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A NON-STRICTLY PSEUDOCONVEX DOMAIN FOR WHICH THE SQUEEZING FUNCTION TENDS TO 1 TOWARDS THE BOUNDARY

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In recent work by Zimmer it was proved that if $\Omega \subset \mathbb{C}^n$ is a bounded convex domain with C^∞-smooth boundary, then Ω is strictly pseudoconvex provided that the squeezing function approaches 1 as one approaches the boundary. We show that this result fails if Ω is only assumed to be C^2-smooth.

1. Introduction

We recall the definition of the squeezing function $S_\Omega(z)$ on a bounded domain $\Omega \subset \mathbb{C}^n$. If $z \in \Omega$, and $f_z : \Omega \to \mathbb{B}^n$ is an embedding with $f_z(z) = 0$, we set

$$S_{\Omega, f_z}(z) := \sup\{r > 0 : B_r(0) \subset f_z(\Omega)\}, \tag{1-1}$$

and then

$$S_\Omega(z) := \sup_{f_z}\{S_{\Omega, f_z}(z)\}. \tag{1-2}$$

A guiding question is the following: which complex analytic properties of Ω are encoded by the behaviour of S_Ω? For instance, if S_Ω is bounded away from 0, then Ω is necessarily pseudoconvex, and the Kobayashi–, Carathéodory–, Bergman– and the Kähler–Einstein metrics are complete, and they are pairwise quasi-isometric; see [Liu, Sun and Yau 2004; Yeung 2009]. Recently, Zimmer [2018b] proved that if

$$\lim_{z \to b\Omega} S_\Omega(z) = 1 \tag{1-3}$$

for a C^∞-smooth, bounded convex domain, then the domain Ω is necessarily strictly

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pseudoconvex.[1] In this short note we will show that this does not hold for C^2-smooth domains.

Theorem 1.1. *There exists a bounded convex C^2-smooth domain $\Omega \subset \mathbb{C}^n$ which is not strongly pseudoconvex, but*

$$\lim_{z \to b\Omega} S_\Omega(z) = 1, \tag{1-4}$$

where $S_\Omega(z)$ denotes the squeezing function on Ω.

For further results about the squeezing function the reader may also consult the references [Diederich, Fornæss and Wold 2016; Deng, Guan and Zhang 2012; 2016; Fornæss and Rong 2016; Fornæss and Wold 2015; Kim and Zhang 2016; Liu, Sun and Yau 2004; Yeung 2009; Zimmer 2018b]. In the last section we will post some open problems.

2. The construction

The construction in $\mathbb{R}^n$ and curvature estimates. We start by describing a construction of a convex domain Ω in $\mathbb{R}^n$ with a single non-strictly convex point. Afterwards we will explain how to make the construction give the conclusion of Theorem 1.1 for each $n = 2m$, when we make the identification with $\mathbb{C}^m$.

Let $x = (x_1, \dots, x_n)$ denote the coordinates on $\mathbb{R}^n$. For any $k \in \mathbb{N}$ we let B_k denote the ball

$$B_k := \{x \in \mathbb{R}^n : x_1^2 + \cdots + x_{n-1}^2 + (x_n - k)^2 < k^2\}. \tag{2-1}$$

On some fixed neighbourhood of the origin, each boundary bB_k may be written as a graph of a function

$$x_n = \psi_k(x') = \psi_k(x_1, \dots, x_{n-1}) = k - \sqrt{k^2 - \|x'\|^2} = \frac{1}{2k}\|x'\|^2 + O(\|x\|^3). \tag{2-2}$$

Fix a smooth cut-off function $\chi(x') = \chi(|x'|)$ with compact support in $\{|x'| < 1\}$ which is one near the origin. We will create a new limit-graphing function $f(x')$ by subsequently gluing the functions ψ_k and ψ_{k+1} by setting

$$g_k(x') = \psi_k(x') + \chi\left(\frac{x'}{\epsilon_k}\right)(\psi_{k+1}(x') - \psi_k(x')), \tag{2-3}$$

where the sequence ϵ_k will converge rapidly to zero, and the boundary of our domain Ω will be defined (locally) as the graph Σ of the function f defined as follows: Start by setting $f_k := \psi_k$ for some $k \in \mathbb{N}$. Then define f_{k+1} inductively by

[1] Added in proof: Zimmer [2018a] has subsequently improved his results to convex domains with $C^{2,\alpha}$-boundary.

setting $f_{k+1} = f_k$ for $\|x'\| \geq \epsilon_k$ and then $f_{k+1} = g_k$ for $\|x'\| < \epsilon_k$. Finally we set $f = \lim_{k\to\infty} f_k$.

To show that the limit function f is C^2-smooth (if the ϵ_k's converge rapidly to 0), we need to show that the sequence $\{f_k\}$ is a Cauchy sequence with respect to the C^2-norm, i.e., we need to estimate the derivatives

$$\sigma_{ij}^k(x') := \frac{\partial^2}{\partial x_i \partial x_j}\left(\chi\left(\frac{x'}{\epsilon_k}\right)(\psi_{k+1}(x') - \psi_k(x'))\right). \tag{2-4}$$

Note first that

$$\psi_{k+1}(x') - \psi_k(x') = \frac{-1}{2k(k+1)}\|x'\|^2 + O(\|x'\|^3). \tag{2-5}$$

We see that

$$|\sigma_{ij}^k(x')| = \left(\frac{1}{\epsilon_k^2}O(\|x'\|^2) + \frac{1}{\epsilon_k}O(\|x'\|)\right)\frac{1}{2k(k+1)} + \frac{1}{\epsilon_k^2}O(\|x'\|^3) + \frac{1}{\epsilon_k}O(\|x'\|^2),$$

and so for $\|x'\| < \epsilon_k$ we have

$$|\sigma_{ij}^k(x')| \leq C \cdot \frac{1}{2k(k+1)} + O(\epsilon_k), \tag{2-6}$$

where the constants are independent of any particular choice of ϵ_k. So if ϵ_k is small enough we see that $|\sigma_{ij}^k|$ is of order of magnitude $1/k^2$, which shows that $\{f_k\}$ will be a Cauchy sequence.

To ensure that Ω is convex we will need to estimate the curvature of Σ, and estimates of the curvature of the partial graphs $\Sigma_k = \{x, g_k(x)\}$ will be necessary to prove Theorem 1.1. Informally our goal is to show the following: *There exist $N, m \in \mathbb{N}$, $N > m$, such that if $k \geq N$ and if ϵ_k is sufficiently small (depending on k), then Σ_k curves, at every point and in all directions, more than bB_{k+m} and less than bB_{k-m}.*

We make this more precise. The surface Σ_k has a defining function $\rho_k(x) = g_k(x') - x_n$. If v_p is a tangent vector to Σ_k at $p = (x', g_k(x))$, the curvature of Σ_k in the direction of v_p is defined as

$$\kappa_p^{\Sigma_k}(v_p) := \frac{H\rho_k(p)(v_p)}{\|\nabla\rho_k(p)\|\|v_p\|^2}, \tag{2-7}$$

where $\nabla\rho_k$ is the gradient, and $H\rho_k$ is the Hessian of ρ_k (which is equal to the Hessian of g_k). The curvature (2-7) depends only on the direction of v_p, and the curvature of bB_k is $\frac{1}{k}$ at all points and in all directions. The precise statement of our goal stated above is this:

Lemma 2.1. *Let ψ_k and χ be defined as above for $k \in \mathbb{N}$. There exist $N, m \in \mathbb{N}$, $N > m$, such that if each ϵ_k is sufficiently small (depending on k), and $k \geq N$, then*

$$\frac{1}{k+m} \leq \kappa_p^{\Sigma_k}(v_p) \leq \frac{1}{k-m}, \tag{2-8}$$

for all v_p tangent to Σ_k.

It is now easy to see that if $\epsilon_k \searrow 0$ sufficiently fast, then Ω is convex, and strictly convex away from the origin. If we let Ω_k denote the domain whose boundary near the origin is given by the graph of f_k, we see that Ω_k is strictly convex, the Hessian being positive definite everywhere. Moreover, $\Omega = \cup_k \Omega_k$, and so Ω is convex.

Proof of Lemma 2.1. When we estimate the curvature we may assume that the functions g_k are simply

$$g_k(x') = \psi_k(x') - \chi\left(\frac{x'}{\epsilon_k}\right)\left(\frac{1}{2k(k+1)}\right)|x'|^2 =: \psi_k(x') + \sigma_k(x'), \tag{2-9}$$

since the higher order terms missing in this expression of g_k can be made insignificant by choosing ϵ_k small enough. Because of the $|x'|^2$ term it is easy to see that

$$dg_k(x') = d\psi_k(x') + \triangle_k(x') \tag{2-10}$$

and

$$Hg_k(x') = H\psi_k(x') + h_k(x'), \tag{2-11}$$

where the coefficients in both $\triangle_k$ and h_k are of order of magnitude $1/k^2$ independently of k and of the choice of a small ϵ_k.

Fix a point x' and a vector $v \in \mathbb{R}^{n-1}$ with $\|v\| = 1$. Then a tangent vector v_p at the point $(x', g_k(x'))$ is given by

$$v_p = (v, dg_k(x')(v)) = (v, d\psi_k(x')(v) + \triangle_k(x')(v)). \tag{2-12}$$

Estimating the curvature we see that

$$\begin{aligned}
\kappa_p^{\Sigma_k}(v_p) &= \frac{(H\psi_k(x') + h_k(x'))(v_p)}{\|\nabla\rho_k(p)\|\,\|v_p\|^2} \\
&= \frac{(H\psi_k(x'))((v, d\psi_k(x')v) + (0', \triangle_k(x')(v)))}{\|-\boldsymbol{e_n} + \nabla\psi_k(p) + \nabla\sigma_k(x')\|\,\|(v, d\psi_k(x')(v)) + (0', \triangle_k(x'))\|^2} + O\left(\frac{1}{k^2}\right) \\
&= \frac{(H\psi_k(x'))((v, d\psi_k(x')v))}{\|-\boldsymbol{e_n} + \nabla\psi_k(x')\|\left(1 + O\left(\frac{1}{k^2}\right)\right)\|(v, d\psi_k(x')(v))\|^2\left(1 + O\left(\frac{1}{k^2}\right)\right)^2} + O\left(\frac{1}{k^2}\right) \\
&= \frac{(H\psi_k(x'))((v, d\psi_k(x')v))}{\|-\boldsymbol{e_n} + \nabla\psi_k(x')\|\,\|(v, d\psi_k(x')(v))\|^2} + O\left(\frac{1}{k^2}\right) = \frac{1}{k} + O\left(\frac{1}{k^2}\right),
\end{aligned}$$

where the term $\frac{1}{k}$ comes from the fact that the expression above is the formula for the curvature of a ball of radius k. From this it is straightforward to deduce the existence of an m such that the lemma holds. □

The squeezing function on Ω. We will now explain why the squeezing function goes to 1 uniformly as we approach $b\Omega$ provided that the ϵ_k's decrease sufficiently fast. Let N, m be as in Lemma 2.1, and start by setting $f_k = \psi_k$ for some $k > N$.

Fix some small $\delta_k > 0$. By Lemma 2.1, if ϵ_k is small enough, we can for each $p = (x', x_n) \in b\Omega_k$, with $\|x'\| < \delta_k$, find a ball B of radius $k+m$ containing Ω_k such that $p \in bB$. By the same lemma we can for each such p also find a local piece of a ball of radius $k-m$ touching p from the inside of Ω_k, and the size of the local ball is uniform. So using Lemma 3.1 we may find a $t_k > 0$ small enough such that

$$S_{\Omega_k}(x', x_n) \geq 1 - \frac{m}{(k+m)} \tag{2-13}$$

if $x_n \leq t_k$.

Next, again by Lemma 2.1, we find a $\delta_{k+1} < \delta_k$ such that if ϵ_{k+1} is small enough, then for each $p = (x', x_n) \in b\Omega_{k+1}$ with $\|x'\| < \delta_{k+1}$, we may oscillate with balls of radius $k+1-m$ and $k+1+m$ respectively. So there is a $t_{k+1} < t_k$ such that

$$S_{\Omega_{k+1}}(x', x_n) \geq 1 - \frac{m}{(k+1+m)} \tag{2-14}$$

if $x_n \leq t_{k+1}$. Furthermore, by further decreasing ϵ_{k+1}, we can keep the estimate (2-13) with Ω_k replaced by Ω_{k+1}. The reason is the following. First of all, by [Fornæss and Wold 2015], there exists a constant C_k such that

$$S_{\Omega_k}(z) \geq 1 - C_k \cdot \operatorname{dist}(z, b\Omega_k), \tag{2-15}$$

and near any compact $K \subset b\Omega_k$ away from 0, this estimate is not going to be disturbed by a small perturbation of $b\Omega_k$ near the point 0; the estimate is obtained by using oscillating balls at points of K whose boundaries will stay bounded away from 0. Furthermore, on any compact subset of Ω_k we have that $S_{\Omega_{k+1}} \to S_{\Omega_k}$ as $\epsilon_{k+1} \to 0$.

Continuing in this fashion, we obtain a decreasing sequence $0 < t_j < t_{j+1}$, $j = k, k+1, \ldots$, and an increasing sequence of domains Ω_j, such that for each j we have

$$S_{\Omega_j}(x', x_n) \geq 1 - \frac{m}{(k+i+m)} \tag{2-16}$$

for $t_{k+i} \leq x_n \leq t_{k+i-1}$, for $i \leq j$. The result now follows from Lemma 3.2.

3. Lemmata

Let $0 < s < 1/2,\ 0 < d < r < 1$, and set $B_s = B(s, 1-s)$, the ball of radius $1-s$ centred at $(s, 0')$. Furthermore, we set

$$B_{s,d} = B_s \cap \{(z_1, z') \in \mathbb{B}^n : \mathcal{R}e(z_1) > d\}. \tag{3-1}$$

Lemma 3.1. *If $B_{s,d} \subset \Omega \subset \mathbb{B}^n$, and if $r > 1 - \frac{sd}{4}$, then $S_\Omega(r, 0) > 1 - s$.*

Proof. Set $\mu = 1 - s$ and $\eta = \frac{d}{2}$, and then

$$B_\eta^\mu = \Big\{(z_1, z') \in \mathbb{C}^n : |z_1 - (1-\eta)|^2 + \frac{\eta}{\mu}|z'|^2 < \eta^2\Big\}. \tag{3-2}$$

Then certainly $\mathcal{R}e(z_1) > d$ on B_η^μ, and we also have that $B_\eta^\mu \subset B_s$. To see the latter, we translate the two balls sending $(1, 0')$ to the origin, where they are defined by

$$\widetilde{B}_s = \{(z_1, z') : 2\mu\, \mathcal{R}e(z_1) + |z|^2 < 0\}, \tag{3-3}$$

and

$$\widetilde{B}_\eta^\mu = \Big\{(z_1, z') : 2\eta\, \mathcal{R}e(z_1) + |z_1|^2 + \frac{\eta}{\mu}|z'|^2 < 0\Big\}. \tag{3-4}$$

Also,

$$\begin{aligned} 2\eta\mathcal{R}e(z_1) + |z_1|^2 + \frac{\eta}{\mu}|z'|^2 < 0 &\Rightarrow 2\eta\, \mathcal{R}e(z_1) + \frac{\eta}{\mu}|z_1|^2 + \frac{\eta}{\mu}|z'|^2 < 0 \\ &\Leftrightarrow 2\mu\mathcal{R}e(z_1) + |z|^2 < 0. \end{aligned}$$

According to Lemma 3.5 in [Fornæss and Wold 2015] we have

$$S_\Omega(r, 0) \geq \sqrt{\mu}\sqrt{1 - 2(1-r)\frac{1}{\eta}} = \sqrt{(1-s)\Big(1 - \frac{4(1-r)}{d}\Big)}, \tag{3-5}$$

from which the lemma follows easily. □

Lemma 3.2. *Let $\Omega_j \subset \Omega_{j+1}$ for $j \in \mathbb{N}$, set $\Omega = \cup_j \Omega_j$, and assume that Ω is bounded. Let $z \in \Omega$, and assume that $S_{\Omega_j}(z) > 1 - \delta$ for all j large enough so that $z \in \Omega_j$. Then $S_\Omega(z) \geq 1 - \delta$.*

Proof. Let $f_j : \Omega_j \to \mathbb{B}^n$ be an embedding such that $f_j(z) = 0$ and $B_{1-\delta}(0) \subset f_j(\Omega_j)$. By passing to a subsequence we may assume that $f_j \to f : \Omega \to \mathbb{B}^n$ uniformly on compact sets, with $f(z) = 0$. Setting $g_j = f_j^{-1} : B_{1-\delta}(0) \to \Omega$ we may also assume that $g_j \to g$ uniformly on compact sets. Then $f|_{g(B_{1-\delta}(0))} = g^{-1}$, from which the result follows. □

4. Some open problems

Problem 4.1. Does Zimmer's result hold for pseudoconvex domains of class C^∞?

Problem 4.2. How much smoothness is needed for Zimmer's result hold for pseudoconvex domains?

Problem 4.3. Let $\Omega \subset \mathbb{C}^2$ be a bounded pseudoconvex domain of class C^∞. Is $S_\Omega(z)$ bounded away from zero?

In light of the result of [Deng, Guan and Zhang 2016], the answer to the last question is affirmative for bounded strictly pseudoconvex domains of class $\mathbb{C}^2$ in all dimensions. For strictly convex domains in $\mathbb{C}^n$, this was proved in [Yeung 2009]. Furthermore, it has been shown in [Kim and Zhang 2016] that the same holds for bounded convex domains without any further regularity assumptions, and by [Nikolov and Andreev 2017], it even holds for bounded $\mathbb{C}$-convex domains in general. On the other hand, by [Fornæss and Rong 2016], the answer is negative in general for $n \geq 3$.

Quantifying the asymptotic behaviour of the squeezing function, we showed in [Fornæss and Wold 2015] that

(i) $S_\Omega(z) \geq 1 - C\,\mathrm{dist}(z, b\Omega)$, and

(ii) $S_\Omega(z) \geq 1 - C\sqrt{\mathrm{dist}(z, b\Omega)}$,

for strongly pseudoconvex domains of class C^4 and C^3 respectively. In [Diederich, Fornæss and Wold 2016] we showed that if the squeezing function approaches 1 essentially faster than in (i), then Ω is biholomorphic to the unit ball.

Problem 4.4. What is the optimal estimate for the squeezing function for strictly pseudoconvex domains of class C^k with $k < 4$?

Let $\phi : \mathbb{B}^2 \to \mathbb{C}^2$ be defined as

$$\phi(z_1, z_2) := (z_1, -z_2 \log(z_1 - 1)).$$

Then $\Omega := \phi(\mathbb{B}^2)$ is of class C^1, and $(1, 0)$ is a non-strictly pseudoconvex boundary point of Ω. So S_Ω being identically equal to 1 does not even imply strict pseudoconvexity in the case of C^1-smooth boundaries.

Problem 4.5. Let $\phi : \mathbb{B}^n \to \Omega$ be a biholomorphism, and assume that Ω is a bounded C^2-smooth domain. Is Ω strictly pseudoconvex?

References

[Deng, Guan and Zhang 2012] F. Deng, Q. Guan, and L. Zhang, "Some properties of squeezing functions on bounded domains", *Pacific J. Math.* **257**:2 (2012), 319–341. MR Zbl

[Deng, Guan and Zhang 2016] F. Deng, Q. Guan, and L. Zhang, "Properties of squeezing functions and global transformations of bounded domains", *Trans. Amer. Math. Soc.* **368**:4 (2016), 2679–2696. MR Zbl

[Diederich, Fornæss and Wold 2016] K. Diederich, J. E. Fornæss, and E. F. Wold, "A characterization of the ball in $\mathbb{C}^n$", *Internat. J. Math.* **27**:9 (2016), art. id. 1650078. MR Zbl

[Fornæss and Rong 2016] J. E. Fornæss and F. Rong, "Estimate of the squeezing function for a class of bounded domains", 2016. To appear in *Math. Ann.* arXiv

[Fornæss and Wold 2015] J. E. Fornæss and E. F. Wold, "An estimate for the squeezing function and estimates of invariant metrics", pp. 135–147 in *Complex analysis and geometry* (Gyeongju, South Korea, 2014), edited by F. Bracci et al., Springer Proc. Math. Stat. **144**, Springer, 2015. MR Zbl

[Kim and Zhang 2016] K.-T. Kim and L. Zhang, "On the uniform squeezing property of bounded convex domains in $\mathbb{C}^n$", *Pacific J. Math.* **282**:2 (2016), 341–358. MR Zbl

[Liu, Sun and Yau 2004] K. Liu, X. Sun, and S.-T. Yau, "Canonical metrics on the moduli space of Riemann surfaces, I", *J. Differential Geom.* **68**:3 (2004), 571–637. MR Zbl

[Nikolov and Andreev 2017] N. Nikolov and L. Andreev, "Boundary behavior of the squeezing functions of $\mathbb{C}$-convex domains and plane domains", *Internat. J. Math* **28**:5 (2017).

[Yeung 2009] S.-K. Yeung, "Geometry of domains with the uniform squeezing property", *Adv. Math.* **221**:2 (2009), 547–569. MR Zbl

[Zimmer 2018a] A. Zimmer, "Characterizing strong pseudoconvexity, obstructions to biholomorphisms, and Lyapunov exponents", *Mathematische Annalen* (2018), 1–34.

[Zimmer 2018b] A. Zimmer, "A gap theorem for the complex geometry of convex domains", *Trans. Amer. Math. Soc.* **370**:10 (2018), 7489–7509. MR

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