# *Pacific Journal of Mathematics*


## A NON-STRICTLY PSEUDOCONVEX DOMAIN FOR WHICH THE SQUEEZING FUNCTION TENDS TO 1 TOWARDS THE BOUNDARY


JOHN ERIK FORNÆSS AND ERLEND FORNÆSS WOLD



Volume 297 No. 1 November 2018

PACIFIC JOURNAL OF MATHEMATICS
Vol. 297, No. 1, 2018
dx.doi.org/10.2140/pjm.2018.297.79

# A NON-STRICTLY PSEUDOCONVEX DOMAIN FOR WHICH THE SQUEEZING FUNCTION TENDS TO 1 TOWARDS THE BOUNDARY

JOHN ERIK FORNÆSS AND ERLEND FORNÆSS WOLD

**In recent work by Zimmer it was proved that if $\Omega \subset \mathbb{C}^n$ is a bounded convex domain with $C^\infty$-smooth boundary, then $\Omega$ is strictly pseudoconvex provided that the squeezing function approaches 1 as one approaches the boundary. We show that this result fails if $\Omega$ is only assumed to be $C^2$-smooth.**

## 1. Introduction

We recall the definition of the squeezing function $S_\Omega(z)$ on a bounded domain $\Omega \subset \mathbb{C}^n$. If $z \in \Omega$, and $f_z : \Omega \to \mathbb{B}^n$ is an embedding with $f_z(z) = 0$, we set

$$S_{\Omega, f_z}(z) := \sup\{r > 0 : B_r(0) \subset f_z(\Omega)\}, \tag{1-1}$$

and then

$$S_\Omega(z) := \sup_{f_z}\{S_{\Omega, f_z}(z)\}. \tag{1-2}$$

A guiding question is the following: which complex analytic properties of $\Omega$ are encoded by the behaviour of $S_\Omega$? For instance, if $S_\Omega$ is bounded away from 0, then $\Omega$ is necessarily pseudoconvex, and the Kobayashi–, Carathéodory–, Bergman– and the Kähler–Einstein metrics are complete, and they are pairwise quasi-isometric; see [Liu, Sun and Yau 2004; Yeung 2009]. Recently, Zimmer [2018b] proved that if

$$\lim_{z \to b\Omega} S_\Omega(z) = 1 \tag{1-3}$$

for a $C^\infty$-smooth, bounded convex domain, then the domain $\Omega$ is necessarily strictly

This article was written as part of the international research program “Several Complex Variables and Complex Dynamics” at the Centre for Advanced Study at the Norwegian Academy of Science and Letters in Oslo during the academic year 2016/2017.
Both authors are supported by the NRC grant number 240569.
*MSC2010:* 32F45, 32H02.
*Keywords:* squeezing function, holomorphic embeddings, holomorphic mappings.

pseudoconvex.[1] In this short note we will show that this does not hold for $C^2$-smooth domains.

**Theorem 1.1.** *There exists a bounded convex $C^2$-smooth domain $\Omega \subset \mathbb{C}^n$ which is not strongly pseudoconvex, but*

$$\lim_{z \to b\Omega} S_\Omega(z) = 1, \tag{1-4}$$

*where $S_\Omega(z)$ denotes the squeezing function on $\Omega$.*

For further results about the squeezing function the reader may also consult the references [Diederich, Fornæss and Wold 2016; Deng, Guan and Zhang 2012; 2016; Fornæss and Rong 2016; Fornæss and Wold 2015; Kim and Zhang 2016; Liu, Sun and Yau 2004; Yeung 2009; Zimmer 2018b]. In the last section we will post some open problems.

## 2. The construction

***The construction in $\mathbb{R}^n$ and curvature estimates.*** We start by describing a construction of a convex domain $\Omega$ in $\mathbb{R}^n$ with a single non-strictly convex point. Afterwards we will explain how to make the construction give the conclusion of Theorem 1.1 for each $n = 2m$, when we make the identification with $\mathbb{C}^m$.

Let $x = (x_1, \dots, x_n)$ denote the coordinates on $\mathbb{R}^n$. For any $k \in \mathbb{N}$ we let $B_k$ denote the ball

$$B_k := \{x \in \mathbb{R}^n : x_1^2 + \dots + x_{n-1}^2 + (x_n - k)^2 < k^2\}. \tag{2-1}$$

On some fixed neighbourhood of the origin, each boundary $bB_k$ may be written as a graph of a function

$$x_n = \psi_k(x') = \psi_k(x_1, \dots, x_{n-1}) = k - \sqrt{k^2 - \|x'\|^2} = \frac{1}{2k}\|x'\|^2 + O(\|x\|^3). \tag{2-2}$$

Fix a smooth cut-off function $\chi(x') = \chi(|x'|)$ with compact support in $\{|x'| < 1\}$ which is one near the origin. We will create a new limit-graphing function $f(x')$ by subsequently gluing the functions $\psi_k$ and $\psi_{k+1}$ by setting

$$g_k(x') = \psi_k(x') + \chi\left(\frac{x'}{\epsilon_k}\right)(\psi_{k+1}(x') - \psi_k(x')), \tag{2-3}$$

where the sequence $\epsilon_k$ will converge rapidly to zero, and the boundary of our domain $\Omega$ will be defined (locally) as the graph $\Sigma$ of the function $f$ defined as follows: Start by setting $f_k := \psi_k$ for some $k \in \mathbb{N}$. Then define $f_{k+1}$ inductively by

[1] Added in proof: Zimmer [2018a] has subsequently improved his results to convex domains with $C^{2,\alpha}$-boundary.

setting $f_{k+1} = f_k$ for $\|x'\| \geq \epsilon_k$ and then $f_{k+1} = g_k$ for $\|x'\| < \epsilon_k$. Finally we set $f = \lim_{k\to\infty} f_k$.

To show that the limit function $f$ is $C^2$-smooth (if the $\epsilon_k$'s converge rapidly to 0), we need to show that the sequence $\{f_k\}$ is a Cauchy sequence with respect to the $C^2$-norm, i.e., we need to estimate the derivatives

$$\sigma_{ij}^k(x') := \frac{\partial^2}{\partial x_i \partial x_j}\left(\chi\left(\frac{x'}{\epsilon_k}\right)(\psi_{k+1}(x') - \psi_k(x'))\right). \tag{2-4}$$

Note first that

$$\psi_{k+1}(x') - \psi_k(x') = \frac{-1}{2k(k+1)}\|x'\|^2 + O(\|x'\|^3). \tag{2-5}$$

We see that

$$|\sigma_{ij}^k(x')| = \left(\frac{1}{\epsilon_k^2}O(\|x'\|^2) + \frac{1}{\epsilon_k}O(\|x'\|)\right)\frac{1}{2k(k+1)} + \frac{1}{\epsilon_k^2}O(\|x'\|^3) + \frac{1}{\epsilon_k}O(\|x'\|^2),$$

and so for $\|x'\| < \epsilon_k$ we have

$$|\sigma_{ij}^k(x')| \leq C \cdot \frac{1}{2k(k+1)} + O(\epsilon_k), \tag{2-6}$$

where the constants are independent of any particular choice of $\epsilon_k$. So if $\epsilon_k$ is small enough we see that $|\sigma_{ij}^k|$ is of order of magnitude $1/k^2$, which shows that $\{f_k\}$ will be a Cauchy sequence.

To ensure that $\Omega$ is convex we will need to estimate the curvature of $\Sigma$, and estimates of the curvature of the partial graphs $\Sigma_k = \{x, g_k(x)\}$ will be necessary to prove Theorem 1.1. Informally our goal is to show the following: *There exist $N, m \in \mathbb{N}$, $N > m$, such that if $k \geq N$ and if $\epsilon_k$ is sufficiently small (depending on $k$), then $\Sigma_k$ curves, at every point and in all directions, more than $bB_{k+m}$ and less than $bB_{k-m}$.*

We make this more precise. The surface $\Sigma_k$ has a defining function $\rho_k(x) = g_k(x') - x_n$. If $v_p$ is a tangent vector to $\Sigma_k$ at $p = (x', g_k(x))$, the curvature of $\Sigma_k$ in the direction of $v_p$ is defined as

$$\kappa_p^{\Sigma_k}(v_p) := \frac{H\rho_k(p)(v_p)}{\|\nabla\rho_k(p)\|\|v_p\|^2}, \tag{2-7}$$

where $\nabla\rho_k$ is the gradient, and $H\rho_k$ is the Hessian of $\rho_k$ (which is equal to the Hessian of $g_k$). The curvature (2-7) depends only on the direction of $v_p$, and the curvature of $bB_k$ is $\frac{1}{k}$ at all points and in all directions. The precise statement of our goal stated above is this:

**Lemma 2.1.** *Let $\psi_k$ and $\chi$ be defined as above for $k \in \mathbb{N}$. There exist $N, m \in \mathbb{N}$, $N > m$, such that if each $\epsilon_k$ is sufficiently small (depending on $k$), and $k \geq N$, then*

$$\frac{1}{k+m} \leq \kappa_p^{\Sigma_k}(v_p) \leq \frac{1}{k-m}, \tag{2-8}$$

*for all $v_p$ tangent to $\Sigma_k$.*

It is now easy to see that if $\epsilon_k \searrow 0$ sufficiently fast, then $\Omega$ is convex, and strictly convex away from the origin. If we let $\Omega_k$ denote the domain whose boundary near the origin is given by the graph of $f_k$, we see that $\Omega_k$ is strictly convex, the Hessian being positive definite everywhere. Moreover, $\Omega = \cup_k \Omega_k$, and so $\Omega$ is convex.

*Proof of Lemma 2.1.* When we estimate the curvature we may assume that the functions $g_k$ are simply

$$g_k(x') = \psi_k(x') - \chi\left(\frac{x'}{\epsilon_k}\right)\left(\frac{1}{2k(k+1)}\right)|x'|^2 =: \psi_k(x') + \sigma_k(x'), \tag{2-9}$$

since the higher order terms missing in this expression of $g_k$ can be made insignificant by choosing $\epsilon_k$ small enough. Because of the $|x'|^2$ term it is easy to see that

$$dg_k(x') = d\psi_k(x') + \triangle_k(x') \tag{2-10}$$

and

$$Hg_k(x') = H\psi_k(x') + h_k(x'), \tag{2-11}$$

where the coefficients in both $\triangle_k$ and $h_k$ are of order of magnitude $1/k^2$ independently of $k$ and of the choice of a small $\epsilon_k$.

Fix a point $x'$ and a vector $v \in \mathbb{R}^{n-1}$ with $\|v\| = 1$. Then a tangent vector $v_p$ at the point $(x', g_k(x'))$ is given by

$$v_p = (v, dg_k(x')(v)) = (v, d\psi_k(x')(v) + \triangle_k(x')(v)). \tag{2-12}$$

Estimating the curvature we see that

$$\begin{aligned}
\kappa_p^{\Sigma_k}(v_p) &= \frac{(H\psi_k(x')+h_k(x'))(v_p)}{\|\nabla\rho_k(p)\|\|v_p\|^2} \\
&= \frac{(H\psi_k(x'))((v, d\psi_k(x')v)+(0', \triangle_k(x')(v)))}{\|-\boldsymbol{e_n}+\nabla\psi_k(p)+\nabla\sigma_k(x')\|\|(v, d\psi_k(x')(v))+(0', \triangle_k(x'))\|^2} + O\Big(\frac{1}{k^2}\Big) \\
&= \frac{(H\psi_k(x'))((v, d\psi_k(x')v))}{\|-\boldsymbol{e_n}+\nabla\psi_k(x')\|\big(1+O\big(\frac{1}{k^2}\big)\big)\|(v, d\psi_k(x')(v))\|^2\big(1+O\big(\frac{1}{k^2}\big)\big)^2} + O\Big(\frac{1}{k^2}\Big) \\
&= \frac{(H\psi_k(x'))((v, d\psi_k(x')v))}{\|-\boldsymbol{e_n}+\nabla\psi_k(x')\|\|(v, d\psi_k(x')(v))\|^2} + O\Big(\frac{1}{k^2}\Big) = \frac{1}{k} + O\Big(\frac{1}{k^2}\Big),
\end{aligned}$$

where the term $\frac{1}{k}$ comes from the fact that the expression above is the formula for the curvature of a ball of radius $k$. From this it is straightforward to deduce the existence of an $m$ such that the lemma holds. □

***The squeezing function on* $\Omega$.** We will now explain why the squeezing function goes to 1 uniformly as we approach $b\Omega$ provided that the $\epsilon_k$'s decrease sufficiently fast. Let $N, m$ be as in Lemma 2.1, and start by setting $f_k = \psi_k$ for some $k > N$.

Fix some small $\delta_k > 0$. By Lemma 2.1, if $\epsilon_k$ is small enough, we can for each $p = (x', x_n) \in b\Omega_k$, with $\|x'\| < \delta_k$, find a ball $B$ of radius $k+m$ containing $\Omega_k$ such that $p \in bB$. By the same lemma we can for each such $p$ also find a local piece of a ball of radius $k-m$ touching $p$ from the inside of $\Omega_k$, and the size of the local ball is uniform. So using Lemma 3.1 we may find a $t_k > 0$ small enough such that

$$S_{\Omega_k}(x', x_n) \geq 1 - \frac{m}{(k+m)} \tag{2-13}$$

if $x_n \leq t_k$.

Next, again by Lemma 2.1, we find a $\delta_{k+1} < \delta_k$ such that if $\epsilon_{k+1}$ is small enough, then for each $p = (x', x_n) \in b\Omega_{k+1}$ with $\|x'\| < \delta_{k+1}$, we may oscillate with balls of radius $k+1-m$ and $k+1+m$ respectively. So there is a $t_{k+1} < t_k$ such that

$$S_{\Omega_{k+1}}(x', x_n) \geq 1 - \frac{m}{(k+1+m)} \tag{2-14}$$

if $x_n \leq t_{k+1}$. Furthermore, by further decreasing $\epsilon_{k+1}$, we can keep the estimate (2-13) with $\Omega_k$ replaced by $\Omega_{k+1}$. The reason is the following. First of all, by [Fornæss and Wold 2015], there exists a constant $C_k$ such that

$$S_{\Omega_k}(z) \geq 1 - C_k \cdot \mathrm{dist}(z, b\Omega_k), \tag{2-15}$$

and near any compact $K \subset b\Omega_k$ away from 0, this estimate is not going to be disturbed by a small perturbation of $b\Omega_k$ near the point 0; the estimate is obtained by using oscillating balls at points of $K$ whose boundaries will stay bounded away from 0. Furthermore, on any compact subset of $\Omega_k$ we have that $S_{\Omega_{k+1}} \to S_{\Omega_k}$ as $\epsilon_{k+1} \to 0$.

Continuing in this fashion, we obtain a decreasing sequence $0 < t_j < t_{j+1}$, $j = k, k+1, \ldots$, and an increasing sequence of domains $\Omega_j$, such that for each $j$ we have

$$S_{\Omega_j}(x', x_n) \geq 1 - \frac{m}{(k+i+m)} \tag{2-16}$$

for $t_{k+i} \leq x_n \leq t_{k+i-1}$, for $i \leq j$. The result now follows from Lemma 3.2.

## 3. Lemmata

Let $0 < s < 1/2,\ 0 < d < r < 1$, and set $B_s = B(s, 1-s)$, the ball of radius $1-s$ centred at $(s, 0')$. Furthermore, we set

$$B_{s,d} = B_s \cap \{(z_1, z') \in \mathbb{B}^n : \mathcal{R}e(z_1) > d\}. \tag{3-1}$$

**Lemma 3.1.** *If $B_{s,d} \subset \Omega \subset \mathbb{B}^n$, and if $r > 1 - \frac{sd}{4}$, then $S_\Omega(r, 0) > 1 - s$.*

*Proof.* Set $\mu = 1 - s$ and $\eta = \frac{d}{2}$, and then

$$B_\eta^\mu = \left\{(z_1, z') \in \mathbb{C}^n : |z_1 - (1-\eta)|^2 + \frac{\eta}{\mu}|z'|^2 < \eta^2\right\}. \tag{3-2}$$

Then certainly $\mathcal{R}e(z_1) > d$ on $B_\eta^\mu$, and we also have that $B_\eta^\mu \subset B_s$. To see the latter, we translate the two balls sending $(1, 0')$ to the origin, where they are defined by

$$\widetilde{B}_s = \{(z_1, z') : 2\mu\, \mathcal{R}e(z_1) + |z|^2 < 0\}, \tag{3-3}$$

and

$$\widetilde{B}_\eta^\mu = \left\{(z_1, z') : 2\eta\, \mathcal{R}e(z_1) + |z_1|^2 + \frac{\eta}{\mu}|z'|^2 < 0\right\}. \tag{3-4}$$

Also,

$$\begin{aligned} 2\eta\mathcal{R}e(z_1) + |z_1|^2 + \frac{\eta}{\mu}|z'|^2 < 0 &\Rightarrow 2\eta\, \mathcal{R}e(z_1) + \frac{\eta}{\mu}|z_1|^2 + \frac{\eta}{\mu}|z'|^2 < 0 \\ &\Leftrightarrow 2\mu\mathcal{R}e(z_1) + |z|^2 < 0. \end{aligned}$$

According to Lemma 3.5 in [Fornæss and Wold 2015] we have

$$S_\Omega(r, 0) \geq \sqrt{\mu}\sqrt{1 - 2(1-r)\frac{1}{\eta}} = \sqrt{(1-s)\Big(1 - \frac{4(1-r)}{d}\Big)}, \tag{3-5}$$

from which the lemma follows easily. □

**Lemma 3.2.** *Let $\Omega_j \subset \Omega_{j+1}$ for $j \in \mathbb{N}$, set $\Omega = \cup_j \Omega_j$, and assume that $\Omega$ is bounded. Let $z \in \Omega$, and assume that $S_{\Omega_j}(z) > 1 - \delta$ for all $j$ large enough so that $z \in \Omega_j$. Then $S_\Omega(z) \geq 1 - \delta$.*

*Proof.* Let $f_j : \Omega_j \to \mathbb{B}^n$ be an embedding such that $f_j(z) = 0$ and $B_{1-\delta}(0) \subset f_j(\Omega_j)$. By passing to a subsequence we may assume that $f_j \to f : \Omega \to \mathbb{B}^n$ uniformly on compact sets, with $f(z) = 0$. Setting $g_j = f_j^{-1} : B_{1-\delta}(0) \to \Omega$ we may also assume that $g_j \to g$ uniformly on compact sets. Then $f|_{g(B_{1-\delta}(0))} = g^{-1}$, from which the result follows. □

## 4. Some open problems

**Problem 4.1.** Does Zimmer's result hold for pseudoconvex domains of class $C^\infty$?

**Problem 4.2.** How much smoothness is needed for Zimmer's result hold for pseudoconvex domains?

**Problem 4.3.** Let $\Omega \subset \mathbb{C}^2$ be a bounded pseudoconvex domain of class $C^\infty$. Is $S_\Omega(z)$ bounded away from zero?

In light of the result of [Deng, Guan and Zhang 2016], the answer to the last question is affirmative for bounded strictly pseudoconvex domains of class $\mathbb{C}^2$ in all dimensions. For strictly convex domains in $\mathbb{C}^n$, this was proved in [Yeung 2009]. Furthermore, it has been shown in [Kim and Zhang 2016] that the same holds for bounded convex domains without any further regularity assumptions, and by [Nikolov and Andreev 2017], it even holds for bounded $\mathbb{C}$-convex domains in general. On the other hand, by [Fornæss and Rong 2016], the answer is negative in general for $n \geq 3$.

Quantifying the asymptotic behaviour of the squeezing function, we showed in [Fornæss and Wold 2015] that

(i) $S_\Omega(z) \geq 1 - C\,\mathrm{dist}(z, b\Omega)$, and

(ii) $S_\Omega(z) \geq 1 - C\sqrt{\mathrm{dist}(z, b\Omega)}$,

for strongly pseudoconvex domains of class $C^4$ and $C^3$ respectively. In [Diederich, Fornæss and Wold 2016] we showed that if the squeezing function approaches 1 essentially faster than in (i), then $\Omega$ is biholomorphic to the unit ball.

**Problem 4.4.** What is the optimal estimate for the squeezing function for strictly pseudoconvex domains of class $C^k$ with $k < 4$?

Let $\phi : \mathbb{B}^2 \to \mathbb{C}^2$ be defined as

$$\phi(z_1, z_2) := (z_1, -z_2 \log(z_1 - 1)).$$

Then $\Omega := \phi(\mathbb{B}^2)$ is of class $C^1$, and $(1, 0)$ is a non-strictly pseudoconvex boundary point of $\Omega$. So $S_\Omega$ being identically equal to 1 does not even imply strict pseudoconvexity in the case of $C^1$-smooth boundaries.

**Problem 4.5.** Let $\phi : \mathbb{B}^n \to \Omega$ be a biholomorphism, and assume that $\Omega$ is a bounded $C^2$-smooth domain. Is $\Omega$ strictly pseudoconvex?

## References

[Deng, Guan and Zhang 2012] F. Deng, Q. Guan, and L. Zhang, "Some properties of squeezing functions on bounded domains", *Pacific J. Math.* **257**:2 (2012), 319–341. MR Zbl

[Deng, Guan and Zhang 2016] F. Deng, Q. Guan, and L. Zhang, "Properties of squeezing functions and global transformations of bounded domains", *Trans. Amer. Math. Soc.* **368**:4 (2016), 2679–2696. MR Zbl

[Diederich, Fornæss and Wold 2016] K. Diederich, J. E. Fornæss, and E. F. Wold, "A characterization of the ball in $\mathbb{C}^n$", *Internat. J. Math.* **27**:9 (2016), art. id. 1650078. MR Zbl

[Fornæss and Rong 2016] J. E. Fornæss and F. Rong, "Estimate of the squeezing function for a class of bounded domains", 2016. To appear in *Math. Ann.* arXiv

[Fornæss and Wold 2015] J. E. Fornæss and E. F. Wold, "An estimate for the squeezing function and estimates of invariant metrics", pp. 135–147 in *Complex analysis and geometry* (Gyeongju, South Korea, 2014), edited by F. Bracci et al., Springer Proc. Math. Stat. **144**, Springer, 2015. MR Zbl

[Kim and Zhang 2016] K.-T. Kim and L. Zhang, "On the uniform squeezing property of bounded convex domains in $\mathbb{C}^n$", *Pacific J. Math.* **282**:2 (2016), 341–358. MR Zbl

[Liu, Sun and Yau 2004] K. Liu, X. Sun, and S.-T. Yau, "Canonical metrics on the moduli space of Riemann surfaces, I", *J. Differential Geom.* **68**:3 (2004), 571–637. MR Zbl

[Nikolov and Andreev 2017] N. Nikolov and L. Andreev, "Boundary behavior of the squeezing functions of $\mathbb{C}$-convex domains and plane domains", *Internat. J. Math* **28**:5 (2017).

[Yeung 2009] S.-K. Yeung, "Geometry of domains with the uniform squeezing property", *Adv. Math.* **221**:2 (2009), 547–569. MR Zbl

[Zimmer 2018a] A. Zimmer, "Characterizing strong pseudoconvexity, obstructions to biholomorphisms, and Lyapunov exponents", *Mathematische Annalen* (2018), 1–34.

[Zimmer 2018b] A. Zimmer, "A gap theorem for the complex geometry of convex domains", *Trans. Amer. Math. Soc.* **370**:10 (2018), 7489–7509. MR

Received December 5, 2016. Revised January 9, 2018.

JOHN ERIK FORNÆSS
DEPARTMENT OF MATHEMATICAL SCIENCES
NORWEGIAN UNIVERSITY OF SCIENCE AND TECHNOLOGY
TRONDHEIM
NORWAY

fornaess@gmail.com

ERLEND FORNÆSS WOLD
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF OSLO
BLINDERN
OSLO
NORWAY

erlendfw@math.uio.no

# PACIFIC JOURNAL OF MATHEMATICS

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

msp.org/pjm




---

See inside back cover or msp.org/pjm for submission instructions.

---


The subscription price for 2018 is US $475/year for the electronic version, and $640/year for print and electronic.
Subscriptions, requests for back issues and changes of subscriber address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFlow® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

# PACIFIC JOURNAL OF MATHEMATICS

Volume 297 No. 1 November 2018

---