ON THE CHOW RING OF THE STACK OF
TRUNCATED BARSOTTI–TATE GROUPS

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We determine the Chow ring of the stack of truncated displays and more generally the Chow ring of the stack of *G*-zips. We also investigate the pullback morphism of the truncated display functor. From this we can determine the Chow ring of the stack of truncated Barsotti–Tate groups over a field of characteristic *p* up to *p*-torsion.

Introduction

Edidin and Graham [1998] developed an equivariant intersection theory for actions of linear algebraic groups G on algebraic spaces X. For such G-spaces they defined G-equivariant Chow groups $A_*^G(X)$ generalizing Totaro's definition [1999] of the G-equivariant Chow ring of a point. They are an invariant of the corresponding quotient stack $[X/G]$, i.e., they are independent of the choice of presentation. Hence they can be used to define the integral Chow group of a quotient stack. If X is smooth these groups carry a ring structure making them into commutative graded rings. Edidin and Graham used their theory to compute the Chow ring of the stacks $\mathscr{M}_{1,1}$ and $\bar{\mathscr{M}}_{1,1}$ of elliptic curves. In an appendix to that paper, Vistoli computed the Chow ring of $\mathscr{M}_2$. Edidin and Fulghesu [2009] computed the integral Chow ring of the stack of hyperelliptic curves of even genus. In this article we investigate the Chow ring of the stack of truncated Barsotti–Tate groups over a field of characteristic $p > 0$.

Let us denote the stack of level-n Barsotti-Tate groups by BT_n. A level-n BT group has a height and a dimension, which are locally constant functions on the base. If $\mathrm{BT}_n^{h,d}$ denotes the stack of level-n BT groups of constant height h and

dimension d we obtain a decomposition $\mathrm{BT}_n = \coprod_{0 \leq d \leq h} \mathrm{BT}_n^{h,d}$ into open and closed substacks. For example, if A is an abelian scheme of relative dimension g then its p^n-torsion subscheme $A[p^n]$ is a level-n BT group of height $2g$ and dimension g.

Although $\mathrm{BT}_n^{h,d}$ has a natural presentation $[X/\mathrm{GL}_{p^{nh}}]$ as a quotient stack with quasiaffine and smooth X (see [Wedhorn 2001]), it seems unlikely that this presentation can be used directly to compute the Chow ring. Instead we compare the stack of truncated Barsotti–Tate groups to a stack whose Chow ring is easier to compute, but still closely related to the Chow ring of BT_n.

Our choice for this stack is the stack $\mathcal{D}\mathrm{isp}_n$ of truncated displays introduced in [Lau 2013]. Displays were first introduced in [Zink 2002] to provide a Dieudonné theory that is valid not only over perfect fields but more generally over $\mathbb{F}_p$-algebras or p-adic rings. While displays are given by an invertible matrix with entries in the ring of Witt vectors $W(R)$, if a basis of the underlying modules is fixed, a truncated display is given by an invertible matrix over the truncated Witt ring $W_n(R)$.

Using crystalline Dieudonné theory one can associate to every p-divisible group a display. This induces a morphism $\phi : \mathrm{BT} \to \mathcal{D}\mathrm{isp}$ from the stack of Barsotti–Tate groups to the stack of displays, which in turn induces a morphism,

$$\phi_n : \mathrm{BT}_n \to \mathcal{D}\mathrm{isp}_n,$$

compatible with the truncations on both sides. By [Lau 2013], this morphism is a smooth morphism of smooth algebraic stacks over k and an equivalence on geometric points.

Theorem A. *The pullback $\phi_n^* : A^*(\mathcal{D}\mathrm{isp}_n) \to A^*(\mathrm{BT}_n)$ is injective and an isomorphism after inverting p.*

Let us sketch the proof. Consider a field L and a morphism $\mathrm{Spec}\, L \to \mathrm{BT}_n$. After base change to a finite field extension of p-power degree the fiber $\phi_n^{-1}(\mathrm{Spec}\, L)$ is equal to the classifying space of an infinitesimal group scheme necessarily of p-power degree. It follows that the pullback map of Bloch's higher Chow groups $A_*(\mathrm{Spec}\, L, m) \to A_*(\phi_n^{-1}(\mathrm{Spec}\, L), m)$ becomes an isomorphism after inverting p. Using the long localization exact sequence the theorem follows from a limit argument and noetherian induction similar to that in [Quillen 1973, Proposition 4.1]. The injectivity assertion follows since $A^*(\mathcal{D}\mathrm{isp}_n)$ is p-torsion free.

Thus to compute the Chow ring of BT_n at least up to p-torsion it suffices to compute the Chow ring of $\mathcal{D}\mathrm{isp}_n$, which is much easier due to the simpler presentation as a quotient stack. More precisely, if $\mathcal{D}\mathrm{isp}_n^{h,d}$ denotes the open and closed substack in $\mathcal{D}\mathrm{isp}_n$ of truncated displays with constant dimension d and height h we have

$$\mathcal{D}\mathrm{isp}_n^{h,d} = [\mathrm{GL}_h(W_n(\,\cdot\,))/G_n^{h,d}],$$

where W_n refers to the ring of truncated Witt vectors and $G_n^{h,d}$ is an extension of

$\mathrm{GL}_d \times \mathrm{GL}_{h-d}$ by a unipotent group. The following result reduces the calculation of $A^*(\mathcal{D}\mathrm{isp}_n)$ to the case $n = 1$.

Theorem B. *The pullback* $\tau_n^* : A^*(\mathcal{D}\mathrm{isp}_1) \to A^*(\mathcal{D}\mathrm{isp}_n)$ *of the truncation map* $\tau_n : \mathcal{D}\mathrm{isp}_n \to \mathcal{D}\mathrm{isp}_1$ *is an isomorphism.*

This is proved using the factorization

$$[\mathrm{GL}_h(W_n(\cdot))/G_n^{h,d}] \to [GL_h/G_n^{h,d}] \to [GL_h/G_1^{h,d}]$$

of τ_n and the fact that the first map is an affine bundle and that $G_n^{h,d}$ is an extension of $G_1^{h,d}$ by a unipotent group.

In a similar way one shows that the Chow ring of $\mathcal{D}\mathrm{isp}_1^{h,d}$ coincides with that of the quotient stack

$$[\mathrm{GL}_h /(\mathrm{GL}_d \times \mathrm{GL}_{h-d})],$$

where the action is given by conjugation with the Frobenius. This situation is a special case of Proposition 2.3.2.

Theorem C. *The following equation holds*:

$$\begin{aligned}
A^*(\mathcal{D}\mathrm{isp}_1^{h,d}) &= A^*_{\mathrm{GL}_d \times \mathrm{GL}_{h-d}}(\mathrm{GL}_h) \\
&= \mathbb{Z}[t_1, \ldots, t_h]^{S_d \times S_{h-d}}/((p-1)c_1, \ldots, (p^h-1)c_h),
\end{aligned}$$

where $c_1, \ldots, c_h$ *are the elementary symmetric polynomials in the variables* $t_1, \ldots, t_h$.

Moreover, $t_1, \ldots, t_d$ and $t_{d+1}, \ldots, t_h$ are the Chern roots of the vector bundle $\mathcal{L}ie$ and ${}^t\mathcal{L}ie^\vee$, respectively, over $\mathcal{D}\mathrm{isp}_1^{h,d}$. Here $\mathcal{L}ie$ is a vector bundle of rank d assigning to a display its Lie algebra, and ${}^t\mathcal{L}ie^\vee$ is of rank $h-d$ assigning to a display the dual Lie algebra of its dual display.

It follows that the $\mathbb{Q}$-vectorspace $A^*(\mathcal{D}\mathrm{isp}_1^{h,d})_{\mathbb{Q}}$ is finite-dimensional of dimension $\binom{h}{d}$, which also equals the number of isomorphism classes of truncated displays of level 1 with height h and dimension d over an algebraically closed field. We show that a basis is given by the cycles of the closures of the respective EO strata. We prove this fact in greater generality for the stack of G-zips [Pink et al. 2011] in Section 2.4. In that section we will also compute the Chow ring of the stack of G-zips for a connected algebraic zip datum. As in the case of displays the computation can be reduced to the situation of Proposition 2.3.2. In fact, truncated displays of level 1 are a special case of G-zips.

To state our result we recall that an algebraic zip datum $\mathcal{Z}$ is a 4-tuple (G, P, Q, φ) consisting of a split reductive group G, parabolic subgroups P and Q, and an isogeny $\varphi : P/R_u(P) \to Q/R_u(Q)$. To $\mathcal{Z}$ one associates the stack of G-zips $[G/E_{\mathcal{Z}}]$, where $E_{\mathcal{Z}}$ is the group $\{(p, q) \in P \times Q \mid \varphi(\pi_P(p)) = \pi_Q(q)\}$ acting on G by the rule $((p,q), g) \mapsto pgq^{-1}$. We also recall that an algebraic group is called *special* if every principal G-bundle is locally trivial for the Zariski topology.

Theorem D. *Let $\mathcal{Z} = (G, P, Q, \varphi)$ be an algebraic zip datum, where G is connected. Let $W_G = W(G, T)$ be the Weyl group of G and $W_L = W(L, T)$ be the Weyl group of a Levi component L of P with respect to a split maximal torus $T \subset L$ of G. Let $g_0 \in G(k)$ be such that $\varphi(T) = {}^{g_0}T$ and let $\tilde{\varphi} : T \to T$ denote the composition of φ followed by conjugation with g_0^{-1}. Then $\tilde{\varphi}$ induces an action on $S = \mathrm{Sym}(\widehat{T})$, which we will also denote by $\tilde{\varphi}$. We then have*

$$A^*([G/E_{\mathcal{Z}}])_{\mathbb{Q}} = S_{\mathbb{Q}}^{W_L}/(f - \tilde{\varphi}f \mid f \in S_+^{W_G})_{\mathbb{Q}}.$$

If G is special we have

$$A^*([G/E_{\mathcal{Z}}]) = S^{W_L}/(f - \tilde{\varphi}f \mid f \in S_+^{W_G}).$$

(*Note that the action of $\tilde{\varphi}$ on S^{W_G} is independent of the choice of g_0 since two choices differ by an element of $N_G(T)$.*)

Gathering the above results we gain the following information on the Chow ring of the stack of truncated Barsotti–Tate groups.

Theorem E.   (i) *We have*

$$A^*(\mathrm{BT}_n^{h,d})_p = \mathbb{Z}[p^{-1}][t_1, \ldots, t_h]^{S_d \times S_{h-d}}/((p-1)c_1, \ldots, (p^h-1)c_h),$$

where c_i denotes the i-th elementary symmetric polynomial in the variables $t_1, \ldots, t_h$, and $t_1, \ldots, t_d$ and $t_{d+1}, \ldots, t_h$ are the Chern roots of $\mathcal{L}ie$ and ${}^t\mathcal{L}ie^\vee$, respectively.

  (ii) *We have $\dim_{\mathbb{Q}} A^*(\mathrm{BT}_n^{h,d})_{\mathbb{Q}} = \binom{h}{d}$ and a basis is given by the cycles of the closures of the EO strata.*

  (iii) $(\mathrm{Pic}\,\mathrm{BT}_n^{h,d})_p = \begin{cases} \mathbb{Z}[p^{-1}]/(p-1) & \text{if } d = 0, h, \\ \mathbb{Z}[p^{-1}] \times \mathbb{Z}[p^{-1}]/(p-1) & \text{otherwise,} \end{cases}$

where the generators for the free and torsion parts are, respectively, $\det(\mathcal{L}ie)$ and $\det(\mathcal{L}ie \otimes {}^t\mathcal{L}ie^\vee)$.

It would be interesting to know if the Chow ring of BT_n has p-torsion, and more specifically if the Picard group of BT_n has p-torsion. However, since ϕ_n^* is injective and the Chow ring of $\mathcal{D}\mathrm{isp}_n$ is p-torsion free, p-torsion in the Chow ring of BT_n cannot be constructed using displays.

Terminology and notation. Every scheme is assumed to be of finite type and separated over the base field k. In Section 2, we assume k to be of characteristic $p > 0$. Algebraic groups are affine smooth group schemes over k. We call an algebraic group G unipotent if G admits a filtration $G = G_0 \supset G_1 \supset \cdots \supset G_e = \{1\}$ by subgroups such that G_i is normal in G_{i-1} with quotient isomorphic to $\mathbb{G}_a$. The character group of an algebraic group G will be denoted by $\widehat{G}$. If X is a scheme, $A^*(X)$ will always denote the operational Chow ring of X [Fulton 1998, Chapter 17]. $A_*(X)$

and $\mathrm{CH}^*(X)$ will be the Chow group of X graded by dimension and codimension, respectively. If X is an algebraic space over k with a left action of an algebraic group G we will refer to X as a G-space. We write $[X/G]$ for the corresponding quotient stack. If G acts freely on X, i.e., the stabilizer of every point is trivial, then $[X/G]$ is an algebraic space. In this case we will write X/G instead of $[X/G]$ and call $X \to X/G$ the principal bundle quotient of X with structure group G.

1. Equivariant intersection theory

1.1. *Equivariant Chow groups.* Consider an algebraic group G over k. By [Edidin and Graham 1998, Lemma 9], we can find a representation V of G, and an open subset U in V such that the complement of U has arbitrary high codimension, and such that the principal bundle quotient U/G exists in the category of schemes. If X is an algebraic space on which G acts then G acts diagonally on $X \times U$ and we will denote the principal bundle quotient $(X \times U)/G$ by X_G.

Convention 1.1.1. We call a pair (V, U) consisting of a G-representation V and an open subset U a good pair for G if G acts freely on U, i.e., the stabilizer of every point is trivial. Sometimes we will call the quotient $X_G = (X \times U)/G$ a mixed space for the G-space X. If (V, U) is a good pair for G with $\mathrm{codim}(U^c, V) > i$ we will also call $(X \times U)/G$ an approximation of $[X/G]$ up to codimension i.

If X has dimension n the i-th equivariant Chow group $A_i^G(X)$ is defined in the following way. Choose a good pair (V, U) for G such that the complement of U has codimension greater than $n - i$. Then one defines

$$A_i^G(X) = A_{i+l-g}(X_G),$$

where l denotes the dimension of V and g is the dimension of G. The definition is independent of the choice of the pair (V, U) as long as $\mathrm{codim}(U^c, V) > n - i$ holds [Edidin and Graham 1998, Definition-Proposition 1].

The equivariant Chow groups have the same functorial properties as ordinary Chow groups [Edidin and Graham 1998, Section 2]. In particular, we have an operational equivariant Chow ring $A_G^*(X)$ [Edidin and Graham 1998, Section 2.6], i.e., an element $c \in A_G^i(X)$ consists of operations $c(Y \to X) : A_*^G(Y) \to A_{*-i}^G(Y)$ for each G-equivariant map $Y \to X$ that are compatible with flat pullback, proper pushforward and Gysin homomorphisms.

We will denote by $\mathrm{CH}_G^*(X)$ the G-equivariant Chow group of X graded by codimension. If X is a pure dimensional G-scheme and (V, U) a good pair for G with $\mathrm{codim}(U^c, V) > i$ then

$$\mathrm{CH}_G^j(X) = \mathrm{CH}^j((X \times U)/G)$$

for all $j \leq i$. This motivates the term "approximation of $[X/G]$ up to codimension i" in Convention 1.1.1.

If X is smooth then $\mathrm{CH}_G^*(X)$ carries a ring structure which makes it into a commutative graded ring with unit element. Moreover, there is a natural isomorphism $A_G^*(X) \cong \mathrm{CH}_G^*(X)$ of graded rings [Edidin and Graham 1998, Proposition 4].

By [Edidin and Graham 1998, Proposition 16], the equivariant Chow groups do not depend on the presentation as a quotient, meaning if X is a G-space and Y is an H-space such that $[X/G] \cong [Y/H]$, then $A_{i+g}^G(X) = A_{i+h}^H(Y)$, where $g = \dim G$ and $h = \dim H$. Hence one can define the Chow group of a quotient stack $[X/G]$ to be

$$A_i([X/G]) = A_{i+g}^G(X)$$

with $g = \dim G$. By [Edidin and Graham 1998, Proposition 19], one has

$$A^*([X/G]) \cong A_*([X/G]),$$

whenever X is smooth.

1.2. *Higher equivariant Chow groups.*
The reason we shall need higher Chow groups is that they extend the localization exact sequence to the left. Higher Chow groups were introduced by Bloch [1986]. For a scheme X, higher Chow groups $A_i(X, m)$ are defined as the homology of the complex $z_i(X, *)$, where $z_i(X, m)$ is the group of cycles of dimension $m + i$ in $X \times \Delta^m$ meeting all faces properly. For $m = 0$ one gets back the usual Chow group $A_*(X)$, and $A_i(X, m)$ may be nontrivial for $-m \leq i \leq \dim X$. The definition of these higher Chow groups also works for algebraic spaces.

In order to define G-equivariant versions $A_*^G(X, m)$ of higher Chow groups we need the homotopy property for the mixed spaces X_G, i.e., the pullback map

$$A_*(X_G, m) \to A_*(\mathcal{E}, m)$$

for a vector bundle $\mathcal{E}$ over X_G is an isomorphism. This is true for any scheme if $\mathcal{E}$ is trivial by [Bloch 1986, Theorem 2.1]. To prove the assertion for arbitrary vector bundles one needs the localization exact sequence of higher Chow groups proved by Bloch in the case of quasiprojective schemes: if X is an equidimensional, quasiprojective scheme over k and $Y \subset X$ is a closed subscheme with complement $U = X - Y$, then there is a long exact sequence of higher Chow groups

$$\cdots \to A_*(Y, m) \to A_*(X, m) \to A_*(U, m) \to A_*(Y, m - 1) \to$$
$$\cdots \to A_*(Y) \to A_*(X) \to A_*(U) \to 0.$$

For a proof see [Edidin and Graham 1998, Lemma 4] and [Bloch 1986, Theorem 3.1].

Remark 1.2.1. Levine extended Bloch's proof of the existence of the long localization exact sequence to all separated schemes of finite type over k [Levine 2001, Theorem 1.7]. Hence for the equivariant higher Chow groups to be well defined

it suffices that we can choose the mixed spaces to be separated schemes over k. However, in all applications we have in mind the conditions of Lemma 1.2.2 will be satisfied.

Lemma 1.2.2. *Let G be an algebraic group and X a normal, quasiprojective G-scheme. Then for any $i > 0$ there is a representation V of G and an invariant open subset $U \subset V$ whose complement has codimension greater than i such that G acts freely on U and the principal bundle quotient $(X \times U)/G$ is a quasiprojective scheme. In other words, the quotient stack $[X/G]$ can be approximated by quasiprojective schemes.*

Proof. Embed G into GL_n for some n. Then there is a representation V of GL_n and an open subset $U \subset V$, whose complement has codimension greater than i such that U/GL_n is a Grassmannian (see [Edidin and Graham 1998, Lemma 9]). Since GL_n is special the GL_n/G-bundle $\pi : U/G \to U/\mathrm{GL}_n$ is locally trivial for the Zariski topology, and we will first show that π is quasiprojective.

Since GL_n/G is quasiprojective and normal there is an ample GL_n-linearizable line bundle $L \to \mathrm{GL}_n/G$ [Thomason 1988, Section 5.7]. Then

$$(U \times L)/\mathrm{GL}_n \to (U \times (\mathrm{GL}_n/G))/\mathrm{GL}_n = U/G$$

is a line bundle relatively ample for π. This shows that π is quasiprojective. The same holds then for U/G. Again by [Thomason 1988, Section 5.7], there is an ample G-linearizable line bundle on X. The pullback to $X \times U$ is then relatively ample for the projection $X \times U \to U$. Applying [Mumford et al. 1994, Proposition 7.1] to this situation yields the claim. $\square$

Definition 1.2.3. (i) A pair (V, U) will be called an admissible pair for a G-scheme X if (V, U) is a good pair for G and if the mixed space X_G is quasiprojective and (locally) equidimensional over k. X will be called an admissible G-scheme if for any i there is an admissible pair (V, U) for X with $\mathrm{codim}(U^c, V) > i$.

(ii) If X is an admissible G-scheme we define its higher equivariant Chow groups to be
$$A_i^G(X, m) = A_{i+l-g}(X_G, m),$$
where $g = \dim G$ and X_G is formed from an l-dimensional admissible pair (V, U) such that $\mathrm{codim}(U^c, V) > \dim X + m - i$.

(iii) We will say that a stack $\mathscr{X}$ admits an admissible presentation if there exists an admissible G-scheme X such that $\mathscr{X} = [X/G]$.

(iv) Let $\mathscr{X}$ be a quotient stack that admits a presentation $\mathscr{X} = [X/G]$ by an admissible G-scheme X. We define the higher equivariant Chow groups of $\mathscr{X}$ as
$$A_*(\mathscr{X}, m) = A_{*+g}^G(X, m),$$
where $g = \dim G$.

Remark 1.2.4. The proof that (ii) and (iv) of Definition 1.2.3 are independent of the choice of the admissible pair (V, U) and the presentation $[X/G]$, respectively, is the same as for ordinary equivariant Chow groups (see Definition-Proposition 1 and Proposition 16, respectively, in [Edidin and Graham 1998]) by using the homotopy property for the mixed spaces.

Remark 1.2.5. We will frequently encounter the situation of a morphism $T \to X$ of G-schemes such that T is open in a G-equivariant vector bundle over X. We remark that, if X is an admissible G-scheme, so is T. This follows since a vector bundle over a quasiprojective scheme is again quasiprojective.

Lemma 1.2.6. *Let* $f : \mathscr{X} \to \mathscr{Y}$ *be a flat map of quotient stacks of relative dimension* r. *Then there is a flat pullback map* $f^* : A_*(\mathscr{Y}) \to A_{*+r}(\mathscr{X})$ *between the Chow groups. If* $\mathscr{X}$ *and* $\mathscr{Y}$ *admit admissible presentations the same assertion holds for the higher Chow groups.*

Furthermore, if $\mathscr{X}$ *and* $\mathscr{Y}$ *are smooth then under the identification* $A_*(\mathscr{X}) = A^*(\mathscr{X})$, *the above morphism is just the natural pullback map between the operational Chow rings.*

Proof. Consider presentations $\mathscr{X} = [X/G]$ and $\mathscr{Y} = [Y/H]$. By definition $A_i(\mathscr{X}) = A_{i+g}^G(X)$ with $g = \dim G$ and similarly for $A_i(\mathscr{Y})$. Choose a good pair (V_1, U_1) for G and a good pair (V_2, U_2) for H. Let $l_i = \dim V_i$. As usual we will write X_G and Y_H for the mixed spaces $(X \times U_1)/G$ and $(Y \times U_2)/H$, respectively. Consider the following fiber square:

$$\begin{array}{ccccc} Z' & \longrightarrow & \mathscr{Z} & \longrightarrow & Y_H \\ \downarrow & & \downarrow & & \downarrow \\ X_G & \longrightarrow & \mathscr{X} & \longrightarrow & \mathscr{Y} \end{array}$$

Then Z' is a bundle over X_G and $\mathscr{Z}$ with fibers U_2 and U_1, respectively, and $Z' \to Y_H$ is a flat map of algebraic spaces of relative dimension $l_1 + r$. Hence

$$A_{i+l_1+l_2+r}(Z') = A_{i+l_1+r}(X_G) = A_{i+r}(\mathscr{X})$$

and we define f^* to be the ordinary pullback of the flat map $Z' \to Y_H$. The exact same construction works for the higher equivariant Chow groups if $\mathscr{X}$ and $\mathscr{Y}$ admit admissible presentations.

For the last part we recall that the isomorphism $A^i(\mathscr{X}) \cong A_{\dim X - i}^G(X)$ maps $c \in A^i(\mathscr{X})$ to $c(X_G \to \mathscr{X}) \cap [X_G] \in A_{\dim X - i}^G(X)$. Thus we need to check the equality

$$f^*(d(Y_H \to \mathscr{Y}) \cap [Y_H]) = d(X_G \to \mathscr{X} \to \mathscr{Y}) \cap [X_G]$$

for $d \in A^i(\mathscr{Y})$. This follows from the compatibility of d with flat pullbacks. $\square$

1.3. *Auxiliary results.*

Lemma 1.3.1. *Let $X \to Y$ be a flat morphism of schemes and $Y' \to Y$ be a finite, flat and surjective map of degree d. Let $X' \to Y'$ be the base change of $X \to Y$ along $Y' \to Y$. Assume the pullback $A_*(Y', m) \to A_*(X', m)$ becomes an isomorphism after inverting some integer d'. Then the pullback $A_*(Y, m) \to A_*(X, m)$ is an isomorphism after inverting dd'.*

Proof. The injectivity of the pullback $A_*(Y, m)_{dd'} \to A_*(X, m)_{dd'}$ follows from the exact diagram:

$$
\begin{array}{ccccc}
0 & \longrightarrow & A_*(Y, m)_{dd'} & \longrightarrow & A_*(Y', m)_{dd'} \\
& & \downarrow & & \cong \downarrow \\
0 & \longrightarrow & A_*(X, m)_{dd'} & \longrightarrow & A_*(X', m)_{dd'}
\end{array}
$$

and the surjectivity from the exact diagram

$$
\begin{array}{ccccc}
A_*(Y', m)_{dd'} & \longrightarrow & A_*(Y, m)_{dd'} & \longrightarrow & 0 \\
\cong \downarrow & & \downarrow & & \\
A_*(X', m)_{dd'} & \longrightarrow & A_*(X, m)_{dd'} & \longrightarrow & 0
\end{array}
$$

where the horizontal maps in the first diagram are induced by pullback and in the second diagram by pushforward. The commutativity of the second diagram is shown by [Fulton 1998, Proposition 1.7]. $\square$
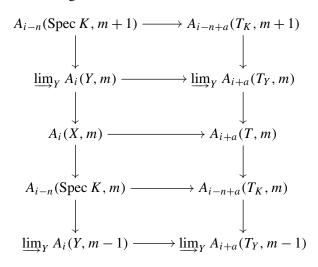
Lemma 1.3.2. *Let $T \to X$ be a morphism of quasiprojective schemes over k. We assume that X is equidimensional and that $T \to X$ is flat of relative dimension a. Let $d, i \in \mathbb{Z}$ and for $x \in X$ let $h(x)$ denote the dimension of the closure of $\{x\}$ in X. If the pullback $A_{i-h(x)}(\operatorname{Spec} k(x), m)_d \to A_{i-h(x)+a}(T_x, m)_d$ is an isomorphism for every $x \in X$ and for any m, then $A_i(X, m)_d \to A_{i+a}(T, m)_d$ is an isomorphism.*

Proof. We follow Quillen's proof of the analogous result in higher K-theory [1973, Proposition 4.1]. First we may assume that X is irreducible for if $X = W_1 \cup \cdots \cup W_r$ is a decomposition into irreducible components we may consider the long localization exact sequence of the pair $(W_1, X - W_1)$. By induction we are thus reduced to the irreducible case. Since the Chow groups only depend on the reduced structure, we may also assume that X is reduced. Let K denote the function field of X. We have

$$
A_{i-n}(\operatorname{Spec} K, m) = \varinjlim_U A_i(U, m),
$$

$$
A_{i-n+a}(T_K, m) = \varinjlim_U A_{i+a}(T_U, m),
$$

where the limit goes over all nonempty open subsets of X and n denotes the dimension of X. In fact, it suffices to go over all nonempty open subsets with

equidimensional complement, since for all nonempty open U in X there exists a nonempty open subset U' contained in U with equidimensional complement. We obtain a commutative diagram

$$
\begin{array}{ccc}
A_{i-n}(\operatorname{Spec} K, m+1) & \longrightarrow & A_{i-n+a}(T_K, m+1) \\
\downarrow & & \downarrow \\
\varinjlim_Y A_i(Y, m) & \longrightarrow & \varinjlim_Y A_{i+a}(T_Y, m) \\
\downarrow & & \downarrow \\
A_i(X, m) & \longrightarrow & A_{i+a}(T, m) \\
\downarrow & & \downarrow \\
A_{i-n}(\operatorname{Spec} K, m) & \longrightarrow & A_{i-n+a}(T_K, m) \\
\downarrow & & \downarrow \\
\varinjlim_Y A_i(Y, m-1) & \longrightarrow & \varinjlim_Y A_{i+a}(T_Y, m-1)
\end{array}
$$

with exact rows, where the limit goes over all proper closed equidimensional subsets of X. After inverting d the first and fourth vertical maps become isomorphisms and we conclude by noetherian induction. $\qquad\square$

Corollary 1.3.3. *Let $T \to X$ be a flat morphism of quasiprojective schemes over k with fibers being affine spaces of some dimension n. Then the pullback $A_*(X, m) \to A_{*+n}(T, m)$ is an isomorphism.*

Proof. This is an immediate consequence of Lemma 1.3.2. $\qquad\square$

Remark 1.3.4. The assertion of the above corollary in the case $m = 0$ also holds without the quasiprojective assumption. One can use the same proof but using Gillet's higher Chow groups. For his higher Chow groups a long localization exact sequence exists for arbitrary schemes. For details see Chapter 8 in [Gillet 1981].

Lemma 1.3.5. *Let K be a unipotent subgroup of an algebraic group G such that the quotient G/K is finite of degree d. Then the pullback $A_G^*(m) \to A_{\{0\}}^*(m)$ is an isomorphism after inverting d.*

Proof. Let (V, U) be an admissible pair for G. Then $U/K \to U/G$ is a G/K-bundle locally trivial for the flat topology. By assumption on G/K the morphism $U/K \to U/G$ is therefore finite, flat and surjective of degree d. It follows that the pullback $A_*(U/G, m) \to A_*(U/K, m) \cong A_*(U, m)$ is injective after inverting d. Also for sufficiently high degree we know that $A_*(\operatorname{Spec} k, m) \to A_*(U, m)$ is surjective. Since we can assume the codimension of U^c in V to be arbitrarily high, we obtain the surjectivity of $A_G^*(m) \to A_{\{0\}}^*(m)$. $\qquad\square$

Lemma 1.3.6. *Let K/k be a Galois extension with Galois group G and let X be a scheme over k. Then pulling back along $X_K \to X$ induces an isomorphism $A_*(X, m)_{\mathbb{Q}} \cong A_*(X_K, m)_{\mathbb{Q}}^G$. If K/k is a finite Galois extension of degree d it suffices to invert d.*

Proof. We first assume that K/k is finite of degree d. Then on the level of cycles we have an injection $z_*(X, \cdot)_d \hookrightarrow z_*(X_K, \cdot)_d^G$ since $X_K \to X$ is finite and flat of degree d. We claim that this map is also surjective. Let $W \subset X_K \times_K \Delta_K^r$ be a subvariety meeting all faces properly. Let $S \subset G$ be the isotropy group of W. It suffices to see that $\sum_{g \in G/S}[gW]$ lies in $z_*(X, \cdot)_d$. For this consider the closed subscheme $V = \cup_{g \in G/S} gW$ (equipped with the reduced structure). Then V is a G-invariant equidimensional subscheme of $X_K \times_K \Delta_K^r$ that meets all faces properly. Thus it has a model $\widetilde{V}$ over k also meeting all faces properly. Finally all components gW have the same multiplicity 1 in the cycle $[V]$ and therefore $\sum_{g \in G/S}[gW] = [\widetilde{V}_K]$. To complete the proof in the finite case it suffices now to note that taking G-invariants is an exact functor on the category of $\mathbb{Z}[\frac{1}{d}]$-modules with G-action, hence $H_i(z_*(X_K, \cdot)_d^G) = H_i(z_*(X_K, \cdot))_d^G$. The general case follows from the finite case and the fact that $A_*(X_K, m)^G = \varinjlim_{L/k} A_*(X_L, m)^{G(L/k)}$, where the limit goes over all finite Galois subextensions L/k of K. $\qquad\square$

1.4. *A pullback lemma.* Throughout we consider the situation of an exact sequence

$$0 \longrightarrow A \longrightarrow G \longrightarrow H \longrightarrow 0$$

of algebraic groups and an admissible H-scheme X such that the induced G-action on X makes X also into an admissible G-scheme. These conditions are always satisfied if X is quasiprojective and normal by Lemma 1.2.2. We are then interested in properties of the pullback homomorphism (Lemma 1.2.6),

$$A_*([X/H], m) \to A_*([X/G], m).$$

Proposition 1.4.1. *Let*

$$0 \longrightarrow A \longrightarrow G \longrightarrow H \longrightarrow 0$$

be an exact sequence of algebraic groups and X an admissible H-scheme such that the induced G-action makes X also into an admissible G-scheme. We also assume H to be special.

   Let $d \in \mathbb{Z}$ be such that $A_{A_L}^*(m) \to A_{\{0\}}^*(m)$ becomes an isomorphism after inverting d for every field extension L of k and every m. Then the pullback $A_*([X/H], m) \to A_*([X/G], m)$ becomes an isomorphism after inverting d.

Proof. First note that the natural map $[X/G] \to [X/H]$ is flat of relative dimension $-a$ with $a = \dim A$. We can choose for any $i \in \mathbb{Z}$ an admissible pair (V, U) for the H-action such that $A_{j+l}([(X \times U)/G], m) = A_j([X/G], m)$ and

$A_{j+l}((X \times U)/H, m) = A_j([X/H], m)$ for all $j > i$. Here l denotes the dimension of V. Note that $X \times U$ is again an admissible G-scheme (see Remark 1.2.5). Replacing X by $X \times U$ we may thus assume that $[X/H]$ is a quasiprojective scheme.

Now, let $(X \times U)/G$ be a quasiprojective mixed space for G. Let $\overline{U}$ be the quotient U/A. Then we can identify $(X \times U)/G$ with the quotient $(X \times \overline{U})/H$ and under this identification the map $(X \times U)/G \to X/H$ corresponds to the $\overline{U}$-bundle $(X \times \overline{U})/H \to X/H$. It is Zariski locally trivial since H is special. We are left to show that the pullback of this map is an isomorphism after inverting d. This will follow from Lemma 1.3.2 once we have seen that the pullback $A_{j-h(x)}(\operatorname{Spec} k(x), m)_d \to A_{j-h(x)+l-a}(\overline{U}_{k(x)}, m)_d$ is an isomorphism for every $x \in X/H$. Here $h(x)$ is the dimension of the closure of $\{x\}$ in X/H. Let us write $L = k(x)$. Assuming the codimension of U^c in V to be sufficiently large we obtain by assumption

$$A_{j-h(x)}(\operatorname{Spec} L, m)_d = A_{j-h(x)+l}(U_L, m)_d = A_{j-h(x)+l-a}(\overline{U}_L, m)_d.$$

For this recall $A_{j+l-a}(\overline{U}_L, m) = A_j^{A_L}(m)$ and $A_{j+l}(U_L, m) = A_j^{\{0\}}(m)$. This proves the claim.   □

The above proposition applies to the following cases.

Corollary 1.4.2. *In the situation of Proposition 1.4.1 the following assertions hold.*

(i) *If A is unipotent then $A_*([X/H], m) \to A_*([X/G], m)$ is an isomorphism.*

(ii) *If A is finite of degree d then $A_*([X/H], m) \to A_*([X/G], m)$ becomes an isomorphism after inverting d.*

Proof. The first part follows from Corollary 1.3.3 and the second part follows from Lemma 1.3.5 applied to the case $K = \{0\}$.   □

The assumption on H to be special is crucial for the proof of Proposition 1.4.1, since we need to know that the fibers of the $\overline{U}$-bundle $(X \times \overline{U})/H \to X/H$ appearing in the proof are given by $\overline{U}$ in order to apply Lemma 1.3.2. However, we have the following version when H is finite.

Proposition 1.4.3. *Let*

$$0 \longrightarrow A \longrightarrow G \longrightarrow H \longrightarrow 0$$

be an exact sequence of algebraic groups and X an admissible H-scheme such that the induced G-action makes X also into an admissible G-scheme. We assume that H is finite of degree d.

Let $d' \in \mathbb{Z}$ be such that $A_{A_L}^*(m) \to A_{\{0\}}^*(m)$ becomes an isomorphism after inverting d' for every field extension L of k and any m. Then the pullback $A_*([X/H], m) \to A_*([X/G], m)$ becomes an isomorphism after inverting dd'.

Proof. We argue the same way as in Proposition 1.4.1 and then have to see that the pullback of $(X \times \overline{U})/H \to X/H$ becomes an isomorphism after inverting dd'. As mentioned earlier we cannot apply Lemma 1.3.2 since the above $\overline{U}$-bundle is not locally trivial for the Zariski topology. Instead it becomes trivial after the finite, flat and surjective base change $X \to X/H$ of degree d, i.e., there is a cartesian diagram

$$
\begin{array}{ccc}
X \times \overline{U} & \longrightarrow & X \\
\downarrow & & \downarrow \\
(X \times \overline{U})/H & \longrightarrow & X/H
\end{array}
$$

The claim thus follows from Lemma 1.3.1. □

Corollary 1.4.4. *In the situation of Proposition 1.4.3 the following assertions hold.*

 (i) *If A is unipotent then $A_*([X/H], m)_d \to A_*([X/G], m)_d$ is an isomorphism.*

 (ii) *If A is finite of degree d' then $A_*([X/H], m)_{dd'} \to A_*([X/G], m)_{dd'}$ is an isomorphism.*

In the next proposition we show that the assertion of Proposition 1.4.1 is valid over $\mathbb{Q}$ for arbitrary H.

Proposition 1.4.5. *Let*

$$
0 \longrightarrow A \longrightarrow G \longrightarrow H \longrightarrow 0
$$

be an exact sequence of algebraic groups and X an admissible H-scheme such that the induced G-action makes X also into an admissible G-scheme.

Assume $A_{A_L}^*(m)_{\mathbb{Q}} \to A_{\{0\}}^*(m)_{\mathbb{Q}}$ is an isomorphism for every field extension L of k and any m. Then the pullback $A_*([X/H], m)_{\mathbb{Q}} \to A_*([X/G], m)_{\mathbb{Q}}$ is an isomorphism.

Proof. Using the notation of the proof of Proposition 1.4.1 we need to see that the pullback of the $\overline{U}$-bundle $T := (X \times \overline{U})/H \to X/H$ is an isomorphism over $\mathbb{Q}$. It suffices to see that $A_*(\operatorname{Spec} k(x), m)_{\mathbb{Q}} \to A_*(T_x, m)_{\mathbb{Q}}$ is an isomorphism for $x \in X/H$. The above $\overline{U}$-bundle may not be trivial for the Zariski topology, but we still have $T_{\bar{x}} = \overline{U}_{\bar{x}}$ and thus $A_*(\operatorname{Spec} k(x)^{\operatorname{sep}}, m)_{\mathbb{Q}} \to A_*(T_{\bar{x}}, m)_{\mathbb{Q}}$ is an isomorphism by assumption. The claim then follows from Lemma 1.3.6 and the fact that the Galois action is compatible with pullback. □

Corollary 1.4.6. *In the situation of Proposition 1.4.5 the following assertions hold.*

 (i) *If A is unipotent then $A_*([X/H], m)_{\mathbb{Q}} \to A_*([X/G], m)_{\mathbb{Q}}$ is an isomorphism.*

 (ii) *If A is finite then $A_*([X/H], m)_{\mathbb{Q}} \to A_*([X/G], m)_{\mathbb{Q}}$ is an isomorphism.*

Lemma 1.4.7. *Let G be a split extension*

$$0 \longrightarrow K \longrightarrow G \longrightarrow H \longrightarrow 0$$

of an algebraic group H by a unipotent group K. Choose a splitting $H \hookrightarrow G$ and let X be a normal, quasiprojective G-scheme. Then the pullback map

$$A_*^G(X, m)_{\mathbb{Q}} \to A_*^H(X, m)_{\mathbb{Q}}$$

is an isomorphism. If G is special, this above map is an isomorphism over $\mathbb{Z}$.

Proof. Let (V, U) be an admissible pair for the G-action on X. It follows from the proof of Lemma 1.2.2 that (V, U) is then also admissible for the induced H-action. The morphism $(X \times U)/H \to (X \times U)/G$ is a G/H-bundle. If G is special this bundle is locally trivial for the Zariski topology. Hence the lemma follows from Corollary 1.3.3 in the special case and Lemmas 1.3.2 and 1.3.6 in the general case. □

1.5. *The restriction map.* We want to describe properties of the restriction map $\mathrm{res}_T^G : A_*^G(X) \to A_*^T(X)$, where T is a split torus in G. This map is defined via flat pullback of the natural map $X_T \to X_G$ between the mixed spaces. Note that more generally one has a restriction map $\mathrm{res}_H^G : A_*^G(X) \to A_*^H(X)$ for every subgroup H of G. We will need the following result.

Theorem 1.5.1. *Let G be a connected reductive group with split maximal torus T and Weyl group $W = W(G, T)$. Let X be a G-scheme.*

  (i) *W acts on $A_*^T(X)$. Furthermore, the restriction morphism $A_*^G(X) \to A_*^T(X)$ induces a map $r : A_*^G(X) \to A_*^T(X)^W$.*

 (ii) *Assume X is smooth. Then r is an isomorphism after tensoring with $\mathbb{Q}$.*

(iii) *Assume X is smooth and that G is special. Then r is injective. Moreover, r is an isomorphism if $A_T^*(X)$ is $\mathbb{Z}$-torsion free (e.g., if $X = \mathrm{Spec}\, k$).*

Part (iii) is basically proved in [Edidin and Graham 1997], where the case $X = \mathrm{Spec}\, k$ is considered. However, there seems to be no complete proof of part (ii) in the literature. We therefore give a proof.

In the following $A^*(X; \mathbb{Q})$ will denote the operational Chow ring of X consisting of characteristic classes with values in rational Chow groups, i.e., an element $c \in A^*(X; \mathbb{Q})$ assigns to each $T \to X$ a morphism

$$c(T \to X) : A_*(T)_{\mathbb{Q}} \to A_*(T)_{\mathbb{Q}}$$

satisfying the usual compatibility conditions [Fulton 1998, Section 17.1]. A proper map $\pi : \widetilde{X} \to X$ is called an envelope if for each irreducible subspace $V \subset X$ there exists an irreducible subspace $\widetilde{V} \subset \widetilde{X}$ such that π maps $\widetilde{V}$ birationally onto V.

Remark 1.5.2. There is a natural map $A^*(X)_{\mathbb{Q}} \to A^*(X; \mathbb{Q})$ and this map is an isomorphism if X is smooth. This follows from

$$
\begin{array}{ccc}
A^*(X)_{\mathbb{Q}} & \xrightarrow[\cong]{\cap[X]} & A_*(X)_{\mathbb{Q}} \\
\downarrow & & \| \\
A^*(X; \mathbb{Q}) & \xrightarrow[\cong]{\cap[X]} & A_*(X)_{\mathbb{Q}}
\end{array}
$$

We recall the following easy lemma.

Lemma 1.5.3. (i) *Let* $\pi : \widetilde{X} \to X$ *be a proper surjective map. Then*

$$
\pi_* : A_*(\widetilde{X})_{\mathbb{Q}} \to A_*(X)_{\mathbb{Q}}
$$

is surjective and $\pi^* : A^*(X; \mathbb{Q}) \to A^*(\widetilde{X}; \mathbb{Q})$ *is injective.*

(ii) *Let* $\pi : \widetilde{X} \to X$ *be a birational envelope. Then* $\pi_* : A_*(\widetilde{X}) \to A_*(X)$ *is surjective and* $\pi^* : A^*(X) \to A^*(\widetilde{X})$ *is injective.*

Proof. The first part of (i) is [Kimura 1992, Proposition 1.3]. The first part of (ii) follows immediately from the definition of an envelope. The second part of (i) and (ii) are formal consequences of their first parts. $\square$

In order to prove Theorem 1.5.1 we consider the following situation: Let G be a connected reductive group with split maximal torus T and Weyl group $W = W(G, T)$. Let M be smooth and $E \to M$ be a principal G-bundle. Consider a Borel subgroup $B \supset T$. Now W acts on $A^*(E/B)$ in the following way. We identify $W = N_G(T)/T$ and choose $w \in N_G(T)$. Then w induces an automorphism $w : E/T \to E/T$ and hence an automorphism $w^* : A^*(E/T) \to A^*(E/T)$. This defines an action of W on $A^*(E/T) = A^*(E/B)$. The following lemma is also mentioned (without proof) in [Vistoli 1989, Section 2.5].

Lemma 1.5.4. *Pullback induces an isomorphism* $A^*(M)_{\mathbb{Q}} \cong A^*(E/B)_{\mathbb{Q}}^{W}$.

Proof. Let $w \in N_G(T)$. Since w lies in G the diagram

$$
\begin{array}{ccc}
E/T & \longrightarrow & E/G = M \\
{\scriptstyle w}\downarrow & \nearrow & \\
E/T & &
\end{array}
$$

commutes and this implies that the image of the pullback $A^*(M) \to A^*(E/B)$ lies in $A^*(E/B)^W$. We are left to show that

$$
A^*(M)_{\mathbb{Q}} \to A^*(E/B)_{\mathbb{Q}}^{W}
$$

is an isomorphism. Let us first show that $A_*(M)_{\mathbb{Q}} \to A_*(E/B)_{\mathbb{Q}}^{W}$ is surjective. For this the smoothness assumption on M is not needed. We recall that every G-torsor

is locally isotrivial by [Raynaud 1970, XIV, Lemma 1.4]. This means that there exists a covering of M by open subsets U with the property that for each U there is a finite, étale and surjective map $U' \to U$ such that $E_{U'} = E \times_M U' \to U'$ becomes a trivial G-torsor. Let V denote the complement of such a U in M and consider the commutative diagram

$$
\begin{array}{ccccccc}
A_*(V)_Q & \longrightarrow & A_*(M)_{\mathbb{Q}} & \longrightarrow & A_*(U)_{\mathbb{Q}} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
A_*(E_V/B)^W_{\mathbb{Q}} & \longrightarrow & A_*(E/B)^W_{\mathbb{Q}} & \longrightarrow & A_*(E_U/B)^W_{\mathbb{Q}} & \longrightarrow & 0
\end{array}
$$

with exact rows. An easy diagram chase shows that if the first and last vertical map are surjective so is $A^*(M)_{\mathbb{Q}} \to A^*(E/B)^W_{\mathbb{Q}}$. Using noetherian induction we are thus reduced to the case that there exists a proper surjective map $M' \to M$ such that $E_{M'} \to M'$ is trivial. Since the diagram

$$
\begin{array}{ccc}
A_*(M')_{\mathbb{Q}} & \longrightarrow & A_*(E_{M'}/B)^W_{\mathbb{Q}} \\
\downarrow & & \downarrow \\
A_*(M)_{\mathbb{Q}} & \longrightarrow & A_*(E/B)^W_{\mathbb{Q}}
\end{array}
$$

commutes [Fulton 1998, Proposition 1.7], and since $A_*(E_{M'}/B)^W_{\mathbb{Q}} \to A_*(E/B)^W_{\mathbb{Q}}$ is surjective by part (i) of the previous lemma we are further reduced to the case of a trivial G-torsor $E = G \times M \to M$. Now G/B has a decomposition into affine cells and therefore we obtain in the case of a trivial G-torsor $A_*(E/B)_{\mathbb{Q}} = A_*(G/B)_{\mathbb{Q}} \otimes A_*(M)_{\mathbb{Q}}$ by [Totaro 2014, Section 3]. From [Demazure 1973, Section 8] we get $A_*(G/B)_{\mathbb{Q}} = S_{\mathbb{Q}}/(S^W_+)$, where $S = \mathrm{Sym}(\widehat{T})$ and S^W_+ denotes the submodule generated by homogeneous W-invariant elements of positive degree. Since $(S_{\mathbb{Q}}/(S^W_+))^W = \mathbb{Q}$ we obtain $A_*(E/B)^W_{\mathbb{Q}} = A_*(M)$ as wanted.

By the previous lemma we know that $A^*(M; \mathbb{Q}) \to A^*(E/B; \mathbb{Q})$ is injective but since M (and therefore E) is smooth we obtain the injectivity of $A^*(M)_{\mathbb{Q}} \to A^*(E/B)_{\mathbb{Q}}$. $\square$

Proof of Theorem 1.5.1. The assertions (i) and (ii) are immediate consequences of Lemma 1.5.4. Under the assumption that $A^*_T(X)$ is $\mathbb{Z}$-torsion free the surjectivity of r follows from part (ii) by using the argumentation of the proof of Lemma 5 in [Edidin and Graham 1997]. $\square$

2. The Chow ring of the stack of level-*n* Barsotti–Tate groups

2.1. *The stack of truncated displays.* Let R be an $\mathbb{F}_p$-algebra. We denote by $W_n(R)$ the ring of truncated Witt vectors of length n. Let $I_{n,R} \subset W_n(R)$ be the image

of the Verschiebung $W_{n-1}(R) \to W_n(R)$ and $J_{n,R} \subset W_n(R)$ be the kernel of the projection $W_n(R) \to W_{n-1}(R)$. The Frobenius on R induces a ring homomorphism $\sigma : W_n(R) \to W_n(R)$ and the inverse of the Verschiebung induces a bijective σ-linear map $\sigma_1 : I_{n+1,R} \to W_n(R)$. Note that $pR = 0$ implies $I_{n,R}J_{n,R} = 0$, hence we may view $I_{n+1,R}$ as a $W_n(R)$-module. We call a σ-linear map $f : M \to N$ between $W_n(R)$-modules a σ-linear isomorphism, if its linearization $f^\sharp : W_n(R) \otimes_{\sigma, W_n(R)} M \to M$ is an isomorphism of $W_n(R)$-modules.

Truncated displays were introduced in [Lau 2013]. Let us recall the necessary notation. We are only going to need the following description of truncated displays.

Definition 2.1.1. A truncated display of level n over an $\mathbb{F}_p$-algebra R is a triple (L, T, Ψ) consisting of projective $W_n(R)$-modules L and T of finite rank and a σ-linear automorphism $\Psi : L \oplus T \to L \oplus T$.

A morphism between truncated displays is defined as follows. First we can use Ψ to define σ-linear maps

$$F : L \oplus T \to L \oplus T, \quad l + t \mapsto p\Psi(l) + \Psi(t),$$

$$F_1 : L \oplus (T \otimes_{W_n(R)} I_{n+1,R}) \to L \oplus T, \quad l + (t \otimes \omega) \mapsto \Psi(l) + \sigma_1(\omega)\Psi(t).$$

Then a morphism between two truncated displays (L, T, Ψ) and (L', T', Ψ') of level n is given by a matrix $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix} \right)$, where $A \in \operatorname{Hom}(L, L')$, $B \in \operatorname{Hom}(T, L')$, $C \in \operatorname{Hom}(L, T' \otimes_{W_n(R)} I_{n+1,R})$ and $D \in \operatorname{Hom}(T, T')$ such that

$$
\begin{array}{ccc}
L \oplus T & \xrightarrow{F} & L \oplus T \\
\downarrow & & \downarrow \\
L' \oplus T' & \xrightarrow{F'} & L' \oplus T'
\end{array}
\quad \text{and} \quad
\begin{array}{ccc}
L \oplus (T \otimes_{W_n(R)} I_{n+1,R}) & \xrightarrow{F_1} & L \oplus T \\
\downarrow & & \downarrow \\
L' \oplus (T' \otimes_{W_n(R)} I_{n+1,R}) & \xrightarrow{F_1'} & L' \oplus T'
\end{array}
$$

commute.

The height of a truncated display is defined as the rank of $L \oplus T$ and the dimension as the rank of T. Both are locally constant functions on $\operatorname{Spec} R$. Let $\mathcal{D}\mathrm{isp}_n \to \operatorname{Spec} \mathbb{F}_p$ denote the stack of truncated displays of level n. That is, for R an $\mathbb{F}_p$-algebra, $\mathcal{D}\mathrm{isp}_n(\operatorname{Spec} R)$ is the groupoid of truncated displays of level n. It is proved in [Lau 2013, Proposition 3.15] that $\mathcal{D}\mathrm{isp}_n$ is a smooth Artin algebraic stack of dimension zero over $\mathbb{F}_p$ with affine diagonal.

For $h \in \mathbb{N}$ and $0 \leq d \leq h$ we denote by $\mathcal{D}\mathrm{isp}_n^{h,d}$ the open and closed substack of truncated displays of level n with constant height h and constant dimension d. Then

$$\mathcal{D}\mathrm{isp}_n = \coprod_{h,d} \mathcal{D}\mathrm{isp}_n^{h,d}.$$

A presentation of $\mathcal{D}\mathrm{isp}_n^{h,d}$. We will adopt the notation of the proof of Proposition 3.15 in [Lau 2013]. Let $X_n^{h,d}$ be the functor on affine $\mathbb{F}_p$-schemes with

$X_n^{h,d}(R) = \mathrm{GL}_h(W_n(R))$. This is an affine open subscheme of $\mathbb{A}^{nh^2}$. Furthermore, let $G_n^{h,d}$ be the functor such that $G_n^{h,d}(R)$ is the group of invertible matrices $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with $A \in \mathrm{GL}_{h-d}(W_n(R))$, $B \in \mathrm{Hom}(W_n(R)^d, W_n(R)^{h-d})$, $C \in \mathrm{Hom}(W_n(R)^{h-d}, I_{n+1,R}^d)$ and $T \in \mathrm{GL}_d(W_n(R))$. Then $G_n^{h,d}$ is a connected algebraic group of dimension nh^2.

Remark 2.1.2. Since $I_{2,R}$ is in bijection to R via σ_1 we may view $G_1^{h,d}(R)$ as the group of invertible matrices with entries in R with respect to the multiplication given by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} = \begin{pmatrix} AA' & AB' + BD' \\ C\sigma(A') + \sigma(D)C' & DD' \end{pmatrix},$$

where in the four blocks we have the usual matrix multiplication.

Let $\pi_n^{h,d} : X_n^{h,d} \to \mathcal{D}\mathrm{isp}_{n,d}$ be the functor that assigns to an invertible matrix $\Psi \in \mathrm{GL}_h(W_n(R))$ the truncated display $(W_n(R)^{h-d}, W_n(R)^d, \Psi)$, where we view Ψ as a σ-linear map $W_n(R)^h \to W_n(R)^h$ via $x \mapsto \Psi \cdot \sigma x$. Now if we let $G_n^{h,d}$ act on $X_n^{h,d}$ via

$$G \cdot \Psi = G\Psi\sigma_1(G)^{-1}$$

where

$$\sigma_1(G) = \begin{pmatrix} \sigma(A) & p\sigma(B) \\ \sigma_1(C) & \sigma(D) \end{pmatrix},$$

then every $G \in G_n^{h,d}$ defines an isomorphism $\pi_n^{h,d}(\Psi) \to \pi_n^{h,d}(G \cdot \Psi)$ of truncated displays. On the other hand, if G defines an isomorphism $\pi_n^{h,d}(\Psi) \to \pi_n^{h,d}(\Psi')$ then necessarily $\Psi' = G\Psi\sigma_1(G)^{-1}$. We thus obtain the following theorem.

Theorem 2.1.3. *The functor $\pi_n^{h,d}$ induces an isomorphism of stacks*

$$[X_n^{h,d}/G_n^{h,d}] \cong \mathcal{D}\mathrm{isp}_n^{h,d}.$$

There are the following two obvious vector bundles on $\mathcal{D}\mathrm{isp}_n^{h,d}$.

Definition 2.1.4. Let $\mathrm{Spec}\, R \to \mathcal{D}\mathrm{isp}_n^{h,d}$ be a map corresponding to a truncated display $\mathcal{P} = (L, T, \Psi)$.

(i) We denote by $\mathcal{L}ie$ the vector bundle of rank d over $\mathcal{D}\mathrm{isp}_n^{h,d}$ that assigns to $\mathrm{Spec}\, R \to \mathcal{D}\mathrm{isp}_n^{h,d}$ the vector bundle $\mathrm{Lie}(\mathcal{P}) = T/I_{n,R}T$ of rank d over R.

(ii) We denote by ${}^t\mathcal{L}ie^\vee$ the vector bundle of rank $h-d$ that assigns to $\mathrm{Spec}\, R \to \mathcal{D}\mathrm{isp}_n^{h,d}$ the vector bundle $L/I_{n,R}L$ of rank $h-d$ over R.

Remark 2.1.5. The notation ${}^t\mathcal{L}ie^\vee$ in the above definition stems from the fact that the dual of $L/I_{n,R}L$ gives the Lie algebra of the dual display $\mathcal{P}^t$. For the definition of the dual display see [Zink 2002, Definition 19].

The truncated display functor. As mentioned in the introduction the strategy for computing the Chow ring of the stack of truncated Barsotti–Tate groups is to relate it to the stack of truncated displays. This happens via the truncated display functor

$$\phi_n : \mathrm{BT}_n \to \mathcal{D}\mathrm{isp}_n$$

constructed in [Lau 2013]. Let us briefly sketch the construction.

Let G be a p-divisible group over an $\mathbb{F}_p$-algebra R. The ring of Witt vectors $W(R)$ is p-adically complete and the ideal I_R in $W(R)$ carries natural divided powers compatible with the canonical divided powers of p. Let $\mathbb{D}(G)$ denote the covariant Dieudonné crystal of G. We can evaluate $\mathbb{D}(G)$ at $W(R) \to R$ and set $P = \mathbb{D}(G)_{W(R)\to R}$ and $Q = \mathrm{Ker}(P \to \mathrm{Lie}(G))$. Furthermore, let $F^\sharp : P^\sigma \to P$ and $V^\sharp : P \to P^\sigma$ be the maps induced by the Frobenius and Verschiebung of G. One can show that there are σ-linear maps $F : P \to P$ and $\dot{F} : Q \to P$ compatible with base change in R such that $(P, Q, F, \dot{F})$ is a display which induces the maps $F^\sharp$ and $V^\sharp$. See [Lau 2013, Proposition 2.4] for the precise statement. This construction yields a 1-morphism

$$\phi : \mathrm{BT} \to \mathcal{D}\mathrm{isp}$$

from the stack of Barsotti–Tate groups to the stack of displays. It is clear from the construction that the Lie algebra of G is equal to the Lie algebra of $\phi(G)$ defined by P/Q.

Moreover, one can prove that for all n there are maps $\phi_n : \mathrm{BT}_n \to \mathcal{D}\mathrm{isp}_n$ compatible with the truncation maps on both sides such that ϕ is the projective limit of the system $(\phi_n)_{n\geq 1}$. The central result in [Lau 2013] is that ϕ_n is a smooth morphism of smooth algebraic stacks over $\mathbb{F}_p$ which is an equivalence on geometric points.

2.2. Group theoretic properties of $G_n^{h,d}$. We denote by $K_{(n,m)}^{h,d}$ the kernel of the projection $G_n^{h,d} \to G_m^{h,d}$ for $m < n$ and by $\widetilde{K}_n^{h,d}$ the kernel of the projection $G_n^{h,d} \to \mathrm{GL}_{h-d} \times \mathrm{GL}_d$. Note that $G_n^{h,0} = \mathrm{GL}_h(W_n(\,\cdot\,))$. We recall the following well known facts about the Witt ring. For an $\mathbb{F}_p$-algebra R we denote by $[\,\cdot\,] : R \to W_n(R)$ the map $r \mapsto (r, 0, \ldots, 0)$ and $^V(\,\cdot\,) : W(R) \to W(R)$ is the Verschiebung.

Lemma 2.2.1. *Let R be an $\mathbb{F}_p$-algebra and $x, y \in R$. Then $[x + y] - [x] - [y]$ lies in $^V W(R)$. Furthermore, $^{V^r}W(R) \cdot {}^{V^s}W(R) \subset {}^{V^{r+s}}W(R)$.*

Proof. The first part follows immediately from the fact that $^V W(R)$ is the kernel of the ring homomorphism $\mathbb{W}_0 : W(R) \to R$ and the fact $\mathbb{W}_0([x]) = x$ for all $x \in R$.

For the second part we may assume $r \geq s$. We then write

$$^{V^r}x\,{}^{V^s}y = {}^{V^r}(x\,{}^{F^r V^s}y) = p^s \cdot {}^{V^r}(x\,{}^{F^{r-s}}y).$$

Since $pR = 0$ we have $p(x_0, x_1, \ldots) = (0, x_0^p, x_1^p, \ldots)$ in $W(R)$ and the lemma follows. $\square$

Lemma 2.2.2. (i) $K_{(n,m)}^{h,d}$ is unipotent.

(ii) $\widetilde{K}_n^{h,d}$ is unipotent.

Proof. (i) First note that $K_{(n,n-1)}^{h,0} = \ker(\mathrm{GL}_h(W_n(\,\cdot\,)) \to \mathrm{GL}_h(W_{n-1}(\,\cdot\,)))$ is unipotent. To see this we consider the Verschiebung ${}^V(\,\cdot\,)$ as a map $W_n(R) \to W_n(R)$. Then by the above lemma the map

$$\mathbb{G}_a^{h^2} \to K_{(n,n-1)}^{h,0}, \qquad A \mapsto I_h + {}^{V^{n-1}}[A]$$

is an isomorphism of algebraic groups.

Next we show that $K_{(n,n-1)}^{h,d}$ is unipotent. This is the group of matrices $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with $A \in K_{(n,n-1)}^{h-d,0}$, $B \in J_n^{(h-d)\times d}$, $C \in J_{n+1}^{d\times(h-d)}$ and $D \in K_{(n,n-1)}^{d,0}$. The multiplication in this group is given by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} = \begin{pmatrix} AA' & AB' + BD' \\ CA' + DC' & DD' \end{pmatrix}.$$

Starting with the normal subgroup

$$\begin{pmatrix} I_{h-d} & J_n^{(h-d)\times d} \\ J_{n+1}^{d\times(h-d)} & I_d \end{pmatrix},$$

which is isomorphic to $\mathbb{G}_a^{2d(h-d)}$, and then using the fact that $K_{(n,n-1)}^{h-d,0}$ and $K_{(n,n-1)}^{d,0}$ are isomorphic to $\mathbb{G}_a^{(h-d)^2}$ and $\mathbb{G}_a^{d^2}$, respectively, one obtains a filtration of $K_{(n,n-1)}^{h,d}$ by normal subgroups, whose successive quotients are isomorphic to a product of copies of $\mathbb{G}_a$. Now we have an exact sequence

$$0 \longrightarrow K_{(n,n-1)}^{h,d} \longrightarrow K_{(n,m)}^{h,d} \longrightarrow K_{(n-1,m)}^{h,d} \longrightarrow 0$$

and by induction we may assume that $K_{(n-1,m)}^{h,d}$ is unipotent. It follows that $K_{(n,m)}^{h,d}$ is unipotent.

(ii) For $n = 1$ the assertion is obvious in view of Remark 2.1.2. For $n > 1$ we use the exact sequence

$$0 \longrightarrow K_{(n,n-1)}^{h,d} \longrightarrow \widetilde{K}_n^{h,d} \longrightarrow \widetilde{K}_{n-1}^{h,d} \longrightarrow 0.$$

By induction and part (i) it follows that $\widetilde{K}_n^{h,d}$ is unipotent. $\qquad\square$

Corollary 2.2.3. (i) $G_n^{h,d}$ is special.

(ii) $\widetilde{K}_n^{h,d}$ is the unipotent radical of $G_n^{h,d}$.

(iii) The projection $X_n^{h,d} \to X_1^{h,d}$ is a trivial $K_{(n,1)}^{h,0}$-torsor.

Proof. We have the exact sequence

$$0 \longrightarrow \widetilde{K}_n^{h,d} \longrightarrow G_n^{h,d} \longrightarrow \mathrm{GL}_{h-d} \times \mathrm{GL}_d \longrightarrow 0.$$

Now $\widetilde{K}_n^{h,d}$ is unipotent, and thus special. Since $\mathrm{GL}_{h-d} \times \mathrm{GL}_d$ is also special part (i) follows.

Clearly the projection $X_n^{h,d} \to X_1^{h,d}$ is a $K_{(n,1)}^{h,0}$-torsor by definition of $K_{(n,1)}^{h,0}$. It is trivial since $K_{(n,1)}^{h,0}$ is unipotent and $X_1^{h,d}$ is affine. $\square$

2.3. The Chow ring of $\mathcal{D}\mathrm{isp}_n$.
We start with the following result which reduces the calculation of $A^*(\mathcal{D}\mathrm{isp}_n)$ to the case $n = 1$.

Theorem 2.3.1. *The pullback*

$$\tau_n^* : A^*(\mathcal{D}\mathrm{isp}_1^{h,d}) \to A^*(\mathcal{D}\mathrm{isp}_n^{h,d})$$

of the truncation $\tau_n : \mathcal{D}\mathrm{isp}_n^{h,d} \to \mathcal{D}\mathrm{isp}_1^{h,d}$ *is an isomorphism.*

Proof. Under the presentation $\mathcal{D}\mathrm{isp}_n^{h,d} = [X_n^{h,d}/G_n^{h,d}]$ the truncation τ_n is induced by the natural projections $X_n^{h,d} \to X_1^{h,d}$ and $G_n^{h,d} \to G_1^{h,d}$. Thus τ_n factors as

$$[X_n^{h,d}/G_n^{h,d}] \to [X_1^{h,d}/G_n^{h,d}] \to [X_1^{h,d}/G_1^{h,d}].$$

By Lemma 2.2.2 and Corollary 1.4.2, the pullback of the second map is an isomorphism. To show that the pullback of the first map is also an isomorphism let us abbreviate $X = X_1^{h,d}$ and $G = G_n^{h,d}$. By part (iii) of Corollary 2.2.3 we know that $X_n^{h,d} = X \times K$ with $K = K_{(n,1)}^{h,0}$, and the projection $X \times K \to X$ is G-equivariant. Moreover, K is an affine space by Lemma 2.2.2. After replacing $[X/G]$ by an appropriate mixed space (see Convention 1.1.1), i.e., replacing X by $X \times U$ where (V, U) is an admissible pair with high codimension, we may assume that $[X/G]$ is a quasiprojective scheme. We claim that $(X \times K)/G \to X/G$ is a Zariski locally trivial affine bundle. Since G is special by part (i) of Corollary 2.2.3 the principal G-bundle $X \to X/G$ is locally trivial for the Zariski topology and after replacing X/G by an appropriate open subset we may assume $X = G \times (X/G)$. We then have an isomorphism $(G \times (X/G) \times K)/G \cong (X/G) \times K$ given by the assignment $(g, x, k) \mapsto (x, k')$, where k' is defined by $g^{-1}(g, x, k) = (1, x, k')$. This proves the claim and hence the pullback of the first map is also an isomorphism by Corollary 1.3.3. $\square$

The main ingredient of the computation of $A^*\mathcal{D}\mathrm{isp}_1^{h,d}$ is the following proposition.

Proposition 2.3.2. *Let G be a connected split reductive group over a field k with split maximal torus T. Consider an isogeny $\varphi : L \to M$, where L and M are Levi components of parabolic subgroups P and Q of G. Assume $T \subset L$ and let $g_0 \in G(k)$ such that $\varphi(T) = {}^{g_0}T$. Let $\tilde{\varphi} : T \to T$ denote the isogeny φ followed by conjugation with g_0^{-1}. We write $S = \mathrm{Sym}(\widehat{T}) = A_T^*$ and $S_+ = A_T^{\geq 1}$. We have a natural action of $\tilde{\varphi}$ on S, that we will also denote by $\tilde{\varphi}$.*

Consider the action of L on G by φ-conjugation. If $W_G = W(G, T)$ and $W_L = W(L, T)$ denote the respective Weyl groups we have

$$A_L^*(G)_{\mathbb{Q}} = S_{\mathbb{Q}}^{W_L}/(f - \tilde{\varphi}f \mid f \in S_+^{W_G})_{\mathbb{Q}}.$$

If G is special we have

$$A_L^*(G) = S^{W_L}/(f - \tilde{\varphi}f \mid f \in S_+^{W_G}).$$

(Note that the action of $\tilde{\varphi}$ on S^{W_G} is independent of the choice of g_0 since two choices differ by an element of $N_G(T)$.)

Proof. The case of special G is proven in [Brokemper 2016, Proposition 1.1]. Let I denote the ideal $(f - \varphi f \mid f \in S_+^{W_G})_{\mathbb{Q}}$ in $S_{\mathbb{Q}}^{W_G}$. It remains to show $A_L^*(G)_{\mathbb{Q}} = S_{\mathbb{Q}}^{W_L}/IS_{\mathbb{Q}}^{W_L}$ in the nonspecial case. Using the same argumentation as in the special case we arrive at

$$A_T^*(G)_{\mathbb{Q}} = S_{\mathbb{Q}}/IS_{\mathbb{Q}}.$$

Now by Theorem 1.5.1 we know $A_L^*(G)_{\mathbb{Q}} = A_T^*(G)_{\mathbb{Q}}^{W_L}$. Since $S_{\mathbb{Q}}^{W_L} \hookrightarrow S_{\mathbb{Q}}$ is finite free [Demazure 1973, Theorem 2(d)], it is also faithfully flat. Hence we obtain $S_{\mathbb{Q}}^{W_L} \cap IS_{\mathbb{Q}} = IS_{\mathbb{Q}}^{W_L}$ and the assertion follows. $\qquad\square$

In the following we will write c_i for the i-th elementary symmetric polynomial in the variables $t_1, \ldots, t_h$ and $c_i^{(j,k)}$ will denote the i-th elementary symmetric polynomial in the variables $t_j, \ldots, t_k$, where $1 \le j < k \le h$ and $1 \le i \le k - j + 1$. We then have $\mathbb{Z}[t_1, \ldots, t_n]^{S_{h-d} \times S_d} = \mathbb{Z}[c_1^{(1,h-d)}, \ldots, c_{h-d}^{(1,h-d)}, c_1^{(h-d+1,h)}, \ldots, c_d^{(h-d+1,h)}]$.

Theorem 2.3.3. $A^*(\mathcal{D}isp_1^{h,d}) = A_{\mathrm{GL}_{h-d} \times \mathrm{GL}_d}^*(\mathrm{GL}_h)$

$$= \mathbb{Z}[t_1, \ldots, t_n]^{S_{h-d} \times S_d}/((p-1)c_1, \ldots, (p^h - 1)c_h),$$

where the $c_i^{(1,h-d)}$ and $c_i^{(h-d+1,h)}$ are the Chern classes of ${}^t\mathcal{L}ie^\vee$ and $\mathcal{L}ie$, respectively.

Proof. We have that $G_1^{h,d}$ is a split extension of the group $\mathrm{GL}_{h-d} \times \mathrm{GL}_d$ by the unipotent group

$$\left\{ \begin{pmatrix} I_{h-d} & * \\ * & I_d \end{pmatrix} \right\},$$

where $*$ denotes an arbitrary matrix (see Remark 2.1.2). The splitting is given by the canonical inclusion $\mathrm{GL}_{h-d} \times \mathrm{GL}_d \hookrightarrow G_1^{h,d}$. Hence by Lemma 1.4.7 we know

$$A^*(\mathcal{D}\mathrm{isp}_1^{h,d}) = A_{\mathrm{GL}_{h-d} \times \mathrm{GL}_d}^*(\mathrm{GL}_h),$$

where the action of $\mathrm{GL}_{h-d} \times \mathrm{GL}_d$ on GL_h is given by σ-conjugation. Since $\mathrm{GL}_{h-d} \times \mathrm{GL}_d$ is special with Weyl group $S_{h-d} \times S_d$ we obtain from Proposition 2.3.2

$$A_{\mathrm{GL}_{h-d} \times \mathrm{GL}_d}^*(\mathrm{GL}_h) = \mathbb{Z}[t_1, \ldots, t_n]^{S_{h-d} \times S_d}/((p-1)c_1, \ldots, (p^h - 1)c_h).$$

For this, note that the Frobenius σ acts on $S = \mathbb{Z}[t_1, \ldots, t_n]$ via $\sigma t_i = pt_i$ and that c_i is a homogenous polynomial in $t_1, \ldots, t_n$ of degree i.

The assertion that the $c_i^{(1,h-d)}$ and $c_i^{(h-d+1,h)}$ are the Chern classes of $\mathcal{L}ie$ and ${}^t\mathcal{L}ie^\vee$, respectively, follows from the following simple fact. Let us write $\mathcal{E}_d$

(resp. $\mathcal{E}_{h-d}$) for the vector bundle over $[*/\operatorname{GL}_d]$ (resp. $[*/\operatorname{GL}_{h-d}]$) that corresponds to the canonical representation of GL_d (resp. GL_{h-d}). Then $\mathcal{L}ie$ is the pullback of $\mathcal{E}_d$ under the natural map

$$\mathcal{D}\mathrm{isp}_1^{h,d} = [GL_h/G_1^{h,d}] \longrightarrow [*/(\operatorname{GL}_d \times \operatorname{GL}_{h-d})] \longrightarrow [*/GL_d]$$

and similarly for ${}^t\mathcal{L}ie^\vee$. $\square$

Corollary 2.3.4. $\operatorname{Pic}(\mathcal{D}\mathrm{isp}_1^{h,d}) = \begin{cases} \mathbb{Z}/(p-1)\mathbb{Z} & \text{if } d = 0, h, \\ \mathbb{Z} \times \mathbb{Z}/(p-1)\mathbb{Z} & \text{otherwise.} \end{cases}$

A generator for the free part is $\det(\mathcal{L}ie)$ *and a generator for the torsion part is* $\det(\mathcal{L}ie \otimes {}^t\mathcal{L}ie^\vee)$.

Proof. Note $\operatorname{Pic} \mathcal{D}\mathrm{isp}_n^{h,d} = A^1 \mathcal{D}\mathrm{isp}_n^{h,d}$ by [Edidin and Graham 1998, Corollary 1]. $\square$

Remark 2.3.5. There is also a more direct approach to compute the above Picard groups. By using a theorem of Rosenlicht, namely that for irreducible varieties X and Y the natural map

$$\mathcal{O}(X)^* \times \mathcal{O}(Y)^* \to \mathcal{O}(X \times Y)^*$$

is surjective, it is not difficult to establish the exact sequence

$$\mathcal{O}(X)^*/k^* \longrightarrow \widehat{G} \longrightarrow \operatorname{Pic}^G(X) \longrightarrow \operatorname{Pic}(X)$$

for G connected and X an irreducible G-scheme. The first map assigns to a nonvanishing regular function on X its eigenvalue. In our case we have $G = \operatorname{GL}_{h-d} \times \operatorname{GL}_d$ and $X = \operatorname{GL}_h$. Then $\mathcal{O}(\operatorname{GL}_h)^*/k^* = \mathbb{Z}$ with generator given by the determinant and eigenvalue given by the character $(p-1)(\det_{\operatorname{GL}_{h-d}} + \det_{\operatorname{GL}_d}) \in \widehat{G}$. Since $\operatorname{Pic}(\operatorname{GL}_h) = 0$ we again obtain the result of the above Corollary.

Remark 2.3.6. The fact that $(\det \mathcal{L}ie \otimes \det {}^t\mathcal{L}ie^\vee)^{p-1}$ is trivial can also be seen directly as follows: $(\det \mathcal{L}ie \otimes \det {}^t\mathcal{L}ie^\vee)^{p-1}$ being trivial means that $\det \mathcal{L}ie \otimes \det {}^t\mathcal{L}ie^\vee$ is fixed under the pullback of the Frobenius map $\mathrm{Frob} : \mathcal{D}\mathrm{isp}_1^{2,1} \to \mathcal{D}\mathrm{isp}_1^{2,1}$ assigning to a display $\mathcal{P}$ over an $\mathbb{F}_p$-algebra R the display $\mathcal{P}^\sigma$ obtained by base change via the Frobenius $\sigma : R \to R$. But by definition of a truncated display we have an isomorphism

$$\Psi : L \oplus T \cong L^\sigma \oplus T^\sigma$$

of R-modules. Taking the determinant of Ψ yields the desired isomorphism

$$\det L \otimes \det T \cong \det L^\sigma \otimes \det T^\sigma.$$

Remark 2.3.7. Let us put this result into context by relating it to the corresponding result for elliptic curves. Let $\mathcal{M}_{1,1} \to \operatorname{Spec} k$ denote the moduli stack of elliptic curves. A morphism $\operatorname{Spec} R \to \mathcal{M}_{1,1}$ corresponds to a pair $(C \to \operatorname{Spec} R, \sigma)$ where

$C \to \operatorname{Spec} R$ is a smooth projective curve of genus 1 and $\sigma : \operatorname{Spec} R \to C$ is a smooth section. We now have the diagram

$$
\begin{array}{ccc}
\mathcal{M}_{1,1} \longrightarrow \mathrm{BT}^{h=2,d=1} & \xrightarrow{\ \phi\ } & \mathcal{D}\mathrm{isp}^{h=2,d=1} \\
\downarrow & & \downarrow \\
\mathrm{BT}^{h=2,d=1}_{n=1} & \xrightarrow[\phi_1]{} & \mathcal{D}\mathrm{isp}^{h=2,d=1}_{n=1}
\end{array}
$$

where $\mathcal{M}_{1,1} \to \mathrm{BT}^{h=2,d=1}$ sends an elliptic curve C to its associated Barsotti–Tate group $C[p^\infty]$. Let us consider the pullback map $A^*(\mathcal{D}\mathrm{isp}^{2,1}_1) \to A^*(\mathcal{M}_{1,1})$. In characteristic p different from 2 and 3, Edidin and Graham computed $A^*(\mathcal{M}_{1,1}) = \mathbb{Z}[t]/(12t)$, where t is given by the first Chern class of the Hodge bundle on $\mathcal{M}_{1,1}$ [Edidin and Graham 1998, Proposition 21].

By construction of the truncated display functor the pullback of $\mathcal{L}ie$ to $\mathcal{M}_{1,1}$ is the dual of the Hodge bundle on $\mathcal{M}_{1,1}$. Since the dual of an elliptic curve is the elliptic curve itself, it follows from Remark 2.1.5 that the pullback of ${}^t\mathcal{L}ie^\vee$ is given by the Hodge bundle. Hence $A^*(\mathcal{D}\mathrm{isp}^{2,1}_1) \to A^*(\mathcal{M}_{1,1})$ is the map

$$
\mathbb{Z}[t_1, t_2]/((p-1)c_1, (p^2-1)c_2) \to \mathbb{Z}[t]/(12t)
$$

that sends t_1 to $-t$ and t_2 to t. Note that $p^2 - 1$ is divisible by 12 if and only if $p \geq 5$. In particular, there can be no such map for $p = 2, 3$, and we deduce that the description $A^*(\mathcal{M}_{1,1}) = \mathbb{Z}[t]/(12t)$ does not hold in characteristic 2 and 3.

2.4. *The Chow ring of the stack of G-zips.* Let us first consider the case of F-zips introduced in [Moonen and Wedhorn 2004]. We denote by F-zip the stack of F-zips over a field k of characteristic $p > 0$. For S a k-scheme F-zip(S) is the groupoid of F-zips over S. If $\tau : \mathbb{Z} \to \mathbb{Z}_{\geq 0}$ is a function with finite support we denote by F-zip$^\tau$ the open and closed substack of F-zips of type τ. Note that

$$
F\text{-zip} = \coprod_\tau F\text{-zip}^\tau.
$$

The stacks F-zip$^\tau$ are smooth Artin algebraic stacks over k which follows for example from the following representation as a quotient stack. Let X_τ denote the k-scheme whose S-valued points are given by

$$
X_\tau(S) = \{\underline{M} = (M, C^\bullet, D_\bullet, \varphi_\bullet) \mid \underline{M} \text{ is an } F\text{-zip of type } \tau, \ M = \mathcal{O}^h_S\}.
$$

This is a smooth scheme of dimension h^2. Here $h = \sum_{i \in \mathbb{Z}} \tau(i)$ is also called the height of $\underline{M}$. The group GL_h acts on X_τ by

$$
G \cdot \underline{M} = (\mathcal{O}^h_S, G(C^\bullet), G(D_\bullet), G\varphi_\bullet(G^{-1})^\sigma).
$$

It is easy to see that two F-zips over S of the above form are isomorphic if and

only if they lie in the same $\mathrm{GL}_h(S)$-orbit. Thus

$$F\text{-zip}^\tau = [X_\tau / \mathrm{GL}_h].$$

An F-zip $\underline{M}$ of type τ with support in $\{0, 1\}$ over an $\mathbb{F}_p$-algebra R is just a tuple

$$\underline{M} = (M, C, D, \varphi_0, \varphi_1),$$

where M is a projective R-module with submodules C and D, which are direct summands of M and isomorphisms

$$\varphi_0 : C^\sigma \to M/D, \quad \varphi_1 : (M/C)^\sigma \to D.$$

Lemma 2.4.1. *Let R be an $\mathbb{F}_p$-algebra. Then we have an equivalence of categories*

$$\mathcal{D}isp_1(R) \to \coprod_{\tau, \mathrm{Supp}(\tau) \in \{0,1\}} F\text{-zip}^\tau(R)$$

given by

$$(L, T, \Psi) \mapsto (L \oplus T, T, \Psi^\sigma(L^\sigma), \Psi^\sigma|_{T^\sigma}, \Psi^\sigma|_{L^\sigma}).$$

The above assignment commutes with pulling back. In particular, we get an isomorphism of stacks

$$F\text{-zip}^\tau \cong \mathcal{D}isp_1^{\tau(0)+\tau(1),\tau(1)}$$

for every type τ with support lying in $\{0, 1\}$.

Proof. An inverse functor is given by the assignment

$$(M, C, D, \varphi_0, \varphi_1) \mapsto (C, M/C, \varphi_0 \oplus \varphi_1). \qquad \square$$

More generally, there is the stack of G-zips introduced in [Pink et al. 2011]. Here G refers to an arbitrary reductive group. It is defined as follows. Let $\mathcal{Z}$ be an algebraic zip datum, i.e., a 4-tuple (G, P, Q, φ) consisting of a split reductive group G, parabolic subgroups P and Q and an isogeny $\varphi : P/R_u(P) \to Q/R_u(Q)$. To $\mathcal{Z}$ one associates the group

$$E_{\mathcal{Z}} = \{(p, q) \in P \times Q \mid \varphi(\pi_P(p)) = \pi_Q(q)\}.$$

Now $E_{\mathcal{Z}}$ acts on G by the rule

$$((p, q), g) \mapsto pgq^{-1}$$

and the quotient stack $[G/E_{\mathcal{Z}}]$ is called the stack of G-zips. If G is connected $\mathcal{Z}$ is called a connected zip datum [Pink et al. 2011, Definition 3.1].

Let us recall how the stack of F-zips is just a special case of this construction. For this let $\tau : \mathbb{Z} \to \mathbb{Z}_{\geq 0}$ be a function with finite support, say $i_1 \leq \cdots \leq i_r$. If we denote $n_k = \tau(i_k)$, then $(n_1, \ldots, n_r)$ defines a partition of $h = \sum_k n_k$. We denote the standard parabolic of type $(n_1, \ldots, n_r)$ in GL_h by P_τ.

Lemma 2.4.2. *Let $\tau : \mathbb{Z} \to \mathbb{Z}_{\geq 0}$ be a function with finite support and let $\mathcal{Z} = (\mathrm{GL}_h, P_\tau, P_\tau^-, \sigma)$ be the algebraic zip datum with P_τ^- the opposite parabolic of P_τ and σ the Frobenius isogeny. Then there is an isomorphism of stacks*

$$[\mathrm{GL}_h / E_{\mathcal{Z}}] \xrightarrow{\sim} F\text{-zip}^\tau.$$

Proof. Let S be an k-scheme. We denote by $C_\tau^\bullet$ the descending filtration

$$C_\tau^\bullet = \mathcal{O}_S^h \supset \mathcal{O}_S^{n_1 + \cdots + n_{r-1}} \supset \cdots \supset$$

in $\mathcal{O}_S^h$ given by the standard flag of type $(n_1, \ldots, n_r)$ and by $D_\bullet^{\tau^-}$ the ascending filtration

$$D_\bullet^{\tau^-} = 0 \subset \mathcal{O}_S^{n_r} \subset \cdots \subset \mathcal{O}_S^{n_r + \cdots + n_2} \subset \mathcal{O}_S^h$$

given by the flag of type opposite to $(n_1, \ldots, n_r)$. To $g \in \mathrm{GL}_h(S)$ we assign the F-zip

$$\underline{M}_g = (\mathcal{O}_S^h, C_\tau^\bullet, g(D_\bullet^{\tau^-}), \varphi_\bullet),$$

where φ is given by the restriction of g to the successive quotients of $C_\tau^\bullet$. Note that we can consider g as a σ-linear map.

If (p, q) is an element of $E_{\mathcal{Z}}$ we get an isomorphism $M_g \to M_{pgq^{-1}}$ of F-zips induced by p. The fact that p commutes with the φ_i is exactly the condition $\sigma(\pi(p)) = \pi(q)$. On the other hand if an isomorphism $p : M_g \to M_{g'}$ of F-zips is given, we see that $g'^{-1} p g$ preserves the flag of type opposite to $(n_1, \ldots, n_r)$. Thus $q = g'^{-1} p g \in P_\tau^-$ and again the compatibility of p with the φ_i implies the condition $\sigma(\pi(p)) = \pi(q)$. $\qquad\square$

We can also use Proposition 2.3.2 to say something about the Chow ring of the stack of G-zips for an arbitrary connected algebraic zip datum.

Definition 2.4.3. We call an algebraic zip datum $\mathcal{Z} = (G, P, Q, \varphi)$ special, if G is special.

Theorem 2.4.4. *Let $\mathcal{Z} = (G, P, Q, \varphi)$ be a connected algebraic zip datum. Let $W_G = W(G, T)$ be the Weyl group of G and $W_L = W(L, T)$ be the Weyl group of a Levi component L of P with respect to a split maximal torus $T \subset L$ of G. Let $g_0 \in G(k)$ be such that $\varphi(T) = {}^{g_0}T$ and let $\tilde{\varphi} : T \to T$ denote the composition of φ followed by conjugation with g_0^{-1}. Then $\tilde{\varphi}$ induces an action on $S = \mathrm{Sym}(\widehat{T})$ that we will also denote by $\tilde{\varphi}$. We then have*

$$A^*([G/E_{\mathcal{Z}}])_{\mathbb{Q}} = S_{\mathbb{Q}}^{W_L} / (f - \tilde{\varphi} f \mid f \in S_+^{W_G})_{\mathbb{Q}}.$$

If $\mathcal{Z}$ is special we have

$$A^*([G/E_{\mathcal{Z}}]) = S^{W_L} / (f - \tilde{\varphi} f \mid f \in S_+^{W_G}).$$

(Note that the action of $\tilde{\varphi}$ on S^{W_G} is independent of the choice of g_0 since two choices differ by an element of $N_G(T)$.)

Proof. By definition of the group $E_{\mathcal{Z}}$ we have a split exact sequence

$$0 \longrightarrow R_u(P) \times R_u(Q) \longrightarrow E_{\mathcal{Z}} \longrightarrow L \longrightarrow 0,$$

where the splitting is given by $L \hookrightarrow E_{\mathcal{Z}}$, $l \mapsto (l, \varphi(l))$. From Lemma 1.4.7 we deduce

$$A^*([G/E_{\mathcal{Z}}])_{\mathbb{Q}} = A_L^*(G)_{\mathbb{Q}},$$

where the action of L on G is given by φ-conjugation. If G is special the above equality holds over $\mathbb{Z}$. We conclude by Proposition 2.3.2. $\square$

Example 2.4.5. We consider the case $\mathcal{Z} = (\mathrm{Sp}(2n), P, P^-, \sigma)$, where σ denotes the q-th power Frobenius. Recall that $\mathrm{Sp}(2n)$ is special and the Weyl group of $\mathrm{Sp}(2n)$ is the wreath product $S_n \wr (\mathbb{Z}/2\mathbb{Z}) = S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$. It acts on $\mathrm{Sym}(\widehat{T}) = \mathbb{Z}[t_1, \ldots, t_n]$ in the following way. S_n acts by permuting the variables $t_1, \ldots, t_n$ and after identifying $\mathbb{Z}/2\mathbb{Z} = \{\pm 1\}$ an element $(\varepsilon_1, \ldots, \varepsilon_n) \in \mathbb{Z}/2\mathbb{Z}^n$ acts by $(\varepsilon_1, \ldots, \varepsilon_n) \cdot t_i = \varepsilon_i t_i$.

If P is a Borel we obtain from the above theorem that

$$A^*([Sp(2n)/E_{\mathcal{Z}}])) = \mathbb{Z}[t_1, \ldots, t_n]/((q^2 - 1)c_1(\underline{t}^2), \ldots, (q^{2n} - 1)c_n(\underline{t}^2)).$$

If P is the maximal parabolic subgroup fixing a maximal isotropic subspace then $L = \mathrm{GL}_n$ and $W_L = S_n$ and therefore

$$A^*([Sp(2n)/E_{\mathcal{Z}}]) = \mathbb{Z}[c_1, \ldots, c_n]/((q^2 - 1)c_1(\underline{t}^2), \ldots, (q^{2n} - 1)c_n(\underline{t}^2)).$$

It turns out that a $\mathbb{Q}$-basis of the Chow ring of the stack of G-zips is given by the closures of the orbits of the action of $E_{\mathcal{Z}}$ on G. To prove this let us introduce the naive Chow group of a quotient stack.

Definition 2.4.6. Let G be an algebraic group and let X be a G-scheme. Let $Z_*([X/G])$ be the free abelian group generated by the set of G-invariant closed subvarieties of X graded by dimension. Let $W_i([X/G])$ be the group $\bigoplus_Y k(Y)^G$, where the sum goes over all G-invariant closed subvarieties of X of dimension $i + 1$. There is the usual divisor map $\mathrm{div} : W_i([X/G]) \to Z_i([X/G])$ and we define the i-th naive Chow group of $[X/G]$ to be

$$A_i^o[X/G] = Z_i([X/G])/\mathrm{div}(W_i([X/G])).$$

Remark 2.4.7. There is more generally a definition of naive Chow groups for arbitrary algebraic stacks ([Kresch 1999, Definition 2.1.4]) which in the case of a quotient stack agrees with the one given above. Thus the above definition is independent of the presentation as a quotient stack.

Remark 2.4.8. There is a natural map $A_*^o[X/G] \to A_*[X/G]$. When X is Deligne–Mumford, i.e., the stabilizer of every point is finite and geometrically reduced, the induced map $A_*^o[X/G]_{\mathbb{Q}} \to A_*[X/G]_{\mathbb{Q}}$ is an isomorphism of groups and an isomorphism of rings if $[X/G]$ is smooth [Kresch 1999, Theorem 2.1.12(ii)].

The stack of G-zips is not Deligne–Mumford. However, we still have the following proposition.

Proposition 2.4.9. *Let G be a connected algebraic group and X be an admissible G-scheme* (see Definition 1.2.3) *with finitely many orbits such that the stabilizer of every point is an extension of a finite group by a unipotent group. Then $A^o_*[X/G]_{\mathbb{Q}} \to A_*[X/G]_{\mathbb{Q}}$ is an isomorphism.*

Proof. We prove this by induction on the number of orbits. Let U denote the open G-orbit and W its complement. We have a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & A^o_*[W/G]_{\mathbb{Q}} & \longrightarrow & A^o_*[X/G]_{\mathbb{Q}} & \longrightarrow & A^o_*[U/G]_{\mathbb{Q}} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_*[W/G]_{\mathbb{Q}} & \longrightarrow & A_*[X/G]_{\mathbb{Q}} & \longrightarrow & A_*[U/G]_{\mathbb{Q}} & \longrightarrow & 0
\end{array}
$$

and we claim that the rows of this diagram are exact. Since there are only finitely many orbits every G-invariant subvariety Y of X is the closure of a G-orbit. Since Y admits a dense G-invariant subset every G-invariant rational function on Y is constant. It follows that $A^o_*[X/G] = \bigoplus_Z \mathbb{Z}[\bar{Z}]$ where the sum goes over all G-orbits Z of X. From this we obtain the exactness of the top row. For the exactness of the lower row we need to see that the pullback map $A_*([X/G], 1)_{\mathbb{Q}} \to A_*([U/G], 1)_{\mathbb{Q}}$ is surjective. But $[U/G]$ is isomorphic to the classifying space of the stabilizer group scheme of U. By assumption and Corollary 1.4.4 we get that $A_*([U/G], m)_{\mathbb{Q}} \to A_*(B\{0\}, m)_{\mathbb{Q}}$ is an isomorphism. Equivalently the pullback of the structure morphism $[U/G] \to \operatorname{Spec} k$ is an isomorphism for the higher Chow groups with rational coefficients and hence the claim follows.

Now the right vertical arrow is an isomorphism since both groups are isomorphic to $\mathbb{Q}$. By induction we may assume that the first vertical arrow is also an isomorphism. $\square$

Recall that an algebraic zip datum $\mathcal{Z}$ is called orbitally finite if G has finitely many $E_{\mathcal{Z}}$-orbits [Pink et al. 2011, Definition 7.2].

Theorem 2.4.10. *Let $\mathcal{Z}$ be an orbitally finite connected algebraic zip datum and $[G/E_{\mathcal{Z}}]$ be the corresponding stack of G-zips. Then the following assertions hold.*

  (i) *$A^o_*[G/E_{\mathcal{Z}}]_{\mathbb{Q}} \to A_*[G/E_{\mathcal{Z}}]_{\mathbb{Q}}$ is an isomorphism.*

  (ii) *$A^o_*[G/E_{\mathcal{Z}}] = \bigoplus_Z \mathbb{Z}[\bar{Z}]$ where the sum goes over all orbits Z.*

In particular, the dimension of $A_*[G/E_{\mathcal{Z}}]_{\mathbb{Q}}$ as a $\mathbb{Q}$-vector space is equal to the number of orbits.

Proof. The assumption of the previous proposition on the stabilizer group schemes hold by [Pink et al. 2011, Theorem 8.1]. $\square$

Corollary 2.4.11. *Let $\mathcal{Z} = (G, P, Q, \varphi)$ be a connected algebraic zip datum and T be a split maximal torus of G in a Levi component L of P. If $\mathcal{Z}$ is orbitally finite the $\mathbb{Q}$-vectorspace $A^*([G/E_\mathcal{Z}])_\mathbb{Q}$ is finite dimensional of dimension $|W_G/W_L|$, where as usual $W_G = W(G, T)$ is the Weyl group of G and $W_L = W(L, T)$ is the Weyl group of L.*

Proof. By the above theorem $\dim_\mathbb{Q} A^*([G/E_\mathcal{Z}])_\mathbb{Q}$ equals the number of $E_\mathcal{Z}$-orbits in G. This number equals $|W_G/W_L|$ by [Pink et al. 2011, Theorem 7.5]. $\qquad\square$

In the case of F-zips the above results read as follows.

Corollary 2.4.12. *Let $\tau : \mathbb{Z} \to \mathbb{Z}_{\geq 0}$ be a function with finite support $i_1 \leq \cdots \leq i_r$ and $n_k = \tau(i_k)$. Let $h = \sum_i n_i$ be its height. Then:*

(i) $\qquad A^* F\text{-zip}^\tau = \mathbb{Z}[t_1, \ldots, t_h]^{S_{n_1} \times \cdots \times S_{n_r}} / ((p-1)c_1, \ldots, (p^h - 1)c_h),$

   with c_i the i-th elementary symmetric polynomial in the variables $t_1, \ldots, t_h$.

(ii) $\qquad \mathrm{Pic}(F\text{-zip}^\tau) = \mathbb{Z}^{r-1} \times \mathbb{Z}/(p-1)\mathbb{Z}.$

(iii) $\qquad \dim_\mathbb{Q} A^*(F\text{-zip}^\tau)_\mathbb{Q} = \dfrac{h!}{n_1! \times \cdots \times n_r!}.$

2.5. *The Chow ring of* BT_n. The goal of this section is to prove the following result.

Theorem 2.5.1. *The pullback $\phi_n^* : A^*(\mathcal{D}isp_n) \to A^*(\mathrm{BT}_n)$ is injective and an isomorphism after inverting p.*

We know that $\mathcal{D}isp_n = \coprod_{d \leq h} \mathcal{D}isp_n^{h,d}$ is a decomposition into open and closed substacks. The same holds for BT_n and the morphism ϕ_n maps $\mathrm{BT}_n^{h,d}$ to $\mathcal{D}isp_n^{h,d}$. It suffices to prove the theorem for the restriction of ϕ_n to $\mathrm{BT}_n^{h,d}$. The following proposition is the crucial point in the proof of Theorem 2.5.1.

Proposition 2.5.2. *Let L be a field extension of k and $\mathrm{Spec}\, L \to \mathcal{D}isp_n$ be a morphism. Then there is a finite field extension L' of L of p-power degree and an infinitesimal commutative group scheme A over L' such that the fiber $\phi_n^{-1}(\mathrm{Spec}\, L')$ is the classifying space of A.*

Proof. The diagonal $\Delta : \mathrm{BT}_n \to \mathrm{BT}_n \times_{\mathcal{D}isp_n} \mathrm{BT}_n$ is flat and surjective by [Lau 2013, Theorem 4.7]. This means that two Barsotti–Tate groups of level n having the same associated display become isomorphic when pulled back to a suitable fppf-covering. It follows that the fiber $(\mathrm{BT}_n)_L$ of a display P over some field L is a gerbe over L. If L is perfect there is a truncated Barsotti–Tate group G over L with $\phi_n(G) = P$, i.e., $(\mathrm{BT}_n)_L$ is a neutral gerbe. In this case $(\mathrm{BT}_n)_L = B\underline{\mathrm{Aut}}^o(G)$ where $\underline{\mathrm{Aut}}^o(G) = \mathrm{Ker}(\underline{\mathrm{Aut}}G \to \underline{\mathrm{Aut}}P)$ is commutative and infinitesimal again by [Lau 2013, Theorem 4.7]. If L is not perfect we may consider the perfect hull $L^{p^{-\infty}}$ in an algebraic closure of L. Then $L \subset L^{p^{-\infty}}$ is purely inseparable and

$(\mathrm{BT}_n)_L(L^{p^{-\infty}})$ is nonempty. Since $(\mathrm{BT}_n)_L(L^{p^{-\infty}}) = \varinjlim_{L'}(\mathrm{BT}_n)_L(L')$, where the limit goes over all finite subextensions $L \subset L' \subset L^{p^{-\infty}}$, we find some L' such that $(\mathrm{BT}_n)_{L'}$ has a section corresponding to a truncated Barsotti–Tate group G over L'. Thus $A = \underline{\mathrm{Aut}}^o(G)$ and L' have the desired properties. $\qquad\square$

Remark 2.5.3. Over the open and closed substack of BT_n consisting of level-n BT-groups of constant dimension d and codimension c the degree of $\mathrm{Aut}^o(G^{\mathrm{univ}})$ is p^{ncd}. See Remark 4.8 in [Lau 2013].

Note that $\mathcal{D}\mathrm{isp}_n^{h,d}$ and $\mathrm{BT}_n^{h,d}$ both admit admissible presentations in the sense of Definition 1.2.3. In the case of $\mathcal{D}\mathrm{isp}_n^{h,d}$ this follows from Theorem 2.1.3 and Lemma 1.2.2. To obtain the assertion for $\mathrm{BT}_n^{h,d}$ we use [Wedhorn 2001, Proposition 1.8] which yields a presentation $\mathrm{BT}_n^h = [Y_n^h / \mathrm{GL}_{p^{nh}}]$ with Y_n^h quasiaffine and of finite type over k. Now BT_n^h is smooth over $\mathrm{Spec}\, k$ [Lau 2013]. Hence Y_n^h is also smooth and in particular normal and equidimensional.

We now consider the flat pullback map

$$\phi_n^* : A_*(\mathcal{D}\mathrm{isp}_n^{h,d}, m) \to A_*(\mathrm{BT}_n^{h,d}, m)$$

from Lemma 1.2.6.

Proposition 2.5.4. $\phi_n^* : A_*(\mathcal{D}\mathrm{isp}_n^{h,d}, m) \to A_*(\mathrm{BT}_n^{h,d}, m)$ *is an isomorphism after inverting* p.

Proof. Let us write $\mathscr{X} = \mathrm{BT}_n^{h,d}$ and $\mathscr{Y} = \mathcal{D}\mathrm{isp}_n^{h,d}$. We fix some $i_o \in \mathbb{Z}$ and show that $\phi_n : A_{i_o}(\mathcal{D}\mathrm{isp}_n^{h,d}, m)_p \to A_{i_o}(\mathrm{BT}_n^{h,d}, m)_p$ is an isomorphism.

Consider an approximation of $\mathscr{Y}$ (see Convention 1.1.1) by a quasiprojective scheme $Y \to \mathscr{Y}$ so that $A_{i_o}(\mathscr{Y}, m) = A_{i_o}(Y, m)$ and similarly an approximation $X \to \mathscr{X}$ of $\mathscr{X}$. Let r denote the relative dimension of $X \to \mathscr{X}$. Let Z be the fiber product $X \times_{\mathscr{Y}} Y$. The morphism $Z \to Y$ is then smooth of relative dimension r and we need to see that the pullback $A_{i_o}(Y, m)_p \to A_{i_o+r}(Z, m)_p$ is an isomorphism. Note that Z is again quasiprojective since it is open in a vector bundle over the quasiprojective scheme X (see Remark 1.2.5). We have the cartesian diagram

$$
\begin{array}{ccccc}
Z_y & \longrightarrow & \mathscr{X}_{k(y)} & \longrightarrow & \mathrm{Spec}\, k(y) \\
\downarrow & & \downarrow & & \downarrow \\
Z & \longrightarrow & \mathscr{X}_Y & \longrightarrow & Y \\
\downarrow & & \downarrow & & \downarrow \\
X & \longrightarrow & \mathscr{X} & \longrightarrow & \mathscr{Y}
\end{array}
$$

By Lemma 1.3.2 it suffices to see that $A_i(\mathrm{Spec}\, k(y), m)_p \to A_{i+r}(Z_y, m)_p$ with $i = i_o - \dim \overline{\{y\}}$ is an isomorphism. According to the previous proposition there is a finite field extension K of $k(y)$ of p-power degree such that $\mathscr{X}_K = BA$ holds for an infinitesimal group scheme A over K.

Since Z_K is open in a vector bundle over $\mathscr{X}_K$ of rank r we have $Z_K = U/A$, where U is open in a representation V of A. Note that V is of dimension r. Hence by choosing codim X^c to be big enough, we may assume $A_i(\operatorname{Spec} K, m) \to A_{i+r}(U, m)$ is an isomorphism. Since A is of p-power degree it follows that the map $A_i(\operatorname{Spec} K, m)_p \to A_{i+r}(Z_K, m)_p$ is an isomorphism. Now since the field extension $K \supset k(y)$ is of p-power degree it follows from Lemma 1.3.1 that

$$A_i(\operatorname{Spec} k(y), m)_p \to A_{i+r}(Z_y, m)_p$$

is also an isomorphism. We are done. $\qquad\square$

Proof of Theorem 2.5.1. Since BT_n and $\mathcal{D}\mathrm{isp}_n$ are smooth the pullback

$$(\phi_n)_p^* : A^*(\mathcal{D}\mathrm{isp}_n)_p \to A^*(\mathrm{BT}_n)_p$$

is an isomorphism by Lemma 1.2.6 and the proposition above. We already know $A^*(\mathcal{D}\mathrm{isp}_n)$ is p-torsion free by Theorems 2.3.1 and 2.3.3. Thus ϕ_n^* is injective. $\quad\square$

Gathering the results of Section 2, we obtain the following theorem:

Theorem 2.5.5.   (i) *We have*

$$A^*(\mathrm{BT}_n^{h,d})_p = \mathbb{Z}[p^{-1}][t_1, \ldots, t_h]^{S_d \times S_{h-d}}/((p-1)c_1, \ldots, (p^h-1)c_h),$$

where c_i denotes the i-th elementary symmetric polynomial in the variables $t_1, \ldots, t_h$, and $t_1, \ldots, t_d$ and $t_{d+1}, \ldots, t_h$ are the Chern roots of $\mathcal{L}ie$ and ${}^t\mathcal{L}ie^\vee$, respectively.

(ii) *We have* $\dim_{\mathbb{Q}} A^*(\mathrm{BT}_n^{h,d})_{\mathbb{Q}} = \binom{h}{d}$ *and a basis is given by the cycles of the closures of the EO strata.*

(iii) $(\operatorname{Pic} \mathrm{BT}_n^{h,d})_p = \begin{cases} \mathbb{Z}[p^{-1}]/(p-1) & \text{if } d = 0, h, \\ \mathbb{Z}[p^{-1}] \times \mathbb{Z}[p^{-1}]/(p-1) & \text{otherwise,} \end{cases}$

where the generator for the free part is $\det(\mathcal{L}ie)$ and for the torsion part is $\det(\mathcal{L}ie \otimes {}^t\mathcal{L}ie^\vee)$.

Proof. By Theorem 2.5.1 we know $A^*(\mathcal{D}\mathrm{isp}_n^{h,d})_p \cong A^*(\mathrm{BT}_n^{h,d})_p$. Further, we have

$$A^*(\mathcal{D}\mathrm{isp}_n^{h,d}) \cong A^*(\mathcal{D}\mathrm{isp}_1^{h,d})$$

by Theorem 2.3.1 and $A^*(\mathcal{D}\mathrm{isp}_1^{h,d})$ was computed in Theorem 2.3.3. This proves part (i). By Lemmas 2.4.1 and 2.4.2 we know that $\mathcal{D}\mathrm{isp}_1^{h,d}$ is isomorphic to the stack $[\mathrm{GL}_h / E_{\mathcal{Z}}]$ corresponding to the Frobenius zip datum $\mathcal{Z} = (\mathrm{GL}_h, P, P^-, \sigma)$, where P is the standard parabolic of type (d, h), P^- is the opposite parabolic and σ is the Frobenius isogeny. Now the dimension of $A^*(\mathcal{D}\mathrm{isp}_1^{h,d})_{\mathbb{Q}}$ as a $\mathbb{Q}$-vectorspace follows from Corollary 2.4.12 and a basis is given by Theorem 2.4.10. This proves (ii). Finally (iii) follows from (i) together with the fact that $\operatorname{Pic} \mathrm{BT}_n^{h,d} = A^1(\mathrm{BT}_n^{h,d})$. $\quad\square$

Acknowledgements

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