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REMARKS ON CRITICAL METRICS OF THE SCALAR CURVATURE AND VOLUME FUNCTIONALS ON COMPACT MANIFOLDS WITH BOUNDARY

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We provide a general Bochner type formula which enables us to prove some rigidity results for *V*-static spaces. In particular, we show that an *n*-dimensional positive static triple with connected boundary and positive scalar curvature must be isometric to the standard hemisphere, provided that the metric has zero radial Weyl curvature and satisfies a suitable pinching condition. Moreover, we classify *V*-static spaces with nonnegative sectional curvature.

1. Introduction

Let (M^n, g) be a connected Riemannian manifold. Following the terminology used by Miao and Tam [2009] as well as Corvino, Eichmair and Miao [Corvino et al. 2013], we say that g is a *V-static metric* if there is a smooth function f on M^n and a constant κ satisfying the V-static equation

$$\mathfrak{L}_g^*(f) = -(\Delta f)g + \operatorname{Hess} f - f \operatorname{Ric} = \kappa g, \tag{1-1}$$

where $\mathfrak{L}_g^*$ is the formal L^2-adjoint of the linearization of the scalar curvature operator $\mathfrak{L}_g$, which plays an important role in problems related to prescribing the scalar curvature function. Here, Ric, Δ and Hess stand, respectively, for the Ricci tensor, the Laplacian operator and the Hessian form on M^n. Such a function f is called *V-static potential*.

It is well known that V-static metrics are important in understanding the interplay between volume and scalar curvature. They arise from the modified problem of finding stationary points for the volume functional on the space of metrics whose scalar curvature is equal to a given constant (see [Corvino et al. 2013; Miao and Tam 2009; 2011; Yuan 2016]). In general, the scalar curvature is not sufficient for controlling the volume. However, Miao and Tam [2012] proved a rigidity result for the upper hemisphere with respect to nondecreasing scalar curvature and volume.

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Corvino et al. [2013] were able to show that when the metric g does not admit a nontrivial solution to (1-1), then one can achieve simultaneously a prescribed perturbation of the scalar curvature that is compactly supported in a bounded domain Ω and a prescribed perturbation of the volume by a small deformation of the metric in $\bar{\Omega}$. We highlight that a Riemannian metric g for which there exists a nontrivial function f satisfying (1-1) must have constant scalar curvature R (see [Corvino et al. 2013, Proposition 2.1; Miao and Tam 2009, Theorem 7]).

The case where $\kappa \neq 0$ in (1-1) and the potential function f vanishes on the boundary was studied by Miao and Tam [2009]. In this approach, a *Miao–Tam critical metric* is a 3-tuple (M^n, g, f), where (M^n, g) is a compact Riemannian manifold of dimension at least 3 with a smooth boundary ∂M and $f : M^n \to \mathbb{R}$ is a smooth function such that $f^{-1}(0) = \partial M$ satisfying the overdetermined-elliptic system

$$\mathfrak{L}_g^*(f) = -(\Delta f)g + \operatorname{Hess} f - f \operatorname{Ric} = g. \tag{1-2}$$

Miao and Tam [2009] showed that these critical metrics arise as critical points of the volume functional on M^n when restricted to the class of metrics g with prescribed constant scalar curvature such that $g_{|T\partial M} = h$ for a prescribed Riemannian metric h on the boundary. Some explicit examples of Miao–Tam critical metrics are in the form of warped products and those examples include the spatial Schwarzschild metrics and AdS–Schwarzschild metrics restricted to certain domains containing their horizon and bounded by two spherically symmetric spheres (see Corollaries 3.1 and 3.2 in [Miao and Tam 2011]). For more details see, for instance, [Baltazar and Ribeiro 2017; Barros et al. 2015; Batista et al. 2017; Corvino et al. 2013; Miao and Tam 2009; 2011; Yuan 2016].

We also remark that (1-1) can be seen as a generalization of the *static equation* $\mathfrak{L}_g^*(f) = 0$ (see [Ambrozio 2017; Corvino 2000]), namely, $\kappa = 0$ in (1-1). We remember that a *positive static triple* is a triple (M^n, g, f) consisting of a connected n-dimensional smooth manifold M with boundary ∂M (possibly empty), a complete Riemannian metric g on M and a nontrivial static potential $f \in C^\infty(M)$ that is nonnegative, vanishes precisely on ∂M and satisfies the static equation

$$\mathfrak{L}_g^*(f) = -(\Delta f)g + \operatorname{Hess} f - f \operatorname{Ric} = 0. \tag{1-3}$$

For the sake of completeness, it is very important to recall the following classical example of a positive static triple with nonempty boundary.

Example 1.1. An example of positive static triple with connected nonempty boundary is given by choosing $(\mathbb{S}^n_+(r), g)$ to be the open upper n-hemisphere $\mathbb{S}^n_+(r)$ of radius r in $\mathbb{R}^{n+1}$ endowed with the Euclidean metric g. Hence, $\partial M = \mathbb{S}^{n-1}(r)$ is the equator and the corresponding height function f on $\mathbb{S}^n_+(r)$ is positive, vanishes along $\partial M = \mathbb{S}^{n-1}(r)$ and satisfies (1-3).

It has been conjectured in 1984 that the only static vacuum spacetime with positive cosmological constant and connected event horizon is the de Sitter space of radius r. This conjecture is the so-called *cosmic no-hair conjecture* and it was formulated by Boucher, Gibbons and Horowitz in [Boucher et al. 1984]. It is closely related to the Fischer–Marsden conjecture (see [Shen 1997]). It should be emphasized that there are positive static triples with double boundary, such as the *Nariai space*. Hence, connectedness of the boundary is essential for Conjecture 1.2 to be true.

Conjecture 1.2 [Boucher et al. 1984, cosmic no-hair conjecture]. *Example 1.1 is the only possible n-dimensional positive static triple with single-horizon* (*connected*) *and positive scalar curvature.*

In the last decades some partial answers to Conjecture 1.2 were achieved. For instance, if (M^n, g) is Einstein it suffices to apply the Obata type theorem due to Reilly [1977] (see also [Obata 1962]) to conclude that Conjecture 1.2 is true. Moreover, Kobayashi [1982] and Lafontaine [1983] proved independently that such a conjecture is also true under the conformally flat condition.

For what follows, we recall that the Bach tensor on a Riemannian manifold (M^n, g), $n \geq 4$, is defined in terms of the components of the Weyl tensor W_{ikjl} as

$$B_{ij} = \frac{1}{n-3}\nabla^k\nabla^l W_{ikjl} + \frac{1}{n-2} R_{kl} W_i{}^k{}_j{}^l, \tag{1-4}$$

while for $n = 3$ it is given by

$$B_{ij} = \nabla^k C_{kij}, \tag{1-5}$$

where C_{ijk} stands for the Cotton tensor. We say that (M^n, g) is Bach-flat when $B_{ij} = 0$.

Qing and Yuan [2013] obtained a classification result for static spaces under Bach-flat assumption. In particular, it is not hard to see that the method used by Qing and Yuan implies that such a conjecture is also true under Bach-flat assumption (see Theorem 1.3 below). Gibbons, Hartnoll and Pope [Gibbons et al. 2003] constructed counterexamples to the cosmic no-hair conjecture in the cases $4 \leq n \leq 8$. However, it remains interesting to show under which conditions such a conjecture remains true. For more details on this subject and further partial answers see, for instance, [Ambrozio 2017; Boucher et al. 1984; Chruściel 2011; Hijazi et al. 2015; Shen 1997]. Next, let us recall the following useful classification.

Theorem 1.3 [Kobayashi 1982; Lafontaine 1983; Qing and Yuan 2013]. *Let (M^n, g, f) be an n-dimensional positive static triple with scalar curvature $R = n(n-1)$. Suppose that (M^n, g) is Bach-flat, then (M^n, g, f) is covered by a static triple equivalent to one of the following static triples*:

(1) *The standard hemisphere with canonical metric*

$$(\mathbb{S}^n_+, g_{\mathbb{S}^{n-1}}, f = x_{n+1}).$$

(2) *The standard cylinder over* $\mathbb{S}^{n-1}$ *with the product metric*

$$\Big(M = \Big[0, \frac{\pi}{\sqrt{n}}\Big] \times \mathbb{S}^{n-1},\ g = dt^2 + \frac{n-2}{n} g_{\mathbb{S}^{n-1}},\ f(t) = \sin(\sqrt{n}t)\Big).$$

(3) *For some constant* $m \in (0, \sqrt{(n-2)^{n-2}/n^n})$ *we consider the Schwarzschild space defined by*

$$\Big(M = [r_1, r_2] \times \mathbb{S}^{n-1},\ g = \frac{1}{1-2mt^{2-n}-t^2} dt^2 + t^2 g_{\mathbb{S}^{n-1}},\ f(t) = \sqrt{1-2mt^{2-n}-t^2}\Big),$$

where $r_1 < r_2$ *are the positive zeroes of* f.

Ambrozio [2017] obtained interesting classification results for static three-dimensional manifolds with positive scalar curvature. To do so, he proved a Bochner type formula for three-dimensional positive static triples involving the traceless Ricci tensor and the Cotton tensor. A similar Bochner type formula was obtained by Batista et al. [2017] for three-dimensional Riemannian manifolds satisfying (1-2). Those formulae may be used to rule out some possible new examples. In this article, we extend such Bochner type formulae for a more general class of metrics and arbitrary dimension $n > 2$. To be precise, we have established the following result.

Theorem 1.4. *Let* (M^n, g, f, κ) *be a connected, smooth Riemannian manifold and* f *is a smooth function on* M^n *satisfying the V-static* (1-1). *Then we have*

$$\frac{1}{2}\operatorname{div}(f\nabla|\mathrm{Ric}|^2) = \Big(\frac{n-2}{n-1}|C_{ijk}|^2 + |\nabla \mathrm{Ric}|^2\Big) f + \frac{n\kappa}{n-1}|\mathring{\mathrm{Ric}}|^2 + \Big(\frac{2}{n-1}R|\mathring{\mathrm{Ric}}|^2 + \frac{2n}{n-2}\operatorname{tr}(\mathring{\mathrm{Ric}}^3)\Big) f - \frac{n-2}{n-1}W_{ijkl}\nabla_l f C_{ijk} - 2f W_{ijkl}R_{ik}R_{jl}, \tag{1-6}$$

where C *stands for the Cotton tensor*, W *is the Weyl tensor and* $\mathring{\mathrm{Ric}}$ *is the traceless Ricci tensor.*

Remembering that three-dimensional Riemannian manifolds have vanishing Weyl tensor, it is easy to see that Theorem 1.4 is a generalization, for any dimension, of Theorem 3 in [Batista et al. 2017] as well as Proposition 12 in [Ambrozio 2017].

Before presenting a couple of applications of the above formula it is fundamental to remember that a Riemannian manifold (M^n, g) has *zero radial Weyl curvature* when

$$W(\cdot, \cdot, \cdot, \nabla f) = 0, \tag{1-7}$$

for a suitable potential function f on M^n. This class of manifolds includes the case of locally conformally flat manifolds. Moreover, this condition has been used to classify gradient Ricci solitons as well as quasi-Einstein manifolds (see [Catino

2012; He et al. 2012; Petersen and Wylie 2010]). Here, we shall use this condition to obtain the following corollary.

Corollary 1.5. *Let (M^n, g, f) be a compact, oriented, connected Miao–Tam critical metric with positive scalar curvature and nonnegative potential function f. Suppose that*

- *M^n has zero radial Weyl curvature and*
- $|\mathring{\mathrm{R}}\mathrm{ic}|^2 \leq \frac{1}{n(n-1)} R^2$.

Then M^n must be isometric to a geodesic ball in $\mathbb{S}^n$.

It is not difficult to see that the above result generalizes Corollary 1 in [Batista et al. 2017]. Next, we get the following result for static spaces.

Corollary 1.6. *Let (M^n, g, f) be a compact, oriented, connected positive static triple with positive scalar curvature. Suppose that*

- *M^n has zero radial Weyl curvature and*
- $|\mathring{\mathrm{R}}\mathrm{ic}|^2 \leq \frac{1}{n(n-1)} R^2$.

Then one of the following assertions holds:

(1) *M^n is equivalent to the standard hemisphere of $\mathbb{S}^n$; or*

(2) $|\mathring{\mathrm{R}}\mathrm{ic}|^2 = \frac{1}{n(n-1)} R^2$ *and (M^n, g, f) is covered by a static triple that is equivalent to the standard cylinder.*

Remark 1.7. It is worthwhile to remark that Corollary 1.6 can be seen as a partial answer to Conjecture 1.2.

Remark 1.8. We also point out that four-dimensional V-static spaces with zero radial Weyl curvature must be locally conformally flat. To prove this claim it suffices to apply the same arguments used in the initial part of the proof of Theorem 2 in [Barros et al. 2015].

In order to proceed, we recall that a classical lemma due to Berger guarantees that any two symmetric tensors T on a Riemannian manifold (M^n, g) with nonnegative sectional curvature must satisfy

$$(\nabla_i \nabla_j T_{ik} - \nabla_j \nabla_i T_{ik}) T_{jk} \geq 0. \tag{1-8}$$

In fact, we have

$$(\nabla_i \nabla_j T_{ik} - \nabla_j \nabla_i T_{ik}) T_{jk} = \sum_{i<j} R_{ijij} (\lambda_i - \lambda_j)^2,$$

where the λ_i's are the eigenvalues of tensor T (see Lemma 4.1 in [Cao 2007]). Here, we shall use these data jointly with Theorem 1.4 to deduce a rigidity result for three-dimensional Miao–Tam critical metrics with nonnegative sectional curvature (see also Proposition 4.2 in Section 4 for a version in arbitrary dimension). More precisely, we have established the following result.

Theorem 1.9. *Let (M^3, g, f) be a three-dimensional compact, oriented, connected Miao–Tam critical metric with smooth boundary ∂M and nonnegative sectional curvature, with f assumed to be nonnegative. Then M^3 is isometric to a geodesic ball in a simply connected space form $\mathbb{R}^3$ or $\mathbb{S}^3$.*

Finally, we get the following result for positive static triples.

Theorem 1.10. *Let (M^n, g, f) be a positive static triple with nonnegative sectional curvature, zero radial Weyl curvature and scalar curvature $R = n(n-1)$. Then, up to a finite quotient, M^n is isometric to either the standard hemisphere $\mathbb{S}^n_+$ or the standard cylinder over $\mathbb{S}^{n-1}$ with the product metric described in Theorem 1.3.*

2. Preliminaries

In this section we shall present some preliminaries which will be useful for the establishment of the desired results. Firstly, we remember that a V-static space is a Riemannian manifold (M^n, g) with a nontrivial solution (f, κ) satisfying the overdetermined-elliptic system

$$-(\Delta f)g + \operatorname{Hess} f - f \operatorname{Ric} = \kappa g,$$

where κ is a constant. As usual, we rewrite in the tensorial language as

$$-(\Delta f)g_{ij} + \nabla_i \nabla_j f - f R_{ij} = \kappa g_{ij}. \tag{2-1}$$

Tracing (2-1) we deduce that f also satisfies the equation

$$\Delta f + \frac{R}{n-1} f + \frac{n\kappa}{n-1} = 0. \tag{2-2}$$

Moreover, by using (2-2) it is not difficult to check that

$$f \,\mathring{\operatorname{Ric}} = \mathring{\operatorname{Hess}} f, \tag{2-3}$$

where $\mathring{T}$ stands for the traceless part of T.

Before proceeding we recall two special tensors in the study of curvature for a Riemannian manifold (M^n, g), $n \geq 3$. The first one is the Weyl tensor W which is defined by the decomposition formula

$$R_{ijkl} = W_{ijkl} + \frac{1}{n-2}(R_{ik}g_{jl} + R_{jl}g_{ik} - R_{il}g_{jk} - R_{jk}g_{il}) - \frac{R}{(n-1)(n-2)}(g_{jl}g_{ik} - g_{il}g_{jk}), \tag{2-4}$$

where R_{ijkl} stands for the Riemann curvature operator Rm, whereas the second one is the Cotton tensor C given by

$$C_{ijk} = \nabla_i R_{jk} - \nabla_j R_{ik} - \frac{1}{2(n-1)}(\nabla_i R g_{jk} - j R g_{ik}). \tag{2-5}$$

Note that C_{ijk} is skew-symmetric in the first two indices and trace-free in any two indices. These two above tensors satisfy

$$C_{ijk} = -\frac{n-2}{n-3}\nabla_l W_{ijkl}, \tag{2-6}$$

provided $n \geq 4$.

For our purpose we also remember that as a consequence of the Bianchi identity

$$(\operatorname{div} Rm)_{jkl} = \nabla_k R_{jl} - \nabla_l R_{jk}. \tag{2-7}$$

Moreover, from commutation formulas (Ricci identities), for any Riemannian manifold (M^n, g) we have

$$\nabla_i \nabla_j R_{kl} - \nabla_j \nabla_i R_{kl} = R_{ijks} R_{sl} + R_{ijls} R_{ks}; \tag{2-8}$$

for more details, see [Chow et al. 2007; Viaclovsky 2011].

To conclude this section, we shall present the following lemma for V-static spaces, which was previously obtained in [Barros et al. 2015] for Miao–Tam critical metrics.

Lemma 2.1. *Let (M^n, g) be a connected, smooth Riemannian manifold and f be a smooth function on M^n satisfying* (1-1). *Then we have*:

$$f(\nabla_i R_{jk} - \nabla_j R_{ik}) = R_{ijkl}\nabla_l f + \frac{R}{n-1}(\nabla_i f g_{jk} - \nabla_j f g_{ik}) - (\nabla_i f R_{jk} - \nabla_j f R_{ik}).$$

Proof. The proof is standard, and it follows the same steps of Lemma 1 in [Barros et al. 2015]. For the sake of completeness we shall sketch it here. Firstly, since g is parallel we may use (2-1) to infer

$$(\nabla_i f) R_{jk} + f \nabla_i R_{jk} = \nabla_i \nabla_j \nabla_k f - (\nabla_i \Delta f) g_{jk}. \tag{2-9}$$

Next, since M^n has constant scalar curvature we have from (2-2) that

$$\nabla_i \Delta f = -\frac{R}{n-1}\nabla_i f,$$

which substituted into (2-9) gives

$$f \nabla_i R_{jk} = -(\nabla_i f) R_{jk} + \nabla_i \nabla_j \nabla_k f + \frac{R}{n-1}\nabla_i f g_{jk}. \tag{2-10}$$

Finally, we apply the Ricci identity to arrive at

$$f(\nabla_i R_{jk} - \nabla_j R_{ik}) = R_{ijkl}\nabla_l f + \frac{R}{n-1}(\nabla_i f g_{jk} - \nabla_j f g_{ik}) - (\nabla_i f R_{jk} - \nabla_j f R_{ik}). \qquad \square$$

3. A Bochner type formula and applications

In this section we shall provide a general Bochner type formula, which enables us to prove some rigidity results for V-static spaces. To do so, we shall obtain some identities involving the Cotton tensor and Weyl tensor on Riemannian manifolds satisfying the V-static equation. Following the notation employed in [Barros et al. 2015], we can use (2-4) jointly with Lemma 2.1 to obtain

$$fC_{ijk} = T_{ijk} + W_{ijkl}\nabla_l f, \tag{3-1}$$

where the auxiliary tensor T_{ijk} is defined as

$$T_{ijk} = \frac{n-1}{n-2}(R_{ik}\nabla_j f - R_{jk}\nabla_i f) + \frac{1}{n-2}(g_{ik}R_{js}\nabla_s f - g_{jk}R_{is}\nabla_s f) - \frac{R}{n-2}(g_{ik}\nabla_j f - g_{jk}\nabla_i f). \tag{3-2}$$

In the sequel, we obtain a divergent formula for any Riemannian manifold (M^n, g) with constant scalar curvature.

Lemma 3.1. *Let (M^n, g) be a connected Riemannian manifold with constant scalar curvature and $f : M \to \mathbb{R}$ be a smooth function defined on M. Then we have*

$$\begin{aligned}\operatorname{div}(f\nabla|\mathrm{Ric}|^2) = -f|C_{ijk}|^2 + 2f|\nabla \mathrm{Ric}|^2 + \langle\nabla f, \nabla|\mathrm{Ric}|^2\rangle + \frac{2n}{n-2} f R_{ij}R_{ik}R_{jk} \\ - \frac{4n-2}{(n-1)(n-2)} f R|\mathring{\mathrm{Ric}}|^2 - \frac{2}{n(n-2)} f R^3 \\ + 2\nabla_i(fC_{ijk}R_{jk}) + 2C_{ijk}\nabla_j f R_{ik} - 2f W_{ijkl}R_{ik}R_{jl}.\end{aligned}$$

Proof. Firstly, since M^n has constant scalar curvature we immediately get

$$f|C_{ijk}|^2 = f|\nabla_i R_{jk} - \nabla_j R_{ik}|^2 = 2f|\nabla \mathrm{Ric}|^2 - 2f\nabla_i R_{jk}\nabla_j R_{ik}.$$

On the other hand, easily one verifies that

$$\nabla_j(f\nabla_i R_{jk}R_{ik}) = \nabla_j f\nabla_i R_{jk}R_{ik} + f\nabla_j\nabla_i R_{jk}R_{ik} + f\nabla_i R_{jk}\nabla_j R_{ik}.$$

Hence, it follows that

$$f|C_{ijk}|^2 = 2f|\nabla \mathrm{Ric}|^2 - 2\nabla_j(f\nabla_i R_{jk}R_{ik}) + 2\nabla_j f\nabla_i R_{jk}R_{ik} + 2f\nabla_j\nabla_i R_{jk}R_{ik}.$$

Next, from the commutation formula for second covariant derivative of the Ricci curvature (see (2-8)) combined with (2-5), we deduce

$$\begin{aligned} f|C_{ijk}|^2 &= 2f|\nabla \mathrm{Ric}|^2 + 2\nabla_j f(C_{ijk} + \nabla_j R_{ik})R_{ik} \\ &\quad + 2f(R_{ij}R_{ik}R_{jl} - R_{ijkl}R_{ik}R_{jl}) - 2\nabla_j(f\nabla_i R_{jk}R_{ik}) \\ &= 2f|\nabla \mathrm{Ric}|^2 + 2C_{ijk}\nabla_j f R_{ik} + \langle\nabla f, \nabla|\mathrm{Ric}|^2\rangle \\ &\quad + 2f(R_{ij}R_{ik}R_{jk} - R_{ijkl}R_{ik}R_{jl}) - 2\nabla_j(f\nabla_i R_{jk}R_{ik}).\end{aligned} \tag{3-3}$$

Now, substituting (2-4) into (3-3) we achieve

$$\begin{aligned}
f|C_{ijk}|^2 &= 2f|\nabla \mathrm{Ric}|^2 + 2C_{ijk}\nabla_j f R_{ik} + \langle \nabla f, \nabla|\mathrm{Ric}|^2\rangle + 2fR_{ij}R_{ik}R_{jk} \\
&\quad - 2fW_{ijkl}R_{ik}R_{jl} - \frac{2f}{n-2}(2R|\mathrm{Ric}|^2 - 2R_{ij}R_{ik}R_{jk}) \\
&\quad + \frac{2Rf}{(n-1)(n-2)}(R^2 - |\mathrm{Ric}|^2) - 2\nabla_j(f\nabla_i R_{jk}R_{ik}) \\
&= 2f|\nabla \mathrm{Ric}|^2 + 2C_{ijk}\nabla_j f R_{ik} + \langle \nabla f, \nabla|\mathrm{Ric}|^2\rangle + \frac{2n}{n-2} fR_{ij}R_{ik}R_{jk} \\
&\quad - 2fW_{ijkl}R_{ik}R_{jl} - \frac{(4n-2)}{(n-1)(n-2)} fR|\mathrm{Ric}|^2 \\
&\quad + \frac{2}{(n-1)(n-2)} fR^3 - 2\nabla_j(f\nabla_i R_{jk}R_{ik}),
\end{aligned}$$

which can be rewritten as

$$\begin{aligned}
f|C_{ijk}|^2 &= 2f|\nabla \mathrm{Ric}|^2 + 2C_{ijk}\nabla_j f R_{ik} + \langle \nabla f, \nabla|\mathrm{Ric}|^2\rangle + \frac{2n}{n-2} fR_{ij}R_{ik}R_{jk} \\
&\quad - 2fW_{ijkl}R_{ik}R_{jl} - \frac{(4n-2)}{(n-1)(n-2)} fR|\mathring{\mathrm{Ric}}|^2 \\
&\quad - \frac{2}{n(n-2)} fR^3 - 2\nabla_j(f\nabla_i R_{jk}R_{ik}) \\
&= 2f|\nabla \mathrm{Ric}|^2 + 2C_{ijk}\nabla_j f R_{ik} + \langle \nabla f, \nabla|\mathrm{Ric}|^2\rangle + \frac{2n}{n-2} fR_{ij}R_{ik}R_{jk} \\
&\quad - 2fW_{ijkl}R_{ik}R_{jl} - \frac{(4n-2)}{(n-1)(n-2)} fR|\mathring{\mathrm{Ric}}|^2 \\
&\quad - \frac{2}{n(n-2)} fR^3 + 2\nabla_i(fC_{ijk}R_{jk}) - \mathrm{div}(f\nabla|\mathrm{Ric}|^2),
\end{aligned}$$

where we used (2-5) to justify the second equality. So, the proof is completed. □

Proceeding, we shall deduce another divergent formula, which plays a crucial role in the proof of Theorem 1.4.

Lemma 3.2. *Let (M^n, g, f, κ) be a V-static space. Then we have*

$$\begin{aligned}
&\tfrac{1}{2}\,\mathrm{div}(f\nabla|\mathrm{Ric}|^2) \\
&\qquad = -f|C_{ijk}|^2 + f|\nabla \mathrm{Ric}|^2 + \langle \nabla f, \nabla|\mathrm{Ric}|^2\rangle - \frac{n\kappa}{n-1}|\mathring{\mathrm{Ric}}|^2 + 2\nabla_i(fC_{ijk}R_{jk}).
\end{aligned}$$

Proof. To start with, we use Lemma 2.1 together with (2-5) to infer

$$\begin{aligned}
&\nabla_i(\nabla_j f R_{ik}R_{jk} + R_{ijkl}\nabla_l f R_{jk}) \\
&\qquad = \nabla_i(\nabla_j f R_{ik}R_{jk}) + \nabla_i\Big[fC_{ijk}R_{jk} - \frac{R}{n-1}(\nabla_i f R - \nabla_j f R_{ji}) \\
&\qquad\qquad + (|\mathrm{Ric}|^2\nabla_i f - \nabla_j f R_{ik}R_{jk})\Big].
\end{aligned}$$

Rearranging the terms we immediately deduce

$$\nabla_i(\nabla_j f R_{ik} R_{jk} + R_{ijkl}\nabla_l f R_{jk}) = \nabla_i(f C_{ijk} R_{jk}) + \nabla_i\Big[-\frac{R^2}{n-1}\nabla_i f + \frac{R}{n-1} R_{ji}\nabla_j f + |\mathrm{Ric}|^2\nabla_i f\Big],$$

and remembering that (M^n, g) has constant scalar curvature we use the twice-contracted second Bianchi identity ($2\,\mathrm{div}\,\mathrm{Ric} = \nabla R = 0$) to get

$$\text{(3-4)}\quad \nabla_i(\nabla_j f R_{ik} R_{jk} + R_{ijkl}\nabla_l f R_{jk}) = \nabla_i(f C_{ijk} R_{jk}) - \frac{R^2}{n-1}\Delta f + \frac{R}{n-1}\nabla_i\nabla_j f R_{ji} + |\mathrm{Ric}|^2\Delta f + \langle\nabla f, \nabla|\mathrm{Ric}|^2\rangle.$$

Therefore, substituting (2-1) and (2-2) into (3-4) we obtain

$$\begin{aligned}\nabla_i(\nabla_j f R_{ik} R_{jk} + R_{ijkl}\nabla_l f R_{jk}) &= \nabla_i(f C_{ijk} R_{jk}) + \langle\nabla f, \nabla|\mathrm{Ric}|^2\rangle - \frac{R^2}{n-1}\Delta f \\ &\quad + \frac{R}{n-1}(f R_{ij} + (\Delta f + \kappa) g_{ij}) R_{ji} + \Delta f |\mathrm{Ric}|^2 \\ &= \nabla_i(f C_{ijk} R_{jk}) + \langle\nabla f, \nabla|\mathrm{Ric}|^2\rangle + \frac{R}{n-1} f |\mathrm{Ric}|^2 + \frac{R^2\kappa}{n-1} + \frac{-Rf - n\kappa}{n-1}|\mathrm{Ric}|^2.\end{aligned}$$

From this, it follows that

$$\text{(3-5)}\quad \nabla_i(\nabla_j f R_{ik} R_{jk} + R_{ijkl}\nabla_l f R_{jk}) = \nabla_i(f C_{ijk} R_{jk}) + \langle\nabla f, \nabla|\mathrm{Ric}|^2\rangle - \frac{n\kappa}{n-1}|\mathring{\mathrm{Ric}}|^2.$$

At the same time, notice that

$$\begin{aligned}\nabla_i(\nabla_j f R_{ik} R_{jk} + R_{ijkl}\nabla_l f R_{jk}) &= \nabla_i\nabla_j f R_{ik} R_{jk} + \nabla_j f R_{ik}\nabla_i R_{jk} + \nabla_i R_{ijkl}\nabla_l f R_{jk} \\ &\quad + R_{ijkl}\nabla_i\nabla_l f R_{jk} + R_{ijkl}\nabla_l f \nabla_i R_{jk}.\end{aligned}$$

Hence, it follows from Lemma 2.1 and (2-1) that

$$\begin{aligned}\nabla_i(\nabla_j f R_{ik} R_{jk} + R_{ijkl}\nabla_l f R_{jk}) &= f(R_{ij} R_{ik} R_{jk} - R_{ijkl} R_{ik} R_{jl}) + \nabla_j f R_{ik}\nabla_i R_{jk} + C_{ijk}\nabla_j f R_{ik} \\ &\quad + f C_{ijk}\nabla_i R_{jk} + (\nabla_i f R_{jk} - \nabla_j f R_{ik})\nabla_i R_{jk} \\ &= f(R_{ij} R_{ik} R_{jk} - R_{ijkl} R_{ik} R_{jl}) + C_{ijk}\nabla_j f R_{ik} \\ &\quad + f C_{ijk}\nabla_i R_{jk} + \tfrac{1}{2}\langle\nabla f, \nabla|\mathrm{Ric}|^2\rangle.\end{aligned}$$

Proceeding, we use that the Cotton tensor is skew-symmetric in the first two indices and (3-3) to infer

$$\begin{aligned}(3\text{-}6)\quad \nabla_i(\nabla_j f R_{ik}R_{jk} + R_{ijkl}\nabla_l f R_{jk}) &= f|C_{ijk}|^2 - f|\nabla \mathrm{Ric}|^2 + \nabla_j(f\nabla_i R_{jk}R_{ik})\\ &= f|C_{ijk}|^2 - f|\nabla \mathrm{Ric}|^2 - \nabla_i(fC_{ijk}R_{jk}) + \tfrac{1}{2}\operatorname{div}(f\nabla|\mathrm{Ric}|^2).\end{aligned}$$

Finally, it suffices to compare (3-5) and (3-6) to get the desired result. □

Proof of Theorem 1.4. A simple computation using (3-1) as well as (3-2) allows us to deduce

$$f|C_{ijk}|^2 = \frac{2(n-1)}{(n-2)}R_{ik}\nabla_j f C_{ijk} + W_{ijkl}\nabla_l f C_{ijk},$$

where we have used that C_{ijk} is skew-symmetric in the first two indices and trace-free in any two indices. Whence, substituting this data into Lemma 3.1 we obtain

$$\begin{aligned}(3\text{-}7)\quad &\operatorname{div}(f\nabla|\mathrm{Ric}|^2)\\ &= 2f|\nabla \mathrm{Ric}|^2 + \langle\nabla f, \nabla|\mathrm{Ric}|^2\rangle + \frac{2n}{(n-2)} f R_{ij}R_{ik}R_{jk}\\ &\quad - \frac{1}{(n-1)} f|C_{ijk}|^2 - \frac{4n-2}{(n-1)(n-2)} f R|\mathring{\mathrm{Ric}}|^2 - \frac{2}{n(n-2)} f R^3\\ &\quad + 2\nabla_i(fC_{ijk}R_{jk}) - \frac{(n-2)}{(n-1)} W_{ijkl}\nabla_l f C_{ijk} - 2f W_{ijkl}R_{ik}R_{jl}.\end{aligned}$$

Now, comparing the expression obtained in (3-7) with Lemma 3.2 we arrive at

$$\begin{aligned}(3\text{-}8)\quad &\tfrac{1}{2}\operatorname{div}(f\nabla|\mathrm{Ric}|^2)\\ &= \Big(\frac{n-2}{n-1}|C_{ijk}|^2 + |\nabla \mathrm{Ric}|^2\Big)f + \frac{n\kappa}{n-1}|\mathring{\mathrm{Ric}}|^2 + \frac{2n}{n-2} f R_{ij}R_{ik}R_{jk}\\ &\quad - \frac{4n-2}{(n-1)(n-2)} f R|\mathring{\mathrm{Ric}}|^2 - \frac{2}{n(n-2)} f R^3\\ &\quad - \frac{n-2}{n-1} W_{ijkl}\nabla_l f C_{ijk} - 2f W_{ijkl}R_{ik}R_{jl}.\end{aligned}$$

On the other hand, recalling that $\mathring{R}_{ij} = R_{ij} - R^2/ng$, it is easy to check that

$$f R_{ij}R_{ik}R_{jk} = f\mathring{R}_{ij}\mathring{R}_{jk}\mathring{R}_{ik} + \frac{3}{n} f R|\mathring{\mathrm{Ric}}|^2 + \frac{fR^3}{n^2}.$$

This substituted into (3-8) gives

$$\begin{aligned}&\tfrac{1}{2}\operatorname{div}(f\nabla|\mathrm{Ric}|^2)\\ &= \Big(\frac{n-2}{n-1}|C_{ijk}|^2 + |\nabla \mathrm{Ric}|^2\Big)f + \frac{n\kappa}{n-1}|\mathring{\mathrm{Ric}}|^2 + \Big(\frac{2}{n-1}R|\mathring{\mathrm{Ric}}|^2 + \frac{2n}{n-2}\operatorname{tr}(\mathring{\mathrm{Ric}}^3)\Big)f\\ &\quad - \frac{n-2}{n-1} W_{ijkl}\nabla_l f C_{ijk} - 2f W_{ijkl}R_{ik}R_{jl},\end{aligned}$$

which finishes the proof of the theorem. □

Proof of Corollaries 1.5 and 1.6. In order to prove Corollaries 1.5 and 1.6, we recall that the Cotton tensor and the divergence of the Weyl tensor are related as follows:

$$C_{ijk} = -\frac{n-2}{n-3}\nabla_l W_{ijkl}. \tag{3-9}$$

Notice also that the zero radial Weyl curvature condition, namely, $W_{ijkl}\nabla_l f = 0$, jointly with (3-9) and (2-1) yields

$$\begin{aligned} 0 &= \nabla_i(W_{ijkl}\nabla_k f R_{jl}) \\ &= \nabla_i W_{ijkl}\nabla_k f R_{jl} + W_{ijkl}\nabla_i\nabla_k f R_{jl} \\ &= \frac{n-3}{n-2}C_{klj}\nabla_k f R_{jl} + f W_{ijkl}R_{ik}R_{jl}. \end{aligned}$$

By using that the Cotton tensor is skew-symmetric in the two first indices we obtain

$$f W_{ijkl}R_{ik}R_{jl} = \frac{n-3}{2(n-2)}C_{ijk}(\nabla_j f R_{ik} - \nabla_i f R_{ik}),$$

which can be succinctly rewritten as

$$f W_{ijkl}R_{ik}R_{jl} = \frac{n-3}{2(n-1)}C_{ijk}T_{ijk}.$$

From this, it follows from (3-1) that

$$f W_{ijkl}R_{ik}R_{jl} = \frac{n-3}{2(n-1)}f|C_{ijk}|^2. \tag{3-10}$$

Now, comparing (3-10) with Theorem 1.4 we achieve

$$\begin{aligned} \frac{1}{2}\operatorname{div}(f\nabla|\mathrm{Ric}|^2) = \Big(\frac{n-2}{n-1}|C_{ijk}|^2 + |\nabla \mathrm{Ric}|^2\Big)f + \frac{n\kappa}{n-1}|\mathring{\mathrm{Ric}}|^2 \\ + \Big(\frac{2}{n-1}R|\mathring{\mathrm{Ric}}|^2 + \frac{2n}{n-2}\operatorname{tr}(\mathring{\mathrm{Ric}}^3)\Big)f - \frac{n-3}{n-1}f|C_{ijk}|^2, \end{aligned}$$

so that

$$\begin{aligned} \tfrac{1}{2}\operatorname{div}(f\nabla|\mathrm{Ric}|^2) = \Big(\frac{1}{n-1}|C_{ijk}|^2 + |\nabla \mathrm{Ric}|^2\Big)f + \frac{n\kappa}{n-1}|\mathring{\mathrm{Ric}}|^2 \\ + \Big(\frac{2}{n-1}R|\mathring{\mathrm{Ric}}|^2 + \frac{2n}{n-2}\operatorname{tr}(\mathring{\mathrm{Ric}}^3)\Big)f. \end{aligned} \tag{3-11}$$

Before proceeding it is important to remember that the classical Okumura's lemma [1974, Lemma 2.1] guarantees

$$\operatorname{tr}(\mathring{\mathrm{Ric}}^3) \geq -\frac{n-2}{\sqrt{n(n-1)}}|\mathring{\mathrm{Ric}}|^3. \tag{3-12}$$

Therefore, upon integrating (3-11) over M we use (3-12) to arrive at

$$0 \geq \int_M \Big(\frac{1}{n-1}|C_{ijk}|^2 + |\nabla \mathrm{Ric}|^2\Big) f\, dM_g + \frac{n\kappa}{n-1}\int_M |\mathring{\mathrm{Ric}}|^2\, dM_g + \int_M \frac{2n}{\sqrt{n(n-1)}}|\mathring{\mathrm{Ric}}|^2\Big(\frac{R}{\sqrt{n(n-1)}} - |\mathring{\mathrm{Ric}}|\Big) f\, dM_g. \tag{3-13}$$

We now suppose that $\kappa = 1$, that is, (M^n, g) is a Miao–Tam critical metric, and we may use our assumption with (3-13) to conclude that $|\mathring{\mathrm{Ric}}|^2 = 0$ and this forces M^n to be Einstein. So, it suffices to apply Theorem 1.1 in [Miao and Tam 2011] to conclude that (M^n, g) is isometric to a geodesic ball in $\mathbb{S}^n$ and this concludes the proof of Corollary 1.5.

From now on we assume that $\kappa = 0$, that is, (M^n, g) is a static space. In this case, our assumption substituted into (3-13) guarantees that either $|\mathring{\mathrm{Ric}}|^2 = 0$ or $|\mathring{\mathrm{Ric}}|^2 = R^2/(n(n-1))$. In the first case, we conclude that (M^n, g) is an Einstein manifold. Then, it suffices to apply Lemma 3 in [Reilly 1977] to conclude that M^n is isometric to a hemisphere of $\mathbb{S}^n$. In the second one, notice that M^n must have a vanishing Cotton tensor and parallel Ricci curvature. From this, we can use (1-4) to infer

$$(n-2)B_{ij} = \nabla_k C_{kij} + W_{ikjl}R_{kl} = W_{ikjl}R_{kl},$$

and consequently, by using the static equation, we deduce

$$(n-2)fB_{ij} = W_{ikjl}\nabla_k\nabla_l f = \nabla_k(W_{ijkl}\nabla_l f) - \nabla_k W_{ikjl}\nabla_l f.$$

Hence, our assumption on Weyl curvature tensor jointly with (3-9) yields

$$(n-2)fB_{ij} = -\nabla_k W_{jlik}\nabla_l f = \frac{n-3}{n-2}C_{jli}\nabla_l f = 0.$$

From here it follows that (M^n, g) is Bach-flat. Hence, the result follows from Corollary 4.4 (see also Theorem 1 in [Ambrozio 2017] for $n = 3$). This is what we wanted to prove. □

4. Critical metrics with nonnegative sectional curvature

In the last decades there have been a lot of interesting results concerning the geometry of manifolds with nonnegative sectional curvature. In this context, as it was previously mentioned, any two symmetric tensors T on a Riemannian manifold (M^n, g) with nonnegative sectional curvature must satisfy

$$(\nabla_i\nabla_j T_{ik} - \nabla_j\nabla_i T_{ik})T_{jk} \geq 0. \tag{4-1}$$

In fact, we have

$$(\nabla_i\nabla_j T_{ik} - \nabla_j\nabla_i T_{ik})T_{jk} = \sum_{i<j} R_{ijij}(\lambda_i - \lambda_j)^2,$$

where the λ_i are the eigenvalues of tensor T (see Lemma 4.1 in [Cao 2007]). In particular, choosing $T = \mathrm{Ric}$ we immediately get

$$(\nabla_i\nabla_j R_{ik} - \nabla_j\nabla_i R_{ik})R_{jk} \geq 0.$$

This combined with (2-8) yields

$$R_{ij}R_{jk}R_{ik} - R_{ijkl}R_{jl}R_{ik} \geq 0. \tag{4-2}$$

In the sequel, we shall deduce a useful expression for $R_{ij}R_{jk}R_{ik} - R_{ijkl}R_{jl}R_{ik}$ on any Riemannian manifold.

Lemma 4.1. *Let (M^n, g) be a Riemannian manifold. Then we have*

$$R_{ij}R_{jk}R_{ik} - R_{ijkl}R_{jl}R_{ik} = \frac{1}{n-1}R|\mathring{\mathrm{Ric}}|^2 + \frac{n}{n-2}\,\mathrm{tr}(\mathring{\mathrm{Ric}}^3) - W_{ijkl}R_{ik}R_{jl}.$$

Proof. By using the definition of the Riemann tensor (2-4) we obtain

$$R_{ij}R_{jk}R_{ik} - R_{ijkl}R_{jl}R_{ik} = \frac{n}{n-2}R_{ij}R_{jk}R_{ik} - W_{ijkl}R_{jl}R_{ik} - \frac{(2n-1)}{(n-1)(n-2)}R|\mathrm{Ric}|^2 + \frac{R^3}{(n-1)(n-2)}$$

so that

$$R_{ij}R_{jk}R_{ik} - R_{ijkl}R_{jl}R_{ik} = \frac{n}{n-2}R_{ij}R_{jk}R_{ik} - W_{ijkl}R_{jl}R_{ik} - \frac{(2n-1)}{(n-1)(n-2)}R|\mathring{\mathrm{Ric}}|^2 - \frac{1}{n(n-2)}R^3. \tag{4-3}$$

On the other hand, we already know that

$$R_{ij}R_{ik}R_{jk} = \mathring{R}_{ij}\mathring{R}_{jk}\mathring{R}_{ik} + \frac{3}{n}R|\mathring{\mathrm{Ric}}|^2 + \frac{R^3}{n^2}.$$

This substituted into (4-3) gives the desired result. □

Since three-dimensional Riemannian manifolds have vanishing Weyl tensor, the proof of Theorem 1.9 follows as an immediate consequence of the following slightly stronger result.

Proposition 4.2. *Let (M^n, g, f) be a compact, oriented, connected Miao–Tam critical metric with smooth boundary ∂M and nonnegative sectional curvature, f, which is also assumed to be nonnegative. Suppose that M^n has zero radial Weyl curvature, then M^n is isometric to a geodesic ball in a simply connected space form $\mathbb{R}^n$ or $\mathbb{S}^n$.*

Proof. To begin with, we multiply by f the expression obtained in Lemma 4.1 and then we use Theorem 1.4 to infer

$$\frac{1}{2}\,\mathrm{div}(f\nabla|\mathrm{Ric}|^2) = \Big(\frac{n-2}{n-1}|C_{ijk}|^2 + |\nabla\,\mathrm{Ric}|^2\Big)f + \frac{n}{n-1}|\mathring{\mathrm{Ric}}|^2 + 2(R_{ij}R_{jk}R_{ik} - R_{ijkl}R_{jl}R_{ik})f - \frac{n-2}{n-1}W_{ijkl}\nabla_l f C_{ijk}$$

and since M^n has zero radial Weyl curvature we get

$$\frac{1}{2}\operatorname{div}(f\nabla|\mathrm{Ric}|^2) = \Big(\frac{n-2}{n-1}|C_{ijk}|^2 + |\nabla\,\mathrm{Ric}|^2\Big)f + \frac{n}{n-1}|\mathring{\mathrm{Ric}}|^2 + 2(R_{ij}R_{jk}R_{ik} - R_{ijkl}R_{jl}R_{ik})f.$$

Finally, upon integrating the above expression over M^n we use (4-2) to conclude that $\mathring{\mathrm{Ric}} = 0$ and then $(M^n g)$ is Einstein. Hence, we apply Theorem 1.1 in [Miao and Tam 2011] to conclude that (M^n, g) is isometric to a geodesic ball in $\mathbb{R}^n$ or $\mathbb{S}^n$. This finishes the proof of the proposition. □

Proceeding, we shall prove Theorem 1.10, which was announced in Section 1.

Theorem 4.3. *Let (M^n, g, f) be a positive static triple with nonnegative sectional curvature, zero radial Weyl curvature and scalar curvature $R = n(n-1)$. Then up to a finite quotient M^n is isometric to either the standard hemisphere $\mathbb{S}^n_+$ or the standard cylinder over $\mathbb{S}^{n-1}$ with the product metric described in Theorem 1.3.*

Proof. The proof looks like the one from the previous theorem. In fact, substituting Lemma 4.1 into Theorem 1.4 we arrive at

$$\frac{1}{2}\operatorname{div}(f\nabla|\mathrm{Ric}|^2) = \Big(\frac{1}{n-1}|C_{ijk}|^2 + |\nabla\,\mathrm{Ric}|^2\Big)f + 2(R_{ij}R_{jk}R_{ik} - R_{ijkl}R_{jl}R_{ik})f.$$

We integrate the above expression over M^n and then use (4-2) to conclude that (M^n, g) must have vanishing Cotton tensor and parallel Ricci curvature. Finally, it suffices to repeat the same arguments applied in the final steps of the proof of Corollary 1.6. So, the proof is completed. □

As an immediate consequence of the previous theorem we get the following result.

Corollary 4.4. *Let (M^3, g, f) be a three-dimensional positive static triple with nonnegative sectional curvature and normalized scalar curvature $R = 6$. Then, up to a finite quotient, M^3 is isometric to either the standard hemisphere $\mathbb{S}^3_+$ or the standard cylinder over $\mathbb{S}^2$ with the product metric described in Theorem 1.3.*

We point out that, by a different approach, Ambrozio [2017] was able to show that a three-dimensional compact positive static triple with scalar curvature 6 and nonnegative Ricci curvature must be equivalent to the standard hemisphere or be covered by the standard cylinder.

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References

[Ambrozio 2017] L. Ambrozio, "On static three-manifolds with positive scalar curvature", *J. Differential Geom.* **107**:1 (2017), 1–45. MR

[Baltazar and Ribeiro 2017] H. Baltazar and E. Ribeiro, Jr., "Critical metrics of the volume functional on manifolds with boundary", *Proc. Amer. Math. Soc.* **145**:8 (2017), 3513–3523. MR Zbl

[Barros et al. 2015] A. Barros, R. Diógenes, and E. Ribeiro, Jr., "Bach-flat critical metrics of the volume functional on 4-dimensional manifolds with boundary", *J. Geom. Anal.* **25**:4 (2015), 2698–2715. MR Zbl

[Batista et al. 2017] R. Batista, R. Diógenes, M. Ranieri, and E. Ribeiro, Jr., "Critical metrics of the volume functional on compact three-manifolds with smooth boundary", *J. Geom. Anal.* **27**:2 (2017), 1530–1547. MR Zbl

[Boucher et al. 1984] W. Boucher, G. W. Gibbons, and G. T. Horowitz, "Uniqueness theorem for anti-de Sitter spacetime", *Phys. Rev. D* (3) **30**:12 (1984), 2447–2451. MR

[Cao 2007] X. Cao, "Compact gradient shrinking Ricci solitons with positive curvature operator", *J. Geom. Anal.* **17**:3 (2007), 425–433. MR Zbl

[Catino 2012] G. Catino, "Generalized quasi-Einstein manifolds with harmonic Weyl tensor", *Math. Z.* **271**:3-4 (2012), 751–756. MR Zbl

[Chow et al. 2007] B. Chow, S.-C. Chu, D. Glickenstein, C. Guenther, J. Isenberg, T. Ivey, D. Knopf, P. Lu, F. Luo, and L. Ni, *The Ricci flow: techniques and applications, I: Geometric aspects*, Mathematical Surveys and Monographs **135**, Amer. Math. Soc., Providence, RI, 2007. MR Zbl

[Chruściel 2011] P. T. Chruściel, "Remarks on rigidity of the de Sitter metric", unpublished manuscript, 2011, https://tinyurl.com/chrusitter.

[Corvino 2000] J. Corvino, "Scalar curvature deformation and a gluing construction for the Einstein constraint equations", *Comm. Math. Phys.* **214**:1 (2000), 137–189. MR Zbl

[Corvino et al. 2013] J. Corvino, M. Eichmair, and P. Miao, "Deformation of scalar curvature and volume", *Math. Ann.* **357**:2 (2013), 551–584. MR Zbl

[Gibbons et al. 2003] G. W. Gibbons, S. A. Hartnoll, and C. N. Pope, "Bohm and Einstein–Sasaki metrics, black holes, and cosmological event horizons", *Phys. Rev. D* (3) **67**:8 (2003), art. id. 084024. MR

[He et al. 2012] C. He, P. Petersen, and W. Wylie, "On the classification of warped product Einstein metrics", *Comm. Anal. Geom.* **20**:2 (2012), 271–311. MR Zbl

[Hijazi et al. 2015] O. Hijazi, S. Montiel, and S. Raulot, "Uniqueness of the de Sitter spacetime among static vacua with positive cosmological constant", *Ann. Global Anal. Geom.* **47**:2 (2015), 167–178. MR Zbl

[Kobayashi 1982] O. Kobayashi, "A differential equation arising from scalar curvature function", *J. Math. Soc. Japan* **34**:4 (1982), 665–675. MR Zbl

[Lafontaine 1983] J. Lafontaine, "Sur la géométrie d'une généralisation de l'équation différentielle d'Obata", *J. Math. Pures Appl.* (9) **62**:1 (1983), 63–72. MR Zbl

[Miao and Tam 2009] P. Miao and L.-F. Tam, "On the volume functional of compact manifolds with boundary with constant scalar curvature", *Calc. Var. Partial Diff. Eq.* **36**:2 (2009), 141–171. MR Zbl

[Miao and Tam 2011] P. Miao and L.-F. Tam, "Einstein and conformally flat critical metrics of the volume functional", *Trans. Amer. Math. Soc.* **363**:6 (2011), 2907–2937. MR Zbl

[Miao and Tam 2012] P. Miao and L.-F. Tam, "Scalar curvature rigidity with a volume constraint", *Comm. Anal. Geom.* **20**:1 (2012), 1–30. MR Zbl

[Obata 1962] M. Obata, "Certain conditions for a Riemannian manifold to be isometric with a sphere", *J. Math. Soc. Japan* **14** (1962), 333–340. MR Zbl

[Okumura 1974] M. Okumura, "Hypersurfaces and a pinching problem on the second fundamental tensor", *Amer. J. Math.* **96** (1974), 207–213. MR Zbl

[Petersen and Wylie 2010] P. Petersen and W. Wylie, "On the classification of gradient Ricci solitons", *Geom. Topol.* **14**:4 (2010), 2277–2300. MR Zbl

[Qing and Yuan 2013] J. Qing and W. Yuan, "A note on static spaces and related problems", *J. Geom. Phys.* **74** (2013), 18–27. MR Zbl

[Reilly 1977] R. C. Reilly, "Applications of the Hessian operator in a Riemannian manifold", *Indiana Univ. Math. J.* **26**:3 (1977), 459–472. MR Zbl

[Shen 1997] Y. Shen, "A note on Fischer–Marsden's conjecture", *Proc. Amer. Math. Soc.* **125**:3 (1997), 901–905. MR Zbl

[Viaclovsky 2011] J. A. Viaclovsky, "Topics in Riemannian geometry", lecture notes, University of Wisconsin, 2011, https://tinyurl.com/math865.

[Yuan 2016] W. Yuan, "Volume comparison with respect to scalar curvature", preprint, 2016. arXiv

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