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Let $\mathbb{Q}_c^{n+1}$ be the complete simply connected $(n+1)$-dimensional space form of curvature c. We obtain a new characterization of geodesic spheres in $\mathbb{Q}_c^{n+1}$ in terms of the higher order mean curvatures. In particular, we prove that the geodesic sphere is the only complete bounded immersed hypersurface in $\mathbb{Q}_c^{n+1}$, $c \le 0$, with constant mean curvature and constant scalar curvature. The proof relies on the well known Omori–Yau maximum principle, a formula of Walter for the Laplacian of the r-th mean curvature of a hypersurface in a space form, and a classical inequality of Gårding for hyperbolic polynomials.

1. Introduction

A question of interest in differential geometry is whether the geodesic sphere is the only compact oriented hypersurface in the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$ with constant r-th mean curvature H_r, for some $r = 1, \ldots, n$. Here H_1, H_2, and H_n are the mean curvature, the scalar curvature, and the Gauss–Kronecker curvature, respectively; see the definitions in Section 2. When $r = 1$ this question is the well known Hopf conjecture, and when $r = 2$ it is a problem proposed by Yau [1982, Problem 31, p. 677].

As proved by Alexandrov [1958] for $r = 1$, and by Ros [1988; 1987] for any r (see also [Montiel and Ros 1991] and the appendix by Korevaar in [Ros 1988]), the above question has an affirmative answer for embedded hypersurfaces. In the immersed case, the question has a negative answer when $r = 1$ — see the examples of nonspherical compact hypersurfaces with constant mean curvature in the Euclidean space constructed by Wente [1986] and Hsiang, Teng and Yu [Hsiang et al. 1983] — and an affirmative answer when $r = n$ (by a theorem of Hadamard). The problem is still unsolved for $1 < r < n$. For partial answers when $r = 2$ (Yau's problem), see [Cheng 2002; Li 1996; Okayasu 2005].

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Because of the difficulty of the above question, it is natural to attempt to obtain the rigidity of the sphere in $\mathbb{R}^{n+1}$ under geometric conditions stronger than requiring that H_r be constant for some r. In this regard, Gardner [1970] proved that if a compact oriented hypersurface M^n in $\mathbb{R}^{n+1}$ has two consecutive mean curvatures H_r and H_{r+1} constant, for some $r = 1, \dots, n-1$, then it is a geodesic sphere. This result was extended to compact hypersurfaces in any space form by Bivens [1983]. For improvements on Bivens' result, see [Koh 1998; Wang 2014].

Cheng and Wan [1994] proved that a complete hypersurface M^3 with constant scalar curvature R and constant mean curvature $H \neq 0$ in $\mathbb{R}^4$ is a generalized cylinder $\mathbb{S}^k(a) \times \mathbb{R}^{3-k}$, for some $k = 1, 2, 3$ and some $a > 0$; see [Núñez 2017] for results of this nature in higher dimensions. From this result one obtains the following improvement, when $n = 3$ and $r = 1$, to the theorem of Gardner referred to above: *The geodesic spheres are the only complete bounded hypersurfaces in $\mathbb{R}^4$ with constant scalar curvature and constant mean curvature* (cf. Corollary 1.2).

Our main result (Theorem 1.1) provides a new characterization of geodesic spheres in space forms. There are many results of this nature in the literature, most of which assure that a compact hypersurface that satisfies certain geometric conditions is a geodesic sphere. What makes the characterization provided by Theorem 1.1 special is that the geometric conditions in it are imposed on a *complete* hypersurface (that is bounded when $c \leq 0$, and contained in a spherical cap when $c > 0$), and not on a *compact* one.

In the theorem below and throughout this work, $\mathbb{Q}_c^{n+1}$ stands for the $(n+1)$-dimensional complete simply connected space of constant sectional curvature c.

Theorem 1.1. *Let M^n be a complete Riemannian manifold with scalar curvature R bounded from below, and let $f : M^n \to \mathbb{Q}_c^{n+1}$ be an isometric immersion. In the case $c \leq 0$, assume that $f(M^n)$ is bounded, and in the case $c > 0$, that $f(M^n)$ lies inside a geodesic ball of radius $\rho < \pi/2\sqrt{c}$. If the mean curvature H is constant and, for some $r = 2, \dots, n$, the r-th mean curvature H_r is constant, then $f(M^n)$ is a geodesic sphere of $\mathbb{Q}_c^{n+1}$.*

The following results follow immediately from the above theorem. Notice that the hypothesis in Theorem 1.1 that the scalar curvature of M^n is bounded from below is superfluous when $r = 2$.

Corollary 1.2. *Let $f : M^n \to \mathbb{Q}_c^{n+1}$ be an isometric immersion of a complete Riemannian manifold M^n in $\mathbb{Q}_c^{n+1}$. In the case $c \leq 0$, assume that $f(M^n)$ is bounded, and in the case $c > 0$, that $f(M^n)$ lies inside a geodesic ball of radius $\rho < \pi/2\sqrt{c}$. If the mean curvature H and the scalar curvature R are constant, then $f(M^n)$ is a geodesic sphere of $\mathbb{Q}_c^{n+1}$.*

Corollary 1.3. *Let $f : M^n \to \mathbb{Q}_c^{n+1}$ be an isometric immersion of a compact Riemannian manifold M^n in $\mathbb{Q}_c^{n+1}$. In the case $c > 0$, assume that $f(M)$ is contained* *in an open hemisphere of* $\mathbb{S}_c^{n+1}$. *If the mean curvature* H *is constant and, for some* $r = 2, \ldots, n$, *the* r*-th mean curvature* H_r *is constant, then* $f(M^n)$ *is a geodesic sphere of* $\mathbb{Q}_c^{n+1}$.

Remark 1.4. The examples of [Wente 1986; Hsiang et al. 1983] referred to in the second paragraph of this section show that the hypothesis that H_r is constant for some r, $2 \le r \le n$, can not be removed from Theorem 1.1. It is surely a difficult question to know whether the theorem holds without the assumption that H is constant (cf. Yau's problem mentioned in the beginning of this section). We do not know whether Theorem 1.1 (for $r \ge 3$) holds without the hypothesis that the scalar curvature of M is bounded below.

The proof of Theorem 1.1 relies on the well known Omori–Yau maximum principle [Cheng and Yau 1975; Omori 1967; Yau 1975], a formula of Walter [1985] for the Laplacian of the r-th mean curvature of a hypersurface in a space form, and a classical inequality of Gårding [1959] for hyperbolic polynomials.

2. Preliminaries

Given an isometric immersion $f : M^n \to N^{n+k}$ of an n-dimensional Riemannian manifold M^n into an $(n+k)$-dimensional Riemannian manifold N^{n+k}, denote by $\sigma : TM \times TM \to TM^{\perp}$ the (vector valued) second fundamental form of f, and by A_ξ the shape operator of the immersion with respect to a (locally defined) unit normal vector field ξ. From the Gauss formula one obtains, for all smooth vector fields X and Y,

$$\langle A_\xi X, Y\rangle = \langle \sigma(X, Y), \xi\rangle. \tag{2-1}$$

In the particular case that M and N are orientable and $k = 1$, one may choose a global unit normal vector field ξ and so define a (symmetric) 2-tensor field h on M by $h(X, Y) = \langle \sigma(X, Y), \xi\rangle$. Then, by (2-1),

$$h(X, Y) = \langle AX, Y\rangle, \qquad X, Y \in \mathfrak{X}(M),$$

where $A = A_\xi$ is the shape operator of the immersion with respect to ξ. If we assume further that N^{n+1} has constant sectional curvature, it follows from the symmetry of h and the Codazzi equation that the covariant derivative ∇h of h is symmetric. From now on we denote by h_{ij} and h_{ijk} the components of h and ∇h, respectively, in a local orthonormal frame field $\{e_1, \ldots, e_n\}$, i.e.,

$$h_{ij} = h(e_i, e_j), \qquad h_{ijk} = \nabla h(e_i, e_j, e_k).$$

Given an isometric immersion $f : M^n \to N^{n+1}$, denote by $\lambda_1, \ldots, \lambda_n$ the principal curvatures of M^n with respect to a global unit normal vector field ξ (i.e., the eigenvalues of the shape operator $A = A_\xi$). It is well known that if we label the principal curvatures at each point by the condition $\lambda_1 \le \cdots \le \lambda_n$, then the resulting functions $\lambda_i : M \to \mathbb{R}$, $i = 1, \dots, n$, are continuous.

The r-th mean curvature H_r, $1 \le r \le n$, of M^n is defined by

$$\binom{n}{r} H_r = \sum_{i_1 < \cdots < i_r} \lambda_{i_1} \cdots \lambda_{i_r}. \tag{2-2}$$

Notice that H_1 is the mean curvature H ($= \frac{1}{n} \operatorname{tr} A$, where $\operatorname{tr} A$ is the trace of A) and $H_n = \lambda_1 \lambda_2 \cdots \lambda_n$ is the Gauss–Kronecker curvature of the immersion. In the particular case that N^{n+1} has constant sectional curvature, the function H_2 is up to a constant the (normalized) scalar curvature R of M^n. In fact, if N^{n+1} has constant sectional curvature c and if $\{e_1, \dots, e_n\}$ is an orthonormal basis for the tangent space at a given point of M^n such that $Ae_i = \lambda_i e_i$, $i = 1, \dots, n$, then the sectional curvature $K(e_i, e_j)$ of the plane spanned by e_i and e_j is given by

$$K(e_i, e_j) = c + \lambda_i \lambda_j$$

by the Gauss equation, and so

$$R = \frac{1}{\binom{n}{2}} \sum_{i<j} K(e_i, e_j) = \frac{1}{\binom{n}{2}} \sum_{i<j} (c + \lambda_i \lambda_j) = c + H_2.$$

The squared norm $|A|^2$ of the shape operator A is defined as the trace of A^2. It is easy to see that

$$|A|^2 = \sum_i \lambda_i^2.$$

From (2-2) and the last two equalities we obtain the following useful relation involving the mean curvature H, the norm $|A|$ of the shape operator A, and the normalized scalar curvature R:

$$n^2 H^2 = \left(\sum_{i=1}^{n} \lambda_i \right)^2 = \sum_{i=1}^{n} \lambda_i^2 + \sum_{i \ne j} \lambda_i \lambda_j = |A|^2 + n(n-1)(R - c). \tag{2-3}$$

In terms of the r-th symmetric function $\sigma_r : \mathbb{R}^n \to \mathbb{R}$,

$$\sigma_r(x_1, \dots, x_n) = \sum_{i_1 < \cdots < i_r} x_{i_1} \cdots x_{i_r}, \tag{2-4}$$

the equality (2-2) can be rewritten as

$$\binom{n}{r} H_r = \sigma_r \circ \vec{\lambda},$$

where $\vec{\lambda} = (\lambda_1, \dots, \lambda_n)$ is the principal curvature vector of the immersion. In order to unify the notation, we define $H_0 = 1 = \sigma_0$ and $H_r = 0 = \sigma_r$ for all $r \ge n + 1$.

As one might expect, the knowledge of the properties of the symmetric functions is very important to the study of the higher order mean curvatures of a hypersurface. In order to state a property of the symmetric functions that is relevant to us, we summarize below some of the results of the classical article [Gårding 1959] on hyperbolic polynomials; see also [Caffarelli et al. 1985, p. 268; Fontenele and Silva 2001, p. 217].

Let $P:\mathbb{R}^n \to \mathbb{R}$ be a homogenous polynomial of degree m and let $a=(a_1,\dots,a_n)$ be a fixed vector of $\mathbb{R}^n$. We say that P is hyperbolic with respect to the vector a, or in short, that P is a-hyperbolic, if for every $x\in\mathbb{R}^n$ the polynomial in s, $P(sa+x)$, has m real roots. Denote by Γ_P the connected component of the set $\{P\neq 0\}$ that contains a. Gårding [1959] proved that Γ_P is an open convex cone, with vertex at the origin, and that the homogenous polynomial of degree $m-1$ defined by

$$Q(x)=\frac{d}{ds}\Big|_{s=0}P(sa+x)=\sum_{j=1}^{n}a_j\frac{\partial P}{\partial x_j}(x)$$

is also a-hyperbolic. Moreover, $\Gamma_P\subset\Gamma_Q$.

As can easily be seen, the n-th symmetric function σ_n is hyperbolic with respect to the vector $a=(1,\dots,1)$. Applying the results of the previous paragraph to σ_n, and observing that

$$\sigma_r(x)=\frac{1}{(n-r)!}\frac{d^{n-r}}{ds^{n-r}}\Big|_{s=0}\sigma_n(sa+x),\qquad r=1,\dots,n-1,$$

one concludes that σ_r, $1\le r\le n$, is hyperbolic with respect to $a=(1,\dots,1)$, and that $\Gamma_1\supset\Gamma_2\supset\cdots\supset\Gamma_n$, where $\Gamma_r:=\Gamma_{\sigma_r}$.

Gårding [1959] established an inequality for hyperbolic polynomials involving their completely polarized forms. A particular case of this inequality, from which the general case is derived, says that

$$\frac{1}{m}\sum_{k=1}^{n}y_k\frac{\partial P}{\partial x_k}(x)\ge P(y)^{1/m}P(x)^{1-1/m},\qquad \forall x,y\in\Gamma_P.$$

As observed in [Caffarelli et al. 1985, p. 269], the above inequality is equivalent to the assertion that $P^{1/m}$ is a concave function on Γ_P. In particular, we have the following result, which plays an important role in the proof of Theorem 1.1.

Proposition 2.1. *For each $r=1,2,\dots,n$, the function $\sigma_r^{1/r}$ is concave on Γ_r.*

3. The Laplacian of the r-th mean curvature

The symmetric functions σ_r, $1\le r\le n$, defined by (2-4), arise naturally from the identity

$$\prod_{s=1}^{n}(x_s+t)=\sum_{r=0}^{n}\sigma_r(x)t^{n-r},$$

which is valid for all $x=(x_1,\ldots,x_n)\in\mathbb{R}^n$ and $t\in\mathbb{R}$. Differentiating this identity with respect to x_j, one obtains

$$\prod_{s\neq j}(x_s+t)=\sum_{r=0}^{n}\frac{\partial\sigma_r}{\partial x_j}(x)t^{n-r},\quad j=1,\ldots,n.$$

Differentiating the above equality with respect to x_i for $i\neq j$ yields

$$\prod_{s\neq i,j}(x_s+t)=\sum_{r=0}^{n}\frac{\partial^2\sigma_r}{\partial x_i\,\partial x_j}(x)t^{n-r},\quad i\neq j.$$

Hence,

$$\frac{\partial^2\sigma_r}{\partial x_i\,\partial x_j}(x)=\begin{cases}\sigma_{r-2}(\widehat{x_i},\widehat{x_j}), & i\neq j,\\ 0, & i=j,\end{cases}\tag{3-1}$$

where $\sigma_{r-2}(\widehat{x_i},\widehat{x_j})=\sigma_{r-2}(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_{j-1},x_{j+1},\ldots,x_n)$.

Walter [1985] established a formula for the Laplacian of the r-th mean curvature of a hypersurface in a space of constant sectional curvature. For the convenience of the reader, we state that formula below. Recall that the Laplacian Δu of a C^2-function u defined on a Riemannian manifold $(M,\langle\cdot,\cdot\rangle)$ is the trace of the 2-tensor field $\operatorname{Hess}u$, called the Hessian of u, defined by $\operatorname{Hess}u(X,Y)=\langle\nabla_X\nabla u,Y\rangle$, for all $X,Y\in\mathfrak{X}(M)$.

Proposition 3.1. *Let M^n be an orientable hypersurface of an orientable Riemannian manifold N_c^{n+1} of constant sectional curvature c. Then for every $r=1,\ldots,n$ and every $p\in M^n$,*

$$\binom{n}{r}\Delta H_r=n\sum_j\frac{\partial\sigma_r}{\partial x_j}(\vec{\lambda})\operatorname{Hess}H(e_j,e_j)-\sum_{i<j}\frac{\partial^2\sigma_r}{\partial x_i\,\partial x_j}(\vec{\lambda})(\lambda_i-\lambda_j)^2K_{ij}+\sum_{i,j,k}\frac{\partial^2\sigma_r}{\partial x_i\,\partial x_j}(\vec{\lambda})(h_{iik}h_{jjk}-h_{ijk}^2),$$

where $\lambda_1,\ldots,\lambda_n$ are the principal curvatures of M^n at p, $\vec{\lambda}=(\lambda_1,\ldots,\lambda_n)$, $\{e_1,\ldots,e_n\}$ is an orthonormal basis of T_pM that diagonalizes the shape operator A, and K_{ij} is the sectional curvature of M^n in the plane spanned by $\{e_i,e_j\}$.

4. Complete and bounded hypersurfaces

In the proof of Theorem 1.1, besides Propositions 2.1 and 3.1, we use the following result.

Proposition 4.1. *Let M^n be a complete Riemannian manifold with sectional curvature K bounded from below and $f : M^n \to \mathbb{Q}_c^{n+k}$ an isometric immersion of M^n into the $(n+k)$-dimensional complete simply connected space $\mathbb{Q}_c^{n+k}$ of constant sectional curvature c. In the case $c \le 0$, assume that $f(M^n)$ is bounded, and in the case $c > 0$, that $f(M^n)$ lies inside a geodesic ball of radius $\rho < \pi/2\sqrt{c}$. Then there exist $p \in M$ and a unit vector $\xi_0 \in (f_* T_p M)^\perp$ such that for any unit vector $v \in T_p M$,*

$$\langle A_{\xi_0} v, v\rangle > \begin{cases} 0, & c \ge 0, \\ \sqrt{-c}, & c < 0. \end{cases} \tag{4-1}$$

We believe that the above proposition is known, but since we were unable to find a reference for it in the literature, we prove it below. The main ingredient in this proof is the following well known maximum principle due to Omori and Yau [Cheng and Yau 1975; Omori 1967; Yau 1975]; see [Fontenele and Xavier 2011, Theorem 3.4] for a conceptual refinement of this principle.

Omori–Yau maximum principle. *Let M^n be a complete Riemannian manifold with sectional curvature or Ricci curvature bounded from below, and let $f : M \to \mathbb{R}$ be a C^2-function bounded from above. Then for every $\varepsilon > 0$, there exists $x_\varepsilon \in M$ such that*

$$f(x_\varepsilon) > \sup f - \varepsilon, \quad \|\nabla f(x_\varepsilon)\| < \varepsilon, \quad \operatorname{Hess} f(x_\varepsilon)(v, v) < \varepsilon \|v\|^2 \quad \forall v \in T_{x_\varepsilon} M - \{0\}$$

or

$$f(x_\varepsilon) > \sup f - \varepsilon, \quad \|\nabla f(x_\varepsilon)\| < \varepsilon, \quad \Delta f(x_\varepsilon) < \varepsilon,$$

respectively.

The following lemma, which is also used in the proof of Proposition 4.1, expresses the gradient and Hessian of the restriction of a function to a submanifold in terms of the space gradient and Hessian; see [Dajczer 1990, p. 46] for a proof. In its statement, we use the symbol ∇ for the gradient of any function involved.

Lemma 4.2. *Let $f : M^n \to N^{n+k}$ be an isometric immersion of a Riemannian manifold M^n into a Riemannian manifold N^{n+k}, and let $g : N \to \mathbb{R}$ be a function of class C^2. Then for all $p \in M$ and $v, w \in T_p M$, one has*

$$f_*(\nabla(g \circ f)(p)) = \big[\nabla g(f(p))\big]^\top,$$

$$\operatorname{Hess}(g \circ f)_p(v, w) = \operatorname{Hess} g_{f(p)}(f_* v, f_* w) + \big\langle \nabla g(f(p)), \sigma_p(v, w)\big\rangle, \tag{4-2}$$

where σ is the second fundamental form of the immersion, f_* is the differential of f and "$\top$" means orthogonal projection onto $f_*(T_p M)$.

Proof of Proposition 4.1. By hypothesis, $f(M)$ is contained in some closed ball $\bar{B}_\rho(q_o)$ of center q_o and radius ρ, with $\rho < \pi/2\sqrt{c}$ if $c > 0$. Let $r(\cdot) = d(\cdot, q_0)$ be the distance function from the point q_0 in $\mathbb{Q}_c^{n+k}$ and let $g = r \circ f$. Since g is bounded from above (for $f(M) \subset \bar{B}_\rho(q_0)$) and the sectional curvatures of M are bounded from below, the Omori–Yau maximum principle assures us that, for every $\varepsilon > 0$, there exist $x_\varepsilon \in M$ such that

$$g(x_\varepsilon) > \sup g - \varepsilon, \quad \|\nabla g(x_\varepsilon)\| < \varepsilon, \quad \operatorname{Hess} g_{x_\varepsilon}(v, v) < \varepsilon \|v\|^2, \quad \forall v \in T_{x_\varepsilon} M.$$

From the last two inequalities and Lemma 4.2, we obtain

$$\varepsilon > \|\nabla g(x_\varepsilon)\| = \|\nabla r(f(x_\varepsilon))^\top\| \tag{4-3}$$

and, for every $v \in T_{x_\varepsilon} M$,

$$\varepsilon \|v\|^2 > \operatorname{Hess} g_{x_\varepsilon}(v, v) = \operatorname{Hess} r_{f(x_\varepsilon)}(f_* v, f_* v) + \langle \sigma_{x_\varepsilon}(v, v), \nabla r(f(x_\varepsilon))\rangle, \tag{4-4}$$

where the superscript "⊤" indicates orthogonal projection on $f_*(T_{x_\varepsilon} M)$.

For every $v \in T_{x_\varepsilon} M$, write

$$f_* v = v_1 + v_2, \tag{4-5}$$

where v_1 and v_2 are the components of $f_* v$ that are parallel and orthogonal, respectively, to $\nabla r(f(x_\varepsilon))$. Recalling that $\overline{\nabla}_{\nabla r} \nabla r = 0$, where $\overline{\nabla}$ is the Riemannian connection of $\mathbb{Q}_c^{n+k}$, one has

$$\begin{aligned} \operatorname{Hess} r_{f(x_\varepsilon)}(f_* v, f_* v) &= \operatorname{Hess} r_{f(x_\varepsilon)}(v_1 + v_2, v_1 + v_2) \\ &= \operatorname{Hess} r_{f(x_\varepsilon)}(v_2, v_2). \end{aligned} \tag{4-6}$$

Note that v_2 is tangent to the geodesic sphere S of $\mathbb{Q}_c^{n+k}$ centered at q_0 that contains $f(x_\epsilon)$. Applying (4-2) for the inclusion $\iota : S \to \mathbb{Q}_c^{n+k}$ and $g = r$, one obtains

$$\operatorname{Hess} r_{f(x_\varepsilon)}(v_2, v_2) = \langle B v_2, v_2 \rangle, \tag{4-7}$$

where B is the shape operator of S with respect to $-\nabla r$. Since the principal curvatures of a geodesic sphere of radius t in $\mathbb{Q}_c^{n+k}$ are constant and given by

$$\mu_c(t) = \begin{cases} \sqrt{c} \cot(\sqrt{c}\, t), & c > 0,\ 0 < t < \pi/\sqrt{c}, \\ 1/t, & c = 0,\ t > 0, \\ \sqrt{-c} \coth(\sqrt{-c}\, t), & c < 0,\ t > 0, \end{cases} \tag{4-8}$$

it follows from (4-6) and (4-7) that

$$\operatorname{Hess} r_{f(x_\varepsilon)}(f_* v, f_* v) = \mu_c(r(f(x_\varepsilon))) \|v_2\|^2. \tag{4-9}$$

As $\|\nabla r\| \equiv 1$, by (4-5) one has $v_1 = \langle f_* v, \nabla r(f(x_\varepsilon))\rangle \nabla r(f(x_\varepsilon))$. Then, by (4-3),

$$\|v_1\| = \left|\langle f_* v, \nabla r(f(x_\varepsilon))^\top \rangle\right| \le \|f_* v\| \, \|\nabla r(f(x_\varepsilon))^\top\| < \varepsilon \|v\|.$$

From (4-5) and the above inequality, we obtain

$$\|v_2\|^2 = \|f_* v\|^2 - \|v_1\|^2 = \|v\|^2 - \|v_1\|^2 > (1 - \varepsilon^2)\|v\|^2. \tag{4-10}$$

Hence, by (4-4), (4-9), and (4-10),

$$\varepsilon\|v\|^2 > \mu_c(r(f(x_\varepsilon)))(1-\varepsilon^2)\|v\|^2 + \big\langle \sigma_{x_\varepsilon}(v,v), \nabla r(f(x_\varepsilon))\big\rangle.$$

Since μ_c is decreasing and $r(f(x_\varepsilon)) \le \rho$, it follows that

$$\begin{aligned}\varepsilon\|v\|^2 &> \mu_c(\rho)(1-\varepsilon^2)\|v\|^2 + \big\langle \sigma_{x_\varepsilon}(v,v), \nabla r(f(x_\varepsilon))\big\rangle \\ &= \mu_c(\rho)(1-\varepsilon^2)\|v\|^2 + \big\langle \sigma_{x_\varepsilon}(v,v), \nabla r(f(x_\varepsilon))^\perp\big\rangle,\end{aligned}$$

where $\nabla r(f(x_\varepsilon))^\perp$ is the component of $\nabla r(f(x_\varepsilon))$ that is orthogonal to $f_*(T_{x_\varepsilon}M)$. Setting $\xi_\varepsilon = -\nabla r(f(x_\varepsilon))^\perp/\|\nabla r(f(x_\varepsilon))^\perp\|$, it follows from (2-1) and the above inequality that

$$\langle A_{\xi_\varepsilon}v, v\rangle = \langle \sigma_{x_\varepsilon}(v,v), \xi_\varepsilon\rangle > \frac{\mu_c(\rho)(1-\varepsilon^2)-\varepsilon}{\|\nabla r(f(x_\varepsilon))^\perp\|} \tag{4-11}$$

for all $v \in T_{x_\varepsilon}M$, $\|v\| = 1$. Since, by (4-3), the term on the right-hand side of (4-11) tends to $\mu_c(\rho)$ when $\varepsilon \to 0$, and, by (4-8), $\mu_c(\rho) > 0$ for $c \ge 0$ and $\mu_c(\rho) > \sqrt{-c}$ for $c < 0$, (4-1) is fulfilled choosing $p = x_\varepsilon$ and $\xi_0 = \xi_\varepsilon$, where ε is any positive number sufficiently small. □

5. Proof of Theorem 1.1

Since H is constant and R is bounded from below, from (2-3) one obtains that $|A|^2$ is bounded, and so that the sectional curvatures of M^n are bounded from below. Then, by Proposition 4.1, there exist a point $p \in M$ and a unit vector $\xi_0 \in (f_*T_pM)^\perp$ such that

$$\langle A_{\xi_0}v, v\rangle > \alpha_c\|v\|^2, \quad v \in T_pM, \tag{5-1}$$

where

$$\alpha_c = \begin{cases} 0, & c \ge 0, \\ \sqrt{-c}, & c < 0.\end{cases}$$

Choosing the unit normal vector field ξ such that $\xi(p) = \xi_0$, by (5-1) the principal curvatures of M at p satisfy

$$\lambda_i(p) > \alpha_c \ge 0, \quad i = 1, \dots, n. \tag{5-2}$$

By Proposition 3.1, as H and H_r are constant one has

$$\sum_{i<j} \frac{\partial^2\sigma_r}{\partial x_i \partial x_j}(\vec{\lambda})(\lambda_i - \lambda_j)^2 K_{ij} = \sum_{i,j,k} \frac{\partial^2\sigma_r}{\partial x_i \partial x_j}(\vec{\lambda})(h_{iik}h_{jjk} - h_{ijk}^2), \tag{5-3}$$

where $\vec{\lambda} = (\lambda_1, \dots, \lambda_n)$. From (5-2) one obtains that $H_r > 0$ and that $\vec{\lambda}(p)$ belongs to the Gårding's cone Γ_r (see Section 2). Then, since M is connected,

$$\vec{\lambda}(q) \in \Gamma_r, \quad \forall q \in M.$$

By Proposition 2.1, $W_r = \sigma_r^{1/r}$ is a concave function on Γ_r. Thus,

$$\sum_{i,j} y_i y_j \frac{\partial^2 W_r}{\partial x_i \partial x_j}(x) \le 0 \tag{5-4}$$

for all $x \in \Gamma_r$ and $(y_1, \dots, y_n) \in \mathbb{R}^n$. A simple computation shows that

$$\frac{\partial^2 W_r}{\partial x_i \partial x_j} = \frac{1}{r}\sigma_r^{(1/r)-2}\Big(\frac{1-r}{r}\frac{\partial \sigma_r}{\partial x_i}\frac{\partial \sigma_r}{\partial x_j} + \sigma_r \frac{\partial^2 \sigma_r}{\partial x_i \partial x_j}\Big).$$

Using the above equality in (5-4), we conclude that

$$\begin{aligned}\sigma_r(x)\sum_{i,j} y_i y_j \frac{\partial^2 \sigma_r}{\partial x_i \partial x_j}(x) &\le \frac{r-1}{r}\sum_{i,j} y_i y_j \frac{\partial \sigma_r}{\partial x_i}(x)\frac{\partial \sigma_r}{\partial x_j}(x) \\ &= \frac{r-1}{r}\Big(\sum_j y_j \frac{\partial \sigma_r}{\partial x_j}(x)\Big)^2,\end{aligned} \tag{5-5}$$

for all $x \in \Gamma_r$ and $(y_1, \dots, y_n) \in \mathbb{R}^n$. Taking $x = \vec{\lambda}$ and $y_i = h_{iik}$, $i = 1, \dots, n$, in (5-5), one obtains

$$\binom{n}{r} H_r \sum_{i,j} h_{iik} h_{jjk} \frac{\partial^2 \sigma_r}{\partial x_i \partial x_j}(\vec{\lambda}) \le \frac{r-1}{r}\Big(\sum_j h_{jjk}\frac{\partial \sigma_r}{\partial x_j}(\vec{\lambda})\Big)^2, \quad \forall k. \tag{5-6}$$

We claim that in a basis that diagonalizes A,

$$\sum_j h_{jjk}\frac{\partial \sigma_r}{\partial x_j}(\vec{\lambda}) = \binom{n}{r} e_k(H_r). \tag{5-7}$$

The claim can be proved using the formula $\binom{n}{r} e_k(H_r) = \operatorname{tr}(P_{r-1}\nabla_{e_k} A)$ [Rosenberg 1993, p. 225], where P_{r-1} is the $(r-1)$-th Newton tensor associated with the shape operator A of M.

Since H_r is a positive constant, from (5-6) and (5-7) one obtains

$$\sum_{i,j} h_{iik} h_{jjk} \frac{\partial^2 \sigma_r}{\partial x_i \partial x_j}(\vec{\lambda}) \le 0, \quad k = 1, \dots, n.$$

Using this information in (5-3), we conclude that the inequality

$$\sum_{i<j} \frac{\partial^2 \sigma_r}{\partial x_i \partial x_j}(\vec{\lambda})(\lambda_i - \lambda_j)^2 K_{ij} \le -\sum_{i,j,k} h_{ijk}^2 \frac{\partial^2 \sigma_r}{\partial x_i \partial x_j}(\vec{\lambda})$$

holds at every point of M. Since, by (3-1) and (5-2),

$$\frac{\partial^2 \sigma_r}{\partial x_i \partial x_j}(\vec{\lambda}(p)) = \begin{cases} \sigma_{r-2}(\widehat{\lambda_i}(p), \widehat{\lambda_j}(p)) > 0, & i \ne j, \\ 0, & i = j, \end{cases} \tag{5-8}$$

it follows that

(5-9) $$\sum_{i<j} \frac{\partial^2 \sigma_r}{\partial x_i \partial x_j}(\vec{\lambda}(p))(\lambda_i(p)-\lambda_j(p))^2 K_{ij}(p) \le 0.$$

Since, by (5-2) and the Gauss equation,

$$K_{ij}(p) = c + \lambda_i(p)\lambda_j(p) > c + \alpha_c^2 \ge 0, \quad i \ne j,$$

it follows from (5-8) and (5-9) that $\lambda_1(p) = \cdots = \lambda_n(p) = H$.

Let U be the set of umbilic points of M. Let B be the set

$$B = \{p \in U : \lambda_i(p) > \alpha_c \text{ for all } i = 1, \dots, n\} \subset U.$$

By the argument above, B is nonempty and open. Assuming $B \ne U$, we can find a point $q \in \partial B \subset U$. By continuity of the principal curvatures, they are all constant and bigger than α_c at q, and hence $B = U$. Since U is closed, open, and nonempty, M is totally umbilical. To finish, it is well known that the only complete totally umbilical hypersurfaces in a space form are the geodesic spheres [Spivak 1975, pp. 75–79].

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