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For a prime number $q \geq 5$ and a positive integer N prime to q, Ribet proved the action of the Hecke algebra on the component group of the Jacobian variety of the modular curve of level Nq at q is "Eisenstein", which means the Hecke operator T_ℓ acts by $\ell + 1$ when ℓ is a prime number not dividing the level. We completely compute the action of the Hecke algebra on this component group by a careful study of supersingular points with extra automorphisms.

1. Introduction

Let $q \geq 5$ be a prime number, and let N be a positive integer. Let $X_0(Nq)$ denote the modular curve over $\mathbb{Q}$ and $J_0(Nq)$ its Jacobian variety. For any integer n, there is the Hecke operator T_n acting on $J_0(Nq)$. Let $\Phi_q(Nq)$ denote the component group of the special fiber $\mathcal{J}$ of the Néron model of $J_0(Nq)$ at q. According to the theorems of Ribet [1988; 1990] (when q does not divide N) and Edixhoven [1991] (in general), the action of the Hecke algebra on $\Phi_q(Nq)$ is "Eisenstein." Here by "Eisenstein" we mean the Hecke operator T_ℓ acts on $\Phi_q(Nq)$ by $\ell + 1$ when a prime number ℓ does not divide Nq.[1] In this article, we compute the action of the Hecke operators T_ℓ on the component group $\Phi_q(Nq)$ when ℓ divides Nq and q does not divide N.

Here is an exotic example[2] which leads us to this study: Let $N = \prod_{i=1}^{\nu} p_i$ be the product of distinct prime numbers with $\nu \geq 1$, and let $q \equiv 2$ or $5 \pmod 9$ be an odd prime number. Assume that $p_i \equiv 4$ or $7 \pmod 9$ for all $1 \leq i \leq \nu$. Let $\mathbb{T}(Nq)$ and $\mathbb{T}(N)$ denote the $\mathbb{Z}$-subalgebras of $\mathrm{End}(J_0(Nq))$ and $\mathrm{End}(J_0(N))$, respectively, generated by all the Hecke operators T_n for $n \geq 1$. Let

$$\mathfrak{m} := \bigl(3,\, T_{p_i} - 1,\, T_q + 1,\, T_\ell - \ell - 1 : \text{ for all } 1 \leq i \leq \nu, \text{ and for primes } \ell \nmid Nq\bigr) \subset \mathbb{T}(Nq)$$

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[1] On the other hand, Ribet and Edixhoven did not proceed to compute the action of the Hecke operator T_p on $\Phi_q(Nq)$ for a prime divisor p of the level Nq because their results were enough for their applications.

[2] This phenomenon cannot occur when the residual characteristic is greater than 3.

and

$$\mathfrak{n} := \big(3,\, T_{p_i} - 1,\, T_\ell - \ell - 1 : \text{ for all } 1 \le i \le \nu, \text{ and for primes } \ell \nmid N\big) \subset \mathbb{T}(N)$$

be Eisenstein ideals. By [Yoo 2016, Theorem 1.4], $\mathfrak{m}$ is maximal. Furthermore, $\mathfrak{n}$ is maximal if and only if $\nu \ge 2$.

The dimension of $J_0(N)[\mathfrak{n}]$ is equal to ν if $\mathfrak{n}$ is maximal, i.e., $\nu \ge 2$. (Here $J_0(N)[\mathfrak{n}] := \{x \in J_0(N)(\overline{\mathbb{Q}}) : Tx = 0 \text{ for all } T \in \mathfrak{n}\}$.) It is an extension of $\mu_3^{\oplus \nu-1}$ by $\mathbb{Z}/3\mathbb{Z}$, and it does not contain a submodule isomorphic to μ_3. On the other hand, the dimension of $J_0(Nq)[\mathfrak{m}]$ is either 2ν or $2\nu+1$. Furthermore $J_0(Nq)[\mathfrak{m}]$ contains a submodule $\mathcal{N}$ isomorphic to $J_0(N)[\mathfrak{n}]$, and it also contains $\mu_3^{\oplus\nu}$ (which is contributed from the Shimura subgroup). As $\mathcal{N}$ is unramified at q, by [Serre and Tate 1968], $\mathcal{N}$ maps injectively into $\mathcal{J}[\mathfrak{m}]$ and it turns out that its image is isomorphic to $\mathcal{J}^0[\mathfrak{m}]$, where $\mathcal{J}^0$ is the identity component of $\mathcal{J}$. (Note that $\Phi_q(Nq)$ is the quotient of $\mathcal{J}$ by $\mathcal{J}^0$.) Since $\mu_3^{\oplus\nu}$ is also unramified at q, it maps into $\mathcal{J}[\mathfrak{m}]$ and therefore its image maps injectively to $\Phi_q(Nq)[\mathfrak{m}]$. (This statement is also true when $\nu = 1$.) The structure of the component group $\Phi_q(Nq)$ is known by the work of Mazur and Rapoport [1977]:[3]

$$\Phi_q(Nq) = \Phi \oplus (\mathbb{Z}/3\mathbb{Z})^{2^\nu - 1},$$

where Φ is cyclic and generated by the image of the cuspidal divisor $(0)-(\infty)$. The action of the Hecke operators on Φ is well known (e.g., [Yoo 2014, Appendix A1]), and so $\Phi[\mathfrak{m}] = 0$. Therefore $(\mathbb{Z}/3\mathbb{Z})^{2^\nu-1}[\mathfrak{m}] \neq 0$ and its dimension is at least ν. Indeed it is equal to $2^{\nu-1}$, which can easily be computed by the theorems below.

Now, we introduce our results.

Theorem 1.1. *For a prime divisor p of N, the Hecke operator T_p acts on $\Phi_q(Nq)$ by p.*

The key idea of the proof is that the two degeneracy maps coincide on the component group (see [Ribet 1988; Edixhoven 1991, §4.2, Lemme 2]).

Now, the missing action is that of the Hecke operator T_q on $\Phi_q(Nq)$. Note that T_q acts on $\Phi_q(Nq)$ by an involution because the action of the Hecke algebra on $\Phi_q(Nq)$ is "q-new." To describe its action more precisely, we define some notation: for $N = \prod_{p \mid N} p^{n_p}$ being the prime factorization of N (i.e., $n_p > 0$), let $\nu := \#\{p : p \neq 2, 3\}$ and let

$$u := \begin{cases} 0 & \text{if } q \equiv 1 \pmod 4 \text{ or } 4 \mid N \text{ or if there exists } p \equiv -1 \pmod 4, \\ 1 & \text{otherwise,} \end{cases}$$

$$v := \begin{cases} 0 & \text{if } q \equiv 1 \pmod 3 \text{ or } 9 \mid N \text{ or if there exists } p \equiv -1 \pmod 3, \\ 1 & \text{otherwise.} \end{cases}$$

[3] There are some minor errors in the paper, which are corrected by Edixhoven [1991, §4.4.1]

Suppose that $(u, v) = (0, 0)$ or $\nu = 0$. Then $\Phi_q(Nq) = \Phi$ and T_q acts on Φ by 1, where Φ is the cyclic subgroup generated by the image of the cuspidal divisor $(0) - (\infty)$ (Proposition 4.1). If $\nu \geq 1$, $\Phi_q(Nq)$ becomes isomorphic to

$$\Phi' \oplus \boldsymbol{A} \oplus \boldsymbol{B},$$

where $\boldsymbol{A} \simeq (\mathbb{Z}/2\mathbb{Z})^{\oplus u(2^\nu-2)}$, $\boldsymbol{B} \simeq (\mathbb{Z}/3\mathbb{Z})^{\oplus v(2^\nu-1)}$ and Φ' is a cyclic group containing Φ and $\Phi'/\Phi \simeq (\mathbb{Z}/2^u\mathbb{Z})$.[4]

Theorem 1.2. *Assume that* $(u, v) \neq (0, 0)$ *and* $\nu \geq 1$.

(1) *Suppose that* $v = 1$. *Then there are distinct subgroups* $B_i \simeq \mathbb{Z}/3\mathbb{Z}$ *of* $\boldsymbol{B}$ *so that* $\boldsymbol{B} = \bigoplus B_i$. *For any* $1 \leq i \leq (2^\nu - 1)$, T_q *acts on* B_i *by* $(-1)^i$.

(2) *Suppose that* $u = 1$. *Then there are distinct subgroups* $A_i \simeq \mathbb{Z}/2\mathbb{Z}$ *of* $\boldsymbol{A}$ *so that* $\boldsymbol{A} = \oplus A_i$. *For any* $1 \leq k \leq (2^{\nu-1} - 2)$, T_q *acts on* $A_{2k-1} \oplus A_{2k}$ *by the matrix* $\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$.[5] *In other words, if* $A_{2k-1} = \langle \boldsymbol{u}_{2k-1} \rangle$ *and* $A_{2k} = \langle \boldsymbol{u}_{2k} \rangle$, *then*

$$T_q(\boldsymbol{u}_{2k-1}) = \boldsymbol{u}_{2k-1} + \boldsymbol{u}_{2k} \quad \textit{and} \quad T_q(\boldsymbol{u}_{2k}) = \boldsymbol{u}_{2k}.$$

For a complete description of the action of T_q on each subgroup, see Section 4.

2. Supersingular points of $X_0(N)$

From now on, we always assume that $q \geq 5$ is a prime number and N is a positive integer which is prime to q. Let p denote a prime divisor of N. Let $\boldsymbol{F}$ be an algebraically closed field of characteristic q.

Let $\Sigma(N)$ denote the set of supersingular points of $X_0(N)(\boldsymbol{F})$. Since we assume that $q \geq 5$, the group of automorphisms of supersingular points is cyclic of order 2, 4 or 6. Let

$$\Sigma_n(N) := \{s \in \Sigma(N) : \#\mathrm{Aut}(s) = n\} \quad \text{and} \quad s_n(N) := \#\Sigma_n(N).$$

Note that $s_4(N) = u \cdot 2^\nu$ and $s_6(N) = v \cdot 2^\nu$ (see [Edixhoven 1991, §4.2, Lemme 1]), where u, v and ν are as in Section 1. Moreover $s_2(N)$ can be computed using Eichler's mass formula [Katz and Mazur 1985, Theorem 12.4.5, Corollary 12.4.6]:

$$\frac{s_2(N)}{2} + \frac{s_4(N)}{4} + \frac{s_6(N)}{6} = \frac{(q-1)Q}{24}, \tag{2-1}$$

where $Q := N \prod_{p \mid N}(1 + p^{-1})$ is the degree of the degeneracy map $X_0(N) \to X_0(1)$.

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[4] The structure of $\Phi_q(Nq)$ is already known by [Mazur and Rapoport 1977] when N is square-free and prime to 6, and by [Edixhoven 1991, §4.4.1] in general.

[5] This reminds us of the result by Mazur [1977]: when N is a prime number, the kernel of the Eisenstein prime of $J_0(N)$ containing a prime number ℓ is completely reducible when ℓ is odd, and is indecomposable when $\ell = 2$.

In the remainder of this section, we study $\Sigma_4(N)$ and $\Sigma_6(N)$ in detail. (See also [Ribet 1988, §2; 1995, §4; Edixhoven 1991, §4.2].) In the section below, we always assume that $\nu \geq 1$, i.e., there is a prime divisor $p \geq 5$ of N. (If $\nu = 0$ then $s_{2e}(N) \leq 1$ for $e = 2$ or 3, and the description is very simple.)

Let $\mathcal{E}$ be a supersingular elliptic curve with $\mathrm{Aut}(\mathcal{E}) = \langle \sigma \rangle$, and let C be a cyclic subgroup of $\mathcal{E}$ of order N. Assume that $q \equiv -1 \pmod 4$ (resp. $q \equiv -1 \pmod 3$) if $\sigma = \sigma_4$ (resp. $\sigma = \sigma_6$), where σ_k is a primitive k-th root of unity.

Proposition 2.1. *Let $N = p^n$ for some $n \geq 1$ with $p \geq 5$. Suppose $\mathrm{Aut}(\mathcal{E}, C) = \langle \sigma \rangle$. Then, there exists another cyclic subgroup D of order N such that $\mathcal{E}[N] \simeq C \oplus D$. Moreover, $\mathrm{Aut}(\mathcal{E}, D) = \langle \sigma \rangle$ and $(\mathcal{E}, C)$ is not isomorphic to $(\mathcal{E}, D)$.*

Proof. Here, we closely follow the argument in the proof of Proposition 1 in [Ribet 1988, §2].

Let R be the subring $\mathbb{Z}[\sigma]$ of $\mathrm{End}(\mathcal{E}, C)$. Since $\mathrm{Aut}(\mathcal{E}, C) = \langle \sigma \rangle$, $p \equiv 1 \pmod 4$ (resp. $p \equiv 1 \pmod 3$) if $\sigma = \sigma_4$ (resp. $\sigma = \sigma_6$). Therefore p splits completely in R. Note that $R = \mathbb{Z}[\sigma]$ is a principal ideal domain and therefore

$$R/pR \simeq R/\gamma R \oplus R/\delta R \simeq \delta R/pR \oplus \gamma R/pR$$

with $p = \gamma\delta$. Moreover,

$$R/NR = R/p^n R \simeq R/\gamma^n R \oplus R/\delta^n R \simeq \delta^n R/NR \oplus \gamma^n R/NR.$$

Note that $\mathcal{E}[N]$ is a free module of rank 1 over R/NR by the action of R on $\mathcal{E}$. We may identify C with the quotient I/NR for some ideal I of R containing N if we fix an R-isomorphism between $\mathcal{E}[N]$ and R/NR. Thus, $I = \delta^n R$ or $\gamma^n R$. Suppose that $I = \delta^n R$. Then, by the fixed isomorphism, $C = \mathcal{E}[\gamma^n]$. Let $D := \mathcal{E}[\delta^n]$ so that its corresponding ideal is $\gamma^n R$. Then, $\mathcal{E}[N] \simeq C \oplus D$. Moreover since $\gamma^n R$ is also an ideal of R, D is also stable under the action of σ. In other words, $\mathrm{Aut}(\mathcal{E}, D) = \langle \sigma \rangle$. Also, $(\mathcal{E}, C)$ cannot be isomorphic to $(\mathcal{E}, D)$ since $\mathrm{Aut}(\mathcal{E}) = \langle \sigma \rangle$ and $\sigma(C) = C$. □

From now on, we use the same notation as in the proof of Proposition 2.1.

Definition 2.2. By the above formulas, for every $n \geq 1$ and $p \equiv 1 \pmod 4$ (resp. $p \equiv 1 \pmod 3$), there are precisely two cyclic subgroups C, D of $\mathcal{E}$ of order p^n such that $\mathrm{Aut}(\mathcal{E}, C) = \mathrm{Aut}(\mathcal{E}, D) = \langle \sigma \rangle$ (and $\mathcal{E}[p^n] \simeq C \oplus D$) if $\sigma = \sigma_4$ (resp. if $\sigma = \sigma_6$). Thus, for each $n \geq 1$ we define $\mathcal{C}_{p^n}$ and $\mathcal{D}_{p^n}$ by

$$\mathcal{C}_{p^n} := \mathcal{E}[\gamma^n] \quad \text{and} \quad \mathcal{D}_{p^n} := \mathcal{E}[\delta^n].$$

Proposition 2.3. *For each $n \geq 1$, $\mathcal{C}_{p^{n+1}}[p^n] = \mathcal{C}_{p^n}$ and $\mathcal{D}_{p^{n+1}}[p^n] = \mathcal{D}_{p^n}$.*

Proof. By the fixed R-isomorphism ι between $\mathcal{E}[p^{n+1}]$ and $R/p^{n+1}R$, we identify $\mathcal{C}_{p^{n+1}}$ with $I/p^{n+1}R$, where $I = \delta^{n+1}R$. As I is an ideal of R, $\gamma I = p(\delta^n R) \subset I$

and $I/\gamma I \simeq R/\gamma R \simeq \mathbb{Z}/p\mathbb{Z}$. Therefore

$$\mathcal{C}_{p^{n+1}}[p^n] \xrightarrow{\iota} (I/p^{n+1}R)[p^n] = \gamma I/p^{n+1}R \xrightarrow[\times 1/p]{\sim} (\delta^n R)/p^n R,$$

which corresponds to $\mathcal{C}_{p^n}$. Similarly, we prove that $\mathcal{D}_{p^{n+1}}[p^n] = \mathcal{D}_{p^n}$, and the proposition follows. □

Let $N = Mp^n$ with $(6M, p) = 1$ and $n \geq 1$. Let L be a cyclic subgroup of $\mathcal{E}$ of order M.

Proposition 2.4. *Suppose that* $\mathrm{Aut}(\mathcal{E}, \mathcal{C}_{p^{n+1}}, L) = \langle\sigma\rangle$. *Then, there is an isomorphism between* $(\mathcal{E}/\mathcal{C}_p, \mathcal{C}_{p^{n+1}}/\mathcal{C}_p, (L \oplus \mathcal{C}_p)/\mathcal{C}_p))$ *and* $(\mathcal{E}, \mathcal{C}_{p^n}, L)$.

Proof. We mostly follow the idea of the proof of Proposition 2 in [Ribet 1988, §2].

The endomorphism γ sends $\mathcal{E}[\gamma^{n+1}] = \mathcal{C}_{p^{n+1}}$ to $\mathcal{E}[\gamma^n] = \mathcal{C}_{p^n}$, and L to itself (because $L \cap \mathcal{E}[p] = 0$). Now we denote by $\bar{\gamma}$ the map $\mathcal{E}/\mathcal{C}_p \to \mathcal{E}$ induced by γ. Note that $\bar{\gamma}$ is an isomorphism because $\mathcal{C}_p$ is $\mathcal{E}[\gamma]$, the kernel of γ. By the above consideration, this isomorphism $\bar{\gamma}$ sends $(\mathcal{C}_{p^{n+1}}/\mathcal{C}_p, (L \oplus \mathcal{C}_p)/\mathcal{C}_p)$ to $(\mathcal{C}_p, L)$ because $\mathcal{C}_{p^{n+1}}/\mathcal{C}_p$ and $(L \oplus \mathcal{C}_p)/\mathcal{C}_p$, respectively, are the images of $\mathcal{C}_{p^{n+1}}$ and L by the quotient map $\mathcal{E} \to \mathcal{E}/\mathcal{C}[p]$. Therefore $\bar{\gamma}$ gives rise to the desired isomorphism between triples. □

Corollary 2.5. *The map* $(\mathcal{E}, C, L) \to (\mathcal{E}, C[p^n], L)$ *induces a bijection between* $\Sigma_{2e}(Np)$ *and* $\Sigma_{2e}(N)$, *where* $\sigma = \sigma_{2e}$. *Moreover if* $(\mathcal{E}, C, L) \in \Sigma_{2e}(Np)$, *we have*

$$(\mathcal{E}, C[p^n], L) \simeq (\mathcal{E}/C[p], C/C[p], (L \oplus C[p])/C[p]).$$

The corollary tells us that two degeneracy maps α_p and β_p in Section 3 coincide on $\Sigma_{2e}(Np)$, which is a generalization of [Edixhoven 1991, §4.2, Lemme 2].

Proposition 2.6. *Suppose that* $\mathrm{Aut}(\mathcal{E}, \mathcal{C}_{p^n}, L) = \langle\sigma\rangle$. *Then,* $\mathrm{Frob}(\mathcal{E}) = \mathcal{E}$ *and* $\mathrm{Frob}(\mathcal{C}_{p^n}) = \mathcal{D}_{p^n}$, *where* Frob *is the Frobenius morphism in characteristic* q. *Furthermore,* $\mathrm{Frob}^2(\mathcal{E}, \mathcal{C}_{p^n}, L) = (\mathcal{E}, \mathcal{C}_{p^n}, L)$.

Proof. Since $\mathcal{E}$ is isomorphic to the reduction of the elliptic curve with j-invariant 1728 (resp. 0) if $\sigma = \sigma_4$ (resp. $\sigma = \sigma_6$), the Frobenius morphism is an endomorphism of $\mathcal{E}$ (see [Silverman 2009, Chapter V, Examples 4.4 and 4.5]). Moreover, the Frobenius morphism and σ generate $\mathrm{End}(\mathcal{E})$, which is a quaternion algebra. (Note that the degree of the Frobenius morphism is q.) Since $\mathrm{End}(\mathcal{E})$ is a quaternion algebra, we have

$$\sigma \circ \mathrm{Frob} = \mathrm{Frob} \circ \bar{\sigma} = \mathrm{Frob} \circ \sigma^{-1},$$

where $\bar{\sigma}$ denotes the complex conjugation in $R = \mathbb{Z}[\sigma]$. Analogously, we have

$$\gamma \circ \mathrm{Frob} = \mathrm{Frob} \circ \bar{\gamma} = \mathrm{Frob} \circ \delta.$$

Since $\sigma(\mathrm{Frob}(\mathcal{C}_{p^n})) = \mathrm{Frob}(\sigma^{-1}(\mathcal{C}_{p^n})) = \mathrm{Frob}(\mathcal{C}_{p^n})$, $\mathrm{Frob}(\mathcal{C}_{p^n})$ is also stable under the action of σ. Moreover $\mathcal{C}_{p^n}$ does not intersect with the kernel of Frob.

Thus, $\mathrm{Frob}(\mathcal{C}_{p^n})$ is either $\mathcal{C}_{p^n}$ or $\mathcal{D}_{p^n}$. As an endomorphism of $\mathcal{E}$, γ sends $\mathcal{C}_{p^n}$ (resp. $\mathcal{D}_{p^n}$) to $\mathcal{C}_{p^{n-1}}$ (resp. $\mathcal{D}_{p^n}$). Similarly, δ maps $\mathcal{C}_{p^n}$ (resp. $\mathcal{D}_{p^n}$) to $\mathcal{C}_{p^n}$ (resp. $\mathcal{D}_{p^{n-1}}$). Therefore if $\mathrm{Frob}(\mathcal{C}_{p^n}) = \mathcal{C}_{p^n}$, then

$$\gamma \circ \mathrm{Frob}(\mathcal{C}_{p^n}) = \gamma(\mathcal{C}_{p^n}) = \mathcal{C}_{p^{n-1}} \quad \text{and} \quad \mathrm{Frob} \circ \delta(\mathcal{C}_{p^n}) = \mathrm{Frob}(\mathcal{C}_{p^n}) = \mathcal{C}_{p^n},$$

which is a contradiction. Thus, we get $\mathrm{Frob}(\mathcal{C}_{p^n}) = \mathcal{D}_{p^n}$.

Since every supersingular point can be defined over $\mathbb{F}_{q^2}$, the quadratic extension of $\mathbb{F}_q$, Frob^2 acts trivially on $\Sigma(N)$ (see [Ribet 1990, Remark 3.5.b]), which proves the last claim. □

Remark 2.7. By taking $H = (\mathbb{Z}/N\mathbb{Z})^*$ in Lemma 1 of [Ribet 1995], we can obtain a similar result if we show that the Atkin–Lehner style involution in [Ribet 1995, §4] is equal to the Frobenius morphism.

3. The action of T_p on the component group

Before discussing the action of the Hecke operators on the component group, we study it on the group of divisors supported on supersingular points, which we denote by $\mathrm{Div}(\Sigma(N))$.

Let $N = Mp^n$ with $(M, p) = 1$ and $n \geq 1$, and assume that $(N, q) = 1$. Let $\alpha_p, \ \beta_p : X_0(Npq) \rightrightarrows X_0(Nq)$ denote two degeneracy maps of degree p, defined by

$$\alpha_p(E, C, L) := (E, C[p^n], L)$$

and

$$\beta_p(E, C, L) := (E/C[p], C/C[p], (L + C[p])/C[p]),$$

where C (resp. L) denotes a cyclic subgroup of order p^{n+1} (resp. Mq) in an elliptic curve E (see [Mazur and Ribet 1991, §13]). Let T_p and ξ_p be two Hecke correspondences defined by the following diagram:

$$\begin{tikzcd}
& X_0(Npq) \arrow[dl, "\alpha_p"'] \arrow[dr, "\beta_p"] & \\
X_0(Nq) \arrow[rr, dashed, shift left, "\xi_p"] & & X_0(Nq) \arrow[ll, dashed, shift left, "T_p"]
\end{tikzcd}$$

By pullback, the Hecke correspondence T_p (resp. ξ_p) induces the Hecke operator $T_p := \beta_{p,*} \circ \alpha_p^*$ (resp. $\xi_p := \alpha_{p,*} \circ \beta_p^*$) on $J_0(Nq)$.

The same description of the Hecke operator T_p on $\mathrm{Div}(\Sigma(N))$ as above works. In other words, we have two degeneracy maps[6] $\alpha_p, \beta_p : \Sigma(Np) \rightrightarrows \Sigma(N)$ of degree p, defined by

$$\alpha_p(E, C, L) := (E, C[p^n], L)$$

[6]Every elliptic curve isogenous to a supersingular one is also supersingular

and

$$\beta_p(E, C, L) := (E/C[p], C/C[p], (L + C[p])/C[p]),$$

where C (resp. L) denotes a cyclic subgroup of order p^{n+1} (resp. M) in a supersingular elliptic curve E over $\boldsymbol{F}$. These maps induce the maps

$$\mathrm{Div}(\Sigma(N)) \underset{\beta_p^*}{\overset{\alpha_p^*}{\rightrightarrows}} \mathrm{Div}(\Sigma(Np)) \underset{\beta_{p,*}}{\overset{\alpha_{p,*}}{\rightrightarrows}} \mathrm{Div}(\Sigma(N))$$

on their divisor groups, and the Hecke operator T_p (resp. ξ_p) can be defined by $\beta_{p,*} \circ \alpha_p^*$ (resp. $\alpha_{p,*} \circ \beta_p^*$). (For the details when $n = 0$, see [Ribet 1990, §3; 1991, pp. 18–22; Edixhoven 1991, §4.1; Emerton 2002, §7]. By the same method, we get the above description without further difficulties.)

Now, let $\Phi_q(Nq)$ denote the component group of the special fiber $\mathcal{J}$ of the Néron model of $J_0(Nq)$ at q. To compute the action of T_p on it, we closely follow the method of Ribet (see [Ribet 1988; 1990, §2, §3; Edixhoven 1991, §1]). Since N is not divisible by q, the identity component $\mathcal{J}^0$ of $\mathcal{J}$ is a semiabelian variety by Deligne and Rapoport [1973] and Raynaud [1970]. Moreover, $\mathcal{J}^0$ is an extension of $J_0(N)_{\boldsymbol{F}} \times J_0(N)_{\boldsymbol{F}}$ by $\mathcal{T}$, the torus of $\mathcal{J}^0$. Let $\mathcal{X}$ be the character group of the torus $\mathcal{T}$. By Grothendieck, there is a (Hecke-equivariant) monodromy exact sequence [SGA 7_{I} 1972] (see also [Ribet 1990, §2, §3; Raynaud 1991; Illusie 2015, §4]),

$$0 \longrightarrow \mathcal{X} \xrightarrow{\ \iota\ } \mathrm{Hom}(\mathcal{X}^t, \mathbb{Z}) \longrightarrow \Phi_q(Nq) \longrightarrow 0.$$

Here $\mathcal{X}^t$ denotes the character group corresponding to the dual abelian variety of $J_0(Nq)$, which is equal to $J_0(Nq)$. Namely, $\mathcal{X}^t = \mathcal{X}$ as sets, but the action of the Hecke operator T_ℓ on $\mathcal{X}^t$ is equal to the action of its dual ξ_ℓ on $\mathcal{X}$ (see [Ribet 1988; 1990, §3; Emerton 2002, §7]). Note that $\mathcal{X}$ is the group of degree 0 elements in $\mathbb{Z}^{\Sigma(N)}$. For $s, t \in \Sigma(N)$, let $e(s) := \frac{1}{2}\#\mathrm{Aut}(s)$ and

$$\phi_s(t) := \begin{cases} e(s) & \text{if } s = t, \\ 0 & \text{otherwise,} \end{cases}$$

and extends via linearity, i.e., $\phi_s\big(\sum a_i t_i\big) = \sum a_i \phi_s(t_i)$. Then, $\iota(s - t) = \phi_s - \phi_t$. Note also that $\mathrm{Hom}(\mathbb{Z}^{\Sigma(N)}, \mathbb{Z})$ is generated by $\psi_s := 1/e(s)\phi_s$, and $\mathrm{Hom}(\mathcal{X}^t, \mathbb{Z})$ is its quotient by the relation

$$\sum_{s \in \Sigma(N)} \psi_s = \sum_{s \in \Sigma(N)} \frac{1}{e(s)} \phi_s = 0.$$

(This is the minimal relation to make $\sum a_w \psi_w$ vanish for all the divisors of the form $s - t$, which are the generators of $\mathcal{X}$.) For more details, see [Ribet 1990, §2, §3, Raynaud 1991].

In conclusion, the component group $\Phi_q(Nq)$ is isomorphic to

$$\mathrm{Hom}(\mathbb{Z}^{\Sigma(N)}, \mathbb{Z})/R,$$

where R is the set of relations

$$R = \Big\{ e(s)\psi_s = e(t)\psi_t \quad \text{for any } s, t \in \Sigma(N), \sum_{t \in \Sigma(N)} \psi_t = 0 \Big\}. \tag{3-1}$$

Let Ψ_s denote the image of ψ_s by the natural projection $\mathrm{Hom}(\mathbb{Z}^{\Sigma(N)}, \mathbb{Z}) \to \Phi_q(Nq)$. The Hecke operator T_p acts on $\mathrm{Hom}(\mathbb{Z}^{\Sigma(N)}, \mathbb{Z})$ via the action of ξ_p on $\mathrm{Div}(\Sigma(N))$, i.e.,

$$T_p(\psi_s)(t) := \psi_s(\xi_p(t)) = \psi_s(\alpha_{p,*} \circ \beta_p^*(t)).$$

For $s \in \Sigma(N)$, we temporarily denote $\alpha_p^*(s) = \sum_{i=1}^p A^i(s)$ and $\beta_p^*(s) = \sum_{i=1}^p B^i(s)$ (allowing repetition). We note that if $e(s) = 1$ then there is no repetition, i.e., $A^i(s) \not\simeq A^j(s)$ and $B^i(s) \not\simeq B^j(s)$ if $i \neq j$. If $e(s) = e > 1$, then after renumbering the index properly we have

$$e(A^i(s)) = 1 \quad \text{for } \ 1 \le i \le p-1 \ \text{ and } \ e(A^p(s)) = e.$$

Moreover, we have

$$A^{e(k-1)+1}(s) \simeq \cdots \simeq A^{ek}(s) \quad \text{for } \ 1 \le k \le \frac{p-1}{e},$$

and

$$A^i(s) \not\simeq A^j(s) \quad \text{if } \ \Big[\frac{i-1}{e}\Big] \neq \Big[\frac{j-1}{e}\Big],$$

where $[x]$ denotes the largest integer less than or equal to x. This can be seen as follows: Let $\sigma = \sigma_{2e}$, and let s represent a pair $(\mathcal{E}, C)$, where C is a cyclic subgroup of E of order N. Since $e(s) = e$, $\sigma(C) = C$. Suppose that $s' \in \Sigma(Np)$ with $\alpha_{p,*}(s') = s$. Then s' represents a pair $(\mathcal{E}, D)$ with $D[N] = C$. If $\sigma(D) = D$, then $\mathrm{Aut}([(\mathcal{E}, D)]) = \langle \sigma \rangle$ and $(\mathcal{E}, D) \not\simeq (\mathcal{E}, D')$ if $D \neq D'$. (Note that there is a unique such D.) On the other hand, if $\sigma(D) \neq D$ then

$$(\mathcal{E}, D) \simeq (\mathcal{E}, \sigma(D)) \simeq \cdots \simeq (\mathcal{E}, \sigma^{e-1}(D)) \simeq (\mathcal{E}, \sigma^e(D)) = (\mathcal{E}, D)$$

and $\mathrm{Aut}([(\mathcal{E}, D)]) = \{\pm 1\}$. Thus, we can rearrange $A^i(s)$ as above. (Note that this can only be possible when $p \equiv 1 \pmod{2e}$, which is true because $e(s) = e$.)

Now, we claim that $\phi_s(\alpha_{p,*}(t)) = \phi_t(\alpha_p^*(s))$. Indeed, $\phi_s(\alpha_{p,*}(t))$ is nonzero if and only if $t \in \{A^1(s), \ldots, A^p(s)\}$. So, it suffices to show this equality when $t \in \{A^1(s), \ldots, A^p(s)\}$. If $e(s) = 1$, then there is no repetition and the claim follows clearly (both are 1). Now, let $e(s) = e > 1$. If $e(t) = 1$, then $t = A^i(s)$ for some $1 \le i \le p-1$. Since the number of repetitions of $t = A^i(s)$ in $\{A^1(s), \ldots, A^p(s)\}$ is e, the above equality holds. If $e(t)=e$, then $t=A^p(s)$ and $\phi_s(\alpha_{p,*}(t))=e=\phi_t(\alpha_p^*(s))$, as claimed. Analogously, we have

$$\phi_t(\beta_{p,*}(s))=\phi_s(\beta_p^*(t)).$$

More generally, we get

$$\begin{aligned}\phi_s(\alpha_{p,*}\circ\beta_p^*(t)) &= \sum_{i=1}^{p}\phi_s(\alpha_{p,*}(B^i(t))) = \sum_{i=1}^{p}\sum_{j=1}^{p}\phi_{B^i(t)}(A^j(s))\\ &= \sum_{j=1}^{p}\sum_{i=1}^{p}\phi_{A^j(s)}(B^i(t)) = \sum_{j=1}^{p}\phi_{A^j(s)}(\beta_p^*(t))\\ &= \sum_{j=1}^{p}\phi_t(\beta_{p,*}(A^j(s))) = \phi_t(\beta_{p,*}\circ\alpha_p^*(s)) = \phi_t(T_p(s)).\end{aligned}$$

If we set $T_p(s)=\sum s_j$, then $\phi_t(T_p(s))=\sum\phi_{s_i}(t)=\sum e(s_i)\psi_{s_i}(t)$ and hence for any $t\in\Sigma(N)$,

$$e(s)T_p(\psi_s)(t)=\phi_s(\alpha_{p,*}\circ\beta_p^*(t))=\phi_t(T_p(s))=e(s_i)\psi_{s_i}(t).$$

In other words, we get

$$T_p(\Psi_s)=\frac{1}{e(s)}\sum e(s_i)\Psi_{s_i}. \tag{3-2}$$

We can also define the action of T_p on the component group via functorialities. Namely, let

$$\Phi_q(Nq) \underset{\beta_p^*}{\overset{\alpha_p^*}{\rightrightarrows}} \Phi_q(Npq) \underset{\beta_{p,*}}{\overset{\alpha_{p,*}}{\rightrightarrows}} \Phi_q(Nq)$$

denote the maps functorially induced from the degeneracy maps.[7] Then, as before, $T_p:=\beta_{p,*}\circ\alpha_p^*$. Note that since the degrees of α_p and β_p are p, we have $\alpha_{p,*}\circ\alpha_p^*=\beta_{p,*}\circ\beta_p^*=p$.

Lemma 3.1. *The operator $\alpha_{p,*}$ is equal to $\beta_{p,*}$ on $\Phi_q(Npq)$.*

Proof. For $s\in\Sigma_{2e}(Npq)$ with $e=2$ or 3, $\alpha_p(s)=\beta_p(s)$ by Corollary 2.5, and hence $\alpha_{p,*}(\Psi_s)=\beta_{p,*}(\Psi_s)$. For $s\in\Sigma_2(Npq)$, let $\alpha_p(s)=t$ and $\beta_p(s)=w$. Then, $\alpha_{p,*}(\Psi_s)=e(t)\Psi_t=e(w)\Psi_w=\beta_{p,*}(\Psi_s)$. In other words, for any $s\in\Sigma(Npq)$, $\alpha_{p,*}(\Psi_s)=\beta_{p,*}(\Psi_s)$. Since Ψ_s's generate $\Phi_q(Npq)$, the result follows. ☐

In fact, Theorem 1.1 is an easy corollary of the above lemma.

[7]If $\alpha_p^*(s)=\sum t_j$ then $\alpha_p^*(\Psi_s)=\sum\Psi_{t_j}$ and if $\alpha_p(t)=s$ then $\alpha_{p,*}(\Psi_t)=e(s)/e(t)\Psi_s$; and similarly for β_p^* and $\beta_{p,*}$.

Proof of Theorem 1.1. Since $\alpha_{p,*} = \beta_{p,*}$ on $\Phi_q(Npq)$, we have

$$T_p(\Psi_s) = \beta_{p,*} \circ \alpha_p^*(\Psi_s) = \alpha_{p,*} \circ \alpha_p^*(\Psi_s) = p\Psi_s,$$

which implies the result. □

4. The action of T_q on the component group

In this section, we provide a complete description of the action of T_q on the component group $\Phi_q(Nq)$. See Propositions 4.2, 4.3 and 4.4, which imply Theorem 1.2.

Note that the Hecke operator T_q acts on $\Sigma(N)$ by the Frobenius morphism [Ribet 1990, Proposition 3.8], and the same is true for ξ_q. Since the Frobenius morphism is an involution on $\Sigma(N)$ (see Proposition 2.6), we have

$$(4\text{-}1)\quad T_q(\psi_s)(t) = \psi_s(\xi_q(t)) = \psi_s(\mathrm{Frob}(t)) = \psi_{\mathrm{Frob}(s)}(t) \quad \text{for any} \quad t \in \Sigma(N),$$

which implies that $T_q(\psi_s) = \psi_{\mathrm{Frob}(s)}$.

From now on, if there is no confusion we remove (N) from the notation for simplicity. Let $n := \frac{1}{12}(q-1)Q$ (which is not necessarily an integer), and let Φ denote the cyclic subgroup of $\Phi_q(Nq)$ generated by $\Psi_{\mathfrak{s}}$ for a fixed $\mathfrak{s} \in \Sigma_2$. (Note that this Φ is the same as that of Mazur and Rapoport [1977], namely, Φ is equal to the cyclic subgroup generated by the image of the cuspidal divisor $(0)-(\infty)$.)

***Case* 1: $(u, v) = (0, 0)$ *or* $v = 0$.** Let $e = 1$ if $(u, v) = (0, 0)$ and $e = 2u + 3v$ if $(u, v) \neq (0, 0)$ and $v = 0$. If $(u, v) = (0, 0)$, $s_2 = n$ and $s_4 = s_6 = 0$. If $(u, v) \neq (0, 0)$ and $v = 0$, then $s_{2e} = 1$ and $s_2 = \frac{1}{e}(en - 1)$. (Note that s_2 is an integer but n is not.)

Proposition 4.1. *The component group $\Phi_q(Nq)$ is equal to Φ, which is cyclic of order en. The Hecke operator T_q acts on it by* 1.

Proof. First, we assume that $(u, v) = (0, 0)$. Then $\Psi_s = \Psi_{\mathfrak{s}}$ for any $s \in \Sigma = \Sigma_2$. Therefore $\Phi_q(Nq) = \Phi$ and $n\Psi_{\mathfrak{s}} = \sum_{s\in\Sigma} \Psi_s = 0$. Moreover, $T_q(\Psi_{\mathfrak{s}}) = \Psi_{s'} = \Psi_{\mathfrak{s}}$, where $s' = \mathrm{Frob}(\mathfrak{s})$.

Now, we assume that $(u, v) \neq (0, 0)$ and $v = 0$. In this case, either $N = 2q$ (with $(u, v) = (1, 0)$ and $e = 2$) or $N = 3q$ (with $(u, v) = (0, 1)$ and $e = 3$). In each case, let $z \in \Sigma_{2e}$. Then

$$\sum_{s\in\Sigma_2} \Psi_s + \Psi_z = s_2\Psi_{\mathfrak{s}} + \Psi_z = 0 \quad \text{and} \quad \Psi_{\mathfrak{s}} = e\Psi_z.$$

Therefore the component group is generated by Ψ_z, and its order is $(es_2 + 1) = en$. Since $en = es_2 + 1$ is prime to e, this group is also generated by $\Psi_{\mathfrak{s}} = e\Psi_z$. (In fact, $\Psi_z = -s_2\Psi_{\mathfrak{s}}$.) Moreover we have $T_q(\Psi_{\mathfrak{s}}) = \Psi_{\mathfrak{s}}$ as above. □

***Case* 2: $(u, v) = (0, 1)$ *and* $v \geq 1$.** In this case, $s_4 = 0$, $s_6 = 2^\nu$, and $s_2 = \frac{1}{3}(3n - 2^\nu)$. Let $\Sigma_6 := \{t_1, t_2, \dots, t_{2^\nu}\}$. Here we assume that $\mathrm{Frob}(t_{2k-1}) = t_{2k}$ for $1 \leq k \leq 2^{\nu-1}$.[8] Let $t := t_{2^\nu-1}$ and $t' := t_{2^\nu}$.

Proposition 4.2. *The component group* $\Phi_q(Nq)$ *decomposes as follows*:

$$\Phi_q(Nq) = \bigoplus_{i=0}^{2^\nu-1} B_i =: B_0 \oplus \boldsymbol{B},$$

where $B_0 = \Phi$ *is cyclic of order* $3n$, *and for* $1 \leq i \leq 2^\nu - 1$, B_i *is cyclic of order* 3. *For* $1 \leq k \leq 2^{\nu-1}$, B_{2k-1} *and* B_{2k} *are generated by*

$$\boldsymbol{v}_{2k-1} := \Psi_{t_{2k-1}} - \Psi_{t_{2k}}$$

and

$$\boldsymbol{v}_{2k} := \Psi_{t_{2k-1}} + \Psi_{t_{2k}} - \Psi_t - \Psi_{t'},$$

respectively. The Hecke operator T_q *acts on* B_i *by* $(-1)^i$.

Proof. Note that $\Psi_s = 3\Psi_{t_i} = 3\Psi_{t_j}$ for all i, j and $\sum_{i=1}^{2^\nu} \Psi_{t_i} + s_2\Psi_s = 0$. Therefore $\Phi_q(Nq)$ is generated by Ψ_{t_i} for $1 \leq i \leq 2^\nu - 1$. The order of each group $\langle \Psi_{t_i} \rangle$ is $9n$ because

$$9n\Psi_{t_i} = 3s_2(3\Psi_{t_i}) + \sum_{i=1}^{2^\nu} 3\Psi_{t_i} = 3\Bigg(\sum_{s\in\Sigma_2} \Psi_s + \sum_{i=1}^{2^\nu} \Psi_{t_i}\Bigg) = 0,$$

and $9n$ is the smallest positive integer to make this happen. Moreover $\langle \Psi_{t_i} \rangle \cap \langle \Psi_{t_j} \rangle$ is of order $3n$ for any $i \neq j$. Since $3n = 3s_2 + 2^\nu$ is prime to 3, we can decompose the component group into

$$\langle 3\Psi_t \rangle \oplus \langle (3s_2 + 2^\nu)\Psi_t \rangle \bigoplus_{i=1}^{2^\nu-2} \langle \Psi_{t_i} - \Psi_t \rangle. \tag{4-2}$$

Since $\Psi_s = 3\Psi_{t_i} = 3\Psi_t = 3\Psi_{t'}$ for any i and

$$\sum_{i=1}^{2^\nu} \Psi_{t_i} = -3s_2\Psi_t,$$

we have

$$\begin{aligned}
\Psi_{2k-1} - \Psi_t &= 2\boldsymbol{v}_{2k-1} + 2\boldsymbol{v}_{2k} + \boldsymbol{v}_{2^\nu-1},\\
\Psi_{2k} - \Psi_t &= \boldsymbol{v}_{2k-1} + 2\boldsymbol{v}_{2k} + \boldsymbol{v}_{2^\nu-1},\\
(3s_2 + 2^\nu)\Psi_t &= \sum_{i=1}^{2^\nu} (\Psi_t - \Psi_{t_i}) = -\sum_{k=1}^{2^{\nu-1}} \boldsymbol{v}_{2k} - (-1)^\nu \boldsymbol{v}_{2^\nu-1}.
\end{aligned}$$

Therefore the decomposition in the proposition is isomorphic to (4-2). The action of T_q on each B_i is obvious from its construction. □

[8] By Proposition 2.6, we know that Frob is an involution of Σ_6 without fixed points.

***Case* 3: $(u, v) = (1, 0)$ *and* $\nu \geq 1$.** Note that $s_4 = 2^\nu$, $s_6 = 0$, and $s_2 = n - 2^{\nu-1}$. Let $\Sigma_4 = \{w_1, w_2, \dots, w_{2^\nu}\}$. As before, we assume that $\mathrm{Frob}(w_{2k-1}) = w_{2k}$ for $1 \leq k \leq 2^{\nu-1}$.[9] Let $w := w_{2^\nu - 1}$ and $w' := w_{2^\nu}$.

Proposition 4.3. *The component group $\Phi_q(Nq)$ decomposes as*

$$\Phi_q(Nq) = \bigoplus_{i=0}^{2^\nu-2} A_i = A_0 \oplus \boldsymbol{A},$$

where A_0 is cyclic of order $4n$ generated by Ψ_w, and for $1 \leq i \leq 2^\nu - 2$, A_i is cyclic of order 2. For $1 \leq k \leq 2^{\nu-1} - 2$, A_{2k-1} and A_{2k} are generated by

$$\boldsymbol{u}_{2k-1} := \Psi_{w_{2k-1}} - \Psi_w \;\; \text{and} \;\; \boldsymbol{u}_{2k} := \Psi_{w_{2k-1}} + \Psi_{w_{2k}} - \Psi_w - \Psi_{w'}, \; \text{respectively.}$$

And $A_{2^\nu-3}$ and $A_{2^\nu-2}$ are generated by

$$\boldsymbol{u}_{2^\nu-3} := \Psi_{w_{2^\nu-3}} - \Psi_w \;\; \text{and} \;\; \boldsymbol{u}_{2^\nu-2} := \Psi_{w_{2^\nu-3}} - \Psi_{w_{2^\nu-2}}, \; \text{respectively.}$$

Moreover, the action of the Hecke operator T_q on each group is as follows:

$$T_q(\Psi_w) = (1+2n)\Psi_w + \sum_{i=1}^{2^{\nu-1}-1} \boldsymbol{u}_{2i},$$

$$T_q(\boldsymbol{u}_{2k-1}) = \boldsymbol{u}_{2k-1} + \boldsymbol{u}_{2k} \quad \text{and} \quad T_q(\boldsymbol{u}_{2k}) = \boldsymbol{u}_{2k} \quad \text{for } 1 \leq k \leq 2^{\nu-1} - 2,$$

$$T_q(\boldsymbol{u}_{2^\nu-3}) = 2n\Psi_w + \boldsymbol{u}_{2^\nu-3} + \sum_{i=1}^{2^{\nu-1}-2} \boldsymbol{u}_{2i} \quad \text{and} \quad T_q(\boldsymbol{u}_{2^\nu-2}) = \boldsymbol{u}_{2^\nu-2}.$$

Proof. The argument in Proposition 4.2 applies *mutatis mutandis*. For instance, when $\nu \geq 2$ an isomorphism between $A_0 \bigoplus_{i=1}^{2^\nu-2} \langle \Psi_{w_i} - \Psi_w \rangle$ and $A_0 \oplus \boldsymbol{A}$ can be given as follows: for $1 \leq k \leq 2^{\nu-1} - 2$,

$$\Psi_{w_{2k}} - \Psi_w = \boldsymbol{u}_{2k} + \boldsymbol{u}_{2k-1} + (\Psi_{w'} - \Psi_w),$$

$$\Psi_w - \Psi_{w'} = 2n\Psi_w + \sum_{i=1}^{2^{\nu-1}-1} \boldsymbol{u}_{2i},$$

$$\Psi_{w_{2^\nu-2}} - \Psi_w = \boldsymbol{u}_{2^\nu-3} + \boldsymbol{u}_{2^\nu-2}.$$

The action of the Hecke operator T_q on each A_i is clear except

$$T_q(\Psi_w) = \Psi_{w'} = \Psi_w - (\Psi_w - \Psi_{w'}) = (1+2n)\Psi_w + \sum_{i=1}^{2^{\nu-1}-1} \boldsymbol{u}_{2i},$$

$$\begin{aligned} T_q(\boldsymbol{u}_{2^\nu-3}) = \Psi_{w_{2^\nu-2}} - \Psi_{w'} &= \boldsymbol{u}_{2^\nu-3} + \boldsymbol{u}_{2^\nu-2} + (\Psi_w - \Psi_{w'}) \\ &= 2n\Psi_w + \boldsymbol{u}_{2^\nu-3} + \sum_{i=1}^{2^{\nu-1}-2} \boldsymbol{u}_{2i}. \quad \square \end{aligned}$$

[9] By Proposition 2.6, we know that Frob is an involution of Σ_4 without fixed points.

***Case* 4: $(u, v) = (1, 1)$ *and* $v \geq 1$.** Note that $s_4 = s_6 = 2^\nu$ and $s_2 = \frac{1}{6}(6n - 5 \cdot 2^\nu)$. Let $\Sigma_4 = \{w_1, \ldots, w_{2^\nu}\}$ and $\Sigma_6 := \{t_1, \ldots, t_{2^\nu}\}$. As before, we assume that $\mathrm{Frob}(w_{2k-1}) = w_{2k}$ and $\mathrm{Frob}(t_{2k-1}) = t_{2k}$ for $1 \leq k \leq 2^{\nu-1}$. Let $w := w_{2^\nu - 1}$ and $w' := w_{2^\nu}$. Also, let $t := t_{2^\nu-1}$ and $t' := t_{2^\nu}$.

Proposition 4.4. *The component group $\Phi_q(Nq)$ decomposes as*

$$\Phi_q(Nq) = A_0 \oplus \boldsymbol{A} \oplus \boldsymbol{B},$$

where A_0 is cyclic of order $12n$ generated by Ψ_w. The structures of $\boldsymbol{A}$ and $\boldsymbol{B}$ are the same as those in Propositions 4.2 and 4.3. The actions of T_q on $\boldsymbol{A}$ and $\boldsymbol{B}$ are the same as before except on $A_{2^\nu-3}$ (when $\nu \geq 2$), where T_q acts by

$$T_q(\boldsymbol{u}_{2^\nu-3}) = 6n\Psi_w + \boldsymbol{u}_{2^\nu-3} + \sum_{i=1}^{2^{\nu-1}-2} \boldsymbol{u}_{2i}.$$

Moreover, the action of T_q on A_0 is analogous to the previous case:

$$T_q(\Psi_w) = (1 + 6n)\Psi_w + \sum_{i=1}^{2^{\nu-1}-1} \boldsymbol{u}_{2i}.$$

Proof. Note that from (3-1), we have

$$s_2\Psi_s + \Psi_{w_1} + \cdots + \Psi_{w'} + \Psi_{t_1} + \cdots + \Psi_{t'} = 0.$$

Multiplying by 3, we have

$$\Psi_{w_1} + \cdots + \Psi_{w'} = -(3s_2 + 2 \cdot 2^\nu)\Psi_s = -(6s_2 + 4 \cdot 2^\nu)\Psi_w. \tag{4-3}$$

Also, multiplying by 4, we have

$$\Psi_{t_1} + \cdots + \Psi_{t'} = -(4s_2 + 3 \cdot 2^\nu)\Psi_s = -(12s_2 + 9 \cdot 2^\nu)\Psi_t. \tag{4-4}$$

Therefore $\Psi_{w_1}, \ldots, \Psi_w, \Psi_{t_1}, \ldots, \Psi_t$ can generate the whole group. By a similar computation, the order of $\langle \Psi_{w_i} \rangle$ is $12n$ and the order of $\langle \Psi_{t_i} \rangle$ is $18n$. All of them contain Φ as a subgroup, which is of order $6n$. Here we note that $\langle \Psi_t \rangle = \langle 3\Psi_t \rangle \oplus \langle 6n\Psi_t \rangle$ because $6n = 6s_2 + 5 \cdot 2^\nu$ is prime to 3. Therefore we can decompose $\Phi_q(Nq)$ into

$$\langle \Psi_w \rangle \bigoplus_{i=1}^{2^\nu-2} \langle \Psi_{w_i} - \Psi_w \rangle \bigoplus_{i=1}^{2^\nu-2} \langle \Psi_{t_i} - \Psi_t \rangle \bigoplus \langle 6n\Psi_t \rangle. \tag{4-5}$$

As in Propositions 4.2 and 4.3, we can find an isomorphism between (4-5) and $A_0 \oplus \boldsymbol{A} \oplus \boldsymbol{B}$, which proves the first part. From (4-3) (and the previous discussions) we have

$$\Psi_w - \Psi_{w'} = (6s_2 + 5 \cdot 2^\nu)\Psi_w + \sum_{i=1}^{2^{\nu-1}-1} \boldsymbol{u}_{2i} = 6n\Psi_w + \sum_{i=1}^{2^{\nu-1}-1} \boldsymbol{u}_{2i}.$$

The action of T_q on each component is also obvious except

$$T_q(\Psi_w) = \Psi_{w'} = \Psi_w - (\Psi_w - \Psi_{w'}) = (1+6n)\Psi_w + \sum_{i=1}^{2^{\nu-1}-1} \boldsymbol{u}_{2i},$$

$$\begin{aligned} T_q(\boldsymbol{u}_{2^\nu-3}) = \Psi_{w_{2^\nu-2}} - \Psi_{w'} &= \boldsymbol{u}_{2^\nu-3} + \boldsymbol{u}_{2^\nu-2} + (\Psi_w - \Psi_{w'}) \\ &= 6n\Psi_w + \boldsymbol{u}_{2^\nu-3} + \sum_{i=1}^{2^{\nu-1}-2} \boldsymbol{u}_{2i}. \qquad \square \end{aligned}$$

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