in the proof of Theorem 1.1, we can obtain the following result of Gursky [2000].

Proposition G [Gursky 2000]. *Let (M^4, g) be a 4-dimensional compact Riemannian manifold with positive Yamabe constant $\mu([g])$. If $b_2^\pm \neq 0$ and*

$$\int_M |W^\pm|^2 = \tfrac{1}{6}\mu^2([g]), \tag{5-2}$$

then (M^4, g) is conformal to a Kähler manifold of positive constant scalar curvature.

Proof. Since $b_2^\pm \neq 0$, there exists a nonzero $\omega^\pm \in H_\pm^2(M)$. Setting $u = |\omega^\pm|$. Based on the first Weitzenböck formulas in Remark 2.2, using the same argument as in the proof of Theorem 1.1, we get

$$\begin{aligned} 0 \geq \Big(2 - \frac{1}{2\alpha}\Big)\frac{1}{6}\mu([g])\Big(\int_M u^{4\alpha}\Big)^{\frac{1}{2}} \\ -\frac{\sqrt{6}}{3}\alpha\Big(\int_M u^{4\alpha}\Big)^{\frac{1}{2}}\Big(\int_M |W^\pm|^2\Big)^{\frac{1}{2}} + \frac{(2\alpha-1)^2}{12\alpha}\int_M Ru^{2\alpha}. \end{aligned} \tag{5-3}$$

We choose $\alpha = \frac{1}{2}$, from (5-3) we get

$$0 \geq \Big[\frac{1}{\sqrt{6}}\mu([g]) - \Big(\int_M |W^\pm|^2\Big)^{\frac{1}{2}}\Big]\Big(\int_M u^2\Big)^{\frac{1}{2}}. \tag{5-4}$$

Equation (5-2) implies that the equality holds in (5-4). When the equality holds in (5-4), all inequalities leading to (5-3) become equalities. From (5-3), the function u^α attains the infimum in the Yamabe functional. Hence the metric $\tilde{g} = u^{2\alpha} g$ is a Yamabe minimizer. Then we get $|\omega^\pm|_{\tilde{g}} = 1$. Since $\int_M |W^\pm|^2$ is conformally invariant, the equality for the Hölder inequality implies that $|W^\pm|_{\tilde{g}}$ is constant. From (5-2), we get $|W^\pm|_{\tilde{g}}^2 = \frac{1}{6}R_{\tilde{g}}^2$. By the first Weitzenböck formula and the maximum principle, we get that $|\omega|$ is constant, thus $\nabla\omega = 0$, i.e., $(M^4, \tilde{g})$ is a Kähler manifold of positive constant scalar curvature. Hence (M^4, g) is conformal to a Kähler manifold of positive constant scalar curvature. □

Based on Propositions F and G, using the same arguments as in the proof of Theorem B*, we can reprove Theorems A and C proved by Gursky by using some results on functional determinant and the Bochner technique which are rewritten as follows:

Theorem A*. *Let (M^4, g) be a 4-dimensional compact Riemannian manifold with positive Yamabe constant $\mu([g])$ and $H^2_+(M) \neq 0$. Then*

$$\int_M |W_g^+|^2 \geq 16 \int_M \sigma_2(A).$$

Furthermore, equality holds in the above inequality if and only if g is conformal to a positive Kähler–Einstein metric.

Theorem C*. *Let (M^4, g) be a 4-dimensional compact Riemannian manifold with positive Yamabe constant and the space of harmonic 1-forms $H^1(M^4) \neq 0$. Then*

$$\int_M |W^+|^2 = 8\pi^2(2\chi(M^4) + 3\sigma(M^4)) - 8\int_M \sigma_2(A) \geq 8\pi^2(2\chi(M^4) + 3\sigma(M^4)).$$

Furthermore, the equality holds in the above inequalities if and only if (M^4, g) is conformal to a quotient of $\mathbb{R}^1 \times \mathbb{S}^3$ with the product metric.

Proof. By Proposition F, $\int_M |\mathring{\mathrm{Ric}}|^2 \geq \frac{1}{12}\mu^2([g])$ for $H^1(M^4) \neq 0$. Since M^4 is compact, there exists a conformally metric $\tilde{g}$ of g such that $\mu^2([g]) = \int_M R^2_{\tilde{g}}$. Hence we get

$$-2\int_M |\mathring{\mathrm{Ric}}_{\tilde{g}}|^2 + \frac{1}{6}\mu^2([g]) = -2\int_M |\mathring{\mathrm{Ric}}_{\tilde{g}}|^2 + \frac{1}{6}\int_M R^2_{\tilde{g}} = -2\int_M |\mathring{\mathrm{Ric}}|^2 + \frac{1}{6}\int_M R^2 \leq 0,$$

i.e.,

$$16\int_M \sigma_2(A) \leq 0.$$

By the Chern–Gauss–Bonnet formula,

$$\int_M |W^+|^2 = 8\pi^2(2\chi(M^4) + 3\sigma(M^4)) - 8\int_M \sigma_2(A).$$

Hence

$$\int_M |W^+|^2 \geq 8\pi^2(2\chi(M^4) + 3\sigma(M^4)).$$

From the proof of Proposition F and the above, the equality holding in the above inequalities implies that $|\nabla|\omega||^2 = \frac{3}{4}|\nabla\omega|^2$ and $\int_M |\mathring{\mathrm{Ric}}|^2 = \frac{1}{12}\mu^2([g]) = \frac{1}{12}\int_M R^2$. By [Bour and Carron 2015, Proposition 5.1 and Section 7.2], (M^4, g) is conformal to a quotient of $\mathbb{R}^1 \times \mathbb{S}^3$ with the product metric. □

It is easy to see from the proof of Theorem A* that the assumption that $H^2_+(M) \neq 0$ in Theorem 1 of [Gursky 1998] can be dropped for metrics with zero Yamabe constant.

Proposition 5.2. Let (M^4, g) be a 4-dimensional compact Riemannian manifold with zero Yamabe constant $\mu([g])$. Then

$$\int_M |W^+|^2 \geq \tfrac{16}{3}\pi^2(2\chi(M^4) + 3\sigma(M^4)).$$

Furthermore, equality holds in the above inequality if and only if g is conformal to a Ricci-flat and anti-self-dual metric, if and only if the universal cover of M is conformal to either $\mathbb{R}^4$ or a $K3$ surface.

Proof of Theorem 1.3. **Case 1:** $\int_M |W^\pm|^2 \leq \int_M |W|^2 < \frac{1}{6}\mu^2([g])$. By Proposition F, we get $b_2^\pm = 0$. From the proof of Theorem D*, we get that (M^4, g) has positive isotropic curvature. According to the main theorem in [Chen et al. 2012], it is diffeomorphic to $\mathbb{S}^4$, $\mathbb{RP}^4$, $\mathbb{S}^3 \times \mathbb{R}/G$ or a connected sum of them. Here G is a cocompact fixed point free discrete subgroup of the isometry group of the standard metric on $\mathbb{S}^3 \times \mathbb{R}$.

Case 2: $W^+ = 0$, $\int_M |W^-|^2 = \frac{1}{6}\mu^2([g])$, or $W^- = 0$, $\int_M |W^+|^2 = \frac{1}{6}\mu^2([g])$. By the Hirzebruch signature formula

$$\frac{1}{48\pi^2}\int_M (|W_g^+|^2 - |W_g^-|^2) = b_2^+ - b_2^- = \sigma(M),$$

we get $b_2^- \neq 0$ or $b_2^+ \neq 0$. By Proposition G, (M^4, g) is conformal to a Kähler manifold of positive constant scalar curvature.

When $W^+ = 0$, by Corollary 1 in [Derdziński 1983], the scalar curvature of $(M^4, \tilde{g})$ is 0, and $\mu([g]) = 0$. This is a contradiction.

When $W^- = 0$, by Lemma 7 in [Derdziński 1983], $(M^4, \tilde{g})$ is locally symmetric. By the result of Bourguignon, $(M^4, \tilde{g})$ is Einstein. Then $\tilde{g}$ is both Einstein and half conformally flat. By the classification theorem of Hitchin, $(M^4, \tilde{g})$ is isometric to either a quotient of $\mathbb{S}^4$ with the round metric or $\mathbb{CP}^2$ with the Fubini–Study metric. Since we are assuming that is not locally conformal flat, $(M^4, \tilde{g})$ is $\mathbb{CP}^2$ with the Fubini–Study metric. □

Remark 5.3. We do not know whether Theorem 1.3 can be deduced directly from [Chen and Zhu 2014, Theorem 1.6], which has weaker conditions. For their proof, Chen and Zhu used Micallef and Wang's result [1993], which we do not use in the proof of Theorem 1.3. Theorem D can be deduced from the proof of Theorem D* and Theorem 1.3.

Proof of Theorem 1.6. From the proof of Theorem 1.4, we have $b_1 = 0$ and $2 \leq \chi(M) \leq 7$.

Case 1: In the case of strict inequality, we have

$$\int_M |W|^2 < \tfrac{1}{3}\mu^2([g]).$$

When $\int_M |W^{\pm}|^2 < \frac{1}{6}\mu^2([g])$, by Proposition F, $b_2^+(M) = b_2^-(M) = 0$. Hence M^4 is covered by a homeomorphism sphere, i.e., M^4 is homeomorphic to the standard sphere $\mathbb{S}^4$ or the real projective space $\mathbb{RP}^4$.

When $\int_M |W^+|^2 < \frac{1}{6}\mu^2([g])$ and $\int_M |W^-|^2 \geq \frac{1}{6}\mu^2([g])$, or $\int_M |W^-|^2 < \frac{1}{6}\mu^2([g])$ and $\int_M |W^+|^2 \geq \frac{1}{6}\mu^2([g])$. By Proposition E and the Hirzebruch signature formula, $b_2(M) = b_2^-(M) \neq 0$ or $b_2(M) = b_2^+(M) \neq 0$. Hence $3 \leq \chi(M) = 2 + b_2 \leq 7$. From the proof of Theorem 1.4, we have $b_2^{\pm} = 1$ and $\chi(M) = 3$. If (M^4, g) has harmonic Weyl tensor, by Theorem 1.4 we have $\int_M |W|^2 = \frac{1}{6}\mu^2([g])$, which contradicts $\int_M |W|^2 > \frac{1}{6}\mu^2([g])$.

Case 2: In the case of equality, we have

$$\int_M |W|^2 = \tfrac{1}{3}\mu^2([g]) = \tfrac{64}{3}\pi^2\chi(M).$$

Hence g is conformal to an Einstein metric $\tilde{g}$. By Corollary 1.8, (M^4, g) is conformal to a quotient of $\mathbb{S}^2 \times \mathbb{S}^2$ with the product metric. □

Proposition 5.4. Let (M^4, g) be a 4-dimensional compact Riemannian manifold with positive Yamabe constant. If

$$\tfrac{1}{6}\mu^2([g]) \leq \int_M |W|^2 \leq \tfrac{1}{3}\mu^2([g]),$$

and the universal cover of (M^4, g) is not homeomorphic to $\mathbb{S}^4$, then one of the following must be true:

(1) (M^4, g) has $\chi(M^4) \leq 4$ and $1 \leq b_2 = b_2^+ \leq 2$.

(2) (M^4, g) has $\chi(M^4) \leq 4$ and $1 \leq b_2 = b_2^- \leq 2$.

(3) The universal cover of (M^4, g) is conformal to a Kähler manifold of positive constant scalar curvature. In particular, (M^4, g) is a quotient of $(\Sigma_1, g_1) \times (\Sigma_2, g_2)$, where the surface (Σ_i, g_i) has constant Gaussian curvature k_i, and $k_1 + k_2 > 0$.

Proof. When $\int_M |W^+|^2 < \frac{1}{6}\mu^2([g])$ and $\int_M |W^-|^2 < \frac{1}{6}\mu^2([g])$. By Proposition F, we have $b_2 = 0$. Hence the universal cover of (M^4, g) is homeomorphic to $\mathbb{S}^4$.

When $\int_M |W^{\mp}|^2 < \frac{1}{6}\mu^2([g])$ and $\int_M |W^{\pm}|^2 \geq \frac{1}{6}\mu^2([g])$. By Proposition E, we have $b_2^{\mp} = 0$. By the Hirzebruch signature formula

$$\text{(5-5)} \qquad \frac{1}{48\pi^2}\int_M (|W_g^+|^2 - |W_g^-|^2) = b_2^+ - b_2^- = \pm b_2^{\pm} = \sigma(M),$$

we get $\pm\sigma(M) = b_2^{\pm} \geq 1$. Since $\int_M |W|^2 \leq \frac{1}{3}\mu^2([g])$, by the fact that $\mu^2([g]) \leq \mu^2(\mathbb{S}^4) = 384\pi^2$ and the inequality is strict unless (M^4, g) is conformal to $\mathbb{S}^4$, we get $b_2^{\pm} \leq 2$. Then we get $\chi(M) \leq 4$.

When $\int_M |W^+|^2 = \int_M |W^-|^2 = \frac{1}{6}\mu^2([g])$. We have that $\sigma(M) = 0$ and $\chi(M)$ is even. For any metric g' of unit volume in the conformal class of g, we have

$$\int_M R_{g'} - \sqrt{6}\int_M |W^{\pm}_{g'}| \geq \mu([g]) - \sqrt{6}\left(\int_M |W^{\pm}_{g'}|^2\right)^{\frac{1}{2}} = \mu([g]) - \sqrt{6}\left(\int_M |W^{\pm}|^2\right)^{\frac{1}{2}} = 0.$$

Case 1: $\int_M R_{g'} - \sqrt{6}\int_M |W^{\pm}_{g'}| > 0$. By [Chen and Zhu 2014, Section 2] and [Gursky 2000, Section 3], there is a metric $\tilde{g}$ of unit volume in the conformal class of g such that

$$\sqrt{6}|W^{\pm}_{\tilde{g}}| < R_{\tilde{g}}.$$

From the proof of Proposition G, we have $b_2 = 0$ for $\sigma(M) = 0$. Hence the universal cover of (M^4, g) is homeomorphic to $\mathbb{S}^4$.

Case 2: $\int_M R_{g'} - \sqrt{6}\int_M |W^{\pm}_{g'}| = 0$. Thus there are two metrics $\tilde{g_1}$ and $\tilde{g_2}$ of unit volume in the conformal class of g such that

$$\sqrt{6}|W^{+}_{\tilde{g_1}}| = R_{\tilde{g_1}}, \qquad \sqrt{6}|W^{-}_{\tilde{g_2}}| = R_{\tilde{g_2}}.$$

We have

$$\int_M |W^{+}_{\tilde{g_1}}|^2 = \int_M |W^{-}_{\tilde{g_2}}|^2 = \frac{1}{6}\int_M R^2_{\tilde{g_1}} = \frac{1}{6}\int_M R^2_{\tilde{g_2}} = \frac{1}{6}\mu^2([g]).$$

Hence $\tilde{g_1}$ and $\tilde{g_2}$ are the Yamabe minimizers of g. So $(M^4, \tilde{g_1})$ has nonnegative isotropic curvature. If $b_2 = 0$, by Theorem 4.10 of [Micallef and Wang 1993], $(M^4, \tilde{g_1})$ becomes positive isotropic curvature. By the proof of Theorem 1.3(1), the universal cover of (M^4, g) is diffeomorphic to $\mathbb{S}^4$. If $b_2 > 0$, from the proof of Proposition G, the universal cover of $(M^4, \tilde{g_1})$ is a Kähler manifold of positive constant scalar curvature. Since the scalar curvature is positive, the universal cover of $(M^4, \tilde{g_1})$ is diffeomorphic to $(\Sigma_1, g_1) \times (\Sigma_2, g_2)$, where (Σ_i, g_i) is a 2-dimensional manifold, and the Gaussian curvature k_i of g_i must be a constant and satisfies $k_1 + k_2 > 0$. □

By Theorems 1.6 and D* and Corollary 1.8, we obtain the following theorem:

Theorem 5.5. *Let (M^4, g) be a 4-dimensional compact Riemannian manifold which is not homeomorphic to $\mathbb{S}^4$ or $\mathbb{RP}^4$ with positive Yamabe constant. If*

$$\int_M |W|^2 + 4\int_M |\mathring{\mathrm{R}}\mathrm{ic}|^2 \leq \tfrac{1}{3}\mu^2([g]),$$

then one of the following must be true:

(1) *(M^4, g) is a quotient of $\mathbb{S}^2 \times \mathbb{S}^2$ with the product metric.*

(2) *(M^4, g) is $\mathbb{CP}^2$ with the Fubini–Study metric.*

(3) (M^4, g) *is conformal to a quotient of* $\mathbb{R}^1 \times \mathbb{S}^3$ *with the product metric.*

(4) (M^4, g) *has* $\chi(M) = 3,\ b_1 = 0$ *and* $b_2 = 1$, *and does not have harmonic Weyl tensor.*

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HAI-PING FU
DEPARTMENT OF MATHEMATICS
NANCHANG UNIVERSITY
NANCHANG
CHINA
mathfu@126.com

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ON THE STRUCTURE OF CYCLOTOMIC NILHECKE ALGEBRAS

JUN HU AND XINFENG LIANG

In this paper we study the structure of the cyclotomic nilHecke algebras $\mathscr{H}_{\ell,n}^{(0)}$, where $\ell, n \in \mathbb{N}$. We construct a monomial basis for $\mathscr{H}_{\ell,n}^{(0)}$ which verifies a conjecture of Mathas. We show that the graded basic algebra of $\mathscr{H}_{\ell,n}^{(0)}$ is commutative and hence isomorphic to the center Z of $\mathscr{H}_{\ell,n}^{(0)}$. We further prove that $\mathscr{H}_{\ell,n}^{(0)}$ is isomorphic to the full matrix algebra over Z and construct an explicit basis for the center Z. We also construct a complete set of pairwise orthogonal primitive idempotents of $\mathscr{H}_{\ell,n}^{(0)}$. Finally, we present a new homogeneous symmetrizing form Tr on $\mathscr{H}_{\ell,n}^{(0)}$ by explicitly specifying its values on a given homogeneous basis of $\mathscr{H}_{\ell,n}^{(0)}$ and show that it coincides with Shan–Varagnolo–Vasserot's symmetrizing form $\mathrm{Tr}^{\mathrm{SVV}}$ on $\mathscr{H}_{\ell,n}^{(0)}$.

1. Introduction

Quiver Hecke algebras $\mathscr{R}_\alpha$ and their finite dimensional quotients $\mathscr{R}_\alpha^\Lambda$ (i.e., cyclotomic quiver Hecke algebras) have been hot topics in recent years. These algebras are remarkable because they can be used to categorify quantum groups and their integrable highest weight modules; see [Kang and Kashiwara 2012; Khovanov and Lauda 2009; Rouquier 2008; 2012; Varagnolo and Vasserot 2011]. These algebras can be regarded as some $\mathbb{Z}$-graded analogues of the affine Hecke algebras and their finite dimensional quotients. Many results concerning the representation theory of the affine Hecke algebras and the cyclotomic Hecke algebras of type A have their $\mathbb{Z}$-graded analogues for the quiver Hecke algebras $\mathscr{R}_\alpha$ and the cyclotomic quotients $\mathscr{R}_\alpha^\Lambda$; see [Brundan and Kleshchev 2009b; Brundan et al. 2011; Lauda and Vazirani 2011]. It is natural to expect that the structure of the affine Hecke algebras and the cyclotomic Hecke algebras of type A also have their $\mathbb{Z}$-graded analogues for the algebras $\mathscr{R}_\alpha$ and $\mathscr{R}_\alpha^\Lambda$. In fact, this is indeed the case for the quiver Hecke algebras $\mathscr{R}_\alpha$. For example, we have faithful polynomial representations, standard basis and a nice description of the center for the algebra $\mathscr{R}_\alpha$ in a similar way as in the case of the affine Hecke algebras of type A. However, the situation turns out to be much more tricky for the cyclotomic quiver Hecke algebras $\mathscr{R}_\alpha^\Lambda$. Only partial

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progress has been made for the structure of the cyclotomic quiver Hecke algebras $\mathscr{R}^\Lambda_\alpha$ so far. For example:

(1) The cyclotomic quiver Hecke algebra of type A has a $\mathbb{Z}$-graded cellular basis by [Hu and Mathas 2010].

(2) The cyclotomic quiver Hecke algebra is a $\mathbb{Z}$-graded symmetric algebra by [Shan et al. 2017].

(3) The center of the cyclotomic quiver Hecke algebra $\mathscr{R}^\Lambda_\alpha$ is the image of the center of the quiver Hecke algebra $\mathscr{R}_\alpha$ whenever the associated Cartan matrix is symmetric of finite type by [Webster 2015].

Apart from the type A case, one does not even know any explicit bases for arbitrary cyclotomic quiver Hecke algebras. On the other hand, for the classical cyclotomic Hecke algebra of type A, we have not only a Dipper–James–Mathas's cellular basis [Dipper et al. 1998] but also a monomial basis (or Ariki–Koike basis [Ariki and Koike 1994]). But even for the cyclotomic quiver Hecke algebra of type A we do not know any explicit monomial basis. This motivates our first question:

Question 1.1. *Can we construct an explicit monomial basis for any cyclotomic quiver Hecke algebra*?

Shan, Varagnolo and Vasserot [Shan et al. 2017] have shown that each cyclotomic quiver Hecke algebra can be endowed with a homogeneous symmetrizing form $\mathrm{Tr}^{\mathrm{SVV}}$ which makes it into a graded symmetric algebra (see Remark 4.7 and [Hu and Mathas 2010, §6.3] for the type A case). However, the SVV symmetrizing form $\mathrm{Tr}^{\mathrm{SVV}}$ is defined in an inductive manner. It is difficult to compute the explicit value of the form $\mathrm{Tr}^{\mathrm{SVV}}$ on any specified homogeneous element. On the other hand, it is well-known that the classical cyclotomic Hecke algebra of type A is symmetric [Malle and Mathas 1998; Brundan and Kleshchev 2008] and the definition of its symmetrizing form is explicit in that it specifies its value on each monomial basis element. This motivates our second question:

Question 1.2. *Can we determine the explicit values of the Shan–Varagnolo–Vasserot symmetrizing form* Tr^{SVV} *on some monomial bases* (*or at least a set of K-linear generators*) *of the cyclotomic quiver Hecke algebra*?

An explicit basis for the center of $\mathscr{R}^\Lambda_\alpha$ is unknown. Even for the classical cyclotomic Hecke algebra of type A, except in the level one case [Geck and Pfeiffer 2000] or in the degenerate case [Brundan 2008], one does not know any explicit basis for the center.

Question 1.3. *Can we give an explicit basis for the center of the cyclotomic quiver Hecke algebra*?

The starting point of this paper is to try to answer the above three questions. As a first step toward this goal, we need to consider the case of the cyclotomic quiver Hecke algebra which corresponds to a quiver with a single vertex and no edges. That is, the cyclotomic nilHecke algebra of type A. Let us recall its definition.

Definition 1.4. Let $\ell, n \in \mathbb{N}$. The nilHecke algebra $\mathscr{H}_n^{(0)}$ of type A is the unital associative K-algebra generated by $\psi_1, \dots, \psi_{n-1}, y_1, \dots, y_n$ which satisfy the following relations:

$$\begin{aligned}
&\psi_r^2 = 0, && \forall\, 1 \le r < n,\\
&\psi_r\psi_k = \psi_k\psi_r, && \forall\, 1 \le k < r-1 < n-1,\\
&\psi_r\psi_{r+1}\psi_r = \psi_{r+1}\psi_r\psi_{r+1}, && \forall\, 1 \le r < n-1,\\
&y_r y_k = y_k y_r, && \forall\, 1 \le r, k \le n,\\
&\psi_r y_{r+1} = y_r\psi_r + 1, \quad y_{r+1}\psi_r = \psi_r y_r + 1, && \forall\, 1 \le r < n,\\
&\psi_r y_k = y_k\psi_r, && \forall\, k \ne r, r+1.
\end{aligned}$$

The cyclotomic nilHecke algebra $\mathscr{H}_{\ell,n}^{(0)}$ of type A is the quotient of $\mathscr{H}_n^{(0)}$ by the two-sided ideal generated by y_1^{ℓ}.

The nilHecke algebras $\mathscr{H}_n^{(0)}$ was introduced by Kostant and Kumar [1986]. It plays an important role in the theory of Schubert calculus; see [Hiller 1982]. Mathas [2015, §2.5] has observed that the Specht module over $\mathscr{H}_{n,n}^{(0)}$ can be realized as the coinvariant algebra with standard bases of Specht modules being identified with the Schubert polynomials of the coinvariant algebras. It is clear that both $\mathscr{H}_n^{(0)}$ and $\mathscr{H}_{\ell,n}^{(0)}$ are $\mathbb{Z}$-graded K-algebras such that each ψ_r is homogeneous with $\deg \psi_r = -2$ and each y_s is homogeneous with $\deg y_s = 2$ for all $1 \le r < n$, $1 \le s \le n$. Mathas [2015, §2.5] has conjectured a monomial basis of the cyclotomic nilHecke algebra $\mathscr{H}_{n,n}^{(0)}$. In this paper, we shall construct a monomial basis of the cyclotomic nilHecke algebra $\mathscr{H}_{\ell,n}^{(0)}$ for arbitrary ℓ (Theorem 2.34) that, in particular, verifies Mathas's conjecture. As an application, we shall construct a basis for the center Z of $\mathscr{H}_{\ell,n}^{(0)}$ (Theorem 3.7). Thus we shall answer Question 1.1 and Question 1.3 for the cyclotomic nilHecke algebra $\mathscr{H}_{\ell,n}^{(0)}$. Furthermore, we shall construct a new homogeneous symmetrizing form Tr (Proposition 4.13) by specifying its values on a homogeneous basis element of $\mathscr{H}_{\ell,n}^{(0)}$. We prove that this new form Tr actually coincides with Shan–Varagnolo–Vasserot's symmetrizing form $\mathrm{Tr}^{\mathrm{SVV}}$ [Shan et al. 2017] on $\mathscr{H}_{\ell,n}^{(0)}$. Thus we also answer Question 1.2 for the cyclotomic nilHecke algebra $\mathscr{H}_{\ell,n}^{(0)}$.

The content of the paper is organized as follows. In Section 2, we shall first review some basic knowledge about the structure and representation of $\mathscr{H}_{\ell,n}^{(0)}$. Lemma 2.12 provides a useful commutator relation which will be used frequently in later discussion. In Corollary 2.18 and 2.19 we determine the graded dimensions of the graded simple modules and their graded projective covers as well as the graded decomposition numbers and the graded Cartan numbers. We construct a monomial basis of the cyclotomic nilHecke algebra $\mathscr{H}_{\ell,n}^{(0)}$ for arbitrary ℓ in Theorem 2.34. We also construct a complete set of pairwise orthogonal primitive idempotents in Corollary 2.25 and Theorem 2.31. In Section 3, we shall first present a basis for the graded basic algebra of $\mathscr{H}_{\ell,n}^{(0)}$ and show that it is isomorphic to the center Z of $\mathscr{H}_{\ell,n}^{(0)}$ in Lemma 3.2. Then we shall give a basis for the center in Theorem 3.7 which consists of certain symmetric polynomials in $y_1, \dots, y_n$. We also show in Proposition 3.8 that $\mathscr{H}_{\ell,n}^{(0)}$ is isomorphic to the full matrix algebra over Z. In Section 4, we shall first show in Lemma 4.4 that the center Z is a graded symmetric algebra by specifying an explicit homogeneous symmetrizing form on Z. Then we shall introduce two homogeneous symmetrizing forms: one is defined by using its isomorphism with the full matrix algebra over the center Z (Lemma 4.6); another is defined by specifying its values on a homogeneous basis element (Definition 4.11 and Proposition 4.13). We show in Proposition 4.14 that these two symmetrizing forms are the same. In Section 5 we show that the form Tr also coincides with Shan–Varagnolo–Vasserot's symmetrizing form $\mathrm{Tr}^{\mathrm{SVV}}$ (which was introduced in [Shan et al. 2017] for general cyclotomic quiver Hecke algebras).

After the submission of this paper, Professor Lauda emailed us that he wonders if our results have some connections with his papers [Khovanov et al. 2012; Lauda 2012]. In the latter paper he proved that the cyclotomic nilHecke algebra is isomorphic to the matrix ring of size $n!$ over the cohomology of a Grassmannian. Combining it with Proposition 3.8 in this paper this implies that the center of the cyclotomic nilHecke algebra is isomorphic to that cohomology of a Grassmannian. He also proposed an interesting question of comparing the trace form Tr in this paper with the natural form on the matrix ring over the cohomology of the Grassmannian which can be defined using integration over the volume form.

2. The structure and representation of $\mathscr{H}_{\ell,n}^{(0)}$

Let $\mathfrak{S}_n$ be the symmetric group on $\{1, 2, \dots, n\}$ and let $s_i := (i, i+1) \in \mathfrak{S}_n$, for $1 \le i < n$. Then $\{s_1, \dots, s_{n-1}\}$ is the standard set of Coxeter generators for $\mathfrak{S}_n$. If $w \in \mathfrak{S}_n$ then the length of w is

$$\ell(w) := \min\{k \in \mathbb{N} \mid w = s_{i_1} \dots s_{i_k} \text{ for some } 1 \le i_1, \dots, i_k < n\}.$$

If $w = s_{i_1} \dots s_{i_k}$ with $k = \ell(w)$ then $s_{i_1} \dots s_{i_k}$ is a reduced expression for w. In this case, we define $\psi_w := \psi_{i_1} \dots \psi_{i_k}$. The braid relation in Definition 1.4 ensures that ψ_w does not depend on the choice of the reduced expression of w. Let $w_{0,n}$ be the unique longest element in $\mathfrak{S}_n$. When n is clear from the context we shall write w_0

instead of $w_{0,n}$ for simplicity. Then $w_0 = w_0^{-1}$ and $\ell(w_0) = n(n-1)/2$. Let $*$ be the unique K-algebra antiautomorphism of $\mathscr{H}_{\ell,n}^{(0)}$ which fixes each of its ψ and y generators.

Lemma 2.1 [Manivel 2001]. *The elements in the set*

$$\{\psi_w y_1^{c_1} \dots y_n^{c_n} \mid w \in \mathfrak{S}_n, c_1, \dots, c_n \in \mathbb{N}\}$$

form a K-basis of the nilHecke algebra $\mathscr{H}_n^{(0)}$ and the center of $\mathscr{H}_n^{(0)}$ is the set of symmetric polynomials in $y_1, \dots, y_n$.

Let $\pi : \mathscr{H}_n^{(0)} \twoheadrightarrow \mathscr{H}_{\ell,n}^{(0)}$ be the canonical surjective homomorphism.

Definition 2.2. An element z in $\mathscr{H}_{\ell,n}^{(0)}$ is said to be symmetric if $z = \pi(f(y_1, \dots, y_n))$ for some symmetric polynomial $f(t_1, \dots, t_n) \in K[t_1, \dots, t_n]$, where $t_1, \dots, t_n$ are n indeterminates over K.

Corollary 2.3. *Any symmetric element in $\mathscr{H}_{\ell,n}^{(0)}$ lies in the center of $\mathscr{H}_{\ell,n}^{(0)}$.*

Proof. This follows from Lemma 2.1 and the surjective homomorphism π. □

Let Γ be a quiver without loops and I its vertex set. For any $i, j \in I$ let d_{ij} be the number of arrows $i \to j$ and set $m_{ij} := d_{ij} + d_{ji}$. This defines a *symmetric* generalized Cartan matrix $(a_{ij})_{i,j\in I}$ by putting $a_{ij} := -m_{ij}$ for $i \neq j$ and $a_{ii} := 2$ for any $i \in I$. Let u, v be two indeterminates over $\mathbb{Z}$. We define $Q_{ij} := (-1)^{d_{ij}}(u-v)^{m_{ij}}$ for any $i \neq j \in I$ and $Q_{ii}(u, v) := 0$ for any $i \in I$. Let $(\mathfrak{h}, \Pi, \Pi^\vee)$ be a realization of the generalized Cartan matrix $(a_{ij})_{i,j\in I}$. Let P be the associated weight lattice which is a finite rank free abelian group and contains $\Pi = \{\alpha_i \mid i \in I\}$, let $P^\vee$ be the associated coweight lattice which is a finite rank free abelian group too and contains $\Pi^\vee = \{\alpha_i^\vee \mid i \in I\}$. Let $Q^+ := \mathbb{N}\Pi \subset P$ be the semigroup generated by Π and $P^+ \subset P$ be the set of integral dominant weights. Let $\Lambda \in P^+$ and $\beta \in Q_n^+$. One can associate it with a quiver Hecke algebra $\mathscr{R}_\beta$ as well as its cyclotomic quotient $\mathscr{R}_\beta^\Lambda$. We refer the readers to [Khovanov and Lauda 2009; Rouquier 2012; Shan et al. 2017] for precise definitions.

Let $\{\Lambda_i \mid i \in I\}$ be the set of fundamental weights. The nilHecke algebra and its cyclotomic quotient can be regarded as a special quiver Hecke algebra and cyclotomic quiver Hecke algebra. That is, the quiver with single one vertex $\{0\}$ and no edges. More precisely, we have

$$\mathscr{H}_n^{(0)} = \mathscr{R}_{n\alpha_0}, \quad \mathscr{H}_{\ell,n}^{(0)} = \mathscr{R}_{n\alpha_0}^{\ell\Lambda_0}. \tag{2.4}$$

Throughout this paper, unless otherwise stated, we shall work in the category of $\mathbb{Z}$-graded $\mathscr{H}_{\ell,n}^{(0)}$-modules. Note that $\mathscr{H}_{\ell,n}^{(0)}$ is a special type A cyclotomic quiver Hecke algebra so that we can apply the theory of graded cellular algebras developed in [Hu and Mathas 2010]. We now recall the definition of graded cellular basis in this special situation (i.e., for $\mathscr{H}_{\ell,n}^{(0)}$).

We use $\varnothing$ to denote the empty partition and (1) to denote the unique partition of 1. Set $|\varnothing| := 0$, $|(1)| := 1$. We define

$$\mathscr{P}_0 := \Big\{ \boldsymbol{\lambda} := (\lambda^{(1)}, \dots, \lambda^{(\ell)}) \;\Big|\; \sum_{i=1}^{\ell} |\lambda^{(i)}| = n,\, \lambda^{(i)} \in \{\varnothing, (1)\},\ \forall\, 1 \le i \le \ell \Big\}.$$

Definition 2.5. If $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(\ell)}) \in \mathscr{P}_0$, then we define $\theta(\boldsymbol{\lambda})$ to be the unique n-tuple $(k_1, \dots, k_n)$ such that $1 \le k_1 < k_2 < \cdots < k_n \le \ell$ and

$$\lambda^{(j)} = \begin{cases} (1) & \text{if } j = k_i \text{ for some } 1 \le i \le n, \\ \varnothing & \text{otherwise.} \end{cases}$$

Given any two n-tuples $(k_1, \dots, k_n)$, $(k'_1, \dots, k'_n)$ of increasing positive integers, we define

$$(k_1, \dots, k_n) \ge (k'_1, \dots, k'_n) \Leftrightarrow k_i \ge k'_i,\ \forall\, 1 \le i \le n,$$

and $(k_1, \dots, k_n) > (k'_1, \dots, k'_n)$ if $(k_1, \dots, k_n) \ge (k'_1, \dots, k'_n)$ and $(k_1, \dots, k_n) \ne (k'_1, \dots, k'_n)$. For any $\boldsymbol{\lambda}, \boldsymbol{\mu} \in \mathscr{P}_0$, we define

$$\boldsymbol{\lambda} > \boldsymbol{\mu} \Leftrightarrow \theta(\boldsymbol{\lambda}) < \theta(\boldsymbol{\mu}).$$

Then ">" is a partial order on $\mathscr{P}_0$.

The following is a special case of [Hu and Mathas 2010, Definition 4.15].

Definition 2.6. Let $\boldsymbol{\lambda} \in \mathscr{P}_0$ with $\theta(\boldsymbol{\lambda}) = (k_1, \dots, k_n)$. We define

$$y_{\boldsymbol{\lambda}} := y_1^{\ell-k_1} \dots y_n^{\ell-k_n}, \quad \deg y_{\boldsymbol{\lambda}} := 2\ell n - 2\sum_{i=1}^{n} k_i.$$

By the main results in [Hu and Mathas 2010], the elements in the set

$$\{\psi^{\boldsymbol{\lambda}}_{w,u} := \psi^*_w y_{\boldsymbol{\lambda}} \psi_u \mid \boldsymbol{\lambda} \in \mathscr{P}_0,\, w, u \in \mathfrak{S}_n\} \tag{2.7}$$

form a graded cellular K-basis of $\mathscr{H}^{(0)}_{\ell,n}$. Each basis element $\psi^{\boldsymbol{\lambda}}_{w,u}$ is homogeneous with degree equal to

$$\deg \psi^{\boldsymbol{\lambda}}_{w,u} := \deg y_{\boldsymbol{\lambda}} - 2\ell(w) - 2\ell(u) = 2\ell n - 2\sum_{i=1}^{n} k_i - 2\ell(w) - 2\ell(u).$$

In particular, $\dim_K \mathscr{H}^{(0)}_{\ell,n} = \ell(\ell-1)\dots(\ell-n+1)n!$. Note that $\mathscr{P}_0 \ne \varnothing$ if and only if $\ell \ge n$. Therefore, $\mathscr{H}^{(0)}_{\ell,n} = 0$ whenever $\ell < n$. Henceforth, we always assume that $\ell \ge n$.

By the general theory of (graded) cellular algebras [Graham and Lehrer 1996; Hu and Mathas 2010], for each $\boldsymbol{\lambda} \in \mathscr{P}_0$, we have a graded Specht module $S^{\boldsymbol{\lambda}}$, which is equipped with an associative homogeneous bilinear form $\langle -,- \rangle_{\boldsymbol{\lambda}}$. Let $\mathrm{rad}\langle -,- \rangle_{\boldsymbol{\lambda}}$ be the radical of that bilinear form. We define $D^{\boldsymbol{\lambda}} := S^{\boldsymbol{\lambda}} / \mathrm{rad}\langle -,- \rangle_{\boldsymbol{\lambda}}$. By [Hu and

Mathas 2010, Corollary 5.11], we know that $D^{\boldsymbol{\lambda}} \neq 0$ if and only if $\boldsymbol{\lambda}$ is a Kleshchev multipartition with respect to $(p; 0, 0, \dots, 0)$, where $p = \operatorname{char} K$.

Let $\boldsymbol{\lambda} \in \mathscr{P}_0$ with $\theta(\boldsymbol{\lambda}) = (k_1, \dots, k_n)$. A $\boldsymbol{\lambda}$-tableau is a bijection $\mathfrak{t} : \{k_1, \dots, k_n\} \to \{1, 2, \dots, n\}$. We use $\operatorname{Tab}(\boldsymbol{\lambda})$ to denote the set of $\boldsymbol{\lambda}$-tableaux. For any $\mathfrak{t} \in \operatorname{Tab}(\boldsymbol{\lambda})$, we define

$$\deg \mathfrak{t} := \sum_{i=1}^{n} \Big(\#\big\{ k_i < j \leq \ell \mid \text{ either } j \notin \{k_1, \dots, k_n\} \text{ or } j = k_b \text{ with } \mathfrak{t}(j) > \mathfrak{t}(k_i) \big\} \\ - \#\big\{ k_i < j \leq \ell \mid j \in \{k_1, \dots, k_n\} \text{ and } \mathfrak{t}(j) < \mathfrak{t}(k_i) \big\} \Big).$$

It is clear that in our special case (i.e., for $\mathscr{P}_0$) the above definition of $\deg \mathfrak{t}$ coincides with that in [Brundan et al. 2011; Hu and Mathas 2010].

Definition 2.8. We define

$$\boldsymbol{\lambda}_{\max} := \big(\underbrace{(1), \dots, (1)}_{n \text{ copies}}, \underbrace{\varnothing, \dots, \varnothing}_{\ell - n \text{ copies}} \big), \quad \boldsymbol{\lambda}_{\min} := \big(\underbrace{\varnothing, \dots, \varnothing}_{\ell - n \text{ copies}}, \underbrace{(1), \dots, (1)}_{n \text{ copies}} \big).$$

It is clear that for any $\boldsymbol{\mu} \in \mathscr{P}_0 \setminus \{\boldsymbol{\lambda}_{\max}, \boldsymbol{\lambda}_{\min}\}$, we have that

$$\boldsymbol{\lambda}_{\min} < \boldsymbol{\mu} < \boldsymbol{\lambda}_{\max}, \quad \deg y_{\boldsymbol{\lambda}_{\min}} < \deg y_{\boldsymbol{\mu}} < \deg y_{\boldsymbol{\lambda}_{\max}}. \tag{2.9}$$

Using [Brundan and Kleshchev 2009a] and the definition of the Kleshchev multipartition in [Ariki and Mathas 2000], it is clear that $\boldsymbol{\lambda}_{\min}$ is the unique Kleshchev multipartition in $\mathscr{P}_0$. Therefore, for any $\boldsymbol{\lambda} \in \mathscr{P}_0$, $D^{\boldsymbol{\lambda}} \neq 0$ if and only if $\boldsymbol{\lambda} = \boldsymbol{\lambda}_{\min}$. Furthermore, $D^{\boldsymbol{\lambda}_{\min}}$ is the unique (self-dual) graded simple module for $\mathscr{H}_{\ell,n}^{(0)}$. Let $P^{\boldsymbol{\lambda}_{\min}}$ be its graded projective cover.

Definition 2.10. We define

$$D_0 := D^{\boldsymbol{\lambda}_{\min}}, \quad P_0 := P^{\boldsymbol{\lambda}_{\min}}.$$

For each $\boldsymbol{\mu} \in \mathscr{P}_0$, we use $(\mathscr{H}_{\ell,n}^{(0)})^{>\boldsymbol{\mu}}$ to denote the K-subspace of $\mathscr{H}_{\ell,n}^{(0)}$ spanned by all the elements of the form $\psi_w^* y_{\boldsymbol{\lambda}} \psi_u$, where $\boldsymbol{\lambda} > \boldsymbol{\mu}$, $w, u \in \mathfrak{S}_n$. Then $(\mathscr{H}_{\ell,n}^{(0)})^{>\boldsymbol{\mu}}$ is a two-sided ideal of $\mathscr{H}_{\ell,n}^{(0)}$. By [Hu and Mathas 2012, Corollary 3.11], for any $1 \leq r \leq n$, if $\theta(\boldsymbol{\mu}) = (k_1, \dots, k_n)$ then

$$y_{\boldsymbol{\mu}} y_r = y_1^{\ell - k_1} \dots y_n^{\ell - k_n} y_r \in (\mathscr{H}_{\ell,n}^{(0)})^{>\boldsymbol{\mu}}. \tag{2.11}$$

Lemma 2.12. *For any $1 \leq i \leq n$, $1 \leq j < n$, there exists elements $h_{i,j}, h'_{i,j} \in \mathscr{H}_{\ell,n}^{(0)}$ such that*

$$\psi_{w_0} y_1^{n-1} y_2^{n-2} \dots y_{n-1} = (-1)^{n(n-1)/2} + \sum_{\substack{1 \leq i \leq n \\ 1 \leq j < n}} y_i h_{i,j} \psi_j. \tag{2.13}$$

Similarly, we have

$$y_1^{n-1}y_2^{n-2}\dots y_{n-1}\psi_{w_0}=(-1)^{n(n-1)/2}+\sum_{\substack{1\le i\le n\\ 1\le j<n}}\psi_j h_{i,j}^* y_i. \tag{2.14}$$

Proof. We only prove the first equality as the second one follows from the first one by applying the anti-involution $*$. We use induction on n. If $n=1$, it is clear that (2.13) holds. Suppose that the lemma holds for the nilHecke algebra $\mathscr{H}_{\ell,n-1}^{(0)}$. We are going to prove (2.13) for $\mathscr{H}_{\ell,n}^{(0)}$.

Recall that the unique longest element $w_0:=w_{0,n}$ of $\mathfrak{S}_n$ has a reduced expression

$$w_0=s_1(s_2s_1)\dots(s_{n-2}s_{n-3}\dots s_1)(s_{n-1}s_{n-2}\dots s_1).$$

Recall that $w_{0,n-1}$ denotes the unique longest element in $\mathfrak{S}_{n-1}$ and

$$w_0=w_{0,n-1}(s_{n-1}s_{n-2}\dots s_1)$$

and $s_1(s_2s_1)\dots(s_{n-2}s_{n-3}\dots s_1)$ is a reduced expression for $w_{0,n-1}$.

We define

$$J_n:=\sum_{i=1}^n y_i\mathscr{H}_{\ell,n}^{(0)}.$$

Then we have, with all congruences modulo J_n,

$$\begin{aligned}
&\psi_{w_0}y_1^{n-1}y_2^{n-2}\dots y_{n-1}\\
&\quad=\psi_{w_0}(y_1y_2\dots y_{n-1})y_1^{n-2}y_2^{n-3}\dots y_{n-2}\\
&\quad=\psi_{w_{0,n-1}}(\psi_{n-1}\psi_{n-2}\dots\psi_1y_1y_2\dots y_{n-1})y_1^{n-2}y_2^{n-3}\dots y_{n-2}\\
&\quad=\psi_{w_{0,n-1}}(\psi_{n-1}y_1y_2\dots y_{n-1}\psi_{n-2}\dots\psi_1)y_1^{n-2}y_2^{n-3}\dots y_{n-2}\quad\text{(by Corollary 2.3)}\\
&\quad=\psi_{w_{0,n-1}}(y_1y_2\dots y_{n-2}\psi_{n-1}y_{n-1}\psi_{n-2}\dots\psi_1)y_1^{n-2}y_2^{n-3}\dots y_{n-2}\\
&\quad=\psi_{w_{0,n-1}}(y_1y_2\dots y_{n-2}(y_n\psi_{n-1}-1)\psi_{n-2}\dots\psi_1)y_1^{n-2}y_2^{n-3}\dots y_{n-2}\\
&\quad\equiv-\psi_{w_{0,n-1}}(y_1y_2\dots y_{n-2}\psi_{n-2}\dots\psi_1)y_1^{n-2}y_2^{n-3}\dots y_{n-2}\quad\text{(by (2.11))}\\
&\quad\equiv-\psi_{w_{0,n-2}}(\psi_{n-2}\psi_{n-3}\dots\psi_1y_1y_2\dots y_{n-2})(\psi_{n-2}\dots\psi_1)y_1^{n-2}y_2^{n-3}\dots y_{n-2}\\
&\quad\equiv-\psi_{w_{0,n-2}}(\psi_{n-2}y_1y_2\dots y_{n-2}\psi_{n-3}\dots\psi_1)(\psi_{n-2}\dots\psi_1)y_1^{n-2}y_2^{n-3}\dots y_{n-2}\\
&\quad\equiv-\psi_{w_{0,n-2}}(y_1y_2\dots y_{n-3}(\psi_{n-2}y_{n-2})\psi_{n-3}\dots\psi_1)(\psi_{n-2}\dots\psi_1)\\
&\qquad\qquad\times y_1^{n-2}y_2^{n-3}\dots y_{n-2}\\
&\quad\equiv-\psi_{w_{0,n-2}}(y_1y_2\dots y_{n-3}(y_{n-1}\psi_{n-2}-1)\psi_{n-3}\dots\psi_1)(\psi_{n-2}\dots\psi_1)\\
&\qquad\qquad\times y_1^{n-2}y_2^{n-3}\dots y_{n-2}\\
&\quad\equiv(-1)^2\psi_{w_{0,n-2}}(y_1y_2\dots y_{n-3}\psi_{n-3}\dots\psi_1)(\psi_{n-2}\dots\psi_1)y_1^{n-2}y_2^{n-3}\dots y_{n-2}\\
&\quad\equiv(-1)^2\psi_{w_{0,n-2}}(y_1y_2\dots y_{n-3})\big((\psi_{n-3}\dots\psi_1)(\psi_{n-2}\dots\psi_1)\big)y_1^{n-2}y_2^{n-3}\dots y_{n-2}
\end{aligned}$$

$$\begin{aligned}
&\equiv (-1)^2 \psi_{w_{0,n-3}} (\psi_{n-3}\psi_{n-4}\dots\psi_1 y_1 y_2 \dots y_{n-3}) \\
&\qquad\qquad \times \big((\psi_{n-3}\dots\psi_1)(\psi_{n-2}\dots\psi_1)(y_1^{n-2}y_2^{n-3}\dots y_{n-2})\big) \\
&\qquad\vdots \\
&\equiv (-1)^{n-1}\big(\psi_1(\psi_2\psi_1)\dots(\psi_{n-3}\dots\psi_1)(\psi_{n-2}\dots\psi_1)(y_1^{n-2}y_2^{n-3}\dots y_{n-2})\big) \\
&\equiv (-1)^{n-1}\psi_{w_{0,n-1}}(y_1^{n-2}y_2^{n-3}\dots y_{n-2}) \\
&\equiv (-1)^{n-1}(-1)^{(n-1)(n-2)/2} \equiv (-1)^{n(n-1)/2},
\end{aligned}$$

as required, where we have used induction in the second-to-last congruence.

Therefore, we have proved that

$$\psi_{w_0} y_1^{n-1} y_2^{n-2} \dots y_{n-1} = (-1)^{n(n-1)/2} + \sum_{\substack{1\le i\le n \\ 1\le j<n}} y_i h_i,$$

where $h_i \in \mathscr{H}_{\ell,n}^{(0)}$. Comparing the degree on both sides, we can assume that each h_i is homogeneous with $h_i \neq 0$ only if $\deg(h_i) = -2 < 0$. On the other hand, we can express each nonzero h_i as a K-linear combination of some monomials of the form $y_1^{c_1}\dots y_n^{c_n}\psi_w$, where $c_1,\dots,c_n \in \mathbb{N}$, $w \in \mathfrak{S}_n$. Since each y_j has degree 2, we can thus deduce that each nonzero h_i must be equal to a K-linear combination of some monomials of the form $y_1^{c_1}\dots y_n^{c_n}\psi_w$ with $c_1,\dots,c_n \in \mathbb{N}$ and $1 \neq w \in \mathfrak{S}_n$. This completes the proof of the lemma. □

Lemma 2.15. (1) *For any $u, w \in \mathfrak{S}_n$, if $\ell(u)+\ell(w) > \ell(uw)$, then $\psi_u\psi_w = 0$.*

(2) *For any $1 \le r < n$, $\psi_r\psi_{w_0} = 0 = \psi_{w_0}\psi_r$.*

Proof. (1) follows from the defining relations for $\mathscr{H}_{\ell,n}^{(0)}$, while (2) follows from the defining relations for $\mathscr{H}_{\ell,n}^{(0)}$ and the fact that w_0 has both a reduced expression which starts with s_r as well as a reduced expression which ends with s_r for any $1 \le r < n$. □

Let $s \in \mathbb{Z}$. For any $\mathbb{Z}$-graded $\mathscr{H}_{\ell,n}^{(0)}$-module M, we define $M\langle s\rangle$ to be a new $\mathbb{Z}$-graded $\mathscr{H}_{\ell,n}^{(0)}$-module as follows:

- $M\langle s\rangle = M$ as an ungraded $\mathscr{H}_{\ell,n}^{(0)}$-module.
- As a $\mathbb{Z}$-graded module, $M\langle s\rangle$ is obtained by shifting the grading on M up by s. That is, $M\langle s\rangle_d = M_{d-s}$, for $d \in \mathbb{Z}$.

Lemma 2.16. *Let $\boldsymbol{\mu} \in \mathscr{P}_0$ with $\theta(\boldsymbol{\mu}) = (k_1,\dots,k_n)$. Then*

$$\dim D_0 = n!, \quad \dim P_0 = \tbinom{\ell}{n} n!, \quad S^{\boldsymbol{\mu}} \cong D_0\Big\langle n\ell - \frac{n(n-1)}{2} - \sum_{i=1}^n k_i\Big\rangle.$$

Proof. By the definitions of $\mathscr{P}_0$ and Specht modules over $\mathscr{H}_{\ell,n}^{(0)}$, it is clear that $S^{\boldsymbol{\mu}} \cong S^{\boldsymbol{\lambda}_{\min}}\langle n\ell - n(n-1)/2 - \sum_{i=1}^n k_i\rangle$. Thus it suffices to show that $S^{\boldsymbol{\lambda}_{\min}} = D^{\boldsymbol{\lambda}_{\min}}$. To this end, we need to compute the bilinear form between standard bases of the Specht module $S^{\boldsymbol{\lambda}_{\min}}$.

By definition, $S^{\boldsymbol{\lambda}_{\min}}$ has a standard basis

$$\{y_1^{n-1}y_2^{n-2}\dots y_{n-1}\psi_w + (\mathscr{H}_{\ell,n}^{(0)})^{>\boldsymbol{\lambda}_{\min}} \mid w \in \mathfrak{S}_n\}.$$

For any $w, u \in \mathfrak{S}_n$, by Lemma 2.15, we see that

$$\begin{aligned}
y_1^{n-1}y_2^{n-2}&\dots y_{n-1}\psi_w\psi_u^* y_1^{n-1}y_2^{n-2}\dots y_{n-1}\\
&= y_1^{n-1}y_2^{n-2}\dots y_{n-1}(\psi_w\psi_{u^{-1}})y_1^{n-1}y_2^{n-2}\dots y_{n-1} = 0
\end{aligned}$$

unless $\ell(wu^{-1}) = \ell(w) + \ell(u^{-1})$.

Now we assume that $\ell(wu^{-1}) = \ell(w) + \ell(u^{-1})$. By the commutator relations between y and ψ generators, (2.11) and the fact that $\ell(w_0) = n(n-1)2$, we can deduce that

$$\begin{aligned}
y_1^{n-1}y_2^{n-2}&\dots y_{n-1}(\psi_w\psi_{u^{-1}})y_1^{n-1}y_2^{n-2}\dots y_{n-1}\\
&= y_1^{n-1}y_2^{n-2}\dots y_{n-1}\psi_{wu^{-1}}y_1^{n-1}y_2^{n-2}\dots y_{n-1} \in (\mathscr{H}_{\ell,n}^{(0)})^{>\boldsymbol{\lambda}_{\min}}
\end{aligned}$$

unless $wu^{-1} = w_0$. In that case, by Lemma 2.12, we have that

$$\begin{aligned}
y_1^{n-1}y_2^{n-2}&\dots y_{n-1}\psi_w\psi_u^* y_1^{n-1}y_2^{n-2}\dots y_{n-1}\\
&= y_1^{n-1}y_2^{n-2}\dots y_{n-1}\psi_{w_0}y_1^{n-1}y_2^{n-2}\dots y_{n-1}\\
&= (-1)^{n(n-1)/2}y_1^{n-1}y_2^{n-2}\dots y_{n-1} \pmod{(\mathscr{H}_{\ell,n}^{(0)})^{>\boldsymbol{\lambda}_{\min}}}.
\end{aligned}$$

Thus we have proved that if $\ell(wu^{-1}) = \ell(w) + \ell(u^{-1})$ and $wu^{-1} = w_0$, then

$$\begin{aligned}
\big\langle y_1^{n-1}y_2^{n-2}\dots y_{n-1}\psi_w + (\mathscr{H}_{\ell,n}^{(0)})^{>\boldsymbol{\lambda}_{\min}}, y_1^{n-1}y_2^{n-2}\dots y_{n-1}\psi_u + (\mathscr{H}_{\ell,n}^{(0)})^{>\boldsymbol{\lambda}_{\min}}\big\rangle_{\boldsymbol{\lambda}_{\min}}\\
= (-1)^{n(n-1)/2};
\end{aligned}$$

otherwise it is equal to 0. This means the Gram matrix of $S^{\boldsymbol{\lambda}_{\min}}$ is invertible and hence the bilinear form $\langle -,-\rangle_{\boldsymbol{\lambda}_{\min}}$ on $S^{\boldsymbol{\lambda}_{\min}}$ is nondegenerate. It follows that $S^{\boldsymbol{\lambda}_{\min}} = D^{\boldsymbol{\lambda}_{\min}} = D_0$ as required. Therefore, $\dim D_0 = \dim S^{\boldsymbol{\lambda}_{\min}} = n!$. Finally, since $\mathscr{H}_{\ell,n}^{(0)} \cong P_0^{\oplus \dim D_0}$, we can deduce that $\dim P_0 = \dim \mathscr{H}_{\ell,n}^{(0)} / \dim D_0 = \binom{\ell}{n}(n!)^2/n! = \binom{\ell}{n}n!$. □

Let q be an indeterminate. The *graded dimension* of M is the Laurent polynomial

$$\dim_q M = \sum_{d\in\mathbb{Z}} (\dim_K M_d)\, q^d \in \mathbb{N}[q, q^{-1}], \tag{2.17}$$

where M_d is the homogeneous component of M which has degree d. In particular,

$\dim_K M = (\dim_q M)|_{q=1}$. As a consequence, we can determine the graded dimension of the unique self-dual graded simple module D_0 and its projective cover P_0, and compute the graded decomposition number $d_{\boldsymbol{\mu},\boldsymbol{\lambda}_{\min}}(q) := [S^{\boldsymbol{\mu}} : D^{\boldsymbol{\lambda}_{\min}}]_q$ and graded Cartan number $c_{\boldsymbol{\lambda}_{\min},\boldsymbol{\lambda}_{\min}}(q) := [P^{\boldsymbol{\lambda}_{\min}} : D^{\boldsymbol{\lambda}_{\min}}]_q$.

Corollary 2.18. *We have*

$$\dim_q D_0 = \sum_{\mathfrak{t} \in \mathrm{Tab}(\boldsymbol{\lambda}_{\min})} q^{\deg \mathfrak{t}},$$

$$\dim_q P_0 = \sum_{\substack{k_1,\dots,k_n \in \mathbb{N} \\ 1 \le k_1 < k_2 < \cdots < k_n \le \ell}} \sum_{\mathfrak{t} \in \mathrm{Tab}(\boldsymbol{\lambda}_{\min})} q^{\deg \mathfrak{t} + 2n\ell - n(n-1) - \sum_{i=1}^n 2k_i}.$$

Corollary 2.19. *Let $\boldsymbol{\mu} \in \mathscr{P}_0$ with $\theta(\boldsymbol{\mu}) = (k_1, \dots, k_n)$. We have*

$$d_{\boldsymbol{\mu},\boldsymbol{\lambda}_{\min}}(q) = q^{n\ell - n(n-1)/2 - \sum_{i=1}^n k_i} \in \delta_{\boldsymbol{\mu},\boldsymbol{\lambda}_{\min}} + q\mathbb{N}[q],$$

$$c_{\boldsymbol{\lambda}_{\min},\boldsymbol{\lambda}_{\min}}(q) = \sum_{\substack{l_1,\dots,l_n \in \mathbb{N} \\ 1 \le l_1 < l_2 < \cdots < l_n \le \ell}} q^{2n\ell - n(n-1) - \sum_{i=1}^n 2l_i} \in 1 + q\mathbb{N}[q].$$

Lemma 2.20 [Hoffnung and Lauda 2010, Proposition 7]. *For any $1 \le s \le n$, we have*

$$\sum_{\substack{l_1,\dots,l_s \in \mathbb{N} \\ l_1 + \cdots + l_s = \ell - s + 1}} y_1^{l_1} y_2^{l_2} \dots y_s^{l_s} = 0.$$

Remark 2.21. Note that one should identify our generator y_r with the generator $-x_{r,\boldsymbol{i}}$ in [Hoffnung and Lauda 2010] so that the relation $\psi_r y_{r+1} = y_r \psi_r + 1$ in Definition 1.4 matches up with the relation $x_{r,\boldsymbol{i}} \delta_{r,i} - \delta_{r,\boldsymbol{i}} x_{r+1,\boldsymbol{i}} = e(\boldsymbol{i})$ when $i_r = i_{r+1}$.

Lemma 2.22 [Hoffnung and Lauda 2010, Proposition 8]. *Let $1 \le m < n$ and $b \in \mathbb{N}$. If $y_{m-1}^b = 0$ then $y_m^b = 0$.*

Lemma 2.23. *For any $2 \le m \le n$ and $\omega_m > \ell - m$, we have*

$$y_1^{\ell-1} y_2^{\ell-2} \dots y_{m-1}^{\ell-m+1} y_m^{\omega_m} = 0. \tag{2.24}$$

Proof. We use induction on m. If $m = 1$, then (2.24) reduces to $y_1^{\omega_1} = 0$ for $\omega_1 > \ell - 1$, which certainly holds by the fact that $y_1^\ell = 0$.

If $m = 2$, then we need to show that $y_1^{\ell-1} y_2^{\omega_2} = 0$ whenever $\omega_2 > \ell - 2$. By Lemma 2.22, we can deduce that $y_2^\ell = 0$ from the equality $y_1^\ell = 0$. Therefore, it remains to show that $y_1^{\ell-1} y_2^{\ell-1} = 0$. In this case, applying Lemma 2.20, we get that

$$y_2^{\ell-1} = \sum_{\substack{l_1, l_2 \in \mathbb{N}, l_1 \ne 0 \\ l_1 + l_2 = \ell - 1}} y_1^{l_1} y_2^{l_2}.$$

It follows that

$$y_1^{\ell-1}y_2^{\ell-1} = -\sum_{\substack{l_1,l_2\in\mathbb{N},l_1\neq 0\\ l_1+l_2=\ell-1}} y_1^{\ell-1+l_1}y_2^{l_2} = 0,$$

as required.

Now assume that (2.24) holds for $2\le k\le m$. Hence $y_1^{\ell-1}y_2^{\ell-2}\dots y_{k-1}^{\ell-k+1}y_k^{\omega_k}=0$ whenever $\omega_k > \ell-k$.

Applying Lemma 2.20 for $s=m+1$, we get that

$$y_{m+1}^{\ell-m} = \sum_{\substack{l_1,\dots,l_{m+1}\in\mathbb{N}\\ l_{m+1}\neq\ell-m,l_1+\cdots+l_{m+1}=\ell-m}} y_1^{l_1}y_2^{l_2}\dots y_{m+1}^{l_{m+1}}.$$

It follows that for any $\omega_{m+1} > \ell-(m+1)$,

$$\begin{aligned}
&y_1^{\ell-1}y_2^{\ell-2}\dots y_{m-1}^{\ell-m+1}y_{m+1}^{\omega_{m+1}}\\
&\qquad = y_1^{\ell-1}y_2^{\ell-2}\dots y_{m-1}^{\ell-m+1}y_{m+1}^{\omega_{m+1}-(\ell-m)}y_{m+1}^{\ell-m}\\
&\qquad = -\sum_{\substack{l_2,\dots,l_{m+1}\in\mathbb{N}\\ l_{m+1}\neq\ell-m,l_2+\cdots+l_{m+1}=\ell-m}} y_1^{\ell-1}y_2^{\ell-2+l_2}\dots y_{m+1}^{\omega_{m+1}-(\ell-m)+l_{m+1}}\\
&\qquad = -\sum_{\substack{l_m,l_{m+1}\in\mathbb{N}\\ l_{m+1}\neq\ell-m,l_m+l_{m+1}=\ell-m}} y_1^{\ell-1}y_2^{\ell-2}\dots y_{m-1}^{\ell-m+1}y_m^{\ell-m+l_m}y_{m+1}^{\omega_{m+1}-(\ell-m)+l_{m+1}}\\
&\qquad = 0,
\end{aligned}$$

where we have used the induction hypothesis in the third and fourth equalities. This completes the proof of the lemma. □

Corollary 2.25. *For any $z_1,z_2\in\mathfrak{S}_n$, we define $F_{z_1,z_2}:=(-1)^{n(n-1)/2}\psi^{\boldsymbol{\lambda}_{\min}}_{w_0z_1,z_2}$. Then $F_{z_1,z_2}\neq 0$ is a homogeneous element of degree $2\ell(z_1)-2\ell(z_2)$. Suppose that $\ell=n$. Then $\sum_{w\in\mathfrak{S}_n}F_{w,w}=1$ and*

$$F_{z_1,z_2}F_{u_1,u_2}=\delta_{z_2,u_1}F_{z_1,u_2},\quad \forall\, u_1,u_2\in\mathfrak{S}_n.$$

In particular, $\mathscr{H}_{n,n}^{(0)}$ is isomorphic to the full matrix algebra $M_{n!\times n!}(K)$ over K with $\{F_{u,w}\}_{u,w\in\mathfrak{S}_n}$ being a complete set of matrix units.

Proof. As a cellular basis element, we know that $\psi^{\boldsymbol{\lambda}_{\min}}_{w_0z_1,z_2}\neq 0$ and hence $F_{z_1,z_2}\neq 0$. By definition, F_{z_1,z_2} is a homogeneous element of degree $2\ell(z_1)-2\ell(z_2)$.

Suppose that $\ell=n$. By Lemma 2.23, for any $1\le r\le n$, we have

$$\begin{aligned}
(2.26)\quad & y_1^{n-1}y_2^{n-2}\dots y_{n-1}y_r\\
&\qquad = (y_1^{n-1}y_2^{n-2}\dots y_{r+1}^{n-r-1}y_r^{n-r+1})y_{r-1}^{n-r+1}y_{r-2}^{n-r+2}\dots y_{n-1}=0.
\end{aligned}$$

For any $u_1, u_2 \in \mathfrak{S}_n$,

$$F_{z_1,z_2}F_{u_1,u_2} = \psi^*_{w_0z_1}y_1^{n-1}y_2^{n-2}\dots y_{n-1}\psi_{z_2}\psi^*_{w_0u_1}y_1^{n-1}y_2^{n-2}\dots y_{n-1}\psi_{u_2}.$$

By Lemma 2.15, this quantity is zero unless $\ell(z_2(w_0u_1)^{-1}) = \ell(z_2)+\ell((w_0u_1)^{-1})$. So we can assume that $\ell(z_2(w_0u_1)^{-1}) = \ell(z_2)+\ell((w_0u_1)^{-1})$. Then we get

$$F_{z_1,z_2}F_{u_1,u_2} = \psi^*_{w_0z_1}y_1^{n-1}y_2^{n-2}\dots y_{n-1}\psi_{z_2u_1^{-1}w_0^{-1}}y_1^{n-1}y_2^{n-2}\dots y_{n-1}\psi_{u_2}.$$

Note that w_0 is the unique longest element in $\mathfrak{S}_n$ with length $(n-1)n/2$. If $z_2u_1^{-1}w_0^{-1} \neq w_0$ then we must have

$$\psi_{z_2u_1^{-1}w_0^{-1}}y_1^{n-1}y_2^{n-2}\dots y_{n-1} \in \sum_{j=1}^{n} y_j\mathscr{H}^{(0)}_{n,n}.$$

In that case, $F_{z_1,z_2}F_{u_1,u_2} = 0$ by (2.26). Therefore, we can further assume that $z_2u_1^{-1}w_0^{-1} = w_0$ and hence $z_2 = u_1$. In the latter case, $F_{z_1,z_2}F_{u_1,u_2} = F_{z_1,u_2}$ by Lemma 2.12 and (2.26). This proves the first part of the corollary.

The second part of the corollary follows from Corollary 2.25 and the fact that $\dim \mathscr{H}^{(0)}_{n,n} = (n!)^2$ and $\{F_{z_1,z_2} \mid z_1, z_2 \in \mathfrak{S}_n\}$ is a basis of $\mathscr{H}^{(0)}_{n,n}$. □

Recall that the weak Bruhat order "$\succeq$" on $\mathfrak{S}_n$ is defined as follows (see [Dipper and James 1986]): For $u, w \in \mathfrak{S}_n$, let $u \succeq w$ if there is a reduced expression $w = s_{j_1}\dots s_{j_k}$ for w and $u = s_{j_1}\dots s_{j_l}$ for some $l \leq k$. We write $u \succ w$ if $u \succeq w$ and $u \neq w$.

Corollary 2.27. *Let $\ell, n \in \mathbb{N}$. For any $z_1, z_2 \in \mathfrak{S}_n$, we define*

$$F'_{z_1,z_2} := \psi^*_{w_0z_1}y_1^{n-1}y_2^{n-2}\dots y_{n-1}\psi_{w_0}y_1^{n-1}y_2^{n-2}\dots y_{n-1}\psi_{z_2}.$$

Then $F'_{z_1,z_2} \neq 0$ is a homogeneous element of degree $2\ell(z_1)-2\ell(z_2)$, and

$$(F'_{z_1,z_1})^2 = F'_{z_1,z_1}, \quad F'_{z_1,z_2} = F'_{z_1,z_1}F'_{z_1,z_2} = F'_{z_1,z_2}F'_{z_2,z_2},$$
$$F'_{z_1,z_2}F'_{z_2,u_2} = F'_{z_1,u_2}, \quad F'_{z_1,z_2}F'_{u_1,u_2} = 0, \quad \forall\, u_1, u_2 \in \mathfrak{S}_n \text{ with } z_2^{-1} \not\succeq u_1^{-1}.$$

Proof. By Lemma 2.12 and (2.11), we have

$$F'_{z_1,z_2} \equiv (-1)^{(n-1)n/2}\psi^{\boldsymbol{\lambda}_{\min}}_{w_0z_1,z_2} \pmod{(\mathscr{H}^{(0)}_{\ell,n})^{>\boldsymbol{\lambda}_{\min}}}. \tag{2.28}$$

In particular, this implies that $F'_{z_1,z_2} \neq 0$ by the cellular structure of $\mathscr{H}^{(0)}_{\ell,n}$. By definition, it is clear that F'_{z_1,z_2} is a homogeneous element of degree $2\ell(z_1)-2\ell(z_2)$.

Again by Lemma 2.12 and Lemma 2.15, we have

$$
\begin{aligned}
(F'_{z_1,z_1})^2 &= \psi^*_{w_0z_1} y_1^{n-1} y_2^{n-2} \dots y_{n-1} \psi_{w_0} y_1^{n-1} y_2^{n-2} \dots y_{n-1} (\psi_{z_1} \psi^*_{w_0z_1}) \\
&\qquad \times y_1^{n-1} y_2^{n-2} \dots y_{n-1} \psi_{w_0} y_1^{n-1} y_2^{n-2} \dots y_{n-1} \psi_{z_1} \\
&= \psi^*_{w_0z_1} y_1^{n-1} y_2^{n-2} \dots y_{n-1} (\psi_{w_0} y_1^{n-1} y_2^{n-2} \dots y_{n-1} \psi_{w_0}) \\
&\qquad \times y_1^{n-1} y_2^{n-2} \dots y_{n-1} \psi_{w_0} y_1^{n-1} y_2^{n-2} \dots y_{n-1} \psi_{z_1} \\
&= (-1)^{(n-1)n/2} \psi^*_{w_0z_1} y_1^{n-1} y_2^{n-2} \dots y_{n-1} (\psi_{w_0} y_1^{n-1} y_2^{n-2} \dots y_{n-1} \psi_{w_0}) \\
&\qquad \times y_1^{n-1} y_2^{n-2} \dots y_{n-1} \psi_{z_1} \\
&= F'_{z_1,z_1}.
\end{aligned}
$$

A similar argument shows that $F'_{z_1,z_2} = F'_{z_1,z_1} F'_{z_1,z_2} = F'_{z_1,z_2} F'_{z_2,z_2}$ and $F'_{z_1,z_2} F'_{z_2,u_2} = F'_{z_1,u_2}$.

Finally, let $u_1, u_2 \in \mathfrak{S}_n$ such that $z_2^{-1} \not\succeq u_1^{-1}$. We have

$$
\begin{aligned}
F'_{z_1,z_2} F'_{u_1,u_2} &= \psi^*_{w_0z_1} y_1^{n-1} y_2^{n-2} \dots y_{n-1} \psi_{w_0} y_1^{n-1} y_2^{n-2} \dots y_{n-1} (\psi_{z_2} \psi^*_{w_0u_1}) \\
&\qquad \times y_1^{n-1} y_2^{n-2} \dots y_{n-1} \psi_{w_0} y_1^{n-1} y_2^{n-2} \dots y_{n-1} \psi_{u_2}.
\end{aligned}
$$

Note that the assumption $z_2^{-1} \not\succeq u_1^{-1}$ implies that $\ell(z_2 u_1^{-1} w_0^{-1}) \neq \ell(z_2) + \ell(u_1^{-1} w_0^{-1})$ because otherwise we would have some $x \in \mathfrak{S}_n$ such that $x z_2 = u_1$ and

$$
\begin{aligned}
\ell(x) &= \ell(w_0) - \ell(z_2 u_1^{-1} w_0^{-1}) = \ell(w_0) - (\ell(z_2) + \ell(u_1^{-1} w_0^{-1})) \\
&= \ell(w_0) - \ell(z_2) - (\ell(w_0) - \ell(u_1^{-1})) = \ell(u_1) - \ell(z_2).
\end{aligned}
$$

By Lemma 2.15, $\ell(z_2 u_1^{-1} w_0^{-1}) \neq \ell(z_2) + \ell(u^{-1} w_0^{-1})$ implies that $\psi_{z_2} \psi^*_{w_0u_1} = 0$. We thus proved that $F'_{z_1,z_2} F'_{u_1,u_2} = 0$ as required. □

Definition 2.29. We fix a total order on $\mathfrak{S}_n$ and list the elements in $\mathfrak{S}_n$ as $1 = w_1, w_2, \dots, w_{n!}$ such that

$$
w_i^{-1} \succ w_j^{-1} \implies i < j.
$$

We define a set of elements $\{\tilde{F}_{w_i,w_j} \mid 1 \le i, j \le n!\}$ in $\mathscr{H}^{(0)}_{\ell,n}$ inductively as follows:

$$
\tilde{F}_{w_1,w_j} = \tilde{F}_{1,w_j} := F'_{1,w_j}, \quad \forall\, 1 \le j \le n!.
$$

Suppose that $\tilde{F}_{w_k,w_j}$ was already defined for any $1 \le k < i$ and $1 \le j \le n!$. Then we define

$$
\tilde{F}_{w_i,w_j} := F'_{w_i,w_j} - \sum_{1 \le k < i} \tilde{F}_{w_k,w_k} F'_{w_i,w_j}, \quad \forall\, 1 \le j \le n!.
$$

By construction and Corollary 2.27, we see that

$$
\tilde{F}_{w_i,w_j} F'_{w_j,w_a} = \tilde{F}_{w_i,w_a}, \quad \forall\, 1 \le a \le n!. \tag{2.30}
$$

Theorem 2.31. *For any* $1 \le i, j \le n!$, *we have that* $\tilde{F}_{w_i,w_j} \neq 0$ *is a homogeneous element of degree* $2\ell(w_i) - 2\ell(w_j)$ *and*

$$\tilde{F}_{w_i,w_j}\tilde{F}_{w_k,w_l} = \delta_{j,k}\tilde{F}_{w_i,w_l}, \quad \forall\, 1 \le k, l \le n!. \tag{2.32}$$

Moreover, for each $1 \le i \le n!$, $\tilde{F}_{w_i,w_i}\mathscr{H}_{\ell,n}^{(0)} \cong P_0$ *is an ungraded right* $\mathscr{H}_{\ell,n}^{(0)}$*-module,* $1 = \sum_{i=1}^{n!} \tilde{F}_{w_i,w_i}$, *and* $\{\tilde{F}_{w_i,w_i} \mid 1 \le i \le n!\}$ *is a complete set of pairwise orthogonal primitive idempotents of* $\mathscr{H}_{\ell,n}^{(0)}$.

Proof. By (2.28), for any $u \in \mathfrak{S}_n$ with $u^{-1} \succ w_1^{-1}$, we have the following relations modulo $\mathscr{H}_{\ell,n}^{(0)})^{>\boldsymbol{\lambda}_{\min}}$:

$$\begin{aligned}
F'_{u,u}F'_{w_1,w_2} &\equiv \psi^{\boldsymbol{\lambda}_{\min}}_{w_0u,u}\psi^{\boldsymbol{\lambda}_{\min}}_{w_0w_1,w_2} \\
&\equiv \psi^*_{w_0u}y_1^{n-1}y_2^{n-2}\dots y_{n-1}(\psi_u(\psi_{w_0w_1})^*)y_1^{n-1}y_2^{n-2}\dots y_{n-1}\psi_{w_2} \\
&\equiv \psi^*_{w_0u}y_1^{n-1}y_2^{n-2}\dots y_{n-1}\psi_{uw_1^{-1}w_0}y_1^{n-1}y_2^{n-2}\dots y_{n-1}\psi_{w_2} \\
&\equiv \sum_{j=1}^{n} r_j\psi^*_{w_0u}y_1^{n-1}y_2^{n-2}\dots y_{n-1}y_jh_j\psi_{w_2} \\
&\equiv 0,
\end{aligned}$$

where $r_j \in K$, $h_j \in \mathscr{H}_{\ell,n}^{(0)}$ for any z, j. Combining this with Corollary 2.27 and (2.28) we can deduce that

$$\tilde{F}_{w_i,w_j} \equiv (-1)^{(n-1)n/2}\psi^{\boldsymbol{\lambda}_{\min}}_{w_0w_i,w_j} \pmod{(\mathscr{H}_{\ell,n}^{(0)})^{>\boldsymbol{\lambda}_{\min}}}. \tag{2.33}$$

In particular, $\tilde{F}_{w_i,w_j} \neq 0$. By definition, Corollary 2.27, and an easy induction, we see that $\tilde{F}_{w_i,w_j}$ is a homogeneous element of degree $2\ell(w_i) - 2\ell(w_j)$.

We are going to prove (2.32). We use induction on k. Suppose that $k = 1$. If $j \neq 1$, then $j > 1$. By construction,

$$\tilde{F}_{w_i,w_j} \in \sum_{w \in \mathfrak{S}_n} \mathscr{H}_{\ell,n}^{(0)}F'_{w,w_j}, \qquad \tilde{F}_{1,w_l} = F'_{1,w_l}.$$

By Corollary 2.27, we have $F'_{w,w_j}F'_{1,u} = 0$. It follows that $\tilde{F}_{w_i,w_j}\tilde{F}_{w_1,w_l} = 0$. If $j = 1$, then by (2.30) we have

$$\tilde{F}_{w_i,w_1}\tilde{F}_{w_1,w_l} = \tilde{F}_{w_i,1}F'_{1,w_l} = \tilde{F}_{w_i,w_l},$$

as required.

In general, suppose that (2.32) holds for any $k < m$. Let us consider the case when $k = m$. By construction, we have

$$\tilde{F}_{w_i,w_j} \in \sum_{w \in \mathfrak{S}_n} \mathscr{H}_{\ell,n}^{(0)}F'_{w,w_j}, \qquad \tilde{F}_{w_m,w_l} \in \sum_{\substack{u \in \mathfrak{S}_n \\ 1 \le a \le m}} F'_{w_a,u}\mathscr{H}_{\ell,n}^{(0)}.$$

Therefore, if $j > m$ then $\tilde{F}_{w_i,w_j}\tilde{F}_{w_m,w_l} = 0$ by Corollary 2.27.

Suppose that $j < m$. Then

$$\begin{aligned}\tilde{F}_{w_i,w_j}\tilde{F}_{w_m,w_l} &= \tilde{F}_{w_i,w_j}\Big(F'_{w_m,w_l} - \sum_{1\le k<m}\tilde{F}_{w_k,w_k}F'_{w_m,w_l}\Big)\\ &= \tilde{F}_{w_i,w_j}\Big(F'_{w_m,w_l} - \sum_{1\le k<m}\delta_{k,j}\tilde{F}_{w_k,w_k}F'_{w_m,w_l}\Big)\\ &= \tilde{F}_{w_i,w_j}F'_{w_m,w_l} - \tilde{F}_{w_i,w_j}F'_{w_m,w_l}\\ &= 0,\end{aligned}$$

as required, where we have used induction hypothesis in the second and the third equalities.

Suppose that $j = m$. Then

$$\begin{aligned}\tilde{F}_{w_i,w_m}\tilde{F}_{w_m,w_l} &= \tilde{F}_{w_i,w_m}\Big(F'_{w_m,w_l} - \sum_{1\le k<m}\tilde{F}_{w_k,w_k}F'_{w_m,w_l}\Big)\\ &= \tilde{F}_{w_i,w_m}F'_{w_m,w_l} - \sum_{1\le k<m}\tilde{F}_{w_i,w_m}\tilde{F}_{w_k,w_k}F'_{w_m,w_l}\\ &= \tilde{F}_{w_i,w_m}F'_{w_m,w_l} - 0 = \tilde{F}_{w_i,w_l},\end{aligned}$$

as required, where we used (2.30) in the last equality, and used the induction hypothesis in the second last equality.

Since

$$P_0^{\oplus \dim D_0} = P_0^{\oplus n!} \cong \mathscr{H}_{\ell,n}^{(0)} \cong \Big(1 - \sum_{w\in\mathfrak{S}_n}\tilde{F}_{w,w}\Big)\mathscr{H}_{\ell,n}^{(0)} \oplus \Big(\bigoplus_{w\in\mathfrak{S}_n}\tilde{F}_{w,w}\mathscr{H}_{\ell,n}^{(0)}\Big),$$

and $\tilde{F}_{w,w}\mathscr{H}_{\ell,n}^{(0)} \neq 0$ for each $w \in \mathfrak{S}_n$. By the Krull–Schmidt theorem we can deduce that for each $w \in \mathfrak{S}_n$, $F_{w,w}\mathscr{H}_{\ell,n}^{(0)} \cong P_0$ is an ungraded right $\mathscr{H}_{\ell,n}^{(0)}$-module and $1 = \sum_{w\in\mathfrak{S}_n}\tilde{F}_{w,w}$. In other words, $\{\tilde{F}_{w_i,w_i} \mid 1 \le i \le n!\}$ is a complete set of pairwise orthogonal primitive idempotents of $\mathscr{H}_{\ell,n}^{(0)}$. □

The following result was first conjectured by A. Mathas [2015, §2.5, before Corollary 2.5.2] in the special case when $\ell = n$.

Theorem 2.34. *The elements in the set*

$$\{\psi_w y_1^{a_1}\dots y_n^{a_n} \mid 0 \le a_i \le \ell - i, \quad \forall\, 1 \le i \le n, w \in \mathfrak{S}_n\} \tag{2.35}$$

form a K-basis of $\mathscr{H}_{\ell,n}^{(0)}$.

Proof. We first claim that for any $b_1, \dots, b_{m-1}, \omega_m \in \mathbb{N}$ with $0 \le b_j \le l - j, \forall\, 1 \le j \le m$,

$$y_1^{b_1}y_2^{b_2}\dots y_{m-1}^{b_{m-1}}y_m^{\omega_m} = \sum_{\substack{c_1,\dots,c_m\in\mathbb{N}\\ 0\le c_i\le \ell-i, \forall\, 1\le i\le m}} r_{c_1,\dots,c_m}y_1^{c_1}y_2^{c_2}\dots y_{m-1}^{c_{m-1}}y_m^{c_m}, \tag{2.36}$$

where $r_{c_1,\dots,c_m} \in K$ for each m-tuple $(c_1, \dots, c_m)$.

We use induction on m. If $m = 1$, there is nothing to prove as $y_1^{\omega_1} = 0$ whenever $\omega_1 > \ell - 1$. Suppose that (2.36) holds for any $1 \le k \le m$.

We now consider the case where $k = m+1$. Applying Lemma 2.20 for $s = m+1$, we get that

$$y_{m+1}^{\ell-m} = -\sum_{\substack{l_1,\dots,l_{m+1}\in\mathbb{N}\\ l_{m+1}\neq\ell-m, l_1+\dots+l_{m+1}=\ell-m}} y_1^{l_1} y_2^{l_2} \dots y_{m+1}^{l_{m+1}}.$$

It follows that

$$\begin{aligned} y_1^{b_1} y_2^{b_2} \dots y_{m-1}^{b_{m-1}} y_m^{b_m} y_{m+1}^{\omega_{m+1}} &= y_1^{b_1} y_2^{b_2} \dots y_{m-1}^{b_{m-1}} y_m^{b_m} y_{m+1}^{\omega_{m+1}-(\ell-m)} y_{m+1}^{\ell-m} \\ &= -\sum_{\substack{l_1,\dots,l_{m+1}\in\mathbb{N}\\ l_{m+1}\neq\ell-m, l_1+\dots+l_{m+1}=\ell-m}} y_1^{b_1+l_1} y_2^{b_2+l_2} \dots y_{m-1}^{b_{m-1}+l_{m-1}} y_m^{b_m+l_m} y_{m+1}^{b'_{m+1}}, \end{aligned}$$

where $b'_{m+1} := \omega_{m+1} - (l-m) + l_{m+1}$.

Our purpose is to show that

$$\begin{aligned} (2.37)\quad & y_1^{b_1} y_2^{b_2} \dots y_m^{b_m} y_{m+1}^{\omega_{m+1}} \\ & \in K\text{-Span}\{y_1^{c_1} y_2^{c_2} \dots y_m^{c_m} y_{m+1}^{c_{m+1}} \mid c_i \in \mathbb{N}, 0 \le c_i \le \ell - i, \forall\, 1 \le i \le m+1\}. \end{aligned}$$

We use induction on ω_{m+1}. Suppose that for any $b_1, \dots, b_m \in \mathbb{N}$ and any $0 \le b < \omega_{m+1}$, we have

$$\begin{aligned} & y_1^{b_1} y_2^{b_2} \dots y_m^{b_m} y_{m+1}^{b} \\ & \in K\text{-Span}\{y_1^{c_1} y_2^{c_2} \dots y_m^{c_m} y_{m+1}^{c_{m+1}} \mid c_i \in \mathbb{N}, 0 \le c_i \le \ell - i, \forall\, 1 \le i \le m+1\}. \end{aligned}$$

We are now going to prove (2.37). If $b'_{m+1} \le l - m$, then by our induction hypothesis we have

$$\begin{aligned} & y_1^{b_1+l_1} y_2^{b_2+l_2} \dots y_{m-1}^{b_{m-1}+l_{m-1}} y_m^{b_m+l_m} \\ & \in K\text{-Span}\{y_1^{c_1} y_2^{c_2} \dots y_{m-1}^{c_{m-1}} y_m^{c_m} \mid c_1, \dots, c_m \in \mathbb{N}, 0 \le c_i \le \ell - i, \forall\, 1 \le i \le m\}, \end{aligned}$$

hence

$$\begin{aligned} & y_1^{b_1+l_1} y_2^{b_2+l_2} \dots y_{m-1}^{b_{m-1}+l_{m-1}} y^{b_m+l_m} y_{m+1}^{b'_{m+1}} \\ & \in K\text{-Span}\{y_1^{c_1} y_2^{c_2} \dots y_m^{c_m} y_{m+1}^{c_{m+1}} \mid c_1, \dots, c_{m+1} \in \mathbb{N}, 0 \le c_i \le \ell - i, \forall\, 1 \le i \le m+1\}. \end{aligned}$$

Therefore, it remains to consider those terms which satisfy $b'_{m+1} > \ell - m$. Since $l_1 + \dots + l_{m+1} = \ell - m$ and $l_{m+1} \neq \ell - m$, we have $0 \le l_{m+1} \le \ell - m - 1$; furthermore, we have $b'_{m+1} \le \omega_{m+1} - 1$. By our induction hypothesis on ω_{m+1}, we have

$$\begin{aligned} & y_1^{b_1+l_1} y_2^{b_2+l_2} \dots y_{m-1}^{b_{m-1}+l_{m-1}} y_m^{b_m+l_m} y_{m+1}^{b'_{m+1}} \\ & \qquad \in K\text{-Span}\{y_1^{c_1} y_2^{c_2} \dots y_m^{c_m} y_{m+1}^{c_{m+1}} \mid c_i \in \mathbb{N}, 0 \le c_i \le \ell - i, \forall\, 1 \le i \le m+1\}. \end{aligned}$$

Therefore, we can conclude that (2.37) always holds. This completes the proof of (2.36).

Now we have proved that the elements in (2.35) form a K-linear generator of $\mathscr{H}_{\ell,n}^{(0)}$. Since the set (2.35) has cardinality equal to $\binom{\ell}{n}(n!)^2$, which is equal to the dimension of $\mathscr{H}_{\ell,n}^{(0)}$, the elements in (2.35) must form a K-basis of $\mathscr{H}_{\ell,n}^{(0)}$. □

Remark 2.38. We shall call the basis (2.35) a *monomial basis* of $\mathscr{H}_{\ell,n}^{(0)}$. It bears much resemblance to the Ariki–Koike basis of the cyclotomic Hecke algebra of type $G(\ell, 1, n)$. For arbitrary cyclotomic quiver Hecke algebras, Question 1.1 (on how to construct a monomial basis) remains open. Anyhow, we regard Theorem 2.34 as a first step in our effort of answering that open question.

3. A basis of the center

The purpose of this section is to give an explicit basis of the center of $\mathscr{H}_{\ell,n}^{(0)}$. Let $Z := Z(\mathscr{H}_{\ell,n}^{(0)})$ be the center of $\mathscr{H}_{\ell,n}^{(0)}$.

Definition 3.1. For each $\boldsymbol{\mu} \in \mathscr{P}_0$, we define

$$b_{\boldsymbol{\mu}} := \psi_{w_0} y_{\boldsymbol{\mu}} \psi_{w_0} y_1^{n-1} y_2^{n-2} \dots y_{n-1}.$$

By Definition 2.29, Corollary 2.27, Lemma 2.12, and Lemma 2.15, we have

$$\begin{aligned} \tilde{F}_{1,1} = F'_{1,1} &= \psi_{w_0}^* y_1^{n-1} y_2^{n-2} \dots y_{n-1} \psi_{w_0} y_1^{n-1} y_2^{n-2} \dots y_{n-1} \\ &= (\psi_{w_0} y_1^{n-1} y_2^{n-2} \dots y_{n-1}) \psi_{w_0} y_1^{n-1} y_2^{n-2} \dots y_{n-1} \\ &= (-1)^{n(n-1)/2} \psi_{w_0} y_1^{n-1} y_2^{n-2} \dots y_{n-1} = F_{1,1}. \end{aligned}$$

Note that each $y_{\boldsymbol{\mu}}$ has a left factor $y_1^{n-1} y_2^{n-2} \dots y_{n-1}$. It follows that

$$b_{\boldsymbol{\mu}} \in \tilde{F}_{1,1} \mathscr{H}_{\ell,n}^{(0)} \tilde{F}_{1,1} \cong \operatorname{End}_{\mathscr{H}_{\ell,n}^{(0)}}(\tilde{F}_{1,1} \mathscr{H}_{\ell,n}^{(0)}) \cong \operatorname{End}_{\mathscr{H}_{\ell,n}^{(0)}}(P_0).$$

Suppose further that $\theta(\boldsymbol{\mu}) = (k_1, \dots, k_n)$, where $1 \le k_1 < k_2 < \dots < k_n \le \ell$. Then by (2.11),

$$\begin{aligned} b_{\boldsymbol{\mu}} &= \psi_{w_0} y_1^{\ell-k_1} y_2^{\ell-k_2} \dots y_n^{\ell-k_n} \psi_{w_0} y_1^{n-1} y_2^{n-2} \dots y_{n-1} \\ &\equiv (-1)^{n(n-1)/2} \psi_{w_0} y_1^{\ell-k_1} y_2^{\ell-k_2} \dots y_n^{\ell-k_n} \pmod{(\mathscr{H}_{\ell,n}^{(0)})^{>\boldsymbol{\mu}}} \\ &\equiv (-1)^{n(n-1)/2} \psi_{w_0,1}^{\boldsymbol{\mu}} \pmod{(\mathscr{H}_{\ell,n}^{(0)})^{>\boldsymbol{\mu}}}. \end{aligned}$$

It follows that $\{b_{\boldsymbol{\mu}} \mid \boldsymbol{\mu} \in \mathscr{P}_0\}$ are K-linearly independent elements in $\tilde{F}_{1,1} \mathscr{H}_{\ell,n}^{(0)} \tilde{F}_{1,1}$.

Lemma 3.2. *The elements in* $\{b_{\boldsymbol{\mu}} \mid \boldsymbol{\mu} \in \mathscr{P}_0\}$ *form a* K*-basis of* $\tilde{F}_{1,1}\mathscr{H}_{\ell,n}^{(0)}\tilde{F}_{1,1}$*. Moreover, the basic algebra* $\operatorname{End}_{\mathscr{H}_{\ell,n}^{(0)}}(P_0)$ *of* $\mathscr{H}_{\ell,n}^{(0)}$ *is commutative and is isomorphic to the center* Z *of* $\mathscr{H}_{\ell,n}^{(0)}$*. In particular,* $\dim_K Z = \binom{\ell}{n}$*.*

Proof. Since $\#\mathscr{P}_0 = \binom{\ell}{n}$ and $\tilde{F}_{1,1}\mathscr{H}_{\ell,n}^{(0)}\tilde{F}_{1,1} \cong \operatorname{End}_{\mathscr{H}_{\ell,n}^{(0)}}(P_0)$, it suffices to show that

$$\dim_K \operatorname{End}_{\mathscr{H}_{\ell,n}^{(0)}}(P_0) = \binom{\ell}{n}.$$

By Lemma 2.16 and Corollary 2.18, we know that $[P_0 : D_0] = \binom{\ell}{n}$ and hence $\dim_K \operatorname{End}_{\mathscr{H}_{\ell,n}^{(0)}}(P_0) = \binom{\ell}{n}$, as required. Thus, the first part of the lemma follows from this together with the discussion in the paragraph above this lemma.

It remains to show that the endomorphism algebra $\operatorname{End}_{\mathscr{H}_{\ell,n}^{(0)}}(P_0)$ is commutative. Once this is proved, and since $\mathscr{H}_{\ell,n}^{(0)}$ is Morita equivalent to $\operatorname{End}_{\mathscr{H}_{\ell,n}^{(0)}}(P_0)$, it will follow from [Curtis and Reiner 1981, (3.54)(iv)] that

$$Z = Z(\mathscr{H}_{\ell,n}^{(0)}) \cong Z(\operatorname{End}_{\mathscr{H}_{\ell,n}^{(0)}}(P_0)) = \operatorname{End}_{\mathscr{H}_{\ell,n}^{(0)}}(P_0),$$

as required.

To show that $\operatorname{End}_{\mathscr{H}_{\ell,n}^{(0)}}(P_0)$ of $\mathscr{H}_{\ell,n}^{(0)}$ is commutative, it suffices to show that $\tilde{F}_{1,1}\mathscr{H}_{\ell,n}^{(0)}\tilde{F}_{1,1}$ is commutative. Furthermore, it is enough to show that $b_{\boldsymbol{\mu}}b_{\boldsymbol{\nu}} = b_{\boldsymbol{\nu}}b_{\boldsymbol{\mu}}$ for any $\boldsymbol{\mu}, \boldsymbol{\nu} \in \mathscr{P}_0$.

By definition,

$$\begin{aligned} b_{\boldsymbol{\mu}}b_{\boldsymbol{\nu}} &= \psi_{w_0}y_{\boldsymbol{\mu}}\psi_{w_0}y_1^{n-1}y_2^{n-2}\dots y_{n-1}\psi_{w_0}y_{\boldsymbol{\nu}}\psi_{w_0}y_1^{n-1}y_2^{n-2}\dots y_{n-1} \\ &= (-1)^{n(n-1)/2}\psi_{w_0}(y_{\boldsymbol{\mu}}\psi_{w_0}y_{\boldsymbol{\nu}})\psi_{w_0}y_1^{n-1}y_2^{n-2}\dots y_{n-1}. \end{aligned}$$

We set

$$J_{1,1} := \sum_{j=1}^{n-1}\psi_j\mathscr{H}_{\ell,n}^{(0)} + \sum_{j=1}^{n-1}\mathscr{H}_{\ell,n}^{(0)}\psi_j.$$

Using the graded cellular basis $\{\psi_{w,u}^{\boldsymbol{\mu}} \mid \boldsymbol{\mu} \in \mathscr{P}_0\}$ of $\mathscr{H}_{\ell,n}^{(0)}$, we can write

$$y_{\boldsymbol{\mu}}\psi_{w_0}y_{\boldsymbol{\nu}} \equiv \sum_{\boldsymbol{\rho}\in\mathscr{P}_0} c_{\boldsymbol{\rho}}y_{\boldsymbol{\rho}} \pmod{J_{1,1}},$$

where $c_\alpha \in K$ for each $\alpha \in \mathscr{P}_0$. Applying the anti-involution "$*$" on both sides of the above equality, we get that

$$y_{\boldsymbol{\nu}}\psi_{w_0}y_{\boldsymbol{\mu}} \equiv \sum_{\boldsymbol{\rho}\in\mathscr{P}_0} c_{\boldsymbol{\rho}}y_{\boldsymbol{\rho}} \pmod{J_{1,1}}.$$

Now using Lemma 2.15 we can deduce that

$$b_{\boldsymbol{\mu}}b_{\boldsymbol{\nu}} = (-1)^{n(n-1)/2}\sum_{\boldsymbol{\rho}\in\mathscr{P}_0} c_{\boldsymbol{\rho}}\psi_{w_0}y_{\boldsymbol{\rho}}\psi_{w_0}y_1^{n-1}y_2^{n-2}\dots y_{n-1} = b_{\boldsymbol{\nu}}b_{\boldsymbol{\mu}},$$

as required. □

Definition 3.3. Let $\boldsymbol{\mu} \in \mathscr{P}_0$ with $\theta(\boldsymbol{\mu}) = (k_1, \dots, k_n)$, where $1 \le k_1 < k_2 < \cdots < k_n \le \ell$. Inside the quiver Hecke algebra $\mathscr{H}_n^{(0)}$, we define $z(\boldsymbol{\mu}) \in K[y_1, \dots, y_n]$ such that

$$y_1^{\ell-k_1} \dots y_n^{\ell-k_n} \psi_{w_0} = z(\boldsymbol{\mu}) + \sum_{r=1}^{n-1} \psi_r h_r,$$

where $h_r \in \mathscr{H}_n^{(0)}$ for each $1 \le r < n$. We define

$$z_{\boldsymbol{\mu}} := \pi(z(\boldsymbol{\mu})) \in \mathscr{H}_{\ell,n}^{(0)}.$$

It is clear that $z_{\boldsymbol{\mu}}$ is a homogeneous element with degree $2\ell n - n(n-1) - 2\sum_{i=1}^{n} k_i$.

Lemma 3.4. *Let $\boldsymbol{\mu} \in \mathscr{P}_0$. Then $z(\boldsymbol{\mu})$ is a symmetric polynomial in $y_1, \dots, y_n$. In particular, $z(\boldsymbol{\mu})$ lives inside the center of $\mathscr{H}_n^{(0)}$ and hence $z_{\boldsymbol{\mu}}$ lives inside the center of $\mathscr{H}_{\ell,n}^{(0)}$. Moreover, $z(\boldsymbol{\lambda}_{\max}) = (-1)^{n(n-1)/2}(y_1 \dots y_n)^{\ell-n}$ and $z(\boldsymbol{\lambda}_{\min}) = (-1)^{n(n-1)/2}$.*

Proof. It suffices to show that $z(\boldsymbol{\mu})$ is symmetric in y_r, y_{r+1} for each $1 \le r < n-1$. In fact, for any $1 \le r < n-1$ and $a, b \in \mathbb{N}$, if $a > b$ then

$$\begin{aligned} y_r^a y_{r+1}^b \psi_r &= y_r^{a-b} (y_r y_{r+1})^b \psi_r = y_r^{a-b} \psi_r (y_r y_{r+1})^b \\ &\equiv -\Bigg(\sum_{k=0}^{a-b-1} y_r^k y_{r+1}^{a+b-1-k} \Bigg) (y_r y_{r+1})^b \Bigg(\mathrm{mod} \sum_{r=1}^{n-1} \psi_r \mathscr{H}_n^{(0)} \Bigg); \end{aligned}$$

if $a < b$, then

$$\begin{aligned} y_r^a y_{r+1}^b \psi_r &= y_{r+1}^{b-a} (y_r y_{r+1})^a \psi_r = y_{r+1}^{b-a} \psi_r (y_r y_{r+1})^a \\ &\equiv \Bigg(\sum_{k=0}^{b-a-1} y_r^k y_{r+1}^{b-a-1-k} \Bigg) (y_r y_{r+1})^a \Bigg(\mathrm{mod} \sum_{r=1}^{n-1} \psi_r \mathscr{H}_n^{(0)} \Bigg); \end{aligned}$$

if $a = b$, then $y_r^a y_{r+1}^b \psi_r = (y_r y_{r+1})^a \psi_r = \psi_r (y_r y_{r+1})^a \in \sum_{r=1}^{n-1} \psi_r \mathscr{H}_n^{(0)}$. This implies that for any monomial $y_1^{c_1} \dots y_n^{c_n} \in \mathscr{H}_n^{(0)}$,

$$y_1^{c_1} \dots y_n^{c_n} \psi_r \equiv f_r(y_1, \dots, y_n) \Bigg(\mathrm{mod} \sum_{r=1}^{n-1} \psi_r \mathscr{H}_n^{(0)} \Bigg),$$

where $f_r(y_1, \dots, y_n) \in K[y_1, \dots, y_n]$ is symmetric in y_r, y_{r+1}.

Since for each $1 \le r < n$, w_0 has a reduced expression which ends with s_r and the element $z(\boldsymbol{\mu})$ is uniquely determined by $\boldsymbol{\mu}$ by Lemma 2.1, it follows that $z(\boldsymbol{\mu})$ is symmetric in y_r, y_{r+1} for any $1 \le r < n-1$. Hence $z(\boldsymbol{\mu})$ is symmetric in $y_1, \dots, y_n$. This completes the proof of the first part of the lemma. The second part of the lemma follows from Lemma 2.12 and direct calculation. □

Lemma 3.5. (1) *For each* $\boldsymbol{\mu} \in \mathscr{P}_0$, *we have that*

$$\psi_{w_0} y_{\boldsymbol{\mu}} \psi_{w_0} y_1^{n-1} \dots y_{n-1} = \psi_{w_0} y_1^{n-1} \dots y_{n-1} z_{\boldsymbol{\mu}}.$$

In particular,

$$\psi_{w_0} y_{\boldsymbol{\mu}} \equiv (-1)^{n(n-1)/2} \psi_{w_0} y_1^{n-1} y_2^{n-2} \dots y_{n-1} z_{\boldsymbol{\mu}} \pmod{(\mathscr{H}_{\ell,n}^{(0)})^{>\boldsymbol{\mu}}}.$$

(2) *As a left* Z*-module*, $P_0 \cong Z^{\oplus n!}$. *In particular*, P_0 *is a free* Z*-module of rank* $n!$.

Proof. First, since $\mathscr{H}_{\ell,n}^{(0)} \cong P_0^{\oplus n!}$, it follows that the center Z must act faithfully on P_0. In other words, the left multiplication defines an injective homomorphism $\iota : Z \hookrightarrow \mathrm{End}_{\mathscr{H}_{\ell,n}^{(0)}}(P_0)$. Comparing the dimensions of both sides, we can deduce that ι is an isomorphism. On the other hand, by Lemma 3.2,

$$0 \neq b_{\boldsymbol{\mu}} \in \tilde{F}_{1,1} \mathscr{H}_{\ell,n}^{(0)} \tilde{F}_{1,1} \cong \mathrm{End}_{\mathscr{H}_{\ell,n}^{(0)}}(P_0).$$

It follows that there exists a unique nonzero homogeneous element $z'_{\boldsymbol{\mu}}$ with degree $2(\ell - k_1 + \cdots + \ell - k_n) - (n-1)n$ such that

$$(3.6)\qquad \begin{aligned} \psi_{w_0} y_{\boldsymbol{\mu}} \psi_{w_0} y_1^{n-1} \dots y_{n-1} &= z'_{\boldsymbol{\mu}} \psi_{w_0} y_1^{n-1} \dots y_{n-1} \\ &= \psi_{w_0} z'_{\boldsymbol{\mu}} y_1^{n-1} \dots y_{n-1} = \psi_{w_0} y_1^{n-1} \dots y_{n-1} z'_{\boldsymbol{\mu}}. \end{aligned}$$

By Lemma 3.4 and Lemma 2.15, we can see that $z'_{\boldsymbol{\mu}} = z_{\boldsymbol{\mu}}$. In particular, $z_{\boldsymbol{\mu}} \neq 0$.

Since

$$\begin{aligned} \psi_{w_0} y_{\boldsymbol{\mu}} \psi_{w_0} y_1^{n-1} \dots y_{n-1} &\equiv (-1)^{n(n-1)/2} \psi_{w_0} y_{\boldsymbol{\mu}} \\ &\equiv (-1)^{n(n-1)/2} \psi_{w_0,1}^{\boldsymbol{\mu}} \pmod{(\mathscr{H}_{\ell,n}^{(0)})^{>\boldsymbol{\mu}}}, \end{aligned}$$

we see that $\psi_{w_0} y_{\boldsymbol{\mu}} \equiv (-1)^{n(n-1)/2} \psi_{w_0} y_1^{n-1} y_2^{n-2} \dots y_{n-1} z_{\boldsymbol{\mu}} \pmod{(\mathscr{H}_{\ell,n}^{(0)})^{>\boldsymbol{\mu}}}$. This proves (1).

Recall that $\tilde{F}_{1,1} = F'_{1,1} = (-1)^{n(n-1)/2} \psi_{w_0} y_1^{n-1} y_2^{n-2} \dots y_{n-1}$. It follows from (1) that for any $\boldsymbol{\mu} \in \mathscr{P}_0$ and $w \in \mathfrak{S}_n$,

$$\tilde{F}_{1,1} z_{\boldsymbol{\mu}} \psi_w \equiv \psi_{w_0,w}^{\boldsymbol{\mu}} \pmod{(\mathscr{H}_{\ell,n}^{(0)})^{>\boldsymbol{\mu}}}.$$

In particular, the elements in the set $\{\tilde{F}_{1,1} z_{\boldsymbol{\mu}} \psi_w \mid \boldsymbol{\mu} \in \mathscr{P}_0, w \in \mathfrak{S}_n\}$ must be K-linearly independent. Since it has the cardinality $\binom{\ell}{n} n!$, we can deduce that it is a K-basis of the right $\mathscr{H}_{\ell,n}^{(0)}$-module $P_0 \cong \tilde{F}_{1,1} \mathscr{H}_{\ell,n}^{(0)}$. Since P_0 is a faithful Z-module, it follows that for any $z \in Z$, $\tilde{F}_{1,1} z = 0$ if and only if $z = 0$. For each $w \in \mathfrak{S}_n$, the subspace spanned by the basis elements in $\{\tilde{F}_{1,1} z_{\boldsymbol{\mu}} \psi_w \mid \boldsymbol{\mu} \in \mathscr{P}_0\}$ is a Z-submodule of P_0 which is isomorphic to Z. This proves that P is a free Z-module with rank $n!$. □

Theorem 3.7. *The elements in the set* $\{z_{\boldsymbol{\mu}} \mid \boldsymbol{\mu} \in \mathscr{P}_0\}$ *form a* K*-basis of the center* $Z := Z(\mathscr{H}_{\ell,n}^{(0)})$ *of* $\mathscr{H}_{\ell,n}^{(0)}$*. In particular, the center of* $\mathscr{H}_{\ell,n}^{(0)}$ *is the set of symmetric polynomials in* $y_1, \dots, y_n$*.*

Proof. Since the elements in $\{b_{\boldsymbol{\mu}} \mid \boldsymbol{\mu} \in \mathscr{P}_0\}$ are K-linearly independent, it follows that the elements in $\{z_{\boldsymbol{\mu}} \mid \boldsymbol{\mu} \in \mathscr{P}_0\}$ are K-linearly independent and hence form a K-basis of the center $Z := Z(\mathscr{H}_{\ell,n}^{(0)})$ by dimension consideration. By Lemma 3.4, each $z_{\boldsymbol{\mu}}$ is a symmetric polynomial in $y_1, \dots, y_n$, hence the center of $\mathscr{H}_{\ell,n}^{(0)}$ is the set of symmetric polynomials in $y_1, \dots, y_n$. □

The following proposition gives a generalization of Corollary 2.25. It can be regarded as a cyclotomic analogue of the results in [Lauda 2010, Proposition 3.5] and [Kleshchev et al. 2013, Theorem 4.5].

Proposition 3.8. *Let* $\{E_{i,j} \mid 1 \le i, j \le n!\}$ *be the matrix units of the full matrix algebra* $M_{n!\times n!}(K)$*. Then the map*

$$E_{i,j} \otimes z \mapsto \tilde{F}_{w_i,w_j} z, \quad \forall\, 1 \le i, j \le n!, z \in Z,$$

extends linearly to a well-defined K*-algebra isomorphism* η *from* $M_{n!\times n!}(K) \otimes_K Z$ *onto* $\mathscr{H}_{\ell,n}^{(0)}$*. In particular,* $\mathscr{H}_{\ell,n}^{(0)} \cong M_{n!\times n!}(Z)$*.*

Proof. In view of Theorem 2.31, it is clear that η is a well-defined K-algebra homomorphism. By Lemma 3.2, it suffices to show that η is an injective map.

Suppose that $\eta(x) = 0$, where $x = \sum_{1\le i,j\le n!} E_{i,j} z_{i,j}$, where $z_{i,j} \in Z$ for each pair (i, j). Then

$$\sum_{1\le i,j\le n!} \tilde{F}_{w_i,w_j} z_{i,j} = \eta(x) = 0.$$

For any pair (i, j) with $1 \le i, j \le n!$, left multiplying with $\tilde{F}_{w_j,w_i}$ and right multiplying with $\tilde{F}_{w_j,w_j}$ we get (by Theorem 2.31) that

$$\begin{aligned}\tilde{F}_{w_j,w_j} z_{i,j} &= \sum_{1\le k,l\le n!} (\tilde{F}_{w_j,w_i} \tilde{F}_{w_k,w_l} \tilde{F}_{w_j,w_j}) z_{k,l} \\ &= \tilde{F}_{w_j,w_i} \Bigg(\sum_{1\le k,l\le n!} \tilde{F}_{w_k,w_l} z_{k,l} \Bigg) \tilde{F}_{w_i,w_j} = 0.\end{aligned}$$

Since $\tilde{F}_{w_j,w_j} \mathscr{H}_{\ell,n}^{(0)} \cong P_0$ is ungraded right $\mathscr{H}_{\ell,n}^{(0)}$-module and Z acts faithfully on P_0, it follows that $z_{i,j} = 0$. This proves that $x = 0$ and hence η is injective. Finally, comparing the dimensions of both sides, we see that η is an isomorphism. □

4. A homogeneous symmetrizing form on $\mathscr{H}_{\ell,n}^{(0)}$

By the work of Shan, Varagnolo and Vasserot [Shan et al. 2017], each cyclotomic quiver Hecke algebra can be endowed with a homogeneous symmetrizing form which makes it into a graded symmetric algebra (see Remark 4.7 and [Hu and

Mathas 2010, §6.3] for the type A case). In particular, the nilHecke algebra $\mathscr{H}_{\ell,n}^{(0)}$ is a graded symmetric algebra. However, the SVV symmetrizing form $\mathrm{Tr}^{\mathrm{SVV}}$ is defined in an inductive manner which relies on some deep results about certain decompositions of the cyclotomic quiver Hecke algebras which come from the biadjointness of the i-induction functors and i-restriction functors in the work of Kang and Kashiwara [2012] and of Kashiwara [2012]. It is rather difficult to compute the explicit value of the form $\mathrm{Tr}^{\mathrm{SVV}}$ on any specified homogeneous element in the cyclotomic quiver Hecke algebra because its inductive definition involves some mysterious correspondence (i.e., $z \mapsto \tilde{z}$, $\ell \mapsto \tilde{\pi}_\ell$ in [Shan et al. 2017, Theorem 3.8]) whose explicit descriptions are not available. In this section, we shall introduce a new homogeneous symmetrizing form Tr such that the value of the form Tr on each graded cellular basis element of $\mathscr{H}_{\ell,n}^{(0)}$ is explicitly given. We will prove in the next section that this form Tr actually coincides with Shan–Varagnolo–Vasserot's symmetrizing form $\mathrm{Tr}^{\mathrm{SVV}}$ on $\mathscr{H}_{\ell,n}^{(0)}$.

The following result seems to be well-known. We add a proof as we can not find a suitable reference.

Lemma 4.1. *Let A, B be two finite dimensional* (*ungraded*) *K-algebras. Suppose that B is Morita equivalent to A. Then there exists a K-linear map $\rho : A^* \to B^*$ such that for any symmetrizing form $\tau \in A^*$ on A, $\rho(\tau) \in B^*$ is a symmetrizing form on B. In particular*, *if A is a symmetric algebra over K*, *then B is a symmetric algebra over K too.*

Proof. By assumption, $B^{\mathrm{op}} \cong \mathrm{End}_A(P)$ for a finite dimensional (ungraded) projective left A-module P. Moreover, there exists a natural number k such that $A^{\oplus k} \cong P \oplus P'$ as left A-modules. Let e be the idempotent of $M_{k\times k}(A)$ which corresponds to the map $A^{\oplus k} \overset{\mathrm{pr}}{\twoheadrightarrow} P \overset{\iota}{\hookrightarrow} A^{\oplus k}$. Then $B^{\mathrm{op}} \cong \mathrm{End}_A(P) \cong eM_{k\times k}(A)e$.

We define $\rho_0 : A^* \to (M_{k\times k}(A))^*$ as follows: for any $f \in A^*$ and $(a_{i,j})_{k\times k} \in M_{k\times k}(A)$,

$$\rho_0(f)((a_{i,j})_{k\times k}) := f\left(\sum_{i=1}^{k} a_{ii}\right).$$

We define $\mathrm{res} : (M_{k\times k}(A))^* \to (eM_{k\times k}(A)e)^*$ as follows: for any $f \in (M_{k\times k}(A))^*$ and $(a_{i,j})_{k\times k} \in M_{k\times k}(A)$,

$$\mathrm{res}(f)(e(a_{i,j})_{k\times k}e) := f(e(a_{i,j})_{k\times k}e).$$

It is easy to check that $\rho := \mathrm{res} \circ \rho_0$ has the property that for any symmetrizing form $\tau \in A^*$ on A, $\rho(\tau) \in B^*$ is a symmetrizing form on $\mathrm{End}_A(P) \cong eM_{k\times k}(A)e \cong B^{\mathrm{op}}$. It is clear that $\rho(\tau)$ is a symmetrizing form on B too. $\square$

The following lemma is clear.

Lemma 4.2. *Let $A = \oplus_{k=0}^{m} A_k$ be a finite dimensional positively $\mathbb{Z}$-graded K-algebra. Let τ be a (not necessarily homogeneous) symmetrizing form on A. We define $\tilde{\tau} : A^* \to K$ as follows: for any homogeneous element $y \in A$,*

$$\tilde{\tau}(y) := \begin{cases} \tau(x) & \text{if } \deg x = m, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\tilde{\tau}$ can be linearly extended to a well-defined homogeneous symmetrizing form on A.

The following definition comes from [Shan et al. 2017, 3.1.5].

Definition 4.3. We define

$$d_\Lambda := 2\ell n - 2n^2.$$

Recall that by Theorem 3.7, the center Z is a positively $\mathbb{Z}$-graded K-algebra with each homogeneous component being one dimensional. In particular, $\deg z \le d_\Lambda$ for all $z \in Z$, and $\deg z_{\boldsymbol{\lambda}_{\max}} = d_\Lambda$.

Lemma 4.4. *The center Z can be endowed with a homogeneous symmetrizing form of degree $-d_\Lambda$ as follows: for any homogeneous element $z \in Z$,*

$$\operatorname{tr}(z) := \begin{cases} 1 & \text{if } z = z_{\boldsymbol{\lambda}_{\max}}, \\ 0 & \text{if } \deg z < d_\Lambda. \end{cases}$$

In particular, Z is a graded symmetric algebra over K.

Proof. By Lemma 3.2, we know that Z is Morita equivalent to $\mathscr{H}_{\ell,n}^{(0)}$. Since $\mathscr{H}_{\ell,n}^{(0)}$ is a symmetric algebra by [Shan et al. 2017], we can deduce from Lemma 4.1 and Lemma 4.2 that Z is a graded symmetric algebra too.

On the other hand, by Lemma 3.2 and Corollary 2.19, we know that the center Z is a positively graded K-algebra with each homogeneous component being one dimensional. Therefore, we are in a position to apply [Hu and Lam 2017, Proposition 3.9] or Lemma 4.1 and Lemma 4.2 to show that tr is a well-defined homogeneous symmetrizing form on Z. □

Since tr is a homogeneous symmetrizing form on Z, for each nonzero homogeneous element $0 \neq z \in Z$, there exists a homogeneous element $\hat{z} \in Z$ with degree $d_\Lambda - \deg z$ such that $\operatorname{tr}(z\hat{z}) \neq 0$. This motivates the following definition.

Definition 4.5. For each $\boldsymbol{\lambda} \in \mathscr{P}_0$, we fix a nonzero homogeneous element $\hat{z}_{\boldsymbol{\lambda}} \in Z$ with degree $d_\Lambda - \deg z_{\boldsymbol{\lambda}}$ such that $\operatorname{tr}(z_{\boldsymbol{\lambda}}\hat{z}_{\boldsymbol{\lambda}}) \neq 0$.

Now we are using Proposition 3.8 and Lemma 4.4 to define a homogeneous symmetrizing form $\widehat{\operatorname{Tr}}$ on $\mathscr{H}_{\ell,n}^{(0)}$ as follows: for any $1 \le i, j \le n!$ and any homogeneous element $z \in Z$,

$$\widehat{\operatorname{Tr}}(\tilde{F}_{w_i,w_j} z) := \begin{cases} c & \text{if } i = j \text{ and } z = c z_{\boldsymbol{\lambda}_{\max}} \text{ for some } c \in K, \\ 0 & \text{if } i \neq j \text{ or } \deg z < d_\Lambda. \end{cases}$$

Lemma 4.6. *The map* $\widehat{\mathrm{Tr}}$ *extends linearly to a well-defined homogeneous symmetrizing form of degree* $-d_\Lambda$ *on* $\mathscr{H}_{\ell,n}^{(0)}$.

Proof. This follows directly from Lemma 4.4 and Proposition 3.8. □

Remark 4.7. Shan, Varagnolo, and Vasserot [Shan et al. 2017] show that each cyclotomic quiver Hecke algebra $\mathscr{R}_\beta^\Lambda$ can be endowed with a homogeneous symmetrizing form $\mathrm{Tr}^{\mathrm{SVV}}$ of degree $d_{\Lambda,\beta}$ which makes it into a graded symmetric algebra, where

$$\beta \in Q_n^+, \quad \Lambda \in P^+, \quad d_{\Lambda,\beta} := 2(\Lambda, \beta) - (\beta, \beta).$$

In the type A case we consider the cyclic quiver or linear quiver with vertices labeled by $\mathbb{Z}/e\mathbb{Z}$, where $e \neq 1$ is a nonnegative integer. In this case, $\mathscr{R}_\beta^\Lambda$ can be identified with the block of the cyclotomic Hecke algebra of type A which corresponds to β by Brundan–Kleshchev's isomorphism [Brundan and Kleshchev 2009a] when the ground field K contains a primitive e-th root of unity or e is equal to the characteristic of the ground field K. There is another homogeneous symmetrizing form $\mathrm{Tr}^{\mathrm{HM}}$ which can be defined (see [Hu and Mathas 2010, §6.3]) as follows: let τ be the ungraded symmetrizing form on $\mathscr{R}_\beta^\Lambda$ defined in [Malle and Mathas 1998] (nondegenerate case) and [Brundan and Kleshchev 2008] (degenerate case). Following [Hu and Mathas 2010, Definition 6.15], for any homogeneous element $x \in \mathscr{R}_\beta^\Lambda$, we define

$$\mathrm{Tr}^{\mathrm{HM}}(x) := \begin{cases} \tau(x) & \text{if } \deg(x) = d_{\Lambda,\beta}, \\ 0 & \text{otherwise.} \end{cases}$$

By the proof of [Hu and Mathas 2010, Theorem 6.17], $\mathrm{Tr}^{\mathrm{HM}}$ is a homogenous symmetrizing form on $\mathscr{R}_\beta^\Lambda$ of degree $-d_{\Lambda,\beta}$. The associated homogenous bilinear form $\langle -,- \rangle$ on $\mathscr{R}_\beta^\Lambda$ of degree $-d_{\Lambda,\beta}$ can be defined as follows: $\langle x, y\rangle := \mathrm{Tr}^{\mathrm{HM}}(xy)$. We take this chance to remark that the bilinear form $\langle -,- \rangle_\beta$ in the paragraph above [Hu and Mathas 2010, Theorem 6.17] should be replaced with the bilinear form $\langle -,- \rangle$ we defined here.

Conjecture 4.8. *The two symmetrizing forms* $\mathrm{Tr}^{\mathrm{SVV}}$ *and* $\mathrm{Tr}^{\mathrm{HM}}$ *on* $\mathscr{R}_\beta^\Lambda$ *differ by a nonzero scalar in* K.

Definition 4.9. For each $\boldsymbol{\mu} \in \mathscr{P}_0$ and $z_1, z_2 \in \mathfrak{S}_n$, we define

$$\phi_{z_1,z_2}^{\boldsymbol{\mu}} := \psi_{z_1}^* y_1^{n-1} y_2^{n-2} \dots y_{n-1} \psi_{w_0} y_{\boldsymbol{\mu}} \psi_{w_0} y_1^{n-1} y_2^{n-2} \dots y_{n-1} \psi_{z_2}.$$

Lemma 4.10. (1) *For each* $\boldsymbol{\mu} \in \mathscr{P}_0$ *and* $z_1, z_2 \in \mathfrak{S}_n$, *we have*

$$\phi_{w_0 z_1,z_2}^{\boldsymbol{\mu}} = F'_{z_1,z_2} z_{\boldsymbol{\mu}} = \psi_{w_0 z_1}^* y_1^{n-1} y_2^{n-2} \dots y_{n-1} \psi_{w_0} y_1^{n-1} y_2^{n-2} \dots y_{n-1} z_{\boldsymbol{\mu}} \psi_{z_2}$$

and

$$\phi_{z_1,z_2}^{\boldsymbol{\mu}} \equiv \psi_{z_1,z_2}^{\boldsymbol{\mu}} \pmod{(\mathscr{H}_{\ell,n}^{(0)})^{>\boldsymbol{\mu}}}.$$

(2) *The elements in the set* $\{\phi^{\boldsymbol{\mu}}_{z_1,z_2} \mid \boldsymbol{\mu} \in \mathscr{P}_0, z_1, z_2 \in \mathfrak{S}_n\}$ *form a homogeneous* K*-basis of* $\mathscr{H}^{(0)}_{\ell,n}$.

Proof. The first part of (1) follows from Lemma 3.5, while the second part of (1) follows from Lemma 2.12. Finally, (2) follows from (1) and (2.7). □

We are going to define another homogeneous symmetrizing form "Tr" on $\mathscr{H}^{(0)}_{\ell,n}$. Let $\boldsymbol{\lambda} \in \mathscr{P}_0$ and $w, u \in \mathfrak{S}_n$. By the same argument used in the proof of Lemma 3.4, there is an element $z_{w,u}$ in the center $Z(\mathscr{H}^{(0)}_{\ell,n})$ of $\mathscr{H}^{(0)}_{\ell,n}$ such that

$$\psi_{w_0} y_1^{n-1} y_2^{n-2} \dots y_{n-1} \psi_u \psi_{w^{-1}w_0} y_1^{n-1} y_2^{n-2} \dots y_{n-1} \psi_{w_0} = \psi_{w_0} z_{w,u}.$$

If $\deg z_{\boldsymbol{\lambda}} + \deg z_{w,u} = d_\Lambda$, then we denote $c_{w,u} \in K$ the unique scalar which satisfies that $z_{w,u} z_{\boldsymbol{\lambda}} = c_{w,u} z_{\boldsymbol{\lambda}_{\max}}$. Note that $\deg z_{\boldsymbol{\lambda}} + \deg z_{w,u} = d_\Lambda$ if and only if $\deg \phi^{\boldsymbol{\lambda}}_{w_0 w,u} = d_\Lambda$.

Definition 4.11. For any $\boldsymbol{\mu} \in \mathscr{P}_0$ and $w, u \in \mathfrak{S}_n$, we define

$$\operatorname{Tr}(F'_{w,u} z_{\boldsymbol{\mu}}) = \operatorname{Tr}(\phi^{\boldsymbol{\mu}}_{w_0 w,u}) := \begin{cases} c_{w,u} & \text{if } \deg F'_{w,u} z_{\boldsymbol{\mu}} = d_\Lambda, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, if $w = u$ and $\boldsymbol{\mu} = \lambda_{\max}$ then $\operatorname{Tr}(\phi^{\boldsymbol{\mu}}_{w,u}) = 1$. Note that

$$\begin{aligned} 1 &= \operatorname{Tr}(\phi^{\boldsymbol{\lambda}_{\max}}_{w_0,1}) = \operatorname{Tr}(F'_{1,1} z_{\boldsymbol{\lambda}_{\max}}) \\ &= \operatorname{Tr}(\psi^*_{w_0} y_1^{n-1} y_2^{n-2} \dots y_{n-1} \psi_{w_0} y_1^{n-1} y_2^{n-2} \dots y_{n-1} z_{\boldsymbol{\lambda}_{\max}}) \\ &= (-1)^{n(n-1)/2} \operatorname{Tr}(\psi^*_{w_0} y_1^{n-1} y_2^{n-2} \dots y_{n-1} z_{\boldsymbol{\lambda}_{\max}}) \\ &= \operatorname{Tr}(\psi^*_{w_0} y_{\boldsymbol{\lambda}_{\max}}), \end{aligned}$$

which implies that

$$\operatorname{Tr}(\psi^*_{w_0} y_{\boldsymbol{\lambda}_{\max}}) = 1. \tag{4.12}$$

Proposition 4.13. *The map* Tr *can be linearly extended to a well-defined homogeneous symmetrizing form of degree* $-d_\Lambda$ *on* $\mathscr{H}^{(0)}_{\ell,n}$.

Proof. By construction, it is clear that the map Tr can be linearly extended to a well-defined homogeneous linear map of degree $-d_\Lambda$ on $\mathscr{H}^{(0)}_{\ell,n}$.

We want to show that $\widehat{\operatorname{Tr}} = \operatorname{Tr}$. Once this is proved, it is automatically proved that Tr is symmetric and nondegenerate. To this end, by Lemma 4.10, it suffices to show that $\widehat{\operatorname{Tr}}(F'_{z_1,z_2} z_{\boldsymbol{\mu}}) = \operatorname{Tr}(F'_{z_1,z_2} z_{\boldsymbol{\mu}})$ for any $\boldsymbol{\mu} \in \mathscr{P}_0$ and $z_1, z_2 \in \mathfrak{S}_n$.

Without loss of generality we can assume that $\deg(F'_{z_1,z_2}z_{\boldsymbol{\mu}}) = d_\Lambda$. Since $\widehat{\mathrm{Tr}}$ is a trace form and $z_{\boldsymbol{\mu}}$ is central, we have

$$\begin{aligned}
&\widehat{\mathrm{Tr}}(F'_{z_1,z_2}z_{\boldsymbol{\mu}})\\
&\quad= \widehat{\mathrm{Tr}}(F'_{z_1,z_1}F'_{z_1,z_2}z_{\boldsymbol{\mu}})\\
&\quad= \widehat{\mathrm{Tr}}\big(\psi^*_{w_0z_1}y_1^{n-1}y_2^{n-2}\dots y_{n-1}\psi_{w_0}y_1^{n-1}y_2^{n-2}\dots\\
&\qquad\qquad \times y_{n-1}\psi_{z_1}\psi^*_{w_0z_1}y_1^{n-1}y_2^{n-2}\dots y_{n-1}\psi_{w_0}y_1^{n-1}y_2^{n-2}\dots y_{n-1}\psi_{z_2}z_{\boldsymbol{\mu}}\big)\\
&\quad= \widehat{\mathrm{Tr}}\big(\psi^*_{w_0z_1}y_1^{n-1}y_2^{n-2}\dots y_{n-1}\psi_{w_0}y_1^{n-1}y_2^{n-2}\dots\\
&\qquad\qquad \times y_{n-1}\psi_{w_0}y_1^{n-1}y_2^{n-2}\dots y_{n-1}\psi_{w_0}y_1^{n-1}y_2^{n-2}\dots y_{n-1}\psi_{z_2}z_{\boldsymbol{\mu}}\big)\\
&\quad= \widehat{\mathrm{Tr}}\big(y_1^{n-1}y_2^{n-2}\dots y_{n-1}\psi_{w_0}y_1^{n-1}y_2^{n-2}\dots\\
&\qquad\qquad \times y_{n-1}\psi_{w_0}y_1^{n-1}y_2^{n-2}\dots y_{n-1}\psi_{z_2}\psi_{z_1^{-1}w_0}y_1^{n-1}y_2^{n-2}\dots y_{n-1}\psi_{w_0}z_{\boldsymbol{\mu}}\big)\\
&\quad= \widehat{\mathrm{Tr}}(y_1^{n-1}y_2^{n-2}\dots y_{n-1}\psi_{w_0}y_1^{n-1}y_2^{n-2}\dots y_{n-1}\psi_{w_0}z_{z_1,z_2}z_{\boldsymbol{\mu}})\\
&\quad= (-1)^{n(n-1)/2}\widehat{\mathrm{Tr}}(y_1^{n-1}y_2^{n-2}\dots y_{n-1}\psi_{w_0}c_{z_1,z_2}z_{\boldsymbol{\lambda}_{\max}})\\
&\quad= (-1)^{n(n-1)/2}c_{z_1,z_2}\widehat{\mathrm{Tr}}(\psi_{w_0}y_1^{n-1}y_2^{n-2}\dots y_{n-1}z_{\boldsymbol{\lambda}_{\max}})\\
&\quad= c_{z_1,z_2}\widehat{\mathrm{Tr}}(\tilde{F}_{1,1}z_{\boldsymbol{\lambda}_{\max}}) = c_{z_1,z_2}\\
&\quad= \mathrm{Tr}(F'_{z_1,z_2}z_{\boldsymbol{\mu}}).
\end{aligned}$$

This completes the proof of $\widehat{\mathrm{Tr}} = \mathrm{Tr}$. In particular, this implies that Tr is symmetric and nondegenerate. That says, Tr can be linearly extended to a well-defined homogeneous symmetrizing form of degree $-d_\Lambda$ on $\mathscr{H}^{(0)}_{\ell,n}$. □

Proposition 4.14. $$\widehat{\mathrm{Tr}} = \mathrm{Tr}\,.$$

Proof. This follows from the proof of Proposition 4.13. □

5. Comparing Tr with the Shan–Varagnolo–Vasserot symmetrizing form $\mathrm{Tr}^{\mathrm{SVV}}$

In this section, we compare the symmetrizing form Tr with the Shan–Varagnolo–Vasserot symmetrizing form $\mathrm{Tr}^{\mathrm{SVV}}$ introduced in [Shan et al. 2017] and show that they are actually the same.

Let A, B be two K-algebras and $i : B \to A$ is a K-algebra homomorphism. Let $A^B := \{x \in A \mid xb = bx, \forall\, b \in B\}$ be the centralizer of B in A. For any $f \in A^B$, we set

$$\mu_f : A \otimes_B A \to A, \qquad a \otimes a' \mapsto afa'.$$

Recall that $\mathscr{H}^{(0)}_{\ell,n} = \mathscr{R}^{\ell\Lambda_0}_{n\alpha_0}$. In the notations of [Shan et al. 2017, §3.1.4], we set

$$\lambda_0 := \langle \ell\Lambda_0 - (n-1)\alpha_0, \alpha_0^\vee\rangle = \ell - 2(n-1). \tag{5.1}$$

We first recall the definition of $\mathrm{Tr}^{\mathrm{SVV}}$ in the case of nilHecke algebra $\mathscr{R}^{\ell\Lambda_0}_{n\alpha_0}$.

Definition 5.2 [Kang and Kashiwara 2012; Shan et al. 2017, Theorem 3.6, (6), (8)]. If $\lambda_0 \geq 0$ then for any $z \in \mathscr{R}^{\ell\Lambda_0}_{n\alpha_0}$ there are unique elements $p_k(z) \in \mathscr{R}^{\ell\Lambda_0}_{(n-1)\alpha_0}$ and $\pi(z) \in \mathscr{R}^{\ell\Lambda_0}_{(n-1)\alpha_0} \otimes_{R^{\ell\Lambda_0}_{(n-2)\alpha_0}} \mathscr{R}^{\ell\Lambda_0}_{(n-1)\alpha_0}$ such that

$$z = \mu_{\psi_{n-1}}(\pi(z)) + \sum_{k=0}^{\lambda_0-1} p_k(z) y_n^k,$$

where the above summation is understood as 0 when $\lambda_0 = 0$.

If $\lambda_0 \leq 0$ then for any $z \in \mathscr{R}^{\ell\Lambda_0}_{n\alpha_0}$, there is a unique element $\tilde{z} \in \mathscr{R}^{\ell\Lambda_0}_{(n-1)\alpha_0} \otimes_{\mathscr{R}^{\ell\Lambda_0}_{(n-2)\alpha_0}} \mathscr{R}^{\ell\Lambda_0}_{(n-1)\alpha_0}$ such that

$$\mu_{\psi_{n-1}}(\tilde{z}) = z \quad \text{and} \quad \mu_{y^k_{n-1}}(\tilde{z}) = 0, \forall\, k \in \{0, 1, \dots, -\lambda_0 - 1\},$$

where the range of k is understood as $\varnothing$ when $\lambda_0 = 0$.

Definition 5.3 [Shan et al. 2017, Theorem 3.8]. For each $n \in \mathbb{N}$, we define $\hat{\varepsilon}_n : \mathscr{R}^{\ell\Lambda_0}_{n\alpha_0} \to \mathscr{R}^{\ell\Lambda_0}_{(n-1)\alpha_0}$ as follows: for any $z \in \mathscr{R}^{\ell\Lambda_0}_{n\alpha_0}$, if $\lambda_0 := \ell - 2(n-1) > 0$ then $\hat{\varepsilon}_n(z) := p_{\ell-2(n-1)-1}(z)$; if $\lambda_0 := \ell - 2(n-1) \leq 0$ then $\hat{\varepsilon}_n(z) := \mu_{y_{n-1}^{-\ell+2(n-1)}}(\tilde{z})$.

Definition 5.4 [Shan et al. 2017, A.3.]. For any $z \in \mathscr{R}^{\ell\Lambda_0}_{n\alpha_0}$,

$$\mathrm{Tr}^{\mathrm{SVV}}(z) := \hat{\varepsilon}_1 \circ \hat{\varepsilon}_2 \circ \cdots \circ \hat{\varepsilon}_n : \mathscr{R}^{\ell\Lambda_0}_{n\alpha_0} \to \mathscr{R}^{\ell\Lambda_0}_{0\alpha_0} = K.$$

Definition 5.5. For each $n \in \mathbb{N}$, we define

$$Z_{0,n} := \psi_{w_{0,n}} y_1^{\ell-1} y_2^{\ell-2} \dots y_n^{\ell-n} \in \mathscr{H}^{(0)}_{\ell,n}.$$

We want to compute the value $\mathrm{Tr}^{\mathrm{SVV}}(Z_{0,n})$. According to Definition 5.2, we need to understand the value $p_{\ell-2(n-1)-1}(Z_{0,n})$ when $\ell > 2(n-1)$ and the value $\mu_{y_{n-1}^{-\ell+2(n-1)}}(\widetilde{Z}_{0,n})$ when $\ell \leq 2(n-1)$.

Lemma 5.6. *Suppose that $\lambda_0 := \ell - 2(n-1) \geq 0$. Then*

$$\begin{aligned}\pi(Z_{0,n}) = ((\psi_1 \dots \psi_{n-2}) y_{n-1}^{\ell-n}) \\ \otimes (\psi_1 \dots \psi_{n-3}\psi_{n-2}) \dots (\psi_1\psi_2)\psi_1 y_1^{\ell-1} y_2^{\ell-2} \dots y_{n-1}^{\ell-n+1} \\ \in \mathscr{R}^{\ell\Lambda_0}_{(n-1)\alpha_0} \otimes_{\mathscr{R}^{\ell\Lambda_0}_{(n-2)\alpha_0}} \mathscr{R}^{\ell\Lambda_0}_{(n-1)\alpha_0},\end{aligned}$$

and for any $k \in \{0, 1, \dots, \lambda_0 - 1\}$,

$$\begin{aligned} p_k(Z_{0,n}) \\ = (\psi_1 \dots \psi_{n-2})(\psi_1 \dots \psi_{n-3}) \dots (\psi_1\psi_2)\psi_1 y_1^{\ell-1} y_2^{\ell-2} \dots y_{n-2}^{\ell-n+2} y_{n-1}^{\ell-n+\lambda_0-k}.\end{aligned}$$

In particular, $p_{\lambda_0-1}(Z_{0,n}) = Z_{0,n-1}$.

Proof. By definition, we have

$$
\begin{aligned}
Z_{0,n} &= \psi_{w_{0,n}} y_1^{\ell-1} y_2^{\ell-2} \dots y_n^{\ell-n} \\
&= (\psi_1 \dots \psi_{n-2}\psi_{n-1})(\psi_1 \dots \psi_{n-3}\psi_{n-2}) \dots (\psi_1\psi_2)\psi_1 y_1^{\ell-1} y_2^{\ell-2} \dots y_n^{\ell-n} \\
&= (\psi_1 \dots \psi_{n-2})(\psi_{n-1} y_n^{\ell-n})\psi_{w_{0,n-1}} y_1^{\ell-1} y_2^{\ell-2} \dots y_{n-1}^{\ell-n+1} \\
&= (\psi_1 \dots \psi_{n-2})\Bigg(y_{n-1}^{\ell-n}\psi_{n-1} + \sum_{\substack{a_1+a_2=\ell-n-1\\ a_1,a_2\ge 0}} y_{n-1}^{a_1} y_n^{a_2}\Bigg) \\
&\qquad\qquad \times \psi_{w_{0,n-1}} y_1^{\ell-1} y_2^{\ell-2} \dots y_{n-1}^{\ell-n+1} \\
&= (\psi_1 \dots \psi_{n-2})(y_{n-1}^{\ell-n}\psi_{n-1})\psi_{w_{0,n-1}} y_1^{\ell-1} y_2^{\ell-2} \dots y_{n-1}^{\ell-n+1} \\
&\qquad + \sum_{\substack{a_1+a_2=\ell-n-1\\ a_1,a_2\ge 0}} (\psi_1 \dots \psi_{n-2} y_{n-1}^{a_1}\psi_{w_{0,n-1}} y_1^{\ell-1} y_2^{\ell-2} \dots y_{n-1}^{\ell-n+1} y_n^{a_2}) \\
&= (\psi_1 \dots \psi_{n-2})(y_{n-1}^{\ell-n}\psi_{n-1})\psi_{w_{0,n-1}} y_1^{\ell-1} y_2^{\ell-2} \dots y_{n-1}^{\ell-n+1} \\
&\quad + \sum_{\substack{a_1+a_2=\ell-n-1\\ a_1,a_2\ge 0}} \big(\psi_1 \dots \psi_{n-2} y_{n-1}^{a_1}(\psi_1 \dots \psi_{n-3}\psi_{n-2})(\psi_1 \dots \psi_{n-4}\psi_{n-3}) \dots \\
&\qquad\qquad \times (\psi_1\psi_2)\psi_1 y_1^{\ell-1} y_2^{\ell-2} \dots y_{n-1}^{\ell-n+1} y_n^{a_2}\big) \\
&= (\psi_1 \dots \psi_{n-2})(y_{n-1}^{\ell-n}\psi_{n-1})\psi_{w_{0,n-1}} y_1^{\ell-1} y_2^{\ell-2} \dots y_{n-1}^{\ell-n+1} \\
&\quad + \sum_{\substack{a_1+a_2=\ell-n-1\\ a_1,a_2\ge 0}} \big((\psi_1 \dots \psi_{n-2})(\psi_1 \dots \psi_{n-3}) \\
&\qquad\qquad \times (\psi_1 \dots \psi_{n-4}) \dots \psi_1 y_{n-1}^{a_1}(\psi_{n-2}\psi_{n-3} \dots \psi_2\psi_1) \\
&\qquad\qquad \times y_1^{\ell-1} y_2^{\ell-2} \dots y_{n-1}^{\ell-n+1} y_n^{a_2}\big) \\
&= \mu_{\psi_{n-1}}\big((\psi_1 \dots \psi_{n-2} y_{n-1}^{\ell-n}) \otimes (\psi_{w_{0,n-1}} y_1^{\ell-1} y_2^{\ell-2} \dots y_{n-1}^{\ell-n+1})\big) \\
&\qquad + \sum_{\substack{a_1+a_2=\ell-n-1\\ a_1,a_2\ge 0}} \psi_{w_{0,n-1}}(y_{n-1}^{a_1}\psi_{n-2} \dots \psi_2\psi_1) y_1^{\ell-1} y_2^{\ell-2} \dots y_{n-1}^{\ell-n+1} y_n^{a_2}.
\end{aligned}
$$

Using the uniqueness in Definition 5.2, we see that to prove the lemma, it suffices to show that

$$
\begin{aligned}
&\sum_{\substack{a_1+a_2=\ell-n-1\\ a_1,a_2\ge 0}} \psi_{w_{0,n-1}}(y_{n-1}^{a_1}\psi_{n-2} \dots \psi_2\psi_1) y_1^{\ell-1} y_2^{\ell-2} \dots y_{n-1}^{\ell-n+1} y_n^{a_2} \\
&\qquad = \sum_{k=0}^{\lambda_0-1} \psi_{w_{0,n-1}} y_1^{\ell-1} y_2^{\ell-2} \dots y_{n-2}^{\ell-n+2} y_{n-1}^{\ell-n+\lambda_0-k} y_n^{k}.
\end{aligned}
$$

In fact,

$$\sum_{\substack{a_1+a_2=\ell-n-1\\ a_1,a_2\geq 0}} \psi_{w_{0,n-1}}(y_{n-1}^{a_1}\psi_{n-2}\dots\psi_2\psi_1)y_1^{\ell-1}y_2^{\ell-2}\dots y_{n-1}^{\ell-n+1}y_n^{a_2}$$

$$= \sum_{\substack{a_1+a_2=\ell-n-1\\ a_1,a_2\geq 0}} \psi_{w_{0,n-1}}(y_{n-1}^{a_1}\psi_{n-2}\dots\psi_2\psi_1)y_1^{\ell-1}y_2^{\ell-2}\dots y_{n-1}^{\ell-n+1}y_n^{a_2}$$

$$= \sum_{\substack{a_1+a_2=\ell-n-1\\ a_1\geq n-2, a_2\geq 0}} \psi_{w_{0,n-1}}(y_{n-1}^{a_1}\psi_{n-2}\dots\psi_2\psi_1)y_1^{\ell-1}y_2^{\ell-2}\dots y_{n-1}^{\ell-n+1}y_n^{a_2}$$

$$\begin{aligned}
&= \psi_{w_{0,n-1}}\psi_{n-2}\dots\psi_2\psi_1 y_1^{\ell-1}y_2^{\ell-2}\dots y_{n-2}^{\ell-n+2}y_{n-1}^{\ell-n+1}y_n^{\ell-2n+1}\\
&\qquad + \psi_{w_{0,n-1}}y_1^{\ell-1}y_2^{\ell-2}\dots y_{n-2}^{\ell-n+2}y_{n-1}^{\ell-n+2}y_n^{\ell-2n}\\
&\qquad + \psi_{w_{0,n-1}}y_1^{\ell-1}y_2^{\ell-2}\dots y_{n-2}^{\ell-n+2}y_{n-1}^{\ell-n+3}y_n^{\ell-2n-1}\\
&\qquad\qquad \vdots\\
&\qquad + \psi_{w_{0,n-1}}y_1^{\ell-1}y_2^{\ell-2}\dots y_{n-2}^{\ell-n+2}y_{n-1}^{2\ell-3n+1}y_n\\
&\qquad + \psi_{w_{0,n-1}}y_1^{\ell-1}y_2^{\ell-2}\dots y_{n-2}^{\ell-n+2}y_{n-1}^{2\ell-3n+2}\\
&= \sum_{k=0}^{\lambda_0-1} \psi_{w_{0,n-1}}y_1^{\ell-1}y_2^{\ell-2}\dots y_{n-2}^{\ell-n+2}y_{n-1}^{\ell-n+\lambda_0-k}y_n^{k},
\end{aligned}$$

where we have used the commutator relations for the ψ and y generators of $\mathscr{H}_{\ell,n}^{(0)}$ and the fact that

$$\psi_{w_{0,n-1}}\psi_r = 0 \quad \text{for any } 1 \leq r < n-1$$

in the second and the last equalities. This completes the proof of the lemma. □

Lemma 5.7. *Suppose that* $\lambda_0 := \ell - 2(n-1) \leq 0$. *Then*

$$\widetilde{Z}_{0,n} = ((\psi_1\psi_2\dots\psi_{n-2})y_{n-1}^{\ell-n}) \otimes ((\psi_1\dots\psi_{n-3}\psi_{n-2})\dots(\psi_1\psi_2)\psi_1 y_1^{\ell-1}y_2^{\ell-2}\dots y_{n-1}^{\ell-n+1}) \in \mathscr{R}^{\ell\Lambda_0}_{(n-1)\alpha_0} \otimes_{R^{\ell\Lambda_0}_{(n-2)\alpha_0}} \mathscr{R}^{\ell\Lambda_0}_{(n-1)\alpha_0}$$

and

$$\begin{aligned}
\mu_{y_{n-1}^{-\lambda_0}}(\widetilde{Z}_{0,n}) &= Z_{0,n-1}\\
&= (\psi_1\dots\psi_{n-2})(\psi_1\dots\psi_{n-3})\dots(\psi_1\psi_2)\psi_1 y_1^{\ell-1}y_2^{\ell-2}\dots y_{n-1}^{\ell-n+1}.
\end{aligned}$$

Proof. By definition, we have

$$\begin{aligned}
Z_{0,n} &= \psi_{w_{0,n}} y_1^{\ell-1} y_2^{\ell-2} \dots y_n^{\ell-n} \\
&= (\psi_1 \dots \psi_{n-2}\psi_{n-1})(\psi_1 \dots \psi_{n-3}\psi_{n-2}) \dots (\psi_1\psi_2)\psi_1 y_1^{\ell-1} y_2^{\ell-2} \dots y_n^{\ell-n} \\
&= (\psi_1 \dots \psi_{n-2})(\psi_{n-1} y_n^{\ell-n})(\psi_1 \dots \psi_{n-3}\psi_{n-2}) \dots \\
&\qquad\qquad \times (\psi_1\psi_2)\psi_1 y_1^{\ell-1} y_2^{\ell-2} \dots y_{n-1}^{\ell-n+1} \\
&= (\psi_1 \dots \psi_{n-2}) \Bigg(y_{n-1}^{\ell-n} \psi_{n-1} + \sum_{\substack{a_1+a_2=\ell-n-1 \\ a_1,a_2 \geq 0}} y_{n-1}^{a_1} y_n^{a_2} \Bigg) \\
&\qquad\qquad \times \psi_{w_{0,n-1}} y_1^{\ell-1} y_2^{\ell-2} \dots y_{n-1}^{\ell-n+1} \\
&= (\psi_1 \dots \psi_{n-2})(y_{n-1}^{\ell-n}\psi_{n-1})\psi_{w_{0,n-1}} y_1^{\ell-1} y_2^{\ell-2} \dots y_{n-1}^{\ell-n+1} \\
&\qquad + \sum_{\substack{a_1+a_2=\ell-n-1 \\ a_1,a_2 \geq 0}} \psi_1 \dots \psi_{n-2}(y_{n-1}^{a_1})\psi_{w_{0,n-1}} y_1^{\ell-1} y_2^{\ell-2} \dots y_{n-1}^{\ell-n+1} y_n^{a_2}.
\end{aligned}$$

We now claim that

$$\sum_{\substack{a_1+a_2=\ell-n-1 \\ a_1,a_2 \geq 0}} \psi_1 \dots \psi_{n-2}(y_{n-1}^{a_1})\psi_{w_{0,n-1}} y_1^{\ell-1} y_2^{\ell-2} \dots y_{n-1}^{\ell-n+1} y_n^{a_2} = 0. \tag{5.8}$$

In fact, we have

$$\begin{aligned}
&\sum_{\substack{a_1+a_2=\ell-n-1 \\ a_1,a_2 \geq 0}} \psi_1 \dots \psi_{n-2}(y_{n-1}^{a_1})\psi_{w_{0,n-1}} y_1^{\ell-1} y_2^{\ell-2} \dots y_{n-1}^{\ell-n+1} y_n^{a_2} \\
&\quad = \sum_{\substack{a_1+a_2=\ell-n-1 \\ a_1,a_2 \geq 0}} \psi_1 \dots \psi_{n-2}(y_{n-1}^{a_1})(\psi_1 \dots \psi_{n-3}\psi_{n-2})(\psi_1 \dots \psi_{n-4}\psi_{n-3}) \dots \\
&\qquad\qquad \times (\psi_1\psi_2)(\psi_1) y_1^{\ell-1} y_2^{\ell-2} \dots y_{n-1}^{\ell-n+1} y_n^{a_2} \\
&\quad = \sum_{\substack{a_1+a_2=\ell-n-1 \\ a_1,a_2 \geq 0}} \psi_{w_{0,n-1}}(y_{n-1}^{a_1}\psi_{n-2} \dots \psi_2\psi_1) y_1^{\ell-1} y_2^{\ell-2} \dots y_{n-1}^{\ell-n+1} y_n^{a_2} \\
&\quad = \sum_{\substack{a_1+a_2=\ell-n-1 \\ a_1>0,a_2 \geq 0}} \psi_{w_{0,n-1}}(y_{n-1}^{a_1}\psi_{n-2} \dots \psi_2\psi_1) y_1^{\ell-1} y_2^{\ell-2} \dots y_{n-1}^{\ell-n+1} y_n^{a_2},
\end{aligned}$$

where the last equality follows from the fact that $\psi_{w_{0,n-1}}\psi_{n-2} = 0$. Now by assumption, $a_1 \leq \ell - n - 1 \leq 2(n-1) - n - 1 = n - 3 < n - 2$. It follows that $y_{n-1}^{a_1}\psi_{n-2} \dots \psi_2\psi_1$ is a sum of some elements which have a left factor of the form ψ_r for some $1 \leq r < n-1$. Therefore, using the fact that $\psi_{w_{0,n-1}}\psi_r = 0$ for any $1 \leq r < n-1$ again, we can deduce that the above sum is 0. This completes the proof of the claim (5.8).

By Definition 5.2, to complete the proof of the lemma, it remains to show that for any $0 \le k \le -\lambda_0 - 1$,

$$\mu_{y_{n-1}^k}\big((\psi_1\psi_2\dots\psi_{n-2}y_{n-1}^{\ell-n})\otimes(\psi_{w_{0,n-1}}y_1^{\ell-1}y_2^{\ell-2}\dots y_{n-1}^{\ell-n+1})\big)=0. \tag{5.9}$$

In fact, we have

$$\begin{aligned}
&\mu_{y_{n-1}^k}\big((\psi_1\psi_2\dots\psi_{n-2}y_{n-1}^{\ell-n})\otimes(\psi_{w_{0,n-1}}y_1^{\ell-1}y_2^{\ell-2}\dots y_{n-1}^{\ell-n+1})\big)\\
&\quad=\mu_{y_{n-1}^k}\big((\psi_1\dots\psi_{n-2}y_{n-1}^{\ell-n})\\
&\qquad\qquad\otimes(\psi_1\dots\psi_{n-3}\psi_{n-2})\dots(\psi_1\psi_2)\psi_1y_1^{\ell-1}y_2^{\ell-2}\dots y_{n-1}^{\ell-n+1}\big)\\
&\quad=(\psi_1\dots\psi_{n-2})(y_{n-1}^{\ell-n+k})(\psi_1\dots\psi_{n-3}\psi_{n-2})(\psi_1\dots\psi_{n-4}\psi_{n-3})\dots\\
&\qquad\qquad\times(\psi_1\psi_2)\psi_1y_1^{\ell-1}y_2^{\ell-2}\dots y_{n-1}^{\ell-n+1}\\
&\quad=(\psi_1\dots\psi_{n-2})(\psi_1\dots\psi_{n-3})\dots\\
&\qquad\qquad\times(\psi_1\psi_2)\psi_1(y_{n-1}^{\ell-n+k}\psi_{n-2}\psi_{n-3}\dots\psi_2\psi_1)y_1^{\ell-1}y_2^{\ell-2}\dots y_{n-1}^{\ell-n+1}\\
&\quad=\psi_{w_{0,n-1}}(y_{n-1}^{\ell-n+k}\psi_{n-2}\psi_{n-3}\dots\psi_1)y_1^{\ell-1}y_2^{\ell-2}\dots y_{n-1}^{\ell-n+1}=0,
\end{aligned}$$

where the last equality follows from the fact that $\psi_{w_{0,n-1}}\psi_r=0$ for any $1\le r<n-1$ and the assumption that

$$\ell-n+k\le\ell-n-\lambda_0-1=\ell-n-(\ell-2(n-1))-1=n-3<n-2$$

so that $y_{n-1}^{\ell-n+k}\psi_{n-2}\psi_{n-3}\dots\psi_1$ is a sum of some elements which have a left factor of the form ψ_r for some $1\le r<n-1$. This completes the proof of (5.9) and hence the proof of the lemma. □

Corollary 5.10. $$\mathrm{Tr}^{\mathrm{SVV}}(Z_{0,n})=1.$$

Proof. This follows from Definition 5.3, Definition 5.4, Lemma 5.6, Lemma 5.7, and an induction on n. □

Theorem 5.11. *The two symmetrizing forms* $\mathrm{Tr}^{\mathrm{SVV}}$ *and* Tr *on the cyclotomic nil-Hecke algebra* $\mathscr{H}_{\ell,n}^{(0)}$ *coincide with each other.*

Proof. Let $1\le i,\ j\le n!$, and $z\in Z$. Suppose that $i\ne j$. Then as $\mathrm{Tr}^{\mathrm{SVV}}$ is a symmetrizing form and z is central, we have

$$\begin{aligned}
\mathrm{Tr}^{\mathrm{SVV}}(\tilde F_{w_i,w_j}z)&=\mathrm{Tr}^{\mathrm{SVV}}(\tilde F_{w_i,w_i}\tilde F_{w_i,w_j}z)=\mathrm{Tr}^{\mathrm{SVV}}(\tilde F_{w_i,w_j}z\tilde F_{w_i,w_i})\\
&=\mathrm{Tr}^{\mathrm{SVV}}(\tilde F_{w_i,w_j}\tilde F_{w_i,w_i}z)=\mathrm{Tr}^{\mathrm{SVV}}(0z)=0.
\end{aligned}$$

It remains to consider the case when $i=j$.

If $\deg z<d_\Lambda$, then as $\mathrm{Tr}^{\mathrm{SVV}}$ is homogeneous of degree $-d_\Lambda$ and $\deg\tilde F_{w_i,w_i}=0$, we have $\mathrm{Tr}^{\mathrm{SVV}}(\tilde F_{w_i,w_i}z)=0$. Therefore, without loss of generality, we can assume that $z=z_{\boldsymbol{\lambda}_{\max}}$. Our purpose is to compare $\mathrm{Tr}^{\mathrm{SVV}}(\tilde F_{w_i,w_i}z_{\boldsymbol{\lambda}_{\max}})$ and $\mathrm{Tr}(\tilde F_{w_i,w_i}z_{\boldsymbol{\lambda}_{\max}})$.

Note that for any $\boldsymbol{\mu} \in \mathscr{P}_0$ with $\boldsymbol{\mu} > \boldsymbol{\lambda}_{\min}$, we have that

$$\deg(y_{\boldsymbol{\mu}} z_{\boldsymbol{\lambda}_{\max}}) > n(n-1) + 2n(\ell - n) = 2\ell n - n(n+1) = \deg(y_1^{\ell-1} y_2^{\ell-2} \dots y_n^{\ell-n}),$$

which implies that $y_{\boldsymbol{\mu}} z_{\boldsymbol{\lambda}_{\max}} = 0$ by Theorem 2.34. By (2.33) and Lemma 3.5, we have

$$\begin{aligned}
\mathrm{Tr}^{\mathrm{SVV}}(\tilde{F}_{w_i,w_i} z_{\boldsymbol{\lambda}\max}) &= (-1)^{n(n-1)/2}\,\mathrm{Tr}^{\mathrm{SVV}}(\psi^{\boldsymbol{\lambda}_{\min}}_{w_0 w_i, w_i} z_{\boldsymbol{\lambda}\max}) = \mathrm{Tr}^{\mathrm{SVV}}(\psi^{\boldsymbol{\lambda}\max}_{w_0 w_i, w_i}) \\
&= \mathrm{Tr}^{\mathrm{SVV}}(\psi_{w_i} \psi^*_{w_0 w_i} y_1^{\ell-1} y_2^{\ell-2} \dots y_n^{\ell-n}) \\
&= \mathrm{Tr}^{\mathrm{SVV}}(\psi_{w_0} y_1^{\ell-1} y_2^{\ell-2} \dots y_n^{\ell-n}) \\
&= \mathrm{Tr}^{\mathrm{SVV}}(Z_{0,n}) = 1, \quad \text{(by Corollary 5.10)} \\
\mathrm{Tr}(\tilde{F}_{w_i,w_i} z_{\boldsymbol{\lambda}\max}) &= (-1)^{n(n-1)/2}\,\mathrm{Tr}(\psi^{\boldsymbol{\lambda}_{\min}}_{w_0 w_i, w_i} z_{\boldsymbol{\lambda}\max}) = \mathrm{Tr}(\psi^{\boldsymbol{\lambda}\max}_{w_0 w_i, w_i}) \\
&= \mathrm{Tr}(\psi_{w_i} \psi^*_{w_0 w_i} y_1^{\ell-1} y_2^{\ell-2} \dots y_n^{\ell-n}) \\
&= \mathrm{Tr}(\psi_{w_0} y_1^{\ell-1} y_2^{\ell-2} \dots y_n^{\ell-n}) = 1. \quad \text{(by (4.12))}
\end{aligned}$$

This shows that $\mathrm{Tr}^{\mathrm{SVV}}(\tilde{F}_{w_i,w_i} z_{\boldsymbol{\lambda}\max}) = \mathrm{Tr}(\tilde{F}_{w_i,w_i} z_{\boldsymbol{\lambda}\max})$.

As a result, we have shown that $\mathrm{Tr}^{\mathrm{SVV}}(\tilde{F}_{w_i,w_j} z) = \mathrm{Tr}(\tilde{F}_{w_i,w_j} z)$ for any $1 \le i, j \le n!$, and $z \in Z$. It follows that $\mathrm{Tr}^{\mathrm{SVV}} = \mathrm{Tr}$, as required. □

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JUN HU
SCHOOL OF MATHEMATICS AND STATISTICS
BEIJING INSTITUTE OF TECHNOLOGY
BEIJING
CHINA
junhu404@bit.edu.cn

XINFENG LIANG
SCHOOL OF MATHEMATICS AND STATISTICS
BEIJING INSTITUTE OF TECHNOLOGY
BEIJING
CHINA
lxfrd@163.com

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TWO APPLICATIONS OF THE SCHWARZ LEMMA

BINGYUAN LIU

We exhibit two applications of Schwarz lemmas in several complex variables. The first application extends Fornæss and Stout's theorem on monotone unions of balls to monotone unions of ellipsoids. The second application extends Yang's theorem on bidiscs to the generalized bidisc defined by the author in his previous work. These applications reveal a connection between the geometry of domains and their curvatures. The proof contains a careful study of biholomorphisms, a detailed analysis on convergences, and a modified argument of Yang.

Introduction

The most striking and influential result in complex analysis of one variable is the Riemann mapping theorem. It asserts that all proper simply connected open subsets in $\mathbb{C}$ are biholomorphic onto the unit disc. Thus, it was hoped that a similar theorem could be proved in $\mathbb{C}^n$ for higher dimensions $n > 1$. In 1906, Poincaré showed the bidisc $\mathbb{D}^2 = \{(z, w) : |z| < 1 \text{ and } |w| < 1\}$ is not biholomorphic to the ball $\mathbb{B}^2 = \{(z, w) : |z|^2 + |w|^2 < 1\}$. This negated the expectation and motivated a new study on biholomorphism in several complex variables.

On the other hand, Fornæss and Stout [1977] showed that a Kobayashi hyperbolic manifold M is biholomorphic onto the unit ball $\mathbb{B}^m$, provided that M admits a monotone union of $\mathbb{B}^m$. Their theorem gives a version of the Riemann mapping theorem in high dimensions under some circumstances. In this paper, we follow this fashion and exhibit a theorem about monotone unions of ellipsoids $E_n := \{(z, w) \in \mathbb{C}^2 : |z|^2 + |w|^{2n} < 1\}$. More precisely, we obtain the following theorem.

Theorem 1. *Let M be a two-dimensional Hermitian manifold with a real bisectional curvature bounded from above by a negative number $-K$, and assume M is a monotone union of ellipsoids E_n for some $n \in \mathbb{Z}^+$. Then M is either biholomorphic onto E_n or onto the unit ball $\mathbb{B}^2$.*

This theorem generalizes Fornæss and Stout's theorem on monotone unions of balls to monotone unions of ellipsoids in dimension 2. We remark that Fornæss

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and Stout's original proof is hard to be adapted into our theorem. Among other difficulties, the situation that biholomorphisms converge to a constant map is hard to be excluded. This difficulty is easy to be resolved in Fornæss and Stout's theorem because of symmetries of balls. However, the shape of ellipsoids is more irregular than balls. Hence, in order to resolve this difficulty we make local estimates around accumulation points and use the estimates to reconstruct biholomorphisms. This new technique is exhibited in Section 2.

The readers are reminded that this theorem does not belong to a classical topic on automorphism groups. For the classical topics on automorphism groups, readers are referred to [Bedford and Pinchuk 1991; 1998; Greene and Krantz 1991; 1993; Wong 1977].

The other application of Schwarz lemmas in this article is about curvature bounds. In the 1970s, Yang [1976] showed that on polydiscs, there do not exist complete Kähler metrics with bounded holomorphic bisectional curvatures. Yang's discovery was recently generalized to product manifolds by Seshadri and Zheng [2008] and Seo [2012]. On the other hand, the author introduced a new type of domains called the generalized bidiscs in [Liu 2017]. It is known that some generalized bidiscs are biholomorphic to bounded domains in $\mathbb{C}^2$. The generalized bidiscs are defined to be $\mathbb{D} \rtimes_\theta \mathbb{H}^+ := \{(z, w) : z \in \mathbb{D} \text{ and } w \in e^{i\theta(z)}\mathbb{H}^+\}$. Here $\mathbb{D}$ denotes the unit disc, $\mathbb{H}^+$ denotes the upper half plane, θ denotes a continuous real function depending on z, and $e^{i\theta(z)}\mathbb{H}^+$ denotes the upper half plane rotated by the angle $\theta(z)$. The generalized bidiscs are, in general, not product manifolds. However, in this paper, we show they share similar geometric properties with bidiscs. That is, some generalized bidiscs do not admit complete Kähler metrics with bounded negative holomorphic bisectional curvatures. More precisely, we show:

Theorem 2. *Let $k \in (0, \pi)$ and $\theta(z) \in [0, k)$ for all $z \in \mathbb{D}$. Then there do not exist two numbers $d > c > 0$ and a complete Kähler metric on $\mathbb{D} \rtimes_\theta \mathbb{H}^+$ such that the holomorphic bisectional curvature is between $-d$ and $-c$.*

These results about curvature bounds are discussed in Section 3.

1. Preliminary and fundamental facts

Let $n \in \mathbb{Z}^+$. It is classical to define ellipsoids $E_n \subset \mathbb{C}^2$ by

$$E_n = \{(z, w) : |z|^2 + |w|^{2n} < 1\}.$$

Let M be a manifold with dimension m. In this paper, we say M is a monotone union of ellipsoids E_n via f_j if

(1) there exists a sequence of open subsets $M_j \subset M$ so that $M_j \Subset M_{j+1}$,

(2) each M_j is biholomorphic, by f_j, onto the ellipsoids E_n, and

(3) $M = \bigcup_j M_j$.

Remark 1.1. We sometimes omit "via f_j" and only say "M is a monotone union of ellipsoids E_n".

Remark 1.2. Similarly, one can define a monotone union of Ω for an arbitrary domain Ω.

We also recall some terminologies on Kähler manifolds. Let (M, J, h) be a Kähler manifold M of dimension m with a Kähler metric h and a complex structure J. The curvature tensor R on (M, J, h) is given by

$$R_{i\bar{j}k\bar{l}} = \frac{\partial^2 h_{i\bar{j}}}{\partial z_k\, \partial \bar{z}_l} - \sum_{\alpha,\beta=1}^{m} h^{\alpha\bar{\beta}} \frac{\partial h_{i\bar{\beta}}}{\partial z_k} \frac{\partial h_{\alpha\bar{j}}}{\partial \bar{z}_l}$$

in local coordinates $(z_1, \dots, z_n)$. The holomorphic bisectional curvature for $X \in T_pM$ at $p \in M$ is given by

$$B(X, Y) = -\frac{\sum_{i,j,k,l=1}^{m} R_{i\bar{j}k\bar{l}} X_i \overline{X}_j Y_k \overline{Y}_l}{\big(\sum_{i,j=1}^{m} h_{i\bar{j}} X_i \overline{X}_j\big)\big(\sum_{i,j=1}^{m} h_{i\bar{j}} Y_i \overline{Y}_j\big)},$$

where

$$X = \sum_{j=1}^{m} X_j \frac{\partial}{\partial z_j} + \sum_{j=1}^{m} \overline{X}_j \frac{\partial}{\partial \bar{z}_j}, \qquad Y = \sum_{j=1}^{m} Y_j \frac{\partial}{\partial z_j} + \sum_{j=1}^{m} \overline{Y}_j \frac{\partial}{\partial \bar{z}_j}.$$

We are going to remind readers with backgrounds on Schwarz lemmas and almost maximal principles.

Theorem 1.3 (the Schwarz lemma of [Yau 1978]). *Let $f : M \to N$ be a holomorphic mapping from a complete Kähler manifold (M, g) with its Ricci curvature bounded from below by a negative constant $-k$ into a Hermitian manifold (N, h) with its holomorphic bisectional curvature bounded from above by a negative constant $-K$. Then*

$$f^*h \le \frac{k}{K} g.$$

Theorem 1.4 (the almost maximal principles of [Yau 1978]; see also [Kim and Lee 2011]). *Let M be a complete Riemannian manifold M with the Ricci curvature bounded from below. Then for any C^2 smooth function $f : M \to \mathbb{R}$ that is bounded from above, there exists a sequence $\{p_k\}$ such that*

$$\lim_{k\to\infty} |\nabla T(p_k)| = 0, \qquad \limsup_{k\to\infty} \Delta T(p_k) \le 0, \qquad \lim_{k\to\infty} T(p_k) = \sup_M T.$$

Recently Yang and Zheng [2016] defined a new notion of curvatures on Hermitian manifolds called real bisectional curvature. We will briefly give the definition and discuss a version of Schwarz lemma in terms of real bisectional curvature as follows.

Definition 1.5 [Yang and Zheng 2016]. Let (M^n, g) be a Hermitian manifold, and denote by R the curvature tensor of the Chern connection. We say the real bisectional curvature of M is bounded from above by a constant C if

$$\sum_{i,j,k,l} R_{i\bar{j}k\bar{l}}\xi_{ij}\xi_{kl} \leq C\,\mathrm{tr}(\xi^2),$$

for all nontrivial, nonnegative, Hermitian $n \times n$ matrices ξ.

Observe Theorem 4.5 in [Yang and Zheng 2016], and use the identity $\Delta v = 2\Box v$ for Kähler manifolds (here Δ is the regular Laplacian, $\Box$ is the complex Laplacian, and v is an arbitrary smooth function). One can easily obtain the following Schwarz lemma as a corollary of Theorem 4.5 in [Yang and Zheng 2016].

Theorem 1.6 (the Schwarz lemma of [Yang and Zheng 2016]). *Let $f : M \to N$ be a holomorphic mapping from a complete Kähler manifold (M, g) with its Ricci curvature bounded from below by a negative constant $-k$ into a Hermitian manifold (N, h) with its real bisectional curvature bounded from above by a negative constant $-K$. If ν is the maximal rank of the map f, then*

$$f^*h \leq \frac{k\nu}{K} g.$$

2. Monotone unions of ellipsoids

We discuss monotone unions of ellipsoids $E_n := \{(z, w) : |z|^2 + |w|^{2n} < 1\}$ in $\mathbb{C}^2$ in this section.

Let M be an m-dimensional complex manifold which is a monotone union of Ω via f_j. Take an arbitrary point $q \in M$, and let $j \to \infty$; then $\{f_j(q)\}_{j=1}^{\infty}$ has a limit point, possibly after passing to a subsequence, because of the boundedness of Ω. Then the location of limit point of $\{f_j(q)\}_{j=1}^{\infty}$ has two possibilities. The limit point of $\{f_j(q)\}_{j=1}^{\infty}$ can be either an interior point of Ω or a boundary point at $\partial\Omega$.

The following lemma settles the case that the limit of $f_j(q)$ is an interior point of Ω. From now on, we will not distinguish between the convergence of sequences and the convergence after passing to subsequences.

Lemma 2.1. *Let M be a m-dimensional Hermitian manifold with a real bisectional curvature bounded from above by a negative number $-K$. Assume M is a monotone union of $\Omega \subset \mathbb{C}^m$ via f_j where Ω is a bounded domain in $\mathbb{C}^m$ with a complete Kähler metric of which the Ricci curvature is bounded from below by a negative number $-k$. We also assume there exists an interior point $q \in M$ so that $f_j(q) \to p \in \Omega$. Then M is biholomorphic onto Ω.*

Proof. Since Ω is bounded, f_j is a normal family of biholomorphisms. Let f_j converge to a holomorphic map F. Considering the inverses $\{f_j^{-1}\}_{j=1}^{\infty}$, we want to show they are locally bounded in a small geodesic ball B_p centered $p \in \Omega$ with radius $\epsilon > 0$. Let d_M be the Hermitian metric of M and d_Ω be the Kähler metric of Ω. Indeed, by Theorem 1.6,

$$(f_j^{-1})^* d_M \leq C d_\Omega$$

for each $j > 0$, where $C = km/K$. Let $N > 0$ be so that $f_j(q) \in B_p$ for all $j > N$. Considering arbitrarily $w \in B_p$, we have

$$d_M(q, f_j^{-1}(w)) \leq C d_\Omega(f_j(q), w) < 2C\epsilon, \tag{1}$$

for $j > N$. This means f_j^{-1} is locally bounded (hence a normal sequence) in B_p. We denote the limit of $\{f_j^{-1}\}_{j=1}^\infty$ by G. One can see that $F \circ G(w) = w$ in B_p because $\{f_j\}_{j=1}^\infty$ is uniformly convergent on compact subsets of M and $\{f_j^{-1}\}_{j=1}^\infty$ is uniformly convergent on compact subsets of B_p.

More generally, $\{f_j^{-1}\}_{j=1}^\infty$ is locally bounded on Ω. Indeed, we consider two interior points $w', w'' \in M$ and use Theorem 1.6 for f_j^{-1} again:

$$d_M(f_j^{-1}(w'), f_j^{-1}(w'')) \leq C d_\Omega(w', w''). \tag{2}$$

From this, we can see that $f_j \circ G$ is well defined everywhere in Ω. Hence, $F \circ G$ is well defined on Ω. Since $F \circ G(w) = w$ for $w \in B_p$, and $F \circ G$ is a holomorphic map, by the identity theorem, we obtain that $F \circ G(w) = w$ for all $w \in \Omega$. This implies F is surjective.

Since $f_j(q) \to p \in \Omega$ as $j \to \infty$, it follows that $\det Jf_j(q) \not\to 0$ by Cartan's theorem. We claim the limit of $(\det Jf_j)(z)$ is nowhere vanishing for arbitrary $z \in M$, where J denotes the Jacobian. The reason is as follows. By the Cauchy estimates, the fact that $\{f_j\}_{j=1}^\infty$ is normal implies that $\{\det Jf_j\}_{j=1}^\infty$ is also normal. But $\{\det Jf_j\}_{j=1}^\infty$ is nowhere zero for each $j > 0$ because f_j is a biholomorphism and then by Hurwitz's theorem, $\det JF$ is a zero function or nowhere zero. And the claim follows by the fact that $\det Jf_j(q) \not\to 0$. Now $\det Jf_j(z) \not\to 0$ for all $z \in M$, and hence, $\det JF(z)$ is nonzero everywhere. This also implies $F(M)$ is open by the open mapping theorem.

We are going to show F is 1-1. For this, we consider two interior points $z', z'' \in M$ and use Theorem 1.6 for f_j^{-1} for each $j > 0$ again:

$$d_M(z', z'') \leq C d_\Omega(f_j(z'), f_j(z'')). \tag{3}$$

Since $\det Jf_j(z)$ does not approach zero for all $z \in M$, $f(z)$ does not approach the boundary $\partial\Omega$ for fixed $z \in M$. In particular, $f_j(z')$ and $f_j(z'')$ do not approach the boundary $\partial\Omega$ where the Kähler metric d_Ω blows up. From this, $F(z') = F(z'')$ implies $d_\Omega(f_j(z'), f_j(z'')) \to 0$. By (3), we obtain that $z' = z''$. Consequently, F is 1-1.

Hence, M is biholomorphic onto Ω via bijective F. $\square$

By a similar argument, we can verify the following corollary. Instead of looking at only the exhaustive subsets of M in the previous lemma, the following corollary considers both exhaustive subsets of M and Ω.

Corollary 2.2. *Let M be an m-dimensional Hermitian manifold with a real bisectional curvature bounded from above by a negative number $-K$. Assume $M = \bigcup_j M_j$ where $M_j \subset M_{j+1}$ and f_j is a biholomorphism from M_j onto $\Omega_j \subset \Omega \subset \mathbb{C}^m$. Suppose Ω is a bounded domain in $\mathbb{C}^m$ and Ω_j is a complete Kähler manifold with the Ricci curvatures bounded from below by a same negative number $-k$ (independent with j). We also assume there exists a point $q \in M$ so that* $\det Jf_j(q) \not\to 0$. *Then F is* 1-1, *and hence*, *M is taut.*

For the sake of completeness, we also include a short outline of the proof.

Outline of proof. Since Ω is bounded, $\Omega_j \subset \Omega$ is bounded too for each $j > 0$. Hence, $\{f_j\}_{j=1}^{\infty}$ is still normal. By $\det Jf_j(q) \not\to 0$, we can see $\det Jf_j(z) \not\to 0$ everywhere for $z \in M$, where $\{\det Jf_j(z)\}_{j=1}^{\infty}$ is normal because of the Cauchy estimates. This means, for any $z \in M$, $f_j(z)$ does not approach $\partial\Omega$. So by Theorem 1.6, we find the limit F of f_j is 1-1. Moreover, this means M is taut. □

Lemma 2.1 and Corollary 2.2 tell us that if there exists one point q such that $f_j(q) \to p \in \Omega$, then for any point $z \in M$, we have $f_j(z)$ approaching an interior point of Ω. Furthermore, the limit of f_j forms a biholomorphism. However, this is not the only case. Indeed, sometimes $f_j(q)$ can approach a boundary point of Ω, and this brings trouble for getting the biholomorphism. For example, the image of $F = \lim_{j\to\infty} f_j$ might be just a constant map into a boundary point under some circumstances. The constant map of course cannot be a biholomorphism. What's behind this phenomenon is that under this situation $\det Jf_j(q) \to 0$ as $j \to \infty$. Thus, we need to compose each f_j with a biholomorphic map ϕ_j so that the resulting map $\det J\phi_j \circ f_j$ has a nonzero limit. To find the appropriate ϕ_j we need to estimate the speed of decay for $\det Jf_j(q)$. It appears the speed of decay can be arbitrary, but indeed, the decay is constrained by the location of $f_j(q)$ due to an application of the Schwarz lemma as follows. The following proposition is one of our main techniques.

Proposition 2.3. *Let M be an m-dimensional Hermitian manifold with a real bisectional curvature bounded from above by a negative number $-K$. Assume M is a monotone union of $\Omega \subset \mathbb{C}^m$ via f_j where Ω is a bounded domain in $\mathbb{C}^m$ with a complete Bergman metric of which the Ricci curvature is bounded from below by a negative number $-k$. We also assume there exists a point $q \in M$ so that $f_j(q) \to p \in \partial\Omega$ where p is strongly pseudoconvex. Then $|Jf_j(q)|/\delta(f_j(q))^{(m+1)/2} \gtrsim \eta$ for some $\eta > 0$,* where δ *is the Euclidean distance function of* Ω, *i.e.*, $\delta(z) = \operatorname{dist}(z, \partial\Omega)$.

Proof. Applying Theorem 1.6 for f_j^{-1}, we have $(f_j^{-1})^* g_M \le C g_\Omega$ for some $C > 0$ where g_M is the metric on M and g_Ω is the Bergman metric of Ω. In local coordinates, we have for any tangent vector $X_o \in T_o\Omega$ at $o \in \Omega$

$$((f_j^{-1})_* X_o)' G_M(f_j^{-1}(o))(f_j^{-1})_* X_o \le C X_o' G_\Omega(o) X_o,$$

where we denote the conjugate transpose by $'$ and matrices of g_M and g_Ω by G_M and G_Ω, respectively. For each $j > 0$, we let $o = f_j(q)$ and have

$$((f_j^{-1})_* X_{f_j(q)})' G_M(f_j^{-1}(f_j(q)))(f_j^{-1})_* X_{f_j(q)} \le C X'_{f_j(q)} G_\Omega(f_j(q)) X_{f_j(q)}.$$

Without loss of generality, we pick up the coordinates on M at q so that G_Ω is the identity matrix at q. Hence, $(Jf_j^{-1}(f_j(q)))' Jf_j^{-1}(f_j(q)) \le C G_\Omega(f_j(q))$ and by the Minkowski determinant theorem, we also have

$$|\det Jf_j^{-1}(f_j(q))|^2 \le C |\det G_\Omega(f_j(q))|. \tag{4}$$

But G_Ω is a metric around a strongly pseudoconvex point p, so by [Fefferman 1974], it is equivalent to the $\partial\bar\partial(\log\delta)$ up to nonzero constant. Moreover, by computing the second-order Taylor expansion of δ at p, we also have

$$|\det G_\Omega(o)| \le \frac{c_0}{\delta(o)^{m+1}}$$

for some $c_0 > 0$, when o is close to p. Again, putting $o = f_j(q)$, we have

$$|\det G_\Omega(f_j(q))| \le \frac{c_0}{\delta(f_j(q))^{m+1}} \tag{5}$$

for sufficiently big $j > 0$. Since $\det Jf_j^{-1}(f_j(q)) \cdot \det Jf_j(q) = 1$, we have, by (4) and (5), that $|\det Jf_j(q)|/\delta(f_j(q))^{(m+1)/2} > 1/\sqrt{c_0 C}$ for sufficient $j > 0$. We let $\eta = 1/\sqrt{c_0 C}$, and thus get the desired result. □

Another technique in this section was motivated by a simple observation in one variable.

Lemma 2.4. *Suppose there is a family of Möbius transforms on the unit disc* $\psi_j(z) = (z + \alpha_j)/(1 + \bar\alpha_j z)$ *where* $\alpha_j \in \mathbb{R}$ *and* $\alpha_j \to 1$. *Fixing* $s \in (0, 1)$, *we define the disc contained in* $\mathbb{D}$:

$$\mathcal{D}_s := \{z \in \mathbb{C} : |z - b| < 1 - b\}$$

where $s = 1 - b$. *Then* $\psi_j^{-1}(\mathcal{D}_s) \to \mathbb{D}$ *as* $j \to \infty$ *in the sense of convergence in increasing subsets.*

Proof. We compute the preimage $\psi_j^{-1}(\mathcal{D}_s)$. By calculation, we see that

$$\left| \frac{z + \alpha_j}{1 + \bar\alpha_j z} - b \right| < 1 - b$$

is equivalent to the inequality

$$\left|z+\frac{(\alpha_j-b)(1-\alpha_j b)-(1-b)^2\alpha_j}{|1-\bar{\alpha}_j b|^2-(1-b)^2|\alpha_j|^2}\right|^2 < \frac{|1-b|^2-|\alpha_j-b|^2}{|1-\bar{\alpha}_j b|^2-(1-b)^2|\alpha_j|^2}+\frac{|(\alpha_j-b)(1-\alpha_j b)-(1-b)^2\alpha_j|^2}{(|1-\bar{\alpha}_j b|^2-(1-b)^2|\alpha_j|^2)^2}.$$

This is a disc centered at

$$o_j=-\frac{(\alpha_j-b)(1-\alpha_j b)-(1-b)^2\alpha_j}{|1-\bar{\alpha}_j b|^2-(1-b)^2|\alpha_j|^2}$$

with radius

$$r_j=\sqrt{\frac{|1-b|^2-|\alpha_j-b|^2}{|1-\bar{\alpha}_j b|^2-(1-b)^2|\alpha_j|^2}+\frac{|(\alpha_j-b)(1-\alpha_j b)-(1-b)^2\alpha_j|^2}{(|1-\bar{\alpha}_j b|^2-(1-b)^2|\alpha_j|^2)^2}}.$$

By a straightforward calculation, $\lim_{j\to\infty} o_j=0$ and $\lim_{j\to\infty} r_j\to 1$. □

The imitation to balls is also available.

Lemma 2.5. *Suppose there is a family of automorphisms*

$$\psi_j(z,w)=\left(\frac{z+a_j}{1+\bar{a}_j z},\frac{\sqrt{1-|a_j|^2}}{1+\bar{a}_j z}w\right)$$

of the unit ball $\mathbb{B}^2$, *where* $a_j\in\mathbb{R}$ *and* $a_j\to 1$. *Fixing* $s\in(0,1)$, *we define a ball contained in* $\mathbb{B}^2$:

$$\mathcal{B}_s:=\{(z,w)\in\mathbb{C}:|(z,w)-(b,0)|<1-b\},$$

where $s=1-b$. *Then* $\psi_j^{-1}(\mathcal{B}_s)\to\mathbb{B}^m$ *as* $j\to\infty$ *in the sense of convergence in increasing subsets.*

Proof. We want to compute the preimage $\psi_j^{-1}(\mathcal{B}_s)$. For this, we need to calculate the $(z,w)\in\mathbb{C}^2$, such that $|\psi_j^{-1}(z,w)-(b,0)|<1-b$. By calculation, this is equivalent to the inequality

$$\left|z+\frac{(a_j-b)(1-a_j b)-(1-b)^2 a_j}{|1-\bar{a}_j b|^2-(1-b)^2|a_j|^2}\right|^2+\left|\frac{\sqrt{1-|a_j|^2}}{\sqrt{|1-\bar{a}_j b|^2-(1-b)^2|a_j|^2}}w\right|^2 < \frac{(1-b)^2-|a_j-b|^2}{|1-\bar{a}_j b|^2-(1-b)^2|a_j|^2}+\frac{|(a_j-b)(1-a_j b)-(1-b)^2 a_j|^2}{(|1-\bar{a}_j b|^2-(1-b)^2|a_j|^2)^2}. \tag{6}$$

Again, as in the previous lemma, one can see the formula in (6) approaches

$$|z|^2+|w|^2<1.$$ □

Due to symmetries of balls, one can see the following lemma is also true.

Lemma 2.6. *Suppose there is a family of automorphisms*

$$\psi_j(z, w) = \left(\frac{\sqrt{1-|a_j|^2}}{1+\bar{a}_j w} z, \frac{w+a_j}{1+\bar{a}_j w} \right)$$

of the unit ball $\mathbb{B}^2$ where $a_j \in \mathbb{R}$ and $a_j \to -1$. Fixing $s \in (0, 1)$, we define a ball contained in $\mathbb{B}^2$:

$$\mathcal{B}_s := \{(z, w) \in \mathbb{C} : |(z, w) - (0, b)| < 1 + b\}$$

where $s = 1 + b$. Then $\psi_j^{-1}(\mathcal{B}_s) \to \mathbb{B}^m$ as $j \to \infty$ in the sense of convergence in increasing subsets.

Proof of Theorem 1. Let $q \in M$ and $f_j(q) \to p$ as $j \to \infty$. There are two possibilities for the location of q: $q \in E_n$ or $q \in \partial E_n$.

If $p \in E_n$, then by Lemma 2.1, M is biholomorphic to E_n. Now we analyze the cases that $p \in \partial E_n$. Suppose that $f_j(q) = (a_j, b_j)$. We define

$$\psi_j(z, w) = \left(\frac{z-a_j}{1-\bar{a}_j z}, e^{-i\theta_j} \frac{\sqrt[2n]{1-|a_j|^2}}{\sqrt[n]{1-\bar{a}_j z}} w \right).$$

Here ψ_j is a family of automorphisms of E_n and θ_j is defined so that $\psi_j \circ f_j(p) = (0, b'_j)$ with $b'_j \in \mathbb{R}$. Since $(0, b'_j) \in E_n$, by the boundedness we have that $(0, b'_j) \to (0, b'_0)$, where $-1 \leq b'_0 \leq 1$. If $b'_0 \in (-1, 1)$, then $(0, b'_0) \in E_n$. And then by Lemma 2.1 for $\psi_j \circ f_j$, we know that M is biholomorphic to E_n. If $b'_0 = 1$ or -1, we discuss it as follows.

Without loss of generality, we now assume $b'_0 = -1$. This means it approaches a strongly pseudoconvex point $p_0 = (0, -1)$. The ellipsoid E_n, by translation, has a defining function

$$\rho(z, w) = |w-1|^2 - \sqrt[n]{1-|z|^2} = -2\,\mathrm{Re}\, w + |w|^2 + \frac{1}{n}|z|^2 + o(|z|^2).$$

Here, the point p_0 has been translated to $(0, 0)$.

On the other hand, we define $\mathcal{B}_s := \{(z, w) \in \mathbb{C}^2 : |z|^2 + |w|^2 - 2\,\mathrm{Re}\, w < 0\}$. It is not hard to see $\mathcal{B}_s$ is a ball centered at $(0, 1)$ with radius 1. We also define $\mathcal{B}_l := \{(z, w) \in \mathbb{C}^2 : (1/n)|z|^2 + (1/n)|w|^2 - 2\,\mathrm{Re}\, w < 0\}$. We can see that $\mathcal{B}_l$ is a ball centered at $(0, n)$ with radius n. So $\mathcal{B}_s \subset E_n \subset \mathcal{B}_l$, and they are tangent to each other at $(0, 0)$. We translate and rescale $\mathcal{B}_l$, $\mathcal{B}_s$, and E_n so that $\mathcal{B}_l = \mathbb{B}^2$. This setup is good for applying Lemma 2.6 to our situation. Due to the translation and rescaling, $\psi_j \circ f_j(p)$ becomes $(0, (b'_j + 1 - n)/n)$ and p_0 is once again relocated at $(0, -1)$. Since p_0 is strongly pseudoconvex, by Proposition 2.3, we see there exists $\eta > 0$ so that

$$|J(\psi_j \circ f_j)(q)| \gtrsim \eta \delta(\psi_j \circ f_j(q))^{3/2}.$$

This implies that

$$|J(\psi_j \circ f_j)(q)| \gtrsim \frac{\eta}{n}(1-|b'_j|)^{3/2},$$

where the n is due to the rescaling $\mathcal{B}_l$ into $\mathbb{B}^2$. We define a family of automorphisms of $\mathcal{B}_l = \mathbb{B}^2$,

$$\phi_j = \left(\frac{\sqrt{1-(b'_j)^2}}{1+b'_j w} z, \frac{w+b'_j}{1+b'_j w}\right).$$

and consequently, their inverses are

$$\phi_j^{-1} = \left(\frac{\sqrt{1-(b'_j)^2}}{1-b'_j w} z, \frac{w-b'_j}{1-b'_j w}\right).$$

Consider

$$J\phi_j^{-1}(z,w) = \frac{(1-(b'_j)^2)^{3/2}}{(1-b'_j w)^3}, \qquad (J\phi_j^{-1})(\psi \circ f_j(q)) = \frac{(1-(b'_j)^2)^{3/2}}{(1-b'_j(b'_j+1-n)/n)^3}.$$

We see that

$$\begin{aligned}\det(J(\phi_j^{-1} \circ \psi_j \circ f_j)(q)) &= \det((J\phi_j^{-1})(\psi_j \circ f_j(q))) \det(J(\psi_j \circ f_j)(q)) \\ &\gtrsim \frac{\eta}{n} \frac{(1-(b'_j)^2)^3}{(1-b'_j(b'_j+1-n)/n)^3},\end{aligned}$$

where the last term is bounded below by a positive number. This can be seen by a calculation using l'Hôpital's rule on $x = b'_j \to -1$.

Thus, the limit F of $\phi_j^{-1} \circ \psi_j \circ f_j$ has nontrivial image. Moreover, the image of F is $\mathbb{B}^2$ because by Lemma 2.6, $\phi_j^{-1}(\mathcal{B}_s) \subset \phi_j^{-1}(E_n)$ and $\phi_j^{-1}(\mathcal{B}_s)$ grows up to $\mathcal{B}_l = \mathbb{B}^2$.

Finally, we check the injectivity of F. The readers are reminded that the Bergman metric on E_n is invariant under ϕ_j^{-1}. Suppose there are $z', z'' \in M$ so that

$$\lim_{j\to\infty} \phi_j^{-1} \circ \psi_j \circ f_j(z') = \lim_{j\to\infty} \phi_j^{-1} \circ \psi_j \circ f_j(z'').$$

We can find big $N > 0$ so that for all $j > N$,

$$\phi_j^{-1} \circ \psi_j \circ f_j(z') \in \phi_N^{-1}(E_n), \qquad \phi_j^{-1} \circ \psi_j \circ f_j(z'') \in \phi_N^{-1}(E_n).$$

Consequently, by Theorem 1.6, we have that

$$d_M(z', z'') \leq C d_{\phi_N^{-1}(E_n)}(\phi_j^{-1} \circ \psi_j \circ f_j(z'), \phi_j^{-1} \circ \psi_j \circ f_j(z'')).$$

The assumption that $\lim_{j\to\infty} \phi_j^{-1} \circ \psi_j \circ f_j(z') = \lim_{j\to\infty} \phi_j^{-1} \circ \psi_j \circ f_j(z'')$ implies $z' = z''$. This proves the injectivity of F. □

Without much effort, one can show the following corollary.

Corollary 2.7. *Let M be an m-dimensional Hermitian manifold with a real bisectional curvature bounded from above by a negative number $-K$, and assume M is a monotone union of balls with the same dimension. Then M is biholomorphic onto $\mathbb{B}^m$.*

3. An application to the generalized bidiscs

In [Liu 2017], the author defined a generalized bidisc $\mathbb{D} \rtimes_\theta \mathbb{H}^+ := \{(z, w) : z \in \mathbb{D}, w \in e^{i\theta(z)}\mathbb{H}^+\}$. It has a noncompact automorphism group and shares some properties with the bidisc. Indeed, when $\theta(z)$ is a zero function, $\mathbb{D} \rtimes_\theta \mathbb{H}^+$ is biholomorphic to a bidisc.

In this section, we prove that the generalized bidisc cannot have a complete Kähler metric with holomorphic bisectional curvature bounded by two negative numbers. This is a result of Yang type. Recall that Yang's theorem [1976] on bidiscs has certain requirements on both variables of the bidisc, but in the proof, we show that it is possible to relax the requirement for one of them. Of course similar results for higher dimensions hold for the same reason. But we will not discuss them here. Our proof is modified from [Seo 2012].

Proof of Theorem 2. We assume the conclusion is not true. Let us denote the Poincaré metric of $\mathbb{D}$ by g and the complete Kähler metric on $\mathbb{D} \rtimes_\theta \mathbb{H}^+$ by h. For each z, we define $i_z(w) = (z, ie^{i\theta(z)}(1+w)/(1-w))$ from $\mathbb{D}$ onto $e^{i\theta(z)}\mathbb{H}^+$. We get $i^*h \le (4/c)g$ by the Schwarz lemma of Yau [1978] because the Ricci curvature of $\mathbb{D}$ is -4. Thus,

$$\begin{pmatrix} 0 & \dfrac{2ie^{i\theta(z)}}{(1-w)^2} \end{pmatrix} \begin{pmatrix} h_{1\bar{1}}\left(z, ie^{i\theta}\dfrac{1+w}{1-w}\right) & h_{1\bar{2}}\left(z, ie^{i\theta}\dfrac{1+w}{1-w}\right) \\ h_{2\bar{1}}\left(z, ie^{i\theta}\dfrac{1+w}{1-w}\right) & h_{2\bar{2}}\left(z, ie^{i\theta}\dfrac{1+w}{1-w}\right) \end{pmatrix} \begin{pmatrix} 0 \\ \dfrac{-2ie^{-i\theta(z)}}{(1-\overline{w})^2} \end{pmatrix} = h_{2\bar{2}}\left(z, ie^{i\theta(z)}\frac{1+w}{1-w}\right)\frac{4}{|1-w|^4} \le \frac{4}{c(1-|w|^2)^2}. \tag{7}$$

The last inequality gives

$$h_{2\bar{2}}\left(z, ie^{i\theta(z)}\frac{1+w}{1-w}\right) \le \frac{|1-w|^4}{c(1-|w|^2)^2} \le \frac{16}{c(1-|w|^2)^2}.$$

Since $k < \pi$, there exists $\epsilon > 0$ such that $k+\epsilon < \pi$. Because of $0 \le \theta(z) < k$, the following is true: $(z, e^{i(k+\epsilon/2)}) \in \mathbb{D} \rtimes_\theta \mathbb{H}^+$. We also have, for all $z \in \mathbb{D}$,

$$\frac{\epsilon}{2} < k + \frac{\epsilon}{2} - \theta(z) < k + \frac{\epsilon}{2} < k + \epsilon < \pi. \tag{8}$$

We fix $w_0 = (e^{i(k+\epsilon/2)-\theta(z)} - i)/(e^{i(k+\epsilon/2-\theta(z))} + i)$ for all $z \in \mathbb{D}$, and by the inequality (8), we can see $|1 - |w_0|| > \eta > 0$ for some positive number η depending on ϵ.

Also by the inequality (7), we have

$$h_{2\bar{2}}\left(z, ie^{i\theta(z)}\frac{1+w_0}{1-w_0}\right) = h_{2\bar{2}}(z, e^{i\theta(z)}e^{i((k+\epsilon/2)-\theta(z))}) = h_{2\bar{2}}(z, e^{i(k+\epsilon/2)}) \leq \frac{16}{c\eta^2}.$$

Let $F(z) := h_{2\bar{2}}(z, e^{i(k+\epsilon/2)})$. We see F is a real bounded positive function on $\mathbb{D}$. We check its Laplacian with respect to Poincaré metric on $\mathbb{D}$; we have (considering the bound of $R_{2\bar{2}1\bar{1}}$)

$$\begin{aligned}
\Delta_g F(z) &= (1-|z|^2)^2 \frac{\partial^2 F}{\partial z\, \partial \bar{z}}(z) \\
&= (1-|z|^2)^2 \left(R_{2\bar{2}1\bar{1}}(z, e^{i(k+\epsilon/2)}) + \sum_{\alpha,\beta=1}^{2} h^{\alpha\bar{\beta}} \frac{\partial h_{2\bar{\beta}}}{\partial z} \frac{\partial h_{\alpha\bar{2}}}{\partial \bar{z}} \right) \\
&\geq c(1-|z|^2)^2 h_{2\bar{2}}(z, e^{i(k+\epsilon/2)}) h_{1\bar{1}}(z, e^{i(k+\epsilon/2)}) \\
&= cF(z)(1-|z|^2)^2 h_{1\bar{1}}(z, e^{i(k+\epsilon/2)}),
\end{aligned}$$

because $\sum_{\alpha,\beta=1}^{2} h^{\alpha\bar{\beta}}(\partial h_{2\bar{\beta}}/\partial z)(\partial h_{\alpha\bar{2}}/\partial \bar{z})$ is nonnegative. Let $\pi : \mathbb{D} \rtimes_\theta \mathbb{H}^+ \to \mathbb{D}$, $\pi(z, w) = z$. We also have $\pi^* g \leq (d/4)h$, which is $(1-|z|^2)^2 h_{1\bar{1}}(z, w) \leq 4/d$. Hence, $\Delta_g F(z) \geq (c/d)F$. Calculate

$$\Delta_g \log F(z) = \frac{\Delta_g F(z)}{F(z)} - \frac{|\nabla_g F(z)|^2}{F(z)^2} \geq \frac{2c}{d} - \frac{|\nabla_g F(z)|^2}{F(z)^2}.$$

By Theorem 1.4, a real function T bounded from above on a complete Riemannian manifold M with Ricci curvature bounded below admits a sequence $\{p_k\}_{k=0}^{\infty} \subset M$ such that

$$\lim_{k\to\infty} |\nabla T(p_k)| = 0, \qquad \limsup_{k\to\infty} \Delta T(p_k) \leq 0, \qquad \lim_{k\to\infty} T(p_k) = \sup_M T.$$

Although $\log F(z)$ is a real function bounded from above on $\mathbb{D}$, it can not have such sequence $\{p_k\}_{k=0}^{\infty} \subset \mathbb{D}$. This contradiction completes the proof. □

A natural question is if we can relax the restriction for $\theta(z)$ in the theorem above.

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BINGYUAN LIU
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA, RIVERSIDE
RIVERSIDE, CA
UNITED STATES

bingyuan@ucr.edu

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MONADS ON PROJECTIVE VARIETIES

SIMONE MARCHESI, PEDRO MACIAS MARQUES AND HELENA SOARES

We generalize Fløystad's theorem on the existence of monads on projective space to a larger set of projective varieties. We consider a variety X, a line bundle L on X, and a basepoint-free linear system of sections of L giving a morphism to projective space whose image is either arithmetically Cohen–Macaulay (ACM) or linearly normal and not contained in a quadric. We give necessary and sufficient conditions on integers a, b and c for a monad of type

$$0 \to (L^\vee)^a \to \mathcal{O}_X^b \to L^c \to 0$$

to exist. We show that under certain conditions there exists a monad whose cohomology sheaf is simple. We furthermore characterize low-rank vector bundles that are the cohomology sheaf of some monad as above.

Finally, we obtain an irreducible family of monads over projective space and make a description on how the same method could be used on an ACM smooth projective variety X. We establish the existence of a coarse moduli space of low-rank vector bundles over an odd-dimensional X and show that in one case this moduli space is irreducible.

1. Introduction

A *monad* over a projective variety X is a complex

$$M_\bullet\colon 0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

of morphisms of coherent sheaves on X, where f is injective and g is surjective. The coherent sheaf $E := \ker g / \operatorname{im} f$ is called the *cohomology sheaf* of the monad $M_\bullet$. This is one of the simplest ways of constructing sheaves, after kernels and cokernels.

The first problem we need to tackle when studying monads is their existence. Fløystad [2000] gave sufficient and necessary conditions for the existence of monads over projective space whose maps are given by linear forms. Costa and Miró-Roig [2009] extended this result to smooth quadric hypersurfaces of dimension at least three, and Jardim [2007] made a further generalization to any hypersurface in projective space. We can find additional partial results on the existence of monads in

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the literature, by means of construction of examples of monads over other projective varieties (for instance, blowups of the projective plane [Buchdahl 2004], Abelian varieties [Gulbrandsen 2013], Fano threefolds [Faenzi 2014] and [Kuznetsov 2012], complete intersection, Calabi–Yau threefolds [Henni and Jardim 2013], and Segre varieties [Macias Marques and Soares 2014]). In [Jardim and Miró-Roig 2008], the authors expressed the wish of having a generalization of the results on existence by Fløystad and by Costa and Miró-Roig to varieties other than projective space and quadric hypersurfaces. Here we generalize Fløystad's theorem to a larger set of projective varieties. We let X be a variety of dimension n and L be a line bundle on X. We consider a linear system $V \subseteq H^0(L)$, without base points, defining a morphism $\varphi : X \to \mathbb{P}(V)$ and suppose that its image $X' \subset \mathbb{P}(V)$ is arithmetically Cohen–Macaulay (ACM) (see Definition 2.2 and Theorem 3.3) or linearly normal and not contained in a quadric hypersurface (Theorem 3.4). Then we give necessary and sufficient conditions on integers a, b and c for the existence of a monad of type

$$0 \to (L^\vee)^a \to \mathcal{O}_X^b \to L^c \to 0. \tag{M}$$

Once existence of a monad over a variety X is proved, we can study its cohomology sheaf. One of the most interesting questions to ask is whether this sheaf is stable and this has been established in special cases (see [Ancona and Ottaviani 1994] and [Jardim and Miró-Roig 2008], for instance). Since stable sheaves are simple, a common approach is to study simplicity (in [Costa and Miró-Roig 2009] the authors show that any mathematical instanton bundle over an odd-dimensional quadric hypersurface is simple, and in particular that it is stable over a quadric threefold). We show that under certain conditions, in the case when X' is ACM, there exists a monad of type (M) whose cohomology sheaf is simple (Proposition 4.1).

As we said, monads are a rather simple way of obtaining new sheaves. When the sheaf we get is locally free, we may consider its associated vector bundle, and by abuse of language we will not distinguish between one and the other. There is a lot of interest in low-rank vector bundles over a projective variety X, i.e., those bundles whose rank is lower than the dimension of X, mainly because they are very hard to find. We characterize low-rank vector bundles that are the cohomology sheaf of a monad of type (M) (Theorem 5.1).

Finally, we would like to be able to describe families of monads, or of sheaves coming from monads. There has been much work done on this since the nineties. Among the properties studied on these families is irreducibility (see for instance [Tikhomirov 2012; 2013] for the case of instanton bundles over projective space). Here we obtain an irreducible family of monads over projective space (Theorem 6.1), and make a description on how the same method could be used on another ACM projective variety. Furthermore, we establish the existence of a coarse moduli space of low-rank vector bundles over an odd-dimensional, ACM projective variety (Theorem 6.5), and show that in one case this moduli space is irreducible (Corollary 6.6).

2. Monads over ACM varieties

Let X be a projective variety of dimension n over an algebraically closed field k, L be a line bundle on X, and $V \subseteq H^0(L)$ yield a linear system without base points, defining a morphism $\varphi : X \to \mathbb{P}(V)$. Our main goal is to study monads over X of type

$$0 \to (L^\vee)^a \to \mathcal{O}_X^b \to L^c \to 0.$$

In this section we recall the concept of monad, as well as the results that were the starting point for the present paper, i.e., Fløystad's work [2000] regarding the existence of monads on projective space.

Let us first fix the notation used throughout the paper.

Notation 2.1. Let $Y \subseteq \mathbb{P}^N$ be a projective variety of dimension n over an algebraically closed field k. Let R_Y be the homogeneous graded coordinate ring of Y and $\mathcal{I}_{Y/\mathbb{P}^N}$ its ideal sheaf.

If $\mathcal{E}$ is a coherent sheaf over Y we will denote its dual by $\mathcal{E}^\vee$. We also denote the graded module $H^i_*(Y, \mathcal{E}) = \oplus_{m\in\mathbb{Z}} H^i(Y, \mathcal{E}(m))$ and $h^i(\mathcal{E}) = \dim H^i(Y, \mathcal{E})$.

Given any k-vector space V, we will write V^* to refer to its dual.

Definition 2.2. Let Y be a projective variety of dimension n over an algebraically closed field k. We say that Y is arithmetically Cohen–Macaulay (ACM) if its graded coordinate ring R_Y is a Cohen–Macaulay ring.

Remark 2.3. If $Y \subseteq \mathbb{P}^N$ is a projective variety then being ACM is equivalent to the following vanishing:

$$H^1_*(\mathbb{P}^N, \mathcal{I}_{Y/\mathbb{P}^N}) = 0, \quad H^i_*(Y, \mathcal{O}_Y) = 0,\ 0 < i < n.$$

Moreover, we note that the notion of ACM variety depends on the embedding.

The first problem we will address concerns the existence of monads on projective varieties (see Section 3) and the generalization of the following result.

Theorem 2.4 [Fløystad 2000, Main Theorem and Corollary 1]. *Let $N \geq 1$. There exists a monad of type*

$$(1) \qquad 0 \to \mathcal{O}_{\mathbb{P}^N}(-1)^a \xrightarrow{f} \mathcal{O}_{\mathbb{P}^N}^b \xrightarrow{g} \mathcal{O}_{\mathbb{P}^N}(1)^c \to 0$$

if and only if one of the following conditions holds:

(i) $b \geq a + c$ *and* $b \geq 2c + N - 1$,

(ii) $b \geq a + c + N$.

If so, there actually exists a monad with the map f degenerating in expected codimension $b-a-c+1$.

If the cohomology of the monad (1) is a vector bundle of rank $< N$ then $N = 2l+1$ is odd and the monad has the form

$$0 \to \mathcal{O}_{\mathbb{P}^{2l+1}}(-1)^c \to \mathcal{O}_{\mathbb{P}^{2l+1}}^{2l+2c} \to \mathcal{O}_{\mathbb{P}^{2l+1}}(1)^c \to 0. \tag{2}$$

Conversely, for every $c, l \geq 0$ there exist monads of type (2) whose cohomology is a vector bundle.

Observe that the vector bundles which are the cohomology of a monad of the form (2) are the so-called instanton bundles.

The next construction uses standard techniques of projective geometry and it explains why we thought Fløystad's case could be generalized to other projective varieties.

Let X' be the image of X in $\mathbb{P}(V)$. Taking $N = \dim V - 1$, let $\mathbb{P}^N := \mathbb{P}(V)$ and $m := \operatorname{codim}_{\mathbb{P}^N} X'$. Consider a monad of type (1) and take a projective linear subspace $\Lambda \subset \mathbb{P}^N$ of dimension $m-1$ such that $\Lambda \cap X' = \varnothing$. Fixing coordinates $z_0, \ldots, z_N$ in $\mathbb{P}^N$ we may assume that $I(\Lambda) = (z_0, \ldots, z_{N-m})$.

Let A and B be the matrices associated to the morphisms f and g, respectively, in (1). Consider the induced morphisms $\tilde{f}$ and $\tilde{g}$ whose matrices are, respectively, $\tilde{A}$ and $\tilde{B}$, obtained from A and B by the vanishing of the linear forms that define a linear complement of Λ, i.e., $\tilde{f} = f|_{\{z_{N-m+1}=\cdots=z_N=0\}}$ and $\tilde{g} = g|_{\{\{z_{N-m+1}=\cdots=z_N=0\}}$.

By construction, $\tilde{B}\tilde{A} = 0$. If $x \in \Lambda$ then the ranks of $\tilde{A}$ and $\tilde{B}$ evaluated at x are no longer maximal, that is, $\operatorname{rk}(\tilde{A}) < a$ and $\operatorname{rk}(\tilde{B}) < c$. In particular, the complex

$$\mathcal{O}_{\mathbb{P}^N}(-1)^a \xrightarrow{\tilde{f}} \mathcal{O}_{\mathbb{P}^N}^b \xrightarrow{\tilde{g}} \mathcal{O}_{\mathbb{P}^N}(1)^c$$

is not a monad on $\mathbb{P}^N$ anymore. Nevertheless, for a general $x \in X'$, the matrices $\tilde{A}(x)$ and $\tilde{B}(x)$ have maximal rank and hence the complex

$$0 \to (L^\vee)^a \xrightarrow{\varphi^*\tilde{f}} \mathcal{O}_X^b \xrightarrow{\varphi^*\tilde{g}} L^c \to 0,$$

where $L = \varphi^*(\mathcal{O}_{\mathbb{P}^N}(1))$, is a monad on X.

3. Existence of monads over ACM varieties

The aim of this section is to prove two characterizations of the existence of monads on projective varieties. We start by giving sufficient conditions for a monad to exist.

Lemma 3.1. *Let X be a variety of dimension n, let L be a line bundle on X, and let $V \subseteq H^0(L)$ be a linear system, with no base points, defining a morphism $X \to \mathbb{P}(V)$. Suppose a, b and c are integers such that one of the following conditions holds*:

(i) *$b \geq a+c$ and $b \geq 2c+n-1$,*

(ii) *$b \geq a+c+n$.*

Then there exists a monad of type

$$\text{(M)} \qquad 0 \to (L^\vee)^a \xrightarrow{f} \mathcal{O}_X^b \xrightarrow{g} L^c \to 0.$$

Moreover, the map f degenerates in expected codimension $b-a-c+1$ and g can be defined by a matrix whose entries are global sections of L that span a subspace of V whose dimension is $\min(b-2c+2, \dim V)$.

The main ideas of the proof follow Fløystad's construction, combined with the projective geometry standard results described at the end of the last section. Observe that, under the hypotheses of the theorem, the existence of a monad (M) is equivalent to the existence of a monad

$$0 \to \mathcal{O}_{X'}(-1)^a \xrightarrow{f} \mathcal{O}_{X'}^b \xrightarrow{g} \mathcal{O}_{X'}(1)^c \to 0.$$

Proof. Let $N = \dim V - 1$ and write $\mathbb{P}^N$ for $\mathbb{P}(V)$. Suppose that one of the conditions (i) and (ii) holds. If b is high enough with respect to a and c so that $b \geq 2c+N-1$ or $b \geq a+c+N$, then by Theorem 2.4, there is a monad

$$0 \to \mathcal{O}_{\mathbb{P}^N}(-1)^a \xrightarrow{f} \mathcal{O}_{\mathbb{P}^N}^b \xrightarrow{g} \mathcal{O}_{\mathbb{P}^N}(1)^c \to 0.$$

By restricting morphisms f and g to X', we get a monad of type (M). So from here on we may assume that $b < a+c+N$ and $b < 2c+N-1$.

Suppose first that condition (i) is satisfied. Then $N-1 > b-2c \geq n-1$, so

$$0 \leq N-(b-2c+2) \leq N-n-1.$$

Therefore we can take a projective linear subspace $\Lambda \subset \mathbb{P}^N$, disjoint from X', of dimension $N-(b-2c+2)$, and choose linearly independent sections $z_0, \dots, z_{b-2c+1} \in H^0(\mathcal{O}_{X'}(1))$ such that $I(\Lambda) = (z_0, \dots, z_{b-2c+1})$. Let us divide the coordinate set $\{z_0, \dots, z_{b-2c+1}\}$ into two subsets: $x_0, \dots, x_p$ and $y_0, \dots, y_q$, with $|p-q| \leq 1$ and such that $b-2c = p+q$. Define the matrices

$$X_{c,c+p} = \begin{bmatrix} x_0 & x_1 & \cdots & x_p & & & \\ & x_0 & x_1 & \cdots & x_p & & \\ & & \ddots & & & \ddots & \\ & & & x_0 & x_1 & \cdots & x_p \end{bmatrix} \quad \text{and} \quad Y_{c,c+q} = \begin{bmatrix} y_0 & y_1 & \cdots & y_q & & & \\ & y_0 & y_1 & \cdots & y_q & & \\ & & \ddots & & & \ddots & \\ & & & y_0 & y_1 & \cdots & y_q \end{bmatrix},$$

of sizes $c \times (c+p)$ and $c \times (c+q)$, respectively. Therefore, the matrices

$$B = \begin{bmatrix} X_{c,c+p} & Y_{c,c+q} \end{bmatrix}, \quad A = \begin{bmatrix} Y_{c+p,c+p+q} \\ -X_{c+q,c+q+p} \end{bmatrix}$$

allow us to construct the following complex on X':

$$\mathcal{O}_{X'}(-1)^{c+p+q} \xrightarrow[A]{f} \mathcal{O}_{X'}^{2c+p+q} \xrightarrow[B]{g} \mathcal{O}_{X'}(1)^c \to 0.$$

By construction $BA = 0$ and $\operatorname{rk} B(x) = c$, for each $x \in X'$.

Our next goal is to construct an injective morphism on X',

$$\mathcal{O}_{X'}(-1)^{c+p+q-s} \xrightarrow{\phi} \mathcal{O}_{X'}(-1)^{c+p+q} \tag{3}$$

so that we are able to compute the expected codimension of the degeneracy locus of the composition $f \circ \phi$, i.e., the codimension of

$$Z_s = \{x \in X' \mid \operatorname{rk}(f \circ \phi)(x) < c + p + q - s\}.$$

Observe that the matrices A and B define two more complexes: one complex on an n-dimensional projective subspace $\mathbb{P}^n \subset \mathbb{P}^N$,

$$\mathcal{O}_{\mathbb{P}^n}(-1)^{c+p+q} \xrightarrow[A]{\hat{f}} \mathcal{O}_{\mathbb{P}^n}^{2c+p+q} \xrightarrow[B]{\hat{g}} \mathcal{O}_{\mathbb{P}^n}(1)^c \to 0,$$

such that $\mathbb{P}^n \cap \Lambda = \varnothing$, and another one on $\mathbb{P}^N$ given by

$$\mathcal{O}_{\mathbb{P}^N}(-1)^{c+p+q} \xrightarrow[A]{\bar{f}} \mathcal{O}_{\mathbb{P}^N}^{2c+p+q} \xrightarrow[B]{\bar{g}} \mathcal{O}_{\mathbb{P}^N}(1)^c.$$

Consider a generic injective morphism

$$\mathcal{O}_{\mathbb{P}^n}(-1)^{c+p+q-s} \xrightarrow{\hat{\phi}} \mathcal{O}_{\mathbb{P}^n}(-1)^{c+p+q},$$

$s \geq 0$, inducing both a morphism $\mathcal{O}_{\mathbb{P}^N}(-1)^{c+p+q-s} \xrightarrow{\bar{\phi}} \mathcal{O}_{\mathbb{P}^N}(-1)^{c+p+q}$ and a morphism ϕ as in (3). Note that the three morphisms are represented by the same matrix.

From Lemmas 2 and 3 in [Fløystad 2000] it follows that the expected codimension of the degeneracy locus $\hat{Z}_s$ of $\hat{f} \circ \hat{\phi}$ is at least $s + 1$. Moreover, denoting the degeneracy locus of $\bar{f} \circ \bar{\phi}$ by $\bar{Z}_s$, we have the following relations:

$$\bar{Z}_s = \bigcup_{x \in \hat{Z}_s} \langle x, \Lambda \rangle, \qquad Z_s = \bar{Z}_s \cap X'.$$

Observe that the fact that $\hat{\phi}$ is injective implies that ϕ is also injective. Computing dimensions, we obtain that $\operatorname{codim}_{\mathbb{P}^N} \bar{Z}_s \geq s + 1$ and thus

$$\operatorname{codim}_{X'} Z_s \geq s + 1.$$

Then, taking $s = c + p + q - a = b - a - c \geq 0$, the complex

$$0 \to \mathcal{O}_{X'}(-1)^a \to \mathcal{O}_{X'}^b \to \mathcal{O}_{X'}(1)^c \to 0$$

is a monad on X' since we have $\operatorname{codim}_{X'} Z_s \geq s+1 = b-c-a+1$ (so, $\dim Z_s \leq n-1$).

Now, suppose condition (ii) holds, i.e., $b \geq a + c + n$, and suppose that $b < 2c + n - 1$ (otherwise we would be again in case (i)). Hence, $c > a + 1$ and $b > 2a + n + 1 > 2a + n - 1$.

Applying case (i) to the inequalities $b \geq a+c+n > a+c$ and $b > 2a+n-1$, we know there is a monad on X' of type

$$0 \to \mathcal{O}_{X'}(-1)^c \to \mathcal{O}_{X'}^b \to \mathcal{O}_{X'}(1)^a \to 0,$$

where the map $\mathcal{O}_{X'}(-1)^c \to \mathcal{O}_{X'}^b$ degenerates in codimension at least $b-a-c+1 \geq n+1$. Dualizing this complex, we get

$$0 \to \mathcal{O}_{X'}(-1)^a \to \mathcal{O}_{X'}^b \to \mathcal{O}_{X'}(1)^c \to 0,$$

which is still a monad on X', for the codimension of the degeneracy locus of $\mathcal{O}_{X'}(-1)^a \to \mathcal{O}_{X'}^b$ is at least $b-a-c+1$. □

Remark 3.2. We could have constructed a monad on X just by taking the pullback of a monad on $\mathbb{P}^n$ and applying Fløystad's result. In fact, we could have defined a finite morphism $X \to \mathbb{P}^n$ by considering precisely $\dim X + 1$ linearly independent global sections of L (and not vanishing simultaneously at any point $x \in X$). The pullback via this morphism of a monad on $\mathbb{P}^n$ would give us a monad on X. Nevertheless, we note that the construction above is far more general. It allows us to use a bigger number of global sections and it also provides an explicit construction of the monad on X.

We next prove the two main results of this section, which generalize Fløystad's theorem on the existence of monads on projective space. We consider a variety X, a line bundle L on X, and a basepoint-free linear system of sections of L giving a morphism to projective space. Each result asks different properties on the image $X' \subset \mathbb{P}(V)$ of the variety X.

Our first result characterizes the existence of monads of type (M) in the case when X' is an ACM projective variety.

Theorem 3.3. *Let X be a variety of dimension n and let L be a line bundle on X. Suppose there is a linear system $V \subseteq H^0(L)$, with no base points, defining a morphism $X \to \mathbb{P}(V)$ whose image $X' \subset \mathbb{P}(V)$ is a projective ACM variety. Then there exists a monad of type*

$$\text{(M)} \qquad 0 \to (L^\vee)^a \xrightarrow{f} \mathcal{O}_X^b \xrightarrow{g} L^c \to 0$$

if and only if one of the following conditions holds:

(i) $b \geq a+c$ *and* $b \geq 2c+n-1$,

(ii) $b \geq a+c+n$.

If so, there actually exists a monad with the map f degenerating in expected codimension $b-a-c+1$. Furthermore, g can be defined by a matrix whose entries are global sections of L that span a subspace of V whose dimension is $\min(b-2c+2, \dim V)$.

Note that if condition (ii) in the above theorem is satisfied then there exists a monad whose cohomology is a vector bundle of rank greater than or equal to the dimension of X.

Proof. The existence of the monad in case conditions (i) or (ii) are satisfied follows from Lemma 3.1. Let us show that these conditions are necessary. Suppose we have a monad on X'

$$0 \to \mathcal{O}_{X'}(-1)^a \xrightarrow{f} \mathcal{O}_{X'}^b \xrightarrow{g} \mathcal{O}_{X'}(1)^c \to 0.$$

This immediately implies that $b \geq a+c$. The image of the induced map $H^0(\mathcal{O}_{X'}^b) \to H^0(\mathcal{O}_{X'}(1)^c)$ defines a vector subspace $U' \subset H^0(\mathcal{O}_{X'}(1)^c)$ which globally generates $\mathcal{O}_{X'}(1)^c$. In particular, there is a diagram

$$\begin{array}{ccccc}
\mathcal{O}_{X'}^b & \xrightarrow{g} & \mathcal{O}_{X'}(1)^c & \longrightarrow & 0 \\
\downarrow & \nearrow_{\tilde{g}} & & & \\
U' \otimes \mathcal{O}_{X'} & & & & \\
\downarrow & & & & \\
0 & & & &
\end{array}$$

Since $\mathcal{O}_{X'}(1)^c$ is globally generated via $\tilde{g}$, we have $\dim U' \geq c+n$, otherwise the degeneracy locus of $\tilde{g}$ would be nonempty.

Let $U \subset U'$ be a general subspace with $\dim U = c+n-1$. Hence the map $\tilde{p} : U \otimes \mathcal{O}_{X'} \to \mathcal{O}_{X'}(1)^c$, induced by $\tilde{g}$, degenerates in dimension zero. Take a splitting

$$H^0(\mathcal{O}_{X'}^b) \rightleftarrows U'$$

and define $W = H^0(\mathcal{O}_{X'}^b)/U$. Denote $\mathcal{I} = \mathcal{I}(X') \subset k[z_0, \dots, z_N]$. Let $S = k[z_0, \dots, z_N]/\mathcal{I}$ be the coordinate ring of X'. Since X' is projectively normal, S is integrally closed and therefore $S = H^0_*(\mathcal{O}_{X'})$, so we have the following commutative diagram of graded S-modules:

$$\begin{array}{ccccc}
 & & U \otimes S & = & U \otimes S \\
 & & \downarrow & & \downarrow p \\
S(-1)^a & \longrightarrow & S^b & \longrightarrow & S(1)^c \\
\| & & \downarrow & & \\
S(-1)^a & \underset{q}{\longrightarrow} & W \otimes S & &
\end{array}$$

Sheafifying the above diagram, we get a surjective map

$$\operatorname{coker} \tilde{q} \to \operatorname{coker} \tilde{p} \to 0.$$

Because $\tilde{p}$ degenerates in the expected codimension we have, by [Buchsbaum and Eisenbud 1977, Theorem 2.3],

$$\operatorname{Fitt}_1(\operatorname{coker} \tilde{p}) = \operatorname{Ann}(\operatorname{coker} \tilde{p}),$$

and so we obtain the following chain of inclusions

$$\mathrm{Fitt}_1(\mathrm{coker}\,\tilde{q}) \subset \mathrm{Ann}(\mathrm{coker}\,\tilde{q}) \subset \mathrm{Ann}(\mathrm{coker}\,\tilde{p}) = \mathrm{Fitt}_1(\mathrm{coker}\,\tilde{p}),$$

where the first inclusion follows from [Eisenbud 1995, Proposition 20.7.a]. Thus,

$$\mathrm{Fitt}_1(\mathrm{coker}\,q) \subset H^0_*(\mathrm{Fitt}_1(\mathrm{coker}\,\tilde{q})) \subset H^0_*(\mathrm{Fitt}_1(\mathrm{coker}\,\tilde{p})).$$

Since p degenerates in expected codimension n and X' is ACM, $S/\mathrm{Fitt}_1(\mathrm{coker}\,p)$ is a Cohen–Macaulay ring of dimension 1; see [Eisenbud 1995, Theorem 18.18]. In particular, $\mathrm{Fitt}_1(\mathrm{coker}\,p)$ is a saturated ideal because the irrelevant maximal ideal $\mathfrak{m} \subset S$ is not an associated prime of it, and thus

$$H^0_*(\mathrm{Fitt}_1(\mathrm{coker}\,\tilde{p})) = \mathrm{Fitt}_1(\mathrm{coker}\,p).$$

By definition, $\mathrm{Fitt}_1(\mathrm{coker}\,p)$ is generated by polynomials of degree at least c, so all polynomials in $\mathrm{Fitt}_1(\mathrm{coker}\,q)$ must also have degree at least c. Note that the map q may be assumed to have generic maximal rank for f is injective and $S^b \to W \otimes S$ is a general quotient. This leads to two possibilities: either $\dim W \geq c$ or $\dim W > a$. Recalling that $\dim W = b - c - n + 1$, we obtain respectively

$$b \geq 2c + n - 1, \quad \text{or} \quad b \geq a + c + n. \qquad \square$$

We now state the second characterization result, with a similar setting as in Theorem 3.3, except that we drop the hypothesis that the image X' of X in $\mathbb{P}(V)$ is ACM, and assume instead that it is linearly normal and not contained in a quadric.

Theorem 3.4. *Let X be a variety of dimension n and let L be a line bundle on X. Suppose there is a linear system $V \subseteq H^0(L)$, with no base points, defining a morphism $X \to \mathbb{P}(V)$ whose image $X' \subset \mathbb{P}(V)$ is linearly normal and not contained in a quadric hypersurface. Then there exists a monad of type*

(M) $$0 \to (L^\vee)^a \xrightarrow{f} \mathcal{O}_X^b \xrightarrow{g} L^c \to 0$$

if and only if one of the following conditions holds:

(i) $b \geq a + c$ *and* $b \geq 2c + n - 1$,

(ii) $b \geq a + c + n$.

If so, there actually exists a monad with the map f degenerating in expected codimension $b - a - c + 1$. Furthermore, g can be defined by a matrix whose entries are global sections of L that span a subspace of V whose dimension is $\min(b - 2c + 2, \dim V)$.

Proof. The proof of the existence of a monad of type (M) follows again from Lemma 3.1. Let us check that at least one of conditions (i) or (ii) is necessary. Let $N = \dim V - 1$ and denote $\mathbb{P}^N = \mathbb{P}(V)$.

Suppose that there is a monad

$$0 \to (L^\vee)^a \xrightarrow{f} \mathcal{O}_X^b \xrightarrow{g} L^c \to 0.$$

Let A and B be matrices defining f and g, respectively. Since the entries of both matrices are elements of $H^0(L)$ and X' is linearly normal, we can choose linear forms on $\mathbb{P}^N$ to represent them, so the entries in the product BA can be regarded as elements of $H^0(\mathcal{O}_{\mathbb{P}^N}(2))$. Since X' is not cut out by any quadric, BA is zero on $\mathbb{P}^N$ yielding a complex

$$\mathcal{O}_{\mathbb{P}^N}(-1)^a \xrightarrow{\tilde{f}} \mathcal{O}_{\mathbb{P}^N}^b \xrightarrow{\tilde{g}} \mathcal{O}_{\mathbb{P}^N}(1)^c.$$

Furthermore, denoting by Z_A and Z_B the degeneracy loci in $\mathbb{P}^N$ of A and B, respectively, we know that $\dim(Z_A \cap X') \le n-1$ and Z_B does not intersect X'. Therefore their dimensions satisfy $\dim Z_A \le N-1$ and $\dim Z_B \le N-n-1$. We can consider a general subspace $\mathbb{P}^n$ that does not meet Z_B and also satisfies $\dim(Z_A \cap \mathbb{P}^n) \le n-1$. So if we consider the complex

$$\mathcal{O}_{\mathbb{P}^n}(-1)^a \xrightarrow{\hat{f}} \mathcal{O}_{\mathbb{P}^n}^b \xrightarrow{\hat{g}} \mathcal{O}_{\mathbb{P}^n}(1)^c,$$

also defined by the matrices A and B, we see that $\hat{f}$ is injective and $\hat{g}$ is surjective, so we have a monad on $\mathbb{P}^n$ and by Theorem 2.4 at least one of conditions (i) and (ii) is satisfied. □

Example 3.5. In [Macias Marques and Soares 2014], two of us presented a collection of examples of monads on Segre varieties. Using the same approach, we can think of similar examples of monads of some varieties that are cut out by quadrics, such as the Grassmannian. The simplest case that is not a hypersurface is $\mathbb{G}(2,5)$, the Grassmannian that parametrizes planes in the projective space $\mathbb{P}^5$, which is embedded in $\mathbb{P}^{19}$ with Plücker coordinates $[X_{j_0j_1j_2}]_{0\le j_0<j_1<j_2\le 5}$ satisfying

$$\sum_{s=0}^{3}(-1)^s X_{j_0j_1l_s}X_{l_0\cdots\widehat{l_s}\cdots l_3}=0 \tag{4}$$

for $0 \le j_0 < j_1 \le 5$ and $0 \le l_0 < l_1 < l_2 < l_3 \le 5$, where $X_{i_0i_1i_2} = (-1)^\sigma X_{i_{\sigma_0}i_{\sigma_1}i_{\sigma_2}}$, for any permutation σ, and $X_{i_0i_1i_2} = 0$ if there are any repeated indices. One of these quadrics is

$$\begin{aligned} &X_{012}X_{345} - X_{013}X_{245} + X_{014}X_{235} - X_{015}X_{234} \\ &\quad = \tfrac{1}{4}\big((X_{012}+X_{345})^2 - (X_{012}-X_{345})^2 - (X_{013}+X_{245})^2 + (X_{013}-X_{245})^2 \\ &\qquad + (X_{014}+X_{235})^2 - (X_{014}-X_{235})^2 - (X_{015}+X_{234})^2 + (X_{015}-X_{234})^2\big), \end{aligned}$$

obtained by using the sextuple $(j_0, j_1, l_0, l_1, l_2, l_3) = (0, 1, 2, 3, 4, 5)$ in (4). Now, for any pair (a, b), with $1 \le a < b \le 5$, consider the linear forms

$$u_{ab} := X_{0ab} + X_{i_1i_2i_3} \quad \text{and} \quad v_{ab} := X_{0ab} - X_{i_1i_2i_3},$$

where i_1, i_2 and i_3 are the unique integers satisfying $\{a, b, i_1, i_2, i_3\} = \{1, 2, 3, 4, 5\}$ and $i_1 < i_2 < i_3$. Then the twenty forms in $\{u_{ab}, v_{ab}\}_{1\le a<b\le 5}$ form a new basis of the coordinate ring of $\mathbb{P}^{19}$ and the above quadric can be rewritten as

$$\tfrac{1}{4}(u_{12}^2 - v_{12}^2 - u_{13}^2 + v_{13}^2 + u_{14}^2 - v_{14}^2 - u_{15}^2 + v_{15}^2).$$

So if seven of the eight linear forms occurring in this quadric vanish at a point of $\mathbb{G}(2, 5)$, so does the eighth. Similarly, using $(0, 3, 1, 2, 4, 5)$, $(0, 4, 1, 2, 3, 5)$, and $(0, 5, 1, 2, 3, 4)$ for $(j_0, j_1, l_0, l_1, l_2, l_3)$ in (4), we see that $\mathbb{G}(2, 5)$ is also cut out by

$$\tfrac{1}{4}(-u_{13}^2 + v_{13}^2 + u_{23}^2 - v_{23}^2 + u_{34}^2 - v_{34}^2 - u_{35}^2 + v_{35}^2),$$
$$\tfrac{1}{4}(-u_{14}^2 + v_{14}^2 + u_{24}^2 - v_{24}^2 - u_{34}^2 + v_{34}^2 - u_{45}^2 + v_{45}^2),$$
$$\tfrac{1}{4}(-u_{15}^2 + v_{15}^2 + u_{25}^2 - v_{25}^2 - u_{35}^2 + v_{35}^2 + u_{45}^2 - v_{45}^2).$$

Therefore, the sixteen linear forms $u_{23}, \dots, u_{45}, v_{12}, \dots, v_{45}$ cannot simultaneously vanish at a point of the Grassmannian, otherwise so would the remaining four u_{12}, u_{13}, u_{14}, and u_{15}. So let us write

$$\begin{array}{llll} w_1 = u_{23}, & w_2 = u_{24}, & w_3 = u_{25}, & w_4 = u_{34}, \\ w_5 = u_{35}, & w_6 = u_{45}, & w_7 = v_{12}, & w_8 = v_{13}, \\ w_9 = v_{14}, & w_{10} = v_{15}, & w_{11} = v_{23}, & w_{12} = v_{24}, \\ w_{13} = v_{25}, & w_{14} = v_{34}, & w_{15} = v_{35}, & w_{16} = v_{45}. \end{array}$$

Let $k \ge 1$ and let $A_1, A_2 \in M_{(k+7)\times k}(S)$ and $B_1, B_2 \in M_{k\times(k+7)}(S)$ be the matrices with entries in $S := K[X_{012}, \dots, X_{345}]$, given by

$$A_1 = \begin{bmatrix} w_8 & & \\ \vdots & \ddots & \\ w_1 & & w_8 \\ & \ddots & \vdots \\ & & w_1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} w_{16} & & \\ \vdots & \ddots & \\ w_9 & & w_{16} \\ & \ddots & \vdots \\ & & w_9 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} w_1 & \cdots & w_8 & & \\ & \ddots & & \ddots & \\ & & w_1 & \cdots & w_8 \end{bmatrix} \quad\text{and}\quad B_2 = \begin{bmatrix} w_9 & \cdots & w_{16} & & \\ & \ddots & & \ddots & \\ & & w_9 & \cdots & w_{16} \end{bmatrix},$$

and note that $B_1 A_2 = B_2 A_1$. Let A and B be the matrices

$$A = \begin{bmatrix} -A_2 \\ A_1 \end{bmatrix} \quad\text{and}\quad B = \begin{bmatrix} B_1 & B_2 \end{bmatrix}, \tag{5}$$

and let

$$0 \to \mathcal{O}_{\mathbb{G}(2,5)}(-1)^k \xrightarrow{\alpha} \mathcal{O}_{\mathbb{G}(2,5)}^{2k+14} \xrightarrow{\beta} \mathcal{O}_{\mathbb{G}(2,5)}(1)^k \to 0$$

be the sequence with maps α and β defined by matrices A and B, respectively. Now A and B fail to have maximal rank k if and only if $w_1, \dots, w_{16}$ are all zero, which, as we have seen, cannot happen in the Grassmannian variety. In particular, α is injective and β is surjective, and since $BA = 0$, this sequence yields a monad. We can reduce the exponent of the middle term in the monad, by using the method we described in the proof of Theorem 3.3, combined with this construction. Let $\Lambda \subset \mathbb{P}^{19}$ be the projective subspace defined by the following ten linear forms:

$$\begin{array}{llll} X_{012} - X_{345}, & X_{013} - X_{245}, & X_{014} - X_{235}, & X_{015} - X_{234}, \\ X_{023} - X_{145}, & X_{024} - X_{135}, & X_{025} - X_{034}, & X_{035} - X_{123}, \\ X_{045} + X_{134} + X_{124}, & \multicolumn{3}{l}{X_{045} + 2X_{134} - 3X_{124} - 5X_{125} + 7X_{012} + 11X_{013}.} \end{array}$$

With the help of a computer algebra system such as Macaulay [Grayson and Stillman $\geq$ 2018], we can check that Λ is disjoint from $\mathbb{G}(2, 5)$, so if $w'_1, \dots, w'_{10}$ are linear forms that complete a basis for the coordinate ring of $\mathbb{P}^{19}$, we can use them to construct matrices analogous to A and B above and obtain a monad

$$0 \to \mathcal{O}_{\mathbb{G}(2,5)}(-1)^k \xrightarrow{\alpha} \mathcal{O}_{\mathbb{G}(2,5)}^{2k+8} \xrightarrow{\beta} \mathcal{O}_{\mathbb{G}(2,5)}(1)^k \to 0.$$

4. Simplicity

Recall that a vector bundle E is said to be simple if its only endomorphisms are the homotheties, i.e., $\operatorname{Hom}(E, E) = \mathbb{C}$. The cohomology of a monad on $\mathbb{P}^N$ of type (1) is known to be simple when it has rank $N - 1$ (see [Ancona and Ottaviani 1994]). Moreover, every instanton bundle on the hyperquadric $Q^{2l+1} \subset \mathbb{P}^{2l+2}$ is simple; see [Costa and Miró-Roig 2009].

We next address the problem of the simplicity of the cohomology of monads on projective varieties of the form (M).

Proposition 4.1. *Let X be a variety of dimension n and let L be a line bundle on X. Suppose there is a linear system $V \subseteq H^0(L)$, with no base points*, $\dim V \geq 3$, *defining a morphism $X \to \mathbb{P}(V)$ such that the ideal sheaf of its image X' satisfies*

$$h^1(\mathcal{I}_{X'}(-1)) = h^2(\mathcal{I}_{X'}(-1)) = h^2(\mathcal{I}_{X'}(-2)) = h^3(\mathcal{I}_{X'}(-2)) = 0.$$

Let a and b be integers such that

$$\max\{n+1, a+1\} \leq b \leq \dim V.$$

Then there exists a monad

$$0 \to (L^\vee)^a \xrightarrow{f} \mathcal{O}_X^b \xrightarrow{g} L \to 0. \tag{6}$$

whose cohomology sheaf is simple.

Moreover, *when b is minimal*, *that is $b = n + 1$*, *then any monad of type* (6) *has a simple cohomology sheaf.*

Proof. Let $N = \dim V - 1$ and write $\mathbb{P}^N$ for $\mathbb{P}(V)$. Since $b \geq \max\{n+1, a+1\}$, Lemma 3.1 guarantees the existence of a monad of type (6). Moreover, since $b \leq \dim V$, we can choose linearly independent linear forms for the matrix that represents g. Consider the display

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & (L^\vee)^a & \longrightarrow & K & \longrightarrow & E & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & (L^\vee)^a & \xrightarrow{f} & \mathcal{O}_X^b & \longrightarrow & Q & \longrightarrow & 0 \\ & & & & \downarrow g & & \downarrow & & \\ & & & & L & = & L & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

Dualizing the first column and tensoring with K we get

$$0 \to K \otimes L^\vee \to K^b \to K \otimes K^\vee \to 0. \tag{7}$$

We claim that K is simple, i.e., $h^0(K \otimes K^\vee) = 1$. To see this, we first observe that, by construction, $L \cong \varphi^* \mathcal{O}_{\mathbb{P}^N}(1)$, where $\varphi : X \to \mathbb{P}^N$ is the morphism given by L. So, considering $\mathcal{O}_{X'}$ as a sheaf over $\mathbb{P}^N$, we have $\varphi_* L \cong \mathcal{O}_{X'}(1)$, and therefore $\varphi_* L^\vee \cong \mathcal{O}_{X'}(-1)$ and $\varphi_*(L^\vee \otimes L^\vee) \cong \mathcal{O}_{X'}(-2)$. Consider the exact sequence on $\mathbb{P}^N$

$$0 \to \mathcal{I}_{X'}(-1) \to \mathcal{O}_{\mathbb{P}^N}(-1) \to \varphi_* L^\vee \to 0.$$

Taking cohomology, we get that $h^0(L^\vee) = h^1(L^\vee) = 0$ from the vanishing of the groups $H^1(\mathcal{I}_{X'}(-1))$, $H^2(\mathcal{I}_{X'}(-1))$ and $H^i(\mathcal{O}_{\mathbb{P}^N}(-1))$. Now, if we tensor the first column of the display by $L^\vee$ and take cohomology, we get $h^1(K \otimes L^\vee) = h^0(\mathcal{O}_X) = 1$. Note also that $H^0(g) : H^0(\mathcal{O}_X^b) \to H^0(L)$ is injective, since the linear forms we chose to construct the matrix for g are linearly independent, hence $h^0(K) = 0$. Therefore we get an injective morphism

$$0 \to H^0(K \otimes K^\vee) \to H^1(K \otimes L^\vee)$$

induced by the exact sequence in (7), and we get $h^0(K \otimes K^\vee) = 1$, as we wished.

We now consider the exact sequence

$$0 \to \mathcal{I}_{X'}(-2) \to \mathcal{O}_{\mathbb{P}^N}(-2) \to \varphi_*(L^\vee \otimes L^\vee) \to 0$$

and take cohomology to get $h^1(L^\vee \otimes L^\vee) = h^2(L^\vee \otimes L^\vee) = 0$, from the vanishing of the groups $H^2(\mathcal{I}_{X'}(-2))$, $H^3(\mathcal{I}_{X'}(-2))$ and $H^i(\mathcal{O}_{\mathbb{P}^N}(-2))$, since $N \geq 2$. We dualize the first row in the display and tensor by E to obtain

$$0 \to E \otimes E^\vee \to E \otimes K^\vee \to E \otimes L^a \to 0, \tag{8}$$

which induces an injective morphism

$$0 \to H^0(E \otimes E^\vee) \to H^0(E \otimes K^\vee).$$

By dualizing the first column and tensoring by $L^\vee$, we may take cohomology and see that $h^0(K^\vee \otimes L^\vee) = h^1(K^\vee \otimes L^\vee) = 0$, for $h^1(L^\vee \otimes L^\vee) = h^2(L^\vee \otimes L^\vee) = 0$ and $h^0(L^\vee) = h^1(L^\vee) = 0$, as we saw above. Now tensoring the first row of the display by $K^\vee$ and taking cohomology we get $h^0(K^\vee \otimes E) = h^0(K^\vee \otimes K) = 1$. Therefore $h^0(E^\vee \otimes E) = 1$, i.e., E is a simple sheaf.

Finally, given any monad of type (6) with $b = n + 1 = \dim X + 1$, the entries of the matrix defining g must be linearly independent, otherwise it would not have maximal rank and g would not be a surjective morphism. Since the linear independence of these linear forms is a key step in the beginning of the proof, we see that in this case, any monad of type (6) has a simple cohomology sheaf. □

The next example shows that the statement in Proposition 4.1 is accurate, that is, there are monads of type (6) whose cohomology is not simple.

Example 4.2. Consider the monad over the quadric $X \subset \mathbb{P}^3$ embedded in $\mathbb{P}^9$ by $L = \mathcal{O}_X(2)$,

$$0 \to \mathcal{O}_X(-2) \xrightarrow[A]{f} \mathcal{O}_X^5 \xrightarrow[B]{g} \mathcal{O}_X(2) \to 0,$$

where

$$B = \begin{bmatrix} x_0^2 & x_1^2 & x_2^2 & x_3^2 & x_3^2 \end{bmatrix}, \quad A = \begin{bmatrix} -x_3^2 & -x_2^2 & x_1^2 & x_0^2 & 0 \end{bmatrix}^T,$$

and x_i are the coordinates in $\mathbb{P}^3$ such that X is defined by the form $x_0^2+x_1^2+x_2^2+x_3^2$. Then $\max\{3, 2\} \le b \le h^0(\mathcal{O}_X(2)) = 10 = N + 1$, however E is not simple. In fact, first note that $K = \ker g$ is not simple since it admits the endomorphism

$$\varphi : (f_1, f_2, f_3, f_4, f_5) \mapsto (f_1, f_2, f_3, f_5, f_4),$$

clearly not a homothety of K: if $f_4 \neq f_5$ then $\varphi(f_1, f_2, f_3, f_4, f_5)$ is not a multiple of $(f_1, f_2, f_3, f_4, f_5)$. Therefore the endomorphism induced on $E \cong K/\operatorname{im} f$ by φ is not a homothety of E (the class of a 5-uple of the same form is not mapped into a multiple of itself).

5. Vector bundles of low rank

In this section we characterize monads whose cohomology is a vector bundle of rank lower than the dimension of X and, in particular, we restrict to the case when X is nonsingular. Moreover, we will deal with the problem of simplicity and stability of this particular case.

Generalizing Fløystad's result, we start by proving the following theorem.

Theorem 5.1. *Let X be a nonsingular, n-dimensional, projective variety, embedded in $\mathbb{P}^N$ by a very ample line bundle L. Let M be a monad as in (M) and E its cohomology. If E is a vector bundle of rank lower than n, then $n = 2k+1$,* $\operatorname{rk} E = 2k$ *and the monad is of type*

$$0 \to (L^\vee)^c \xrightarrow{f} \mathcal{O}_X^{2k+2c} \xrightarrow{g} L^c \to 0. \tag{9}$$

Conversely, for each odd dimensional variety X with an associated ACM embedding given by a line bundle L and for each $c \geq 1$ there exists a vector bundle which is cohomology of a monad of type (9).

Proof. Suppose we have a vector bundle E of rank lower than $\dim X = n$ which is the cohomology of a monad M of type (M). Then its dual $E^\vee$ is the cohomology of the dual monad $M^\vee$. Since both E and $E^\vee$ are vector bundles which do not satisfy condition (ii) of Theorem 3.3, we must have

$$b \geq 2c+n-1 \quad \text{and} \quad b \geq 2a+n-1.$$

On the other hand, the hypothesis $\operatorname{rk} E < n$ implies that

$$b \leq a+c+n-1.$$

Combining the three inequalities we get that

$$a = c \quad \text{and} \quad b = 2c+n-1.$$

Then the monad M is of type

$$0 \to (L^\vee)^c \xrightarrow{f} \mathcal{O}_X^{2c+n-1} \xrightarrow{g} L^c \to 0,$$

therefore $\operatorname{rk} E = n-1$ which implies that $c_n(E) = 0$.

Hence, since the Chern polynomial of $\mathcal{O}_X$ is $c_t(\mathcal{O}_X) = 1$ (for X is nonsingular), we have

$$c_t(E) = \frac{1}{(1-lt)^c(1+lt)^c} = (1+l^2t^2+l^4t^4+\cdots)^c,$$

where l denotes $c_1(L)$. If $n = 2k$, for some $k \in \mathbb{Z}$, then $c_{2k}(E) = \alpha_{2k} l^{2k}$, where $\alpha_{2k} > 0$ is the binomial coefficient of the expansion of the series of $c_t(E)$. Observe that $l^{2k} = c_{2k}(L^{2k})$ and, by the projection formula, see [Fulton 1998, Theorem 3.2 (c)], this Chern class cannot be zero, contradicting the assertion above. So we conclude that n is odd and that the monad is of type (9).

Conversely, for any $c \geq 1$ and $(2k+1)$-dimensional variety X, there exists a monad of type (9) whose cohomology is a vector bundle E of rank $2k$ constructed using the technique described in the proof of Lemma 3.1. □

Minimal rank bundles defined using "many" global sections. In Theorem 3.3 we showed that the morphism g in (M) can be defined by a matrix B whose entries are global sections of L that span a subspace of $H^0(L)$ of dimension $\min(b-2c+2, h^0(L))$. This was done by giving an example of such a matrix, but surely there are others. Moreover, the dimension of the subspace spanned by the entries of these matrices can be bigger as we shall see in the following examples.

Take the quadric hypersurface $Q_3 \subset \mathbb{P}^4$, defined by $x_0^2+x_1^2+x_2^2+x_3^2+x_4^2=0$, and $L=\mathcal{O}_{Q_3}(2)$. Following the techniques used in Section 3 to construct a monad, we are able to obtain

$$\mathcal{O}_{Q_3}(-2)^2 \xrightarrow{A} \mathcal{O}_{Q_3}^6 \xrightarrow{B} \mathcal{O}_{Q_3}(2)^2, \tag{10}$$

where

$$A=\begin{bmatrix} -x_2^2 & -x_3^2 \\ 0 & -x_2^2 \\ -x_3^2 & 0 \\ x_0^2 & x_1^2 \\ 0 & x_0^2 \\ x_1^2 & 0 \end{bmatrix} \quad \text{and} \quad B=\begin{bmatrix} x_0^2 & x_1^2 & 0 & x_2^2 & x_3^2 & 0 \\ 0 & x_0^2 & x_1^2 & 0 & x_2^2 & x_3^2 \end{bmatrix}. \tag{11}$$

We have $BA=0$, and A and B have maximal rank when evaluated at every point of Q_3. Indeed, the rank of both A and B is not maximal only when evaluated at the point $(0:0:0:0:1)\in\mathbb{P}^4$, that does not belong to the quadric.

In order to use more global sections in the matrices defining the monad, we could simply "add another diagonal" whose entries involve an additional global section. Unfortunately, this method will increase the rank of the sheaf. For example, take the monad

$$\mathcal{O}_{Q_3}(-2)^2 \xrightarrow{A'} \mathcal{O}_{Q_3}^7 \xrightarrow{B'} \mathcal{O}_{Q_3}(2)^2 \tag{12}$$

given by the matrices

$$A'=\begin{bmatrix} -x_2^2-x_3^2 & -x_4^2 \\ -x_2^2 & -x_3^2 \\ 0 & -x_2^2-x_4^2 \\ x_0^2+x_1^2 & 0 \\ x_0^2 & x_1^2 \\ 0 & x_0^2 \\ 0 & x_1^2 \end{bmatrix} \quad \text{and} \quad B'=\begin{bmatrix} x_0^2 & x_1^2 & 0 & x_2^2 & x_3^2 & x_4^2 & 0 \\ 0 & x_0^2 & x_1^2 & 0 & x_2^2 & x_3^2 & x_4^2 \end{bmatrix} \tag{13}$$

with maximal rank evaluated at each point of Q_3. The cohomology of this monad is a rank 3 vector bundle on the quadric.

Therefore, our goal is to construct examples of minimal rank vector bundles whose monads are defined by matrices using a number of independent global sections of L strictly bigger than $\dim X + 1$. Indeed, the monads obtained this way cannot be the pullback of some monad over a projective space via a finite morphism (as described in Remark 3.2).

In the following two examples we will achieve such a goal in the particular case of the quadric considered above. However, the technique is easily reproducible for other varieties. The key point is to consider two matrices such that the union of their respective standard determinantal varieties does not intersect the base variety.

We get such examples by slightly modifying the matrices A and B. Consider a monad of type (10) but defined by the matrices

$$A_1 = \begin{bmatrix} -x_2^2 & -x_3^2 \\ 0 & -x_2^2 \\ -x_3x_4 & 0 \\ x_0^2 & x_1^2 \\ 0 & x_0^2 \\ x_1^2 & 0 \end{bmatrix} \quad \text{and} \quad B_1 = \begin{bmatrix} x_0^2 & x_1^2 & 0 & x_2^2 & x_3^2 & 0 \\ 0 & x_0^2 & x_1^2 & 0 & x_2^2 & x_3x_4 \end{bmatrix}. \tag{14}$$

Then, $B_1A_1 = 0$, and both A_1 and B_1 have maximal rank at every point of $\mathbb{P}^4$ except at points $(0:0:0:1:0)$ and $(0:0:0:0:1)$, neither belonging to the quadric.

It is possible to insert an additional global section in the previous matrices, by considering, for example, the monad defined by the matrices

$$A_2 = \begin{bmatrix} -x_2^2 & -x_3^2 \\ 0 & -x_2^2 \\ -x_3x_4 & 0 \\ x_0^2 & x_1^2 \\ 0 & x_0^2 \\ x_1^2 + x_1x_4 & 0 \end{bmatrix} \quad \text{and} \quad B_2 = \begin{bmatrix} x_0^2 & x_1^2 & 0 & x_2^2 & x_3^2 & 0 \\ 0 & x_0^2 & x_1^2 + x_1x_4 & 0 & x_2^2 & x_3x_4 \end{bmatrix}. \tag{15}$$

Again, $B_2A_2 = 0$, and A_2 and B_2 have maximal rank when evaluated at all points of projective space except at $(0:0:0:1:0)$, $(0:0:0:0:1)$ and $(0:1:0:0:-1)$, that do not belong to the quadric.

As we wanted, in both examples we used a number of global sections strictly bigger than $\dim Q_3 + 1$; it would be interesting to determine all the possible matrices obtained with this technique, once one fixes the base variety and the monad.

Simplicity and stability. We note that it is straightforward to construct examples of vector bundles on X, with $\operatorname{Pic} X = \mathbb{Z}$, satisfying properties of simplicity and stability. In fact, it is enough to consider, as observed in Remark 3.2, $\dim X + 1$

generic sections of L in order to get a finite morphism $\varphi : X \to \mathbb{P}^{2n+1}$. Using the *flatness miracle* and the projection formula, it is possible to prove that $\varphi_*\mathcal{O}_X$ is locally free and, moreover, $\varphi_*\mathcal{O}_X = \bigoplus_{i=0}^{p} \mathcal{O}_{\mathbb{P}^{2n+1}}(-a_i)$, for some positive p and nonnegative a_i; see [Barth et al. 1984, Lemma I.17.2]. Finally, using once again the projection formula (to the cohomology bundle) as well as commutativity of the tensor product with the pullback, we can conclude that the pullback of a simple (respectively stable) bundle E on $\mathbb{P}^{2n+1}$ is a simple (respectively stable) bundle on the projective variety X.

Nevertheless, we always have the following property.

Theorem 5.2. *Let X be a variety of dimension n and let L be a line bundle on X. Suppose there is a linear system $V \subseteq H^0(L)$, with no base points, defining a morphism $X \to \mathbb{P}(V)$ whose image $X' \subset \mathbb{P}(V)$ satisfies $h^2(\mathcal{I}_{X'}(-1)) = 0$ and at least one of the following conditions*:

(1) *X' is a projective ACM variety*;

(2) *X' is linearly normal and is not contained in a quadric hypersurface.*

Suppose in addition that there is a monad of type (9) over X whose cohomology sheaf E is locally free. Then $H^0(E) = 0$.

Proof. From the hypotheses, we see that X satisfies the conditions in Theorem 3.3 or Theorem 3.4. A monad of type (9) over X admits the following display:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (L^\vee)^c & \longrightarrow & K & \longrightarrow & E & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (L^\vee)^c & \xrightarrow{f} & \mathcal{O}_X^{2k+2c} & \longrightarrow & Q & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle g} & & \downarrow & & \\
 & & & & L^c & = & L^c & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

Taking cohomology on the exact sequence

$$0 \to \mathcal{I}_{X'}(-1) \to \mathcal{O}_{\mathbb{P}^N}(-1) \to \varphi_* L^\vee \to 0,$$

we get that $h^0(L^\vee) = h^1(L^\vee) = 0$, since $h^1(\mathcal{I}_{X'}(-1)) = h^2(\mathcal{I}_{X'}(-1)) = 0$. Therefore, taking cohomology on the first row of this display we have that $H^0(E) = H^0(K)$. Let us suppose that $H^0(K) \neq 0$, and let $\delta = h^0(K)$. Applying Lemma 1.6 in [Arrondo et al. 2016], we see that $K \simeq K' \oplus \mathcal{O}_X^\delta$, since $K^\vee$ is an $(L^\vee, \mathcal{O}_X)$-Steiner bundle (see Definition 1.3 in [Arrondo et al. 2016]). Therefore, the matrix defining g, with a suitable change of variables, may be assumed to have δ zero columns. So, again by Lemma 1.6, $(K')^\vee$ is itself an $(L^\vee, \mathcal{O}_X)$-Steiner bundle, sitting on a short exact sequence

$$0 \to (L^\vee)^c \to \mathcal{O}_X^{2k+2c-\delta} \to (K')^\vee \to 0.$$

Dualizing this, we get

$$0 \to K' \to \mathcal{O}_X^{2k+2c-\delta} \to L^c \to 0,$$

with $H^0(K') = 0$. Therefore, we would get a new monad, whose cohomology might be a sheaf, defined as

$$0 \to (L^\vee)^c \to \mathcal{O}_X^{2k+2c-\delta} \to L^c \to 0.$$

But this contradicts the conditions of existence of Theorems 3.3 and 3.4, thus proving the statement. □

Corollary 5.3. *Every rank* 2 *vector bundle* E *on a three dimensional ACM smooth projective variety* X *with* $\mathrm{Pic}(X) = \mathbb{Z}$, *defined by a monad of type* (9), *is stable.*

Proof. The result follows directly from the previous theorem and the Hoppe's criterion for stability; see [Hoppe 1984, Theorem 12]. □

6. The set of monads and the moduli problem

The existence part in Theorem 3.3 is proved by explicitly constructing a monad on a given projective variety X. The construction therein does not, however, give an answer to the question of "how many" monads of type (M) exist. We would like to know more about the algebraic structure of the set of pairs of morphisms which define a monad over a projective variety. In the case of projective space we prove the following:

Theorem 6.1. *Let* a, b, c *satisfy the conditions of Theorem 2.4, and suppose that* $1 \le c \le 2$. *Then for any surjective morphism* $g \in \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^n}^b, \mathcal{O}_{\mathbb{P}^n}(1)^c)$ *there is a morphism* $f \in \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^n}(-1)^a, \mathcal{O}_{\mathbb{P}^n}^b)$ *yielding a monad of type* (1).

Furthermore, the set of pairs

$$(f, g) \in \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^n}(-1)^a, \mathcal{O}_{\mathbb{P}^n}^b) \times \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^n}^b, \mathcal{O}_{\mathbb{P}^n}(1)^c)$$

yielding such a monad is an irreducible algebraic variety.

Proof. Let $g \in \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^n}^b, \mathcal{O}_{\mathbb{P}^n}(1)^c)$ be a surjective morphism and let $K_g := \ker g$. Then for any injective morphism $f \in H^0(K_g(1))^a$, the pair (f, g) yields a monad of type (1). If we consider the exact sequence

$$0 \to K_g \to \mathcal{O}_{\mathbb{P}^n}^b \xrightarrow{g} \mathcal{O}_{\mathbb{P}^n}(1)^c \to 0,$$

tensor by $\mathcal{O}_{\mathbb{P}^n}(1)$ and take cohomology, we can see that

$$h^0(K_g(1)) = b(n+1) - c\binom{n+2}{2} + h^1(K_g(1)).$$

Now following the arguments in the proof of Theorem 3.2 in [Costa and Miró-Roig 2007] we see that K_g is m-regular for any $m \geq c$. Therefore, since $c \leq 2$, K_g is 2-regular, i.e., $h^1(K_g(1)) = 0$. Since an injective morphism $f \in H^0(K_g(1))^a$ comes from a choice of a independent elements in $H^0(K_g(1))$, we wish to show that $h^0(K_g(1)) \geq a$, i.e., $b(n+1) - c\binom{n+2}{2} - a \geq 0$. We can check that the conditions in Theorem 2.4 imply this inequality.

The irreducibility of the set of pairs (f, g) that yield a monad of type (1) comes from the fact that the subset of surjective morphisms $g \in \mathrm{Hom}(\mathcal{O}^b_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n}(1)^c)$ is irreducible and the fiber of the projection

$$\mathrm{Hom}(\mathcal{O}_{\mathbb{P}^n}(-1)^a, \mathcal{O}^b_{\mathbb{P}^n}) \times \mathrm{Hom}(\mathcal{O}^b_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n}(1)^c) \to \mathrm{Hom}(\mathcal{O}^b_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n}(1)^c),$$

at a point corresponding to the surjective morphism g is the irreducible set of injective morphisms in $H^0(K_g(1))^a$, which has fixed dimension $a\bigl(b(n+1) - c\binom{n+2}{2}\bigr)$. □

Before discussing the more general setting of monads on ACM smooth projective varieties we give an example of reducibility with $c = 5$ on projective space.

Example 6.2. Consider the set of instanton bundles defined by a monad of the form

$$0 \to \mathcal{O}_{\mathbb{P}^3}(-1)^5 \xrightarrow{f} \mathcal{O}^{12}_{\mathbb{P}^3} \xrightarrow{g} \mathcal{O}_{\mathbb{P}^3}(1)^5 \to 0.$$

It was proved in [Jardim et al. 2018] that the moduli space of instanton sheaves of rank 2 and charge 5 is reducible. Furthermore, the set of pairs

$$(f, g) \in \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^3}(-1)^5, \mathcal{O}^{12}_{\mathbb{P}^3}) \times \mathrm{Hom}(\mathcal{O}^{12}_{\mathbb{P}^3}, \mathcal{O}_{\mathbb{P}^3}(1)^5)$$

yielding such a monad is a reducible algebraic variety.

Remark 6.3. It was brought to our attention that the Main Theorem in [Jardim et al. 2017] shows that the moduli space of instanton sheaves of rank 2 and charge 3 is reducible, which means that Theorem 6.1 is sharp.

The general setting. When X is a projective variety the general setting is the following. Let X be a projective variety embedded on $\mathbb{P}^N$ by a very ample line bundle L. Consider the set of all morphisms $g : \mathcal{O}^b_X \to L^c$, described by the vector space $B^* \otimes C \otimes H^0(L)$, where B and C are, respectively, k-vector spaces of dimensions b and c.

Denote $\mathbb{P}(B^* \otimes C \otimes H^0(L))$ by $\mathbb{P}$ and consider the map

(16) $$\mathcal{O}_{\mathbb{P}}(-1) \to B^* \otimes C \otimes H^0(L) \otimes \mathcal{O}_{\mathbb{P}}$$

of sheaves over $\mathbb{P}$, whose fiber at a point in $\mathbb{P}(B^* \otimes C \otimes H^0(L))$ corresponds to the natural inclusion. So, from (16) we get a map

$$B \otimes H^0(L) \otimes \mathcal{O}_{\mathbb{P}}(-1) \to C \otimes H^0(L) \otimes H^0(L) \otimes \mathcal{O}_{\mathbb{P}},$$

and hence also a map

$$B \otimes H^0(L) \otimes \mathcal{O}_{\mathbb{P}}(-1) \xrightarrow{\varphi} C \otimes H^0(L \otimes L) \otimes \mathcal{O}_{\mathbb{P}},$$

induced by the natural morphism $H^0(L) \otimes H^0(L) \to H^0(L \otimes L)$.

Now recall that $h^0(L) = N + 1$ and suppose that a, b, c are positive integers that satisfy the conditions of Theorem 3.3. Then the degeneracy locus

$$Z = \{g \in \mathbb{P}(B^* \otimes C \otimes H^0(L)) \mid \mathrm{rk}_g(\varphi) \leq b(N+1) - a\}$$

describes the set of morphisms g in a short exact sequence

$$0 \to K_g \to \mathcal{O}_X^b \xrightarrow{g} L^c \to 0$$

such that $h^0(K_g \otimes L) \geq a$ and for which it is thus possible to construct a monad of type (M). Note furthermore that

$$\operatorname{codim} Z \leq a\bigl(c\tbinom{N+2}{2} - b(N+1) + a\bigr).$$

Hence, whenever Z is irreducible (for example when $\operatorname{codim} Z < 0$) and $h^0(K_g)$ is constant for every morphism g we see that the set of the pairs (f, g) yielding a monad (M) on X is an irreducible algebraic variety. In this case, Theorem 6.1 can be extended to ACM varieties.

The moduli space of vector bundles of low rank. Let X be an ACM smooth projective variety of odd dimension $2k+1$, for some $k \in \mathbb{N}$, with an embedding in $\mathbb{P}^N$ by a very ample line bundle L on X, where $h^0(L) = N + 1$.

Consider the set $\mathcal{V}_{2k,c}$ of rank $2k$ vector bundles which are the cohomology of a monad of type

$$0 \to (L^\vee)^c \xrightarrow{f} \mathcal{O}_X^{2k+2c} \xrightarrow{g} L^c \to 0, \tag{17}$$

with $1 \leq c \leq 2$.

Remark 6.4. Observe that the hypotheses in Corollary 1, §4 Chapter 2, in [Okonek et al. 1980], hold for monads defined by (17). Hence the isomorphisms of monads of this type correspond bijectively to the isomorphisms of the corresponding cohomology bundles. In particular, the two categories are equivalent and we will not distinguish between their corresponding objects.

We want to construct a moduli space $M(\mathcal{V}_{2k,c})$ of vector bundles in $\mathcal{V}_{2k,c}$. In order to do this we will use King's framework of moduli spaces of representations of finite dimensional algebras in [King 1994].

We first note that according to [Jardim and Prata 2015, Theorem 1.3], the category $\mathcal{M}_{k,c}$ of monads of type (17) is equivalent to the full subcategory $\mathcal{G}_{k,c}^{gis}$ of the category $\mathcal{R}(Q_{k,c})$ of representations $R = (\{\mathbb{C}^c, \mathbb{C}^{2k+2c}, \mathbb{C}^c\}, \{A_i\}_{i=1}^{N+1}, \{B_j\}_{j=1}^{N+1})$ of the quiver $Q_{k,c}$ of the form

$$\bullet \underset{(N+1)}{\overset{\longrightarrow}{\underset{\longrightarrow}{\cdots}}} \bullet \underset{(N+1)}{\overset{\longrightarrow}{\underset{\longrightarrow}{\cdots}}} \bullet$$

which are (σ, γ)-globally injective and surjective and satisfy

$$\sum (B_i A_j + B_j A_i) \otimes (\sigma_i \gamma_j) = 0. \tag{18}$$

Let us briefly recall here the definitions of (σ, γ)-globally injective and surjective (see [Jardim and Prata 2015] for more details). Given a monad as in (17), choose bases $\gamma = (\gamma_1, \dots, \gamma_{N+1})$ of $\operatorname{Hom}(L^\vee, \mathcal{O}_X)$ and $\sigma = (\sigma_1, \dots, \sigma_{N+1})$ of $\operatorname{Hom}(\mathcal{O}_X, L)$. Set

$$\alpha = \sum_{i=1}^{N+1} A_i \otimes \gamma_i \quad \text{and} \quad \beta = \sum_{j=1}^{N+1} B_j \otimes \sigma_j.$$

The monad conditions of injectivity of f and surjectivity of g are reinterpreted in the language of the associated representation $R = (\{\mathbb{C}^c, \mathbb{C}^{2k+2c}, \mathbb{C}^c\}, \{A_i\}_{i=1}^{N+1}, \{B_j\}_{i=1}^{N+1})$ in $\mathcal{G}_{k,c}^{gis}$ as $\alpha(P) = \sum_{i=1}^{N+1} A_i \otimes \gamma_i(P)$ is injective and $\beta(P) = \sum_{j=1}^{N+1} B_j \otimes \sigma_j(P)$ is surjective, respectively, for all $P \in X$. In this case, we say that R is (σ, γ)-globally injective and surjective. The monad condition $g \circ f = 0$ is rewritten as in (18).

For the sake of simplicity, we will write $R = (c, 2k+2c, c)$ when we refer to the representation $R = (\{\mathbb{C}^c, \mathbb{C}^{2k+2c}, \mathbb{C}^c\}, \{A_i\}, \{B_j\})$. The notion of semistability for representations in $\mathcal{G}_{k,c}^{gis}$, as defined by King, is the following: a representation $R = (c, 2k+2c, c)$ is λ-*semistable* is there is a triple $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{Z}^3$ such that

$$\langle (\lambda_1, \lambda_2, \lambda_3), (c, 2k+2c, c) \rangle = 0, \quad \langle (\lambda_1, \lambda_2, \lambda_3), (a', b', c') \rangle \geq 0,$$

for all subrepresentations $R' = (a', b', c')$ of the representation R ($\langle \cdot , \cdot \rangle$ denotes the usual dot product). The representation is λ-*stable* if the only subrepresentations R' with $\langle (\lambda_1, \lambda_2, \lambda_3), (a', b', c') \rangle = 0$ are R and 0.

Moreover, by King's central result [1994, Theorem 4.1], the existence of such a λ guarantees the existence of a coarse moduli space for families of λ-semistable representations up to S-equivalence (two λ-semistable representations are S-equivalent if they have the same composition factors in the full abelian subcategory of λ-semistable representations).

Given the equivalences of the categories $\mathcal{M}_{k,c}$ and $\mathcal{G}_{k,c}^{gis}$, and after Remark 6.4, we see that we can define a moduli space $\mathcal{M}(\mathcal{V}_{2k,c})$ whenever we can construct a moduli space of the abelian category $\mathcal{G}_{k,c}^{gis}$.

When $c = 1$ we prove:

Theorem 6.5. *There is a coarse moduli space $\mathcal{M}(\mathcal{V}_{2k,1})$ of λ-semistable vector bundles in $\mathcal{V}_{2k,1}$.*

Proof. Let $R = (1, 2k+2, 1)$ be a representation in $\mathcal{G}_{k,1}^{gis}$, let $R' = (a', b', c')$ be any subrepresentation of R, and let $R'' = (a'', b'', c'')$ be the corresponding quotient representation. Then, we have a diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \bullet^{a'} & \longrightarrow & \bullet^{b'} & \longrightarrow & \bullet^{c'} & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \bullet^{1} & \longrightarrow & \bullet^{2k+2} & \longrightarrow & \bullet^{1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \bullet^{a''} & \longrightarrow & \bullet^{b''} & \longrightarrow & \bullet^{c''} & \longrightarrow & 0
\end{array}$$

(the fact that R is (σ, γ)-globally injective and surjective implies that R' is still injective, though not necessarily surjective, and that the quotient representation R'' preserves surjectivity).

R is λ-semistable if we can find $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{Z}^3$ such that

$$\langle(\lambda_1, \lambda_2, \lambda_3), (1, 2k+2, 1)\rangle = \lambda_1 + (2k+2)\lambda_2 + \lambda_3 = 0$$

and

$$\langle(\lambda_1, \lambda_2, \lambda_3), (a', b', c')\rangle \geq 0.$$

It is immediate from the diagram that either $a' = 0$ or $a' = 1$.

When $a' = 0$ and $b' = 2k+1$, we see that $b'' = 1,\ c'' = 0$ and hence $c' = 1$. So, R is λ-semistable if

$$\langle(\lambda_1, \lambda_2, \lambda_3), (0, 2k+1, 1)\rangle = (2k+1)\lambda_2 + \lambda_3 > 0.$$

When $a' = 0$ and $b' = 2k+2$, we see again that $b'' = c'' = 0$ and so $c' = 1$. The λ-semistability of R implies

$$\langle(\lambda_1, \lambda_2, \lambda_3), (0, 2k+2, 1)\rangle = (2k+2)\lambda_2 + \lambda_3 > 0.$$

Now suppose $a' = 1$. In this case $b' = 2k+2$, so that $b'' = c'' = 0$ and thus $c' = 1$, that is, $R' = R$ and we must have

$$\lambda_1 + (2k+2)\lambda_2 + \lambda_3 = 0.$$

Hence, we can choose the triple $\lambda = (-1, 0, 1)$ satisfying all the required inequalities in order for R to be λ-semistable.

The only subrepresentations left to consider are the ones of the form $(0, b', 0)$, but also for these ones, the choice of the triple $\lambda = (-1, 0, 1)$ satisfies the semistability condition.

The irreducibility statement follows from Theorem 6.1 and the general setting described above. □

The following is a consequence of Theorems 6.1 and 6.5.

Corollary 6.6. *Let $\mathcal{V}_{2k,1}$ be the set of rank $2k$ vector bundles which are the cohomology of a monad of type*

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-1) \xrightarrow{f} \mathcal{O}_{\mathbb{P}^n}^{2k+2} \xrightarrow{g} \mathcal{O}_{\mathbb{P}^n}(1) \to 0.$$

Then the coarse moduli space $\mathcal{M}(\mathcal{V}_{2k,1})$ of λ-semistable vector bundles in $\mathcal{V}_{2k,1}$ is irreducible.

Naturally, irreducibility of the moduli space will be guaranteed in each case where we get an irreducible family, as mentioned in the general setting described after Theorem 6.1.

Remark 6.7. When $c = 2$ an analogous study leads us to the conclusion that there is no λ such that a representation $R = (2, 2k + 2, 2)$ is λ-semistable. Therefore, in this case we are not able to construct the moduli space $\mathcal{M}(\mathcal{V}_{2k,2})$ with the help of King's construction.

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SIMONE MARCHESI
INSTITUTO DE MATEMÁTICA, ESTATÍSTICA E COMPUTAÇÃO CIENTÍFICA
UNIVERSIDADE ESTADUAL DE CAMPINAS
CIDADE UNIVERSITÁRIA "ZEFERINO VAZ"
CAMPINAS
BRAZIL
marchesi@ime.unicamp.br

PEDRO MACIAS MARQUES
DEPARTAMENTO DE MATEMÁTICA
ESCOLA DE CIÊNCIAS E TECNOLOGIA
CENTRO DE INVESTIGAÇÃO EM MATEMÁTICA E APLICAÇÕES
INSTITUTO DE INVESTIGAÇÃO E FORMAÇÃO AVANÇADA
UNIVERSIDADE DE ÉVORA
ÉVORA
PORTUGAL
pmm@uevora.pt

HELENA SOARES
DEPARTAMENTO DE MATEMÁTICA
INSTITUTO UNIVERSITÁRIO DE LISBOA (ISCTE-IUL)
UNIDE (BRU-BUSINESS RESEARCH UNIT)
LISBOA
PORTUGAL
helena.soares@iscte.pt

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MINIMAL REGULARITY SOLUTIONS OF SEMILINEAR GENERALIZED TRICOMI EQUATIONS

ZHUOPING RUAN, INGO WITT AND HUICHENG YIN

We prove the local existence and uniqueness of minimal regularity solutions u of the semilinear generalized Tricomi equation $\partial_t^2 u - t^m \Delta u = F(u)$ with initial data $(u(0,\cdot), \partial_t u(0,\cdot)) \in \dot{H}^\gamma(\mathbb{R}^n) \times \dot{H}^{\gamma-2/(m+2)}(\mathbb{R}^n)$ under the assumptions that $|F(u)| \lesssim |u|^\kappa$ and $|F'(u)| \lesssim |u|^{\kappa-1}$ for some $\kappa > 1$. Our results improve previous results of M. Beals and ourselves. We establish Strichartz-type estimates for the linear generalized Tricomi operator $\partial_t^2 - t^m \Delta$ from which the semilinear results are derived.

1. Introduction

In this paper, we are concerned with the local well-posedness problem for minimal regularity solutions u of the semilinear generalized Tricomi equation

$$\begin{aligned} &\partial_t^2 u - t^m \Delta u = F(u) \quad \text{in } [0,T] \times \mathbb{R}^n, \\ &u(0,\cdot) = \varphi \in \dot{H}^\gamma(\mathbb{R}^n), \quad \partial_t u(0,\cdot) = \psi \in \dot{H}^{\gamma-2/(m+2)}(\mathbb{R}^n), \end{aligned} \tag{1-1}$$

where $n \geq 2$, $m \in \mathbb{N}$, $\gamma \in \mathbb{R}$, $\Delta = \sum_{i=1}^n \partial_i^2$, and $T > 0$. The nonlinearity $F \in C^1(\mathbb{R})$ obeys the estimates

$$|F(u)| \lesssim |u|^\kappa, \quad |F'(u)| \lesssim |u|^{\kappa-1} \tag{1-2}$$

for some $\kappa > 1$. For $n \geq 3$ and $\kappa > \kappa_3$ (see below) we further assume that $\kappa \in \mathbb{N}$ and $F(u) = \pm u^\kappa$.

The main objective of this paper is to find the minimal number γ for which (1-1) under assumption (1-2) possesses a unique local solution

$$u \in C([0,T], \dot{H}^\gamma(\mathbb{R}^n)) \cap L^s((0,T); L^q(\mathbb{R}^n))$$

for certain s, q with $\min\{s,q\} \geq \kappa$. Then $F(u) \in L^{s/\kappa}((0,T); L^{q/\kappa}(\mathbb{R}^n)) \subseteq L^1_{\text{loc}}((0,T) \times \mathbb{R}^n)$ holds, and (1-1) is understood in distributions.

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We first introduce notation used throughout this paper. Set

$$\mu_* = \frac{(m+2)n+2}{2}, \qquad \kappa_* = \frac{\mu_*+2}{\mu_*-2} = \frac{(m+2)n+6}{(m+2)n-2},$$

$$\kappa_0 = 1 + \frac{6\mu_* + m}{\mu_*(m+2)n} \quad \text{if } n \geq 3 \text{ or } n = 2,\, m \geq 3,$$

$$\kappa_1 = \begin{cases} 2 & \text{if } n = 2,\, m = 1; \\ \dfrac{(\mu_*+2)(m+2)(n-1)+8}{(\mu_*-2)(m+2)(n-1)+8} & \text{if } n \geq 3 \text{ or } n = 2,\, m \geq 2; \end{cases}$$

$$\kappa_2 = \frac{\mu_*(\mu_*+2)(n-1)-2(n+1)}{\mu_*(\mu_*-2)(n-1)-2(n+1)},$$

$$\kappa_3 = \frac{\mu_*-m}{\mu_*-m-4} \quad \text{if } n \geq 3.$$

Note that μ_* is the homogeneous dimension of the degenerate differential operator $\partial_t^2 - t^m\Delta$ and κ_* is the power κ for which the equation $\partial_t^2 u - t^m \Delta u = \pm |u|^{\kappa-1}u$ is conformally invariant.

Note further that $1 < \kappa_0 < \kappa_1 < \kappa_* < \kappa_2 < \kappa_3$ whenever it applies.

Next we state the main results of this paper.

Theorem 1.1. *Let $n \geq 2$ and F be as above. Suppose further $\kappa > \kappa_1$ and $(\varphi, \psi) \in \dot H^\gamma(\mathbb{R}^n) \times \dot H^{\gamma-2/(m+2)}(\mathbb{R}^n)$, where*

$$\text{(1-3)} \quad \gamma = \gamma(\kappa, m, n) = \begin{cases} \frac14(n+1) - \dfrac{n+1}{\mu_*(\kappa-1)} - \dfrac{m}{2\mu_*(m+2)} & \text{if } \kappa_1 < \kappa \leq \kappa_*, \\ \frac12 n - \dfrac{4}{(m+2)(\kappa-1)} & \text{if } \kappa \geq \kappa_*. \end{cases}$$

Then problem (1-1) possesses a unique solution

$$u \in C([0,T]; \dot H^\gamma(\mathbb{R}^n)) \cap L^s((0,T); L^q(\mathbb{R}^n))$$

for some $T > 0$, where

$$\text{(1-4)} \quad \|u\|_{C([0,T];\dot H^\gamma(\mathbb{R}^n))} + \|u\|_{L^s((0,T);L^q(\mathbb{R}^n))} \lesssim \|\varphi\|_{\dot H^\gamma(\mathbb{R}^n)} + \|\psi\|_{\dot H^{\gamma-2/(m+2)}(\mathbb{R}^n)}$$

and $q = \mu_*(\kappa-1)/2$,

$$\frac1s = \begin{cases} \frac14(m+2)(n-1)\left(\frac12 - \frac1q\right) + \dfrac{m}{4\mu_*} & \text{if } \kappa_1 < \kappa \leq \kappa_*, \\ \frac1q & \text{if } \kappa \geq \kappa_*. \end{cases}$$

Remark 1.2. As a byproduct of the proof of Theorem 1.1, we see that problem (1-1) admits a unique global solution $u \in C([0,\infty); \dot H^\gamma(\mathbb{R}^n)) \cap L^\infty((0,\infty); \dot H^\gamma(\mathbb{R}^n)) \cap L^{\mu_*(\kappa-1)/2}(\mathbb{R}_+ \times \mathbb{R}^n)$ in case $n \geq 2$, $\kappa \geq \kappa_*$ if $(\varphi,\psi) = \varepsilon(u_0,u_1)$, $(u_0,u_1) \in$

$\dot{H}^{\gamma}(\mathbb{R}^n)\times\dot{H}^{\gamma-2/(m+2)}(\mathbb{R}^n)$, and $\varepsilon>0$ is small (see Section 5.1.2 and Section 5.1.3 in the proof of Theorem 1.1 below). With a different argument, the global result $u\in L^{\mu_*(\kappa-1)/2}(\mathbb{R}_+\times\mathbb{R}^n)$ for problem (1-1) was obtained in [He et al. 2017].

Remark 1.3. For $\gamma<n/2-4/((m+2)(\kappa-1))$, one obtains ill-posedness for problem (1-1) by scaling. More specifically, if $u=u(t,x)$ solves the Cauchy problem (1-1), where $F(u)=\pm|u|^{\kappa-1}u$, then

$$u_\varepsilon(t,x)=\varepsilon^{-2/(\kappa-1)}u(\varepsilon^{-1}t,\varepsilon^{-(m+2)/2}x),\quad \varepsilon>0,$$

also solves (1-1), with $u_\varepsilon(0,x)=\varphi_\varepsilon(x)$, $\partial_t u_\varepsilon(0,x)=\psi_\varepsilon(x)$ for some resulting φ_ε, ψ_ε. Observe that

$$\frac{\|\varphi_\varepsilon\|_{\dot{H}^{\gamma}(\mathbb{R}^n)}}{\|\varphi\|_{\dot{H}^{\gamma}(\mathbb{R}^n)}}=\frac{\|\psi_\varepsilon\|_{\dot{H}^{\gamma}(\mathbb{R}^n)}}{\|\psi\|_{\dot{H}^{\gamma}(\mathbb{R}^n)}}=\varepsilon^{\frac{1}{2}(m+2)\left(\frac{1}{2}n-\gamma\right)-\frac{2}{\kappa-1}},$$

and $\frac{1}{2}(m+2)(\frac{n}{2}-\gamma)-2/(\kappa-1)>0$ for $\gamma<n/2-4/((m+2)(\kappa-1))$. Hence, $\gamma<n/2-4/((m+2)(\kappa-1))$ implies that both the norm of the data $(\varphi_\varepsilon,\psi_\varepsilon)$ and the lifespan $T_\varepsilon=\varepsilon T$ of the solution u_ε go to zero as $\varepsilon\to 0$, where T is the lifespan of the solution u.

In case $\kappa_*\le\kappa<\kappa_2$, as a supplement to Theorem 1.1, we consider the local existence and uniqueness of solutions u of problem (1-1) in the space $C([0,T];\dot{H}^{\gamma}(\mathbb{R}^n))\cap L^s((0,T);L^q(\mathbb{R}^n))$ for certain $s\neq q$.

Theorem 1.4. *Let $n\ge 2$, F be as above, $\gamma=\gamma(\kappa,m,n)$ be as in Theorem 1.1, and suppose that $\kappa_*\le\kappa<\kappa_2$. Then the unique solution u of problem (1-1) also belongs to the space $L^s((0,T);L^q(\mathbb{R}^n))$, where*

$$\frac{1}{q}=\frac{1}{(m+2)(n-1)}\left(\frac{8}{\kappa-1}-\frac{m}{\mu_*}\right)-\frac{n-1}{2(n+1)}$$

and

$$\frac{1}{s}=\frac{(m+2)(n-1)}{4}\left(\frac{1}{2}-\frac{1}{q}\right)+\frac{m}{4\mu_*}.$$

Moreover, estimate (1-4) is satisfied.

If $n\ge 3$ or $n=2$, $m\ge 3$, then we find a number $\gamma(\kappa,m,n)$ also for certain κ in the range $\kappa_0\le\kappa<\kappa_1$.

Theorem 1.5. *Let $n\ge 3$ or $n=2$ with $m\ge 3$. Let F be as above and $\kappa_0\le\kappa<\kappa_1$. In addition, let the exponent $\gamma=\gamma(\kappa,m,n)$ in (1-1) be given by*

$$\text{(1-5)}\quad \gamma(\kappa,m,n)=\frac{n+1}{4}-\frac{n+1}{4\mu_*(m+2)}\cdot\frac{\mu_*(m+2)(n-1)+12\mu_*+2m}{2n\kappa-(n+1)}-\frac{m}{2\mu_*(m+2)}.$$

Then problem (1-1) possesses a unique solution $u \in C([0,T]; \dot{H}^{\gamma}(\mathbb{R}^n)) \cap L^s((0,T); L^q(\mathbb{R}^n))$ *for some* $T > 0$, *where*

$$\frac{1}{q} = \frac{1}{2n\kappa-(n+1)}\Big(\tfrac{1}{2}(n-1) + \frac{6}{m+2} + \frac{m}{\mu_*(m+2)}\Big)$$

and

$$\frac{1}{s} = \tfrac{1}{4}(m+2)(n-1)\Big(\frac{1}{2} - \frac{1}{q}\Big) + \frac{m}{4\mu_*}.$$

Moreover, estimate (1-4) is satisfied.

Remark 1.6. Other than for the wave equation when $m = 0$ (see also Remark 1.8 below), here γ can be negative in certain situations. In fact, $\gamma(\kappa, m, n) < 0$ holds in the following cases:

(i) $\kappa_1 < \kappa < \frac{35}{17}$ $(< \kappa_*)$ if $n = 2$, $m = 1$ and $\kappa_1 < \kappa < \frac{13}{7}$ $(< \kappa_*)$ if $n = 2$, $m = 2$ (see Theorem 1.1);

(ii) $$\kappa_0 < \kappa < \frac{\mu_*(\mu_*+2)(n+1)}{\mu_*(\mu_*-1)(n+1)-mn} \quad (\le \kappa_1)$$
if $n \ge 3$ or $n = 2$, $m \ge 3$ (see Theorem 1.5).

Remark 1.7. For initial data (φ, ψ) belonging to $H^{\gamma}(\mathbb{R}^n) \times H^{\gamma-2/(m+2)}(\mathbb{R}^n)$, where $\gamma \ge \gamma(\kappa, m, n)$, Theorems 1.1, 1.4, and 1.5 remain valid.

Remark 1.8. For $m = 0$, (1-1) becomes

$$\partial_t^2 u - \Delta u = F(u) \quad \text{in } (0,T) \times \mathbb{R}^n,$$
$$u(0,\cdot) = \varphi \in \dot{H}^{\gamma}(\mathbb{R}^n), \quad \partial_t u(0,\cdot) = \psi \in \dot{H}^{\gamma-1}(\mathbb{R}^n),$$

while the exponents $\kappa_*, \kappa_0, \kappa_1, \kappa_2$, and κ_3 are

$$\kappa_* = \frac{n+3}{n-1}, \quad \kappa_2 = \frac{(n+1)^2-6}{(n-1)^2-2}, \quad \kappa_1 = \frac{(n+1)^2}{(n-1)^2+4} \quad \text{if } n \ge 3,$$
$$\kappa_0 = \frac{n+3}{n}, \quad \kappa_3 = \frac{n+1}{n-3} \quad \text{if } n \ge 4.$$

For $n \ge 3$, γ defined in (1-3) equals

$$\gamma(\kappa,0,n) = \begin{cases} \frac{1}{4}(n+1) - 1/(\kappa-1) & \text{if } \kappa_1 < \kappa \le \kappa_*, \\ \frac{1}{2}n - 2/(\kappa-1) & \text{if } \kappa \ge \kappa_*, \end{cases} \tag{1-6}$$

whereas, for $n \ge 4$, γ defined in (1-5) equals

$$\gamma(\kappa,0,n) = \tfrac{1}{4}(n+1) - \tfrac{1}{4}(n+1)(n+5)\,\frac{1}{2n\kappa-(n+1)}. \tag{1-7}$$

Note that the numbers in (1-6) and (1-7) are exactly those in [Lindblad and Sogge 1995, (2.1) and (2.5)]. In that paper, the local existence problem for minimal regularity solutions of the semilinear wave equation was systematically studied.

The results were achieved by establishing Strichartz-type estimates for the linear wave operator $\partial_t^2 - \Delta$. Under certain restrictions on the nonlinearity $F(u, \nabla u)$, for the more general semilinear wave equation

$$\partial_t^2 u - \Delta u = F(u, \nabla u), \quad u(0, x) = \varphi(x), \quad \partial_t u(0, x) = \psi(x),$$

many remarkable results on the ill-posedness or well-posedness problem on the local existence of low regularity solutions have been obtained; see [Kapitanski 1994; Lindblad 1998; Lindblad and Sogge 1995; Ponce and Sideris 1993; Smith and Tataru 2005; Struwe 1992].

Remark 1.9. There are some essential differences between degenerate hyperbolic equations and strictly hyperbolic equations. Amongst others, the symmetry group is smaller (see [Lupo and Payne 2005]) and there is a loss of regularity for the linear Cauchy problem (see, e.g., [Dreher and Witt 2005; Taniguchi and Tozaki 1980]). Therefore, when compared to the semilinear wave equation, a more delicate analysis is required when one studies minimal regularity results for the semilinear generalized Tricomi equation in the degenerate hyperbolic region.

The Tricomi equation (i.e., (1-1) for $n = 1$, $m = 1$) was first studied by Tricomi [1923], who initiated work on boundary value problems for linear partial differential operators of mixed elliptic-hyperbolic type. So far, these equations have been extensively studied in bounded domains under suitable boundary conditions and several applications to transonic flow problems were given (see [Bers 1958; Germain 1954; Tricomi 1923; Morawetz 2004]). Conservation laws for equations of mixed type were derived by Lupo and Payne [2003; 2005]. In [Ruan et al. 2015b], we established the local solvability for low regularity solutions of the semilinear equation $\partial_t^2 u - t^m \Delta u = F(u)$, where $n \geq 2$, $m \in \mathbb{N}$ is odd, in the domain $(-T, T) \times \mathbb{R}^n$ for some $T > 0$. In [Barros-Neto and Gelfand 1999; 2002; Yagdjian 2004; 2015], fundamental solutions for the linear Tricomi operator and the linear generalized Tricomi operator have been explicitly computed. In the case $n = 2$ and $m = 1$, Beals [1992] obtained the local existence of the solution u of the equation $\partial_t^2 u - t\Delta u = F(u)$ with initial data of H^s-regularity, where $s > \frac{1}{2}n$. For the equation $\partial_t^2 u - t^m \Delta u = a(t)F(u)$, where $n \geq 2$, $m \in \mathbb{N}$ is even, and both a and F are of power type, Yadgjian [2006] obtained global existence and uniqueness for small data solutions provided the solution v of the linear problem $\partial_t^2 v - t^m \Delta v = 0$ fulfills $t^\beta v \in C([0, \infty); L^q(\mathbb{R}^n))$ for certain β, q depending on n, m, and the powers occurring in a and F.

In [Ruan et al. 2014; 2015a], for the semilinear generalized Tricomi equation $\partial_t^2 u - t^m \Delta u = F(u)$ with initial data of a special structure, i.e., homogeneous of degree 0 or piecewise smooth along a hyperplane, we obtained local existence and uniqueness via establishing L^∞ estimates on the solutions v of the linear equation $\partial_t^2 v - t^m \Delta v = g$. Note that when the nonlinear term $F(u)$ is of power type, for higher and higher powers of κ, these L^∞ estimates are basically required to guarantee existence. In this paper, where the initial data in $\dot{H}^\gamma(\mathbb{R}^n)$ is of no special structure and γ is minimal to guarantee local well-posedness of problem (1-1), the arguments of [Ruan et al. 2014; 2015a] fail. Inspired by the methods in [Lindblad and Sogge 1995], however, we are able to overcome the technical difficulties related to degeneracy and low regularity and eventually obtain the local well-posedness of problem (1-1).

We first study the linear problem

$$\begin{aligned} \partial_t^2 u - t^m \Delta u &= f(t,x) \quad \text{in } (0,T)\times\mathbb{R}^n, \\ u(0,\cdot) = \varphi(x), &\quad \partial_t u(0,\cdot) = \psi(x), \end{aligned} \tag{1-8}$$

and establish Strichartz-type estimates of the form

$$\begin{aligned} \|u\|_{C_t^0 \dot{H}_x^\gamma(S_T)} &+ \|u\|_{L_t^s L_x^q(S_T)} \\ &\le C\big(\|\varphi\|_{\dot{H}^\gamma(\mathbb{R}^n)} + \|\psi\|_{\dot{H}^{\gamma-2/(m+2)}(\mathbb{R}^n)} + \|f\|_{L_t^r L_x^p(S_T)}\big) \end{aligned} \tag{1-9}$$

for certain s, q, r, p (see below) and some constant $C = C(T,\gamma,s,q,r,p) > 0$, where $S_T = (0,T)\times\mathbb{R}^n$. Note that, by scaling, a necessary condition for this estimate in case $T=\infty$ to hold is

$$\tfrac{1}{2}(m+2)n\Big(\frac{1}{p}-\frac{1}{q}\Big)+\frac{1}{r}-\frac{1}{s}=2. \tag{1-10}$$

In doing so, in Section 2, we introduce certain Fourier integral operators W $(=W^0)$ and W^α for $\alpha\in\mathbb{C}$. These operators depend on a parameter $\mu\ge 2$, introduced in (2-15), which plays an auxiliary role for the linear problems and agrees with the homogeneous dimension μ_* when applied to the semilinear problems. Along with the operators W and W^α we also consider their dyadic parts W_j and W_j^α, respectively, resulting from a dyadic decomposition of frequency space. Continuity of the operators W_j and W_j^α between function spaces which holds uniformly in j ultimately provides linear estimates on the solutions u of (1-8).

In Section 3, we prove boundedness of the operators W_j^α from $L_t^r L_x^p(\mathbb{R}_+^{1+n})$ to $L_t^{r'} L_x^{p'}(\mathbb{R}_+^{1+n})$ (see Theorem 3.1) and from $L_t^r L_x^p(\mathbb{R}_+^{1+n})$ to $L_t^\infty L_x^2(\mathbb{R}_+^{1+n})$ (see Theorem 3.4), where μ has to satisfy the lower bound $\mu\ge\max\{2,m/2\}$. Combining Theorem 3.1 and Stein's analytic interpolation theorem, we show boundedness of the operators W_j^α from $L^q(\mathbb{R}_+^{1+n})$ to $L^{p_0}(\mathbb{R}_+^{1+n})$, where $q_0\le q\le\infty$ (see Theorem 3.6). Through an additional dyadic decomposition now with respect to the time variable t, using Theorems 3.1 and 3.6 together with interpolation, we prove boundedness of the operators W_j from $L_t^r L_x^p((0,T)\times\mathbb{R}^n)$ to $L_t^s L_x^q((0,T)\times\mathbb{R}^n)$ for any $T>0$ (see Theorems 3.7 and 3.8), where μ has to satisfy the new lower bounds $\mu\ge\mu_*$ (Theorem 3.7) and $\mu\ge\max\{2,mn/2\}$ (Theorem 3.8), respectively.

In the sequel, we shall use the following notation:

$$\frac{1}{p_0}=\frac{1}{2}+\frac{2\mu-m}{\mu(2\mu_*-m)},\quad \frac{1}{p_1}=\frac{1}{2}+\frac{2\mu-m}{\mu(m+2)(n-1)},\quad \frac{1}{p_2}=\frac{2}{p_0}-\frac{1}{p_1}.$$

Note that

$$1<p_1\le p_0\le p_2\le 2\quad \text{if } n\ge 3 \text{ or } n=2,\, m\ge 2,$$

while $1\le p_1$ in case of $n=2$ and $m=1$ requires $\mu=2$ (and then $p_1=1$). For $1\le p\le 2$, p' denotes the conjugate exponent of p defined by $1/p+1/p'=1$. Further, q_ℓ denotes p'_ℓ for $\ell=0,1,2$, while q_0^* equals q_0 when $\mu=\mu_*$ (see Remark 4.2). We often abbreviate function spaces $C_t^0\dot{H}_x^\gamma(S_T)=C([0,T];\dot{H}^\gamma(\mathbb{R}^n))$ and $L_t^rL_x^p(S_T)=L^r((0,T);L^p(\mathbb{R}^n))$, and $A\lesssim B$ means that $A\le CB$ holds for some generic constant $C>0$.

The paper is organized as follows: In Section 2, we define a class of Fourier integral operators associated with the linear generalized Tricomi operator $\partial_t^2-t^m\Delta$ in $\mathbb{R}_+\times\mathbb{R}^n$. Then, in Section 3, we establish a series of mixed-norm space-time estimates for those Fourier integral operators. These estimates are applied, in Section 4, to obtain Strichartz-type estimates for the solutions of the linear generalized Tricomi equation which in turn, in Section 5, allow us to prove the local existence and uniqueness results for problem (1-1).

2. Some preliminaries

In this section, we first recall an explicit formula for the solution of the linear generalized Tricomi equation obtained in [Taniguchi and Tozaki 1980] and then apply it to define a class of Fourier integral operators which will play a key role in proving our main results.

Consider the Cauchy problem of the linear generalized Tricomi equation

$$\partial_t^2u-t^m\Delta u=f(t,x)\quad \text{in } \mathbb{R}_+\times\mathbb{R}^n,\quad u(0,\cdot)=\varphi,\quad \partial_tu(0,\cdot)=\psi. \tag{2-1}$$

Its solution u can be written as $u=v+w$, where v solves the Cauchy problem of the homogeneous equation

$$\partial_t^2v-t^m\Delta v=0\quad \text{in } \mathbb{R}_+\times\mathbb{R}^n,\quad v(0,\cdot)=\varphi,\quad \partial_tv(0,\cdot)=\psi, \tag{2-2}$$

and w solves the inhomogeneous equation with zero initial data:

$$\partial_t^2w-t^m\Delta w=f(t,x)\quad \text{in } \mathbb{R}_+\times\mathbb{R}^n,\quad w(0,\cdot)=\partial_tw(0,\cdot)=0. \tag{2-3}$$

Recall that (see [Taniguchi and Tozaki 1980] or [Yagdjian 2006]) the solutions v and w of problems (2-2) and (2-3) can be expressed as

$$v(t,x)=V_0(t,D_x)\varphi(x)+V_1(t,D_x)\psi(x)$$

and

$$w(t,x)=\int_0^t \big(V_1(t,D_x)V_0(\tau,D_x)-V_0(t,D_x)V_1(\tau,D_x)\big)f(\tau,x)\,d\tau, \tag{2-4}$$

where the symbols $V_j(t,\xi)$ $(j=0,1)$ of the Fourier integral operators $V_j(t,D_x)$ are

$$\begin{aligned} V_0(t,\xi)&=e^{-z/2}\,\Phi\Big(\frac{m}{2(m+2)},\frac{m}{m+2};z\Big),\\ V_1(t,\xi)&=te^{-z/2}\,\Phi\Big(\frac{m+4}{2(m+2)},\frac{m+4}{m+2};z\Big), \end{aligned} \tag{2-5}$$

with $z=2i\phi(t)|\xi|$ and $\phi(t)=(2/(m+2))t^{(m+2)/2}$. Here, $\Phi(a,c;z)$ is the confluent hypergeometric function which is an analytic function of z. Recall (see [Erdélyi et al. 1953, p. 254]) that

$$\frac{d^n}{dz^n}\Phi(a,c;z)=\frac{(a)_n}{(c)_n}\Phi(a+n,c+n;z), \tag{2-6}$$

where $(a)_0=1$, $(a)_n=a(a+1)\dots(a+n-1)$. In addition, for $0<\arg(z)<\pi$, one has that (see [Yagdjian 2006, (3.5)–(3.7)])

$$e^{-z/2}\,\Phi(a,c;z)=\frac{\Gamma(c)}{\Gamma(a)}\,e^{z/2}H_+(a,c;z)+\frac{\Gamma(c)}{\Gamma(c-a)}\,e^{-z/2}H_-(a,c;z), \tag{2-7}$$

where

$$\begin{aligned} H_+(a,c;z)&\\ &=\frac{e^{-i\pi(c-a)}}{e^{i\pi(c-a)}-e^{-i\pi(c-a)}}\frac{1}{\Gamma(c-a)}\,z^{a-c}\int_\infty^{(0+)}e^{-\theta}\theta^{c-a-1}\Big(1-\frac{\theta}{z}\Big)^{a-1}d\theta,\\ H_-(a,c;z)&=\frac{1}{e^{i\pi a}-e^{-i\pi a}}\frac{1}{\Gamma(a)}\,z^{-a}\int_\infty^{(0+)}e^{-\theta}\theta^{a-1}\Big(1+\frac{\theta}{z}\Big)^{c-a-1}d\theta. \end{aligned}$$

Moreover, it holds that

$$\begin{aligned} \big|\partial_\xi^\beta\big(H_+(a,c;2i\phi(t)|\xi|)\big)\big| &\lesssim(\phi(t)|\xi|)^{a-c}(1+|\xi|)^{-|\beta|} \quad \text{if } \phi(t)|\xi|\ge 1,\\ \big|\partial_\xi^\beta\big(H_-(a,c;2i\phi(t)|\xi|)\big)\big| &\lesssim(\phi(t)|\xi|)^{-a}(1+|\xi|)^{-|\beta|} \quad \text{if } \phi(t)|\xi|\ge 1. \end{aligned} \tag{2-8}$$

Choose $\eta\in C_c^\infty(\mathbb{R}_+)$ such that $0\le\eta\le 1$ with $\eta(r)=1$ if $r\le 1$ and $\eta(r)=0$ if $r\ge 2$. Then from (2-5) and (2-7), we can write

$$\begin{aligned} &V_0(t,D_x)\varphi(x)\\ &\quad=\int_{\mathbb{R}^n}e^{i(x\cdot\xi-\phi(t)|\xi|)}b_1(t,\xi)\hat{\varphi}(\xi)\,đ\xi+\int_{\mathbb{R}^n}e^{i(x\cdot\xi+\phi(t)|\xi|)}b_2(t,\xi)\hat{\varphi}(\xi)\,đ\xi \end{aligned} \tag{2-9}$$

and

$$V_1(t,D_x)\psi(x) = \int_{\mathbb{R}^n} e^{i(x\cdot\xi-\phi(t)|\xi|)} b_3(t,\xi)\hat{\psi}(\xi)\,đ\xi + \int_{\mathbb{R}^n} e^{i(x\cdot\xi+\phi(t)|\xi|)} b_4(t,\xi)\hat{\psi}(\xi)\,đ\xi, \tag{2-10}$$

where

$$b_1(t,\xi) = \eta(\phi(t)|\xi|)\Phi\Big(\frac{m}{2(m+2)},\frac{m}{m+2};z\Big) + \big(1-\eta(\phi(t)|\xi|)\big)H_-\Big(\frac{m}{2(m+2)},\frac{m}{m+2};z\Big),$$

$$b_2(t,\xi) = \big(1-\eta(\phi(t)|\xi|)\big)H_+\Big(\frac{m}{2(m+2)},\frac{m}{m+2};z\Big),$$

$$b_3(t,\xi) = t\eta(\phi(t)|\xi|)\Phi\Big(\frac{m+4}{2(m+2)},\frac{m+4}{m+2};z\Big) + t\big(1-\eta(\phi(t)|\xi|)\big)H_-\Big(\frac{m+4}{2(m+2)},\frac{m+4}{m+2};z\Big),$$

$$b_4(t,\xi) = t\big(1-\eta(\phi(t)|\xi|)\big)H_+\Big(\frac{m+4}{2(m+2)},\frac{m+4}{m+2};z\Big),$$

and $đ\xi = (2\pi)^{-n}\,d\xi$. We can also write

$$\begin{aligned}\int_0^t V_0(t,D_x)V_1(\tau,D_x)f(\tau,x)\,d\tau &= \int_0^t\int_{\mathbb{R}^n} e^{i(x\cdot\xi+(\phi(t)+\phi(\tau))|\xi|)} b_2(t,\xi)b_4(\tau,\xi)\hat{f}(\tau,\xi)\,đ\xi\,d\tau \\ &+ \int_0^t\int_{\mathbb{R}^n} e^{i(x\cdot\xi+(\phi(t)-\phi(\tau))|\xi|)} b_2(t,\xi)b_3(\tau,\xi)\hat{f}(\tau,\xi)\,đ\xi\,d\tau \\ &+ \int_0^t\int_{\mathbb{R}^n} e^{i(x\cdot\xi-(\phi(t)+\phi(\tau))|\xi|)} b_1(t,\xi)b_3(\tau,\xi)\hat{f}(\tau,\xi)\,đ\xi\,d\tau \\ &+ \int_0^t\int_{\mathbb{R}^n} e^{i(x\cdot\xi-(\phi(t)-\phi(\tau))|\xi|)} b_1(t,\xi)b_4(\tau,\xi)\hat{f}(\tau,\xi)\,đ\xi\,d\tau\end{aligned} \tag{2-11}$$

and

$$\begin{aligned}\int_0^t V_1(t,D_x)V_0(\tau,D_x)f(\tau,x)\,d\tau &= \int_0^t\int_{\mathbb{R}^n} e^{i(x\cdot\xi+(\phi(t)+\phi(\tau))|\xi|)} b_4(t,\xi)b_2(\tau,\xi)\hat{f}(\tau,\xi)\,đ\xi\,d\tau \\ &+ \int_0^t\int_{\mathbb{R}^n} e^{i(x\cdot\xi-(\phi(t)-\phi(\tau))|\xi|)} b_3(t,\xi)b_2(\tau,\xi)\hat{f}(\tau,\xi)\,đ\xi\,d\tau \\ &+ \int_0^t\int_{\mathbb{R}^n} e^{i(x\cdot\xi-(\phi(t)+\phi(\tau))|\xi|)} b_3(t,\xi)b_1(\tau,\xi)\hat{f}(\tau,\xi)\,đ\xi\,d\tau\end{aligned} \tag{2-12}$$

$$+\int_0^t \int_{\mathbb{R}^n} e^{i(x\cdot\xi+(\phi(t)-\phi(\tau))|\xi|)} b_4(t,\xi) b_1(\tau,\xi) \hat{f}(\tau,\xi)\, đ\xi\, d\tau,$$

where $\hat{f}(\tau,\xi)$ is the Fourier transform of $f(\tau,x)$ with respect to the variable x.

In view of the analyticity of $\Phi(a,c;z)$ with respect to the variable z, identity (2-6), and estimates (2-8), we have that, for $(t,\xi)\in\mathbb{R}_+^{1+n}$,

$$|\partial_\xi^\beta b_\ell(t,\xi)| \lesssim (1+\phi(t)|\xi|)^{-\frac{m}{2(m+2)}} |\xi|^{-|\beta|}, \quad \ell=1,2, \tag{2-13}$$

and

$$|\partial_\xi^\beta b_\ell(t,\xi)| \lesssim t(1+\phi(t)|\xi|)^{-\frac{m+4}{2(m+2)}} |\xi|^{-|\beta|}, \quad \ell=3,4. \tag{2-14}$$

Thus, for $\ell=1,2$, $k=3,4$, $\mu\ge 2$, $t,\tau>0$, and $\xi\in\mathbb{R}^n$, one has from (2-13) and (2-14) that

$$\begin{aligned}
&\left|\partial_\xi^\beta\big(b_k(t,\xi)b_\ell(\tau,\xi)\big)\right| \\
&\qquad \lesssim t(1+\phi(t)|\xi|)^{-\frac{m+4}{2(m+2)}}(1+\phi(\tau)|\xi|)^{-\frac{m}{2(m+2)}}|\xi|^{-|\beta|} \\
&\qquad \lesssim (1+\phi(t)|\xi|)^{-\frac{m}{2(m+2)}}(1+\phi(\tau)|\xi|)^{-\frac{m}{2(m+2)}}|\xi|^{-\frac{2}{m+2}-|\beta|} \\
&\qquad \lesssim (1+|\phi(t)-\phi(\tau)||\xi|)^{-\frac{m}{\mu(m+2)}}|\xi|^{-\frac{2}{m+2}-|\beta|}.
\end{aligned} \tag{2-15}$$

Furthermore, estimates (2-13)–(2-15) yield that, for $\ell=1,2$, $k=3,4$, or $\ell=3,4$, $k=1,2$ and for $\mu\ge 2$, $t,s>0$, and $\xi\in\mathbb{R}^n$, one has

$$\begin{aligned}
&\left|\partial_\xi^\beta\left(\int_t^\infty \overline{b_\ell(\tau,\xi)b_k(t,\xi)}\,\partial_\tau(b_\ell(\tau,\xi)b_k(s,\xi))\,d\tau\right)\right| \\
&\qquad\qquad \lesssim (1+|\phi(t)-\phi(s)||\xi|)^{-\frac{m}{\mu(m+2)}}|\xi|^{-\frac{4}{m+2}-|\beta|}
\end{aligned} \tag{2-16}$$

and

$$\begin{aligned}
&\left|\partial_\xi^\beta\left(\int_s^\infty \overline{b_\ell(\tau,\xi)b_k(t,\xi)}\,\partial_\tau(b_\ell(\tau,\xi)b_k(s,\xi))\,d\tau\right)\right| \\
&\qquad\qquad \lesssim (1+|\phi(t)-\phi(s)||\xi|)^{-\frac{m}{\mu(m+2)}}|\xi|^{-\frac{4}{m+2}-|\beta|}.
\end{aligned} \tag{2-17}$$

In order to study the function w in (2-4), in view of (2-11), (2-12), and (2-15)–(2-17), it suffices to consider, for a given $\mu\ge 2$, the Fourier integral operator W:

$$Wf(t,x)=\int_0^t\int_{\mathbb{R}^n} e^{i(x\cdot\xi+(\phi(t)-\phi(s))|\xi|)} b(t,s,\xi)\hat{f}(s,\xi)\,đ\xi\,ds, \tag{2-18}$$

where $b\in C^\infty(\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}^n)$ satisfies the following:

(i) for $t,s>0$ and $\xi\in\mathbb{R}^n$,

$$|\partial_\xi^\beta b(t,s,\xi)| \lesssim (1+|\phi(t)-\phi(s)||\xi|)^{-\frac{m}{\mu(m+2)}}|\xi|^{-\frac{2}{m+2}-|\beta|}; \tag{2-19}$$

(ii) for $t, s > 0$ and $\xi \in \mathbb{R}^n$,

$$\left|\partial_\xi^\beta \left(\int_t^\infty \overline{b(\tau,t,\xi)}\, \partial_\tau b(\tau,s,\xi)\, d\tau \right)\right| \lesssim (1+|\phi(t)-\phi(s)||\xi|)^{-\frac{m}{\mu(m+2)}} |\xi|^{-\frac{4}{m+2}-|\beta|} \tag{2-20}$$

and

$$\left|\partial_\xi^\beta \left(\int_s^\infty \overline{b(\tau,t,\xi)}\, \partial_\tau b(\tau,s,\xi)\, d\tau \right)\right| \lesssim (1+|\phi(t)-\phi(s)||\xi|)^{-\frac{m}{\mu(m+2)}} |\xi|^{-\frac{4}{m+2}-|\beta|}. \tag{2-21}$$

Let $\Theta \in C_c^\infty(\mathbb{R}_+)$ satisfy $\operatorname{supp}\Theta \subseteq \left[\frac{1}{2}, 2\right]$ and

$$\sum_{j=-\infty}^{\infty} \Theta(t/2^j) = 1 \quad \text{for } t > 0.$$

Then, as in [Lindblad and Sogge 1995], for $j \in \mathbb{Z}$ and $\alpha \in \mathbb{C}$, we define dyadic operators W_j and W_j^α as

$$W_j f(t,x) = \int_0^t \int_{\mathbb{R}^n} e^{i(x\cdot\xi+(\phi(t)-\phi(s))|\xi|)} b_j(t,s,\xi)\hat{f}(s,\xi)\, đ\xi\, ds$$

and

$$W_j^\alpha f(t,x) = \int_0^t \int_{\mathbb{R}^n} e^{i(x\cdot\xi+(\phi(t)-\phi(s))|\xi|)} b_j(t,s,\xi)\hat{f}(s,\xi)\, \frac{đ\xi}{|\xi|^\alpha}\, ds, \tag{2-22}$$

where $b_j(t,s,\xi) = \Theta(|\xi|/2^j) b(t,s,\xi)$. Here, $b \in C^\infty(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^n)$ satisfies estimates (2-19)–(2-21).

Littlewood–Paley theory gives us a relationship between Wf and $W_j f$ $(= W_j^0 f)$, which will play an important role in our arguments in Section 4.

Proposition 2.1. *Let $n \geq 2$. For $1 < p \leq 2$, $1 \leq r \leq 2$, $2 \leq q < \infty$, and $2 \leq s \leq \infty$, let*

$$\|W_j f\|_{L_t^s L_x^q} \lesssim \|f\|_{L_t^r L_x^p} \tag{2-23}$$

hold uniformly in j. Then

$$\|Wf\|_{L_t^s L_x^q} \lesssim \|f\|_{L_t^r L_x^p}.$$

Proof. This is actually an application of [Lindblad and Sogge 1995, Lemma 3.8]. For the sake of completeness, we give the proof here. By Littlewood–Paley theory (see, e.g., [Stein 1970]), for any $1 < \rho < \infty$,

$$\|Wf(t,\cdot)\|_{L^\rho(\mathbb{R}^n)} \lesssim \left\| \left(\sum_{j=-\infty}^{\infty} |W_j f(t,\cdot)|^2 \right)^{1/2} \right\|_{L^\rho(\mathbb{R}^n)} \lesssim \|Wf(t,\cdot)\|_{L^\rho(\mathbb{R}^n)}.$$

Together with the Minkowski inequality, this yields

$$\|Wf\|_{L_t^s L_x^q} \lesssim \Big(\sum_{j=-\infty}^{\infty} \|W_j f\|^2_{L_t^s L_x^q} \Big)^{1/2} \tag{2-24}$$

and

$$\Big(\sum_{j=-\infty}^{\infty} \|W_j f\|^2_{L_t^r L_x^p} \Big)^{1/2} \lesssim \|Wf\|_{L_t^r L_x^p}. \tag{2-25}$$

Notice that

$$f = \sum_{k=-\infty}^{\infty} f_k,$$

where $f_k(\tau, x) = \Theta(\tau/2^k) f(\tau, x)$. Therefore, for some $M_0 \in \mathbb{N}$,

$$\begin{aligned}
\|Wf\|^2_{L_t^s L_x^q}
&\lesssim \sum_{j=-\infty}^{\infty} \|W_j f\|^2_{L_t^s L_x^q} && \text{(by (2-24))}\\
&= \sum_{j=-\infty}^{\infty} \Big\| W_j \Big(\sum_{|j-k| \le M_0} f_k \Big) \Big\|^2_{L_t^s L_x^q} && \text{(due to the compact support of } \Theta)\\
&\lesssim \sum_{j=-\infty}^{\infty} \Big(\sum_{|j-k| \le M_0} \|W_j f_k\|_{L_t^s L_x^q} \Big)^2 && \text{(by Minkowski inequality)}\\
&\lesssim \sum_{j=-\infty}^{\infty} \sum_{|j-k| \le M_0} \|f_k\|^2_{L_t^r L_x^p} && \text{(by (2-23))}\\
&\lesssim \sum_{j=-\infty}^{\infty} \|f_j\|^2_{L_t^r L_x^p} \lesssim \|f\|^2_{L_t^r L_x^p} && \text{(by (2-25)),}
\end{aligned}$$

which completes the proof of Proposition 2.1. □

3. Mixed-norm estimates for a class of Fourier integral operators

In this section, for $j \in \mathbb{Z}$, $\alpha \in \mathbb{C}$, and $\mu \ge 2$, we shall study mixed norm estimates for the class of Fourier integral operators W_j^α defined in (2-22).

We start by considering the boundedness of the operator W_j^α from $L_t^r L_x^p$ to $L_t^{r'} L_x^{p'}$, where $1 < r,\ p \le 2$. We denote $\lambda_j = 2^j$. *All the following estimates hold uniformly in* j.

Theorem 3.1. *Let* $n \ge 2$ *and* $\mu \ge \max\{2, m/2\}$. *Then*:

(i) *For* $\max\{p_1, 1\} < p \le 2$ *and*

$$\frac{1}{r} = 1 - \frac{m}{4\mu} - \tfrac{1}{4}(m+2)(n-1)\Big(\frac{1}{p} - \frac{1}{2}\Big), \tag{3-1}$$

we have that

$$\|W_j^\alpha f\|_{L_t^{r'} L_x^{p'}(\mathbb{R}_+^{1+n})} \lesssim \lambda_j^{\left(\frac{1}{p}-\frac{1}{2}\right)(n+1)-\frac{m}{\mu(m+2)}-\frac{2}{m+2}-\operatorname{Re}\alpha} \|f\|_{L_t^r L_x^p(\mathbb{R}_+^{1+n})}. \tag{3-2}$$

Consequently,

$$\|W_j^\alpha f\|_{L_t^{r'} L_x^{p'}(\mathbb{R}_+^{1+n})} \lesssim \|f\|_{L_t^r L_x^p(\mathbb{R}_+^{1+n})} \quad \text{if } \operatorname{Re}\alpha = \Big(\frac{1}{p} - \frac{1}{2}\Big)(n+1) - \frac{m}{\mu(m+2)} - \frac{2}{m+2}. \tag{3-3}$$

(ii) *For* $p_1 > 1$ *and* $1 < p < p_1$, *we have that*

$$\|W_j^\alpha f\|_{L_t^2 L_x^{p'}(\mathbb{R}_+^{1+n})} \lesssim \lambda_j^{n\left(\frac{2}{p}-1\right)-\frac{4}{m+2}-\operatorname{Re}\alpha} \|f\|_{L_t^2 L_x^p(\mathbb{R}_+^{1+n})}. \tag{3-4}$$

In particular,

$$\|W_j^\alpha f\|_{L_t^2 L_x^{p'}(\mathbb{R}_+^{1+n})} \lesssim \|f\|_{L_t^2 L_x^p(\mathbb{R}_+^{1+n})} \quad \text{if } \operatorname{Re}\alpha = n\Big(\frac{2}{p} - 1\Big) - \frac{4}{m+2}. \tag{3-5}$$

To prove Theorem 3.1, for fixed $t,\ \tau > 0$, we first consider the operator B_j^α:

$$B_j^\alpha f(t,\tau,x) = \int_{\mathbb{R}^n} e^{i(x\cdot\xi + (\phi(t)-\phi(\tau))|\xi|)}\, b_j(t,\tau,\xi)\, \hat{f}(\tau,\xi)\, \frac{d\!\!{}^-\xi}{|\xi|^\alpha}.$$

Lemma 3.2. *Let* $n \ge 2$ *and* $1 \le p \le 2$. *Then, for* $t, \tau > 0$,

$$\begin{aligned} \|B_j^\alpha f(t,\tau,\cdot)\|_{L^{p'}(\mathbb{R}^n)} &\lesssim \lambda_j^{\left(\frac{1}{p}-\frac{1}{2}\right)(n+1)-\frac{m}{\mu(m+2)}-\frac{2}{m+2}-\operatorname{Re}\alpha} \\ &\quad \times \big(\lambda_j^{-\frac{2}{m+2}} + |t-\tau|\big)^{-(m+2)\left(\frac{1}{p}-\frac{1}{2}\right)\frac{n-1}{2}-\frac{m}{2\mu}} \|f(\tau,\cdot)\|_{L^p(\mathbb{R}^n)}. \end{aligned} \tag{3-6}$$

Proof. Denote

$$K_j^\alpha(t,\tau,x,y) = \int_{\mathbb{R}^n} e^{i((x-y)\cdot\xi + (\phi(t)-\phi(\tau))|\xi|)}\, b_j(t,\tau,\xi)\, \frac{d\!\!{}^-\xi}{|\xi|^\alpha}. \tag{3-7}$$

Then $B_j^\alpha f$ can be written as

$$B_j^\alpha f(t,\tau,x) = \int_{\mathbb{R}^n} K_j^\alpha(t,\tau,x,y) f(\tau,y)\, dy.$$

Since $\operatorname{supp}_\xi b_j \subseteq \{\xi \in \mathbb{R}^n \mid \lambda_j/2 \le |\xi| \le 2\lambda_j\}$, we have from (2-19) that

$$|\partial_\xi^\beta b_j(t,\tau,\xi)| \lesssim \lambda_j^{-\frac{m}{\mu(m+2)}-\frac{2}{m+2}-|\beta|}(\lambda_j^{-\frac{2}{m+2}}+|t-\tau|)^{-\frac{m}{2\mu}}. \tag{3-8}$$

We now apply (3-8) to derive estimate (3-6) by Plancherel's theorem when $p=2$ and by the stationary phase method when $p=1$. By interpolation, we then obtain (3-6) for $1<p<2$.

Indeed, it follows from Plancherel's theorem that

$$\begin{aligned}\|B_j^\alpha f(t,\tau,\cdot)\|_{L_x^2(\mathbb{R}^n)} &= \|e^{i(\phi(t)-\phi(\tau))|\xi|}b_j(t,\tau,\xi)\hat{f}(\tau,\xi)|\xi|^{-\alpha}\|_{L_\xi^2(\mathbb{R}^n)}\\ &\lesssim \lambda_j^{-\frac{m}{\mu(m+2)}-\frac{2}{m+2}-\operatorname{Re}\alpha}(\lambda_j^{-\frac{2}{m+2}}+|t-\tau|)^{-\frac{m}{2\mu}}\|f(\tau,\cdot)\|_{L^2(\mathbb{R}^n)}.\end{aligned} \tag{3-9}$$

On the other hand, by the stationary phase method (see, e.g., [Sogge 1993, Lemma 7.2.4]), we have that, for any $N \ge 0$,

$$\begin{aligned}|K_j^\alpha(t,\tau,x,y)| &\lesssim \lambda_j^n(1+|\phi(t)-\phi(\tau)|\lambda_j)^{-\frac{n-1}{2}}(\lambda_j^{-\frac{2}{m+2}}+|t-\tau|)^{-\frac{m}{2\mu}}\\ &\qquad\times\lambda_j^{-\frac{m}{\mu(m+2)}-\frac{2}{m+2}-\operatorname{Re}\alpha}\big(1+\lambda_j\big||x-y|-|\phi(t)-\phi(\tau)|\big|\big)^{-N}\\ &\lesssim \lambda_j^{\frac{n+1}{2}-\frac{m}{\mu(m+2)}-\frac{2}{m+2}-\operatorname{Re}\alpha}(\lambda_j^{-\frac{2}{m+2}}+|t-\tau|)^{-\frac{(m+2)(n-1)}{4}-\frac{m}{2\mu}}\\ &\qquad\times\big(1+\lambda_j\big||x-y|-|\phi(t)-\phi(\tau)|\big|\big)^{-N}.\end{aligned} \tag{3-10}$$

Choosing $N=0$ in (3-10) gives

$$\begin{aligned}&\|(B_j^\alpha f)(t,\tau,\cdot)\|_{L^\infty(\mathbb{R}^n)}\\ &\quad\le \|K_j^\alpha(t,\tau,\cdot,\cdot)\|_{L_{x,y}^\infty}\|f(\tau,\cdot)\|_{L^1(\mathbb{R}^n)}\\ &\quad\lesssim \lambda_j^{\frac{n+1}{2}-\frac{m}{\mu(m+2)}-\frac{2}{m+2}-\operatorname{Re}\alpha}(\lambda_j^{-\frac{2}{m+2}}+|t-\tau|)^{-\frac{1}{4}(m+2)(n-1)-\frac{m}{2\mu}}\|f(\tau,\cdot)\|_{L^1(\mathbb{R}^n)}.\end{aligned}$$

Interpolation between (3-9) and this last estimate yields (3-6) in case $1 \le p \le 2$, which completes the proof of estimate (3-6). □

Proof of Theorem 3.1. Now we return to the proof of Theorem 3.1. From (3-7), we have

$$W_j^\alpha f(t,x) = \int_0^t (B_j^\alpha f)(t,\tau,x)\,d\tau. \tag{3-11}$$

Using Minkowski's inequality and estimate (3-6), we thus have that

$$\begin{aligned} (3\text{-}12)\quad &\|W_j^\alpha f(t,\cdot)\|_{L^{p'}(\mathbb{R}^n)} \\ &\lesssim \lambda_j^{\left(\frac{1}{p}-\frac{1}{2}\right)(n+1)-\frac{m}{\mu(m+2)}-\frac{2}{m+2}-\operatorname{Re}\alpha} \\ &\quad\times \int_0^\infty (\lambda_j^{-\frac{2}{m+2}}+|t-\tau|)^{-(m+2)\left(\frac{1}{p}-\frac{1}{2}\right)\frac{n-1}{2}-\frac{m}{2\mu}} \|f(\tau,\cdot)\|_{L^p(\mathbb{R}^n)}\,d\tau. \end{aligned}$$

Case 1: $\max\{p_1,1\}<p\le 2$. In this case, we have $1<r<2$. Note that

$$\frac{1}{r}-\frac{1}{r'}=-(m+2)\left(\frac{1}{p}-\frac{1}{2}\right)\frac{n-1}{2}-\frac{m}{2\mu}+1.$$

Then it follows from the Hardy–Littlewood–Sobolev theorem and (3-12) that estimate (3-2) holds.

Case 2: $p_1>1$ *and* $1<p<p_1$. In this case,

$$(m+2)\left(\frac{1}{p}-\frac{1}{2}\right)\frac{n-1}{2}+\frac{m}{2\mu}>1.$$

Thus,

$$\sup_{t>0}\int_0^\infty (\lambda_j^{-\frac{2}{m+2}}+|t-\tau|)^{-(m+2)\left(\frac{1}{p}-\frac{1}{2}\right)\frac{n-1}{2}-\frac{m}{2\mu}}\,d\tau<\infty,$$

which together with Schur's lemma and (3-12) yields (3-4). □

We would like to stress that in the proof of Theorem 3.1 only condition (2-19) on the function $b\in C^\infty(\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}^n)$ was used, whereas the conditions (2-20) and (2-21) were not required,

Remark 3.3. Note that the adjoint operator $(W_j^\alpha)^*$ of W_j^α is of the form

$$(3\text{-}13)\quad (W_j^\alpha)^* f(t,x)=\int_t^\infty\int_{\mathbb{R}^n} e^{i(x\cdot\xi+(\phi(t)-\phi(\tau))|\xi|)}\,\overline{b_j(\tau,t,\xi)}\,\hat{f}(\tau,\xi)\frac{d\!\!\!^{-}\xi}{|\xi|^\alpha}\,d\tau.$$

By duality, we infer from Theorem 3.1 that

$$\begin{aligned} (3\text{-}14)\quad &\|(W_j^\alpha)^* f\|_{L_t^{r'}L_x^{p'}(\mathbb{R}_+^{1+n})} \\ &\qquad\lesssim \lambda_j^{\left(\frac{1}{p}-\frac{1}{2}\right)(n+1)-\frac{m}{\mu(m+2)}-\frac{2}{m+2}-\operatorname{Re}\alpha}\|f\|_{L_t^rL_x^p(\mathbb{R}_+^{1+n})} \end{aligned}$$

if $\max\{p_1,1\}<p\le 2$ and

$$(3\text{-}15)\qquad \|(W_j^\alpha f)^*\|_{L_t^2L_x^{p'}(\mathbb{R}_+^{1+n})}\lesssim \lambda_j^{n\left(\frac{2}{p}-1\right)-\frac{4}{m+2}-\operatorname{Re}\alpha}\|f\|_{L_t^2L_x^p(\mathbb{R}_+^{1+n})}$$

if $p_1>1$ and $1<p<p_1$. Here, r is given in (3-1).

As an application of Theorem 3.1, we obtain the boundedness of the operator W_j^α from $L_t^rL_x^p$ to $L_t^\infty L_x^2$, where $1<r,p\le 2$.

Theorem 3.4. *Let $n \geq 2$ and $\mu \geq \max\{2, m/2\}$. Then*:

(i) *For* $\max\{p_1, 1\} < p \leq 2$ *and r as in* (3-1), *we have that*

$$\|W_j^\alpha f\|_{L_t^\infty L_x^2(\mathbb{R}_+^{1+n})} \lesssim \lambda_j^{(\frac{1}{p}-\frac{1}{2})\frac{n+1}{2}-\frac{m}{2\mu(m+2)}-\frac{2}{m+2}-\operatorname{Re}\alpha} \|f\|_{L_t^r L_x^p(\mathbb{R}_+^{1+n})}. \tag{3-16}$$

Consequently,

$$\|W_j^\alpha f\|_{L_t^\infty L_x^2(\mathbb{R}_+^{1+n})} \lesssim \|f\|_{L_t^r L_x^p(\mathbb{R}_+^{1+n})} \quad \text{if } \operatorname{Re}\alpha = \Big(\frac{1}{p}-\frac{1}{2}\Big)\frac{n+1}{2} - \frac{m}{2\mu(m+2)} - \frac{2}{m+2}. \tag{3-17}$$

(ii) *For $p_1 > 1$ and $1 < p < p_1$, we have that*

$$\|W_j^\alpha f\|_{L_t^\infty L_x^2(\mathbb{R}_+^{1+n})} \lesssim \lambda_j^{n\left(\frac{1}{p}-\frac{1}{2}\right)-\frac{3}{m+2}-\operatorname{Re}\alpha} \|f\|_{L_t^2 L_x^p(\mathbb{R}_+^{1+n})}. \tag{3-18}$$

In particular,

$$\|W_j^\alpha f\|_{L_t^\infty L_x^2(\mathbb{R}_+^{1+n})} \lesssim \|f\|_{L_t^2 L_x^p(\mathbb{R}_+^{1+n})} \quad \text{if } \operatorname{Re}\alpha = n\Big(\frac{1}{p}-\frac{1}{2}\Big) - \frac{3}{m+2}. \tag{3-19}$$

Proof. For given $j \in \mathbb{Z}$ and $\alpha \in \mathbb{C}$, denote $U = W_j^\alpha f$. Then from (2-22) we have

$$U(t) = \int_0^t e^{i(\phi(t)-\phi(\tau))\sqrt{-\Delta}} b_j(t,\tau,D_x)(-\Delta)^{-\alpha/2} f(\tau)\,d\tau,$$

where $b_j(t,\tau,D_x)$ is the pseudodifferential operator with full symbol $b_j(t,\tau,\xi)$. Then $U(t)$ solves the Cauchy problem

$$\begin{aligned} i\partial_t U(t) &= -t^{m/2}\sqrt{-\Delta}U(t) + i b_j(t,t,D_x)(-\Delta)^{-\alpha/2} f(t) \\ &\qquad + i\int_0^t e^{i(\phi(t)-\phi(\tau))\sqrt{-\Delta}} \partial_t b_j(t,\tau,D_x)(-\Delta)^{-\alpha/2} f(\tau)\,d\tau, \\ U(0) &= 0. \end{aligned}$$

Multiplying by $\overline{U(t)}$ and then integrating over $\mathbb{R}^n$ yields

$$\begin{aligned} &i\langle \partial_t U(t), U(t)\rangle \\ &\quad = -t^{m/2}\langle \sqrt{-\Delta}U(t), U(t)\rangle + i\langle b_j(t,t,D_x)(-\Delta)^{-\alpha/2} f(t), U(t)\rangle \\ &\qquad + i\Big\langle \int_0^t e^{i(\phi(t)-\phi(\tau))\sqrt{-\Delta}} \partial_t b_j(t,\tau,D_x)(-\Delta)^{-\alpha/2} f(\tau)\,d\tau, U(t)\Big\rangle, \end{aligned}$$

and, therefore,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|U(t)\|^2 &= \operatorname{Re}\left\langle \int_0^t e^{i(\phi(t)-\phi(\tau))\sqrt{-\Delta}}\partial_t b_j(t,\tau,D_x)(-\Delta)^{-\alpha/2} f(\tau)\,d\tau, U(t)\right\rangle \\ &\qquad + \operatorname{Re}\langle b_j^*(t,t,D_x)(-\Delta)^{-\alpha/2}U(t), f(t)\rangle.\end{aligned}$$

Consequently,

$$\begin{aligned}\|U(s)\|^2 &= 2\operatorname{Re}\int_0^s\left\langle \int_0^t e^{i(\phi(t)-\phi(\tau))\sqrt{-\Delta}}\partial_t b_j(t,\tau,D_x)(-\Delta)^{-\alpha/2} f(\tau)\,d\tau, U(t)\right\rangle dt \\ &\qquad + 2\operatorname{Re}\int_0^s \langle b_j^*(t,t,D_x)(-\Delta)^{-\alpha/2}U(t), f(t)\rangle\,dt \\ &\lesssim \left|\int_0^s\int_{\mathbb{R}^n} L_j^\alpha f(t,x)\overline{W_j^\alpha f(t,x)}\,dx\,dt\right| \\ &\qquad + \left|\int_0^s\int_{\mathbb{R}^n} b_j^*(t,t,D_x)W_j^{2\alpha} f(t,x)\overline{f(t,x)}\,dx\,dt\right| \\ &= \mathrm{I}+\mathrm{II},\end{aligned}$$

where

$$\mathrm{I} = \left|\int_0^s\int_{\mathbb{R}^n} L_j^\alpha f(t,x)\overline{W_j^\alpha f(t,x)}\,dx\,dt\right|,$$

$$\mathrm{II} = \left|\int_0^s\int_{\mathbb{R}^n} b_j^*(t,t,D_x)W_j^{2\alpha} f(t,x)\overline{f(t,x)}\,dx\,dt\right|,$$

and

$$L_j^\alpha f(t,x) = \int_0^t\int_{\mathbb{R}^n} e^{i(x\cdot\xi+(\phi(t)-\phi(\tau))|\xi|)}\partial_t b_j(t,\tau,\xi)\hat f(\tau,\xi)\,\frac{đ\xi}{|\xi|^\alpha}\,d\tau.$$

From (2-19), one has that, for any fixed $t>0$, $b_j(t,t,D_x)\in\Psi^{-2/(m+2)}(\mathbb{R}^n)$, and then $b_j^*(t,t,D_x)\in\Psi^{-2/(m+2)}(\mathbb{R}^n)$, which yields that the term II is essentially

$$\left|\int_0^s\int_{\mathbb{R}^n} (W_j^{2\alpha+2/(m+2)} f)(t,x)\overline{f(t,x)}\,dx\,dt\right|,$$

and thus by application of Theorem 3.1 it follows that

$$\text{(3-20)}\quad \mathrm{II} \lesssim \begin{cases} \lambda_j^{(n+1)\left(\frac{1}{p}-\frac{1}{2}\right)-\frac{m}{\mu(m+2)}-\frac{4}{m+2}-2\operatorname{Re}\alpha}\|f\|^2_{L_t^r L_x^p(\mathbb{R}_+^{1+n})} & \text{if } \max\{p_1,1\}<p\le 2, \\ \lambda_j^{n\left(\frac{2}{p}-1\right)-\frac{6}{m+2}-2\operatorname{Re}\alpha}\|f\|^2_{L_t^2 L_x^p(\mathbb{R}_+^{1+n})} & \text{if } 1<p<p_1. \end{cases}$$

As for the term I, note that

$$\mathrm{I} = \left| \int_0^s \int_{\mathbb{R}^n} (W_j^{\alpha})^* L_j^{\alpha} f(t,x) \overline{f(t,x)} \, dx \, dt \right| \le \|(W_j^{\alpha})^* L_j^{\alpha} f\|_{L_t^{\rho'} L_x^{p'}(\mathbb{R}_+^{1+n})} \|f\|_{L_t^{\rho} L_x^{p}(\mathbb{R}_+^{1+n})}.$$

For any $t > 0$, we have from (3-13) that

$$(W_j^{\alpha})^* L_j^{\alpha} f(t,x) = \int_t^{\infty} \int_0^{\tau} \int_{\mathbb{R}^n} e^{i(x\cdot\xi + (\phi(t)-\phi(s))|\xi|)} \times \overline{b_j(\tau,t,\xi)} \partial_\tau b_j(\tau,s,\xi) \hat{f}(s,\xi) \frac{đ\xi}{|\xi|^{2\alpha}} \, ds \, d\tau$$
$$= \int_0^t \int_{\mathbb{R}^n} e^{i(x\cdot\xi + (\phi(t)-\phi(s))|\xi|)} \times \left(\int_t^{\infty} \overline{b_j(\tau,t,\xi)} \partial_\tau b_j(\tau,s,\xi) \, d\tau \right) \hat{f}(s,\xi) \frac{đ\xi}{|\xi|^{2\alpha}} \, ds$$
$$+ \int_t^{\infty} \int_{\mathbb{R}^n} e^{i(x\cdot\xi + (\phi(t)-\phi(s))|\xi|)} \times \left(\int_s^{\infty} \overline{b_j(\tau,t,\xi)} \partial_\tau b_j(\tau,s,\xi) \, d\tau \right) \hat{f}(s,\xi) \frac{đ\xi}{|\xi|^{2\alpha}} \, ds. \tag{3-21}$$

Due to conditions (2-19)–(2-21), one has that the first and second term in (3-21) are essentially $W_j^{2\alpha+2/(m+2)} f$ and $(W_j^{2\alpha+2/(m+2)})^* f$, respectively, where $b \in C^{\infty}(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^n)$ satisfies condition (2-19). Then, by applying Theorem 3.1 and estimates (3-14) and (3-15), we have that

$$\mathrm{I} \lesssim \begin{cases} \lambda_j^{(n+1)\left(\frac{1}{p}-\frac{1}{2}\right) - \frac{m}{\mu(m+2)} - \frac{4}{m+2} - 2\operatorname{Re}\alpha} \|f\|^2_{L_t^r L_x^p(\mathbb{R}_+^{1+n})} & \text{if } \max\{p_1, 1\} < p \le 2, \\ \lambda_j^{n\left(\frac{2}{p}-1\right) - \frac{6}{m+2} - 2\operatorname{Re}\alpha} \|f\|^2_{L_t^2 L_x^p(\mathbb{R}_+^{1+n})} & \text{if } p_1 > 1 \text{ and } 1 < p < p_1, \end{cases}$$

which together with (3-20) yields that

$$\|U(t)\|^2 \lesssim \begin{cases} \lambda_j^{(n+1)\left(\frac{1}{p}-\frac{1}{2}\right) - \frac{m}{\mu(m+2)} - \frac{4}{m+2} - 2\operatorname{Re}\alpha} \|f\|^2_{L_t^r L_x^p(\mathbb{R}_+^{1+n})} & \text{if } \max\{p_1, 1\} < p \le 2, \\ \lambda_j^{n\left(\frac{2}{p}-1\right) - \frac{6}{m+2} - 2\operatorname{Re}\alpha} \|f\|^2_{L_t^2 L_x^p(\mathbb{R}_+^{1+n})} & \text{if } p_1 > 1 \text{ and } 1 < p < p_1. \end{cases}$$

Note that $\|W_j^\alpha f(t,\cdot)\|_{L^2(\mathbb{R}^n)} = \|U(t)\|$. Therefore, we have obtained estimates (3-16)–(3-19), which completes the proof of Theorem 3.4. □

Remark 3.5. With similar arguments as in the proof of Theorem 3.4, we have from Theorem 3.1 and estimates (3-14) and (3-15) that the operator $(W_j^\alpha)^*$ also satisfies the estimates (3-16)–(3-19).

Note that if $r = p$ for r defined in (3-1), then $r = p = p_0$. Combining Theorem 3.1 and the kernel estimate (3-10), we obtain boundedness of the operator W_j^α from $L^{p_0}(\mathbb{R}_+^{1+n})$ to $L^q(\mathbb{R}_+^{1+n})$ for certain $\alpha \in \mathbb{C}$ when $q_0 \le q \le \infty$.

Theorem 3.6. *Let* $\mu \ge \max\{2, m/2\}$ *and* $q_0 \le q \le \infty$. *Then*

$$\|W_j^\alpha f\|_{L^q(\mathbb{R}_+^{1+n})} \lesssim \|f\|_{L^{p_0}(\mathbb{R}_+^{1+n})}, \tag{3-22}$$

where

$$\operatorname{Re}\alpha = n - \frac{2}{m+2} - \Big(n + \frac{2}{m+2}\Big)\Big(\frac{1}{q} + \frac{1}{q_0}\Big).$$

Proof. Case (i): $q = q_0$. Note that

$$n - \frac{2}{q_0}\Big(n + \frac{2}{m+2}\Big) = \Big(\frac{1}{p_0} - \frac{1}{2}\Big)(n+1) - \frac{m}{\mu(m+2)}.$$

An application of (3-3) with $r = p$ yields that

$$\|W_j^\alpha f\|_{L^{q_0}(\mathbb{R}_+^{1+n})} \lesssim \|f\|_{L^{p_0}(\mathbb{R}_+^{1+n})}, \quad \operatorname{Re}\alpha = n - \frac{2}{m+2} - \frac{2}{q_0}\Big(n + \frac{2}{m+2}\Big). \tag{3-23}$$

Case (ii): $q = \infty$. In order to derive (3-22), it suffices to show that the integral kernel K_j^α defined in (3-7) satisfies

$$\begin{gathered} \sup_{(t,x)\in\mathbb{R}_+^{1+n}} \int_{\mathbb{R}_+^{1+n}} |K_j^\alpha(t,\tau,x,y)|^{q_0}\, d\tau\, dy < \infty, \\ \operatorname{Re}\alpha = n - \frac{2}{m+2} - \frac{1}{q_0}\Big(n + \frac{2}{m+2}\Big). \end{gathered} \tag{3-24}$$

In fact, from (3-7) we have

$$W_j^\alpha f(t,x) = \int_0^t \int_{\mathbb{R}^n} K_j^\alpha(t,\tau,x,y) f(\tau,y)\, dy\, d\tau.$$

By Hölder's inequality, then

$$\|W_j^\alpha f\|_{L^\infty(\mathbb{R}_+^{1+n})} \lesssim \|f\|_{L^{p_0}(\mathbb{R}_+^{1+n})}, \quad \operatorname{Re}\alpha = n - \frac{2}{m+2} - \frac{1}{q_0}\Big(n + \frac{2}{m+2}\Big). \tag{3-25}$$

Now it remains to derive estimate (3-24). In fact, due to the kernel estimate (3-10), for any $N > n$ and $\alpha \in \mathbb{C}$ with $\operatorname{Re}\alpha = n - 2/(m+2) - 1/q_0(n + 2/(m+2))$, we have by (3-10)

$$
\begin{aligned}
\int_{\mathbb{R}^{1+n}_+} |K^\alpha_j(t,\tau,x,y)|^{q_0}\,d\tau\,dy
&\lesssim \lambda_j^{\left(\frac{n+1}{2}-\operatorname{Re}\alpha-\frac{m}{\mu(m+2)}-\frac{2}{m+2}\right)q_0} \\
&\qquad\times\int_0^\infty (\lambda_j^{-\frac{2}{m+2}}+|t-\tau|)^{-\left(\frac{(m+2)(n-1)}{4}+\frac{m}{2\mu}\right)q_0}\,d\tau \\
&\qquad\times\int_{\mathbb{R}^n}(1+\lambda_j\big||x-y|-|\phi(t)-\phi(\tau)|\big|)^{-N}\,dy \\
&\lesssim \lambda_j^{\left(\frac{n+1}{2}-\operatorname{Re}\alpha-\frac{m}{\mu(m+2)}-\frac{2}{m+2}\right)q_0} \\
&\qquad\times\int_0^\infty (\lambda_j^{-\frac{2}{m+2}}+|t-\tau|)^{-\left(\frac{(m+2)(n-1)}{4}+\frac{m}{2\mu}\right)q_0}\,d\tau \\
&\qquad\times\lambda_j^{-1}\int_0^\infty (1+r)^{-N}(\lambda_j^{-1}r+|\phi(t)-\phi(\tau)|)^{n-1}\,dr \\
&= \lambda_j^{\left(\frac{n+1}{2}-\operatorname{Re}\alpha-\frac{m}{\mu(m+2)}-\frac{2}{m+2}\right)q_0-1} \\
&\qquad\times\int_0^\infty (\lambda_j^{-\frac{2}{m+2}}+|t-\tau|)^{-\left(\frac{(m+2)(n-1)}{4}+\frac{m}{2\mu}\right)q_0} \\
&\qquad\qquad\qquad(\lambda_j^{-1}+|\phi(t)-\phi(\tau)|)^{n-1}\,d\tau \\
&\qquad\times\int_0^\infty (1+r)^{-N}\Big(\frac{r+\lambda_j|\phi(t)-\phi(\tau)|}{1+\lambda_j|\phi(t)-\phi(\tau)|}\Big)^{n-1}\,dr \\
&\lesssim \lambda_j^{\left(\frac{n+1}{2}-\operatorname{Re}\alpha-\frac{m}{\mu(m+2)}-\frac{2}{m+2}\right)q_0-1} \\
&\qquad\times\int_0^\infty (\lambda_j^{-\frac{2}{m+2}}+|t-\tau|)^{-\left(\frac{(m+2)(n-1)}{4}+\frac{m}{2\mu}\right)q_0+\frac{(m+2)(n-1)}{2}}\,d\tau \\
&\lesssim \lambda_j^{\left(n-\operatorname{Re}\alpha-\frac{2}{m+2}\right)q_0-n-\frac{2}{m+2}}=1,
\end{aligned}
$$

and hence (3-24) holds.

Case (iii): $q_0<q<\infty$. Applying Stein's interpolation theorem, one obtains that estimate (3-22) holds by interpolating between estimates (3-23) and (3-25). □

Now we consider boundedness of the operator W_j from $L^r_tL^p_x(S_T)$ to $L^s_tL^q_x(S_T)$, where $1/p$ is symmetric around $1/p_0$.

Theorem 3.7. *Let $n\ge 2$. Further let $p_1<p<p_2$ if $n=2$, $m\ge 2$, or if $n\ge 3$, and $1<p<7\mu/(4\mu-2)$ if $n=2$, $m=1$. Then, for any $\mu\ge\mu_*$ and $T>0$,*

$$\|W_jf\|_{L^s_tL^q_x(S_T)}\lesssim\|f\|_{L^r_tL^p_x(S_T)}, \tag{3-26}$$

where r is defined as in (3-1) and

$$\frac{1}{q}=\frac{1}{p}-\frac{4}{(m+2)(n+1)}\Big(1+\frac{m}{2\mu}\Big),$$
$$\frac{1}{s}=\frac{(m+2)(n-1)}{4}\Big(\frac{1}{2}-\frac{1}{q}\Big)+\frac{m}{4\mu}. \tag{3-27}$$

Proof. Since $1/p$ is symmetric around $1/p_0$, by duality it suffices to consider the case $\max\{p_1,1\}<p\le p_0$.

In order to derive (3-26), we now need a further dyadic decomposition with respect to the time variable t. Choose a function $\eta\in C_c^\infty(\mathbb{R}_+)$ such that $0\le\eta\le 1$, $\operatorname{supp}\eta\subseteq\big[\frac{1}{2},2\big]$, and

$$\sum_{\ell=-\infty}^{\infty}\eta(2^{-\ell}t)=1.$$

Let us fix $\lambda=2^j$ and set

$$\eta_0(t)=\sum_{k\le 0}\eta(\lambda 2^{-k}t),\quad \eta_\ell(t)=\eta(\lambda 2^{-\ell}t)\quad\text{for } \ell\in\mathbb{N}.$$

Then,

$$W_jf(t,x)=\sum_{k=0}^{\infty}G_kf(t,x),$$

where

$$G_kf(t,x)=\int_0^t\int_{\mathbb{R}^n}e^{i(x\cdot\xi+(\phi(t)-\phi(\tau))|\xi|)}\eta_k(t-\tau)\,b_j(t,\tau,\xi)\,\hat{f}(\tau,\xi)\,đ\xi\,d\tau. \tag{3-28}$$

Hence, to derive (3-26), it suffices to show that, for any $k\in\mathbb{N}_0$,

$$\|G_kf\|_{L_t^sL_x^q(S_T)}\lesssim 2^{-\varepsilon_pk}\|f\|_{L_t^rL_x^p(S_T)} \tag{3-29}$$

for some $\varepsilon_p>0$. From (3-1) and (3-27), we know that

$$\frac{(m+2)n}{2}\Big(\frac{1}{p}-\frac{1}{q}\Big)+\frac{1}{r}-\frac{1}{s}=2.$$

Due to scaling invariance, we need to consider only the case $\lambda=1$ (by a change of variable if $\lambda\ne 1$). Repeating the arguments which are used to prove (3-2), we get that, for any $k\in\mathbb{N}_0$,

$$\|G_kf\|_{L_t^{r'}L_x^{p'}(S_T)}\lesssim 2^{-k((m+2)(1/p-1/2)(n-1)/2+m/(2\mu))}\|f\|_{L_t^rL_x^p(S_T)}. \tag{3-30}$$

Note that $(m+2)\big(1/p-\frac{1}{2}\big)\frac{1}{2}(n-1)+m/(2\mu)>\frac{1}{3}$, since $p\le p_0$.

Furthermore, an immediate consequence of (3-16) for $\alpha = 0$ is

$$\|G_k f\|_{L_t^\infty L_x^2(S_T)} \lesssim \|f\|_{L_t^r L_x^p(S_T)},$$

and thus, for any $1 < \rho < \infty$,

$$\|G_k f\|_{L_t^\rho L_x^2(S_T)} \lesssim \|f\|_{L_t^r L_x^p(S_T)}. \tag{3-31}$$

Choose

$$\theta = \frac{4p(2\mu+m)}{\mu(m+2)(n+1)(2-p)} - 1. \tag{3-32}$$

Then $0 \le \theta \le 1$ and, for the number q from (3-27),

$$\frac{1}{q} = \frac{\theta}{p'} + \frac{1-\theta}{2}.$$

For s from (3-27) and θ from (3-32), we define s_0 by

$$2\Big(\frac{1}{s} - \frac{1}{s_0}\Big) = \theta\Big((m+2)\Big(\frac{1}{p} - \frac{1}{2}\Big)\frac{n-1}{2} + \frac{m}{2\mu}\Big)$$

and then set $\rho = \rho_*$ such that

$$\frac{1}{s_0} = \frac{\theta}{r'} + \frac{1-\theta}{\rho_*}.$$

Since $2 < s < s_0$, by interpolating between (3-30) and (3-31) when $\rho = \rho_*$, we obtain that

$$\|G_k f\|_{L_t^{s_0} L_x^q(S_T)} \lesssim 2^{-2k(1/s-1/s_0)} \|f\|_{L_t^r L_x^p(S_T)}. \tag{3-33}$$

Let $\{I_\ell\}$ be nonoverlapping intervals of side length 2^k and $\bigcup_\ell I_\ell = \mathbb{R}_+$, and denote by χ_I the characteristic function of I. In view of (3-28) and the compact support of η_k, we have that if $f(t,x) = 0$ for $t \notin I_\ell$, then $G_k f(t,x) = 0$ for $t \notin I_\ell^*$, where I_ℓ^* is the interval with the same center as I_ℓ but of side length $C_0 2^k$ with some constant $C_0 = C_0(\eta) > 0$. Thus, from Minkowski's inequality,

$$\begin{aligned} \|G_k f(t,\cdot)\|^s_{L^q(\mathbb{R}^n)} &\le \Big(\sum_\ell \|G_k(\chi_{I_\ell} f)(t,\cdot)\|_{L^q(\mathbb{R}^n)}\Big)^s \\ &\lesssim \sum_\ell \|G_k(\chi_{I_\ell} f)(t,\cdot)\|^s_{L^q(\mathbb{R}^n)}. \end{aligned} \tag{3-34}$$

Denote $\overline{I_\ell^*} = I_\ell^* \cap (0,T)$. Estimate (3-34) together with Hölder's inequality and (3-33) yields that, for any $k \in \mathbb{N}_0$,

$$\begin{aligned}\|G_k f\|^s_{L^s_t L^q_x(S_T)} &\lesssim \sum_\ell \|G_k(\chi_{I_\ell} f)\|^s_{L^s_t L^q_x(\overline{I_\ell^*}\times\mathbb{R}^n)} \\ &\lesssim \sum_\ell |\overline{I_\ell^*}|^{1-s/s_0} \|G_k(\chi_{I_\ell} f)\|^s_{L^{s_0}_t L^q_x(\overline{I_\ell^*}\times\mathbb{R}^n)} \\ &\lesssim 2^{k(1-s/s_0)} 2^{-2ks(1/s-1/s_0)} \sum_\ell \|\chi_{I_\ell} f\|^s_{L^r_t L^p_x(S_T)} \\ &\lesssim 2^{-k(1-s/s_0)} \|f\|_{L^r_t L^p_x(S_T)}.\end{aligned}$$

Therefore, we get estimate (3-29) with $\varepsilon_p = 1-s/s_0$ and, hence, (3-26) holds. □

By a similar argument as in the proof of Theorem 3.7, we obtain the boundedness of operator W_j from $L^2_t L^p_x(S_T)$ to $L^s_t L^q_x(S_T)$ when $p_1 > 1$ and $1 < p < p_1$.

Theorem 3.8. *Let $n \ge 3$ or $n = 2$, $m \ge 2$. Suppose $1 < p < p_1$. Then, for $\mu \ge \max\{2, mn/2\}$ and $T > 0$, we have that*

$$\|W_j f\|_{L^s_t L^q_x(S_T)} \lesssim \|f\|_{L^2_t L^p_x(S_T)}, \tag{3-35}$$

where

$$\begin{aligned}\frac{1}{q} &= \frac{2n}{p(n+1)} - \frac{n-1}{2(n+1)} - \frac{m+6\mu}{\mu(m+2)(n+1)}, \\ \frac{1}{s} &= (m+2)\Big(\frac{1}{2}-\frac{1}{q}\Big)\Big(\frac{n-1}{4}\Big) + \frac{m}{4\mu}.\end{aligned} \tag{3-36}$$

Proof. Note that when $1 < p < p_1$, we have

$$(m+2)\Big(\frac{1}{p}-\frac{1}{2}\Big)\Big(\frac{n-1}{2}\Big) + \frac{m}{2\mu} > 1.$$

Then we can apply similar arguments as in the proof of Theorem 3.7 to obtain (3-35). We omit the details. □

Remark 3.9. By similar arguments as above one can show that under assumptions (3-27) and (3-36), adjoints $(W_j)^*$ of W_j also satisfy estimates (3-26) and (3-35), respectively.

4. Mixed-norm estimates for the linear generalized Tricomi equation

In this section, based on the mixed-norm space-time estimates of the Fourier integral operators W_j^α obtained in Section 3, we shall establish Strichartz-type estimates for the linear generalized Tricomi equation.

First we consider the inhomogeneous equation with zero initial data, i.e., problem (2-3).

Theorem 4.1. *Let $n \geq 2$. Suppose w is a solution of (2-3) in S_T for some $T > 0$. Then*:

(i) *For $\mu \geq \mu_*$,*

$$\|w\|_{L_t^s L_x^q(S_T)} \lesssim \|f\|_{L_t^r L_x^p(S_T)}, \tag{4-1}$$

provided that $p_1 < p < p_2$ if $n \geq 3$ or $n = 2$, $m \geq 2$; and $1 < p < 7\mu/(4\mu - 2)$ if $n = 2$ and $m = 1$. Here $r = r(p, \mu)$ is as in (3-1) and q and s are taken from (3-27).

(ii) *For $\mu \geq \max\{2, m/2\}$,*

$$\|w\|_{L^q(S_T)} \lesssim \||D_x|^{\gamma-\gamma_0} f\|_{L^{p_0}(S_T)}, \quad q_0 \leq q < \infty, \tag{4-2}$$

where

$$\begin{aligned} \gamma = \gamma(m, n, q) &= \frac{n}{2} - \frac{1}{q}\Big(n + \frac{2}{m+2}\Big), \\ \gamma_0 = \gamma_0(m, n, \mu) &= \frac{1}{q_0}\Big(n + \frac{2}{m+2}\Big) + \frac{2}{m+2} - \frac{n}{2}. \end{aligned} \tag{4-3}$$

(iii) *For $\mu \geq \max\{2, m/2\}$, $\max\{p_1, 1\} < p \leq 2$, and $0 < t \leq T$,*

$$\|w(t, \cdot)\|_{\dot{H}^\gamma(\mathbb{R}^n)} \lesssim \|f\|_{L_t^r L_x^p(S_T)}, \tag{4-4}$$

where $r = r(m, n, p, \mu)$ is defined in (3-1) and

$$\gamma = \gamma(m, n, \mu, p) = \frac{2}{m+2} + \frac{m}{2\mu(m+2)} - \Big(\frac{1}{p} - \frac{1}{2}\Big)\frac{n+1}{2}.$$

(iv) *For $\mu \geq \max\{2, m/2\}$, $\gamma \in \mathbb{R}$, and $0 \leq t \leq T$,*

$$\|w(t, \cdot)\|_{\dot{H}^\gamma(\mathbb{R}^n)} \lesssim \big\||D_x|^{\gamma-\gamma_0} f\big\|_{L^{p_0}(S_T)}, \tag{4-5}$$

where γ_0 is from (4-3).

Remark 4.2. If we choose $\mu = \mu_*$, then

$$p_0 = p_0^* = \frac{2\mu_*}{\mu_* + 2}, \quad q_0 = q_0^* = \frac{2\mu_*}{\mu_* - 2},$$

and for γ and γ_0 defined in (4-3),

$$\gamma(m, n, q_0^*) = \gamma_0(m, n, \mu_*) = \frac{1}{m+2}.$$

Thus, we have from (4-2) that

$$\|w\|_{L^{q_0^*}(S_T)} \lesssim \|f\|_{L^{p_0^*}(S_T)},$$

which, for any $\rho \in \mathbb{R}$, together with $[|D_x|^\rho, \partial_t^2 - t^m \Delta] = 0$ implies that

$$\big\| |D_x|^\rho w \big\|_{L^{q_0^*}(S_T)} \lesssim \big\| |D_x|^\rho f \big\|_{L^{p_0^*}(S_T)}.$$

Proof of Theorem 4.1. (i): One obtains (4-1) by applying Proposition 2.1 and Theorem 3.7 directly.

(ii): For $\alpha \in \mathbb{C}$, the Fourier transform of $|D_x|^\alpha f(t,x)$ with respect to the variable x is $|\xi|^\alpha \hat{f}(t,\xi)$. Thus, we can write $W_j f$ as

$$W_j f(t,x) \\ = \int_0^t \int_{\mathbb{R}^n} e^{i(x\cdot\xi + (\phi(t)-\phi(\tau))|\xi|)} \Theta(|\xi|/2^j)\, b(t,\tau,\xi) (\widehat{|D_x|^\alpha f})(\tau,\xi) |\xi|^{-\alpha}\, đ\xi\, d\tau$$

and $W_j(f) = W_j^\alpha(|D_x|^\alpha f)$.

Therefore, applying Theorem 3.6, we get that

$$\|W_j f\|_{L^q(S_T)} = \big\| W_j^{\gamma-\gamma_0}(|D_x|^{\gamma-\gamma_0} f) \big\|_{L^q(S_T)} \lesssim \big\| |D_x|^{\gamma-\gamma_0} f \big\|_{L^{p_0}(S_T)},$$

which together with Proposition 2.1 yields (4-2).

(iii): Note that $[|D_x|^\gamma, \partial_t^2 - t^m \Delta] = 0$ and then

$$(\partial_t^2 - t^m \Delta)(|D_x|^\gamma w) = |D_x|^\gamma f. \tag{4-6}$$

From (ii) we know that $W_j(|D_x|^\gamma f) = W_j^{-\gamma}(f)$. Thus, for $\gamma = 2/(m+2) + m/(2\mu(m+2)) - (1/p - 1/2)(n+1)/2$, we have from estimate (3-17) that

$$\big\| W_j(|D_x|^\gamma f)(t,\cdot) \big\|_{L^2(\mathbb{R}^n)} = \|W_j^{-\gamma} f(t,\cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \|f\|_{L_t^r L_x^p}.$$

Thus, by (4-6) and Proposition 2.1 it follows that

$$\big\| (|D_x|^\gamma w)(t,\cdot) \big\|_{L^2(\mathbb{R}^n)} \lesssim \|f\|_{L_t^r L_x^p},$$

which together with Plancherel's theorem implies that

$$\|w(t,\cdot)\|_{\dot{H}^\gamma(\mathbb{R}^n)} = \big\| |\xi|^\gamma \hat{w}(t,\xi) \big\|_{L_\xi^2(\mathbb{R}^n)} = \big\| (|D_x|^\gamma w)(t,\cdot) \big\|_{L_x^2(\mathbb{R}^n)} \lesssim \|f\|_{L_t^r L_x^p},$$

and estimate (4-4) holds.

(iv): From (ii) we also know that

$$W_j(g) = W_j^{-\gamma_0}(|D_x|^{-\gamma_0} g).$$

In (3-1), we have $r = p = p_0$ when $r = p$. The estimate (3-17) for

$$\alpha = -\gamma_0 = \Big(\frac{1}{p_0} - \frac{1}{2}\Big)\frac{n+1}{2} - \frac{m}{2\mu(m+2)} - \frac{2}{m+2}$$

with $p = p_0$ yields that

$$\|W_j(g)(t,\cdot)\|_{L^2(\mathbb{R}^n)} = \left\|W_j^{-\gamma_0}(|D_x|^{-\gamma_0} g)(t,\cdot)\right\|_{L^2(\mathbb{R}^n)} \lesssim \left\||D_x|^{-\gamma_0} g\right\|_{L^{p_0}(S_T)},$$

and then, for $g = |D_x|^\gamma f$, where $\gamma \in \mathbb{R}$,

$$\left\|W_j(|D_x|^\gamma f)(t,\cdot)\right\|_{L^2(\mathbb{R}^n)} \lesssim \left\||D_x|^{\gamma-\gamma_0} f\right\|_{L^{p_0}(S_T)}. \tag{4-7}$$

Therefore, one has from Plancherel's theorem, Proposition 2.1, (4-6), and (4-7) that

$$\|w(t,\cdot)\|_{\dot{H}^\gamma(\mathbb{R}^n)} = \left\|(|D_x|^\gamma w)(t,\cdot)\right\|_{L^2(\mathbb{R}^n)} \lesssim \left\||D_x|^{\gamma-\gamma_0} f\right\|_{L^{p_0}(S_T)}$$

Hence, estimate (4-5) holds. □

In case $n \geq 2$ and $m \geq 2$ if $n = 2$, we have a more complete set of inequalities for the solution of the linear generalized Tricomi equation.

Theorem 4.3. *Let $n \geq 3$ or $n = 2$ with $m \geq 2$. Suppose w solves (2-3) in S_T. Then*:

(i) *For $\mu \geq \max\{2, mn/2\}$ and $1/p_1 < 1/p \leq \frac{1}{2} + (m+6\mu)/(2\mu n(m+2))$,*

$$\|w\|_{L_t^s L_x^q(S_T)} \lesssim \|f\|_{L_t^2 L_x^p(S_T)}, \tag{4-8}$$

where q and s are defined in (3-36).

(ii) *For $\mu \geq \max\{2, mn/2\}$ and $\frac{1}{2} \leq 1/p < \frac{1}{2} + (2\mu(n-3)+m(3n-1))/(\mu(m+2)(n^2-1))$,*

$$\|w\|_{L_t^2 L_x^q(S_T)} \lesssim \|f\|_{L_t^r L_x^p(S_T)}, \tag{4-9}$$

where r is defined in (3-1) and

$$\frac{1}{q} = \frac{n+1}{2np} + \frac{n-1}{4n} - \frac{m+6\mu}{2\mu(m+2)n}. \tag{4-10}$$

(iii) *For $\mu \geq \max\{2, m/2\}$ and $1 < p < p_1$ and $\gamma = 3/(m+2) - n\big(1/p - \frac{1}{2}\big)$,*

$$\|w(t,\cdot)\|_{\dot{H}^\gamma(\mathbb{R}^n)} \lesssim \|f\|_{L_t^2 L_x^p(S_T)}. \tag{4-11}$$

Proof. (i) Note that, under these assumptions,

$$1 < \frac{2\mu n(m+2)}{\mu n(m+2)+6\mu+m} \leq p < p_1, \quad 2 \leq q < \infty, \quad 2 \leq s < \infty.$$

Thus, we get estimate (4-8) by applying Proposition 2.1 and Theorem 3.8.

(ii): This will follow from the dual version of Theorem 3.8. Indeed, when

$$\frac{1}{2} \leq \frac{1}{p} < \frac{1}{2} + \frac{2\mu(n-3)+m(3n-1)}{\mu(m+2)(n^2-1)},$$

then, for q defined in (4-10),

$$1 < \frac{2\mu(m+2)n}{\mu(m+2)n+6\mu+m} \le q' < p_1$$

and

$$\frac{1}{p'} = \frac{2n}{q'(n+1)} - \frac{n-1}{2(n+1)} - \frac{m+6\mu}{\mu(m+2)(n+1)}.$$

For r defined by (3-1), the conjugate exponent r' can be expressed by

$$r' = \frac{8\mu p'}{\mu(m+2)(n-1)(p'-2)+2mp'}.$$

Thus, from Remark 3.9, we have that

$$\|W_j^* f\|_{L_t^{r'} L_x^{p'}(S_T)} \lesssim \|f\|_{L_t^2 L_x^{q'}(S_T)},$$

and then, by duality,

$$\|W_j f\|_{L_t^2 L_x^q(S_T)} \lesssim \|f\|_{L_t^r L_x^p(S_T)}.$$

Therefore, from Proposition 2.1 we have that estimate (4-9) holds.

(iii): Note again that $W_j(|D_x|^\gamma f) = W_j^{-\gamma}(f)$. Then, in view of (4-6) and estimate (3-19) for $\alpha = -\gamma = n\big(1/p - \frac{1}{2}\big) - 3/(m+2)$, one has that estimate (4-11) holds. □

Now we consider the Cauchy problem (2-2).

Theorem 4.4. *Let $n \ge 2$ and $\mu \ge \max\{2, m/2\}$. Suppose v solves the Cauchy problem (2-2). Then:*

(i) *For $q_0 \le q < \infty$,*

$$\|v\|_{L^q(\mathbb{R}_+^{1+n})} \lesssim \|\varphi\|_{\dot H^\gamma(\mathbb{R}^n)} + \|\psi\|_{\dot H^{\gamma-2/(m+2)}(\mathbb{R}^n)}, \tag{4-12}$$

where $\gamma = n/2 - ((m+2)n+2)/(q(m+2))$.

(ii) *For $2 \le q < \infty$ when $n = 2$ and $m = 1$, and $2 \le q < q_1$ when $n \ge 2$ and $m \ge 2$ if $n = 2$,*

$$\|v\|_{L_t^s L_x^q(\mathbb{R}_+^{1+n})} \lesssim \|\varphi\|_{\dot H^\gamma(\mathbb{R}^n)} + \|\psi\|_{\dot H^{\gamma-2/(m+2)}(\mathbb{R}^n)}, \tag{4-13}$$

where

$$\frac{1}{s} = \frac{(m+2)(n-1)}{4}\Big(\frac{1}{2} - \frac{1}{q}\Big) + \frac{m}{4\mu}, \quad \gamma = \frac{n+1}{2}\Big(\frac{1}{2} - \frac{1}{q}\Big) - \frac{m}{2\mu(m+2)}.$$

(iii) *For $q_1 < q < \infty$ as well as $n \ge 2$ and $m \ge 2$ if $n = 2$,*

$$\|v\|_{L_t^2 L_x^q(\mathbb{R}_+^{1+n})} \lesssim \|\varphi\|_{\dot H^\gamma(\mathbb{R}^n)} + \|\psi\|_{\dot H^{\gamma-2/(m+2)}(\mathbb{R}^n)}, \tag{4-14}$$

where $\gamma = n\big(\frac{1}{2} - 1/q\big) - 1/(m+2)$.

Proof. The goal is to prove that

$$\|v\|_{L_t^\sigma L_x^\rho(\mathbb{R}_+^{1+n})} \lesssim \|\varphi\|_{\dot{H}^\gamma(\mathbb{R}^n)} + \|\psi\|_{\dot{H}^{\gamma-2/(m+2)}(\mathbb{R}^n)} \tag{4-15}$$

for certain $2 \le \sigma \le \infty$ and $2 \le \rho < \infty$.

Note that

$$\begin{aligned} t(1+\phi(t)|\xi|)^{-\frac{m+4}{2(m+2)}} &\le (1+\phi(t)|\xi|)^{-\frac{m}{2(m+2)}}|\xi|^{-\frac{2}{m+2}} \\ &\le (1+\phi(t)|\xi|)^{-\frac{m}{\mu(m+2)}}|\xi|^{-\frac{2}{m+2}}. \end{aligned}$$

In order to establish (4-15), from the expression of the function v in (4-22) together with (2-9) and (2-10) and the estimates of $b_\ell(t,\xi)(1 \le \ell \le 4)$ in (2-13) and (2-14), it suffices to show that

$$\|P\varphi\|_{L_t^\sigma L_x^\rho(\mathbb{R}_+^{1+n})} \lesssim \|\varphi\|_{\dot{H}^\gamma(\mathbb{R}^n)}, \tag{4-16}$$

where the operator P is of the form

$$(P\varphi)(t,x) = \int_{\mathbb{R}^n} e^{i(x\cdot\xi+\phi(t)|\xi|)} a(t,\xi)\hat{\varphi}(\xi)\, đ\xi$$

with $a \in C^\infty(\mathbb{R}_+ \times \mathbb{R}^n)$ and, for any $(t,\xi) \in \mathbb{R}_+^{1+n}$,

$$|\partial_\xi^\beta a(t,\xi)| \lesssim (1+\phi(t)|\xi|)^{-m/(\mu(m+2))}|\xi|^{-|\beta|}. \tag{4-17}$$

Note that $P\varphi$ can be written as

$$(P\varphi)(t,x) = \int_{\mathbb{R}^n} e^{i(x\cdot\xi+\phi(t)|\xi|)} a(t,\xi)\widehat{|D_x|^\gamma\varphi}(\xi)\,\frac{đ\xi}{|\xi|^\gamma},$$

and, for $h = |D_x|^\gamma\varphi$, by Plancherel's theorem,

$$\|h\|_{L^2(\mathbb{R}^n)} = \||\xi|^\gamma\hat{\varphi}\|_{L^2(\mathbb{R}^n)} = \|\varphi\|_{\dot{H}^\gamma(\mathbb{R}^n)}.$$

Therefore, in order to prove (4-16), it suffices to show that the operator Q, where

$$(Qh)(t,x) = \int_{\mathbb{R}^n} e^{i(x\cdot\xi+\phi(t)|\xi|)} a(t,\xi)\hat{h}(\xi)\,\frac{đ\xi}{|\xi|^\gamma}, \tag{4-18}$$

is bounded from $L^2(\mathbb{R}^n)$ to $L_t^\sigma L_x^\rho(\mathbb{R}_+^{1+n})$. By duality, it suffices to show that the adjoint Q^* of Q,

$$(Q^*f)(x) = \int_0^\infty \int_{\mathbb{R}^n} e^{i(x\cdot\xi-\phi(\tau)|\xi|)}\overline{a(\tau,\xi)}|\xi|^{-\gamma}\hat{f}(\tau,\xi)\, đ\xi\, d\tau, \tag{4-19}$$

satisfies

$$\|Q^*f\|_{L^2(\mathbb{R}^n)} \lesssim \|f\|_{L_t^{\sigma'} L_x^{\rho'}(\mathbb{R}_+^{1+n})}. \tag{4-20}$$

Note that

$$\|Q^* f\|^2_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} (Q^* f)(x)\overline{(Q^* f)(x)}\,dx$$
$$= \int_{\mathbb{R}^{1+n}_+} QQ^* f(t,x)\overline{f(t,x)}\,dt\,dx \le \|QQ^* f\|_{L^\sigma_t L^\rho_x} \|f\|_{L^{\sigma'}_t L^{\rho'}_x}.$$

Thus, in order to get (4-20), we only need to show that

$$\|QQ^* f\|_{L^\sigma_t L^\rho_x} \lesssim \|f\|_{L^{\sigma'}_t L^{\rho'}_x}. \tag{4-21}$$

From (4-18) and (4-19), we have that

$$QQ^* f(t,x) = \int_0^\infty \int_{\mathbb{R}^n} e^{i(x\cdot\xi + (\phi(t)-\phi(\tau))|\xi|)} a(t,\xi)\overline{a(\tau,\xi)}\hat{f}(\tau,\xi)\,\frac{đ\xi}{|\xi|^{2\gamma}}\,d\tau.$$

By (4-17), we further have that

$$\left|\partial^\beta_\xi (a(t,\xi)\overline{a(\tau,\xi)})\right| \lesssim (1 + |\phi(t)-\phi(\tau)||\xi|)^{-\frac{m}{\mu(m+2)}} |\xi|^{-|\beta|}.$$

Thus, by Proposition 2.1, in order to get (4-21), it suffices to show that

$$\|G_j f\|_{L^\sigma_t L^\rho_x} \lesssim \|f\|_{L^{\sigma'}_t L^{\rho'}_x},$$

where the operator G_j is defined as

$$G_j f(t,x) = \int_0^\infty \int_{\mathbb{R}^n} e^{i(x\cdot\xi + (\phi(t)-\phi(\tau))|\xi|)} \Theta(|\xi|/2^j) a(t,\xi)\overline{a(\tau,\xi)}\hat{f}(\tau,\xi)\,\frac{đ\xi}{|\xi|^{2\gamma}}\,d\tau.$$

Note that $G_j f$ is essentially $W_j^{2\gamma-2/(m+2)} f$. Therefore, in order to get (4-14), it suffices to show that

$$\|W_j^{2\gamma-2/(m+2)} f\|_{L^\sigma_t L^\rho_x} \lesssim \|f\|_{L^{\sigma'}_t L^{\rho'}_x}. \tag{4-22}$$

We first show (4-12): For $\gamma = n/2 - (n(m+2)+2)/(q(m+2))$ and $q = q_0$, we have that

$$\left(2\gamma - \frac{2}{m+2}\right) = \left(\frac{1}{p_0} - \frac{1}{2}\right)(n+1) - \frac{m}{\mu(m+2)} - \frac{2}{m+2}.$$

Thus, we have from estimate (3-3) when $r = p = p_0$ that

$$\left\|W_j^{2\gamma-2/(m+2)}\right\|_{L^{q_0}(\mathbb{R}^{1+n}_+)} \lesssim \|f\|_{L^{p_0}(\mathbb{R}^{1+n}_+)}. \tag{4-23}$$

On the other hand, from (2-22) and the compact support of Θ,

$$\|W_j^{2\gamma-2/(m+2)} f\|_{L^\infty(\mathbb{R}^{1+n}_+)} \lesssim \|f\|_{L^1(\mathbb{R}^{1+n}_+)}. \tag{4-24}$$

By interpolation between (4-23) and (4-24), we obtain that

$$\|W_j^{2\gamma-2/(m+2)} f\|_{L^q(\mathbb{R}_+^{1+n})} \lesssim \|f\|_{L^{q'}(\mathbb{R}_+^{1+n})}, \quad q_0 \le q \le \infty,$$

where q' is the conjugate exponent q. Therefore, we get estimate (4-12).

Next we derive (4-13). Since

$$\frac{1}{s} = \frac{(m+2)(n-1)}{4}\Big(\frac{1}{2}-\frac{1}{q}\Big) + \frac{m}{4\mu},$$

we can write

$$\frac{1}{s'} = 1 - \frac{(m+2)(n-1)}{4}\Big(\frac{1}{q'}-\frac{1}{2}\Big) - \frac{m}{4\mu}.$$

Thus, when $\gamma = (n+1)/2\,\big(\frac{1}{2}-1/q\big) - m/(2\mu(m+2))$, applying estimate (3-3) for $\max\{p_1, 1\} < q' \le 2$, we have

$$\|W_j^{2\gamma-2/(m+2)} f\|_{L_t^s L_x^q(\mathbb{R}_+^{1+n})} \lesssim \|f\|_{L_t^{s'} L_x^{q'}(\mathbb{R}_+^{1+n})},$$

and, therefore, estimate (4-13) holds.

Finally we prove (4-14). When $\gamma = n\big(\frac{1}{2}-1/q\big) - 1/(m+2)$, we have from (3-5) that, for $p_1 > 1$ and $1 < q' < p_1$,

$$\|W_j^{2\gamma-2/(m+2)} f\|_{L_t^2 L_x^q(\mathbb{R}_+^{1+n})} \lesssim \|f\|_{L_t^2 L_x^{q'}(\mathbb{R}_+^{1+n})}.$$

Thus, estimate (4-14) holds. □

Combining Theorems 4.1, 4.3, and 4.4, we obtain the following results:

Theorem 4.5. *Let u solve the Cauchy problem (2-1) in the strip S_T. Then*

$$\|u\|_{C_t^0 \dot H_x^\gamma(S_T)} + \|u\|_{L_t^s L_x^q(S_T)} \lesssim \|\varphi\|_{\dot H^\gamma(\mathbb{R}^n)} + \|\psi\|_{\dot H^{\gamma-2/(m+2)}(\mathbb{R}^n)} + \|f\|_{L_t^r L_x^p(S_T)}, \tag{4-25}$$

provided that the exponents p, q, r, and s satisfy scaling invariance condition (1-10) and one of the following sets of conditions:

(i)
$$\frac{1}{p} - \frac{1}{q} = \frac{4}{(m+2)(n+1)}\Big(1 + \frac{m}{2\mu}\Big),$$
$$\frac{1}{s} = \frac{(m+2)(n-1)}{4}\Big(\frac{1}{2}-\frac{1}{q}\Big) + \frac{m}{4\mu},$$
$$\gamma = \frac{n+1}{2}\Big(\frac{1}{2}-\frac{1}{q}\Big) - \frac{m}{2\mu(m+2)},$$

where $\mu \ge \mu_*$,

$$-\frac{1}{6\mu} < \gamma < \frac{47}{84} + \frac{25}{42\mu} \qquad \text{if } n = 2, m = 1,$$

$$|\gamma - \gamma_*| < \gamma_d = \frac{2(2\mu-m)(n+1)}{\mu(m+2)(n-1)(2\mu_*-m)} \qquad \text{if } n \ge 3 \text{ or } n = 2, m \ge 2,$$

and

$$\gamma_* = \frac{2}{m+2} + \frac{m}{2\mu(m+2)} - \frac{(2\mu-m)(n+1)}{2\mu(2\mu_*-m)}.$$

(ii) $n \geq 3$ *or* $n = 2, m \geq 2$ *and* $r = 2$,

$$\frac{1}{s} = \frac{(m+2)(n-1)}{4}\Big(\frac{1}{2}-\frac{1}{q}\Big) + \frac{m}{4\mu}, \quad \gamma = \frac{n+1}{2}\Big(\frac{1}{2}-\frac{1}{q}\Big) - \frac{m}{2\mu(m+2)},$$

where $\mu \geq \max\{2, mn/2\}$ *and*

$$-\frac{m}{2\mu(m+2)} \leq \gamma < \frac{3}{m+2} - \frac{n(2\mu-m)}{\mu(m+2)(n-1)}.$$

(iii) $n \geq 3$ *or* $n = 2, m \geq 2$ *and* $s = 2$,

$$\frac{1}{r} = 1 - \frac{m}{4\mu} - \frac{(m+2)(n-1)}{4}\Big(\frac{1}{p}-\frac{1}{2}\Big), \quad \gamma = n\Big(\frac{1}{2}-\frac{1}{q}\Big) - \frac{1}{m+2},$$

where $\mu \geq \max\{2, mn/2\}$ *and*

$$\frac{\mu(n+1)-mn}{\mu(m+2)(n-1)} < \gamma < \frac{2}{m+2} + \frac{m}{2\mu(m+2)}.$$

Remark 4.6. We can rewrite the conditions of (4-5) in terms of q.

(i) For $\mu \geq \mu_*$,

$$\text{(4-26)} \quad \begin{aligned} &\frac{8}{63}\Big(1-\frac{4}{\mu}\Big) < \frac{1}{q} \leq \frac{1}{2} && \text{if } n = 2, m = 1,\\ &\frac{1}{p_2} < \frac{1}{q} + \frac{4}{(m+2)(n+1)}\Big(1+\frac{m}{2\mu}\Big) < \frac{1}{p_1} && \text{if } n \geq 3 \text{ or } n = 2,\ m \geq 2. \end{aligned}$$

(ii) For $\mu \geq \max\{2, mn/2\}$,

$$\text{(4-27)} \quad \frac{2n}{(n+1)p_1} - \frac{n-1}{2(n+1)} - \frac{1}{(m+2)(n+1)}\Big(6+\frac{m}{\mu}\Big) < \frac{1}{q} \leq \frac{1}{2}.$$

(iii) For $\mu \geq \max\{2, mn/2\}$,

$$\text{(4-28)} \quad \frac{1}{2} - \frac{1}{2(m+2)n}\Big(6+\frac{m}{\mu}\Big) < \frac{1}{q} < \frac{1}{q_1}.$$

Theorem 4.7. *Let u solve the Cauchy problem (2-1) in the strip S_T. Then*

$$\text{(4-29)} \quad \|u\|_{C_t^0\dot H_x^\gamma(S_T)} + \|u\|_{L^q(S_T)} \lesssim \|\varphi\|_{\dot H^\gamma(\mathbb{R}^n)} + \|\psi\|_{\dot H^{\gamma-2/(m+2)}(\mathbb{R}^n)} + \||D_x|^{\gamma-\gamma_0} f\|_{L^{p_0}(S_T)}$$

provided that the exponents p, q, r, and s satisfy (1-10) *and* $\mu \geq \max\{2, m/2\}$, $q_0 \leq q < \infty$, *where*

$$\gamma = \tfrac{1}{2}n - \frac{n(m+2)+2}{q(m+2)}, \quad \gamma_0 = \frac{2}{m+2} + \frac{m}{2\mu(m+2)} - \frac{n+1}{2}\Big(\frac{1}{p_0}-\frac{1}{2}\Big).$$

Corollary 4.8. *Under the conditions of Theorem 4.7, one has*

$$\|u\|_{C_t^0\dot H_x^\gamma(S_T)}+\|u\|_{L^q(S_T)}+\||D_x|^{\gamma-1/(m+2)}u\|_{L^{q_0^*}(S_T)} \lesssim \|\varphi\|_{\dot H^\gamma(\mathbb{R}^n)}+\|\psi\|_{\dot H^{\gamma-2/(m+2)}(\mathbb{R}^n)}+\||D_x|^{\gamma-1/(m+2)}f\|_{L^{p_0^*}(S_T)}, \tag{4-30}$$

where $\gamma=n/2-((m+2)n+2)/(q(m+2))$ *and* $q_0^*\le q<\infty$.

Proof. This follows by combining estimate (4-29) and Remark 4.2 when $\mu=\mu_*$. □

An application of Theorem 4.5 yields the following:

Corollary 4.9. *Let u solve the Cauchy problem*

$$\partial_t^2u-t^m\Delta u=f(t,x)g(t,x)\quad \text{in } S_T,$$
$$u(0,\cdot)=\partial_tu(0,\cdot)=0.$$

Then, for any $\mu\ge\mu_*$ *and* $0<R\le\infty$,

$$\|u\|_{C_t^0\dot H_x^\gamma(S_T\cap\Lambda_R)}+\|u\|_{L_t^sL_x^q(S_T\cap\Lambda_R)}+\|u\|_{L_t^\infty L_x^\delta(S_T\cap\Lambda_R)} \lesssim \|f\|_{L_t^\sigma L_x^\rho(S_T\cap\Lambda_R)}\|g\|_{L_t^sL_x^q(S_T\cap\Lambda_R)}, \tag{4-31}$$

where q is as in (4-26),

$$\rho=\frac{\mu(m+2)(n+1)}{2(2\mu+m)},\qquad \sigma=\frac{\mu(n+1)}{2\mu-mn}, \tag{4-32}$$

$$\frac1s=\frac{(m+2)(n-1)}{4}\Big(\frac12-\frac1q\Big)+\frac{m}{4\mu},\qquad \frac{n}{\delta}=\frac{n}{q}+\frac{2}{m+2}\Big(\frac1s-\frac{m}{4\mu}\Big), \tag{4-33}$$

and

$$\Lambda_R=\big\{(t,x)\in\mathbb{R}_+\times\mathbb{R}^n\mid |x|+\phi(t)<R\big\}.$$

Proof. First we study the case $R=\infty$. Note that (4-33) gives that

$$n\Big(\frac12-\frac1\delta\Big)=\frac{n+1}{2}\Big(\frac12-\frac1q\Big)-\frac{m}{2\mu(m+2)}.$$

Applying estimate (4-25) in case (i) together with the Sobolev embedding

$$\dot H^{n(1/2-1/\delta)}(\mathbb{R}^n)\hookrightarrow L^\delta(\mathbb{R}^n),$$

we have

$$\|u\|_{C_t^0\dot H_x^\gamma(S_T)}+\|u\|_{L_t^sL_x^q(S_T)}+\|u\|_{L_t^\infty L_x^\delta(S_T)}\lesssim\|fg\|_{L_t^rL_x^p(S_T)},$$

where $1/p=1/q+1/\rho$ and $1/r=1/s+1/\sigma$. In addition, from Hölder's inequality,

$$\|fg\|_{L_t^rL_x^p(S_T)}\le\|f\|_{L_t^\sigma L_x^\rho(S_T)}\|g\|_{L_t^sL_x^q(S_T)}. \tag{4-34}$$

Thus, estimate (4-31) holds for $R=\infty$.

Now let $R < \infty$. Let χ denote the characteristic function of $S_T \cap \Lambda_R$. If u solves $\partial_t^2 u - t^m \Delta u = fg$ with vanishing initial data and u_χ solves $\partial_t^2 u_\chi - t^m \Delta u_\chi = \chi f g$ with vanishing initial data, then $u = u_\chi$ in $S_T \cap \Lambda_R$ due to finite propagation speed (see [Taniguchi and Tozaki 1980]). Therefore,

$$\begin{aligned}\|u\|_{C_t^0 \dot{H}_x^\gamma(S_T\cap\Lambda_R)} + \|u\|_{L_t^s L_x^q(S_T\cap\Lambda_R)} + \|u\|_{L_t^\infty L_x^\delta(S_T\cap\Lambda_R)} \\ = \|u_\chi\|_{C_t^0 \dot{H}_x^\gamma(S_T)} + \|u_\chi\|_{L_t^s L_x^q(S_T)} + \|u_\chi\|_{L_t^\infty L_x^\delta(S_T)} \\ \le \|\chi f\|_{L_t^\sigma L_x^\rho(S_T)} \|\chi g\|_{L_t^s L_x^q(S_T)}.\end{aligned}$$

Consequently, estimate (4-31) holds. □

As another application of Theorem 4.5 we have the following:

Corollary 4.10. *Let u be a solution of*

$$\begin{aligned}\partial_t^2 u - t^m \Delta u = F(v) \quad \text{in } S_T, \\ u(0,\cdot) = \partial_t u(0,\cdot) = 0.\end{aligned}$$

If $q < \infty$ and $1/(m+2) \le \gamma = n/2 - (n(m+2)+2)/(q(m+2)) \le (m+3)/(m+2)$, then

$$\|u\|_{C_t^0 \dot{H}_x^\gamma(S_T)} + \|u\|_{L^q(S_T)} + \||D_x|^{\gamma-1/(m+2)} u\|_{L^{q_0^*}(S_T)} \lesssim \|F'(v)\|_{L^{\mu_*/2}(S_T)} \||D_x|^{\gamma-1/(m+2)} v\|_{L^{q_0^*}(S_T)}. \tag{4-35}$$

Proof. This follows from estimate (4-30) by taking fractional derivatives. Indeed, for $0 \le \gamma - 1/(m+2) \le 1$, one has

$$\begin{aligned}\|u\|_{C_t^0 \dot{H}_x^\gamma(S_T)} + \|u\|_{L^q(S_T)} + \||D_x|^{\gamma-1/(m+2)} u\|_{L^{q_0^*}(S_T)} \\ \lesssim \left\||D_x|^{\gamma-1/(m+2)}(F(v))\right\|_{L^{p_0^*}(S_T)} \\ \lesssim \|F'(v)\|_{L^{\mu_*/2}(S_T)} \left\||D_x|^{\gamma-1/(m+2)} v\right\|_{L^{q_0^*}(S_T)}.\end{aligned}$$ □

5. Solvability of the semilinear generalized Tricomi equation

In this section, we will apply Theorems 4.5 and 4.7 and Corollaries 4.8–4.10 with $\mu = \mu_*$ to establish the existence and uniqueness of the solution u of problem (1-1). Thereby, we will use the following iteration scheme: For $j \in \mathbb{N}_0$, let u_j be the solution of

$$\begin{aligned}\partial_t^2 u_j - t^m \Delta u_j = F(u_{j-1}) \quad \text{in } \mathbb{R}_+ \times \mathbb{R}^n, \\ u_j(0,\cdot) = \varphi, \quad \partial_t u_j(0,\cdot) = \psi,\end{aligned} \tag{5-1}$$

where $u_{-1} = 0$.

Notice that, for $\mu = \mu_*$, the exponents from (4-25) in case (i) are

$$\gamma_* = \frac{1}{m+2}, \quad \gamma_d = \frac{2(n+1)}{\mu_*(m+2)(n-1)}.$$

In order to get the existence of solutions of the Cauchy problem (1-1) as stated in Theorems 1.1, 1.4, and 1.5, we need to show that, for the sequences $\{u_j\}_{j=0}^{\infty}$ and $\{F(u_j)\}_{j=0}^{\infty}$ defined by (5-1), there exist a $T > 0$ and a function u such that

$$u_j \to u \quad \text{in } L^1_{\text{loc}}(S_T) \quad \text{as } j \to \infty, \tag{5-2}$$

$$F(u_j) \to F(u) \quad \text{in } L^1_{\text{loc}}(S_T) \quad \text{as } j \to \infty. \tag{5-3}$$

From (5-2) and (5-3), one obviously has that the limit function u solves problem (1-1) in S_T.

Furthermore, let u, $\tilde{u}$ both solve the Cauchy problem (1-1) in S_T. Then $v = u - \tilde{u}$ satisfies

$$\begin{aligned} \partial_t^2 v - t^m \Delta v &= G(u,\tilde{u})v \quad \text{in } S_T,\\ v(0,\cdot) &= \partial_t v(0,\cdot) = 0, \end{aligned} \tag{5-4}$$

where $G(u,\tilde{u}) = (F(u) - F(\tilde{u}))/(u-\tilde{u})$ if $u \neq \tilde{u}$ and $G(u,u) = F'(u)$. For certain $s, q \geq 2$, we will show that $v \in L^s_t L^q_x(S_T)$ and

$$\|v\|_{L^s_t L^q_x(S_T)} \leq \tfrac{1}{2} \|v\|_{L^s_t L^q_x(S_T)}. \tag{5-5}$$

Uniqueness of the solution of the Cauchy problem (1-1) in S_T follows.

5.1. *Proof of Theorem 1.1.*

5.1.1. *Case $\kappa_1 < \kappa < \kappa_*$.* From the assumptions of Theorem 1.1, we have

$$\gamma = \frac{n+1}{4} - \frac{n+1}{\mu_*(\kappa-1)} - \frac{m}{2\mu_*(m+2)}$$

and

$$q = \frac{\mu_*(\kappa-1)}{2}, \quad \frac{1}{s} = \frac{(m+2)(n-1)}{4}\left(\frac{1}{2} - \frac{1}{q}\right) + \frac{m}{4\mu_*}. \tag{5-6}$$

Thus,

$$\gamma = \frac{n+1}{2}\left(\frac{1}{2} - \frac{1}{q}\right) - \frac{m}{2\mu_*(m+2)}, \quad \frac{1}{m+2} - \frac{2(n+1)}{\mu_*(m+2)(n-1)} < \gamma < \frac{1}{m+2}.$$

Existence. In order to show (5-2), set

$$\begin{aligned} H_j(T) &= \|u_j\|_{C^0_t \dot{H}^\gamma_x(S_T)} + \|u_j\|_{L^s_t L^q_x(S_T)},\\ N_j(T) &= \|u_j - u_{j-1}\|_{L^s_t L^q_x(S_T)}. \end{aligned} \tag{5-7}$$

We claim that there exists a constant $\varepsilon_0 > 0$ small such that

$$2T^{1/q-1/s}H_0(T) \le \varepsilon_0 \tag{5-8}$$

and

$$H_j(T) \le 2H_0(T), \quad N_j(T) \le \tfrac{1}{2} N_{j-1}(T). \tag{5-9}$$

Indeed, from the iteration scheme (5-1), we have

$$(\partial_t^2 - t^m \Delta)(u_{j+1} - u_{k+1}) = G(u_j, u_k)(u_j - u_k). \tag{5-10}$$

Note that in (4-32),

$$\rho = \sigma = \tfrac{1}{2}\mu_*$$

when $\mu = \mu_*$. Thus, from (4-31) and condition (1-2),

$$\begin{aligned} \|u_{j+1} - u_{k+1}\|_{C_t^0 \dot{H}_x^\gamma(S_T)} + \|u_{j+1} - u_{k+1}\|_{L_t^s L_x^q(S_T)} &\lesssim \|G(u_j, u_k)\|_{L^{\mu_*/2}(S_T)} \|u_j - u_k\|_{L_t^s L_x^q(S_T)} \\ &\lesssim \big(\|u_j\|^{\kappa-1}_{L^q(S_T)} + \|u_k\|^{\kappa-1}_{L^q(S_T)}\big) \|u_j - u_k\|_{L_t^s L_x^q(S_T)}. \end{aligned} \tag{5-11}$$

Note that $s > q$ for $\kappa < \kappa_*$. By Hölder's inequality, we arrive at

$$\|u_j\|_{L^q(S_T)} \le T^{1/q-1/s} \|u_j\|_{L_t^s L_x^q(S_T)}. \tag{5-12}$$

Since $u_{-1} = 0$, (5-11) together with (5-12) implies that

$$\|u_{j+1} - u_0\|_{L_t^s L_x^q(S_T)} + \|u_{j+1} - u_0\|_{C_t^0 \dot{H}_x^\gamma(S_T)} \lesssim T^{(\kappa-1)(1/q-1/s)} \|u_j\|^\kappa_{L_t^s L_x^q(S_T)}.$$

From the Minkowski inequality, we have that there exists an ε_0 with $0 < \varepsilon_0 \le 2^{-2/(\kappa-1)}$ such that

$$H_{j+1}(T) \le H_0(T) + \tfrac{1}{2} H_j(T) \quad \text{if } T^{1/q-1/s} H_j(T) \le \varepsilon_0.$$

Therefore, by induction on j,

$$H_j(T) \le 2H_0(T) \quad \text{if } 2T^{1/q-1/s} H_0(T) \le \varepsilon_0. \tag{5-13}$$

Taking $k = j - 1$ in (5-10), estimates (5-11)–(5-13) yield that

$$N_{j+1}(T) \le \tfrac{1}{2} N_j(T) \quad \text{if } 2H_0(T) T^{1/q-1/s} \le \varepsilon_0,$$

which together with (5-13) implies that (5-9) holds as long as (5-8) holds.

Since $u_{-1} \equiv 0$ and u_0 is a solution of problem (2-2), we have from (4-13) that, for $\varphi \in \dot{H}^\gamma(\mathbb{R}^n)$ and $\psi \in \dot{H}^{\gamma-2/(m+2)}(\mathbb{R}^n)$,

$$N_0(T) \le H_0(T) \lesssim \|\varphi\|_{\dot{H}^\gamma(\mathbb{R}^n)} + \|\psi\|_{\dot{H}^{\gamma-2/(m+2)}(\mathbb{R}^n)}.$$

Thus, by choosing $T > 0$ small, (5-8) holds. Consequently, there is a function $u \in C_t^0 \dot{H}_x^\gamma(S_T) \cap L_t^s L_x^q(S_T)$ such that

$$u_j \to u \quad \text{in } L_t^s L_x^q(S_T) \text{ as } j \to \infty, \tag{5-14}$$

and, therefore, (5-2) holds. It also follows that u_j converges to u almost everywhere. By Fatou's lemma, it follows that

$$\begin{aligned} &\|u\|_{C_t^0 \dot{H}_x^\gamma(S_T)} + \|u\|_{L_t^s L_x^q(S_T)} \\ &\qquad \le \liminf_{j\to\infty} \big(\|u_j\|_{C_t^0 \dot{H}_x^\gamma(S_T)} + \|u_j\|_{L_t^s L_x^q(S_T)}\big) \le 2H_0(T), \end{aligned} \tag{5-15}$$

which shows that estimate (1-4) holds.

Now we prove (5-3). It suffices to show that $F(u)$ is bounded in $L_t^r L_x^p(S_T)$ and $F(u_j)$ converges to $F(u)$ in $L_t^r L_x^p(S_T)$ as $j \to \infty$, where $p = q/\kappa$ and $1/r = 1 - m/(4\mu_*) - (m+2)(n-1)/4\,\big(1/p - \tfrac{1}{2}\big)$. In fact, $r\kappa < s$ if $\kappa < \kappa_*$, thus, for $q = p\kappa$, by condition (1-2) and Hölder's inequality, we have

$$\|F(u)\|_{L_t^r L_x^p(S_T)} \lesssim \|u\|^\kappa_{L_t^{r\kappa} L_x^{p\kappa}(S_T)} \lesssim T^{1/r-\kappa/s} \|u\|^\kappa_{L_t^s L_x^q(S_T)}.$$

Moreover, in view of $1/p - 1/q = 1/r - 1/s = 2/\mu_*$, by Hölder's inequality and estimates (5-11)–(5-13) and (5-15), we have

$$\begin{aligned} \|F(u_j) - F(u)\|_{L_t^r L_x^p(S_T)} &\le \|G(u_j, u)\|_{L^{\mu_*/2}(S_T)} \|u_j - u\|_{L_t^s L_x^q(S_T)} \\ &\lesssim T^{(\kappa-1)(1/q-1/s)} H_0(T)^{\kappa-1} \|u_j - u\|_{L_t^s L_x^q(S_T)} \\ &\lesssim \|u_j - u\|_{L_t^s L_x^q(S_T)}. \end{aligned}$$

Applying (5-14), we have that $F(u_j)$ converges to $F(u)$ in $L_t^r L_x^p(S_T)$ and, therefore, (5-3) holds.

From (5-2) and (5-3), we have that the limit function $u \in C_t^0 \dot{H}_x^\gamma(S_T)) \cap L_t^s L_x^q(S_T)$ solves the Cauchy problem (1-1) in S_T.

Uniqueness. Suppose u, $\tilde{u} \in C([0,T], \dot{H}^\gamma(\mathbb{R}^n)) \cap L_t^s L_x^q(S_T)$ solve the Cauchy problem (1-1) in S_T. Then $v = u - \tilde{u} \in C([0,T], \dot{H}^\gamma(\mathbb{R}^n)) \cap L_t^s L_x^q(S_T)$ is a solution of problem (5-4). From Corollary 4.9, we have that

$$\begin{aligned} &\|v\|_{L_t^s L_x^q(S_T)} \\ &\quad \le C\big(\|u\|^{\kappa-1}_{L^q(S_T)} + \|\tilde{u}\|^{\kappa-1}_{L^q(S_T)}\big)\|v\|_{L_t^s L_x^q(S_T)} && \text{(by (4-31) and (1-2))} \\ &\quad \le C T^{(\kappa-1)(1/q-1/s)} \\ &\qquad\quad \times \big(\|u\|^{\kappa-1}_{L_t^s L_x^q(S_T)} + \|\tilde{u}\|^{\kappa-1}_{L_t^s L_x^q(S_T)}\big)\|v\|_{L_t^s L_x^q(S_T)} && \text{(by Hölder's inequality)} \\ &\quad \le C 2^\kappa \big(T^{1/q-1/s} H_0(T)\big)^{\kappa-1} \|v\|_{L_t^s L_x^q(S_T)} && \text{(by (5-15))} \\ &\quad \le \tfrac{1}{2}\|v\|_{L_t^s L_x^q(S_T)} && \text{(by (5-8)).} \end{aligned}$$

Thus (5-5) holds and $u=\tilde u$ in S_T.

5.1.2. *Case $\kappa_*\le\kappa$ if $n=2$ or $\kappa_*\le\kappa\le\kappa_3$ if $n\ge 3$.*

Existence. From the assumptions of Theorem 1.1, we have

$$\gamma=\tfrac12 n-\frac{4}{(m+2)(\kappa-1)},\qquad s=q=\frac{\mu_*(\kappa-1)}{2}.$$

Thus,

$$\frac{1}{m+2}\le\gamma=\tfrac12 n-\frac{(m+2)n+2}{q(m+2)}\le\frac{m+3}{m+2}.$$

To show (5-2), we set

$$H_j(T)=\|u_j\|_{C_t^0\dot H_x^\gamma(S_T)}+\|u_j\|_{L^q(S_T)}+\||D_x|^{\gamma-1/(m+2)}u_j\|_{L^{q_0^*}(S_T)},$$

and

$$N_j(T)=\|u_j-u_{j-1}\|_{L^{q_0^*}(S_T\cap\Lambda_R)}. \tag{5-16}$$

We claim that there exists a constant $\varepsilon_0>0$ such that

$$H_0(T)\le\varepsilon_0, \tag{5-17}$$

and

$$H_j(T)\le 2H_0(T),\qquad N_j(T)\le\tfrac12 N_{j-1}(T). \tag{5-18}$$

Indeed, since $u_{-1}=0$, from the iteration scheme (5-1), we have

$$(\partial_t^2-t^m\Delta)(u_{j+1}-u_0)=F(u_j). \tag{5-19}$$

Thus, estimate (4-35) together with condition (1-2) yields, for $0\le\gamma-1/(m+2)\le 1$,

$$\begin{aligned}H_{j+1}(T)&\le H_0(T)+C\|F'(u_j)\|_{L^{\mu_*/2}(S_T)}\||D_x|^{\gamma-1/(m+2)}u_j\|_{L^{q_0^*}(S_T)}\\&\le H_0(T)+C\|u_j\|^{\kappa-1}_{L^q(S_T)}\||D_x|^{\gamma-1/(m+2)}u_j\|_{L^{q_0^*}(S_T)}\\&\le H_0(T)+CH_j(T)^\kappa.\end{aligned}$$

Therefore, by induction, we have that

$$H_j(T)\le 2H_0(T)\quad\text{if } C2^\kappa H_0(T)^{\kappa-1}<1.$$

Consequently,

$$H_j(T)\le 2H_0(T)\quad\text{if } H_0(T)\le\varepsilon_0 \tag{5-20}$$

for some $\varepsilon_0>0$ small. Notice that, for q and s from (5-6), when $q=s$, so $q=s=q_0^*$. Hence, by using estimates (5-11)–(5-13) together with (5-20), we get that for N_j defined in (5-16),

$$N_j(T)\le\tfrac12 N_{j-1}(T)\quad\text{if } H_0(T)\le\varepsilon_0. \tag{5-21}$$

Estimates (5-20) and (5-21) tell us that (5-18) holds as long as (5-17) holds. To get (5-17), from estimate (4-30) (with $f=0$) we have that, for $\varphi \in \dot{H}^{\gamma}(\mathbb{R}^n)$ and $\psi \in \dot{H}^{\gamma-2/(m+2)}(\mathbb{R}^n)$,

$$H_0(T) \lesssim \|\varphi\|_{\dot{H}^{\gamma}(\mathbb{R}^n)} + \|\psi\|_{\dot{H}^{\gamma-2/(m+2)}(\mathbb{R}^n)}. \tag{5-22}$$

Due to the continuity of the norm in $L^q(S_T)$, (5-17) holds for some $T>0$ small. (If $\|\varphi\|_{\dot{H}^{\gamma}(\mathbb{R}^n)} + \|\psi\|_{\dot{H}^{\gamma-2/(m+2)}(\mathbb{R}^n)}$ is small, then (5-17) holds for any $T>0$, consequently, we get global existence.)

Note that $q=\mu_*(\kappa-1)/2 \geq q_0^*$ when $\kappa \geq \kappa_*$. Thus, from Hölder's inequality and (5-22),

$$N_0(T) = \|u_0\|_{L^{q_0^*}(S_T\cap\Lambda_R)} \lesssim \|u_0\|_{L^q(S_T)} \lesssim H_0(T). \tag{5-23}$$

From estimates (5-17), (5-18), and (5-23), we get that there exists a function $u \in C_t^0\dot{H}_x^{\gamma}(S_T) \cap L^q(S_T)$ with $|D_x|^{\gamma-1/(m+2)}u \in L^{q_0^*}(S_T)$ such that

$$u_j \to u \quad \text{in } L^{q_0^*}(S_T\cap\Lambda_R) \text{ as } j\to\infty, \tag{5-24}$$

and (5-2) holds. Thus, from Fatou's lemma and (5-18), it follows that

$$\|u\|_{C_t^0\dot{H}_x^{\gamma}(S_T)} + \|u\|_{L^q(S_T)} + \||D_x|^{\gamma-1/(m+2)}u\|_{L^{q_0^*}(S_T)} \leq 2H_0(T) \tag{5-25}$$

and u satisfies estimate (1-4).

Since $q=\mu_*(\kappa-1)/2 \geq \kappa$ when $\kappa\geq\kappa_*$, we have from condition (1-2) that $F(u)$ is locally integrable for $u\in L^q(S_T)$. By Hölder's inequality,

$$\begin{aligned}\int_{S_T\cap\Lambda_R} |F(u_j)-F(u)|\,dt\,dx &= \int_{S_T\cap\Lambda_R} |G(u_j,u)||u_j-u|\,dt\,dx \\ &\leq \|G(u_j,u)\|_{L^{p_0^*}(S_T\cap\Lambda_R)}\|u_j-u\|_{L^{q_0^*}(S_T\cap\Lambda_R)}.\end{aligned}$$

Note that $p_0^* < \mu_*/2$. Thus, from condition (1-2) we have that

$$\begin{aligned}\|G(u_j,u)\|_{L^{p_0^*}(S_T\cap\Lambda_R)} &\lesssim \|u_j\|^{\kappa-1}_{L^{p_0^*(\kappa-1)}(S_T\cap\Lambda_R)} + \|u\|^{\kappa-1}_{L^{p_0^*(\kappa-1)}(S_T\cap\Lambda_R)} \\ &\lesssim \|u_j\|^{\kappa-1}_{L^q(S_T\cap\Lambda_R)} + \|u\|^{\kappa-1}_{L^q(S_T\cap\Lambda_R)} \lesssim H_0(T)^{\kappa-1},\end{aligned}$$

which together with (5-24) implies that $F(u_j)\to F(u)$ in $L^1_{\mathrm{loc}}(S_T)$. Hence, (5-3) holds.

From (5-2) and (5-3), we have that the limit function $u\in C_t^0\dot{H}_x^{\gamma}(S_T)\cap L^q(S_T)$ with $|D_x|^{\gamma-1/(m+2)}u\in L^{q_0^*}(S_T)$ is a weak solution of the Cauchy problem (1-1) in S_T.

Uniqueness. Suppose $u, \tilde{u} \in C_t^0 \dot{H}_x^\gamma(S_T) \cap L^q(S_T)$ with $|D_x|^{\gamma-1/(m+2)} u$ and $|D_x|^{\gamma-1/(m+2)} \tilde{u} \in L^{q_0^*}(S_T)$ solving the Cauchy problem (1-1) in S_T. Then $v = u-\tilde{u} \in C_t^0 \dot{H}_x^\gamma(S_T) \cap L^q(S_T)$ is a weak solution of problem (5-4). Thus, it follows from Corollary 4.9 that

$$\begin{aligned} \|v\|_{L^q(S_T)} &\le C\big(\|u\|_{L^q(S_T)}^{\kappa-1} + \|\tilde{u}\|_{L^q(S_T)}^{\kappa-1}\big)\|v\|_{L^q(S_T)} && \text{(by (4-31) and (1-2))} \\ &\le C 2^\kappa H_0(T)^{\kappa-1} \|v\|_{L^q(S_T)} && \text{(by (5-25))} \\ &\le \tfrac{1}{2}\|v\|_{L_t^s L_x^q(S_T)} && \text{(by (5-17)).} \end{aligned}$$

Thus (5-5) holds and $u = \tilde{u}$ in S_T.

5.1.3. *Case $n \ge 3$ and $\kappa > \kappa_3$, $\kappa \in \mathbb{N}$.*

Existence. From the assumptions of Theorem 1.1, we have

$$\gamma = \tfrac{1}{2}n - \frac{4}{(m+2)(\kappa-1)}, \quad s = q = \frac{\mu_*(\kappa-1)}{2}, \quad F(u) = \pm u^\kappa,$$

and

$$\gamma = \tfrac{1}{2}n - \frac{(m+2)n+2}{q(m+2)} > 1 + \frac{1}{m+2}.$$

To verify (5-2), we set

$$H_j(T) = \|u_j\|_{C_t^0 \dot{H}_x^\gamma(S_T)} + \sup_{q_0^* \le \tau \le \frac{1}{2}\mu_*(\kappa-1)} \Big\| |D_x|^{\frac{(m+2)n+2}{\tau(m+2)} - \frac{4}{(m+2)(\kappa-1)}} u_j \Big\|_{L^\tau(S_T)}$$

and

$$N_j(T) = \|u_j - u_{j-1}\|_{L^{q_0^*}(S_T \cap \Lambda_R)}.$$

We claim that there exists a constant $\varepsilon_0 > 0$ such that

$$H_0(T) \le \varepsilon_0 \tag{5-26}$$

and

$$H_j(T) \le 2H_0(T), \quad N_j(T) \le \tfrac{1}{2} N_{j-1}(T). \tag{5-27}$$

In fact, applying Minkowski's inequality and estimate (4-30) (with $\varphi = \psi = 0$),

$$\begin{aligned} H_{j+1}(T) &\le H_0(T) \\ &\quad + C \sup_{q_0^* \le \tau \le \mu_*(\kappa-1)/2} \big\| |D_x|^{\frac{1}{2}n - \frac{1}{m+2} - \frac{4}{(m+2)(\kappa-1)}} (u_j^\kappa) \big\|_{L^{p_0^*}(S_T)}. \end{aligned} \tag{5-28}$$

Note that $\alpha = n/2 - 1/(m+2) - 4/((m+2)(\kappa-1)) > 1$ when $\kappa > \kappa_3$. Thus, $|D_x|^\alpha(u_j^\kappa)$ can be expressed as a finite linear combination of $\prod_{\ell=1}^{\kappa} |D_x|^{\alpha_\ell} u_j$, where $0 \le \alpha_\ell \le \alpha$ $(1 \le \ell \le \kappa)$ and $\sum_{\ell=1}^{\kappa} \alpha_\ell = \alpha$. By Hölder's inequality, $\| |D_x|^\alpha (u_j^\kappa) \|_{L^{p_0^*}(S_T)}$ is dominated by a finite sum of terms of the form

$$\prod_{\ell=1}^{\kappa} \| |D_x|^{\alpha_\ell} u_j \|_{L^{\tau_\ell}(S_T)},$$

where $\sum_{\ell=1}^{\kappa} 1/\tau_\ell = 1/p_0^*$. We choose τ_ℓ so that

$$\alpha_\ell = \frac{n(m+2)+2}{\tau_\ell(m+2)} - \frac{4}{(m+2)(\kappa-1)}.$$

Then

$$q_0^* \le \tau_\ell \le \frac{\mu_*(\kappa-1)}{2}, \qquad \sum_{\ell=1}^{\kappa} \frac{1}{\tau_\ell} = \frac{1}{p_0^*},$$

and, therefore,

$$\| |D_x|^{\alpha_\ell} u_j \|_{L^{\tau_\ell}(S_T)} \le H_j(T),$$

which together with (5-28) yields that

$$H_{j+1}(T) \le H_0(T) + C_\kappa H_j(T)^\kappa.$$

By induction, we have that

$$H_j(T) \le 2H_0(T) \quad \text{if } H_0(T) \le \varepsilon_0. \tag{5-29}$$

For q and s from (5-6), when $q = s$, then $q = s = q_0^*$. Hence, by estimates (5-11)–(5-13) and together with (5-29), we get that

$$N_j(T) \le \tfrac{1}{2} N_{j-1}(T) \quad \text{if } H_0(T) \le \varepsilon_0. \tag{5-30}$$

From (5-29) and (5-30), we get that (5-27) holds as long as (5-26) holds.

Note that

$$\frac{n(m+2)+2}{\tau(m+2)} - \frac{4}{(m+2)(\kappa-1)} = 0, \tag{5-31}$$

for $\tau = \mu_*(\kappa-1)/2$ and

$$\frac{n(m+2)+2}{\tau(m+2)} - \frac{4}{(m+2)(\kappa-1)} = \gamma - \frac{1}{m+2}. \tag{5-32}$$

for $\tau = q_0^*$. On the other hand, we have from (4-30) (with $f = 0$) that, for $\varphi \in \dot H^\gamma(\mathbb{R}^n)$ and $\psi \in \dot H^{\gamma-2/(m+2)}(\mathbb{R}^n)$,

$$\|u_0\|_{C_t^0 \dot H_x^\gamma(S_T)} + \|u_0\|_{L^{\mu_*(\kappa-1)/2}(S_T)} + \| |D_x|^{\gamma-1/(m+2)} u_0 \|_{L^{p_0^*}(S_T)} \lesssim \|\varphi\|_{\dot H^\gamma(\mathbb{R}^n)} + \|\psi\|_{\dot H^{\gamma-2/(m+2)}(\mathbb{R}^n)}. \tag{5-33}$$

By interpolation together with (5-31)–(5-33), we conclude that

$$H_0(T) \lesssim \|\varphi\|_{\dot H^\gamma(\mathbb{R}^n)} + \|\psi\|_{\dot H^{\gamma-2/(m+2)}(\mathbb{R}^n)}.$$

It follows that (5-26) holds by choosing $T > 0$ small. (We can take $T = \infty$ if $\|\varphi\|_{\dot H^{\gamma}(\mathbb{R}^n)} + \|\psi\|_{\dot H^{\gamma-2/(m+2)}(\mathbb{R}^n)}$ is small which then yields global existence.)

From Hölder's inequality and (5-31),

$$N_0(T) = \|u_0\|_{L^{q_0^*}(S_T\cap\Lambda_R)} \le C_R\|u_0\|_{L^{\mu_*(\kappa-1)/2}(S_T)} \le C_R H_0(T) < \infty. \tag{5-34}$$

Therefore, we have from (5-27), (5-26), and (5-34) that there exists a function $u \in C_t^0\dot H_x^{\gamma}(S_T)\cap L^q(S_T)$ with $|D_x|^{\gamma-1/(m+2)}u \in L^{q_0^*}(S_T)$ such that

$$u_j \to u \quad \text{in } L^{q_0^*}(S_T\cap\Lambda_R) \text{ as } j\to\infty,$$

and, therefore, (5-2) holds. Thus, from Fatou's lemma and (5-27),

$$\|u\|_{C_t^0\dot H_x^{\gamma}(S_T)} + \|u\|_{L^q(S_T)} + \big\||D_x|^{\gamma-1/(m+2)}u\big\|_{L^{q_0^*}(S_T)} \le 2H_0(T) \tag{5-35}$$

and u satisfies estimate (1-4).

Note that $q = \mu_*(\kappa-1)/2 \ge \kappa$ when $\kappa > \kappa_3$. Thus, for $u \in L^q(S_T)$, by Hölder's inequality and condition (1-2), we get that $F(u)$ is locally integrable and $F(u_j)$ converges to $F(u)$ in $L^1_{\mathrm{loc}}(S_T)$, and hence (5-3) holds.

Applying (5-2) and (5-3), it follows that the limit function $u \in C_t^0\dot H_x^{\gamma}(S_T)\cap L^q(S_T)$ with $|D_x|^{\gamma-1/(m+2)}u \in L^{q_0^*}(S_T)$ is a weak solution of the Cauchy problem (1-1) in S_T.

Uniqueness. This follows from the same arguments as in 5.1.2. □

5.2. ***Proof of Theorem 1.4.*** From the assumption of Theorem 1.4, we have

$$\gamma = \frac{n}{2} - \frac{4}{(m+2)(\kappa-1)},$$
$$\frac{1}{q} = \frac{1}{(m+2)(n+1)}\Big(\frac{8}{\kappa-1} - \frac{m}{\mu_*}\Big) - \frac{n-1}{2(n+1)},$$

and

$$\frac{1}{s} = \frac{(m+2)(n-1)}{4}\Big(\frac{1}{2}-\frac{1}{q}\Big) + \frac{m}{4\mu_*}.$$

Thus,

$$\gamma = \Big(\frac{n+1}{2}\Big)\Big(\frac{1}{2}-\frac{1}{q}\Big) - \frac{m}{2\mu_*(m+2)}$$

and

$$\frac{1}{m+2} \le \gamma < \frac{1}{m+2} + \frac{2(n+1)}{\mu_*(m+2)(n-1)},$$

where $\kappa_* \le \kappa < \kappa_2$.

To show (5-2), we set

$$H_j(T) = \|u_j\|_{C_t^0\dot H_x^{\gamma}(S_T)} + \|u_j\|_{L_t^sL_x^q(S_T)} + \|u_j-u_0\|_{L_t^\infty L_x^\delta(S_T)}$$

and

$$N_j(T) = \|u_j - u_{j-1}\|_{L_t^s L_x^q(S_T)},$$

where

$$\frac{1}{s} + \frac{(m+2)n}{2q} = \frac{(m+2)n}{2\delta} = \frac{m+2}{2}\Big(\frac{n}{2} - \gamma\Big). \tag{5-36}$$

We claim that there exist a constant $\varepsilon_0 > 0$ and a $\theta \in [0, 1]$ such that

$$2H_0(T)^\theta (2H_0(T) + \|u_0\|_{L_t^\infty L_x^\delta(S_T)})^{1-\theta} \le \varepsilon_0 \tag{5-37}$$

and

$$H_j(T) \le 2H_0(T), \qquad N_j(T) \le \tfrac{1}{2} N_{j-1}(T). \tag{5-38}$$

Indeed, due to (5-36), from Sobolev's embedding theorem we have that

$$\|u(t,\cdot)\|_{L^\delta(\mathbb{R}^n)} \lesssim \|u(t,\cdot)\|_{\dot{H}^\gamma(\mathbb{R}^n)}.$$

Applying Hölder's inequality, we get that

$$\|u_j\|_{L^{\mu_*(\kappa-1)/2}(S_T)} \le \|u_j\|^\theta_{L_t^s L_x^q(S_T)} \|u_j\|^{1-\theta}_{L_t^\infty L_x^\delta(S_T)},$$

where $\theta = 2/(n(m+2)+2) + 4n(m+2)/(\mu_*(m+2)(n-1)(q-2)+2mq)$. Note that $0 \le \theta \le 1$ for $\gamma \ge 1/(m+2)$.

By the same arguments as in the proof of Theorem 1.1, we get that (5-37) and (5-38) hold. Consequently, (5-2) and (5-3) also hold. Hence, the limit $u \in C_t^0 \dot{H}_x^\gamma(S_T) \cap L_t^s L_x^q(S_T)$ of the sequence $\{u_j\}$ is a solution of the Cauchy problem (1-1) in S_T. Moreover, by Fatou's lemma and (5-38), we have that

$$\|u\|_{C_t^0 \dot{H}_x^\gamma(S_T)} + \|u\|_{L_t^s L_x^q(S_T)} \le 2H_0(T),$$

which together with (5-37) yields that u satisfies estimate (1-4).

Further, by the same arguments as in the proof of Theorem 1.1, it follows that if both $u, \tilde{u}$ solve the Cauchy problem (1-1) in S_T, then $u = \tilde{u}$ in S_T. □

5.3. ***Proof of Theorem 1.5.*** From the assumptions of Theorem 1.5, we have

$$\gamma = \frac{n+1}{2}\Big(\frac{1}{2} - \frac{1}{q}\Big) - \frac{m}{2\mu_*(m+2)}$$

and

$$-\frac{m}{2\mu_*(m+2)} \le \gamma < \frac{1}{m+2} - \frac{2(n+1)}{\mu_*(m+2)(n-1)} = \frac{3}{m+2} - \frac{n(2\mu_* - m)}{\mu_*(m+2)(n-1)}.$$

To verify (5-2), we set

$$H_j(T) = \|u_j\|_{C_t^0 \dot{H}_x^\gamma(S_T)} + \|u_j\|_{L_t^s L_x^q(S_T)}, \quad N_j(T) = \|u_j - u_{j-1}\|_{L_t^s L_x^q(S_T)}.$$

Let $p = q/\kappa$. Then

$$\frac{2n}{(n+1)p} = \frac{1}{q} + \frac{6\mu+m}{\mu(m+2)(n+1)} - \frac{n-1}{2(n+1)}.$$

Thus we can apply Theorem 4.5 in case (ii) together with Hölder's inequality to find that

$$\begin{aligned}\|u_{j+1}-u_{k+1}\|_{C_t^0\dot H_x^\gamma(S_T)} + \|u_{j+1}-u_{k+1}\|_{L_t^sL_x^q(S_T)} &\\ \lesssim \|F(u_j)-F(u_k)\|_{L_t^2L_x^p(S_T)}&\\ \lesssim \|G(u_j,u_k)\|_{L_t^\rho L_x^\sigma(S_T)}\|u_j-u_k\|_{L_t^sL_x^q(S_T)},&\end{aligned}$$

where $1/\rho = \frac{1}{2} - 1/s$, and $1/\sigma = 1/p - 1/q = (\kappa-1)/q$.

Note that $s > (\kappa-1)\rho$ when $\gamma < 1/(m+2) - 2(n+1)/(\mu_*(m+2)(n-1))$. Due to condition (1-2) and Hölder's inequality,

$$\begin{aligned}\|G(u_j,u_k)\|_{L_t^\rho L_x^\sigma(S_T)} &\lesssim \|u_j\|^{\kappa-1}_{L_t^{\rho(\kappa-1)}L_x^q(S_T)} + \|u_k\|^{\kappa-1}_{L_t^{\rho(\kappa-1)}L_x^q(S_T)}\\ &\lesssim T^{1/2-1/s}(\|u_j\|^{\kappa-1}_{L_t^sL_x^q(S_T)} + \|u_k\|^{\kappa-1}_{L_t^sL_x^q(S_T)}).\end{aligned}$$

As in the proof of Theorem 1.1, we get that

$$H_j(T) \le 2H_0(T), \quad N_j(T) \le \tfrac{1}{2}N_{j-1}(T), \tag{5-39}$$

and

$$N_0(T) \le H_0(T)T^{1/2-\kappa/s} \le \varepsilon_0, \tag{5-40}$$

for $\varepsilon_0 > 0$ small by choosing $T > 0$ small. Therefore, there is a function $u \in C_t^0\dot H_x^\gamma(S_T) \cap L_t^sL_x^q(S_T)$ such that

$$u_j \to u \quad \text{in } L_t^sL_x^q(S_T) \text{ as } j \to \infty$$

and (5-2) holds. Combining Fatou's lemma and (5-39), we see that

$$\|u\|_{C_t^0\dot H_x^\gamma(S_T)} + \|u\|_{L_t^sL_x^q(S_T)} \le 2H_0(T).$$

Together with (5-40) we get that u satisfies estimate (1-4).

Moreover, since $2\kappa > s$, by condition (1-2) and Hölder's inequality, we have that, for $p = q/\kappa$,

$$\begin{aligned}\|F(u)\|_{L_t^2L_x^p(S_T)} &\lesssim \|u\|^\kappa_{L_t^{2\kappa}L_x^q(S_T)}\\ &\lesssim T^{1/2-\kappa/s}\|u\|^\kappa_{L_t^sL_x^q(S_T)}\end{aligned}$$

and

$$\begin{aligned}\|F(u_j)-F(u)\|_{L_t^2L_x^p(S_T)} &\lesssim T^{1/2-1/s}(\|u_j\|^{\kappa-1}_{L_t^sL_x^q(S_T)}+\|u\|^{\kappa-1}_{L_t^sL_x^q(S_T)})\|u_j-u\|_{L_t^sL_x^q(S_T)}\\ &\lesssim T^{1/2-1/s}H_0(T)^{\kappa-1}\|u_j-u\|_{L_t^sL_x^q(S_T)}.\end{aligned}$$

Therefore, $F(u)\in L_t^2L_x^{q/\kappa}(S_T)$ and $F(u_j)\to F(u)$ in $L_t^2L_x^{q/\kappa}(S_T)$ as $j\to\infty$, hence (5-3) holds. Consequently, the limit function $u\in C_t^0\dot{H}_x^\gamma(S_T)\cap L_t^sL_x^q(S_T)$ solves the Cauchy problem (1-1) in S_T.

Now suppose $u,\tilde{u}\in C_t^0\dot{H}_x^\gamma(S_T)\cap L_t^sL_x^q(S_T)$ both solve the Cauchy problem (1-1) in S_T. Then $v=u-\tilde{u}\in C_t^0\dot{H}_x^\gamma(S_T)\cap L_t^sL_x^q(S_T)$ is a solution of (5-4). Applying Theorem 4.5 in case (ii) and Hölder's inequality, it follows that

$$\begin{aligned}\|v\|_{L_t^sL_x^q(S_T)} &\le C\|G(u,\tilde{u})v\|_{L_t^2L_x^p(S_T)}\\ &\le CT^{1/2-1/s}(\|u\|^{\kappa-1}_{L_t^sL_x^q(S_T)}+\|\tilde{u}\|^{\kappa-1}_{L_t^sL_x^q(S_T)})\|v\|_{L_t^sL_x^q(S_T)}\\ &\le CT^{1/2-1/s}H_0(T)^{\kappa-1}\|v\|_{L_t^sL_x^q(S_T)}\le \tfrac{1}{2}\|v\|_{L_t^sL_x^q(S_T)}.\end{aligned}$$

Thus (5-5) holds and $u=\tilde{u}$ in S_T. □

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ZHUOPING RUAN
DEPARTMENT OF MATHEMATICS
NANJING UNIVERSITY
NANJING
CHINA
zhuopingruan@nju.edu.cn

INGO WITT
MATHEMATICAL INSTITUTE
UNIVERSITY OF GÖTTINGEN
GÖTTINGEN
GERMANY
iwitt@mathematik.uni-goettingen.de

HUICHENG YIN
SCHOOL OF MATHEMATICAL SCIENCES AND MATHEMATICAL INSTITUTE
NANJING NORMAL UNIVERSITY
NANJING
CHINA
huicheng@nju.edu.cn

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TEMPEREDNESS OF MEASURES DEFINED BY POLYNOMIAL EQUATIONS OVER LOCAL FIELDS

David Taylor, V. S. Varadarajan,
Jukka Virtanen and David Weisbart

Dedicated to the memory of Professor Jun-Ichi Igusa

We investigate the asymptotic growth of the canonical measures on the fibers of morphisms between vector spaces over local fields of arbitrary characteristic. For a single polynomial over $\mathbb{R}$, this is due to Igusa and Raghavan. For nonarchimedean local fields we use a version of the Łojasiewicz inequality which follows from work of Greenberg, together with the theory of the Brauer group of local fields to construct definite forms of arbitrarily high degree, and to transfer questions at infinity to questions near the origin. We then use these to generalize results of Hörmander on estimating the growth of polynomials at infinity in terms of the distance to their zero loci. Specifically, when a fiber corresponds to a noncritical value which is stable, i.e., remains noncritical under small perturbations, we show that the canonical measure on the fiber is tempered, which generalizes results of Igusa and Raghavan, and Virtanen and Weisbart.

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1. Introduction

Let V be a finite dimensional vector space over $\mathbb{R}$ and $f\colon V \to \mathbb{R}$ a smooth nonconstant function. In the physics and mathematics literature the measure denoted by $\delta(f-c)$ figures prominently; it is a measure living on the smooth part of the zero locus $Z(f-c)$ of $f-c$, $c \in \mathbb{R}$ [Gel'fand and Shilov 1964]. Given f and choices of Haar measures on V and $\mathbb{R}$, $\delta(f-c)$ is uniquely defined for all c. Similarly if $\boldsymbol{f} = (f_1, f_2, \dots, f_r)\colon V \to \mathbb{R}^r$ is a smooth map with $df_1 \wedge \cdots \wedge df_r \neq 0$, for given Haar measures on V and $\mathbb{R}^r$, there is a canonical measure on the smooth part of the common zero locus $Z(\boldsymbol{f}-\boldsymbol{c}) = Z(f_1-c_1, f_2-c_2, \dots, f_r-c_r)$ of the $f_i - c_i$ for all $\boldsymbol{c} = (c_1, c_2, \dots, c_r)$. We denote this measure by $\mu_{\boldsymbol{f},\boldsymbol{c}}$. In this context, the finiteness of $\mu_{\boldsymbol{f},\boldsymbol{c}}$ around the singular points of $Z(f_1-c_1, f_2-c_2, \dots, f_r-c_r)$, as well as the behavior at infinity of the extended measure, viewed as a Borel measure on V, are interesting questions. If the f_i are polynomials and $Z(\boldsymbol{f}-\boldsymbol{c})$ is smooth, then it is natural to expect that $\mu_{\boldsymbol{f},\boldsymbol{c}}$ is tempered. That is,

Definition 1.1 (tempered measure). Let V be any finite dimensional k-vector space, k a *local field*. A Borel measure μ on V is *tempered* if

$$\int_V (1+\|x\|^2)^{-\alpha}\, d\mu(x) < \infty$$

for some integer α (in any norm).

This is equivalent to saying that there are constants $A > 0$, $b \geq 0$ such that

$$\text{(G)} \qquad \mu(B_R) \leq AR^b$$

for all $R \geq 1$, B_R being the closed ball in V of radius R and center $\mathbf{0}$ (in any norm).

In [Igusa 1978] Igusa and Raghavan proved that if $k = \mathbb{R}$ and f is a nonconstant polynomial on V and $c \in \mathbb{R}$ is a noncritical value of f, i.e., the locus $Z(f-c)$ is smooth, then $\mu_{f,c}$ is tempered, and further that the growth estimate G for the measure is uniform in a neighborhood of c; here we must remember that by the algebraic Sard's theorem (Proposition 2.4), f has only finitely many critical values, so that every noncritical value c has neighborhoods consisting only of noncritical values.

The measures $\mu_{f,c}$, $\mu_{\boldsymbol{f},\boldsymbol{c}}$ can be defined over any local field. Throughout this paper by local field we mean a locally compact nondiscrete field of any characteristic, other than $\mathbb{C}$; measure theoretic questions over $\mathbb{C}$ usually reduce to $\mathbb{R}$, and so we do not treat the case of $\mathbb{C}$ separately. In [Igusa 1978] Igusa and Raghavan define the measures $\mu_{f,c}$ for any local field but do not consider their behavior at infinity, the reason being that over a nonarchimedean field they were concerned only with integrating Schwartz–Bruhat functions (i.e., compactly supported complex-valued locally constant functions). However the work of Harish-Chandra [1973] shows the necessity as well as utility of working with locally constant functions that do not vanish outside a compact set. The question of extending the results of [Igusa 1978] to the nonarchimedean case and for $r > 1$ is certainly a natural one. In [Virtanen and Weisbart 2014] the measures $\mu_{f,c}$ were shown to be tempered when f is a nondegenerate quadratic form and $c \neq 0$; moreover for the case $c = 0$ the locus $Z(f)$ has 0 as its only singularity, and it was shown that the measure $\mu_{f,0}$ is finite in the neighborhood of 0 if $\dim V \geq 3$, and the extended measure is tempered in V. The work of [Virtanen and Weisbart 2014] was motivated by physical questions arising in the theory of elementary particles over p-adic spacetimes. In this paper we generalize the results of [Igusa 1978] and [Virtanen and Weisbart 2014] to the measures $\mu_{\boldsymbol{f},\boldsymbol{c}}$ where the f_i $(1 \leq i \leq r)$ are polynomials on a vector space V over a local field k, with $\dim(V) = m$ and $df_1 \wedge df_2 \wedge \cdots \wedge df_r \not\equiv 0$, so that $m \geq r$. Note that for $r > 1$ and $k = \mathbb{R}$ this question is already more general than the one treated in [Igusa 1978].

We now describe our main result using the above notation. Let $\boldsymbol{f} : V \to k^r$ be the polynomial map whose components are the f_i, with $df_1 \wedge \cdots \wedge df_r \not\equiv 0$. A point $x \in V$ is called a critical point (CP) of $\boldsymbol{f}$ if the differentials $df_{i,x}$ are linearly dependent. We write $C(\boldsymbol{f})$ for the set of critical points of $\boldsymbol{f}$; the image $\boldsymbol{f}(C(\boldsymbol{f}))$ in k^r is called the set of critical values of $\boldsymbol{f}$, and is denoted by $CV(\boldsymbol{f})$. By the algebraic Sard's theorem (Proposition 2.4) one knows that *in characteristic zero* the Zariski closure in k^r of $CV(\boldsymbol{f})$ is a *proper* algebraic subset of k^r. A point $c \in k^r$ is called *stably noncritical* if it has an open neighborhood (in the k-topology) consisting only of noncritical values. This is the same as saying that the fibers above points sufficiently close to c are smooth. If k has characteristic zero, stably noncritical values exist and form a nonempty open set in k^r whose complement in the image of $\boldsymbol{f}$ has measure 0. Then the following is our main result. For $r = 1$ and $k = \mathbb{R}$ it was proved in [Igusa 1978]. Note that in this case the characteristic is 0 and there are only finitely many critical values and so every noncritical value is stably noncritical.

Theorem 1.2. *Fix $\boldsymbol{f}$ and write $\mu_{\boldsymbol{c}} = \mu_{\boldsymbol{f},\boldsymbol{c}}$. Suppose $\boldsymbol{c}$ is stably noncritical. Then $\mu_{\boldsymbol{c}}$ is tempered and there are constants $A > 0$, $\gamma \geq 0$ such that for all $\boldsymbol{d}$ in an open neighborhood of $\boldsymbol{c}$*

$$\mu_{\boldsymbol{d}}(B_R) \leq AR^{m-r+\gamma} \qquad (R \geq 1, \boldsymbol{d} \in U).$$

Suppose k has characteristic 0; then stably noncritical values form a nonempty dense open set whose complement in the image of $\boldsymbol{f}$ has measure 0; for $r = 1$, the critical set is finite and all noncritical values are stably noncritical.

Remark 1.3. In view of the failure of Sard's theorem over characteristic $p > 0$ (see page 233), we do not know if stably noncritical values of $\boldsymbol{c}$ always exist when k is a local field of positive characteristic.

Remark 1.4. The results and ideas in the paper lie at the interface of analysis of geometry over local fields and are motivated by the themes from quantum theory over p-adic spacetimes. We do not know what, if any, are the arithmetic consequences of our results.

As an application of our theory we prove that if k has characteristic 0, the orbits of regular semisimple elements of a semisimple Lie algebra over k are closed, and the invariant measures on them are tempered. For $k = \mathbb{R}$ this is a result of Harish-Chandra [1957].

2. Canonical measures on level sets of polynomial maps

Canonical measures on the fibers of submersive maps. The construction below is well known and our treatment is a very mild variant of Harish-Chandra's [1964] for the case $k = \mathbb{R}$ (see also [Varadarajan 1977]). Serre's book [2006] is a good reference for the theory of analytic manifolds and maps over a local field of arbitrary characteristic. (All of our manifolds are second countable.)

Lemma 2.1. *Let V, W be vector spaces of finite dimension m, r respectively, and $L\colon V \to W$ be a surjective linear map. Let $U = \ker L$. Let σ, τ be exterior forms on V, W of degrees m, r respectively, with $\tau \neq 0$. Then there exists a unique exterior $(m-r)$-form ρ on U such that if $\{u_1, u_2, \ldots, u_{m-r}\}$ is a basis for U, then*

$$\rho(u_1, u_2, \ldots, u_{m-r}) = \frac{\sigma(u_1, \ldots, u_{m-r}, v_1, \ldots, v_r)}{\tau(Lv_1, \ldots, Lv_r)},$$

where $v_i \in V$ are such that $\{u_1, \ldots, u_{m-r}, v_1, \ldots, v_r\}$ is a basis for V.

Proof. For fixed v_i it is obvious that this defines an exterior $(m-r)$-form on U. Its independence of the choice of the v_i is easy to check. □

We write $\rho = \sigma/\tau$. Note that this definition is relative to L.

Theorem 2.2. *Let k be a local field of arbitrary characteristic and M, N be analytic manifolds over k of dimensions m, r respectively, and $\pi : M \to N$ be an analytic map, surjective, and submersive everywhere. Let σ_M (resp. τ_N) be an analytic exterior m-form (resp. r-form) on M (resp. N), with $\tau_N \neq 0$ everywhere on N. Then there is a unique analytic exterior form $\rho := \rho_{M/N}$ on M such that for any $y \in N$, the pull back of ρ to the fiber $\pi^{-1}(y)$ is the exterior $(m-r)$-form $x \mapsto \sigma_x/\tau_y$ relative to $d\pi_x : T_x(M) \to T_y(N)$.*

Proof. The pointwise definition of ρ is clear after the preceding lemma. For analyticity we use local coordinates around x and $y = \pi(x)$, say $x_1, \ldots, x_m$, such that π is the projection $(x_1, \ldots, x_m) \mapsto (x_1, \ldots, x_r)$. Then

$$\sigma_M = s(x_1, \ldots, x_m)\, dx_1 \cdots dx_m, \qquad \tau = t(x_1, \ldots, x_r)\, dx_1 \cdots dx_r,$$

and

$$\rho = \bigl(s(x_1, \ldots, x_m)/t(x_1, \ldots, x_r)\bigr)\, dx_{n+1} \cdots dx_m.$$ □

Remark 2.3. Let s_M (resp. t_N) be the measures defined on M (resp. N) by $|\sigma_M|$ (resp. $|\tau_N|$). We denote by $r_{M/N,y}$ the measures defined on $\pi^{-1}(y)$ by $|\rho|$. The smooth functions in the nonarchimedean case are the locally constant functions. Then we have [Harish-Chandra 1964]

$$\int_M \alpha\, ds_M = \int_N f_\alpha\, dt_N, \qquad f_\alpha(y) = \int_{\pi^{-1}(y)} \alpha\, dr_{M/N,y}$$

for all smooth compactly supported complex-valued functions α on M.

It is easy to show, using partitions of unity that the map $\alpha \mapsto f_\alpha$ is surjective, and continuous when $k = \mathbb{R}$. This gives rise to an injection of the space of distributions on N into the space of distributions on M, say $T \mapsto T^*$. Then $r_{M/N,y} = \delta(y)^*$, $\delta(y)$ being the Dirac distribution at $y \in N$. Replacing $\delta(y)$ by its derivatives, we get distributions on M, supported by $\pi^{-1}(y)$. If F is a locally integrable function on N, it defines a distribution on N, say T_F, and T_F^* is $T_{F\circ\pi}$ where $F \circ \pi$ is a locally integrable function on M. Thus the map $T \mapsto T^*$ is the natural extension of the map $F \mapsto F\circ\pi$ from the space of locally integrable functions on N to the corresponding space on M. The map $T \mapsto T^*$ plays a fundamental role in Harish-Chandra's theory [1964] of characters on real semisimple Lie groups. Finally, in algebrogeometric terminology, ρ above is the top *relative* exterior form.

We shall now apply this result to polynomial maps $\boldsymbol{f}: V \to k^r$ where V is a vector space of finite dimension m over a local field k of arbitrary characteristic such that $df_1 \wedge \cdots \wedge df_r \not\equiv 0$ on V, the f_i being the components of $\boldsymbol{f}$; let $V^\times$ be the set of points where this exterior form is nonzero in V, so that $V^\times$ is nonempty Zariski open in V; let $N(\boldsymbol{f}) = \boldsymbol{f}(V^\times)$. Clearly $m \geq r$ and $N(\boldsymbol{f})$ is nonempty open (in the k-topology) in k^r. Then, by Theorem 2.2 with $M = V^\times$, $N = N(\boldsymbol{f})$, we have a measure $\mu_{\boldsymbol{c}}$ for $\boldsymbol{c} \in N$ on $L'_{\boldsymbol{c}} := L_{\boldsymbol{c}} \cap V^\times$ where $L_{\boldsymbol{c}}$ is the level set

(2-1) $$L_{\boldsymbol{c}} = Z(f_1 - c_1, \ldots, f_r - c_r) = \{x \in V \mid f_1(x) = c_1, \ldots, f_r(x) = c_r\}.$$

Exactly as before, we may view the $\mu_{\boldsymbol{f},\boldsymbol{c}}$ as distributions living on $L'_{\boldsymbol{c}}$ which is all of $L_{\boldsymbol{c}}$ if $\boldsymbol{c}$ is a noncritical value. The derivatives of $\mu_{\boldsymbol{f},\boldsymbol{c}}$ with respect to the differential operators of k^m (when $k = \mathbb{R}$) then yield distributions supported by $L_{\boldsymbol{c}}$. Examples of such distributions have important applications ([Gel'fand and Shilov 1964],[Kolk and Varadarajan 1992]) in analysis and physics.

Fix a noncritical value $\boldsymbol{c}$ of $\boldsymbol{f}$. Let $J = \{i_1 < i_2, \ldots, < i_r\}$ be an ordered subset of r elements in $\{1, 2, \ldots, m\}$. Let

(2-2) $$\partial_J := \frac{\partial(f_1, \ldots, f_r)}{\partial(x_{i_1}, \ldots, x_{i_r})}.$$

Then $L_{\boldsymbol{c}}$ is smooth and $L_{\boldsymbol{c}} = \bigcup_J L_{\boldsymbol{c},J}$ where the sum is over all sets J as above and

(2-3) $$L_{\boldsymbol{c},J} := \{x \in L_{\boldsymbol{c}} \mid \partial_J(x) \neq 0\}.$$

Locally on $L_{\boldsymbol{c},J}$, $(f_1, \dots, f_r, y_1, \dots, y_{m-r})$ is a new coordinate system, the y_j being some enumeration of the $x_i (i \neq i_\nu)$. Obviously $dy_1 \cdots dy_m = \varepsilon \partial_J(x)\, dx_1 \cdots dx_m$, where ε is locally constant and equal to ± 1. Another way of interpreting this formula is the following: if π_J is the projection map from $L_{\boldsymbol{c},J}$ that takes x to $(y_1, \dots, y_{m-r})$, then π_J is a local analytic isomorphism and

$$\rho_{\boldsymbol{c}} = \varepsilon \frac{1}{\partial_J(x)} \pi_J^*(dy_1 \cdots dy_{m-r}), \tag{2-4}$$

where ε is locally constant and ± 1-valued. Hence to control the growth of the measure defined by $|\rho|$ at infinity, we must find *lower bounds* of the $\|\partial_J(x)\|$ on $L_{\boldsymbol{c},J}$ for $\|x\| \geq 1$. Let

$$\nabla_r(x) = (\partial_J(x)).$$

We call ∇_r the generalized gradient of $(f_1, \dots, f_r)$. Then we must find lower bounds for $\|\nabla_r(x)\| := \max_J \|\partial_J(x)\|$ for $\|x\| \geq 1$ on $L_{\boldsymbol{c},J}$. In this quest we follow [Igusa 1978], and our techniques force us to assume $\boldsymbol{c}$ to be stably noncritical. For $r = 1$, ∇_1 is just the gradient ∇, and that work reduces the issue of the lower bounds for the gradient field by replacing ∇f (for $k = \mathbb{R}$) by $\sum_{1 \leq j \leq m} |\partial_j f|^2$, where $\partial_j f = \partial f / \partial x_j$. For nonarchimedean local fields and for $r > 1$ we have to replace the sum of squares by suitable *definite* forms whose degrees will grow with m. Igusa and Raghavan find lower bounds for $|\nabla|$ using Hörmander's inequalities [1958] over $\mathbb{R}$. We generalize Hörmander's inequalities to any local field and use them with the existence of definite forms of sufficiently high degree to get lower bounds for $\|\nabla_r\|$ on the level sets $L_{\boldsymbol{c},J}$.

The Hörmander inequalities over $\mathbb{R}$ are of two types: H1 and H2. H1 is local and is essentially the Łojasiewicz inequality [1959]; Hörmander derives H2 from H1 by inversion. Over nonarchimedean k, H1 turns out to be a consequence of a Henselization lemma of Greenberg [1966], as observed in [Bollaerts 1990]. The reduction of H2 to H1 is more subtle in the nonarchimedean case. We prove it by embedding V in a division algebra D, central over k, prove H2 for D, and then deduce H2 for V. The descent from D to V is elementary. To prove H2 in D we use the map $x \mapsto x^{-1}$ on $D \setminus \{0\}$ to reduce H2 to H1. The existence of central division algebras over k of arbitrarily high dimension is nontrivial and follows from the theory of the Brauer group of k. The lower bounds of $\nabla_r f$ obtained from these arguments allow us to prove that when $\boldsymbol{c}$ is a *stably noncritical* value of $\boldsymbol{f}$, $\mu_{\boldsymbol{f},\boldsymbol{c}}(B_r) = O(R^{m-r+\gamma})$ for some $\gamma \geq 0$, uniformly near $\boldsymbol{c}$. We do not know if we can take $\gamma = 0$ always. If $\|\nabla_r \boldsymbol{f}\|$ is bounded away from zero at infinity on $L_{\boldsymbol{c}}$, then it is obvious that we may take $\gamma = 0$; but $\inf \|\nabla_r \boldsymbol{f}\|$ may be zero on $L_{\boldsymbol{c}}$. (See page 252).

Algebraic Sard's theorem in characteristic* 0 *for polynomial maps. Let V be a vector space over k of finite dimension m. Recall the definitions of $C(\boldsymbol{f})$ and $CV(\boldsymbol{f})$.

Proposition 2.4. *Let k be of characteristic* 0. *The Zariski closure*, $Cl(CV(\boldsymbol{f}))$ *is a proper subset of* k^r; *in particular*, *if* $r = 1$, *then* $CV(\boldsymbol{f})$ *is finite.*

Proof. Fix a basis of V so that $V \simeq k^m$. The field generated by the coefficients of the f_j, say $k_1 \supset k$, can be embedded in $\mathbb{C}$. It is thus enough to prove Proposition 2.4 over $\mathbb{C}$ itself, where it is just the statement that the fibers of $\boldsymbol{f}$ are generically smooth. Over $\mathbb{C}$ this is essentially Sard's lemma for affine algebraic varieties treated by Mumford [1995]. □

Analytic Sard's theorem in characteristic p > 0. In characteristic $p > 0$, the algebraic Sard's lemma fails abysmally [Mumford and Oda 2015, p. 179] over algebraically closed fields. Indeed, let f be a polynomial in two variables X, Y giving rise to a map $K^2 \to K$ where K is algebraically closed and of characteristic $p > 0$, for example,

$$f = X^{p+1} + X^p Y + Y^p.$$

Then the gradient of f vanishes precisely on the Y-axis, and f on the Y-axis is the map $y \mapsto y^p$ which is surjective. So the image of the singular set is all of K, and every fiber has a singular point. But if we replace K by a local field, then $y \mapsto y^p$ is *not* surjective, and in fact the image under f of the singular set is k^p which is a closed proper subset of k (in the k-topology), and is of measure zero in k. Thus the generic fiber (in the k-topology) is smooth in k.

We shall now consider the situation over local fields of characteristic $p > 0$. From Sard [1942] we know that when $k = \mathbb{R}$ and the map is of class $C^{(a)}$ $(a > 0)$, $f(C)$ has measure zero when $a > m - r$. Now, when k has characteristic $p > 0$, the derivatives of f are not enough to determine the coefficients of the power series expansion of f whose order is greater than $p - 1$. So there is an analogy with the case of $C^{(p-1)}$ over $\mathbb{R}$, suggesting that over k the condition $p > m - r + 1$ would be sufficient to guarantee that $f(C)$ is a null set. This suggestion, which leads to Theorem 2.5, is due to Professor Pierre Deligne (personal communication, 2016), which we gratefully acknowledge.

Theorem 2.5. *Let X, Y be analytic manifolds over a local field k of characteristic $p > 0$, of dimensions m, r respectively. Let $f : X \to Y$ be an analytic. Let C be the critical set for f. Then $f(C)$ has measure zero in Y if $p > m - n + 1$.*

Proof. The proof that $f(C)$ has measure zero in Y when $p > m - r + 1$ is a minor adaptation of [Guillemin and Pollack 1974], needed because we have an additional restriction on p.

The result is local and so we may take X to be a compact open set $U \in k^m$. We use induction on m. We define the filtration $C = C_0 \supset C_1 \supset \cdots \supset C_{p-1}$, where C_s $(1 \le s \le p - 1)$ is the set where all derivatives of the components of f of order $\le s$ vanish. The sets C, C_s are compact while $C_s \setminus C_{s+1}$ is locally compact and second countable, hence a countable union of compact sets. So $f(C)$, $f(C_s)$ are compact, and $f(C_s \setminus C_{s+1})$ is a countable union of compact sets.

The inductive proof that $f(C \setminus C_1)$ is a null set reduces to the case when (m, r) becomes $(m-1, r-1)$. Since $m-r = (m-1)-(r-1)$, the condition on p remains the same and induction applies.

The inductive proof that $f(C_s \setminus C_{s+1})$ is a null set reduces to the case when (m, r) becomes $(m-1, r)$. Since $p > m-r+1 > (m-1)-r+1$, induction applies again.

It remains to show that $f(C_{p-1})$ is a null set when $p > m-r+1$. We shall show actually that $f(C_{p-1})$ is a null set when $p > m/r$. This is enough since $m/r \le m-r+1$. This is a local result and so we may work around a point of C_{p-1} which can be taken to be the origin. We use the max norm on k^m and k^r so that the norms take values in $q^{\mathbb{Z}}$, where $q > 1$ is the cardinality of the residue field of k. By scaling, if necessary, we may assume that all components of f are given by power series expansions, absolutely convergent on the ball $B(q) := \{x \in k^m \mid \|x\| \le q\}$. Note that $B(1) = R^m$, where R is the ring of integers of k. In order to estimate the growth of these series we need a lemma:

Lemma 2.6. *Let g be an analytic function on $B(q)$ given by an absolutely convergent power series expansion about* 0 *on $B(q)$. Let D be the set in $B(1)$ where $\partial^\beta f = 0$ for all β with $|\beta| \le p-1$. Then we have*

$$|g(x+h) - g(x)| \le A\|h\|^p$$

uniformly for $x \in D$, $\|h\| \le 1 \le q-1$, the constant $A > 0$ depending only on g.

Proof. We use [Serre 2006, pp. 67–75]. We have

$$g(x) = \sum_\alpha c_\alpha X^\alpha, \quad \sum_\alpha |c_\alpha| = A < \infty.$$

For $x \in B(1)$ we have $g(x+h) = \sum_\beta \Delta^\beta g(x) h^\beta$, where

$$\Delta^\beta g(x) = \sum_{\alpha \ge \beta} c_\alpha \binom{\alpha}{\beta} x^{\alpha-\beta}, \quad \beta! \Delta^\beta g(x) = \partial^\beta g(x).$$

Then $|\Delta^\beta g(x)| \le A$ on $B(1)$. If $x \in D$, $\|h\| \le 1 \le q-1$, then $x+h \in B(1)$. Moreover, for $|\beta| \le p-1$, $\beta! \Delta^\beta g(x) = 0$ so that $\Delta^\beta g(x) = 0$. Hence,

$$g(x+h) = g(x) + \sum_{|\beta| \ge p} (\Delta^\beta g)(x) h^\beta.$$

But, for $y \in B(1)$,

$$|\Delta^\beta g(y)| \le \sum |c_\alpha| = A.$$

So,

$$|g(x+h) - g(x)| \le A\|h\|^p \qquad (x \in D, \|h\| \le 1),$$

proving the lemma. □

We now divide $B(1)^m$ into very small "cells". Let P be the maximal ideal in R. Let N be any integer ≥ 1. Then $B(1)$ is the disjoint union of q^N cosets of P^N each of which is a compact open set that has diameter $\leq q^{-N}$ and volume q^{-N}. This gives a partition of $B(1)^m$ into q^{mN} compact open sets ("cells") of diameter $\leq q^{-N}$ and volume q^{-Nm}. By the above lemma, if $x, x+h \in D$ and are in one of these cells, say γ, then

$$\|f(x+h) - f(x)\| \leq A\|h\|^p \leq Aq^{-Np}, \tag{2-5}$$

where A is a constant independent of x. Hence, $f(\gamma)$ is contained in a set of diameter $\leq q^{-Np}$ and hence volume $\leq q^{-Npr}$. Thus $f(D \cap C_{p-1})$ is enclosed in a set of volume $\leq q^{mN-Npr} = q^{-N(pr-m)}$. If $p > m/r$ this expression goes to 0 as $N \to \infty$, and we are done. □

Remark 2.7. If $\boldsymbol{f} = (f_1, \ldots, f_r)$ is a polynomial map of k^m into k^r such that $df_1 \wedge \cdots \wedge df_r \neq 0$, then $\boldsymbol{f}(k^m)$ is open and Sard's theorem shows that almost every fiber of f is smooth in k. So there are always noncritical values. Whether some of them are stable is not known to us.

Remark 2.8. When $r = 1$, the above condition reduces to $p > m$. Both this condition and the fact that when $m \geq p+1$ it is possible that the image of the critical set can be all of k were communicated to us by Professor Pierre Deligne (2016). We are grateful for his generosity and for giving us permission to discuss his example.

Example 2.9 (Deligne). We take $m = p+1$ with coordinates $y, x_1, \ldots, x_p$. The field $k := \mathbb{F}[[t]][1/t]$, where $\mathbb{F}$ is a finite field of characteristic p, is a local field of characteristic p. Then k is a vector space of dimension p over $k^{(p)} := \{x^p \mid x \in k\}$. Let $(a_i)_{1 \leq i \leq p}$ be a basis for $k/k^{(p)}$, for instance $a_i = t^{i-1}$, $(1 \leq i \leq p)$. Consider, for an integer $n > 1$, prime to p,

$$f = y^n + a_1 x_1^p + \cdots + a_p x_p^p.$$

Then the critical locus is given by $y = 0$. Its image under f is obviously all of k. If we do not insist that $df \neq 0$, we can omit y so that f maps the critical set k^p onto k.

This example is easily modified for the case $r > 1$. We consider k^{p+r} with coordinates $y_1, \ldots, y_{r-1}, y, x_1 \ldots, x_p$ and take the map $f : k^{p+r} \to k^r$ defined by

$$f : (y_1, \ldots, y_{r-1}, y, x_1 \ldots, x_p) \mapsto \left(y_1, \ldots, y_{r-1}, y^n + \sum_{i=1}^{p} a_i x_i^p\right),$$

where the notation is as before. The critical set is again given by $y = 0$, and the map restricted to this set is

$$f : (y_1, \ldots, y_{r-1}, 0, x_1 \ldots, x_p) \mapsto \left(y_1, \ldots, y_{r-1}, \sum_{i=1}^{p} a_i x_i^p\right),$$

whose range is k^r. Exactly as before, if we omit y, we get a map where $df_1 \wedge \cdots \wedge df_r$ is zero but f maps the critical set k^{p+r-1} onto k^r.

3. Construction of definite forms and their associated norms

As mentioned in Remark 2.3 we begin by discussing the construction of definite forms in an arbitrary number of variables over k.

Proposition 3.1. *Let V be a finite dimensional vector space over a local field. If $k = \mathbb{R}$, and $\nu(x)$ is a positive definite quadratic form on V, then $|\nu(x)|^{1/2}$ is a norm on V. If k is nonarchimedean, and r is an integer such that $r^2 \geq m$, then there is a homogeneous polynomial $\nu\colon V \to k$ of degree r such that*

(a) *ν is definite, i.e., for $x \in V$, $\nu(x) = 0$ if and only if $x = 0$;*

(b) *$|\nu(x)|^{1/r}$ is a nonarchimedean norm on V.*

Proof. We deal only with the case of nonarchimedean k. By the theory of the Brauer group of k [Weil 1967, chapter XII, theorem 1] and its corollary we can find a division algebra D over k which is central over k and $\dim_k(D) = r^2$. Since $V \hookrightarrow D$, it is enough to prove the proposition for $V = D$. The advantage is that we can use the algebraic structure of D.

Let ν be the reduced norm [Weil 1967, chapter IX, proposition 6] of D. Then, $\nu\colon D \to k$ is a homogeneous polynomial function on D of degree r, and $\nu(x)^r = \det(\lambda(x))$ where $\lambda(x)$ is the endomorphism $y \mapsto xy$ of D. Note that $\det(\lambda)$ is a polynomial function on D with values in k, homogeneous of degree r^2. As $\lambda(x)$ is invertible for any $x \neq 0$ in D, $\det(\lambda(x))$ and hence $\nu(x)$, is nonzero for $x \neq 0$ in D. Hence, ν is a definite form of degree r on D. It remains to prove that $N(x) := |\nu(x)|^{1/r}$ is a nonarchimedean norm on D. This reduces to showing that $N(1+u) \leq 1$ if $u \in D$ and $N(u) \leq 1$, or equivalently, that $|\lambda(1+u)| \leq 1$ if $u \in D$ and $|\lambda(u)| \leq 1$, which follows from [Weil 1967, chapter I, section 4]. □

Remark 3.2. Actually, $\nu(x)^r = \det \lambda(x)$ will serve our purposes as well and is obviously a homogeneous polynomial of degree r^2, Then $|\nu(x)|^{1/r} = |\det \lambda(x)|^{1/r^2}$. We introduced ν because it is of smaller degree and this may be of use in other contexts.

4. Hörmander's inequalities over nonarchimedean local fields

Let V be a finite dimensional vector space over a local, nonarchimedean field k, with its canonical norm $|\cdot|$. Let $\|\cdot\|$ be a nonarchimedean norm on V. We may assume that the norms on k and V take values in the set $\{0, q^{\pm 1}, q^{\pm 2}, \dots\}$, where q is the cardinality of the residue field of k. Also, let $f\colon V \to k$ be a polynomial function, and let $Z(f)$ denote its zero locus. For $x \in V$ and nonempty $E \subset V$ let $\operatorname{dist}(x, E) := \inf_{y \in E} \|x - y\|$.

Theorem 4.1 (H1). *Let $f: V \to k$ be a polynomial function on V. Suppose that $Z(f) \neq \varnothing$. Then there exist constants $C > 0$, $\alpha \geq 0$ such that*

$$|f(x)| \geq C \cdot \operatorname{dist}(x, Z(f))^{\alpha} \tag{4-1}$$

for all $x \in V$ with $\|x\| \leq 1$.

Theorem 4.2 (H2). *Let $f: V \to k$ be a polynomial function, $Z(f)$ as above. Then*

(a) *if $Z(f) = \varnothing$, then there exist constants $C > 0$ and $\beta \geq 0$ such that*

$$|f(x)| \geq C \cdot \frac{1}{\|x\|^{\beta}} \quad (x \in V,\ \|x\| \geq 1); \tag{4-2}$$

(b) *if $Z(f) \neq \varnothing$, then there exist constants $C > 0$ and $\alpha, \beta \geq 0$ such that*

$$\|f(x)\| \geq C \cdot \frac{\operatorname{dist}(x, Z(f))^{\alpha}}{\|x\|^{\beta}} \quad (x \in V,\ \|x\| \geq 1); \tag{4-3}$$

Remark 4.3. Theorem 4.1 and 4.2 were proved by Hörmander [1958] when $k = \mathbb{R}$. Also, H1 is a special case of the Łojasiewicz inequality for f a real analytic function [Łojasiewicz 1959].

In proving H1 we may assume that $V = k^m$ and $f \in R[x_1, \ldots, x_m]$, R being the ring of integers in k. Let $P \subset R$ be the maximal ideal of R. Suppose that $Z(f) \neq \varnothing$ but $Z(f) \cap R^m = \varnothing$. Then there exists a constant $b > 0$ such that $|f(x)| \geq b > 0$ for $x \in R^m$. On the other hand, as R^m is compact, there exists $b_1 > 0$ such that $\operatorname{dist}(x, Z(f)) \leq b_1$ for all $x \in R^m$. Hence $|f(x)| \geq bb_1^{-1} b_1 \geq bb_1^{-1} \operatorname{dist}(x, Z(f))$ for all $x \in R^m$. Hence we may assume in addition that $Z(f) \cap R^m \neq \varnothing$ in the proof of H1.

Proof of H1: k nonarchimedean. We follow [Greenberg 1966], specialized to the case of a single polynomial.

Proof. By theorem 1 there, applied to the single polynomial f, we can find integers, $N, c \geq 1$ and $s \geq 0$ such that if $\nu \geq N$ and $f(x) \equiv 0 \pmod{P^{\nu}}$, and $x \in R^m$, then there exists $y \in R^m$ such that $f(y) = 0$ and $x_i - y_i \equiv 0 \pmod{P^{[\nu/c]-s}}$ for all i.

Assume $|f(x)| = q^{-(N+\ell)}$, $\ell \geq 0$. Then there exists $y \in Z(f) \cap R^m$ such that

$$\|x - y\| \leq q^{-[(N+\ell)/c]+s} \leq q^{-[(N+\ell)/c-1]+s} \leq q^{s+1} |f(x)|^{1/c},$$

which implies that

$$\operatorname{dist}(x, Z(f) \cap R^m) \leq q^{s+1} |f(x)|^{1/c},$$

so that

$$|f(x)| \geq \frac{\operatorname{dist}(x, Z(f) \cap R^m)^c}{q^{c(s+1)}} \geq \frac{\operatorname{dist}(x, Z(f))^c}{q^{c(s+1)}}.$$

Thus, H1 is proved for $x \in R^m$ with $|f(x)| \le q^{-N}$. For x in R^m with $|f(x)| > q^{-N}$, we have $q^{-N} < |f(x)| \le 1$, while $\mathrm{dist}(x, Z(f) \cap R^m) \le 1$ since $\|x - y\| \le 1$ for $x, y \in R^m$. Hence

$$|f(x)| \ge q^{-N} \mathrm{dist}(x, Z(f) \cap R^m) \ge q^{-N} \mathrm{dist}(x, Z(f) \cap R^m)^c \ge q^{-N} \mathrm{dist}(x, Z(f))^c.$$

If $C = \min(q^{-N}, q^{-(s+1)c})$, then we have H1 with $\alpha = c$. □

Remark 4.4. That the local version of the Łojasiewicz inequality comes out of [Greenberg 1966] has been observed in [Bollaerts 1990]; we give this proof since it includes the case when k has characteristic > 0. Greenberg's result is applicable here because R is then complete ($k^* = k$ in his notation).

Proof of H2.

Lemma 4.5. *If H2 is true for a k-vector space V, then it is also true for any subspace W of V. In particular, for a central division algebra, D_r over k, of dimension $r^2 \ge \dim_k V$, it is enough to prove H2 for D_r.*

Proof. Let $W \subseteq V$ be a subspace, and $U \subseteq V$ such that $V = W \oplus U \simeq W \times U$. Let f be a polynomial on W. Define the polynomial g on V by $g(w+u) := f(w)$. For $w \in W, u \in U$, we take $\|w+u\| = \max(\|u\|, \|w\|)$; because U and W are complementary, this is nonarchimedean. Clearly $Z(g) = Z(f) \times U$.

Suppose $Z(f) = \varnothing$. Then $Z(g) = \varnothing$. Since H2 is true for V and $W \subset V$, there exist constants $C > 0$, $\beta \ge 0$ such that $|f(w)| \ge C\|w\|^{-\beta}$ for $w \in W$, $\|w\| \ge 1$. We may therefore assume that $Z(f) \ne \varnothing$, so $Z(g) \ne \varnothing$.

Then, $|g(x)| \ge C \mathrm{dist}(x, Z(g))^\alpha \|x\|^{-\beta}$ for $x \in V$, $\|x\| \ge 1$ where $C > 0$, $\alpha, \beta \ge 0$ are constants. If $x = w \in W$, $\mathrm{dist}(w, Z(g)) = \mathrm{dist}(w, Z(f))$. □

Now we prove H2 for D_r. Our proof is inspired by Hörmander's [1958]. It replaces the inversion in his proof by the involution $x \mapsto x^{-1}$ of $D_r^\times := D_r \setminus \{0\}$.

For a division algebra D_r of dimension r^2, central over k, let us recall $\nu := \nu_r : D_r \to k$ of Proposition 3.1, and note that it has the following property: if k' is any field containing k such that there exists an isomorphism $F : k' \otimes_k D_r \xrightarrow{\sim} M_r(k') = M_r$ where M_r is the algebra of $r \times r$ matrices over k', then $\nu(a) = \det F(a)$ for $a \in D_r$ [Weil 1967, Proposition 6, p. 168],

Lemma 4.6. *For any polynomial function $f : D \to k$ of degree d, f not necessarily homogeneous, let $f^*(x) := f(x^{-1})\nu(x)^d$ for $x \ne 0$; then $f^*(x)$ extends uniquely to a polynomial function $D_r \to k$. Moreover, for nonzero x, $x \in Z(f)$ if and only if $x^{-1} \in Z(f^*)$.*

Proof. Uniqueness is obvious. To prove that f^* has a polynomial extension it suffices to prove it for $k' \otimes_k D_r$, where k' is a separable extension of k such that $k' \otimes_k D_r \simeq M_r(k')$. The required result is compatible with addition and multiplication of the f so that it is enough to verify it for $f = 1$ (obvious) and $f = a_{ij}$, a matrix entry; then $f^* = a^{ij} \det = A_{ij}$, the corresponding cofactor. The last statement of the lemma is obvious □

Remark 4.7. From now on we use the norm $\|x\| = |\nu(x)|^{1/r}$ for D_r, $r \geq 2$.

Lemma 4.8. *If* $x, y, x - y$ *are all nonzero, then* $\|x - y\| = \|x^{-1} - y^{-1}\| \|x\| \|y\|$

Proof. Use $y - x = x(x^{-1} - y^{-1})y$ and the multiplicativity of $\|\cdot\|$. □

The next two lemmas are auxiliary before we prove H2 for D_r.

Lemma 4.9. *If* $Z(f)$ *is nonempty, there exists a constant* $A \geq 1$ *such that*

$$\operatorname{dist}(x, Z(f)) \leq A\|x\| \quad \textit{for all } x \textit{ with } \|x\| \geq 1.$$

Proof. Choose $z_0 \in Z(f)$. Then $\operatorname{dist}(x, Z(f)) \leq \|x - z_0\| \leq \max(\|x\|, \|z_0\|)$. If $\|x\| \geq \|z_0\|$, then $\operatorname{dist}(x, Z(f)) \leq \|x\|$ and we can take $A = 1$. If $\|x\| < \|z_0\|$ then $\|x - z_0\| = \|z_0\| \leq \|z_0\| \|x\|$ for $\|x\| \geq 1$; and as $\|z_0\| \geq 1$, the lemma is proved if we take $A = 1 + \|z_0\|$. □

Lemma 4.10. *Suppose* $Z(f)$ *contains a nonzero element. Then there exists a constant* $C > 0$ *such that*

$$\operatorname{dist}(x^{-1}, Z(f^*)) \geq C \frac{\operatorname{dist}(x, Z(f))}{\|x\|^2} \qquad (\|x\| \geq 1). \tag{4-4}$$

Proof. First assume $0 \notin Z(f^*)$. Then $Z(f^*) = Z(f^*) \setminus \{0\} \neq \varnothing$. Then, with $\|x\| \geq 1$,

$$\operatorname{dist}(x^{-1}, Z(f^*) \setminus \{0\}) = \inf_{0 \neq z \in Z(f^*)} \|x^{-1} - z\| = \inf_{0 \neq y \in Z(f)} \|x^{-1} - y^{-1}\| = \inf_{0 \neq y \in Z(f)} E,$$

where $E := \|x - y\| \|x\|^{-1} \|y\|^{-1}$.

We consider cases: (a) $\|y\| > \|x\|$ and (b) $\|y\| \leq \|x\|$. In case (a) $\|x - y\| = \|y\|$ so that $E = \|x\|^{-1} = \|x\| \|x\|^{-2} \geq A^{-1} \operatorname{dist}(x, Z(f)) \|x\|^{-2}$, where $A \geq 1$ is as in Lemma 4.9. In case (b) $E \geq \|x - y\| \|x\|^{-2}$ so that $\inf E \geq \operatorname{dist}(x, Z(f)) \|x\|^{-2}$. These give (4-4) with $C = 1/A$.

If $0 \in Z(f^*)$, then $\operatorname{dist}(x^{-1}, Z(f^*)) = \min(\operatorname{dist}(x^{-1}, Z(f^*) \setminus \{0\}), \|x^{-1}\|)$. Now $\|x\|^{-1} = \|x\| \|x\|^{-2} \geq C \|x\|^{-2} \operatorname{dist}(x, Z(f))$ by Lemma 4.9 where $C = 1/A$, while $\operatorname{dist}(x^{-1}, Z(f^*) \setminus \{0\}) \geq C \|x\|^{-2} \operatorname{dist}(x, Z(f))$, by above. □

Proof of H2 *for* D_r. We consider two cases: (a) $Z(f) = \varnothing$, (b) $Z(f) \neq \varnothing$.

Case (a): Then $Z(f^*) = \varnothing$ or $\{0\}$. If $Z(f^*) = \varnothing$, then there exists a constant $C > 0$ such that $|f^*(x)| \geq C > 0$ with $\|x\| \leq 1$. So, $|f^*(y)| = |f(y^{-1})| \|y\|^{rd} \geq C > 0$ for $0 < \|y\| \leq 1$, which becomes $|f(x)| \geq C \|x\|^{rd} \geq C > 0$ for $\|x\| \geq 1$.

If $Z(f^*) = \{0\}$, then $\operatorname{dist}(z, Z(f^*)) = \|z\|$, and $|f^*(y)| \geq C \|y\|^{\beta}$ with $0 < \|y\| \leq 1$ for constants $C > 0$, $\beta \geq 0$ by Theorem 4.1. Then $|f(y^{-1})| \|y\|^{rd} \geq C \|y\|^{\beta}$ with $\|y\| \leq 1$ or $|f(x)| \geq C \|x\|^{rd} \|x\|^{-\beta} \geq C \|x\|^{-\beta}$ with $\|x\| \geq 1$.

Case (b): $Z(f)$ is now nonempty, and hence either $Z(f) = \{0\}$ or $Z(f)$ contains a nonzero element. If $Z(f) = \{0\}$, then $Z(f^*) = \varnothing$ or $\{0\}$. This comes under case (a), above, and we have $|f(x)| \geq C\|x\|^{-\beta}$ with $\|x\| \geq 1$ which gives (a).

Suppose $Z(f)$ contains a nonzero element. By H1, there exists constants $C_1 > 0$, $\alpha \geq 0$ such that $|f^*(x^{-1})| \geq C_1 \operatorname{dist}(x^{-1}, Z(f^*))^\alpha$ with $\|x\| \geq 1$. So by Lemma 4.10, for $C_2 = C_1 C^\alpha$, $|f(x)| \geq C_2 \operatorname{dist}(x, Z(f))^\alpha \|x\|^{-2\alpha}$ for $\|x\| \geq 1$, proving (b). □

Criterion for a polynomial not to be rapidly decreasing on a set S. In [Igusa 1978] Igusa and Raghavan develop what is essentially a criterion for a polynomial on an real vector space *not to be rapidly decreasing* on a set of vectors of norm ≥ 1. In this section we generalize that method to all local fields, introducing several polynomials in the criterion.

Lemma 4.11. *Let $\boldsymbol{f} : V \to k^r$ be a polynomial map and d the maximum of the degrees of its components. Then there exists a constant $C > 0$ such that for all $x, y \in V$ with $\|x\| \geq 1$,*

$$\|\boldsymbol{f}(x) - \boldsymbol{f}(y)\| \leq C\|x\|^{d-1} \max_{0 \leq r \leq d} (\|x - y\|^r).$$

Proof. It is enough to prove this for $r = 1$, $\boldsymbol{f} = f$. The estimate is compatible with addition in f and so we may assume f to be a monomial of degree d in some coordinate system on V. Assume the result for all monomials of degree $d - 1$. Then $f = x_i g$, where g is a monomial of degree $d - 1$. We have

$$x_i g(x) - y_i g(y) = x_i(g(x) - g(y)) + (x_i - y_i)(g(y) - g(x)) + (x_i - y_i)g(x),$$

and the estimate is obvious for each of the three terms. □

Proposition 4.12. *Let $S \subseteq V$ be a set with $\|x\| \geq 1$ for all $x \in S$. Let g be polynomial on V. If $Z(g) = \varnothing$, we have*

$$|g(x)| \geq \frac{C}{\|x\|^\gamma} \qquad (\|x\| \geq 1)$$

for some $C > 0$, $\gamma \geq 0$. Suppose $Z(g) \neq \varnothing$ and suppose that there exist polynomials $f_i : V \to k$, $i = 1, \ldots r$, and a constant $b > 0$ such that $\max |f_i(x) - f_i(y)| \geq b > 0$ *for all $x \in S$, $y \in Z(g)$. Then there exist constants $C > 0$ and $\gamma \geq 0$ such that*

$$|g(x)| \geq \frac{C}{\|x\|^\gamma} \qquad (x \in S). \tag{4-5}$$

Proof. The first statement is (a) of H2. We now assume $Z(g) \neq \varnothing$. We identify $V \simeq k^m$ and work in coordinates. Set $d := \max_i(\deg(f_i))$. In what follows, $C_1, C_2, \ldots,$ are constants > 0.

For all $x \in S$ and $y \in Z(g)$, by Lemma 4.11 for some constant $C > 0$, we have $0 < b \leq \max_{1 \leq i \leq r} |f_i(x) - f_i(y)| \leq C\|x\|^{d-1} \max_{1 \leq r \leq r} \|x - y\|^r$ for all $x \in S$, $y \in Z(g)$.

Choose $y \in Z(g)$ such that $\|x-y\| = \mathrm{dist}(x, Z(g))$. Then for all $x \in S$, we have

$$0 < b \le C_1 \|x\|^{d-1} \max_{1 \le r \le d} (\mathrm{dist}(x, Z(g))^r).$$

We consider the two cases (a) $\mathrm{dist}(x, Z(g)) \le 1$, so the maximum above is $\mathrm{dist}(x, Z(g))$, and (b) $\mathrm{dist}(x, Z(g)) > 1$, so the maximum is $\mathrm{dist}(x, Z(g))^d$.

By H2, there exist constants $C_2 > 0,\ \alpha, \beta \ge 0$ such that

$$|g(x)| \ge C_2 \mathrm{dist}(x, Z(g))^\alpha \|x\|^{-\beta},$$

so $\mathrm{dist}(x, Z(g)) \le C_3 |g(x)|^{1/\alpha} \|x\|^{\beta/\alpha}$. In case (a), $0 < b \le C_3 |g(x)|^{1/\alpha} \|x\|^{\beta/\alpha + (d-1)}$, and in case (b), $0 < b \le C_4 |g(x)|^{d/\alpha} \|x\|^{d\beta/\alpha + (d-1)}$. So in both cases, with $\delta = d\beta/\alpha + (d-1)$, one has

$$0 < b \le C_5 \|x\|^\delta \max(|g(x)|^{1/\alpha}, |g(x)|^{d/\alpha}).$$

Hence, $\max(|g(x)|, |g(x)|^d) \ge C_6 \|x\|^{-\delta\alpha}$, giving in all cases $|g(x)| \ge C_7 \|x\|^{-\delta\alpha}$ with $x \in S$. □

Lower bounds of $\|\nabla_r \boldsymbol{f}\|$ on stably noncritical level sets. Let V and $\boldsymbol{f} =: V \to k^r$ ($\boldsymbol{f} = (f_1, \dots f_r)$, $r \le m = \dim_k V$) be as usual. Let $C(\boldsymbol{f})$ be the critical set of $\boldsymbol{f}$, and $CV(\boldsymbol{f}) = \boldsymbol{f}(C(\boldsymbol{f}))$ have their usual meanings. Write $W = CV(\boldsymbol{f})$. We assume that the closure $\overline{W}$, in the k-topology of k^r, of W is a proper subset of k^r. Our assumption is equivalent to assuming that stably noncritical values of $\boldsymbol{f}$ exist, which is true in characteristic zero (see page 232). Let $L_{\boldsymbol{c}}$, $\nabla_r \boldsymbol{f}$, and $\partial_J \boldsymbol{f}$ be defined as in Section 2.

If $\omega \subset k^r \setminus \overline{W}$ is a compact set, then there exists $b > 0$ such that $\|u - v\| \ge b > 0$ for $u \in \omega$, $v \in \overline{W}$. This means $\max_i |f_i(x) - f_i(y)| \ge b > 0$, with $\boldsymbol{c} \in \omega$, $x \in L_{\boldsymbol{c}}$, $y \in C(\boldsymbol{f})$.

Proposition 4.13. *Let $\omega \subset k^r$ be an open set whose closure consists entirely of noncritical values of $\boldsymbol{f} = (f_1, \dots, f_r)$. For $\boldsymbol{c} \in \omega$, let $L_{\boldsymbol{c}}$ be defined as above. Then there exist constants, $C, \gamma > 0$ such that*

$$\|\nabla_r \boldsymbol{f}(x)\| \ge \frac{C}{\|x\|^\gamma} \qquad (x \in L_{\boldsymbol{c}}, \boldsymbol{c} \in \omega, \|x\| \ge 1) \tag{4-6}$$

Proof. We write (y_J) for the coordinates on $k^{\binom{m}{r}}$ and select a definite homogeneous form ν, which is positive definite of degree 2 if k archimedean, and of degree R on $k^{\binom{m}{r}}$, where R is any integer ≥ 2 such that $R^2 \ge \binom{m}{r}$, with the property that $|\nu(y)|^{1/R}$ is a norm on $k^{\binom{m}{r}}$, if k is nonarchimedean. Then $\nu(\nabla_r \boldsymbol{f}(x)) = 0$ if and only if $\nabla_r \boldsymbol{f}(x) = 0$, i.e., if and only if x is a critical point of $\boldsymbol{f}$. Let $g(x) = \nu(\nabla_r \boldsymbol{f}(x))$. Then $Z(g)$ is the set of critical points of $\boldsymbol{f}$. Suppose first that $Z(g) \ne \varnothing$. Now there exists $b > 0$ such that

$$\|u - v\| = \max_{1 \le i \le r} |u_i - v_i| \ge b > 0 \quad (u \in \omega, v \in \overline{W})$$

Hence, as $\boldsymbol{f}(x) \in \omega$ for $x \in L_{\boldsymbol{c}}$ $(c \in \omega)$ and $\boldsymbol{f}(y) \in \overline{W}$ for $y \in Z(g)$, $\|\boldsymbol{f}(x) - \boldsymbol{f}(y)\| \ge b > 0$. So by Proposition 4.12 there exist constants $C > 0$, $\delta \ge 0$ such that

$$|\nu(\nabla_r \boldsymbol{f}(x))| = |g(x)| \ge \frac{C}{\|x\|^\delta} \quad (x \in L_{\boldsymbol{c}},\ \boldsymbol{c} \in \omega,\ \|x\| \ge 1).$$

But ν is homogeneous of degree d ($d = 2$ for archimedean and R for nonarchimedean k) and definite. So there exist constants $C_1, C_2 > 0$ such that

$$C_1 \|\nabla_r \boldsymbol{f}(x)\|^d \le |\nu(\nabla_r \boldsymbol{f}(x))| = |g(x)| \le C_2 \|\nabla_r \boldsymbol{f}(x)\|^d.$$

So for suitable $C > 0$, $\gamma \ge 0$, we have $\|\nabla_r \boldsymbol{f}(x)\| \ge C\|x\|^{-\gamma}$. The case $Z(g) = \varnothing$ is taken care of by the first statement of Proposition 4.12. □

Remark 4.14. We cannot make $\gamma = 0$ in all cases. For instance, let $\operatorname{char} k = 0$ and $r = 1$, $f(x, y, z) = x^2z^2 + y^3z$ and $c = -1$. Consider $x_n = n$, $z_n = 1/n$, $y_n = -(2n)^{1/3}$. Then $F(x_n, y_n, z_n) = 1 - 2 = -1$, $\partial F/\partial X(x_n, y_n, z_n) = 2x_n z_n^2 \to 0$, and $\partial F/\partial Y(x_n, y_n, z_n) = 3y_n^2 z_n \to 0$, $\partial F/\partial Z(x_n, y_n, z_n) = 2x_n^2 z_n + y_n^3 = 2n - 2n = 0$. But $\|(x_n, y_n, z_n)\| = n$, $\|\nabla f(x_n, y_n, z_m)\| \sim \text{Const} \cdot 1/n^{1/3}$. So $\gamma \ge 1/3$. We do not know the minimal value of γ.

5. Proof of temperedness of canonical measures on stably noncritical level sets

Consequences of Krasner's lemma. The well-known lemma of Krasner [Artin 1967] has an important consequence (Corollary 5.3). Let k be a local field of arbitrary characteristic and K its algebraic closure. The following lemma must be well known, but we prove it in this form.

Lemma 5.1. *We can find a countable family $\{k_n\}$ of finite extensions of k with the property that any finite extension of k is contained in one of the k_n. In particular $K = \bigcup_n k_n$.*

Proof. We first work with separable extensions of fixed degree n over k. Let S_n be the set of monic, irreducible and separable elements of $k[X]$ of degree n. Then it follows from Krasner's lemma that if $f \in S_n$, there is an $\varepsilon = \varepsilon(f) > 0$ with the following property: if g is monic and $\|f - g\| < \varepsilon$, then $g \in S_n$ and $K(f) = K(g)$ in K, where $K(h)$ denotes the splitting field of h. Since S_n is a separable metric space, it follows that there are at most a countable number of these splitting fields, and any separable extension of degree n over k is contained in one of these. Let us enumerate these splitting fields as $\{k_{nj}\}$ $(j = 1, 2, \dots)$. If k has characteristic 0 we are already finished. Suppose k has characteristic $p > 0$. Let $F(x \mapsto x^p)$ be the Frobenius automorphism of K. Define the extension $k_{njr} = F^{-r}(k_{nj})$ for $r = 1, 2, \ldots,$ which are clearly finite over k. Clearly, any finite extension of k of finite degree is contained in one of the k_{njr}. □

Remark 5.2. If k has characteristic 0, then there are only a finite number of extensions of fixed degree n. But in prime characteristic this is not true: the field $k = F_2[\![X]\!][X^{-1}]$ of Laurent series in X with F_2 a finite field of characteristic 2 has a countably infinite number of separable quadratic extensions. Indeed, the extensions defined by $T^2 - T - c = 0$ are distinct for infinitely many values of c.

Corollary 5.3. *If M is an affine subvariety of some A^n_K and $M(k')$ is countable for all finite extensions k' of k, then M is finite.*

Proof. By Lemma 5.1, $M(K) = \bigcup_{k'} M(k')$ is countable, hence finite. □

A consequence of the refined Bézout's theorem. The refinement of Bézout's theorem due to Fulton [1998, Example 8.4.7, p. 148, and Section 12.3] (see also [Vogel 1984, Corollary 2.26, p. 85]), is the statement that if Z_i $(1 \le i \le r)$ are r $(r \ge 2)$ pure dimensional varieties in $\mathbb{P}^m_K$, then the number of irreducible components of $\bigcap_i Z_i$ is bounded by the Bézout number $\prod_i \deg(Z_i)$. It has the following simple consequence.

Lemma 5.4. *Let U be a nonempty Zariski open subset of $\mathbb{A}^r_K$ so that $U \subset \mathbb{A}^r_K \subset \mathbb{P}^r_K$. Let h_i $(i = 1, 2, \ldots, r)$ be polynomials on A^r_K with $\deg h_i =: d_i$, and let Z_i be the zero locus of h_i. Let $Z_i^\times = Z_i \cap U$ and $\bar{Z}_i$ the closure of Z_i in $\mathbb{P}^r_K$. If $\bigcap_i Z_i^\times = F$ is nonempty and finite, then F has at most $D := \prod_i d_i$ elements.*

Proof. Since $\mathbb{A}^r_K$ is Zariski dense in $\mathbb{P}_K$ we have $\bar{Z}_i \cap A^r_K = Z_i$; moreover, $\bar{Z}_i$ is of pure degree d_i. Let W_0 be an irreducible component of $W := \bigcap \bar{Z}_i$ that meets U. Since W_0 is irreducible and $W_0 \cap U$ is nonempty open in W_0, it is dense in W_0. Let $w \in W_0 \cap U$. Then w is in each of the $\bar{Z}_i \cap U$ and so $w \in F$. So $W_0 \cap U$ is finite and contained in F. Since $W_0 \cap U$ is dense in W_0, it follows that $W_0 \cap U$ must consist of a single element of F and W_0 itself consists of that point. Moreover all points of F are accounted for in this manner as F is contained in the union of irreducible components of W which meet U. Hence the cardinality of F is at most the number of irreducible components of W, which is at most D. □

The maps π_J and a universal bound for the cardinality of their fibers. Let $V \simeq k^m$ so that $\boldsymbol{f} = (f_1, \ldots, f_r)$ with $f_j \in k[x_1, \ldots, x_m]$. Assume that $\boldsymbol{c}$ is a noncritical value of $\boldsymbol{f}$ so that $L_{\boldsymbol{c}}$ has no singularities. Fix $J \subset \underline{m} := \{1, \ldots, m\}$, and let $\pi_J : k^m \to k^{m-r}$ map $(x_1, \ldots, x_m)$ to $(y_1, \ldots, y_{m-r})$, where $\{y_j\}_{j=1}^{m-r} = \{x_i \mid i \in \underline{m} \setminus J\}$. We wish to prove that the map π_J restricted to $L_{\boldsymbol{c}}$ has fibers of cardinality $\le D := d_1 \cdots d_r$, where $d_i := \deg(f_i)$. Without loss of generality assume $J = \{1, \ldots, r\}$, so that

$\pi_J : (x_1, \dots, x_m) \mapsto (x_{r+1}, \dots, x_m)$. Write $x = (x_1, \dots, x_m)$ and $y = (x_{r+1}, \dots, x_m)$. Define z so that $x = (z, y)$.

We regard $L_{\boldsymbol{c}}$ as an affine variety and $L_{\boldsymbol{c},J}$ as an affine open subvariety. For any k' with $k \subset k' \subset K$ we have the respective sets of k'-points, $L_{\boldsymbol{c}}(k')$ and $L_{\boldsymbol{c},J}(k')$. Denote the restriction of π_J to $L_{\boldsymbol{c},J}$ by $\bar{\pi}_J$.

Proposition 5.5. *Let $D = \prod_{1 \le i \le r} d_i$. Then the fibers of $\bar{\pi}_J$ are all of cardinality $\le D$.*

Proof. Note that $d\bar{\pi}_J$ is an isomorphism on $L_{\boldsymbol{c},J}(k)$. Hence $U_J(k) := \bar{\pi}_J(L_{\boldsymbol{c},J}(k))$ is open in k^{m-r} and $\bar{\pi}_J$ is a local analytic isomorphism of $L_{\boldsymbol{c},J}(k)$ onto $U_J(k)$. For any field k' between k and K, we write again $\bar{\pi}_J$ for the map $L_{\boldsymbol{c},J}(k') \to k'^{m-r}$, and $U_J(k')$ for its image. If k' is a *finite* extension of k, then k' is again a local field; exactly as for k, we have $d\bar{\pi}_J : L_{\boldsymbol{c},J}(k') \to U_J(k')$ is an analytic isomorphism. For any k', $k \subset k' \subset K$ with k'/k finite, $U_J(k')$ is open in k'^{m-r} and the fibers of $\bar{\pi}_J$ on $L_{\boldsymbol{c},j}(k')$ are discrete and at most countable. If we then fix $y \in U_J(k)$, and write W_y for the affine variety $\bar{\pi}_J^{-1}(y)$, then $W_y(k')$ is at most countable for all finite extensions k'/k. Hence, by Corollary 5.3, $W_y(K)$ is finite. Let $F := W_y(K)$.

On the other hand, $\pi_J^{-1}(y)(K) = K^r \times \{y\} \simeq K^r$. Let $h_i(z) := f_i(z, y) - c_i$. Then h_i is a polynomial on K^r of degree $\le d_i$. Moreover, since $\bar{\pi}_J^{-1}(y)(k)$ is nonempty, $\partial(h_1, \dots, h_r)/\partial(x_1, \dots, x_r) = \partial_J(z, y)$ is not identically zero on K^r. Thus, $\{z | \partial_J(z, y) \ne 0\}$ is a nonempty affine open U_1 in K^r. Moreover, $F = \bigcap_{1 \le i \le r} Z(h_i)^\times$ where $Z(h_i)^\times := Z(h_i) \cap U_1$. So Lemma 5.4 applies and proves that $\# F \le D$. $\square$

Lemma 5.6. *Let ∂_J be as on page 232. Then if ω_{m-r} is the exterior form corresponding to the Haar measure on k^{m-r}, the exterior form*

$$\rho_{\boldsymbol{c}} := \frac{1}{\partial_J(x)} \bar{\pi}_J^*(\omega_{m-r})$$

on $L_{\boldsymbol{c},J}$ has the property that $|\rho_{\boldsymbol{c}}|$ generates the measure $\mu_{\boldsymbol{c}} := \mu_{\boldsymbol{f},\boldsymbol{c}}$. In particular, if λ is the Haar measure on k^{m-r} and ν is the measure generated by $|\bar{\pi}_J^*(\omega_{m-r})|$, then $\bar{\pi}_J$ takes ν to λ in small open neighborhoods of each point of $L_{\boldsymbol{c},J}(k)$, and $d\mu_{\boldsymbol{c}} = |\partial_J(x)|^{-1} \, d\nu$.

Proof. This is clear from (2-4). $\square$

Proof of Theorem 1.2. This follows from three things: the lower bounds on $\|\nabla_r\|$ when $\boldsymbol{c}$ is a stably noncritical value of $\boldsymbol{f}$, the relationship between λ, ν, $\mu_{\boldsymbol{f},\boldsymbol{c}}$, and the temperedness of λ. The simple measure-theoretic lemma below explains this. Let R, S be locally compact metric spaces which are second countable, with Borel measures r, s respectively on them, and $\pi : R \to S$ a continuous surjective map which is a local homeomorphism, and takes r to s in a small neighborhood of each point of R: this means that for each $x \in R$ there are open sets M_x, $N_{\pi(x)}$ containing x and $\pi(x)$ respectively, such that π is a homeomorphism of M_x with $N_{\pi(x)}$ and takes r to s.

Lemma 5.7. *If there is a natural number d such that all fibers of π have cardinality at most d, then for each Borel set $E \subset R$, $\pi(E)$ is a Borel set in S, and we have*

$$r(E) \leq d \cdot s(\pi(E)).$$

Moreover if $f \geq 0$ is a continuous function on R and t is the Borel measure on R defined by $dt = f\,dr$, then for any Borel set $E \subset R$ we have

$$t(E) \leq \sup_E |f| \cdot d \cdot s(\pi(E)).$$

Proof. The second inequality follows trivially from the first, so that we need only prove the first. We use induction on d. For $d = 1$, π is a continuous bijection of R with S; being a local homeomorphism, it is then a global homeomorphism. It is easy to see that it takes r to s globally, and so the results are trivial. Let $d > 1$, assume the results for $d - 1$, and suppose that there are points of S the fibers over which have cardinality exactly d. Let S_d be the set of such points in S. Now, if the fiber above a point has e elements, the fibers of neighboring points have cardinality $\geq e$, and so S_d is open in S. Let $R_d = \pi^{-1}(S_d)$. Then $\pi : R_d \to S_d$ is a d-sheeted covering map. If $x \in R_d$, we can find an open set M containing $\pi(x)$ such that $N := \pi^{-1}(M) = \bigsqcup_{1 \leq j \leq d} N_j$ where $\pi : N_j \to M$ is a homeomorphism taking r to s. If $E \subset N$ is a Borel set, then $E = \bigsqcup_j E \cap N_j$, so that $\pi(E) = \bigcup_j \pi(E \cap N_j)$ is Borel as π is a homeomorphism on each N_j. Moreover,

$$r(E) = \sum_j r(E \cap N_j) = \sum_j s(\pi(E \cap N_j) \leq d \cdot s(\pi(E)).$$

These two properties are true with any Borel $M' \subset M$ and $N' = \pi'(M')$ replacing M, N respectively. Write now $S_d = \bigcup_n M_n$ where the M_n are open and have the properties described above for M. Then $S_d = \bigsqcup_n M'_n$ where $M'_n \subset M_n$, so that $R_d = \bigsqcup_n \pi^{-1}(M'_n)$. The two properties above are valid for any Borel set contained in any $\pi^{-1}(M'_n)$, hence they follow for any Borel set $E \subset R_d$. Write $S' = S \setminus S_d$, $R' = \pi^{-1}(S') = R \setminus R_d$. Then (R', S', π) inherit the properties of (R, S, π) with $d - 1$ instead of d. The result is valid for (R', S', π) and hence for (R, S, π), as is easily seen.

We are now ready to prove Theorem 1.2. Assume that $\boldsymbol{c}$ is a stably noncritical value of $\boldsymbol{f}$. For simplicity of notation we will suppress mentioning $\boldsymbol{c}$, because all of our estimates are locally uniform in $\boldsymbol{c}$. On $L_{\boldsymbol{c}} = L$ we have the estimate

$$\|\nabla_r(x)\| = \max_J |\partial_J(x)| > \frac{C}{\|x\|^\gamma} \qquad (\|x\| \geq 1),$$

where $C > 0$, $\gamma \geq 0$ are constants that remain the same when $\boldsymbol{c}$ is varied in a small neighborhood of $\boldsymbol{c}$. Let us write L^+ for the subset of L where $\|x\| > 1$. Now, at each point $x \in L^+$ some $|\partial_J(x)|$ equals $\|\nabla_r(x)\|$. Hence if we write

$$M_J = \{x \in L^+ \mid |\partial_J(x)| > C\|x\|^{-\gamma}\},$$

then

$$L^+ = \bigcup_J M_J.$$

The map $\bar{\pi}_J$ is open on M_J onto its image W_J and is a local analytic isomorphism. Moreover, if λ, ν, $\mu = \mu_{\boldsymbol{c}}$ have the same meaning as before, we have, on M_J,

$$d\mu = |\partial_J(x)|^{-1}\, d\nu$$

and hence, for any Borel set $E \subset M_J$, with D as in Lemma 5.4,

$$\mu(E) \le D \cdot \sup_E |\partial_J(x)^{-1}| \cdot \lambda(\pi_J(E)).$$

Remembering that $|\partial_J(x)|^{-1} < C^{-1}\|x\|^{\gamma}$, we get from this that

$$\mu(E) \le DC^{-1} \cdot \sup_E \|x\|^{\gamma} \cdot \lambda(\pi_J(E)).$$

If we take $E = B_R \cap M_J$ where $B_R = \{x \in k^m \mid \|x\| < R\}$, we see that $\pi_J(E)$ is a subset of the open ball of k^{m-r} of radius R, and hence $\lambda(\pi_J(E)) \le AR^{m-r}$ where A is a universal constant. Hence

$$\mu(B_R \cap M_J) \le ADC^{-1} \cdot R^{m-r+\gamma}.$$

Since this is true for all J, the temperedness of μ together with the growth estimate is proved, as well as the assertion that the last estimate remains unchanged if $\boldsymbol{c}$ varies in a small neighborhood of its original value. This finishes the proof of Theorem 1.2. □

6. Invariant measures on regular adjoint orbits of a semisimple Lie algebra

As an application of our Theorem 1.2 we shall prove that the invariant measures on *regular* semisimple orbits of a semisimple Lie algebra $\mathfrak{g} := \mathfrak{g}_K$ over a local field k of *characteristic* 0 are tempered.

The restriction to regular orbits is a consequence of the methods we use; the result is expected to be true without any condition on the orbit of the adjoint action.

For the moment let k be any field of characteristic 0 and K the algebraic closure of k. We write $\mathfrak{g}_K = K \otimes_k \mathfrak{g}_k$. Let $P(K)$ be the K-algebra of polynomial functions on $\mathfrak{g}_K$ with values in K. Since such a polynomial is determined by its restriction to $\mathfrak{g}_k$, the restriction to k defines an isomorphism of $P(K)$ with the K-algebra $P_k(K)$ of K-valued polynomial functions on $\mathfrak{g}_k$.

Let G be the connected adjoint group of $\mathfrak{g}_k$. It is a linear algebraic group defined over k and we write $G(k')$ for the group of its points over k', $k \subset k' \subset K$. We regard $G(k')$ as a subset of $G = G(K)$. From [Borel 1991] we know that $G(k)$ is Zariski-dense in $G(K)$. Now $G(K)$ acts on $P(K)$ and we denote by $J(K)$ the K-algebra of invariants of this action, which is a graded algebra in the obvious way. By a theorem of Chevalley, $J(K)$ is freely generated by homogeneous elements $p_1, \dots, p_r$ of degrees $d_1, \dots, d_r$ respectively, where r is the rank of $\mathfrak{g}_k$. In view of our remarks above, $J(K)$ is isomorphic to the graded K-algebra of invariants of $G(k)$ in $P_k(K)$. The action by $G(k)$ leaves $P_k(k)$ invariant, and we write $J(k)$ for the graded k-subalgebra of $G(k)$- invariants in $P_k(k)$. It is clear that

$$J(k) \simeq J(K)^{\mathrm{Gal}(K/k)}$$

as graded k-algebras.

The following lemma is surely known but we include it for the sake of completeness.

Lemma 6.1. *The graded k-algebra $J(k)$ is freely generated by homogeneous elements $q_1, \dots, q_r$ of degrees $d_1, \dots, d_r$ respectively.*

Proof. There is a finite extension k' of k with $k \subset k' \subset K$ such that the free homogeneous generators p_i of $J(K)$ have their coefficients in k'. Hence we may come down from K to k'. Let (e_α) be a k-basis for k'. Then we can write each p_i as

$$p_i = \sum_\alpha p_{i,\alpha} e_\alpha \quad (p_{i,\alpha} \in P_k(k)).$$

Since the $p_{i,\alpha}$ are k-valued, the $G(k)$-invariance of the p_i implies that the $p_{i,\alpha}$ are in $J(k)$. Now the p_i are algebraically independent, and so, $\omega := dp_1 \wedge \cdots \wedge dp_r \not\equiv 0$. Let

$$\omega_{\alpha_1,\dots,\alpha_r} = dp_{1,\alpha_1} \wedge \cdots \wedge dp_{r,\alpha_r}.$$

Then

$$\omega = \sum_{\alpha_1,\dots,\alpha_r} \omega_{\alpha_1,\dots,\alpha_r} e_{\alpha_1} \wedge e_{\alpha_2} \cdots \wedge e_{\alpha_r} \not\equiv 0.$$

Hence we can choose $\alpha_1, \dots, \alpha_r$ such that $\omega_{\alpha_1,\dots,\alpha_r} \not\equiv 0$. With this choice, let

$$q_i = p_{i,\alpha_i} \qquad (1 \le i \le r).$$

Then the q_i are homogeneous elements of $J(k)$ and $\deg(q_i) = d_i$, and they are algebraically independent.

Now $J(k')$ is freely generated by the p_i of degree d_i. Hence, its Poincaré series is $\prod_i (1 - T^{d_i})^{-1}$. For any integer $m \ge 1$ let D_m be the dimension of $J(k')_m$, the subspace of degree m in $J(k')$. So $\dim(J(k)_m) \le D_m$. On the other hand, let $J_1(k)$ be the subalgebra of $J(k)$ generated by the q_i. Since the q_i are homogeneous, this is a graded subalgebra of $J(k)$, and it has the same Poincaré series as $J(k')$. Now $J_1(k)_m \subset J(k)_m$ for all m, and so

$$D_m = \dim J_1(k)_m \le \dim J(k)_m \le \dim J(k')_m = D_m.$$

This proves that $J_1(k)_m = J(k)m$ for all m, so that $J_1(k) = J(k)$. This finishes the proof of the lemma. □

Let $r = \operatorname{rank}(\mathfrak{g})$. Then by assumption we can choose $g_1, \dots, g_r \in J(k)$ freely generating $J(k)$, hence also $J(K)$ (over K). An element $H \in \mathfrak{g}_K$ is semisimple (resp. nilpotent) if ad X is semisimple (resp. nilpotent). A semisimple element H is called *regular* if its centralizer is a Cartan subalgebra (CSA) of $\mathfrak{g}_K$. There is an invariant polynomial $D \in J(k)$, called the discriminant of $\mathfrak{g}$, such that if $X \in \mathfrak{g}_k$, X is semisimple and regular if and only if $D(X) \neq 0$. If $Y \in \mathfrak{g}$ is any element, we can write $Y = H + X$ where H is semisimple and X is a nilpotent in the derived algebra of the centralizer of H in $\mathfrak{g}_K$ (which is semisimple). It is known [Kostant 1963] that the orbit of $H + X$ has H in its closure, and so, for any $g \in J(K)$, we have $g(H) = g(H+X)$. If $\mathfrak{h}_K$ is a CSA of $\mathfrak{g}_K$, it is further known that the restriction map from $\mathfrak{g}_K$ to $\mathfrak{h}_K$ is an isomorphism of $J(K)$ with the algebra $J(\mathfrak{h}_K)$ of polynomials on $\mathfrak{h}_K$ invariant under the Weyl group W_K of $\mathfrak{h}_K$. It is known that the differentials $dg_1, \dots, dg_r$ are linearly independent at an element Y of $\mathfrak{g}_K$ if and only if Y lies in an adjoint orbit of maximal dimension, which is $\dim(\mathfrak{g}_K) - \operatorname{rank}(\mathfrak{g}_K) = n - r$, where $n = \dim(\mathfrak{g}_K)$ [Kostant 1963]. If Y is semisimple, this happens if and only if Y is regular. Let $\mathfrak{g}'_K$ be the invariant open set of regular semisimple elements. We write

$$\boldsymbol{F} = (g_1, \dots, g_r) \colon \mathfrak{g}_K \mapsto K^r$$

and view it as a polynomial map of $\mathfrak{g}_K$ into K^r commuting with the action of the adjoint group. Before we apply Theorem 1.2 to this set up, we need some preliminary discussion. Let $\mathcal{R} = \boldsymbol{F}(\mathfrak{g}'_K)$. The next lemma deals with the situation over K.

Lemma 6.2. *We have $\mathfrak{g}'_K = \boldsymbol{F}^{-1}(\mathcal{R})$. Moreover $\mathcal{R}$ is Zariski open in K^r, and is precisely the set of noncritical values of $\boldsymbol{F}$, so that all the noncritical values are also stably noncritical. Moreover, for any $\boldsymbol{c} \in \mathcal{R}$, the preimage $\boldsymbol{F}^{-1}(\boldsymbol{c})$ is an orbit under the adjoint group, consisting entirely of regular semisimple elements, hence smooth.*

Proof. Since $dg_1 \wedge \cdots \wedge dg_r \neq 0$ everywhere on $\mathfrak{g}'_K$, the map $\boldsymbol{F}$ is smooth on $\mathfrak{g}'_K$. Hence it is an open map [Görtz and Wedhorn 2010, Corollary 14.34], showing that $\boldsymbol{F}(\mathfrak{g}'_K) = \mathcal{R}$ is open in K^r.

We shall prove that if $Y \in \mathfrak{g}_K$ and $X \in \mathfrak{g}'_K$ are such that $\boldsymbol{F}(Y) = \boldsymbol{F}(X)$, then Y is regular semisimple, and is conjugate to X under the adjoint group. Suppose Y is not regular semisimple. Write $Y = Z + N$, where Z is semisimple and N is a nilpotent in the derived algebra of the centralizer of Z. The $\boldsymbol{F}(Y) = \boldsymbol{F}(Z) = \boldsymbol{F}(X)$.

Using the action of the adjoint group separately on X and Z we may assume that $X, Z \in \mathfrak{h}_K$ where $\mathfrak{h}_K$ is a CSA, and $\boldsymbol{F}(X) = \boldsymbol{F}(Z)$. Then all Weyl group invariant polynomials take the same value at Z and X and so Z and X are conjugate under the Weyl group. But as X is regular, so is Z, hence $N = 0$ or Y itself is regular semisimple. So, $\mathfrak{g}'_K = \boldsymbol{F}^{-1}(\mathcal{R})$. But then the above argument already shows that Y and X are conjugate under the adjoint group. Since the fibers of $\boldsymbol{F}$ above points of $\mathcal{R}$ are smooth, all points of $\mathcal{R}$ are stably noncritical. It remains to show that there are no other noncritical values. Suppose $Y \in \mathfrak{g}_K$ is such that $\boldsymbol{d} = \boldsymbol{F}(Y)$ is a noncritical value where $\boldsymbol{d} \notin \mathcal{R}$. Then $Y \notin \mathfrak{g}'_K$. Now $Y = Z + N$ as before, where Z is no longer regular (it is semisimple still). Then $\boldsymbol{F}(Z) = \boldsymbol{F}(Y)$ and so $Z \in \boldsymbol{F}^{-1}(\boldsymbol{d})$. But as Z is semisimple but not regular, $dg_1 \wedge \cdots \wedge dg_r$ is zero at Z [Kostant 1963]. Thus Z is a singular point of $\boldsymbol{F}^{-1}(\boldsymbol{d})$, contradicting the fact that $\boldsymbol{d}$ is noncritical. The lemma is thus completely proved. □

We now come to the case where the ground field is k, a local field of characteristic 0. We assume that the g_i have coefficients in k. Fix a regular semisimple element H_0 in $\mathfrak{g}_k$. Let

$$W(k) := W_{H_0}(k) = \{X \in \mathfrak{g}(k) \mid g_i(X) = g_i(H_0)(1 \le i \le r)\}.$$

Theorem 6.3. *Then the canonical measure on $W(k)$ is tempered, and the growth estimate* G (*see Section 1*) *is uniform when H varies in a neighborhood of H_0.*

Proof. For the map $\boldsymbol{F}$ on $\mathfrak{g}_k$ we know that $(g_1(H_0), \ldots, g_r(H_0))$ is a stably noncritical value and so the theorem follows at once from Theorem 1.2. □

Although $W(K)$ is a single orbit under $G(K)$, this may no longer true over k. $W(k)$ is a k-analytic manifold of dimension $n - r$. On the other hand, the stabilizer in $G(k)$ of any point of $W(k)$ has dimension r and so its orbit under $G(k)$ is an open submanifold of $W(k)$. If we do this at every point of $W(k)$ we obtain a decomposition of $W(k)$ into a disjoint union of $G(k)$-orbits which are open submanifolds of dimension $n - r$ and so all these submanifolds are closed also. Thus the orbit $G(k).H_0$ is an open and closed submanifold of $W(k)$ of dimension $n - r$. Now the canonical measure on $W(k)$ is invariant under $G(k)$ and so on the orbit $G(k).H_0$ it is a multiple of the invariant measure on the orbit. Note that the orbit being closed, the invariant measure on it is a Borel measure on $\mathfrak{g}_k$. Since the canonical measure is tempered on $W(k)$ by Theorem 6.3, it is immediate that the invariant measure on the orbit $G(k).H_0$ is also tempered. Hence we have proved the following theorem:

Theorem 6.4. *The orbits of regular semisimple elements of $\mathfrak{g}_k$ are closed, and the invariant measures on them are tempered.*

For temperedness of invariant measures on semisimple symmetric spaces at the Lie algebra level over $\mathbb{R}$; see [Heckman 1982].

Remark 6.5. Ranga Rao [1972] and Deligne have independently shown that for any $X \in \mathfrak{g}_k$, there is an invariant measure on the adjoint orbit of X, and this measure extends to a Borel measure on the k-closure of the adjoint orbit of X. It is natural to ask if these are tempered in our sense when k is nonarchimedean. We shall consider this question in another paper since it does not follow from the results proved here.

7. Examples

In this section we give some examples. We consider only single polynomials ($r = 1$) of degree $d \geq 3$, defined over a local field k of characteristic 0. Let $f \in k[x_1, \dots, x_m]$.

Elementary methods when r = 1 and f is homogeneous. For f homogeneous we have Euler's theorem on homogeneous functions, which asserts that $\sum_i x_i \partial f/\partial x_i = d \cdot f$. Let $L_c = \{x \in k^m \mid f(x) = c\}$ for $c \in k$. Then, for any critical point x of f, we have $f(x) = 0$, i.e., L_0 contains all the critical points. So every $c \in k \setminus \{0\}$ is a noncritical value and so is also stably noncritical. Moreover, Euler's identity for $x \in L_c$, $c \neq 0$, gives $\sum_i x_i \partial f/\partial x_i = dc$, so that we have

$$|d||c| = \left|\sum_i x_i \frac{\partial f}{\partial x_i}\right| \leq C\|x\|\|\nabla f(x)\| \quad (C > 0),$$

giving the estimate, with A a constant > 0,

$$\|\nabla f(x)\| \geq A\|x\|^{-1}, \quad \|x\| \geq 1,\ x \in L_c.$$

Moreover the projection $(x_1, \dots, x_m) \mapsto (x_1, \dots, \hat{x}_i, \dots x_m)$ has the property that all fibers have cardinality $\leq d$. We thus have Theorem 1.2 with

$$\mu_{f,c} = O(R^m) \qquad (R \to \infty),$$

where O is uniform locally around c. We can actually say more.

Proposition 7.1. *Suppose* $\mathbf{0}$ *is the only singularity in* L_0, *i.e., the projective locus of* L_0 *is smooth. Then for any compact set* $W \subset k \setminus \{\mathbf{0}\}$, *we have*

(7-1)
$$\inf_{c \in W, x \in L_c, \|x\| \geq 1} \|\nabla f(x)\| > 0.$$

Moreover, the measure $\mu_{f,0}$ *defined on* $L_0 \setminus \{0\}$ *is finite in open neighborhoods of* $\mathbf{0}$ *if* $m > d$, *so that it extends to a Borel measure on* L_0. *Finally, for all* $c \in k$,

$$\mu_{f,c}(B_r) = O(R^{m-1}).$$

If $m \leq d$, *there are examples where* $\mu_{f,0}$ *is not finite in neighborhoods of* $\mathbf{0}$.

Proof. To prove (7-1) assume (7-1) is not true. Then we can find sequences $c_n \in W$, $x_n \in L_{c_n}$ such that $c_n \to c \in W$, $\nabla f(x_n) \to 0$ as $n \to \infty$. By passing to a subsequence and permuting the coordinates we may assume that $x_n = (x_{n1}, \ldots, x_{nm})$ where $|x_{n1}| \geq |x_{nj}|$ ($j \geq 2$) and $|x_{n1}| \to \infty$. Now,

$$f(x_{n1}, \ldots, x_{nm}) = x_{n1}^d f(1, x_{n1}^{-1} x_{n2}, \ldots, x_{n1}^{-1} x_{nm}) = c_n \to c$$

and

$$(\nabla f)(x_{n1}, \ldots, x_{nm}) = x_{n1}^{d-1} (\nabla f)(1, x_{n1}^{-1} x_{n2}, \ldots, x_{n1}^{-1} x_{nm}) \to 0.$$

Now $|x_{n1}^{-1} x_{nj}| \leq 1$ for $2 \leq j \leq m$ and so, passing to a subsequence, we may assume that $x_{n1}^{-1} x_{nj} \to v_j$ for $j \geq 2$. Hence,

$$f(1, v_2, \ldots, v_m) = 0 \quad \text{and} \quad (\nabla f)(1, v_2, \ldots, v_m) = 0,$$

showing that $(1, v_2, \ldots, v_m) \neq (0, \ldots, 0)$ is a singularity of L_0. Then (7-1) leads to the conclusion

$$\mu_{f,c}(B_R) = O(R^{m-1}) \quad (R \to \infty)$$

locally uniformly at each $c \neq 0$.

For $\mu_{f,0}$ defined on $L_0 \setminus \{0\}$, one must first show that it is finite on small neighborhoods of 0, i.e., it extends to a Borel measure on L_0, if $m > d$. Let $S = \{u \in L_0 \mid \|u\| = 1\}$. Then there exist constants $a, b > 0$ such that $a \leq \|\nabla f(x)\| \leq b$ for all $x \in S$. Hence, by homogeneity,

$$a\|x\|^{d-1} \leq \|\nabla f(x)\| \leq b\|x\|^{d-1} \qquad (x \in L_0 \setminus \{0\}).$$

Hence

$$\|\nabla f(x)\| \geq a > 0 \qquad (x \in L_0, \|x\| \geq 1).$$

As before, this leads to $\mu_{f,0}(B_R \setminus B_1) = O(R^{m-1})$ as $R \to \infty$. Around $\mathbf{0}$ we obtain the finiteness of $\mu_{f,0}$ from the estimate $b^{-1}\|x\|^{-(d-1)} \leq \|\nabla f(x)\|^{-1} \leq a^{-1}\|x\|^{-(d-1)}$ and the fact that

$$\int_{x \in k^{m-1}, 0 < \|x\| < 1} \|x\|^{-(d-1)} \, d^{m-1}x < \infty$$

if $m > d$ for both $k = \mathbb{R}$ and k nonarchimedean. We shall now suppose that $f = X^4 + Y^4 - Z^4$. Then $\mathbf{0}$ is the only critical point. The map $(x, y, z) \mapsto (x, y)$ on $L_0 \cap \{(x, y, z) \mid x > 0\}$ is a diffeomorphism and the measure $\mu_{f,0}$ is

$$\frac{1}{|\partial f / \partial z|} \, dx \, dy = \frac{1}{4} \frac{dx \, dy}{(x^4 + y^4)^{3/4}}$$

and it is easy to verify that

$$\iint_N \frac{dx \, dy}{(x^4 + y^4)^{3/4}} = \infty$$

for any neighborhood N of $(0, 0)$. ☐

Remark 7.2. It follows from Proposition 7.1 that to have

$$\inf_{x\in L_c, \|x\|\geq 1} \|\nabla f(x)\| = 0 \quad (c \neq 0) \tag{7-2}$$

we must look for f such that L_0 has singular points $\neq \mathbf{0}$. In the next section we describe some of these examples.

Some hypersurfaces in $\mathbb{P}_k^{m-1}$ with $[1:0:\cdots:0]$ as an isolated singularity. We do not try to give a "normal form" for such hypersurfaces; nevertheless large families of these can be described. We work in k^m, k a local field of characteristic 0. Since the first coordinate axis in k^m is chosen to be an isolated critical line (ICL), the first variable will be distinguished in what follows. Let us write $X, Y_1, \dots, Y_{m-1}$ as the variables. Write $\boldsymbol{Y} = (Y_1, \dots, Y_{m-1})$. Let $C(\varepsilon) = \{(X, \boldsymbol{Y}) \mid \|\boldsymbol{Y}\| \leq \varepsilon|X|\}$

Lemma 7.3. *Suppose $(X_n, \boldsymbol{Y}_n)$ is a sequence of points in L_c $(c \neq 0)$ such that they are in $C(\varepsilon)$ for some $\varepsilon < 1$. Let $F(X_n, \boldsymbol{Y}_n) = c \neq 0$ and $\nabla F(X_n, \boldsymbol{Y}_n) \to 0$. Then if the X-axis is an (ICL) for F, we must have $X_n \to \infty$, $1/X_n\boldsymbol{Y}_n \to \mathbf{0}$ as $n \to \infty$.*

Proof. By Euler's theorem, there is no singularity on L_c $(c \neq 0)$. Hence $\|\nabla F\|$ is bounded away from 0 on each compact subset of L_c. Hence, item 2 above implies $\|(X_n, \boldsymbol{Y}_n)\| = |X_n| \to \infty$. Then $\|{X_n}^{-1}\boldsymbol{Y}_n\| \leq 1$ and has a limit point $\boldsymbol{\eta}$. Passing to a subsequence, if necessary, we have ${X_n}^{-1}\boldsymbol{Y}_n \to \boldsymbol{\eta}$ as $n \to \infty$. If $d = \deg(F)$ we have $X_n^d F(1, {X_n}^{-1}\boldsymbol{Y}_n) = c$, $X_n^{d-1}\partial_X F(1, {X_n}^{-1}\boldsymbol{Y}_n) \to 0$, and $X_n^{d-1}\partial_{Y_i} F(1, {X_n}^{-1}\boldsymbol{Y}_n) \to 0$. So $F(1, \boldsymbol{\eta}) = 0$ and $\nabla F(1, \eta) = 0$, while $\boldsymbol{\eta} \in C(\varepsilon)$. Hence $\boldsymbol{\eta} = \mathbf{0}$ since ε can be arbitrarily small. □

Lemma 7.4. *If $(1, \mathbf{0})$ is a critical point of F, then F has the form*

$$F = X^{d-2}p_2 + X^{d-3}p_3 + \cdots + p_d$$

where p_r is a homogeneous polynomial in $\boldsymbol{Y}$ of degree r.

Proof. Write $F = X^{d-2}p_2 + X^{d-3}p_3 + \cdots + p_d$. Then p_0 is a constant, and $F(1, \mathbf{0}) = \mathbf{0}$ gives $p_0 = 0$. Then, $\partial F/\partial Y_i(1, \mathbf{0}) = 0$ gives $p_1 = 0$. □

From now on we let $d \geq 3$ and write

$$F = X^{d-2}p_2 + \cdots + p_d, \quad G = p_2 + \cdots + p_d.$$

Note that G is a polynomial in $\boldsymbol{Y}$, but *not necessarily* homogeneous.

Lemma 7.5. *If $\mathbf{0}$ is an isolated critical point (ICP) of G, then the X-axis is an ICL of F. In particular, this is so if the quadratic form p_2 is nondegenerate.*

Proof. We must prove that if $(1, \boldsymbol{Y}_n)$ is a CP for F with $\boldsymbol{Y}_n \to \mathbf{0}$, then $\boldsymbol{Y}_n = 0$ for $n \geq 1$. The conditions for $(1, \boldsymbol{Y}_n)$ to be a CP of F are

$$F(1, \boldsymbol{Y}_n) = 0, \quad \frac{\partial}{\partial X}F(1, \boldsymbol{Y}_n) = 0, \quad \frac{\partial}{\partial \boldsymbol{Y}_i}F(1, \boldsymbol{Y}_n) = 0 \text{ for all } i.$$

Consequently $G(\boldsymbol{Y}_n) = 0$ and $\partial G/\partial \boldsymbol{Y}_i(\boldsymbol{Y}_n) = 0$ for all i. Since $\boldsymbol{Y}_n \to \mathbf{0}$ and $\mathbf{0}$ is an ICP for G, $\boldsymbol{Y}_n = 0$ for all $n \gg 1$.

For the second statement, suppose p_2 is nondegenerate. By Morse's lemma [Duistermaat 1973] for local fields k, ch. $= 0$, there is a local diffeomorphism of k^{m-1} fixing $\mathbf{0}$ taking G to p_2. But $\mathbf{0}$ is an isolated CP for p_2, which makes it isolated for G. □

We remark that Duistermaat's proof [1973] of Morse's lemma is over $\mathbb{R}$, but its proof applies to the nonarchimedean case without any change, so we omit it.

Lemma 7.6. *The converse to the first statement of Lemma 7.5 is true if*

$$F = X^{d-r} p_r + p_d \quad (r \geq 2).$$

Proof. We must show that $G = p_r + p_d$ has $\mathbf{0}$ as an ICP if $(1, \mathbf{0})$ is an ICP for F. Suppose $\boldsymbol{w}_n$ are CPs for $G = p_r + p_d$ with $\boldsymbol{w}_n \to \mathbf{0}$. Then $G(w_n) = F(1, \boldsymbol{w}_n) = 0$ for all n, and $G_i(\boldsymbol{w}_n) = \partial F/\partial Y_i(1, \boldsymbol{w}_n) = 0$ for all n. Hence, $p_{r,i}(\boldsymbol{w}_n) + p_{d,i}(\boldsymbol{w}_n) = 0$ for all n. By Euler's theorem, $r p_r(\boldsymbol{w}_n) + d p_d(\boldsymbol{w}_n) = 0$ for all n. But, $p_r(\boldsymbol{w}_n) + p_d(\boldsymbol{w}_n) = 0$ for all n as well. So, $p_r(\boldsymbol{w}_n) = p_d(\boldsymbol{w}_n) = 0$ for all n. Hence, $\partial F/\partial X(1, \boldsymbol{w}_n) = (d-r) p_r(\boldsymbol{w}_n) = 0$ for all n. So $(1, \boldsymbol{w}_n)$ is a CP of F for all n. As $(1, \mathbf{0})$ is assumed to be an ICP for F, $\boldsymbol{w}_n = \mathbf{0}$ for $n \gg 1$. So $\mathbf{0}$ is an ICP for F. □

Study of condition (7-2) for $F = X^{d-2} p_2 + p_d$ where $G = p_2 + p_d$ has $\mathbf{0}$ as an ICP. Let us consider $F = X^2 + P_4(Y)$ where P_4 is a homogeneous quartic polynomial in Y, Z. For this to have $(t, 0, 0)$ as and ICL we must have $(0, 0)$ as an ICP for $G = Z^2 + P_4(Y, Z)$.

Lemma 7.7. *$G = Z^2 + P_4(Y, Z)$ has $\mathbf{0}$ as an ICP if and only if $Z^2 \nmid P_4(Y, Z)$, i.e.,*

$$P_4(Y, Z) = a_0 Y^4 + a_1 Y^3 Z + a_2 Y^2 Z^2 + a_3 Y Z^3 + a_4 Z^4$$

where at least one of a_0, a_1 is nonzero. In this case $\mathbf{0}$ is its only CP.

Proof. The equations which determine whether (y, z) is a CP of G are

$$z^2 + P_4(y, z) = 0, \quad \frac{\partial P_4}{\partial Y}(y, z) = 0 \quad \text{and} \quad 2Z + \frac{\partial P_4}{\partial Z}(y, z) = 0.$$

From the second and third equations just defined, using Euler's theorem, we have $2z^2 + 4P_4(y, z) = 0$, which implies $z^2 = 0$ and $P_4(yz) = 0$.

So the only critical points are of the form $(y, 0)$. Then $(0, 0)$ is certainly a CP. If $(y, 0)$ is a critical point for some $y \neq 0$, then $4a_0 y^3 = 0$, $a_1 y^3 = 0$ which implies a_0, a_1 both vanish. The entire Y-axis consists of critical points, and so for $(0, 0)$ to be an ICP, at least one of $a_0, a_1 \neq 0$. in which case $(0, 0)$ is the only CP.

We consider the cases (I) $a_0 \neq 0$ and (II) $a_0 = 0$, $a_1 \neq 0$. We consider case (I). We shall now verify that $\inf_{\|\boldsymbol{u}\|>1} \|\nabla F(\boldsymbol{u})\| > 0$ if $\boldsymbol{u} \in L_c$, $\|\boldsymbol{u}\| \geq 1$. Assume $F = X^2 Z^2 + P_4(Y, Z)$, and in view of Lemma 7.3, choose a sequence (x_n, y_n, z_n) such that $x_n \to \infty$, $y_n/x_n \to 0$, $z_n/x_n \to 0$ and:

(i) $x_n^2 y_n^2 + P_4(y_n, z_n) = c$,

(ii) $\partial F/\partial X = 2x_n z_n^2 \to 0$,

(iii) $\partial P_4/\partial Y(y_n, z_n) \to 0$,

(iv) $2x_n^2 z_n + \partial P_4/\partial Z(y_n, z_n) \to 0$.

From (ii) we get $z_n \to 0$. Assuming we are in case (I), y_n is bounded. Otherwise, by passing to a subsequence we may assume $y_n \to \infty$ giving $\partial P_4/\partial Y(y_n, z_n) = 4a_0 y_n^3 + 3a_1 y_n^2 z_n + \cdots \to 0$. If $a_0 \neq 0$, then $\partial P_4/\partial Y(y_n, z_n) = 4a_0 y_n^3(1+o(z_n/y_n)) \to \infty$, which is a contradiction. But if $\eta \neq 0$ is a limit point of y_n, then

$$\frac{\partial P_4}{\partial Y}(z_n, y_n) \to 4a_0\eta^3 \neq 0$$

which is a contradiction. So, $y_n \to 0$ necessarily. Then, $P_4(y_n, z_n) \to 0$ and $\partial P_4/\partial Z(y_n, z_n) \to 0$. Hence by (iv), $x_n^2 z_n \to 0$, by (i) $x_n^2 z_n^2 \to c \neq 0$, a contradiction. This finishes case (I).

Assuming we are in case (II), $a_0 = 0$, $a_1 \neq 0$, we claim $y_n \to \infty$. Otherwise, by passing to a subsequence, we may assume $y_n \to \eta$. Then $P_4(y_n, z_n) = a_1 y_n^3 z_n + \cdots$ so that $P_4(y_n, z_n) \to 0$. Hence, $x_n^2 z_n^2 \to c$. But $\partial P_4/\partial Z(y_n, z_n) = a_1 y_n^3 + \cdots \to a_1\eta^3$. Hence, by (iv), $x_n^2 z_n = o(1)$. So, as $z_n \to 0$, we have $x_n^2 y_n^2 \to 0$. Hence, $c = 0$ is a contradiction.

We are left with the case $x_n \to \infty$, $y_n \to \infty$, $z_n \to 0$, $(y_n/x_n)(z_n/x_n) \to 0$, and $P_4(Y, Z) = a_1 Y^3 Z + \cdots$, for $a_1 \neq 0$. But $\partial P_4/\partial Y(y_n, z_n) = 3a_1 y_n^2 z_n(1+o(z_n/y_n)) \to 0$ if and only if $y_n^2 z_n \to 0$. In this case may we have a counterexample to statement (7-2). Remark 4.14 gives an example of this kind. Note that case (I) is generic among the families we consider. □

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DAVID TAYLOR
DEPARTMENT OF MATHEMATICS
UCLA
LOS ANGELES, CA
UNITED STATES
dwtaylor@math.ucla.edu

V. S. VARADARAJAN
DEPARTMENT OF MATHEMATICS
UCLA
LOS ANGELES, CA
UNITED STATES
vsv@math.ucla.edu

JUKKA VIRTANEN
DEPARTMENT OF MATHEMATICS
UCLA
LOS ANGELES, CA
UNITED STATES
semisimplemath@gmail.com

DAVID WEISBART
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA, RIVERSIDE
RIVERSIDE, CA
UNITED STATES
dweisbart@gmail.com

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