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LOCALIZATION FUNCTORS AND COSUPPORT IN DERIVED CATEGORIES OF COMMUTATIVE NOETHERIAN RINGS

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Let R be a commutative Noetherian ring. We introduce the notion of localization functors λ^W with cosupports in arbitrary subsets W of Spec$R$; it is a common generalization of localizations with respect to multiplicatively closed subsets and left derived functors of ideal-adic completion functors. We prove several results about the localization functors λ^W, including an explicit way to calculate λ^W using the notion of Čech complexes. As an application, we can give a simpler proof of a classical theorem by Gruson and Raynaud, which states that the projective dimension of a flat R-module is at most the Krull dimension of R. As another application, it is possible to give a functorial way to replace complexes of flat R-modules or complexes of finitely generated R-modules by complexes of pure-injective R-modules.

1. Introduction

Throughout this paper, we assume that R is a commutative Noetherian ring. We denote by $\mathcal{D} = D(\operatorname{Mod} R)$ the derived category of all complexes of R-modules, by which we mean that $\mathcal{D}$ is the unbounded derived category. For a triangulated subcategory $\mathcal{T}$ of $\mathcal{D}$, its left and right orthogonal subcategories are defined as

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${}^{\perp}\mathcal{T}=\{X\in\mathcal{D}\mid \mathrm{Hom}_{\mathcal{D}}(X,\mathcal{T})=0\}$ and $\mathcal{T}^{\perp}=\{Y\in\mathcal{D}\mid \mathrm{Hom}_{\mathcal{D}}(\mathcal{T},Y)=0\}$, respectively. Moreover, $\mathcal{T}$ is called localizing if $\mathcal{T}$ is closed under arbitrary direct sums, and colocalizing if it is closed under arbitrary direct products.

Recall that the support of a complex $X\in\mathcal{D}$ is defined as

$$\operatorname{supp} X=\{\mathfrak{p}\in\operatorname{Spec} R\mid X\otimes_R^{\mathrm{L}}\kappa(\mathfrak{p})\neq 0\},$$

where $\kappa(\mathfrak{p})=R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$. We write $\mathcal{L}_W=\{X\in\mathcal{D}\mid \operatorname{supp} X\subseteq W\}$ for a subset W of $\operatorname{Spec} R$. Then $\mathcal{L}_W$ is a localizing subcategory of $\mathcal{D}$. Neeman [1992] proved that any localizing subcategory of $\mathcal{D}$ is obtained in this way. The localization theory of triangulated categories [Krause 2010] yields a couple of adjoint pairs (i_W,γ_W) and (λ_W,j_W) as it is indicated in the following diagram:

$$\mathcal{L}_W \underset{\gamma_W}{\overset{i_W}{\rightleftarrows}} \mathcal{D} \underset{j_W}{\overset{\lambda_W}{\rightleftarrows}} \mathcal{L}_W^{\perp} \tag{1.1}$$

Here, i_W and j_W are the inclusion functors $\mathcal{L}_W\hookrightarrow\mathcal{D}$ and $\mathcal{L}_W^{\perp}\hookrightarrow\mathcal{D}$, respectively. In [Nakamura and Yoshino 2018], we introduced the colocalization functor with support in W as the functor γ_W. If V is a specialization-closed subset of $\operatorname{Spec} R$, then γ_V coincides with the right derived functor $\mathrm{R}\Gamma_V$ of the section functor Γ_V with support in V; it induces the local cohomology functors $H_V^i(-)=H^i(\mathrm{R}\Gamma_V(-))$. In [loc. cit.], we established some methods to compute γ_W for general subsets W of $\operatorname{Spec} R$. Furthermore, the local duality theorem and Grothendieck type vanishing theorem of local cohomology were extended to the case of γ_W.

On the other hand, in this paper, we introduce the notion of localization functors with cosupports in arbitrary subsets W of $\operatorname{Spec} R$. Recall that the cosupport of a complex $X\in\mathcal{D}$ is defined as

$$\operatorname{cosupp} X=\{\mathfrak{p}\in\operatorname{Spec} R\mid \mathrm{RHom}_R(\kappa(\mathfrak{p}),X)\neq 0\}.$$

We write $\mathcal{C}^W=\{X\in\mathcal{D}\mid \operatorname{cosupp} X\subseteq W\}$ for a subset W of $\operatorname{Spec} R$. Then $\mathcal{C}^W$ is a colocalizing subcategory of $\mathcal{D}$. Neeman [2011] proved that any colocalizing subcategory of $\mathcal{D}$ is obtained in this way.[1]

We remark that there are equalities

$${}^{\perp}\mathcal{C}^W=\mathcal{L}_{W^c},\qquad \mathcal{C}^W=\mathcal{L}_{W^c}^{\perp}, \tag{1.2}$$

where $W^c=\operatorname{Spec} R\setminus W$. The second equality follows from [Neeman 1992, Theorem 2.8], which states that $\mathcal{L}_{W^c}$ is equal to the smallest localizing subcategory of $\mathcal{D}$ containing the set $\{\kappa(\mathfrak{p})\mid \mathfrak{p}\in W^c\}$. Then it is seen that the first equality holds, since ${}^{\perp}(\mathcal{L}_{W^c}^{\perp})=\mathcal{L}_{W^c}$ (see [Krause 2010, §4.9]).

[1]This result is not needed in this work.

Now we write $\lambda^W = \lambda_{W^c}$ and $j^W = j_{W^c}$. By (1.1) and (1.2), there is a diagram of adjoint pairs:

$$
\begin{tikzcd}
{}^{\perp}\mathcal{C}^W = \mathcal{L}_{W^c} \arrow[r, shift left, "i_{W^c}"] & \mathcal{D} \arrow[l, shift left, "\gamma_{W^c}"] \arrow[r, shift left, "\lambda^W"] & \mathcal{C}^W = \mathcal{L}_{W^c}^{\perp} \arrow[l, shift left, "j^W"]
\end{tikzcd}
$$

We call λ^W *the localization functor with cosupport in W.*

For a multiplicatively closed subset S of R, the localization functor λ^{U_S} with cosupport in U_S is nothing but $(-) \otimes_R S^{-1}R$, where $U_S = \{\mathfrak{p} \in \operatorname{Spec} R \mid \mathfrak{p} \cap S = \varnothing\}$. Moreover, for an ideal $\mathfrak{a}$ of R, the localization functor $\lambda^{V(\mathfrak{a})}$ with cosupport in $V(\mathfrak{a})$ is isomorphic to the left derived functor $\mathrm{L}\Lambda^{V(\mathfrak{a})}$ of the $\mathfrak{a}$-adic completion functor $\Lambda^{V(\mathfrak{a})} = \varprojlim(- \otimes_R R/\mathfrak{a}^n)$ defined on $\operatorname{Mod} R$. See Section 2 for details.

In this paper, we establish several results about the localization functor λ^W with cosupport in a general subset W of $\operatorname{Spec} R$.

In Section 3, we prove that λ^W is isomorphic to $\prod_{\mathfrak{p} \in W} \mathrm{L}\Lambda^{V(\mathfrak{p})}(- \otimes_R R_{\mathfrak{p}})$ if there is no inclusion relation between two distinct prime ideals in W. Furthermore, we give a method to compute λ^W for a general subset W. We write $\eta^W : \mathrm{id}_{\mathcal{D}} \to \lambda^W$ $(= j^W\lambda^W)$ for the natural morphism given by the adjointness of (λ^W, j^W). In addition, note that when $W_0 \subseteq W$, there is a morphism $\eta^{W_0}\lambda^W : \lambda^W \to \lambda^{W_0}\lambda^W \cong \lambda^{W_0}$. The following theorem is one of the main results of this paper.

Theorem 1.3 (Theorem 3.15). *Let W, W_0 and W_1 be subsets of* $\operatorname{Spec} R$ *with $W = W_0 \cup W_1$. We denote by $\overline{W_0}^s$ (resp. $\overline{W_1}^g$) the specialization (resp. generalization) closure of W. Suppose that one of the following conditions holds*:

(1) $W_0 = \overline{W_0}^s \cap W$.

(2) $W_1 = W \cap \overline{W_1}^g$.

Then, for any $X \in \mathcal{D}$, there is a triangle

$$\lambda^W X \xrightarrow{f} \lambda^{W_1} X \oplus \lambda^{W_0} X \xrightarrow{g} \lambda^{W_1}\lambda^{W_0} X \longrightarrow \lambda^W X[1],$$

where

$$f = \begin{pmatrix} \eta^{W_1}\lambda^W X \\ \eta^{W_0}\lambda^W X \end{pmatrix}, \quad g = \begin{pmatrix} \lambda^{W_1}\eta^{W_0} X & (-1)\cdot \eta^{W_1}\lambda^{W_0} X \end{pmatrix}.$$

This theorem enables us to compute λ^W by using λ^{W_0} and λ^{W_1} for smaller subsets W_0 and W_1. Furthermore, as long as we consider the derived category $\mathcal{D}$, this theorem and Theorem 3.22 generalize Mayer–Vietoris triangles by Benson, Iyengar and Krause [Benson et al. 2008, Theorem 7.5].

In Section 4, as an application, we give a simpler proof of a classical theorem due to Gruson and Raynaud. The theorem states that the projective dimension of a flat R-module is at most the Krull dimension of R.

Section 5 contains some basic facts about cotorsion flat R-modules.

Section 6 is devoted to studying the cosupport of a complex X consisting of cotorsion flat R-modules. As a consequence, we can calculate $\gamma_{V^c}X$ and $\lambda^V X$ explicitly for a specialization-closed subset V of $\operatorname{Spec} R$.

In Section 7, using Theorem 1.3 above, we give a new way to get λ^W. In fact, provided that $d = \dim R$ is finite, we are able to calculate λ^W by a Čech complex of functors of the form

$$\prod_{0\le i\le d} \bar{\lambda}^{W_i} \longrightarrow \prod_{0\le i<j\le d} \bar{\lambda}^{W_j}\bar{\lambda}^{W_i} \longrightarrow \cdots \longrightarrow \bar{\lambda}^{W_d}\cdots\bar{\lambda}^{W_0},$$

where $W_i = \{\mathfrak{p} \in W \mid \dim R/\mathfrak{p} = i\}$ and $\bar{\lambda}^{W_i} = \prod_{\mathfrak{p}\in W_i} \Lambda^{V(\mathfrak{p})}(- \otimes_R R_{\mathfrak{p}})$ for $0 \le i \le d$. This Čech complex sends a complex X of R-modules to a double complex in a natural way. We shall prove that $\lambda^W X$ is isomorphic to the total complex of the double complex if X consists of flat R-modules.

Section 8 treats commutativity of λ^W with tensor products. Consequently, we show that $\lambda^W Y$ can be computed by using the Čech complex above if Y is a complex of finitely generated R-modules.

In Section 9, as an application, we give a functorial way to construct quasi-isomorphisms from complexes of flat R-modules, or complexes of finitely generated R-modules to complexes of pure-injective R-modules.

2. Localization functors

In this section, we summarize some notions and basic facts used in the later sections.

We write $\operatorname{Mod} R$ for the category of all modules over a commutative Noetherian ring R. For an ideal $\mathfrak{a}$ of R, $\Lambda^{V(\mathfrak{a})}$ denotes the $\mathfrak{a}$-adic completion functor $\varprojlim(- \otimes_R R/\mathfrak{a}^n)$ defined on $\operatorname{Mod} R$. Moreover, we also denote by $M_{\mathfrak{a}}^{\wedge}$ the $\mathfrak{a}$-adic completion $\Lambda^{V(\mathfrak{a})} M = \varprojlim M/\mathfrak{a}^n M$ of an R-module M. If the natural map $M \to M_{\mathfrak{a}}^{\wedge}$ is an isomorphism, then M is called $\mathfrak{a}$-adically complete. In addition, when R is a local ring with maximal ideal $\mathfrak{m}$, we simply write $\widehat{M}$ for the $\mathfrak{m}$-adic completion of M.

We start with the following proposition.

Proposition 2.1. *Let $\mathfrak{a}$ be an ideal of R. If F is a flat R-module, then so is $F_{\mathfrak{a}}^{\wedge}$.*

As stated in [Simon 1990, 2.4], this fact is known. For the reader's convenience, we mention that this proposition follows from the two lemmas below.

Lemma 2.2. *Let $\mathfrak{a}$ be an ideal of R and F be a flat R-module. We consider a short exact sequence of finitely generated R-modules*

$$0 \to L \to M \to N \to 0.$$

Then

$$0 \to (F \otimes_R L)_{\mathfrak{a}}^{\wedge} \to (F \otimes_R M)_{\mathfrak{a}}^{\wedge} \to (F \otimes_R N)_{\mathfrak{a}}^{\wedge} \to 0$$

is exact.

Lemma 2.3. *Let* $\mathfrak{a}$ *and* F *be as above. Then we have a natural isomorphism,*

$$(F \otimes_R M)^{\wedge}_{\mathfrak{a}} \cong F^{\wedge}_{\mathfrak{a}} \otimes_R M,$$

for any finitely generated R*-module* M*.*

Using the Artin–Rees lemma and [Bourbaki 1961, Chap. I, §2.6, Proposition 6], we can prove Lemma 2.2, from which we obtain Lemma 2.3. Furthermore, Lemmas 2.2 and 2.3 imply that $F^{\wedge}_{\mathfrak{a}} \otimes_R (-)$ is an exact functor from the category of finitely generated R-modules to $\operatorname{Mod} R$. Therefore Proposition 2.1 holds. □

It is also possible to show that $F^{\wedge}_{\mathfrak{a}}$ is flat over $R^{\wedge}_{\mathfrak{a}}$ by the same argument as above.

If R is a local ring with maximal ideal $\mathfrak{m}$, then $\mathfrak{m}$-adically complete flat R-modules are characterized as follows:

Lemma 2.4. *Let* $(R, \mathfrak{m}, k)$ *be a local ring and* F *a flat* R*-module. Set* $B = \dim_k F/\mathfrak{m}F$*. Then there is an isomorphism*

$$\widehat{F} \cong \widehat{\bigoplus_B R},$$

where $\bigoplus_B R$ *is the direct sum of* B*-copies of* R*.*

This lemma is proved in [Raynaud and Gruson 1971, Part. II, Proposition 2.4.3.1]. See also [Enochs and Jenda 2000, Lemma 6.7.4].

As in the introduction, we denote by $\mathcal{D} = D(\operatorname{Mod} R)$ the derived category of all complexes of R-modules. We write complexes X cohomologically:

$$X = (\cdots \to X^{i-1} \to X^i \to X^{i+1} \to \cdots).$$

For a complex P of R-modules, we say that P is K-projective if $\operatorname{Hom}_R(P, -)$ preserves acyclicity of complexes, where a complex is called acyclic if all its cohomology modules are zero. Similarly, for a complex F of R-modules, we say that F is K-flat if $(-) \otimes_R F$ preserves acyclicity of complexes.

Let $\mathfrak{a}$ be an ideal of R and $X \in \mathcal{D}$. If P is a K-projective resolution of X, then we have $\mathrm{L}\Lambda^{V(\mathfrak{a})} X \cong \Lambda^{V(\mathfrak{a})} P$. Moreover, $\mathrm{L}\Lambda^{V(\mathfrak{a})} X$ is also isomorphic to $\Lambda^{V(\mathfrak{a})} F$ if F is a K-flat resolution of X. Further, it is known that the following proposition holds.

Proposition 2.5. *Let* $\mathfrak{a}$ *be an ideal of* R *and* X *be a complex of flat* R*-modules. Then* $\mathrm{L}\Lambda^{V(\mathfrak{a})} X$ *is isomorphic to* $\Lambda^{V(\mathfrak{a})} X$*.*

Proof. To show this, we note there is an integer $n \geq 0$ such that $H^i(\mathrm{L}\Lambda^{V(\mathfrak{a})} M) = 0$ for all $i > n$ and all R-modules M, see [Greenlees and May 1992, Theorem 1.9] or [Alonso Tarrío et al. 1997, p. 15]. Using this fact, we can show that $\Lambda^{V(\mathfrak{a})}$ preserves acyclicity of complexes of flat R-modules. Then it is straightforward to see that $\mathrm{L}\Lambda^{V(\mathfrak{a})} X$ is isomorphic to $\Lambda^{V(\mathfrak{a})} X$. □

Let W be any subset of $\operatorname{Spec} R$. Recall that γ_W denotes a right adjoint to the inclusion functor $i_W : \mathcal{L}_W \hookrightarrow \mathcal{D}$, and λ^W denotes a left adjoint to the inclusion functor $j^W : \mathcal{C}^W \hookrightarrow \mathcal{D}$. Moreover, γ_W and λ^W are identified with $i_W\gamma_W$ and $j^W\lambda^W$, respectively. We write $\varepsilon_W : \gamma_W \to \mathrm{id}_{\mathcal{D}}$ and $\eta^W : \mathrm{id}_{\mathcal{D}} \to \lambda^W$ for the natural morphisms induced by the adjointness of (i_W, γ_W) and (λ^W, j^W), respectively.

Note that $\lambda^W\eta^W$ (resp. $\gamma_W\varepsilon_W$) is invertible, and the equality $\lambda^W\eta^W = \eta^W\lambda^W$ (resp. $\gamma_W\varepsilon_W = \varepsilon_W\gamma_W$) holds, i.e., λ^W (resp. γ_W) is a localization (resp. colocalization) functor on $\mathcal{D}$. See [Krause 2010] for more details. In this paper, we call λ^W the localization functor with cosupport in W.

Using (1.2), we restate [Nakamura and Yoshino 2018, Lemma 2.1] as follows.

Lemma 2.6. *Let W be a subset of* $\operatorname{Spec} R$. *For any $X \in \mathcal{D}$, there is a triangle of the following form*:

$$\gamma_{W^c}X \xrightarrow{\varepsilon_{W^c}X} X \xrightarrow{\eta^W X} \lambda^W X \longrightarrow \gamma_{W^c}X[1].$$

Furthermore, *if*

$$X' \longrightarrow X \longrightarrow X'' \longrightarrow X'[1]$$

is a triangle with $X' \in {}^{\perp}\mathcal{C}^W = \mathcal{L}_{W^c}$ and $X'' \in \mathcal{C}^W = \mathcal{L}_{W^c}^{\perp}$, then there exist unique isomorphisms $a : \gamma_{W^c}X \to X'$ and $b : \lambda^W X \to X''$ such that the following diagram is commutative:

$$\begin{tikzcd}
\gamma_{W^c}X \arrow[r, "\varepsilon_{W^c}X"] \arrow[d, "a"] & X \arrow[r, "\eta^W X"] \arrow[d, equal] & \lambda^W X \arrow[r] \arrow[d, "b"] & \gamma_{W^c}X[1] \arrow[d, "{a[1]}"] \\
X' \arrow[r] & X \arrow[r] & X'' \arrow[r] & X'[1]
\end{tikzcd}$$

Remark 2.7. (i) Let $X \in \mathcal{D}$ and W be a subset of $\operatorname{Spec} R$. By Lemma 2.6, X belongs to ${}^{\perp}\mathcal{C}^W = \mathcal{L}_{W^c}$ if and only if $\lambda^W X = 0$. This is equivalent to saying that $\lambda^{\{\mathfrak{p}\}}X = 0$ for all $\mathfrak{p} \in W$, since ${}^{\perp}\mathcal{C}^W = \mathcal{L}_{W^c} = \bigcap_{\mathfrak{p}\in W} \mathcal{L}_{\{\mathfrak{p}\}^c} = \bigcap_{\mathfrak{p}\in W} {}^{\perp}\mathcal{C}^{\{\mathfrak{p}\}}$.

(ii) Let W_0 and W be subsets of $\operatorname{Spec} R$ with $W_0 \subseteq W$. It follows from the uniqueness of adjoint functors that

$$\lambda^{W_0}\lambda^W \cong \lambda^{W_0} \cong \lambda^W\lambda^{W_0};$$

see also [Nakamura and Yoshino 2018, Remark 3.7(i)].

Now we give a typical example of localization functors. Let S be a multiplicatively closed subset S of R, and set $U_S = \{\mathfrak{p} \in \operatorname{Spec} R \mid \mathfrak{p} \cap S = \varnothing\}$. It is known that the localization functor λ^{U_S} with cosupport in U_S is nothing but $(-) \otimes_R S^{-1}R$. For the reader's convenience, we give a proof of this fact. Let $X \in \mathcal{D}$. It is clear that $\operatorname{cosupp} X \otimes_R S^{-1}R \subseteq U_S$, or equivalently, $X \otimes_R S^{-1}R \in \mathcal{C}^{U_S}$. Moreover, embedding the natural morphism $X \to X \otimes_R S^{-1}R$ into a triangle,

$$C \longrightarrow X \longrightarrow X \otimes_R S^{-1}R \longrightarrow C[1],$$

we have $C \otimes_R S^{-1}R = 0$. This yields an inclusion relation $\operatorname{supp} C \subseteq (U_S)^c$. Hence it holds that $C \in \mathcal{L}_{(U_S)^c}$. Since we have shown that $C \in \mathcal{L}_{(U_S)^c}$ and $X \otimes_R S^{-1}R \in \mathcal{C}^{U_S}$, it follows from Lemma 2.6 that $\lambda^{U_S} X \cong X \otimes_R S^{-1}R$. Therefore we obtain the isomorphism

$$\lambda^{U_S} \cong (-) \otimes_R S^{-1}R. \tag{2.8}$$

For $\mathfrak{p} \in \operatorname{Spec} R$, we write $U(\mathfrak{p}) = \{\mathfrak{q} \in \operatorname{Spec} R \mid \mathfrak{q} \subseteq \mathfrak{p}\}$. If $S = R \setminus \mathfrak{p}$, then $U(\mathfrak{p})$ is equal to U_S, so that $\lambda^{U(\mathfrak{p})} \cong (-) \otimes_R R_{\mathfrak{p}}$ by (2.8). We remark that $\lambda^{U(\mathfrak{p})} = \lambda_{U(\mathfrak{p})^c}$ is written as $L_{Z(\mathfrak{p})}$ in [Benson et al. 2008], where $Z(\mathfrak{p}) = U(\mathfrak{p})^c$.

There is another important example of localization functors. Let $\mathfrak{a}$ be an ideal of R. It was proved by [Greenlees and May 1992] and [Alonso Tarrío et al. 1997] that $\mathrm{L}\Lambda^{V(\mathfrak{a})} : \mathcal{D} \to \mathcal{D}$ is a right adjoint to $\mathrm{R}\Gamma_{V(\mathfrak{a})} : \mathcal{D} \to \mathcal{D}$. In [Nakamura and Yoshino 2018, Proposition 5.1], using the adjointness property of $(\mathrm{R}\Gamma_{V(\mathfrak{a})}, \mathrm{L}\Lambda^{V(\mathfrak{a})})$, we proved that $\lambda^{V(\mathfrak{a})} = \lambda_{V(\mathfrak{a})^c}$ coincides with $\mathrm{L}\Lambda^{V(\mathfrak{a})}$. Hence there is an isomorphism

$$\lambda^{V(\mathfrak{a})} \cong \mathrm{L}\Lambda^{V(\mathfrak{a})}. \tag{2.9}$$

The functor $H_i^{\mathfrak{a}}(-) = H^{-i}(\mathrm{L}\Lambda^{V(\mathfrak{a})}(-))$ is called the i-th local homology functor with respect to $\mathfrak{a}$.

A subset W of $\operatorname{Spec} R$ is said to be specialization-closed (resp. generalization-closed) provided that the following condition holds: if $\mathfrak{p} \in W$ and $\mathfrak{q} \in \operatorname{Spec} R$ with $\mathfrak{p} \subseteq \mathfrak{q}$ (resp. $\mathfrak{p} \supseteq \mathfrak{q}$), then $\mathfrak{q} \in W$.

If V is a specialization-closed subset, then we have

$$\gamma_V \cong \mathrm{R}\Gamma_V; \tag{2.10}$$

see [Lipman 2002, Appendix 3.5].

3. Auxiliary results on localization functors

In this section, we give several results to compute localization functors λ^W with cosupports in arbitrary subsets W of $\operatorname{Spec} R$.

We first give the following lemma.

Lemma 3.1. *Let V be a specialization-closed subset of* Spec R. *Then we have the following equalities*;

$$^{\perp}\mathcal{C}^V = \mathcal{L}_{V^c} = \mathcal{L}_V^{\perp} = \mathcal{C}^{V^c}.$$

Proof. This follows from [Nakamura and Yoshino 2018, Lemma 4.3] and (1.2). □

Let W be a subset of $\operatorname{Spec} R$. We denote by $\overline{W}^s$ the specialization closure of W, which is the smallest specialization-closed subset of $\operatorname{Spec} R$ containing W. Moreover, for a subset W_0 of W, we say that W_0 is specialization-closed in W if $V(\mathfrak{p}) \cap W \subseteq W_0$ for any $\mathfrak{p} \in W_0$ (see [Nakamura and Yoshino 2018, Definition 3.10]). This is equivalent to saying that $\overline{W_0}^s \cap W = W_0$.

Corollary 3.2. *Let $W_0 \subseteq W \subseteq \operatorname{Spec} R$ be sets. Suppose that W_0 is specialization-closed in W. Setting $W_1 = W \setminus W_0$, we have $\mathcal{C}^{W_1} \subseteq {}^{\perp}\mathcal{C}^{W_0}$.*

Proof. Note that $W_1 \subseteq (\overline{W_0}^s)^c$. Further, we have ${}^{\perp}\mathcal{C}^{\overline{W_0}^s} = \mathcal{C}^{(\overline{W_0}^s)^c}$ by Lemma 3.1. Hence it holds that $\mathcal{C}^{W_1} \subseteq \mathcal{C}^{(\overline{W_0}^s)^c} = {}^{\perp}\mathcal{C}^{\overline{W_0}^s} \subseteq {}^{\perp}\mathcal{C}^{W_0}$. □

Remark 3.3. For an ideal $\mathfrak{a}$ of R, $\lambda^{V(\mathfrak{a})}$ is a right adjoint to $\gamma_{V(\mathfrak{a})}$ by (2.9) and (2.10). More generally, it is known that for any specialization-closed subset V, $\lambda^V : \mathcal{D} \to \mathcal{D}$ is a right adjoint to $\gamma_V : \mathcal{D} \to \mathcal{D}$. We now prove this fact, which will be used in the next proposition. Let $X, Y \in \mathcal{D}$, and consider the following triangles:

$$\gamma_V X \longrightarrow X \longrightarrow \lambda^{V^c} X \longrightarrow \gamma_V X[1],$$
$$\gamma_{V^c} Y \longrightarrow Y \longrightarrow \lambda^V Y \longrightarrow \gamma_{V^c} Y[1].$$

Since $\lambda^{V^c} X \in \mathcal{C}^{V^c} = {}^{\perp}\mathcal{C}^V$ by Lemma 3.1, applying $\operatorname{Hom}_{\mathcal{D}}(-, \lambda^V Y)$ to the first triangle, we have $\operatorname{Hom}_{\mathcal{D}}(\gamma_V X, \lambda^V Y) \cong \operatorname{Hom}_{\mathcal{D}}(X, \lambda^V Y)$. Moreover, Lemma 3.1 implies that $\gamma_{V^c} Y \in \mathcal{L}_{V^c} = \mathcal{L}_V^{\perp}$. Hence, applying $\operatorname{Hom}_{\mathcal{D}}(\gamma_V X, -)$ to the second triangle, we have $\operatorname{Hom}_{\mathcal{D}}(\gamma_V X, Y) \cong \operatorname{Hom}_{\mathcal{D}}(\gamma_V X, \lambda^V Y)$. Thus there is a natural isomorphism $\operatorname{Hom}_{\mathcal{D}}(\gamma_V X, Y) \cong \operatorname{Hom}_{\mathcal{D}}(X, \lambda^V Y)$, so that (γ_V, λ^V) is an adjoint pair. See also [Nakamura and Yoshino 2018, Remark 5.2].

Proposition 3.4. *Let V and U be arbitrary subsets of* $\operatorname{Spec} R$. *Suppose that one of the following conditions holds*:

(1) *V is specialization-closed.*

(2) *U is generalization-closed.*

Then we have an isomorphism

$$\lambda^V \lambda^U \cong \lambda^{V \cap U}.$$

Proof. Let $X \in \mathcal{D}$ and $Y \in \mathcal{C}^{V \cap U} = \mathcal{C}^V \cap \mathcal{C}^U$. Then there are natural isomorphisms

$$\operatorname{Hom}_{\mathcal{D}}(\lambda^V \lambda^U X, Y) \cong \operatorname{Hom}_{\mathcal{D}}(\lambda^U X, Y) \cong \operatorname{Hom}_{\mathcal{D}}(X, Y).$$

Recall that $\lambda^{V \cap U}$ is a left adjoint to the inclusion functor $\mathcal{C}^{V \cap U} \hookrightarrow \mathcal{D}$. Hence, by the uniqueness of adjoint functors, we only have to verify that $\lambda^V \lambda^U X \in \mathcal{C}^{V \cap U}$. Since $\lambda^V \lambda^U X \in \mathcal{C}^V$, it remains to show that $\lambda^V \lambda^U X \in \mathcal{C}^U$.

Case 1: Let $\mathfrak{p} \in U^c$. Since $\operatorname{supp} \gamma_V \kappa(\mathfrak{p}) \subseteq \{\mathfrak{p}\}$, it follows from (1.2) that $\gamma_V \kappa(\mathfrak{p}) \in \mathcal{L}_{U^c} = {}^{\perp}\mathcal{C}^U$. Thus, by the adjointness of (γ_V, λ^V), we have

$$\operatorname{RHom}_R(\kappa(\mathfrak{p}), \lambda^V \lambda^U X) \cong \operatorname{RHom}_R(\gamma_V \kappa(\mathfrak{p}), \lambda^U X) = 0.$$

This implies that $\operatorname{cosupp} \lambda^V \lambda^U X \subseteq U$, i.e., $\lambda^V \lambda^U X \in \mathcal{C}^U$.

Case 2: Since U^c is specialization-closed, Case 1 yields an isomorphism $\lambda^{U^c}\lambda^V \cong \lambda^{U^c\cap V}$. Furthermore, setting $W=(U^c\cap V)\cup U$, we see that $U^c\cap V$ is specialization-closed in W, and $W\setminus(U^c\cap V)=U$. Hence we have $\lambda^{U^c}(\lambda^V\lambda^U X)\cong\lambda^{U^c\cap V}\lambda^U X=0$, by Corollary 3.2. It then follows from Lemma 3.1 that $\lambda^V\lambda^U X\in{}^{\perp}\mathcal{C}^{U^c}=\mathcal{C}^U$. □

Remark 3.5. For arbitrary subsets W_0 and W_1 of $\operatorname{Spec} R$, Remark 2.7(ii) and Proposition 3.4 yield the isomorphisms

$$\lambda^{W_0}\lambda^{W_1}\cong\lambda^{W_0}\lambda^{\overline{W_0}^s}\lambda^{W_1}\cong\lambda^{W_0}\lambda^{\overline{W_0}^s\cap W_1},$$

$$\lambda^{W_0}\lambda^{W_1}\cong\lambda^{W_0}\lambda^{\overline{W_1}^g}\lambda^{W_1}\cong\lambda^{W_0\cap\overline{W_1}^g}\lambda^{W_1}.$$

The next result is a corollary of (2.8), (2.9) and Proposition 3.4.

Corollary 3.6. *Let S be a multiplicatively closed subset of R and $\mathfrak{a}$ be an ideal of R. We set $W=V(\mathfrak{a})\cap U_S$. Then we have*

$$\lambda^W\cong \mathrm{L}\Lambda^{V(\mathfrak{a})}(-\otimes_R S^{-1}R).$$

Since $V(\mathfrak{p})\cap U(\mathfrak{p})=\{\mathfrak{p}\}$ for $\mathfrak{p}\in\operatorname{Spec} R$, as a special case of this corollary, we have the following result.

Corollary 3.7. *Let $\mathfrak{p}$ be a prime ideal of R. Then we have*

$$\lambda^{\{\mathfrak{p}\}}\cong \mathrm{L}\Lambda^{V(\mathfrak{p})}(-\otimes_R R_{\mathfrak{p}}).$$

The next lemma follows from this corollary and Lemma 2.4.

Lemma 3.8. *Let $\mathfrak{p}$ be a prime ideal of R and F be a flat R-module. Then $\lambda^{\{\mathfrak{p}\}}F$ is isomorphic to $\left(\bigoplus_B R_{\mathfrak{p}}\right)^{\wedge}_{\mathfrak{p}}$, where $\bigoplus_B R_{\mathfrak{p}}$ is the direct sum of B-copies of $R_{\mathfrak{p}}$ and $B=\dim_{\kappa(\mathfrak{p})} F\otimes_R\kappa(\mathfrak{p})$.*

Remark 3.9. If W_1 and W_2 are both specialization-closed or both generalization-closed, then Proposition 3.4 implies that $\lambda^{W_1}\lambda^{W_2}\cong\lambda^{W_2}\lambda^{W_1}$. However, in general, λ^{W_1} and λ^{W_2} need not commute. For example, let $\mathfrak{p},\mathfrak{q}\in\operatorname{Spec} R$ with $\mathfrak{p}\subsetneq\mathfrak{q}$. Then $(\lambda^{\{\mathfrak{p}\}}R)\otimes_R\kappa(\mathfrak{q})=\widehat{R_{\mathfrak{p}}}\otimes_R\kappa(\mathfrak{q})=0$ and $(\lambda^{\{\mathfrak{q}\}}R)\otimes_R\kappa(\mathfrak{p})=\widehat{R_{\mathfrak{q}}}\otimes_R\kappa(\mathfrak{p})\neq 0$. Then we see from Lemma 3.8 that $\lambda^{\{\mathfrak{q}\}}\lambda^{\{\mathfrak{p}\}}R=0$ and $\lambda^{\{\mathfrak{p}\}}\lambda^{\{\mathfrak{q}\}}R\neq 0$.

Compare this remark with [Benson et al. 2008, Example 3.5]. See also [Nakamura and Yoshino 2018, Remark 3.7(ii)].

Let $\mathfrak{p}$ be a prime ideal which is not maximal. Then $\lambda^{\{\mathfrak{p}\}}$ is distinct from $\varLambda^{\mathfrak{p}}=\mathrm{L}\Lambda^{V(\mathfrak{p})}\operatorname{RHom}_R(R_{\mathfrak{p}},-)$, which is introduced in [Benson et al. 2012]. To see this, let $\mathfrak{q}$ be a prime ideal with $\mathfrak{p}\subsetneq\mathfrak{q}$. Then it holds that $\operatorname{cosupp}\widehat{R_{\mathfrak{q}}}=\{\mathfrak{q}\}\subseteq U(\mathfrak{p})^c$. Hence $\widehat{R_{\mathfrak{q}}}$ belongs to $\mathcal{C}^{U(\mathfrak{p})^c}$. Then we have $\operatorname{RHom}_R(R_{\mathfrak{p}},\widehat{R_{\mathfrak{q}}})=0$ since $R_{\mathfrak{p}}\in\mathcal{L}_{U(\mathfrak{p})}={}^{\perp}\mathcal{C}^{U(\mathfrak{p})^c}$ by (1.2). This implies that $\varLambda^{\mathfrak{p}}\widehat{R_{\mathfrak{q}}}=\mathrm{L}\Lambda^{V(\mathfrak{p})}\operatorname{RHom}_R(R_{\mathfrak{p}},\widehat{R_{\mathfrak{q}}})=0$, while $\lambda^{\{\mathfrak{p}\}}\widehat{R_{\mathfrak{q}}}\cong\lambda^{\{\mathfrak{p}\}}\lambda^{\{\mathfrak{q}\}}R\neq 0$ by Remark 3.9.

Let $X \in \mathcal{D}$, and write $\varGamma_{\mathfrak{p}} = \mathrm{R}\Gamma_{V(\mathfrak{p})}(- \otimes_R R_{\mathfrak{p}})$ (see [Benson et al. 2008]). Recall that $\mathfrak{p} \in \operatorname{supp} X$ (resp. $\mathfrak{p} \in \operatorname{cosupp} X$) if and only if $\varGamma_{\mathfrak{p}} X \neq 0$ (resp. $\varLambda^{\mathfrak{p}} X \neq 0$); see [Foxby and Iyengar 2003, Theorems 2.1 and 4.1] and [Benson et al. 2012, §4]. In contrast, $\mathfrak{p} \in \operatorname{cosupp} X$ (resp. $\mathfrak{p} \in \operatorname{supp} X$) if and only if $\gamma_{\{\mathfrak{p}\}} X \neq 0$ (resp. $\lambda^{\{\mathfrak{p}\}} X \neq 0$), by Lemma 2.6. Here, $\gamma_{\{\mathfrak{p}\}} \cong \mathrm{R}\Gamma_{V(\mathfrak{p})} \operatorname{RHom}_R(R_{\mathfrak{p}}, -)$ by [Nakamura and Yoshino 2018, Corollary 3.3]. See also [Sather-Wagstaff and Wicklein 2017, Propositions 3.6 and 4.4].

Let W be a subset of $\operatorname{Spec} R$. We denote by $\dim W$ the supremum of lengths of chains of distinct prime ideals in W (see [Nakamura and Yoshino 2018, Definition 3.6]).

Theorem 3.10. *Let W be a subset of* $\operatorname{Spec} R$. *We assume that* $\dim W = 0$. *Then there are isomorphisms*

$$\lambda^W \cong \prod_{\mathfrak{p} \in W} \lambda^{\{\mathfrak{p}\}} \cong \prod_{\mathfrak{p} \in W} \mathrm{L}\Lambda^{V(\mathfrak{p})}(- \otimes_R R_{\mathfrak{p}}).$$

Proof. Let $X \in \mathcal{D}$, and consider the natural morphisms $\eta^{\{\mathfrak{p}\}} X : X \to \lambda^{\{\mathfrak{p}\}} X$ for $\mathfrak{p} \in W$. Take the product of the morphisms, and we obtain a morphism $f : X \to \prod_{\mathfrak{p} \in W} \lambda^{\{\mathfrak{p}\}} X$. Embed f into a triangle

$$C \longrightarrow X \xrightarrow{\ f\ } \prod_{\mathfrak{p} \in W} \lambda^{\{\mathfrak{p}\}} X \longrightarrow C[1].$$

Note that $\prod_{\mathfrak{p} \in W} \lambda^{\{\mathfrak{p}\}} X \in \mathcal{C}^W$. We have to prove that $C \in {}^{\perp}\mathcal{C}^W$. For this purpose, take any prime ideal $\mathfrak{q} \in W$. Then $\{\mathfrak{q}\}$ is specialization-closed in W, because $\dim W = 0$. Hence we have

$$\prod_{\mathfrak{p} \in W \setminus \{\mathfrak{q}\}} \lambda^{\{\mathfrak{p}\}} X \in \mathcal{C}^{W \setminus \{\mathfrak{q}\}} \subseteq {}^{\perp}\mathcal{C}^{\{\mathfrak{q}\}},$$

by Corollary 3.2. Thus an isomorphism $\lambda^{\{\mathfrak{q}\}}\big(\prod_{\mathfrak{p} \in W} \lambda^{\{\mathfrak{p}\}} X\big) \cong \lambda^{\{\mathfrak{q}\}} X$ holds. Then it is seen from the triangle above that $\lambda^{\{\mathfrak{q}\}} C = 0$ for all $\mathfrak{q} \in W$, so that $C \in {}^{\perp}\mathcal{C}^W$; see Remark 2.7(i). Therefore Lemma 2.6 yields $\lambda^W X \cong \prod_{\mathfrak{p} \in W} \lambda^{\{\mathfrak{p}\}} X$. The second isomorphism in the theorem follows from Corollary 3.7. ☐

Example 3.11. Let W be a subset of $\operatorname{Spec} R$ such that W is an infinite set with $\dim W = 0$. Let $X^{\{\mathfrak{p}\}}$ be a complex with $\operatorname{cosupp} X^{\{\mathfrak{p}\}} = \{\mathfrak{p}\}$ for each $\mathfrak{p} \in W$. We take $\mathfrak{p} \in W$. Since $\dim W = 0$, it holds that $X^{\{\mathfrak{q}\}} \in \mathcal{C}^{V(\mathfrak{p})^c}$ for any $\mathfrak{q} \in W \setminus \{\mathfrak{p}\}$. Furthermore, Lemma 3.1 implies that $\mathcal{C}^{V(\mathfrak{p})^c}$ is equal to ${}^{\perp}\mathcal{C}^{V(\mathfrak{p})}$, which is closed under arbitrary direct sums. Thus it holds that

$$\bigoplus_{\mathfrak{q} \in W \setminus \{\mathfrak{p}\}} X^{\{\mathfrak{q}\}} \in \mathcal{C}^{V(\mathfrak{p})^c} = {}^{\perp}\mathcal{C}^{V(\mathfrak{p})} \subseteq {}^{\perp}\mathcal{C}^{\{\mathfrak{p}\}}.$$

Therefore, setting $Y = \bigoplus_{\mathfrak{p}\in W} X^{\{\mathfrak{p}\}}$, we have $\lambda^{\{\mathfrak{p}\}} Y \cong X^{\{\mathfrak{p}\}}$. It then follows from Theorem 3.10 that

$$\lambda^W Y \cong \prod_{\mathfrak{p}\in W} \lambda^{\{\mathfrak{p}\}} Y \cong \prod_{\mathfrak{p}\in W} X^{\{\mathfrak{p}\}}.$$

Under this identification, the natural morphism $Y \to \lambda^W Y$ coincides with the canonical morphism $\bigoplus_{\mathfrak{p}\in W} X^{\{\mathfrak{p}\}} \to \prod_{\mathfrak{p}\in W} X^{\{\mathfrak{p}\}}$.

Remark 3.12. Let W, $X^{\{\mathfrak{p}\}}$ be as in Example 3.11, and suppose that each $X^{\{\mathfrak{p}\}}$ is an R-module. Then $\bigoplus_{\mathfrak{p}\in W} X^{\{\mathfrak{p}\}}$ is not in $\mathcal{C}^W$, because the natural morphism $\bigoplus_{\mathfrak{p}\in W} X^{\{\mathfrak{p}\}} \to \lambda^W\big(\bigoplus_{\mathfrak{p}\in W} X^{\{\mathfrak{p}\}}\big)$ is not an isomorphism. Hence the cosupport of $\bigoplus_{\mathfrak{p}\in W} X^{\{\mathfrak{p}\}}$ properly contains W. In particular, setting $X^{\{\mathfrak{p}\}} = \kappa(\mathfrak{p})$, we have $W \subsetneq \operatorname{cosupp} \bigoplus_{\mathfrak{p}\in W} \kappa(\mathfrak{p})$. Similarly, we can prove that $W \subsetneq \operatorname{supp} \prod_{\mathfrak{p}\in W} \kappa(\mathfrak{p})$. Nakamura noticed these facts through discussion with Srikanth Iyengar.

It is possible to give another type of example, by which we also see that a colocalizing subcategory of $\mathcal{D}$ is not necessarily closed under arbitrary direct sums. Suppose that $(R, \mathfrak{m})$ is a complete local ring with $\dim R \geq 1$. Then we have $R \cong \widehat{R} \in \mathcal{C}^{V(\mathfrak{m})}$. However the free module $\bigoplus_{\mathbb{N}} R$ is never $\mathfrak{m}$-adically complete, so that $\bigoplus_{\mathbb{N}} R$ is not isomorphic to $\lambda^{V(\mathfrak{m})}\big(\bigoplus_{\mathbb{N}} R\big)$. Hence $\bigoplus_{\mathbb{N}} R$ is not in $\mathcal{C}^{V(\mathfrak{m})}$.

For a subset W of $\operatorname{Spec} R$, $\overline{W}^g$ denotes the generalization closure of W, which is the smallest generalization-closed subset of $\operatorname{Spec} R$ containing W. In addition, for a subset $W_1 \subseteq W$, we say that W_1 is generalization-closed in W if $W \cap U(\mathfrak{p}) \subseteq W_1$ for any $\mathfrak{p} \in W_1$. This is equivalent to saying that $W \cap \overline{W_1}^g = W_1$.

We extend Proposition 3.4 to the following corollary, which will be used in Theorem 3.15.

Corollary 3.13. *Let W_0 and W_1 be arbitrary subsets of* $\operatorname{Spec} R$. *Suppose that one of the following conditions hold*:

(1) *W_0 is specialization-closed in $W_0 \cup W_1$.*

(2) *W_1 is generalization-closed in $W_0 \cup W_1$.*

Then we have an isomorphism

$$\lambda^{W_0}\lambda^{W_1} \cong \lambda^{W_0 \cap W_1}.$$

Proof. Set $W = W_0 \cup W_1$. By the assumption, we have

$$\overline{W_0}^s \cap W = W_0 \;\text{ or }\; W \cap \overline{W_1}^g = W_1.$$

Therefore, it holds that

$$\overline{W_0}^s \cap W_1 = W_0 \cap W_1 \;\text{ or }\; W_0 \cap \overline{W_1}^g = W_0 \cap W_1.$$

Hence this proposition follows from Remark 3.5 and Remark 2.7(ii). $\square$

Remark 3.14. (i) Let W_0 and W be subsets of $\operatorname{Spec} R$ with $W_0 \subseteq W$. Under the isomorphism $\lambda^{W_0}\lambda^W \cong \lambda^{W_0}$ by Remark 2.7(ii), there is a morphism $\eta^{W_0}\lambda^W : \lambda^W \to \lambda^{W_0}$.

(ii) Let W_0 and W_1 be subsets of $\operatorname{Spec} R$. Let $X \in \mathcal{D}$. Since $\eta^{W_1} : \mathrm{id}_{\mathcal{D}} \to \lambda^{W_1}$ is a morphism of functors, there is a commutative diagram of the following form:

$$\begin{tikzcd} X \arrow[r, "\eta^{W_0}X"] \arrow[d, "\eta^{W_1}X"] & \lambda^{W_0}X \arrow[d, "\eta^{W_1}\lambda^{W_0}X"] \\ \lambda^{W_1}X \arrow[r, "\lambda^{W_1}\eta^{W_0}X"] & \lambda^{W_1}\lambda^{W_0}X \end{tikzcd}$$

Now we prove the following result, which is the main theorem of this section.

Theorem 3.15. *Let W, W_0 and W_1 be subsets of* $\operatorname{Spec} R$ *with $W = W_0 \cup W_1$. Suppose that one of the following conditions holds*:

(1) *W_0 is specialization-closed in W.*

(2) *W_1 is generalization-closed in W.*

Then, for any $X \in \mathcal{D}$, there is a triangle of the form

$$\lambda^W X \xrightarrow{f} \lambda^{W_1}X \oplus \lambda^{W_0}X \xrightarrow{g} \lambda^{W_1}\lambda^{W_0}X \longrightarrow \lambda^W X[1],$$

where f and g are morphisms represented by the matrices

$$f = \begin{pmatrix} \eta^{W_1}\lambda^W X \\ \eta^{W_0}\lambda^W X \end{pmatrix}, \qquad g = \begin{pmatrix} \lambda^{W_1}\eta^{W_0}X & (-1)\cdot\eta^{W_1}\lambda^{W_0}X \end{pmatrix}.$$

Proof. We embed the morphism g into a triangle

$$C \xrightarrow{a} \lambda^{W_1}X \oplus \lambda^{W_0}X \xrightarrow{g} \lambda^{W_1}\lambda^{W_0}X \longrightarrow C[1].$$

Notice that $C \in \mathcal{C}^W$ since $\mathcal{C}^{W_0}, \mathcal{C}^{W_1} \subseteq \mathcal{C}^W$. By Remark 3.14, it is easily seen that $g \cdot f = 0$. Thus there is a morphism $b : \lambda^W X \to C$ making the following diagram commutative:

$$\begin{tikzcd} \lambda^W X \arrow[r, equal] \arrow[d, "b"] & \lambda^W X \arrow[r] \arrow[d, "f"] & 0 \arrow[r] \arrow[d] & \lambda^W X[1] \arrow[d, "b[1]"] \\ C \arrow[r, "a"] & \lambda^{W_1}X \oplus \lambda^{W_0}X \arrow[r, "g"] & \lambda^{W_1}\lambda^{W_0}X \arrow[r] & C[1] \end{tikzcd} \tag{3.16}$$

We only have to show that b is an isomorphism. To do this, embedding the morphism b into a triangle

$$Z \longrightarrow \lambda^W X \xrightarrow{b} C \longrightarrow Z[1], \tag{3.17}$$

we prove that $Z = 0$. Since $\lambda^W X, C \in \mathcal{C}^W$, Z belongs to $\mathcal{C}^W$. Hence it suffices to show that $Z \in {}^{\perp}\mathcal{C}^W$.

First, we prove that $\lambda^{W_1} b$ is an isomorphism. We employ a similar argument to [Benson et al. 2008, Theorem 7.5]. Consider the sequence

$$\lambda^W X \xrightarrow{f} \lambda^{W_1} X \oplus \lambda^{W_0} X \xrightarrow{g} \lambda^{W_1}\lambda^{W_0} X, \tag{3.18}$$

and apply λ^{W_1} to it. Then we obtain a sequence which can be completed to a split triangle. The triangle appears in the first row of the diagram below. Moreover, λ^{W_1} sends the second row of the diagram (3.16) to a split triangle, which appears in the second row of the diagram below:

$$\begin{tikzcd}
\lambda^{W_1} X \arrow[r, "\lambda^{W_1} f"] \arrow[d, "\lambda^{W_1} b"] & \lambda^{W_1} X \oplus \lambda^{W_1}\lambda^{W_0} X \arrow[r, "\lambda^{W_1} g"] \arrow[d, equal] & \lambda^{W_1}\lambda^{W_0} X \arrow[r, "0"] \arrow[d, equal] & \lambda^{W_1} X[1] \arrow[d, "\lambda^{W_1} b[1]"] \\
\lambda^{W_1} C \arrow[r, "\lambda^{W_1} a"] & \lambda^{W_1} X \oplus \lambda^{W_1}\lambda^{W_0} X \arrow[r, "\lambda^{W_1} g"] & \lambda^{W_1}\lambda^{W_0} X \arrow[r, "0"] & \lambda^{W_1} C[1]
\end{tikzcd}$$

Since this diagram is commutative, we conclude that $\lambda^{W_1} b$ is an isomorphism.

Next, we prove that $\lambda^{W_0} b$ is an isomorphism. Thanks to Corollary 3.13, we are able to follow the same process as above. In fact, the corollary implies that $\lambda^{W_0}\lambda^{W_1} \cong \lambda^{W_0 \cap W_1}$. Thus, applying λ^{W_0} to the sequence (3.18), we obtain a sequence which can be completed into a split triangle. Furthermore, λ^{W_0} sends the second row of the diagram (3.16) to a split triangle. Consequently we see that there is a morphism of triangles:

$$\begin{tikzcd}
\lambda^{W_0} X \arrow[r, "\lambda^{W_0} f"] \arrow[d, "\lambda^{W_0} b"] & \lambda^{W_0 \cap W_1} X \oplus \lambda^{W_0} X \arrow[r, "\lambda^{W_0} g"] \arrow[d, equal] & \lambda^{W_0 \cap W_1} X \arrow[r, "0"] \arrow[d, equal] & \lambda^{W_0} X[1] \arrow[d, "\lambda^{W_0} b[1]"] \\
\lambda^{W_0} C \arrow[r, "\lambda^{W_0} a"] & \lambda^{W_0 \cap W_1} X \oplus \lambda^{W_0} X \arrow[r, "\lambda^{W_0} g"] & \lambda^{W_0 \cap W_1} X \arrow[r, "0"] & \lambda^{W_0} C[1]
\end{tikzcd}$$

Therefore $\lambda^{W_0} b$ is an isomorphism.

Since we have shown that $\lambda^{W_0} b$ and $\lambda^{W_1} b$ are isomorphisms, it follows from the triangle (3.17) that $\lambda^{W_0} Z = \lambda^{W_1} Z = 0$. Thus we have $Z \in {}^{\perp}\mathcal{C}^W$ by Remark 2.7(i). □

Remark 3.19. Let f, g and a be as above. Let $h : X \to \lambda^{W_1} X \oplus \lambda^{W_0} X$ be a morphism induced by $\eta^{W_1} X$ and $\eta^{W_0} X$. Then $g \cdot h = 0$ by Remark 3.14(ii). Hence there is a morphism $b' : X \to C$ such that the following diagram is commutative:

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, "b'"] & X \arrow[r] \arrow[d, "h"] & 0 \arrow[r] \arrow[d] & X[1] \arrow[d, "b'[1]"] \\
C \arrow[r, "a"] & \lambda^{W_1} X \oplus \lambda^{W_0} X \arrow[r, "g"] & \lambda^{W_1}\lambda^{W_0} X \arrow[r] & C[1]
\end{tikzcd}$$

We can regard any morphism b' making this diagram commutative as the natural morphism $\eta^W X$. In fact, since $\lambda^W h = f$, applying λ^W to this diagram, and setting $\lambda^W b' = b$, we obtain the diagram (3.16). Note that $b \cdot \eta^W X = b'$. Moreover, the above proof implies that $b : \lambda^W X \to C$ is an isomorphism. Thus we can identify b' with $\eta^W X$ under the isomorphism b.

We give some examples of Theorem 3.15.

Example 3.20. (1) Let x be an element of R. Recall that $\lambda^{V(x)} \cong \mathrm{L}\Lambda^{V(x)}$ by (2.9). We put $S = \{1, x, x^2, \dots\}$. Since $V(x)^c = U_S$, it holds that $\lambda^{V(x)^c} = \lambda^{U_S} \cong (-) \otimes_R R_x$ by (2.8). Set $W = \operatorname{Spec} R$, $W_0 = V(x)$ and $W_1 = V(x)^c$. Then the theorem yields the triangle

$$R \longrightarrow R_x \oplus R^{\wedge}_{(x)} \longrightarrow (R^{\wedge}_{(x)})_x \longrightarrow R[1].$$

(2) Suppose that $(R, \mathfrak{m})$ is a local ring with $\mathfrak{p} \in \operatorname{Spec} R$ and having $\dim R/\mathfrak{p} = 1$. Setting $W = V(\mathfrak{p})$, $W_0 = V(\mathfrak{m})$ and $W_1 = \{\mathfrak{p}\}$, we see from the theorem and Corollary 3.7 that there is a short exact sequence,

$$0 \longrightarrow R^{\wedge}_{\mathfrak{p}} \longrightarrow \widehat{R_{\mathfrak{p}}} \oplus \widehat{R} \longrightarrow \widehat{(\widehat{R})_{\mathfrak{p}}} \longrightarrow 0.$$

Actually, this gives a pure-injective resolution of $R^{\wedge}_{\mathfrak{p}}$; see Section 9. Moreover, if R is a 1-dimensional local domain with quotient field Q, then this short exact sequence is of the form

$$0 \longrightarrow R \longrightarrow Q \oplus \widehat{R} \longrightarrow \widehat{R} \otimes_R Q \longrightarrow 0.$$

By similar arguments to Proposition 3.4 and Corollary 3.13, one can prove the following proposition, which is a generalized form of [Nakamura and Yoshino 2018, Proposition 3.1].

Proposition 3.21. *Let W_0 and W_1 be arbitrary subsets of* $\operatorname{Spec} R$. *Suppose that one of the following conditions hold*:

(1) *W_0 is specialization-closed in $W_0 \cup W_1$.*

(2) *W_1 is generalization-closed in $W_0 \cup W_1$.*

Then we have an isomorphism

$$\gamma_{W_0}\gamma_{W_1} \cong \gamma_{W_0 \cap W_1}.$$

As with Theorem 3.15, it is possible to prove the following theorem, in which we implicitly use the fact that $\gamma_{W_0}\gamma_W \cong \gamma_{W_0}$ if $W_0 \subseteq W$ (see [Nakamura and Yoshino 2018, Remark 3.7(i)]).

Theorem 3.22. *Let W, W_0 and W_1 be subsets of* $\operatorname{Spec} R$ *with $W = W_0 \cup W_1$. Suppose that one of the following conditions holds*:

(1) *W_0 is specialization-closed in W.*

(2) *W_1 is generalization-closed in W.*

Then, for any $X \in \mathcal{D}$, there is a triangle of the form

$$\gamma_{W_1}\gamma_{W_0} X \xrightarrow{f} \gamma_{W_1} X \oplus \gamma_{W_0} X \xrightarrow{g} \gamma_W X \longrightarrow \gamma_{W_1}\gamma_{W_0} X[1],$$

where f and g are morphisms represented by the following matrices;

$$f = \begin{pmatrix} \gamma_{W_1}\varepsilon_{W_0}X \\ (-1)\cdot\varepsilon_{W_1}\gamma_{W_0}X \end{pmatrix}, \quad g = \begin{pmatrix} \varepsilon_{W_1}\gamma_W X & \varepsilon_{W_0}\gamma_W X \end{pmatrix}.$$

Remark 3.23. As long as we work on the derived category $\mathcal{D}$, Theorem 3.15 and Theorem 3.22 generalize Mayer–Vietoris triangles in the sense of [Benson et al. 2008, Theorem 7.5], in which γ_V and λ_V are written as $\varGamma_V$ and L_V, respectively, for a specialization-closed subset V of Spec R.

4. Projective dimension of flat modules

As an application of results in Section 3, we give a simpler proof of a classical theorem due to Gruson and Raynaud.

Theorem 4.1 [Raynaud and Gruson 1971, Part. II, Corollary 3.2.7]. *Let F be a flat R-module. Then the projective dimension of F is at most* $\dim R$.

We start by showing the following lemma.

Lemma 4.2. *Let F be a flat R-module and $\mathfrak{p}$ be a prime ideal of R. Suppose that $X \in \mathcal{C}^{\{\mathfrak{p}\}}$. Then there is an isomorphism*

$$\mathrm{RHom}_R(F, X) \cong \prod_B X,$$

where $B = \dim_{\kappa(\mathfrak{p})} F \otimes_R \kappa(\mathfrak{p})$.

Proof. Since $\lambda^{\{\mathfrak{p}\}} : \mathcal{D} \to \mathcal{C}^{\{\mathfrak{p}\}}$ is a left adjoint to the inclusion functor $\mathcal{C}^{\{\mathfrak{p}\}} \hookrightarrow \mathcal{D}$, we have $\mathrm{RHom}_R(F, X) \cong \mathrm{RHom}_R(\lambda^{\{\mathfrak{p}\}} F, X)$. Moreover, it follows from Lemma 3.8 that $\lambda^{\{\mathfrak{p}\}} F \cong \big(\bigoplus_B R_{\mathfrak{p}}\big)^{\wedge}_{\mathfrak{p}} \cong \lambda^{\{\mathfrak{p}\}}\big(\bigoplus_B R\big)$, where $B = \dim_{\kappa(\mathfrak{p})} F \otimes_R \kappa(\mathfrak{p})$. Therefore we obtain isomorphisms

$$\mathrm{RHom}_R(F, X) \cong \mathrm{RHom}_R\Big(\lambda^{\{\mathfrak{p}\}}\Big(\bigoplus_B R\Big), X\Big) \cong \mathrm{RHom}_R\Big(\bigoplus_B R, X\Big) \cong \prod_B X. \quad \square$$

Let $a, b \in \mathbb{Z} \cup \{\pm\infty\}$ with $a \le b$. We write $\mathcal{D}^{[a,b]}$ for the full subcategory of $\mathcal{D}$ consisting of all complexes X of R-modules such that $H^i(X) = 0$ for $i \notin [a, b]$ (see [Kashiwara and Schapira 2006, Notation 13.1.11]). For a subset W of Spec R, $\max W$ denotes the set of prime ideals $\mathfrak{p} \in W$ which are maximal with respect to inclusion in W.

Proposition 4.3. *Let F be a flat R-module and $X \in \mathcal{D}^{[-\infty,0]}$. Suppose that W is a subset of* Spec R *such that $n = \dim W$ is finite. Then we have* $\mathrm{Ext}^i_R(F, \lambda^W X) = 0$ *for $i > n$.*

Proof. We use induction on n. First, we suppose that $n = 0$. It then holds that

$$\lambda^W X \cong \prod_{\mathfrak{p}\in W} \lambda^{\{\mathfrak{p}\}} X \cong \prod_{\mathfrak{p}\in W} \mathrm{L}\Lambda^{V(\mathfrak{p})}\, X_{\mathfrak{p}} \in \mathcal{D}^{[-\infty,0]},$$

by Theorem 3.10. Hence, noting that

$$\mathrm{RHom}_R(F, \lambda^W X) \cong \prod_{\mathfrak{p}\in W} \mathrm{RHom}_R(F, \lambda^{\{\mathfrak{p}\}} X),$$

we have $\mathrm{Ext}^i_R(F, \lambda^W X) = 0$ for $i > 0$, by Lemma 4.2.

Next, we suppose $n > 0$. Set $W_0 = \max W$ and $W_1 = W \setminus W_0$. By Theorem 3.15, there is a triangle

$$\lambda^W X \longrightarrow \lambda^{W_1} X \oplus \lambda^{W_0} X \longrightarrow \lambda^{W_1}\lambda^{W_0} X \longrightarrow \lambda^W X[1].$$

Note that $\dim W_0 = 0$ and $\dim W_1 = n-1$. By the argument above, it holds that $\mathrm{Ext}^i_R(F, \lambda^{W_0} X) = 0$ for $i > 0$. Furthermore, since $X, \lambda^{W_0} X \in \mathcal{D}^{[-\infty,0]}$, we have $\mathrm{Ext}^i_R(F, \lambda^{W_1} X) = \mathrm{Ext}^i_R(F, \lambda^{W_1}\lambda^{W_0} X) = 0$ for $i > n-1$, by the inductive hypothesis. Hence it is seen from the triangle that $\mathrm{Ext}^i_R(F, \lambda^W X) = 0$ for $i > n$. □

Proof of Theorem 4.1. We may assume that $d = \dim R$ is finite. Let M be any R-module. We only have to show that $\mathrm{Ext}^i_R(F, M) = 0$ for $i > d$. Setting $W = \mathrm{Spec}\, R$, we have $\dim W = d$ and $M \cong \lambda^W M$. It then follows from Proposition 4.3 that $\mathrm{Ext}^i_R(F, M) \cong \mathrm{Ext}^i_R(F, \lambda^W M) = 0$ for $i > d$. □

5. Cotorsion flat modules and cosupport

In this section, we summarize some basic facts about cotorsion flat R-modules.

Recall that an R-module M is called cotorsion if $\mathrm{Ext}^1_R(F, M) = 0$ for any flat R-module F. This is equivalent to saying that $\mathrm{Ext}^i_R(F, M) = 0$ for any flat R-module F and any $i > 0$. Clearly, all injective R-modules are cotorsion.

A cotorsion flat R-module means an R-module which is cotorsion and flat. If F is a flat R-module and $\mathfrak{p} \in \mathrm{Spec}\, R$, then Corollary 3.7 implies that $\lambda^{\{\mathfrak{p}\}} F$ is isomorphic to $\widehat{F_{\mathfrak{p}}}$, which is a cotorsion flat R-module by Lemma 4.2 and Proposition 2.1. Moreover, recall that $\widehat{F_{\mathfrak{p}}}$ is isomorphic to the $\mathfrak{p}$-adic completion of a free $R_{\mathfrak{p}}$-module by Lemma 3.8.

We remark that arbitrary direct products of flat R-modules are flat, since R is Noetherian. Hence, if $T_{\mathfrak{p}}$ is the $\mathfrak{p}$-adic completion of a free $R_{\mathfrak{p}}$ module for each $\mathfrak{p} \in \mathrm{Spec}\, R$, then $\prod_{\mathfrak{p}\in\mathrm{Spec}\, R} T_{\mathfrak{p}}$ is a cotorsion flat R-module. Conversely, the following fact holds.

Proposition 5.1 [Enochs 1984]. *Let F be a cotorsion flat R-module. Then there is an isomorphism*

$$F \cong \prod_{\mathfrak{p}\in\mathrm{Spec}\, R} T_{\mathfrak{p}},$$

where $T_{\mathfrak{p}}$ is the $\mathfrak{p}$-adic completion of a free $R_{\mathfrak{p}}$ module.

Proof. See [Enochs 1984, Theorem; Enochs and Jenda 2000, Theorem 5.3.28]. □

Let S be a multiplicatively closed subset of R and $\mathfrak{a}$ be an ideal of R. For a cotorsion flat R-module F, we have $\mathrm{RHom}_R(S^{-1}R, F) \cong \mathrm{Hom}_R(S^{-1}R, F)$ and $\mathrm{L}\Lambda^{V(\mathfrak{a})} F \cong \Lambda^{V(\mathfrak{a})} F$. Moreover, by Proposition 5.1, we may regard F as an R-module of the form $\prod_{\mathfrak{p}\in\operatorname{Spec} R} T_{\mathfrak{p}}$. Then it holds that

$$\mathrm{RHom}_R\Big(S^{-1}R, \prod_{\mathfrak{p}\in\operatorname{Spec} R} T_{\mathfrak{p}}\Big) \cong \mathrm{Hom}_R\Big(S^{-1}R, \prod_{\mathfrak{p}\in\operatorname{Spec} R} T_{\mathfrak{p}}\Big) \cong \prod_{\mathfrak{p}\in U_S} T_{\mathfrak{p}}. \tag{5.2}$$

This fact appears implicitly in [Xu 1996, §5.2]. Furthermore we have

$$\mathrm{L}\Lambda^{V(\mathfrak{a})} \prod_{\mathfrak{p}\in\operatorname{Spec} R} T_{\mathfrak{p}} \cong \Lambda^{V(\mathfrak{a})} \prod_{\mathfrak{p}\in\operatorname{Spec} R} T_{\mathfrak{p}} \cong \prod_{\mathfrak{p}\in V(\mathfrak{a})} T_{\mathfrak{p}}. \tag{5.3}$$

One can show (5.2) and (5.3) by Lemma 3.1 and (2.9). See also Thompson's recent lemma [2017b, Lemma 2.2].

Let F be a cotorsion flat R-module with $\operatorname{cosupp} F \subseteq W$ for a subset W of $\operatorname{Spec} R$. Then it follows from Proposition 5.1 that F is isomorphic to an R-module of the form $\prod_{\mathfrak{p}\in W} T_{\mathfrak{p}}$. More precisely, using Lemma 2.4, (5.2) and (5.3), one can show the following corollary, which is essentially proved in [Enochs and Jenda 2000, Lemma 8.5.25].

Corollary 5.4. *Let F be a cotorsion flat R-module, and set $W = \operatorname{cosupp} F$. Then we have an isomorphism*

$$F \cong \prod_{\mathfrak{p}\in W} T_{\mathfrak{p}},$$

where $T_{\mathfrak{p}}$ is of the form $\big(\bigoplus_{B_{\mathfrak{p}}} R_{\mathfrak{p}}\big)^{\wedge}_{\mathfrak{p}}$ with $B_{\mathfrak{p}} = \dim_{\kappa(\mathfrak{p})} \mathrm{Hom}_R(R_{\mathfrak{p}}, F) \otimes_R \kappa(\mathfrak{p})$.

6. Complexes of cotorsion flat modules and cosupport

In this section, we study the cosupport of a complex X consisting of cotorsion flat R-modules. As a consequence, we obtain an explicit way to calculate $\gamma_{V^c} X$ and $\lambda^V X$ for a specialization-closed subset V of $\operatorname{Spec} R$.

Notation 6.1. Let W be a subset of $\operatorname{Spec} R$. Let X be a complex of cotorsion flat R-modules such that $\operatorname{cosupp} X^i \subseteq W$ for all $i \in \mathbb{Z}$. Under Corollary 5.4, we use a presentation of the form

$$X = \Big(\cdots \to \prod_{\mathfrak{p}\in W} T^i_{\mathfrak{p}} \to \prod_{\mathfrak{p}\in W} T^{i+1}_{\mathfrak{p}} \to \cdots\Big),$$

where $X^i = \prod_{\mathfrak{p}\in\operatorname{Spec} R} T^i_{\mathfrak{p}}$ and $T^i_{\mathfrak{p}}$ is the $\mathfrak{p}$-adic completion of a free $R_{\mathfrak{p}}$-module.

Remark 6.2. Let $X = \left(\cdots \to \prod_{\mathfrak{p}\in \operatorname{Spec} R} T_{\mathfrak{p}}^{i} \to \prod_{\mathfrak{p}\in \operatorname{Spec} R} T_{\mathfrak{p}}^{i+1} \to \cdots\right)$ be a complex of cotorsion flat R-modules. Let V be a specialization-closed subset of $\operatorname{Spec} R$. By Lemma 3.1, we have $\operatorname{Hom}_R\left(\prod_{\mathfrak{p}\in V^c} T_{\mathfrak{p}}^{i}, \prod_{\mathfrak{p}\in V} T_{\mathfrak{p}}^{i+1}\right) = 0$ for all $i \in \mathbb{Z}$. Therefore $Y = \left(\cdots \to \prod_{\mathfrak{p}\in V^c} T_{\mathfrak{p}}^{i} \to \prod_{\mathfrak{p}\in V^c} T_{\mathfrak{p}}^{i+1} \to \cdots\right)$ is a subcomplex of X, where the differentials in Y are the restrictions of ones in X.

We say that a complex X of R-modules is left (resp. right) bounded if $X^i = 0$ for $i \ll 0$ (resp. $i \gg 0$). When X is left and right bounded, X is called bounded.

Proposition 6.3. *Let W be a subset of* $\operatorname{Spec} R$ *and X be a complex of cotorsion flat R-modules such that* $\operatorname{cosupp} X^i \subseteq W$ *for all $i \in \mathbb{Z}$. Suppose that one of the following conditions holds*:

1. *X is left bounded.*
2. *W is equal to $V(\mathfrak{a})$ for an ideal $\mathfrak{a}$ of R.*
3. *W is generalization-closed.*
4. $\dim W$ *is finite.*

Then it holds that $\operatorname{cosupp} X \subseteq W$, *i.e., $X \in \mathcal{C}^W$.*

To show this, we use the elementary lemma below. Therein, for a complex X and $n \in \mathbb{Z}$, we define the truncations $\tau_{\le n} X$ and $\tau_{>n} X$ as follows (see [Hartshorne 1966, Chapter I, §7]):

$$\begin{aligned}\tau_{\le n} X &= (\cdots \to X^{n-1} \to X^{n} \to 0 \to \cdots),\\ \tau_{>n} X &= (\cdots \to 0 \to X^{n+1} \to X^{n+2} \to \cdots).\end{aligned}$$

Lemma 6.4. *Let W be a subset of* $\operatorname{Spec} R$. *We assume that $\tau_{\le n} X \in \mathcal{C}^W$ (resp. $\tau_{>n} X \in \mathcal{L}_W$) for all $n \ge 0$ (resp. $n < 0$). Then we have $X \in \mathcal{C}^W$ (resp. $X \in \mathcal{L}_W$).*

Recall that $\mathcal{C}^W$ (resp. $\mathcal{L}_W$) is closed under arbitrary direct products (resp. sums). Then one can show this lemma by using homotopy limits (resp. colimits), see [Bökstedt and Neeman 1993, Remarks 2.2 and 2.3]. □

Proof of Proposition 6.3. Case 1: We have $\tau_{\le n} X \in \mathcal{C}^W$ for all $n \ge 0$, since $\tau_{\le n} X$ are bounded. Thus Lemma 6.4 implies that $X \in \mathcal{C}^W$.

Case 2: By (2.9), Proposition 2.5 and (5.3), it holds that $\lambda^{V(\mathfrak{a})} X \cong \mathrm{L}\Lambda^{V(\mathfrak{a})} X \cong \Lambda^{V(\mathfrak{a})} X \cong X$. Hence X belongs to $\mathcal{C}^{V(\mathfrak{a})}$.

Case 3: It follows from Case 1 that $\tau_{>n} X \in \mathcal{C}^W$ for all $n < 0$. Moreover, we have $\mathcal{C}^W = \mathcal{L}_W$ by Lemma 3.1. Thus Lemma 6.4 implies that $X \in \mathcal{L}_W = \mathcal{C}^W$.

Case 4: Under Notation 6.1, we write $X^i = \prod_{\mathfrak{p}\in W} T_{\mathfrak{p}}^{i}$ for $i \in \mathbb{Z}$. Set $n = \dim W$, and use induction on n. First, suppose that $n = 0$. It is seen from Remark 6.2 that X is the direct product of complexes of the form $Y^{\{\mathfrak{p}\}} = (\cdots \to T_{\mathfrak{p}}^{i} \to T_{\mathfrak{p}}^{i+1} \to \cdots)$ for $\mathfrak{p} \in W$.

Furthermore, by Cases 2 and 3, we have $\operatorname{cosupp} Y^{\{\mathfrak{p}\}} \subseteq V(\mathfrak{p}) \cap U(\mathfrak{p}) = \{\mathfrak{p}\}$. Thus it holds that $X \cong \prod_{\mathfrak{p}\in W} Y^{\{\mathfrak{p}\}} \in \mathcal{C}^W$.

Next, suppose that $n > 0$. Set $W_0 = \max W$ and $W_1 = W \setminus W_0$. We write $Y = \left(\cdots \to \prod_{\mathfrak{p}\in W_1} T_{\mathfrak{p}}^i \to \prod_{\mathfrak{p}\in W_1} T_{\mathfrak{p}}^{i+1} \to \cdots\right)$, which is a subcomplex of X by Remark 6.2. Hence there is a short exact sequence of complexes,

$$0 \longrightarrow Y \longrightarrow X \longrightarrow X/Y \longrightarrow 0,$$

where $X/Y = \left(\cdots \to \prod_{\mathfrak{p}\in W_0} T_{\mathfrak{p}}^i \to \prod_{\mathfrak{p}\in W_0} T_{\mathfrak{p}}^{i+1} \to \cdots\right)$. Note that $\dim W_0 = 0$ and $\dim W_1 = n - 1$. Then we have $\operatorname{cosupp} X/Y \subseteq W_0$, by the argument above. Moreover the inductive hypothesis implies that $\operatorname{cosupp} Y \subseteq W_1$. Hence it holds that $\operatorname{cosupp} X \subseteq W_0 \cup W_1 = W$. □

Under some assumption, it is possible to extend condition (4) in Proposition 6.3 to the case where $\dim W$ is infinite; see Remark 7.15. See also [Thompson 2017a, Theorem 2.7].

Corollary 6.5. *Let X be a complex of cotorsion flat R-modules and W be a specialization-closed subset of* $\operatorname{Spec} R$. *Under Notation 6.1, we write*

$$X = \left(\cdots \to \prod_{\mathfrak{p}\in \operatorname{Spec} R} T_{\mathfrak{p}}^i \to \prod_{\mathfrak{p}\in \operatorname{Spec} R} T_{\mathfrak{p}}^{i+1} \to \cdots\right).$$

Suppose that one of the conditions in Proposition 6.3 holds. Then it holds that

$$\begin{aligned} \gamma_{W^c} X &\cong \left(\cdots \to \prod_{\mathfrak{p}\in W^c} T_{\mathfrak{p}}^i \to \prod_{\mathfrak{p}\in W^c} T_{\mathfrak{p}}^{i+1} \to \cdots\right), \\ \lambda^W X &\cong \left(\cdots \to \prod_{\mathfrak{p}\in W} T_{\mathfrak{p}}^i \to \prod_{\mathfrak{p}\in W} T_{\mathfrak{p}}^{i+1} \to \cdots\right). \end{aligned} \tag{6.6}$$

Proof. Since $Y = \left(\cdots \to \prod_{\mathfrak{p}\in W^c} T_{\mathfrak{p}}^i \to \prod_{\mathfrak{p}\in W^c} T_{\mathfrak{p}}^{i+1} \to \cdots\right)$ is a subcomplex of X by Remark 6.2, there is a triangle in $\mathcal{D}$:

$$Y \longrightarrow X \longrightarrow X/Y \longrightarrow Y[1],$$

where $X/Y = \left(\cdots \to \prod_{\mathfrak{p}\in W} T_{\mathfrak{p}}^i \to \prod_{\mathfrak{p}\in W} T_{\mathfrak{p}}^{i+1} \to \cdots\right)$. By Proposition 6.3, we have $X/Y \in \mathcal{C}^W$. Moreover, since W^c is generalization-closed, it holds that $Y \in \mathcal{C}^{W^c} = {}^{\perp}\mathcal{C}^W$ by Proposition 6.3 and Lemma 3.1. Therefore we conclude that $\gamma_{W^c} X \cong Y$ and $\lambda^W X \cong X/Y$ by Lemma 2.6. □

Let X be a complex of cotorsion flat R-modules and S be a multiplicatively closed subset of R. We assume that X is left bounded, or $\dim R$ is finite. It then follows from the corollary and (5.2) that

$$\gamma_{U_S} X \cong \left(\cdots \to \prod_{\mathfrak{p} \in U_S} T^i_{\mathfrak{p}} \to \prod_{\mathfrak{p} \in U_S} T^{i+1}_{\mathfrak{p}} \to \cdots \right) \cong \mathrm{Hom}_R(S^{-1}R, X).$$

We now recall that $\gamma_{U_S} \cong \mathrm{RHom}_R(S^{-1}R, -)$; see [Nakamura and Yoshino 2018, Proposition 3.1]. Hence it holds that $\mathrm{RHom}_R(S^{-1}R, X) \cong \mathrm{Hom}_R(S^{-1}R, X)$. This fact also follows from Lemma 9.1.

7. Localization functors via Čech complexes

In this section, we introduce a new notion of Čech complexes to calculate $\lambda^W X$, where W is a general subset W of $\mathrm{Spec}\, R$ and X is a complex of flat R-modules.

We first set the following notation.

Notation 7.1. Let W be a subset of $\mathrm{Spec}\, R$ with $\dim W = 0$. We define a functor $\bar{\lambda}^W : \mathrm{Mod}\, R \to \mathrm{Mod}\, R$ by

$$\bar{\lambda}^W = \prod_{\mathfrak{p} \in W} \Lambda^{V(\mathfrak{p})}(- \otimes_R R_{\mathfrak{p}}).$$

For a prime ideal $\mathfrak{p}$ in W, we write

$$\bar{\eta}^{\{\mathfrak{p}\}} : \mathrm{id}_{\mathrm{Mod}\, R} \to \bar{\lambda}^{\{\mathfrak{p}\}} = \Lambda^{V(\mathfrak{p})}(- \otimes_R R_{\mathfrak{p}})$$

for the composition of the natural morphisms $\mathrm{id}_{\mathrm{Mod}\, R} \to (-) \otimes_R R_{\mathfrak{p}}$ and $(-) \otimes_R R_{\mathfrak{p}} \to \Lambda^{V(\mathfrak{p})}(- \otimes_R R_{\mathfrak{p}})$. Moreover, $\bar{\eta}^W : \mathrm{id}_{\mathrm{Mod}\, R} \to \bar{\lambda}^W = \prod_{\mathfrak{p} \in W} \bar{\lambda}^{\{\mathfrak{p}\}}$ denotes the product of the morphisms $\bar{\eta}^{\{\mathfrak{p}\}}$ for $\mathfrak{p} \in W$.

Notation 7.2. Let $\{W_i\}_{0 \le i \le n}$ be a family of subsets of $\mathrm{Spec}\, R$, and suppose that $\dim W_i = 0$ for $0 \le i \le n$. For a sequence $(i_m, \dots, i_1, i_0)$ of integers with $0 \le i_0 < i_1 < \cdots < i_m \le n$, we write

$$\bar{\lambda}^{(i_m, \dots, i_1, i_0)} = \bar{\lambda}^{W_{i_m}} \cdots \bar{\lambda}^{W_{i_1}} \bar{\lambda}^{W_{i_0}}.$$

If the sequence is empty, then we use the general convention that $\lambda^{(\,)} = \mathrm{id}_{\mathrm{Mod}\, R}$. For an integer s with $0 \le s \le m$, $\bar{\eta}^{W_{i_s}} : \mathrm{id}_{\mathrm{Mod}\, R} \to \bar{\lambda}^{(i_s)}$ induces a morphism

$$\bar{\lambda}^{(i_m, \dots, i_{s+1})} \bar{\eta}^{W_{i_s}} \bar{\lambda}^{(i_{s-1}, \dots, i_0)} : \bar{\lambda}^{(i_m, \dots, \hat{i}_s, \dots, i_0)} \to \bar{\lambda}^{(i_m, \dots, i_0)},$$

where we mean by $\hat{i}_s$ that i_s is omitted. We set

$$\partial^{m-1} : \prod_{0 \le i_0 < \cdots < i_{m-1} \le n} \bar{\lambda}^{(i_{m-1}, \dots, i_0)} \to \prod_{0 \le i_0 < \cdots < i_m \le n} \bar{\lambda}^{(i_m, \dots, i_0)}$$

to be the product of the morphisms $\bar{\lambda}^{(i_m, \dots, \hat{i}_s, \dots, i_0)} \to \bar{\lambda}^{(i_m, \dots, i_0)}$ multiplied by $(-1)^s$.

Remark 7.3. Let $W_0, W_1 \subseteq \operatorname{Spec} R$ be subsets such that $\dim W_0 = \dim W_1 = 0$. As with Remark 3.14(ii), the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{id}_{\operatorname{Mod} R} & \xrightarrow{\bar{\eta}^{W_0}} & \bar{\lambda}^{W_0} \\ \big\downarrow{\scriptstyle \bar{\eta}^{W_1}} & & \big\downarrow{\scriptstyle \bar{\eta}^{W_1}\bar{\lambda}^{W_0}} \\ \bar{\lambda}^{W_1} & \xrightarrow{\bar{\lambda}^{W_1}\bar{\eta}^{W_0}} & \bar{\lambda}^{W_1}\bar{\lambda}^{W_0} \end{array}$$

Definition 7.4. Let $\mathbb{W} = \{W_i\}_{0\le i\le n}$ be a family of subsets of $\operatorname{Spec} R$, and suppose that $\dim W_i = 0$ for $0 \le i \le n$. By Remark 7.3, it is possible to construct a Čech complex of functors of the form

$$\prod_{0\le i_0\le n} \bar{\lambda}^{(i_0)} \xrightarrow{\partial^0} \prod_{0\le i_0<i_1\le n} \bar{\lambda}^{(i_1,i_0)} \to \cdots \to \prod_{0\le i_0<\cdots<i_{n-1}\le n} \bar{\lambda}^{(i_{n-1},\dots,i_0)} \xrightarrow{\partial^{n-1}} \bar{\lambda}^{(n,\dots,0)},$$

which we denote by $L^{\mathbb{W}}$ and call it *the Čech complex with respect to* $\mathbb{W}$.

For an R-module M, $L^{\mathbb{W}}M$ denotes the complex of R-modules obtained by $L^{\mathbb{W}}$ in a natural way, where it is concentrated in degrees from 0 to n. We call $L^{\mathbb{W}}M$ *the Čech complex of M with respect to* $\mathbb{W}$. Note that there is a chain map $\ell^{\mathbb{W}}M : M \to L^{\mathbb{W}}M$ induced by the map $M \to \prod_{0\le i_0\le n} \bar{\lambda}^{(i_0)}M$ in degree 0, which is the product of $\bar{\eta}^{W_{i_0}}M : M \to \bar{\lambda}^{(i_0)}M$ for $0 \le i_0 \le n$.

More generally, we regard every term of $L^{\mathbb{W}}$ as a functor $C(\operatorname{Mod} R) \to C(\operatorname{Mod} R)$, where $C(\operatorname{Mod} R)$ denotes the category of complexes of R-modules. Then $L^{\mathbb{W}}$ naturally sends a complex X to a double complex, which we denote by $L^{\mathbb{W}}X$. Furthermore, we write $\operatorname{tot} L^{\mathbb{W}}X$ for the total complex of $L^{\mathbb{W}}X$. The family of chain maps $\ell^{\mathbb{W}}X^j : X^j \to L^{\mathbb{W}}X^j$ for $j \in \mathbb{Z}$ induces a morphism $X \to L^{\mathbb{W}}X$ as double complexes, from which we obtain a chain map $\ell^{\mathbb{W}}X : X \to \operatorname{tot} L^{\mathbb{W}}X$.

Remark 7.5. (i) We regard $\operatorname{tot} L^{\mathbb{W}}$ as a functor $C(\operatorname{Mod} R) \to C(\operatorname{Mod} R)$. Then $\ell^{\mathbb{W}}$ is a morphism $\mathrm{id}_{C(\operatorname{Mod} R)} \to \operatorname{tot} L^{\mathbb{W}}$ of functors. Moreover, if M is an R-module, then $\operatorname{tot} L^{\mathbb{W}}M = L^{\mathbb{W}}M$.

(ii) Let $a, b \in \mathbb{Z} \cup \{\pm\infty\}$ with $a \le b$ and X be a complex of R-modules such that $X^i = 0$ for $i \notin [a, b]$. Then it holds that $(\operatorname{tot} L^{\mathbb{W}}X)^i = 0$ for $i \notin [a, b+n]$, where n is the number given to $\mathbb{W} = \{W_i\}_{0\le i\le n}$.

(iii) Let X be a complex of flat R-modules. Then we see that $\operatorname{tot} L^{\mathbb{W}}X$ consists of cotorsion flat R-modules with cosupports in $\bigcup_{0\le i\le n} W_i$.

Definition 7.6. Let W be a nonempty subset of $\operatorname{Spec} R$ and $\{W_i\}_{0\le i\le n}$ be a family of subsets of W. We say that $\{W_i\}_{0\le i\le n}$ is *a system of slices of* W if the following conditions hold:

(1) $W = \bigcup_{0\le i\le n} W_i$.

(2) $W_i \cap W_j = \varnothing$ if $i \ne j$.

(3) $\dim W_i = 0$ for $0 \le i \le n$.

(4) W_i is specialization-closed in $\bigcup_{i \le j \le n} W_j$ for each $0 \le i \le n$.

Compare this definition with the filtrations in [Hartshorne 1966, Chapter IV, §3].

If $\dim W$ is finite, then there exists at least one system of slices of W. Conversely, if there is a system of slices of W, then $\dim W$ is finite.

Proposition 7.7. *Let W be a subset of* $\operatorname{Spec} R$ *and $\mathbb{W} = \{W_i\}_{0 \le i \le n}$ be a system of slices of W. Then, for any flat R-module F, there is an isomorphism in $\mathcal{D}$;*

$$\lambda^W F \cong L^{\mathbb{W}} F.$$

Under this isomorphism, $\ell^{\mathbb{W}} F : F \to L^{\mathbb{W}} F$ coincides with $\eta^W F : F \to \lambda^W F$ in $\mathcal{D}$.

Proof. We use induction on n, which is the number given to $\mathbb{W} = \{W_i\}_{0 \le i \le n}$. Suppose that $n = 0$. It then holds that $L^{\mathbb{W}} F = \bar{\lambda}^{W_0} F = \bar{\lambda}^W F$ and $\ell^{\mathbb{W}} F = \bar{\eta}^{W_0} F = \bar{\eta}^W F$. Hence this proposition follows from Theorem 3.10.

Next, suppose that $n > 0$, and write $U = \bigcup_{1 \le i \le n} W_i$. Setting $U_{i-1} = W_i$, we obtain a system of slices $\mathbb{U} = \{U_i\}_{0 \le i \le n-1}$ of U. Consider the following two squares, where the first and second are in $C(\operatorname{Mod} R)$ and $\mathcal{D}$, respectively:

$$\begin{array}{ccc} F & \xrightarrow{\bar{\eta}^{W_0} F} & \bar{\lambda}^{W_0} F \\ \downarrow{\scriptstyle \ell^{\mathbb{U}} F} & & \downarrow{\scriptstyle \ell^{\mathbb{U}} \bar{\lambda}^{W_0} F} \\ L^{\mathbb{U}} F & \xrightarrow{L^{\mathbb{U}} \bar{\eta}^{W_0} F} & L^{\mathbb{U}} \bar{\lambda}^{W_0} F \end{array} \qquad \begin{array}{ccc} F & \xrightarrow{\eta^{W_0} F} & \lambda^{W_0} F \\ \downarrow{\scriptstyle \eta^U F} & & \downarrow{\scriptstyle \eta^U \lambda^{W_0} F} \\ \lambda^U F & \xrightarrow{\lambda^U \eta^{W_0} F} & \lambda^U \lambda^{W_0} F \end{array}$$

By Remarks 7.5(i) and 3.14(ii), both of them are commutative. Moreover, $\lambda^U \eta^{W_0} F$ is the unique morphism which makes the right square commutative, because λ^U is a left adjoint to the inclusion functor $\mathcal{C}^U \hookrightarrow \mathcal{D}$. Then, regarding the left square as being in $\mathcal{D}$, we see from the inductive hypothesis that the left and right squares coincide in $\mathcal{D}$.

Let $\bar{g} : L^{\mathbb{U}} F \oplus \bar{\lambda}^{W_0} F \to L^{\mathbb{U}} \bar{\lambda}^{W_0} F$ and $\bar{h} : F \to L^{\mathbb{U}} F \oplus \bar{\lambda}^{W_0} F$ be chain maps represented by the matrices

$$\bar{g} = \begin{pmatrix} L^{\mathbb{U}} \bar{\eta}^{W_0} F & (-1) \cdot \ell^{\mathbb{U}} \bar{\lambda}^{W_0} X \end{pmatrix}, \quad \bar{h} = \begin{pmatrix} \ell^{\mathbb{U}} F \\ \bar{\eta}^{W_0} F \end{pmatrix}.$$

Notice that the mapping cone of $\bar{g}[-1]$ is nothing but $L^{\mathbb{W}} F$. Then we can obtain the following morphism of triangles, regarded as being in $\mathcal{D}$:

$$\begin{array}{ccccccc} F[-1] & \longrightarrow & F[-1] & \longrightarrow & 0 & \longrightarrow & F \\ \downarrow{\scriptstyle \ell^{\mathbb{W}} F[-1]} & & \downarrow{\scriptstyle \bar{h}[-1]} & & \downarrow & & \downarrow{\scriptstyle \ell^{\mathbb{W}} F} \\ L^{\mathbb{W}} F[-1] & \longrightarrow & (L^{\mathbb{U}} F \oplus \bar{\lambda}^{W_0} F)[-1] & \xrightarrow{\bar{g}[-1]} & L^{\mathbb{U}} \bar{\lambda}^{W_0} F[-1] & \longrightarrow & L^{\mathbb{W}} F \end{array} \tag{7.8}$$

Therefore, by Theorem 3.15 and Remark 3.19, there is an isomorphism $\lambda^W F \cong L^{\mathbb{W}} F$ such that $\ell^{\mathbb{W}} F$ coincides with $\eta^W F$ under this isomorphism. $\square$

The following corollary is one of the main results of this paper.

Corollary 7.9. *Let W and $\mathbb{W} = \{W_i\}_{0 \le i \le n}$ be as above. Let X be a complex of flat R-modules. Then there is an isomorphism in $\mathcal{D}$;*

$$\lambda^W X \cong \operatorname{tot} L^{\mathbb{W}} X.$$

Under this isomorphism, $\ell^{\mathbb{W}} X : X \to \operatorname{tot} L^{\mathbb{W}} X$ coincides with $\eta^W X : X \to \lambda^W X$ in $\mathcal{D}$.

Proof. We embed $\ell^{\mathbb{W}} X : X \to \operatorname{tot} L^{\mathbb{W}} X$ into a triangle

$$C \longrightarrow X \xrightarrow{\ell^{\mathbb{W}} X} \operatorname{tot} L^{\mathbb{W}} X \longrightarrow C[1].$$

Proposition 6.3 and Remark 7.5(iii) imply that $\operatorname{tot} L^{\mathbb{W}} X \in \mathcal{C}^W$. Thus it suffices to show that $\lambda^{W_i} C = 0$ for each i, by Lemma 2.6 and Remark 2.7(i). For this purpose, we prove that $\lambda^{W_i} \ell^{\mathbb{W}} X$ is an isomorphism in $\mathcal{D}$. This is equivalent to showing that $\bar{\lambda}^{W_i} \ell^{\mathbb{W}} X$ is a quasi-isomorphism, since X and $\operatorname{tot} L^{\mathbb{W}} X$ consist of flat R-modules.

Consider the natural morphism $X \to L^{\mathbb{W}} X$ of double complexes, which is induced by the chain maps $\ell^{\mathbb{W}} X^j : X^j \to L^{\mathbb{W}} X^j$ for $j \in \mathbb{Z}$. To prove that $\bar{\lambda}^{W_i} \ell^{\mathbb{W}} X$ is a quasi-isomorphism, it is enough to show that $\bar{\lambda}^{W_i} \ell^{\mathbb{W}} X^j$ is a quasi-isomorphism for each $j \in \mathbb{Z}$; see [Kashiwara and Schapira 2006, Theorem 12.5.4]. Furthermore, by Proposition 7.7, each $\ell^{\mathbb{W}} X^j$ coincides with $\eta^W X^j : X^j \to \lambda^W X^j$ in $\mathcal{D}$. Since $W_i \subseteq W$, it follows from Remark 2.7(ii) that $\lambda^{W_i} \eta^W X^j$ is an isomorphism in $\mathcal{D}$. This means that $\bar{\lambda}^{W_i} \ell^{\mathbb{W}} X^j$ is a quasi-isomorphism. ☐

Let W be a subset of $\operatorname{Spec} R$, and suppose that $n = \dim W$ is finite. Then Corollary 7.9 implies $\lambda^W R \in \mathcal{D}^{[0,n]}$. We give an example such that $H^n(\lambda^W R) \neq 0$.

Example 7.10. Let $(R, \mathfrak{m})$ be a local ring of dimension $d \geq 1$. Then we have $\dim V(\mathfrak{m})^c = d - 1$. By Lemma 2.6, there is a triangle

$$\gamma_{V(\mathfrak{m})} R \longrightarrow R \longrightarrow \lambda^{V(\mathfrak{m})^c} R \longrightarrow \gamma_{V(\mathfrak{m})} R[1].$$

Since $\mathrm{R}\Gamma_{V(\mathfrak{m})} \cong \gamma_{V(\mathfrak{m})}$ by (2.10), Grothendieck's nonvanishing theorem implies that $H^d(\gamma_{V(\mathfrak{m})} R)$ is nonzero. Then we see from the triangle that $H^{d-1}(\lambda^{V(\mathfrak{m})^c} R) \neq 0$.

We denote by $\mathcal{D}^-$ the full subcategory of $\mathcal{D}$ consisting of complexes X such that $H^i(X) = 0$ for $i \gg 0$. Let W be a subset of $\operatorname{Spec} R$ and $X \in \mathcal{D}^-$. If $\dim W$ is finite, then we have $\lambda^W R \in \mathcal{D}^-$ by Corollary 7.9. However, as shown in the following example, it can happen that $\lambda^W R \notin \mathcal{D}^-$ when $\dim W$ is infinite.

Example 7.11. Assume that $\dim R = +\infty$, and set $W = \max(\operatorname{Spec} R)$. Then it holds that $\dim W = 0$ and $\dim W^c = +\infty$. Since each $\mathfrak{m} \in W$ is maximal, there are isomorphisms

$$\gamma_W \cong \mathrm{R}\Gamma_W \cong \bigoplus_{\mathfrak{m} \in W} \mathrm{R}\Gamma_{V(\mathfrak{m})} .$$

Thus we see from Example 7.10 that $\gamma_W R \notin \mathcal{D}^-$. Then, considering the triangle

$$\gamma_W R \longrightarrow R \longrightarrow \lambda^{W^c} R \longrightarrow \gamma_W R[1],$$

we have $\lambda^{W^c} R \notin \mathcal{D}^-$.

Let W be a subset of $\operatorname{Spec} R$ and $X \in \mathcal{C}^W$. Then $\eta^W X : X \to \lambda^W X$ is an isomorphism in $\mathcal{D}$. Thus Remark 7.5(iii) and Corollary 7.9 yield the following result.

Corollary 7.12. *Let W be a subset of* $\operatorname{Spec} R$, *and $\mathbb{W} = \{W_i\}_{0 \le i \le n}$ be a system of slices of W. Let X be a complex of flat R-modules with* $\operatorname{cosupp} X \subseteq W$. *Then the chain map $\ell^{\mathbb{W}} X : X \to \operatorname{tot} L^{\mathbb{W}} X$ is a quasi-isomorphism, where* $\operatorname{tot} L^{\mathbb{W}} X$ *consists of cotorsion flat R-modules with cosupports in W.*

Remark 7.13. If $d = \dim R$ is finite, then any complex Y is quasi-isomorphic to a K-flat complex consisting of cotorsion flat R-modules. To see this, set

$$W_i = \{\mathfrak{p} \in \operatorname{Spec} R \mid \dim R/\mathfrak{p} = i\}$$

for $0 \le i \le d$. Then $\mathbb{W} = \{W_i\}_{0 \le i \le d}$ is a system of slices of $\operatorname{Spec} R$. We take a K-flat resolution X of Y such that X consists of flat R-modules. Corollary 7.12 implies that $\ell^{\mathbb{W}} X : X \to \operatorname{tot} L^{\mathbb{W}} X$ is a quasi-isomorphism, and $\operatorname{tot} L^{\mathbb{W}} X$ consists of cotorsion flat R-modules. At the same time, the chain maps $\ell^{\mathbb{W}} X^i : X^i \to L^{\mathbb{W}} X^i$ are quasi-isomorphisms for all $i \in \mathbb{Z}$. Then it is not hard to see that the mapping cone of $\ell^{\mathbb{W}} X$ is K-flat. Thus $\operatorname{tot} L^{\mathbb{W}} X$ is K-flat.

By Proposition 6.3 and Corollary 7.12, we have the next result.

Corollary 7.14. *Let W be a subset of* $\operatorname{Spec} R$ *such that* $\dim W$ *is finite. Then a complex $X \in \mathcal{D}$ belongs to $\mathcal{C}^W$ if and only if X is isomorphic to a complex Z of cotorsion flat R-modules such that* $\operatorname{cosupp} Z^i \subseteq W$ *for all $i \in \mathbb{Z}$.*

Remark 7.15. If $\dim W$ is infinite, it is possible to construct a similar family to systems of slices. We first put $W_0 = \max W$. Let $i > 0$ be an ordinal, and suppose that subsets W_j of W are defined for all $j < i$. Then we put $W_i = \max\big(W \setminus \bigcup_{j<i} W_j\big)$. In this way, we obtain the smallest ordinal $o(W)$ satisfying the following conditions:

(1) $W = \bigcup_{0 \le i < o(W)} W_i$.

(2) $W_i \cap W_j = \varnothing$ if $i \ne j$.

(3) $\dim W_i \le 0$ for $0 \le i < o(W)$.

(4) W_i is specialization-closed in $\bigcup_{i \le j < o(W)} W_j$ for each $0 \le i < o(W)$.

One should remark that the ordinal $o(W)$ can be uncountable in general; see [Gordon and Robson 1973, p. 48, Theorem 9.8]. However, if R is an infinite-dimensional commutative Noetherian ring given by Nagata [1962, Appendix A1, Example 1], then $o(W)$ is at most countable. Moreover, using transfinite induction, it is possible to extend condition (4) in Proposition 6.3 and Corollary 6.5 to the case where $o(W)$ is countable. One can also extend Corollary 7.14 to the case where $o(W)$ is countable.

Using Theorem 3.22 and results in [Nakamura and Yoshino 2018, §3], it is possible to give a similar result to Corollary 7.9, for colocalization functors γ_W and complexes of injective R-modules.

8. Čech complexes and complexes of finitely generated modules

Let W be a subset of $\operatorname{Spec} R$ and $\mathbb{W} = \{W_i\}_{0\le i\le n}$ be a system of slices of W. In this section, we prove that $\lambda^W Y$ is isomorphic to $\operatorname{tot} L^{\mathbb{W}} Y$ if Y is a complex of finitely generated R-modules.

We denote by $\mathcal{D}_{\mathrm{fg}}$ the full subcategory of $\mathcal{D}$ consisting of all complexes with finitely generated cohomology modules, and set $\mathcal{D}^-_{\mathrm{fg}} = \mathcal{D}^- \cap \mathcal{D}_{\mathrm{fg}}$. We first prove the following proposition.

Proposition 8.1. *Let W be a subset of* $\operatorname{Spec} R$ *such that* $\dim W$ *is finite. Let $X, Y \in \mathcal{D}$. We suppose that one of the following conditions holds*:

(1) *$X \in \mathcal{D}^-$ and $Y \in \mathcal{D}^-_{\mathrm{fg}}$.*

(2) *X is a bounded complex of flat R-modules and $Y \in \mathcal{D}_{\mathrm{fg}}$.*

Then there are natural isomorphisms

$$(\gamma_{W^c} X) \otimes^{\mathrm{L}}_R Y \cong \gamma_{W^c}(X \otimes^{\mathrm{L}}_R Y), \quad (\lambda^W X) \otimes^{\mathrm{L}}_R Y \cong \lambda^W (X \otimes^{\mathrm{L}}_R Y).$$

For $X \in \mathcal{D}$ and $n \in \mathbb{Z}$, we define the cohomological truncations $\sigma_{\le n} X$ and $\sigma_{>n} X$ as follows (see [Hartshorne 1966, Chapter I, §7]):

$$\sigma_{\le n} X = (\cdots \to X^{n-2} \to X^{n-1} \to \operatorname{Ker} d_X^n \to 0 \to \cdots),$$
$$\sigma_{>n} X = (\cdots \to 0 \to \operatorname{Im} d_X^n \to X^{n+1} \to X^{n+2} \to \cdots).$$

Proof of Proposition 8.1. Applying $(-) \otimes^{\mathrm{L}}_R Y$ to the triangle

$$\gamma_{W^c} X \to X \to \lambda^W X \to \gamma_{W^c} X[1],$$

we obtain the triangle

$$(\gamma_{W^c} X) \otimes^{\mathrm{L}}_R Y \longrightarrow X \otimes^{\mathrm{L}}_R Y \longrightarrow (\lambda^W X) \otimes^{\mathrm{L}}_R Y \longrightarrow (\gamma_{W^c} X) \otimes^{\mathrm{L}}_R Y[1].$$

Since $\operatorname{supp} \gamma_{W^c} X \subseteq W^c$, we have $\operatorname{supp}(\gamma_{W^c} X) \otimes^{\mathrm{L}}_R Y \subseteq W^c$, i.e., $(\gamma_{W^c} X) \otimes^{\mathrm{L}}_R Y \in \mathcal{L}_{W^c}$. Hence it remains to show that $(\lambda^W X) \otimes^{\mathrm{L}}_R Y \in \mathcal{C}^W$; see Lemma 2.6.

Case 1: We remark that X is isomorphic to a right bounded complex of flat R-modules. Then it is seen from Corollary 7.9 that $\lambda^W X$ is isomorphic to a right bounded complex Z of cotorsion flat R-modules such that $\operatorname{cosupp} Z^i \subseteq W$ for all $i \in \mathbb{Z}$. Furthermore, Y is isomorphic to a right bounded complex P of finite free R-modules. Hence it follows that $X \otimes_R^{\mathrm{L}} Y \cong Z \otimes_R P$, where the second one consists of cotorsion flat R-modules with cosupports in W. Then we have $X \otimes_R^{\mathrm{L}} Y \cong Z \otimes_R P \in \mathcal{C}^W$ by Proposition 6.3.

Case 2: By Corollary 7.9, $\lambda^W X$ is isomorphic to a bounded complex consisting of cotorsion flat R-modules with cosupports in W. Thus it is enough to prove that $Z \otimes_R Y \in \mathcal{C}^W$ for a cotorsion flat R-module Z with $\operatorname{cosupp} Z \subseteq W$.

We consider the triangle $\sigma_{\leq n} Y \to Y \to \sigma_{>n} Y \to \sigma_{\leq n} Y[1]$ for an integer n. Applying $Z \otimes_R (-)$ to this triangle, we obtain the following one:

$$Z \otimes_R \sigma_{\leq n} Y \longrightarrow Z \otimes_R Y \longrightarrow Z \otimes_R \sigma_{>n} Y \longrightarrow Z \otimes_R \sigma_{\leq n} Y[1].$$

Let $\mathfrak{p} \in W^c$. Case 1 implies that $Z \otimes_R \sigma_{\leq n} Y \in \mathcal{C}^W$ for any $n \in \mathbb{Z}$, since $\lambda^W Z \cong Z$. Thus, applying $\mathrm{RHom}_R(\kappa(\mathfrak{p}), -)$ to the triangle above, we have

$$\mathrm{RHom}_R(\kappa(\mathfrak{p}), Z \otimes_R Y) \cong \mathrm{RHom}_R(\kappa(\mathfrak{p}), Z \otimes_R \sigma_{>n} Y).$$

Furthermore, taking a projective resolution P of $\kappa(\mathfrak{p})$, we have

$$\mathrm{RHom}_R(\kappa(\mathfrak{p}), Z \otimes_R \sigma_{>n} Y) \cong \mathrm{Hom}_R(P, Z \otimes_R \sigma_{>n} Y).$$

Let j be any integer. To see that $\mathrm{RHom}_R(\kappa(\mathfrak{p}), Z \otimes_R Y) = 0$, it suffices to show that there exists an integer n such that $H^0(\mathrm{Hom}_R(P[j], Z \otimes_R \sigma_{>n} Y)) = 0$. Note that $P^i = 0$ for $i > 0$. Moreover, each element of $H^0(\mathrm{Hom}_R(P[j], Z \otimes_R \sigma_{>n} Y)) \cong \mathrm{Hom}_{\mathcal{D}}(P[j], Z \otimes_R \sigma_{>n} Y)$ is represented by a chain map $P[j] \to Z \otimes_R \sigma_{>n} Y$. Therefore it holds that $H^0(\mathrm{Hom}_R(P[j], Z \otimes_R \sigma_{>n} Y)) = 0$ if $n > -j$. $\square$

Remark 8.2. (i) In the proposition, we can remove the finiteness condition on $\dim W$ if $W = V(\mathfrak{a})$ for an ideal $\mathfrak{a}$. In such cases, we need only use $\mathfrak{a}$-adic completions of free R-modules instead of cotorsion flat R-modules.

(ii) If W is a generalization-closed subset of $\operatorname{Spec} R$, then the isomorphisms in the proposition hold for any $X, Y \in \mathcal{D}$ because γ_{W^c} is isomorphic to $\mathrm{R}\Gamma_{W^c}$.

Let W be a subset of $\operatorname{Spec} R$ and $\mathbb{W} = \{W_i\}_{0 \leq i \leq n}$ be a system of slices of W. Let $Y \in \mathcal{D}_{\mathrm{fg}}$. By Propositions 8.1 and 7.7, we have

$$\lambda^W Y \cong (\lambda^W R) \otimes_R^{\mathrm{L}} Y \cong (L^{\mathbb{W}} R) \otimes_R Y. \tag{8.3}$$

Let F be a flat R-module and M be a finitely generated R-module. Then we see from Lemma 2.3 that

$$(\bar{\lambda}^{W_i} F) \otimes_R M \cong \bar{\lambda}^{W_i}(F \otimes_R M).$$

This fact ensures that $(\bar{\lambda}^{(i_m, \ldots, i_1, i_0)} R) \otimes_R M \cong \bar{\lambda}^{(i_m, \ldots, i_1, i_0)} M$. Thus, if Y is a complex of finitely generated R-modules, then there is a natural isomorphism

$$(L^{\mathbb{W}} R) \otimes_R Y \cong \operatorname{tot} L^{\mathbb{W}} Y \tag{8.4}$$

in $C(\operatorname{Mod} R)$. By (8.3) and (8.4), we have shown the following proposition.

Proposition 8.5. *Let W be a subset of* $\operatorname{Spec} R$ *and* $\mathbb{W} = \{W_i\}_{0 \le i \le n}$ *be a system of slices of W. Let Y be a complex of finitely generated R-modules. Then there is an isomorphism in $\mathcal{D}$*;

$$\lambda^W Y \cong \operatorname{tot} L^{\mathbb{W}} Y.$$

Under this identification, $\ell^{\mathbb{W}} Y : Y \to \operatorname{tot} L^{\mathbb{W}} Y$ coincides with $\eta^W Y : Y \to \lambda^W Y$ in $\mathcal{D}$.

We see from (8.4) and the remark below that it is also possible to give a quick proof of this proposition, provided that Y is a right bounded complex of finitely generated R-modules.

Remark 8.6. Let W be a subset of $\operatorname{Spec} R$ and $\mathbb{W} = \{W_i\}_{0 \le i \le n}$ be a system of slices of W. We denote by $K(\operatorname{Mod} R)$ the homotopy category of complexes of R-modules. Note that $\operatorname{tot} L^{\mathbb{W}}$ induces a triangulated functor $K(\operatorname{Mod} R) \to K(\operatorname{Mod} R)$, which we also write $\operatorname{tot} L^{\mathbb{W}}$. Then it is seen from Corollary 7.9 that $\lambda^W : \mathcal{D} \to \mathcal{D}$ is isomorphic to the left derived functor of $\operatorname{tot} L^{\mathbb{W}} : K(\operatorname{Mod} R) \to K(\operatorname{Mod} R)$.

Let W be a subset of $\operatorname{Spec} R$ such that $n = \dim W$ is finite. By Proposition 8.5, if an R-module M is finitely generated, then $\lambda^W M \in \mathcal{D}^{[0,n]}$. On the other hand, since $\lambda^{V(\mathfrak{a})} \cong \mathrm{L}\Lambda^{V(\mathfrak{a})}$ for an ideal $\mathfrak{a}$, it can happen that $H^i(\lambda^W M) \neq 0$ for some $i < 0$ when M is not finitely generated; see [Nakamura and Yoshino 2018, Example 5.3].

Remark 8.7. Let $n \ge 0$ be an integer. Let $\mathfrak{a}_i$ be ideals of R and S_i be multiplicatively closed subsets of R for $0 \le i \le n$. In Notation 7.2 and Definition 7.4, one can replace $\bar{\lambda}^{(i)} = \bar{\lambda}^{W_i}$ by $\Lambda^{V(\mathfrak{a}_i)}(- \otimes_R S_i^{-1} R)$, and construct a kind of Čech complex. For this Čech complex and λ^W with $W = \bigcup_{0 \le i \le n} (V(\mathfrak{a}_i) \cap U_{S_i})$, it is possible to show similar results to Corollary 7.9 and Proposition 8.5, provided that one of the following conditions holds:

(1) $V(\mathfrak{a}_i) \cap U_{S_i}$ is specialization-closed in $\bigcup_{i \le j \le n} (V(\mathfrak{a}_j) \cap U_{S_j})$ for each $0 \le i \le n$.

(2) $V(\mathfrak{a}_i) \cap U_{S_i}$ is generalization-closed in $\bigcup_{0 \le j \le i} (V(\mathfrak{a}_j) \cap U_{S_j})$ for each $0 \le i \le n$.

9. Čech complexes and complexes of pure-injective modules

In this section, as an application, we give a functorial way to construct a quasi-isomorphism from a complex of flat R-modules or a complex of finitely generated R-modules to a complex of pure-injective R-modules.

We start with the following well known fact.

Lemma 9.1. *Let X be a complex of flat R-modules and Y be a complex of cotorsion R-modules. We assume that one of the following conditions holds*:

(1) *X is right bounded and Y is left bounded.*

(2) *X is bounded and* $\dim R$ *is finite.*

Then we have $\operatorname{RHom}_R(X, Y) \cong \operatorname{Hom}_R(X, Y)$.

One can prove this lemma by [Kashiwara and Schapira 2006, Theorem 12.5.4] and Theorem 4.1.

Next, we recall the notion of pure-injective modules and resolutions. We say that a morphism $f : M \to N$ of R-modules is pure if $f \otimes_R L$ is a monomorphism in $\operatorname{Mod} R$ for any R-module L. Moreover, an R-module P is called pure-injective if $\operatorname{Hom}_R(f, P)$ is an epimorphism in $\operatorname{Mod} R$ for any pure morphism $f : M \to N$ of R-modules. Clearly, all injective R-modules are pure-injective. Furthermore, all pure-injective R-modules are cotorsion; see [Enochs and Jenda 2000, Lemma 5.3.23].

Let M be an R-module. A complex P together with a quasi-isomorphism $M \to P$ is called a pure-injective resolution of M if P consists of pure-injective R-modules and $P^i = 0$ for $i < 0$. It is known that any R-module has a minimal pure-injective resolution, which is constructed by using pure-injective envelopes, see [Enochs 1987] and [Enochs and Jenda 2000, Example 6.6.5, Definition 8.1.4]. Moreover, if F is a flat R-module and P is a pure-injective resolution of M, then we have $\operatorname{RHom}_R(F, M) \cong \operatorname{Hom}_R(F, P)$ by Lemma 9.1.

Now we observe that any cotorsion flat R-module is pure-injective. Consider an R-module of the form $\big(\bigoplus_B R_{\mathfrak{p}}\big)^{\wedge}_{\mathfrak{p}}$ with some index set B and a prime ideal $\mathfrak{p}$, which is a cotorsion flat R-module. Writing $E_R(R/\mathfrak{p})$ for the injective hull of $R/\mathfrak{p}$, we have

$$\Big(\bigoplus_B R_{\mathfrak{p}}\Big)^{\wedge}_{\mathfrak{p}} \cong \operatorname{Hom}_R\Big(E_R(R/\mathfrak{p}), \bigoplus_B E_R(R/\mathfrak{p})\Big);$$

see [Enochs and Jenda 2000, Theorem 3.4.1]. It follows from tensor-hom adjunction that $\operatorname{Hom}_R(M, I)$ is pure-injective for any R-module M and any injective R-module I. Hence $\big(\bigoplus_B R_{\mathfrak{p}}\big)^{\wedge}_{\mathfrak{p}}$ is pure-injective. Thus any cotorsion flat R-module is pure-injective; see Proposition 5.1.

There is another example of pure-injective R-modules. Let M be a finitely generated R-module. Using the Five Lemma, we are able to prove an isomorphism

$$\operatorname{Hom}_R\Big(E_R(R/\mathfrak{p}), \bigoplus_B E_R(R/\mathfrak{p})\Big) \otimes_R M \cong \operatorname{Hom}_R\Big(\operatorname{Hom}_R(M, E_R(R/\mathfrak{p})), \bigoplus_B E_R(R/\mathfrak{p})\Big).$$

Therefore $\big(\bigoplus_B R_{\mathfrak{p}}\big)^{\wedge}_{\mathfrak{p}} \otimes_R M$ is pure-injective; it is also isomorphic to $\big(\bigoplus_B M_{\mathfrak{p}}\big)^{\wedge}_{\mathfrak{p}}$ by Lemma 2.3. Further, Proposition 8.1 implies that $\operatorname{cosupp}\big(\bigoplus_B M_{\mathfrak{p}}\big)^{\wedge}_{\mathfrak{p}} \subseteq \{\mathfrak{p}\}$.

By the above observation, we see that Corollary 7.12, (8.4) and Proposition 8.5 yield the following theorem, which is one of the main results of this paper.

Theorem 9.2. *Let W be a subset of* $\operatorname{Spec} R$ *and $\mathbb{W} = \{W_i\}_{0 \le i \le n}$ be a system of slices of W. Let Z be a complex of flat R-modules or a complex of finitely generated*

R-modules. We assume that $\operatorname{cosupp} Z \subseteq W$. *Then* $\ell^{\mathbb{W}} Z : Z \to \operatorname{tot} L^{\mathbb{W}} Z$ *is a quasi-isomorphism, where* $\operatorname{tot} L^{\mathbb{W}} Z$ *consists of pure-injective R-modules with cosupports in W.*

Remark 9.3. Let N be a flat or finitely generated R-module. Suppose that $d = \dim R$ is finite. Set $W_i = \{\mathfrak{p} \in \operatorname{Spec} R \mid \dim R/\mathfrak{p} = i\}$ and $\mathbb{W} = \{W_i\}_{0 \le i \le d}$. By Theorem 9.2, we obtain a pure-injective resolution $\ell^{\mathbb{W}} N : N \to L^{\mathbb{W}} N$ of N, that is, there is an exact sequence of R-modules of the form

$$0 \to N \to \prod_{0 \le i_0 \le d} \bar{\lambda}^{(i_0)} N \to \prod_{0 \le i_0 < i_1 \le d} \bar{\lambda}^{(i_1, i_0)} N \to \cdots \to \bar{\lambda}^{(d, \dots, 0)} N \to 0.$$

We remark that, in $C(\operatorname{Mod} R)$, $L^{\mathbb{W}} N$ need not be isomorphic to a minimal pure-injective resolution P of N. In fact, when N is a projective or finitely generated R-module, it holds that $P^0 \cong \prod_{\mathfrak{m} \in W_0} \widehat{N_{\mathfrak{m}}} = \bar{\lambda}^{(0)} N$ (see [Warfield 1969, Theorem 3] and [Enochs and Jenda 2000, Remark 6.7.12]), while $(L^{\mathbb{W}} N)^0 = \prod_{0 \le i_0 \le d} \bar{\lambda}^{(i_0)} N$. Furthermore, Enochs [1987, Theorem 2.1] proved that if N is a flat R-module, then P^i is of the form $\prod_{\mathfrak{p} \in W_{\ge i}} T_{\mathfrak{p}}^i$ for $0 \le i \le d$ (see Notation 6.1), where

$$W_{\ge i} = \{\mathfrak{p} \in \operatorname{Spec} R \mid \dim R/\mathfrak{p} \ge i\}.$$

On the other hand, for a flat or finitely generated R-module N, the differential maps in the pure-injective resolution $L^{\mathbb{W}} N$ are concretely described. In addition, our approach based on the localization functor λ^W and the Čech complex $L^{\mathbb{W}}$ provide a natural morphism $\ell^{\mathbb{W}} : \operatorname{id}_{C(\operatorname{Mod} R)} \to \operatorname{tot} L^{\mathbb{W}}$ which induces isomorphisms in $\mathcal{D}$ for all complexes of flat R-modules and complexes of finitely generated R-modules. The reader should also compare Theorem 9.2 with [Thompson 2017b, Theorem 5.2].

We close this paper with the following example of Theorem 9.2.

Example 9.4. Let R be a 2-dimensional local domain with quotient field Q. Let $\mathbb{W} = \{W_i\}_{0 \le i \le 2}$ be as in Remark 9.3. Then $L^{\mathbb{W}} R$ is a pure-injective resolution of R, and $L^{\mathbb{W}} R$ is of the following form:

$$0 \to Q \oplus \Big(\prod_{\mathfrak{p} \in W_1} \widehat{R_{\mathfrak{p}}} \Big) \oplus \widehat{R} \to \Big(\prod_{\mathfrak{p} \in W_1} \widehat{R_{\mathfrak{p}}} \Big)_{(0)} \oplus (\widehat{R})_{(0)} \oplus \prod_{\mathfrak{p} \in W_1} \widehat{(\widehat{R})_{\mathfrak{p}}} \to \Big(\prod_{\mathfrak{p} \in W_1} \widehat{(\widehat{R})_{\mathfrak{p}}} \Big)_{(0)} \to 0$$

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References

[Alonso Tarrío et al. 1997] L. Alonso Tarrío, A. Jeremías López, and J. Lipman, "Local homology and cohomology on schemes", *Ann. Sci. Éc. Norm. Supér* (4) **30**:1 (1997), 1–39. Correction available at https://tinyurl.com/lipmanfix. MR Zbl

[Benson et al. 2008] D. Benson, S. B. Iyengar, and H. Krause, "Local cohomology and support for triangulated categories", *Ann. Sci. Éc. Norm. Supér.* (4) **41**:4 (2008), 573–619. MR Zbl

[Benson et al. 2012] D. J. Benson, S. B. Iyengar, and H. Krause, "Colocalizing subcategories and cosupport", *J. Reine Angew. Math.* **673** (2012), 161–207. MR Zbl

[Bökstedt and Neeman 1993] M. Bökstedt and A. Neeman, "Homotopy limits in triangulated categories", *Compositio Math.* **86**:2 (1993), 209–234. MR Zbl

[Bourbaki 1961] N. Bourbaki, *Algèbre commutative: Chapitre 1: Modules plats; Chapitre 2: Localisation*, Hermann, Paris, 1961. MR Zbl

[Enochs 1984] E. Enochs, "Flat covers and flat cotorsion modules", *Proc. Amer. Math. Soc.* **92**:2 (1984), 179–184. MR Zbl

[Enochs 1987] E. E. Enochs, "Minimal pure injective resolutions of flat modules", *J. Algebra* **105**:2 (1987), 351–364. MR Zbl

[Enochs and Jenda 2000] E. E. Enochs and O. M. G. Jenda, *Relative homological algebra*, de Gruyter Expositions in Math. **30**, de Gruyter, Berlin, 2000. MR Zbl

[Foxby and Iyengar 2003] H.-B. Foxby and S. Iyengar, "Depth and amplitude for unbounded complexes", pp. 119–137 in *Commutative algebra* (Grenoble/Lyon, 2001), edited by L. L. Avramov et al., Contemp. Math. **331**, Amer. Math. Soc., Providence, RI, 2003. MR Zbl

[Gordon and Robson 1973] R. Gordon and J. C. Robson, *Krull dimension*, Mem. Amer. Math. Soc. **133**, Amer. Math. Soc., Providence, RI, 1973. MR Zbl

[Greenlees and May 1992] J. P. C. Greenlees and J. P. May, "Derived functors of I-adic completion and local homology", *J. Algebra* **149**:2 (1992), 438–453. MR Zbl

[Hartshorne 1966] R. Hartshorne, *Residues and duality*, Lecture Notes in Math. **20**, Springer, 1966. MR Zbl

[Kashiwara and Schapira 2006] M. Kashiwara and P. Schapira, *Categories and sheaves*, Grundlehren der Math. Wissenschaften **332**, Springer, 2006. MR Zbl

[Krause 2010] H. Krause, "Localization theory for triangulated categories", pp. 161–235 in *Triangulated categories*, edited by T. Holm et al., London Math. Soc. Lecture Note Ser. **375**, Cambridge Univ. Press, 2010. MR Zbl

[Lipman 2002] J. Lipman, "Lectures on local cohomology and duality", pp. 39–89 in *Local cohomology and its applications* (Guanajuato, 1999), edited by G. Lyubeznik, Lecture Notes in Pure and Appl. Math. **226**, Dekker, New York, 2002. MR Zbl

[Nagata 1962] M. Nagata, *Local rings*, Interscience Tracts in Pure and Appl. Math. **13**, Interscience, New York, 1962. MR Zbl

[Nakamura and Yoshino 2018] T. Nakamura and Y. Yoshino, "A local duality principle in derived categories of commutative Noetherian rings", *J. Pure Appl. Algebra* **222**:9 (2018), 2580–2595.

[Neeman 1992] A. Neeman, "The chromatic tower for $D(R)$", *Topology* **31**:3 (1992), 519–532. MR Zbl

[Neeman 2011] A. Neeman, "Colocalizing subcategories of $D(R)$", *J. Reine Angew. Math.* **653** (2011), 221–243. MR Zbl

[Raynaud and Gruson 1971] M. Raynaud and L. Gruson, "Critères de platitude et de projectivité: techniques de 'platification' d'un module", *Invent. Math.* **13** (1971), 1–89. MR Zbl

[Sather-Wagstaff and Wicklein 2017] S. Sather-Wagstaff and R. Wicklein, "Support and adic finiteness for complexes", *Comm. Algebra* **45**:6 (2017), 2569–2592. MR Zbl

[Simon 1990] A.-M. Simon, "Some homological properties of complete modules", *Math. Proc. Cambridge Philos. Soc.* **108**:2 (1990), 231–246. MR Zbl

[Thompson 2017a] P. Thompson, "Cosupport computations for finitely generated modules over commutative Noetherian rings", preprint, 2017. arXiv

[Thompson 2017b] P. Thompson, "Minimal complexes of cotorsion flat modules", preprint, 2017. arXiv

[Warfield 1969] R. B. Warfield, Jr., "Purity and algebraic compactness for modules", *Pacific J. Math.* **28** (1969), 699–719. MR Zbl

[Xu 1996] J. Xu, *Flat covers of modules*, Lecture Notes in Math. **1634**, Springer, 1996. MR Zbl

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