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We study volume growth, a Liouville theorem for f-harmonic functions, and a volume comparison property of unit balls in complete noncompact gradient Ricci shrinkers and gradient steady Ricci solitons. We also study integral properties of f-harmonic functions and harmonic functions on complete manifolds, such as the Ricci–Einstein solitons.

1. Introduction

In the study of Ricci flow on a compact Riemannian manifold, because of its complicated nonlinearity, one meets singularities of the flow in finite time. After blowing up, one expects to get self-similar Ricci shrinkers or steady Ricci solitons [Hamilton 1995]. In the case of a type-I singularity in dimension three, one gets nontrivial gradient Ricci shrinkers via the use of the Ivey–Hamilton pinching estimate, and the classification of this type of self-similar Ricci shrinker was done by G. Perelman [2000]. In the case of a type-I singularity of dimension four, A. Naber [2010] showed that one gets a gradient Ricci shrinker, and nontrivial properties of this Ricci shrinker have been studied by others. These solitons can be considered as special examples of weighted Riemannian manifolds or metric measure spaces [Bakry and Émery 1985; Wei and Wylie 2009; Lott and Villani 2009; Chen 2009; Chow et al. 2006; Lichnerowicz 1970; Lott 2003; Ma 2013; Sturm 2006a; 2006b; Yang 2009; Munteanu and Wang 2011; 2014; 2015]. Because of the importance of four-dimensional Ricci shrinkers, many people study various properties of them; see for example [Ma 2013; Cao and Zhou 2010; Cao 2007; 2010; Carrillo and Ni 2009; Munteanu and Wang 2015; Haslhofer and Müller 2011]. In this paper, we examine three questions about Ricci shrinking and steady solitons, in particular for Ricci shrinkers which are the complete Riemannian manifold (M, g) such that $\mathrm{Ric}_f := \mathrm{Ric} + D^2 f = \lambda g$ on M, where Ric is the Ricci curvature of (M, g), $f : M \to R$ is a smooth function in M, and $\lambda > 0$ is a constant. One question under

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consideration in this paper is about the volume comparison of unit balls on Ricci shrinkers. The other two are about f-harmonic functions and harmonic functions with finite energy. The volume comparison of unit balls is an important step to understanding the volume growth of geodesic balls in gradient Ricci shrinkers. The Liouville theorems for f-harmonic and harmonic functions with finite energy are important to understanding the connectivity at infinity about gradient Ricci solitons; see [Munteanu and Wang 2015].

We have the following new results. The first is the volume comparison of unit balls at any point $x \in M$ with a unit ball at a fixed point $p \in M$, which deals with the injectivity radius decay from the point p to the point x and is important to understanding the topology of the underlying manifold at infinity.

Theorem 1. *On the complete noncompact gradient Ricci shrinker (M, g, f) with Ricci curvature bounded by $-(n-1)k^2$ for some constant $k \geq 0$, we have*

$$\mathrm{Vol}(B_x(1)) \geq \exp(-\sqrt{c_0(n-1)R})\, \mathrm{Vol}(B_p(1)),$$

where c_0 is a uniform constant which does not depend on x and $R = d(p, x) > R_0$ for some uniform constant $R_0 > 1$.

The above result is motivated by the work of Ovidiu Munteanu and Jiaping Wang [2014] and the result is not sharp, as pointed out by the referee. In fact, if the Ricci curvature is bounded from below, the logarithmic Sobolev inequality of [Carrillo and Ni 2009] implies a uniform lower bound of the volume of unit balls, $\mathrm{Vol}(B_x(1)) \geq \frac{1}{c}\mathrm{Vol}(B_p(1))$. However, from our argument, our result is true on a complete smooth measure space of dimension n with $\mathrm{Ric}_f \geq \frac{1}{2}g$ and $|\nabla f|^2 \leq f$ and with Ricci curvature bounded below by $-(n-1)k^2$ for some constant $k \geq 0$ on M. Notice that with the extra bound on Ricci curvature, our result improves the result of Lemma 5.2 in [Munteanu and Wang 2014]. We remark that the above result is still true for steady Ricci solitons. See Theorem 5 in Section 2.

We want to understand the topology and geometry from properties of (super- or sub-) harmonic functions and f-harmonic functions on Ricci solitons. By a well-known argument, we know that there is no nontrivial positive f-harmonic function on a gradient Ricci shrinker. In fact there is no nonconstant positive f-superharmonic function u ($\Delta_f u := \Delta u - \nabla f . \nabla u \leq 0$) on the complete Riemannian manifold (M, g) with $\mathrm{Ric}_f \geq \frac{1}{2}g$ on M. The process of proving this is below. Recall that the weighted volume of (M, g) is finite [Morgan 2005]; i.e., $V_f(M) := \int_M e^{-f}\, dv_g < \infty$. Assume $u > 0$ is such a positive f-superharmonic function on M. Let $v = \log u$. Then we have

$$\Delta_f v = \frac{\Delta_f u}{u} - |\nabla v|^2 \leq -|\nabla v|^2.$$

Thus, for any cut-off function $\phi \geq 0$ on the ball $B_p(2R) \subset M$ with $\phi = 1$ in $B_p(R)$, we have

$$\int_M |\nabla v|^2 \phi^2 e^{-f}\, dv_g \leq 2 \int_M \phi \nabla \phi . \nabla v e^{-f}\, dv_g.$$

By the Cauchy–Schwarz inequality we get

$$\int_M |\nabla v|^2 \phi^2 e^{-f}\, dv_g \leq 4 \int_M |\nabla \phi|^2 e^{-f}\, dv_g \leq \frac{16}{R^2} V_f(M) \to 0$$

as $R \to \infty$. Hence, $|\nabla v|^2 = 0$ in M, which implies that u is a constant on M.

Although this result is known to experts, we cannot find it in the literature. We formulate it as below.

Proposition 2. *Let (M, g, f) be a complete Riemannian manifold (M, g) with potential function f satisfying* $\mathrm{Ric}_f \geq \frac{1}{2} g$ *on M. Then there is no nonconstant positive f-superharmonic function on (M, g).*

We are now trying to find another kind of Liouville theorem for an f-harmonic function with weighted finite energy on a gradient Ricci shrinker. We show that as a direct consequence of a Bochner-type formula, see [Ma and Du 2010], we have the following Liouville-type theorem.

Theorem 3. *Let (M, g) be a complete noncompact Riemannian manifold such that* $\mathrm{Ric}_f \geq h(x) g$ *for some potential functions $f(x)$ and some nontrivial nonnegative function $h(x)$ in M. There is no nontrivial f-harmonic function u defined in (M, g) with weighted finite energy*; *i.e.*,

$$\int_M |\nabla u|^2 e^{-f}\, dv_g < \infty.$$

The proof of this result is given in Section 3. We remark that when one studies self-similar solutions to the Ricci–Einstein-type flow

$$\partial_t g = -2\,\mathrm{Rc}(g) + \gamma R g,$$

where γ is a physical constant and $R = R(g)$ is the scalar curvature of the manifold (M, g), one gets the soliton equation of it in the form

$$\mathrm{Ric}_f = \gamma R g + \lambda g,$$

which gives what may be called Ricci–Einstein solitons. Theorem 3 can be used for these solitons.

Proposition 4. *Fix any $p \in M$. Assume that the complete noncompact Riemannian manifold (M, g) satisfies* $\mathrm{Ric}_f \geq h(x) g$ *for some potential functions $f(x)$ and some smooth function $h(x)$ with*

$$|\nabla f(x)| \leq \alpha d(x, p) + b$$

for some uniform constants $\alpha \geq 0$ and $b > 0$ in M. Then for any harmonic function u with finite energy, i.e.,

$$\int_M |\nabla u|^2 < \infty,$$

we have the integral inequality

$$\int_M |\nabla^2 u|^2 + \int_M h(x)|\nabla u|^2 \leq \frac{1}{2}\int_M \Delta f |\nabla u|^2.$$

As a consequence, when $\Delta f(x) \leq 2h(x)$ on M, we have $D^2 u = 0$ in M; i.e., ∇u is a parallel vector field on M.

We now give a few remarks related to this result.

(1) Naber [2010] proved that for the weighted smooth metric space (M, g, f) satisfying $\mathrm{Ric}_f \leq \frac{1}{2}g$ and $|\mathrm{Ric}| \leq C$, there exists $\alpha > 0$ such that if $\Delta_f u := \Delta u - \nabla f.\nabla u = 0$ on M with $|u(x)| \leq A\exp(\alpha d(x, p)^2)$ for some $A > 0$ and $p \in M$, then u is a constant.

(2) Munteanu and Sesum [2013] proved that for gradient shrinking Kähler–Ricci solitons, if the harmonic function u has finite energy, i.e., $\int_M |\nabla u|^2 < \infty$, then u is a constant. As a consequence of this result, they showed that such a manifold has at most one nonparabolic end; see [Munteanu and Sesum 2013] for the definition of nonparabolic end. In the earlier work [Munteanu and Wang 2011], they proved that on a weighted smooth metric space (M, g, f) satisfying $\mathrm{Ric}_f \geq 0$ and f is a bounded function, any sublinear growth f-harmonic function on M must be a constant.

(3) Some consequences of Proposition 4 are given in Section 3.

Here is the plan of the paper. We study the volume comparison of unit balls of gradient Ricci shrinkers and gradient steady Ricci solitons in Section 2. We prove Proposition 2 and Theorem 3 in Section 3. In the last section, we consider integral properties for harmonic functions on Ricci solitons and prove Proposition 4.

2. Volume comparison of unit balls

We now prove Theorems 1 and 5. We use an idea similar to the proof of Theorem 2.3 in [Munteanu and Wang 2014].

We first give a proof of an improved volume comparison of unit balls on the weighted Riemannian manifold of shrinking type.

Proof of Theorem 1. Again, we take any point $x \in M$ and express the volume form in the geodesic polar coordinates centered at x as

$$dV|_{\exp_x(r\xi)} = J(x, r, \xi)\, dr\, d\xi$$

for $r > 0$ and $\xi \in S_x M$ a unit tangent vector at x. We let $R = d(p, x)$ and omit the dependence of the geometric quantities on ξ. Let $R = d(p, x) > 2$. Let $\gamma(s)$ be the minimizing geodesic starting from x connecting $\gamma(0) = x$ to any point $\gamma(T) \in B_p(1)$. By the triangle inequality we know that $T \in [R-1, R+1]$. It is well known [Li 1993] that along the minimizing geodesic curve γ, we have

$$m'(r) + \frac{1}{n-1} m^2(r) + \mathrm{Ric}(\partial_r, \partial_r) \le 0,$$

where $r > 0$ and $m = m(r) = \frac{d}{dr}(\log J)(r)$. Using the Ricci soliton equation $\mathrm{Ric}_f = \frac{1}{2} g$ we immediately obtain

$$m'(r) + \frac{1}{n-1} m^2(r) \le -\frac{1}{2} + f''(r).$$

Integrating this relation, we get for $r > 1$,

$$m(r) + \frac{1}{n-1} \int_1^r m^2(t)\, dt \le -\frac{r-1}{2} + f'(r) - f'(1) + m(1).$$

Recall the well-known fact (see (2.3) and (2.8) in [Cao and Zhou 2010] in different notation) that for any $T > \tau > 0$, we have

$$-\tfrac{1}{2}(T-\tau) - c \le f'(\tau) \le -\tfrac{1}{2}(T-\tau) + c \tag{1}$$

and $|f'(T)| \le c$ since $\gamma(T) \in B_p(1)$. Here and everywhere in the proofs, c denotes a constant depending only on the dimension n and $f(p)$. By this we know

$$-\frac{r-1}{2} + f'(r) - f'(1) \le c_0$$

for some uniform constant $c_0 > 0$.

Using the Ricci curvature lower bound, a standard argument shows, see (1.1.8) in [Schoen and Yau 1994], that there is a uniform constant $c_1 > 0$ such that $m(s) \le c_1$ for $s \ge \frac{1}{2}$. Then we have for another uniform constant $c_0 > 0$ and for any $r > 1$,

$$m(r) + \frac{1}{n-1} \int_1^r m^2(t)\, dt \le c_0.$$

By the Cauchy–Schwarz inequality we obtain

$$m(r) + \frac{1}{(n-1)r} \left(\int_1^r m(t)\, dt \right)^2 < c \tag{2}$$

for any $c > c_0$ and $r > 1$.

Claim. For any $r > 1$,

$$\int_1^r m(t)\, dt \le \sqrt{c(n-1)r}. \tag{3}$$

In fact, let

$$v(t)=\sqrt{c(n-1)t}-\int_1^t m(r)\,dr.$$

Then

$$v'(t)=\frac{\sqrt{c(n-1)}}{2\sqrt{t}}-m(t).$$

Clearly $v(1)>0$ by the fact that $c>c_0$. Suppose that v is negative somewhere for $t>1$. Let $d>1$ be the first zero point of v; i.e., $v(d)=0$. Then by the choice of d, we have $v'(d)\le 0$. That is,

$$\int_1^d m(t)\,dt=\sqrt{c(n-1)d}$$

and

$$m(d)\ge\frac{\sqrt{c(n-1)}}{2\sqrt{d}}.$$

By direct computation we know

$$m(d)+\frac{1}{(n-1)d}\left(\int_1^d m(t)\,dta\right)^2\ge\frac{\sqrt{c(n-1)}}{2\sqrt{d}}+c,$$

which is a contradiction with (2) at $r=d$.

The relation (3) implies

$$\log J(x,r,\xi)/J(x,1,\xi)\le\sqrt{c(n-1)r}$$

for any $r>1$ and we have at $r=R$,

$$J(x,1,\xi)\ge\exp(-\sqrt{c(n-1)R})J(x,R,\xi).$$

Integrating over the unit tangent vectors ξ we get

$$\text{Area}(\partial B_x(1))\ge\exp(-\sqrt{c(n-1)R})\,\text{Vol}(B_p(1)),$$

where $R=d(p,x)>2$. Similarly we have

$$\text{Area}(\partial B_x(s))\ge\exp(-\sqrt{c(n-1)R})\,\text{Vol}(B_p(1))$$

for any $s\in\left[\frac{1}{2},1\right]$. Hence, we have

$$\text{Vol}(B_x(1))\ge\exp(-\sqrt{c(n-1)R})\,\text{Vol}(B_p(1)).\qquad\square$$

A similar argument gives us the result below for gradient steady Ricci solitons.

Theorem 5. *On the complete noncompact Riemannian manifold (M,g,f) with Ricci curvature bounded below by $-(n-1)k^2$ for some constant $k\ge 0$ and $\text{Rc}_f\ge 0$ on M and $|\nabla f|\le C$ on M for some uniform constant $C>0$, we have*

$$\text{Vol}(B_x(1))\ge\exp(-\sqrt{c_0(n-1)R})\,\text{Vol}(B_p(1)),$$

where c_0 is a uniform constant which does not depends on p and $R = d(p, x) > R_0$ for some uniform constant $R_0 > 1$.

The proof of Theorem 5 is almost the same as the previous proof. So we only present the necessary modification and use the same notation as above. Again we have along the minimizing geodesic curve γ,

$$m'(r) + \frac{1}{n-1} m^2(r) + \mathrm{Ric}(\partial_r, \partial_r) \le 0.$$

Using the relation $\mathrm{Ric}_f \ge 0$ we immediately obtain

$$m'(r) + \frac{1}{n-1} m^2(r) \le f''(r).$$

Integrating this relation we get for $t > 1$ and $s \in \left[\frac{1}{2}, 1\right]$,

$$m(t) + \frac{1}{n-1} \int_s^t m^2(r)\, dr \le f'(t) - f'(1) + m(s).$$

This means that for any $t > 1$ and $s = 1$,

$$m(t) + \frac{1}{n-1} \int_1^t m^2(r)\, dr \le C_0$$

for some uniform constant C_0. The remaining part of the proof is the same as the proof of Theorem 1 and we omit the details.

We now consider the volume growth of geodesic balls in manifolds with density and we show that for $(M, g, e^{-f}\, dv)$ a complete smooth metric measure space of dimension n with $\mathrm{Ric}_f \ge \frac{1}{2}$, $|\nabla f|^2 \le f$, and also with both Ricci curvature and Δf bounded from above, the volume growth of geodesic balls is in polynomial order (which may be smaller than n).

Proposition 6. *Let $(M, g, e^{-f} dv)$ be a complete smooth metric measure space of dimension n. Assume that* $\mathrm{Ric}_f \ge \frac{1}{2}$, $|\nabla f|^2 \le f$. *Assume further that $\Delta f \le K$ and* $\mathrm{Ric} \le K_1$ *for some constants $K > 0$ and $K_1 > 0$. Then for some $p \in M$, the volume growth of geodesic balls $B_p(r)$ is of polynomial order*; *i.e., there is a uniform constant $C > 0$ such that*

$$V(r) \le C r^{2K}.$$

Proof. Recall that under the conditions $\mathrm{Ric}_f \ge \frac{1}{2}$ and $|\nabla f|^2 \le f$, there are a point $p \in M$, two constants $r_0 > 0$ and a depending only on n and $f(p)$ such that

$$\left(\tfrac{1}{2} d(x, p) - a\right)^2 \le f(x) \le \left(\tfrac{1}{2} d(x, p) + a\right)^2. \tag{4}$$

This is from Proposition 4.2 in the interesting paper [Munteanu and Wang 2014]. By this we know that $|\nabla f(x)| \le \frac{1}{2} d(x, p) + a$. We may assume that $d(x, p) > 2$. Consider any minimizing normal geodesic $\gamma(s)$, $0 \le s \le r := d(x, p)$, starting from the point $\gamma(0)=p$ to the point $\gamma(r)=x$. Let $X=\dot\gamma(s)$. By the second variation formula of arc length we know

$$\int_0^r \phi^2 \operatorname{Ric}(X,X)\,ds \le (n-1)\int_0^r |\dot\phi(s)|^2\,ds$$

for any $\phi \in C_0^{1-}([0,r])$. Let $\phi(s)=s$ on $[0,1]$, $\phi(s)=r-s$ on $[r-1,r]$, and $\phi(s)=1$ on $[1,r-1]$. Then we have

$$\int_0^r \operatorname{Ric}(X,X)\,ds = \int_0^r \phi^2 \operatorname{Ric}(X,X)\,ds + \int_0^r (1-\phi^2)\operatorname{Ric}(X,X)\,ds.$$

We derive via the use of $\operatorname{Ric}\le K_1$, and by an argument similar to the proof before (2.8) in [Cao and Zhou 2010], that

$$\int_0^r \operatorname{Ric}(X,X)\,ds \le 2(n-1)+2K_1.$$

Since

$$\nabla_X \dot f = \nabla^2 f(X,X) \ge \tfrac12 - \operatorname{Ric}(X,X),$$

integrating it from 0 to r, we get

$$\dot f(r) = \tfrac12 r - \int_0^r \operatorname{Ric}(X,X)\,ds \ge \tfrac12 r - c \tag{5}$$

for some constant c depending only on K_1, n and $f(p)$. Hence,

$$|\nabla f|(x) \ge \dot f(r) \ge \tfrac12 d(x,p) - c.$$

Define

$$\rho(x) = 2\sqrt{f(x)}.$$

Then,

$$|\nabla\rho| = \frac{|\nabla f|}{\sqrt f} \le 1.$$

Let, for $r>0$ large,

$$D(r)=\{x\in M;\ \rho(x)\le r\},\qquad V(r)=\operatorname{Vol}(D(r)).$$

As in [Cao and Zhou 2010], by using the coarea formula we have

$$V(r)=\int_0^r ds\int_{\partial D(r)} \frac{1}{|\nabla\rho|}\,dA,$$

$$V'(r)=\int_{\partial D(r)} \frac{1}{|\nabla\rho|}\,dA = \frac{r}{2}\int_{\partial D(r)} \frac{1}{|\nabla f|}\,dA.$$

By the assumption $\Delta f \le K$ and the divergence theorem we have

$$2KV(r) \ge 2\int_{D(r)} \Delta f = 2\int_{\partial D(r)} |\nabla f|\,dA.$$

By (5) we know that on $\partial D(r)$, there is a uniform constant $C > 2$ such that for $r \geq 2C$,

$$|\nabla f|^2 \geq f - C.$$

Then we have

$$2\int_{\partial D(r)} |\nabla f|\, dA \geq 2\int_{\partial D(r)} \frac{f - C}{|\nabla f|}\, dA.$$

The right side of above inequality is greater than or equal to

$$(r-2)V'(r).$$

Hence we have

$$2KV(r) \geq (r-2)V'(r),$$

which then implies

$$V(r) \leq V(2C)r^{2K}$$

for $r > 2C$. □

We remark that the above argument is motivated by the proof of the volume growth estimate in [Cao and Zhou 2010]. Our result is different from the deep result Theorem 1.4 in [Munteanu and Wang 2014] in the case when the constant $2K$ is smaller than n.

3. Harmonic and f-harmonic functions on Ricci–Einstein solitons

We now prove Theorem 3. We wish that we could use the Caccioppoli argument (see the proof of Proposition 8.1 in [Naber 2010], with the use of Lemma 2.2 replaced by Proposition 4.2 in [Munteanu and Wang 2014]) to conclude that with some decay assumption such as finite energy, an f-harmonic function u is a constant function on M. However, we have a simpler proof of this result below.

Proof of Theorem 3. Recall the Bochner formula for the f-harmonic function $u : M \to R$,

$$\tfrac{1}{2}\Delta_f|\nabla u|^2 = |\nabla^2 u|^2 + \mathrm{Rc}_f(\nabla u, \nabla u).$$

By our assumption that $\mathrm{Ric}_f \geq h(x)g$, we know

$$\tfrac{1}{2}\Delta_f|\nabla u|^2 \geq |\nabla^2 u|^2 + h(x)|\nabla u|^2.$$

Let ϕ be the standard cut-off function on $B_p(2r)$ and let $dm = \exp(-f)\, dv_g$. Then we have

$$\int_M (|\nabla^2 u|^2 + h(x)|\nabla u|^2)\phi\, dm \leq \int_M \left(\tfrac{1}{2}\Delta_f \phi\right)|\nabla u|^2\, dm.$$

The right side goes to zero as $r \to \infty$. Hence we have

$$\int_M (|\nabla^2 u|^2 + h(x)|\nabla u|^2)\, dm = 0,$$

which implies that u is a constant. □

Let (M, g) be a complete noncompact Riemannian manifold of dimension n. Fix $p \in M$. In this section we always assume that (M, g) satisfies $\mathrm{Ric}_f \ge h(x)g$ for some function $h(x)$ and $|\nabla f| \le \alpha d(x, p) + b$. Then we have $R + \Delta f \ge nh(x)$ on M. We study the L^2 estimate for the Hessian matrix for harmonic functions with finite energy.

Proof of Proposition 4. Let $u : M \to R$ be a harmonic function on (M, g, f) with finite energy

$$\int_M |\nabla u|^2 < \infty.$$

Recall the Bochner formula for the harmonic function $u : M \to R$,

$$\tfrac{1}{2}\Delta|\nabla u|^2 = |\nabla^2 u|^2 + \mathrm{Rc}(\nabla u, \nabla u).$$

Using the assumption $\mathrm{Ric}_f \ge h(x)g$ we have

$$\tfrac{1}{2}\Delta|\nabla u|^2 \ge |\nabla^2 u|^2 + h(x)|\nabla u|^2 - \nabla^2 f(\nabla u, \nabla u). \tag{6}$$

Recall the Hessian matrix $\nabla^2 f = (f_{ij})$ in local coordinates (x_i) in M.

Let $\phi = \phi_r$ be the cut-off function on $B_{2r}(p)$. We write by $o(1)$ the quantities such that $o(1) \to 0$ as $r \to \infty$. Then, we have

$$\int_M (|\nabla^2 u|^2 + h(x)|\nabla u|^2)\phi^2 \le \int_M \big(\tfrac{1}{2}\Delta|\nabla u|^2 + \nabla^2 f(\nabla u, \nabla u)\big)\phi^2. \tag{7}$$

By direct computation we have, for $\epsilon > 0$ small,

$$\int_M \big(\tfrac{1}{2}\Delta|\nabla u|^2\big)\phi^2 = -2\int_M \phi D^2 u(\nabla u, \nabla\phi) \le \epsilon\int_M |\nabla^2 u|^2\phi^2 + o(1).$$

Using $|\nabla f| \le \alpha d(x, p) + b$ and integrating by parts, we obtain

$$\int_M f_{ij}u_i u_j\phi^2 = -\int f_i u_{ij} u_j\phi^2 + o(1).$$

Furthermore, we have

$$\int_M f_{ij}u_i u_j\phi^2 = \frac{1}{2}\int_M \Delta f|\nabla u|^2\phi^2 + o(1).$$

Hence by (6) we have

$$\int_M \big((1-\epsilon)|\nabla^2 u|^2 + h(x)|\nabla u|^2\big)\phi^2 - \frac{1}{2}\int_M \Delta f|\nabla u|^2\phi^2 \le o(1).$$

That is,

$$\int_M (1-\epsilon)|\nabla^2 u|^2\phi^2 + \int_M h(x)|\nabla u|^2\phi^2 \le \frac{1}{2}\int_M \Delta f|\nabla u|^2\phi^2 + o(1).$$

Sending $r \to \infty$ and letting $\epsilon \to 0$, we obtain

$$\int_M |\nabla^2 u|^2 + \int_M h(x)|\nabla u|^2 \le \frac{1}{2}\int_M \Delta f|\nabla u|^2. \tag{8}$$

Note that when $\Delta f \le 2h(x)$ on M, by (8) we have $D^2u = 0$ in M; i.e., ∇u is a parallel vector field on M. □

We remark that when $\mathrm{Ric}_f = \lambda g$ (and $R + \Delta f = n\lambda$) with λ being a constant, by the proof of Proposition 4 we have

$$\int_M |\nabla^2 u|^2 + \frac{1}{2}\int_M R|\nabla u|^2 \le \int_M \frac{(n-2)\lambda}{2}|\nabla u|^2.$$

Note that the assumption about the potential function f in Proposition 4 is true on the steady soliton (M, g); see [Hamilton 1995]. We now give an application of this integral inequality (8). A special case of the Liouville-type theorem below, due to Munteanu and Sesum [2013, Theorem 4.1], can be derived from the integral estimate (8).

Proposition 7. *Let $n = 2$. Assume that the complete noncompact surface (M, g, f) satisfies* $\mathrm{Ric}_f = h(x)g$ *on M with $R \ge 0$, and $|\nabla f| \le b$ for some $b > 0$ in M. Then there is no nontrivial harmonic function on (M, g) with finite energy on (M, g).*

Proof. Note that $R + \Delta f = nh(x) = 2h(x)$ in M. Then $\Delta f = 2h(x) - R$ in M. By (8) we have

$$\int_M |\nabla^2 u|^2 + \frac{1}{2}\int_M R|\nabla u|^2 \le 0.$$

If $R = 0$, then (M, g) is flat and the result follows from Theorem 4.1 in [Munteanu and Sesum 2013].

We may assume $R > 0$ in M. We argue by contradiction. Assume that there is a nontrivial harmonic function with finite energy on (M, g). By (8) we know

$$\int_M |\nabla^2 u|^2 + \frac{1}{2}\int_M R|\nabla u|^2 = 0.$$

Hence ∇u is a parallel vector field on M and $R = 0$, a contradiction with $R > 0$. □

We remark that we can give a new proof of Theorem 4.1 in [Munteanu and Sesum 2013], which is on a gradient steady Ricci soliton. It says that there is no nontrivial harmonic function with finite energy on the steady Ricci soliton (M, g). The proof is below. We may assume that (M, g, f) is a nontrivial steady Ricci soliton. Recall that it is well known that either $R > 0$ or $R = 0$ on M. By (8), we have $R = 0$, and then

$$\Delta_f R = -2|\mathrm{Ric}|^2,$$

and we know that $\mathrm{Ric} = 0$ on M. By [Schoen and Yau 1994], we know that there is no nontrivial harmonic function with finite energy.

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