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AN AMIR–CAMBERN THEOREM FOR QUASI-ISOMETRIES OF $C_0(K, X)$ SPACES

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Let X be a finite-dimensional Banach space. We prove that if K and S are locally compact Hausdorff spaces and there exists a bijective map $T : C_0(K, X) \to C_0(S, X)$ such that

$$\frac{1}{M}\|f-g\| - L \leq \|T(f) - T(g)\| \leq M\|f-g\| + L,$$

for every $f, g \in C_0(K, X)$ then K and S are homeomorphic, whenever $L \geq 0$ and $1 \leq M^2 < S(X)$, where $S(X)$ denotes the Schäffer constant of X.

This nonlinear vector-valued extension of the Amir–Cambern theorem via quasi-isometries T with large M was previously unknown even for the classical ℓ_p^n spaces, $1 < p < \infty$, $p \neq 2$ and $n \geq 2$.

1. Introduction

If K is a locally compact Hausdorff space and X is a Banach space, we denote by $C_0(K, X)$ the Banach space of continuous functions vanishing at infinity on K, taking values in X, and provided with the usual supremum norm. If X is the scalar field ($\mathbb{R}$ or $\mathbb{C}$) we will denote this space by $C_0(K)$. In the case where K is a compact Hausdorff space we write $C(K, X)$ instead of $C_0(K, X)$.

The well-known Banach–Stone theorem states that if K and S are locally compact Hausdorff spaces, then the existence of a linear isometry T from $C_0(K)$ onto $C_0(S)$ implies that K and S are homeomorphic [Banach 1932; Behrends 1979; Stone 1937]. Amir [1965] and Cambern [1967] independently generalized this theorem by proving that if $C_0(K)$ and $C_0(S)$ are isomorphic under a linear isomorphism T satisfying $\|T\|\|T^{-1}\| < 2$, then K and S must also be homeomorphic. The constant 2 is the best possible for the formulation of this result [Cambern 1970; Cohen 1975].

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Various authors, beginning with Jerison [1950], have considered the problem of determining geometric properties of X which allow generalizations of these theorems to the $C_0(K, X)$ spaces; see for instance [Cidral et al. 2015].

In the present paper we strengthen the Amir–Cambern theorem by showing that the conclusion holds if the requirement that T be a linear isomorphism with $\|T\|\|T^{-1}\| < 2$ is replaced by the requirement that T be a *bijective coarse* (M, L)-*quasi-isometry* on $C_0(K, X)$ spaces for finite-dimensional spaces X with $L \geq 0$ and M satisfying $1 \leq M^2 < S(X)$, where $S(X)$ is the following parameter introduced by Schäffer [Gao and Lau 1990; Schäffer 1976] for Banach spaces X:

$$S(X) = \inf\bigl\{\max\{\|x+y\|, \|x-y\|\} : \|x\| = 1 \text{ and } \|y\| = 1\bigr\}.$$

Recall that a Banach space X is called uniformly nonsquare [James 1964, Definition 1.1] if there exists $0 < \delta < 1$ such that for any $x, y \in X$, with $\|x\| = 1$ and $\|y\| = 1$, we have

$$\min\{\|x+y\|, \|x-y\|\} \leq 2(1-\delta).$$

Then, X is uniformly nonsquare if and only if $S(X) > 1$ [Kato et al. 2001, Proposition 1]. Moreover, $S(\mathbb{R}) = 2$, $S(\mathbb{C}) = \sqrt{2}$, and $1 \leq S(X) \leq \sqrt{2}$ for every Banach space with dimension greater than or equal to 2 [Gao and Lau 1990, Theorem 2.5]. If X is a Hilbert space with dimension at least 2 then $S(X) = \sqrt{2}$, but this equality does not characterize the Hilbert spaces X; see for instance [Komuro et al. 2016, p.1].

A bijective map $T : C_0(K, X) \to C_0(S, X)$ is said to be a coarse (M, L)-quasi-isometry or simply an (M, L)-quasi-isometry if for some constants $M \geq 1$ and $L \geq 0$ the inequalities

$$\frac{1}{M}\|f-g\| - L \leq \|T(f) - T(g)\| \leq M\|f-g\| + L$$

are satisfied for all $f, g \in C_0(K, X)$. This notion includes some important concepts used in the nonlinear classification of Banach spaces [Benyamini and Lindenstrauss 2000; Godefroy et al. 2014; Górak 2011; Kalton 2008].

Thus, the main aim of this work is to prove the following nonlinear vector-valued extension of the Amir–Cambern theorem via quasi-isometries.

Theorem 1.1. *Let X be a finite-dimensional Banach space with $S(X) > 1$. Suppose that K and S are locally compact Hausdorff spaces and there exists a bijective (M, L)-quasi-isometry T from functions $C_0(K, X)$ onto $C_0(S, X)$ satisfying*

$$M^2 < S(X),$$

then K and S are homeomorphic.

The starting point of our research toward proving Theorem 1.1 was the fact that, for the particular case where X is a *finite-dimensional strictly convex space* [Clarkson 1936] and $M < 1 + \epsilon_0$ for some $\epsilon_0 > 0$, the theorem was proved by Jarosz [1989, Theorem 4]. However, even in the case where $X = \mathbb{R}$, the arguments presented in the proof of [Jarosz 1989, Theorem 1] require ϵ_0 to be very small, namely $\epsilon_0 < 10^{-30}$. In addition, if X has dimension at least 2, ϵ_0 depends on the modulus of convexity of X and nothing is established about it beyond its existence.

This result of Jarosz naturally leads us to the following problem.

Problem 1.2. When can the Amir–Cambern theorem be extended for $C_0(K, X)$ spaces to (M, L)-quasi-isometries with $M > 1$?

Theorem 1.1 states that every finite-dimensional uniformly nonsquare space and, in particular, every finite-dimensional strictly convex space provides a positive solution to the above problem for a range M depending on a geometrical property of X. Notice also that in the special case where $X = \ell_p^n$ (the real n-dimensional l_p space, $1 < p < \infty$ and $n \geq 2$), the following immediate corollary of Theorem 1.1 was only known when $p = 2$ [Galego and Da Silva 2018, Main Theorem].

Corollary 1.3. *Let $1 < p < \infty$ and $n \geq 2$. Suppose that K and S are locally compact Hausdorff spaces and T is a bijective (M, L)-quasi-isometry from $C_0(K, \ell_p^n)$ onto $C_0(S, \ell_p^n)$ satisfying*

$$M^2 < \min\{2^{1/p}, 2^{1-1/p}\},$$

then K and S are homeomorphic.

Proof. It suffices to recall that by [Gao and Lau 1990, Theorem 3.1], for every $1 < p < \infty$ and $n \geq 2$, we know that

$$S(\ell_p^n) = \min\{2^{1/p}, 2^{1-1/p}\}. \qquad \square$$

The case $X = \mathbb{R}$ of Theorem 1.1 was proved in [Galego and Porto da Silva 2016, Main Theorem]. On the other hand, Theorem 1.1 does not apply to $X = \ell_\infty^n$, $n \geq 2$, the real n-dimensional l_∞ space, because in this case $S(X) = 1$ and moreover by a well-known result of Sundaresan [1973, p.22] there are nonhomeomorphic compact Hausdorff spaces K and S such that $C(K, X)$ is isometric with $C(S, X)$.

Notice that, in view of Problem 1.2 and in connection with Theorem 1.1, the following question arises naturally.

Problem 1.4. Suppose that X is a Banach space such that there exists $c > 1$ satisfying the following property: for any locally compact Hausdorff spaces K and S and bijective (M, L)-quasi-isometry T from $C_0(K, X)$ onto $C_0(S, X)$ with

$$M^2 < c,$$

it follows that K and S are homeomorphic.

Then:

(1) Is it true that $S(X) > 1$?

(2) Is $c \leq S(X)$?

(3) Does it follow that X is a finite-dimensional space?

2. An inequality involving the Schäffer constant

We begin the proof of Theorem 1.1 by establishing an inequality related to the Schäffer constant that will be very useful later. The constant

$$J(X)=\sup\{\min\{\|x+y\|,\|x-y\|\}:\|x\|=1 \text{ and } \|y\|=1\}, \tag{2-1}$$

is called the nonsquare or James constant of X.

If X is a real Banach space of finite dimension at least 2, then according to [Casini 1986, Proposition 2.1] or [Gao and Lau 1990, Theorem 2.5]

$$J(X)S(X)=2. \tag{2-2}$$

This fact also holds if X is a complex Banach space, for if $X_{\mathbb{R}}$ is its natural real Banach space structure, we have that $J(X)=J(X_{\mathbb{R}})$ and $S(X)=S(X_{\mathbb{R}})$. So, from now on, we shall not distinguish the scalar field of X.

Lemma 2.1. *Let X be a Banach space and $x, y \in X$. Then*

$$\min\{\|x+y\|,\|x-y\|\}\le\frac{2}{S(X)}\max\{\|x\|,\|y\|\}.$$

Proof. First assume that $X=\mathbb{R}$. Then $S(X)=2$ and for each pair $x, y\in\mathbb{R}$,

$$\min\bigl\{|x+y|,|x-y|\bigr\}=\bigl||x|-|y|\bigr|\le\max\bigl\{|x|,|y|\bigr\},$$

so we are done.

Suppose now that the dimension of X is at least 2 and fix $x, y\in X$. Since $S(X)\le 2$, the lemma follows trivially in the case that $x=0$ or $y=0$. Thus, we assume that $x, y\neq 0$ and put $\hat{x}=x/\|x\|$, $\hat{y}=y/\|y\|$. We may also assume that

$$\|x\|=\max\{\|x\|,\|y\|\} \quad\text{and}\quad \|\hat{x}+\hat{y}\|=\min\{\|\hat{x}+\hat{y}\|,\|\hat{x}-\hat{y}\|\}. \tag{2-3}$$

Next, by (2-1) and (2-2) we infer that

$$\frac{2}{S(X)}=J(X)=\sup\{\min\{\|u+v\|,\|u-v\|\}:\|u\|=1 \text{ and } \|v\|=1\}.$$

Then, by (2-3) it follows that

$$\|\hat{x}+\hat{y}\|\le\frac{2}{S(X)},$$

and putting $\alpha=\|y\|/\|x\|\in(0,1]$, we note that

$$\frac{\|x+y\|}{\|x\|}=\|\hat{x}+\alpha\hat{y}\|\le(1-\alpha)\|\hat{x}\|+\alpha\|\hat{x}+\hat{y}\|\le\frac{2}{S(X)},$$

and we are also done. □

3. Special sets associated to (M, L)-quasi-isometries

In a recent study of (M, L)-quasi-isometries between the spaces $C_0(K)$ and $C_0(S)$ [Galego and Porto da Silva 2016], subsets $\Gamma_{\mathrm{w}}(k, \mathrm{v})$ and $\Gamma_{\mathrm{v}}(s, \mathrm{w})$ of S and K, respectively, where $k \in K,\ s \in S$ and v and w are suitable elements of $\mathbb{R}$, were introduced. In this section, we introduce the definitions and a property (Proposition 3.1) of an analogous class of these sets for $\mathrm{v}, \mathrm{w} \in X$ instead of $\mathbb{R}$.

From now on, as in the hypotheses of Theorem 1.1, we fix a finite-dimensional Banach space X with $S(X) > 1$ and a bijective coarse (M, L)-quasi-isometry

$$T : C_0(K, X) \to C_0(S, X)$$

satisfying $M^2 < S(X)$ and $L \geq 0$. One can easily see that for any $\alpha > 0$, the map $f \mapsto \alpha T(f/\alpha)$ is a bijective coarse $(M, \alpha L)$-quasi-isometry, so it is possible to change the value of L as we wish. Then, we may suppose that $L = 1$. Moreover, notice that the map $f \mapsto T(f) - T(0)$ is a bijective coarse quasi-isometry with the same constants (M, L) of T, with the additional property that it maps 0 to 0. For that reason we may suppose that $T(0) = 0$. Finally, notice that the map T^{-1} does not necessarily have the same constants (M, L) of T; in fact we can only guarantee that it is a bijective coarse (M, ML)-quasi-isometry. Thus, we may actually suppose that $L = 1/M$, and this will ensure that both the maps T and T^{-1} are bijective coarse $(M, 1)$-quasi-isometries.

Let H be a locally compact Hausdorff space, $k \in H,\ f \in C_0(H, X)$ and $\mathrm{v} \in X$. Following [Galego and Porto da Silva 2016, Definition 2.2] we set

$$\omega(k, f, \mathrm{v}) = \max\{\|f\|, \|f(k) - \mathrm{v}\|\}.$$

Let $\mathrm{v}, \mathrm{w} \in X$ satisfy $\|\mathrm{v}\| \geq M$ and $\|\mathrm{w}\| = \|\mathrm{v}\|/M - 1$. Following [Galego and Porto da Silva 2016, Definition 3.1], we set

$$\Gamma_{\mathrm{w}}(k, \mathrm{v}) = \big\{s \in S \;:\; \|Tf(s) - \mathrm{w}\| \leq M\omega(k, f, \mathrm{v}) + 1, \quad \text{for all } f \in C_0(K, X)\big\}.$$

Similarly, for $s \in S,\ \mathrm{w}, \mathrm{z} \in X$ with $\|\mathrm{w}\| \geq M$ and $\|\mathrm{z}\| = \|\mathrm{w}\|/M - 1$, we also set

$$\Lambda_{\mathrm{z}}(s, \mathrm{w}) = \big\{k \in K \;:\; \|T^{-1}g(k) - \mathrm{z}\| \leq M\omega(s, g, \mathrm{w}) + 1, \quad \text{for all } g \in C_0(S, X)\big\}.$$

Since it is required in the definition of the sets $\Gamma_{\mathrm{w}}(k, \mathrm{v})$ and $\Lambda_{\mathrm{z}}(s, \mathrm{w})$ that $\|\mathrm{v}\| \geq M$ and $\|\mathrm{w}\| = \|\mathrm{v}\|/M - 1$ and, respectively, $\|\mathrm{w}\| \geq M$ and $\|\mathrm{z}\| = \|\mathrm{w}\|/M - 1$, these restrictions on the norms of the parameters will be implicit in every usage of these sets.

It is important to have in mind that, since both T and T^{-1} are bijective coarse $(M, 1)$-quasi-isometries, for any result involving the sets $\Gamma_{\mathrm{w}}(k, \mathrm{v})$, a similar result holds for the sets $\Lambda_{\mathrm{z}}(s, \mathrm{w})$. We will use the same result label to refer to either case.

The proof of the following proposition is essentially the same as the proof of [Galego and Porto da Silva 2016, Proposition 3.2].

Proposition 3.1. *There exists* $r_0 > 0$, *depending only on* M, *such that*, *for all* $k \in K$ *and* $\mathrm{v} \in X$ *with* $\|\mathrm{v}\| \geq r_0$, *there exists* $\mathrm{w} \in X$ *such that* $\Gamma_{\mathrm{w}}(k, \mathrm{v}) \neq \varnothing$.

Proof. Let $\mathcal{V}_k$ denote the set of open neighborhoods of k. For each $U \in \mathcal{V}_k$ we fix $f_U \in C_0(K)$ such that the image of f_U is contained in $[0, 1]$, $f_U(k) = 1$ and $f_U|_{K \setminus U} \equiv 0$. We consider the net $(\mathrm{v} \cdot f_U)_{U \in \mathcal{V}_k}$.

Claim. $\limsup \|f - \mathrm{v} \cdot f_U\| \leq \omega(k, f, \mathrm{v}), \quad \text{for all } f \in C_0(K, X).$

Given $\varepsilon > 0$, fix $U_\varepsilon \in \mathcal{V}_k$ such that

$$\|f(u) - f(k)\| < \varepsilon, \quad \text{for all } u \in U_\varepsilon.$$

Pick $U \in \mathcal{V}_k$ such that $U \subset U_\varepsilon$, and we shall evaluate $\|f - \mathrm{v} \cdot f_U\|$. If $u \in U$, then

$$\|f(u) - \mathrm{v} \cdot f_U(u)\| \leq \|f(k) - \mathrm{v} \cdot f_U(u)\| + \varepsilon.$$

Notice that $\mathrm{v} \cdot f_U$ has its image contained in the segment $\{\alpha \mathrm{v} : \alpha \in [0, 1]\}$, and then

$$\|f(k) - \mathrm{v} \cdot f_U(u)\| \leq \max\{\|f(k)\|, \|f(k) - \mathrm{v}\|\}.$$

Therefore,

$$\|f(u) - \mathrm{v} \cdot f_U(u)\| \leq \max\{\|f(k)\|, \|f(k) - \mathrm{v}\|\} + \varepsilon, \quad \text{for all } u \in U. \tag{3-1}$$

On the other hand, if $u \in K \setminus U$, then $f_U(u) = 0$, and consequently,

$$\|f(u) - \mathrm{v} \cdot f_U(u)\| = \|f(u)\| \leq \|f\|, \quad \text{for all } u \in K \setminus U. \tag{3-2}$$

By (3-1) and (3-2), we conclude that

$$\|f - \mathrm{v} \cdot f_U\| \leq \max\{\|f\|, \|f(k) - \mathrm{v}\|\} + \varepsilon,$$

and the claim is proved.

For each U, we fix $s_U \in S$ such that

$$\|T(\mathrm{v} \cdot f_U)(s_U)\| = \|T(\mathrm{v} \cdot f_U)\|.$$

Since $\|\mathrm{v} \cdot f_U\| = \|\mathrm{v}\|$ and $T(0) = 0$, we have that

$$\|\mathrm{v}\|/M - 1 \leq \|T(\mathrm{v} \cdot f_U)(s_U)\| \leq M\|\mathrm{v}\| + 1.$$

Then, the net $(T(\mathrm{v} \cdot f_U)(s_U))_{U \in \mathcal{V}_k} \subset X$ is bounded and since X is finite-dimensional, we may assume that $T(\mathrm{v} \cdot f_U)(s_U) \to \mathrm{w}_0$, for some $\mathrm{w}_0 \in X$. Moreover, we have

$$\|\mathrm{w}_0\| \geq \|\mathrm{v}\|/M - 1. \tag{3-3}$$

The vector w_0 will be later used to define w.

Now, let us see that $(s_U)_{U\in\mathcal{V}_k}$ admits a convergent subnet. It follows by the claim that

$$\text{(3-4)}\quad \begin{aligned}\limsup\|Tf(s_U)-T(\mathrm{v}\cdot f_U)(s_U)\| &\le \limsup\|Tf-T(\mathrm{v}\cdot f_U)\|\\ &\le M\limsup\|f-\mathrm{v}\cdot f_U\|+1\\ &\le M\omega(k,f,\mathrm{v})+1,\quad \text{for all } f\in C_0(K,X).\end{aligned}$$

Fix $f_0 \in C_0(K, X)$ such that $\|f_0\| = \|\mathrm{v}\|/2$ and $f_0(k) = \mathrm{v}/2$. Then $\omega(k, f_0, \mathrm{v}) = \|\mathrm{v}\|/2$ and, by (3-3) and (3-4), we have

$$\text{(3-5)}\quad \begin{aligned}&\liminf\|Tf_0(s_U)\|\\ &\ge \liminf\|T(\mathrm{v}\cdot f_U)(s_U)\| - \limsup\|Tf_0(s_U)-T(\mathrm{v}\cdot f_U)(s_U)\|\\ &\ge \|\mathrm{w}_0\| - (M\omega(k,f_0,\mathrm{v})+1)\\ &\ge \|\mathrm{v}\|/M - 1 - (M\|\mathrm{v}\|/2+1)\\ &= \|\mathrm{v}\|(1/M - M/2) - 2.\end{aligned}$$

Since $M^2 < S(X) \le 2$, we have that $1/M - M/2 > 0$, and then, there exists r_0 depending only on M such that for $\|\mathrm{v}\| > r_0$, we have

$$\liminf\|Tf_0(s_U)\| > 0.$$

Since Tf_0 vanishes at infinity, it follows that $(s_U)_{U\in\mathcal{V}_k}$ admits a convergent subnet, so we may assume that $s_U \to s$. By (3-4), we derive that

$$\text{(3-6)}\qquad \|Tf(s)-\mathrm{w}_0\| \le M\omega(k,f,\mathrm{v})+1,\quad \text{for all } f\in C_0(K,X).$$

Define $\mathrm{w} = \alpha_0\mathrm{w}_0$, with $\alpha_0 = (\|\mathrm{v}\|/M - 1)/\|\mathrm{w}_0\|$. We have that $\|\mathrm{w}\| = \|\mathrm{v}\|/M - 1$ and, by (3-3), $\alpha_0 \le 1$.

We will conclude the proof by showing that (3-6) is also satisfied for w instead of w_0. Given $f \in C_0(K, X)$, notice that

$$\|Tf(s)\| \le \|Tf\| \le M\|f\| + 1 \le M\omega(k,f,\mathrm{v}) + 1,$$

then, by (3-6),

$$\|Tf(s)-\mathrm{w}\| \le \alpha_0\|Tf(s)-\mathrm{w}_0\| + (1-\alpha_0)\|Tf(s)\| \le M\omega(k,f,\mathrm{v})+1. \quad\square$$

From now on, we consider r_0 given by Proposition 3.1 to be fixed. Since r_0 depends only on M, this same constant works for the sets $\Lambda_{\mathrm{v}}(s, \mathrm{w})$.

4. The special sets $\Gamma_{\mathrm{w}}(k, \mathrm{v})$ when $M^2 < S(X)$

In this section we state a fundamental proposition concerning the special sets $\Gamma_{\mathrm{w}}(k, \mathrm{v})$ associated to the $(M, 1)$-quasi-isometry T that we are considering.

Proposition 4.1. *There exists $r_1 > r_0$, depending only on M and $S(X)$, such that, for all $k \in K$, $\mathrm{v} \in X$ and $\mathrm{v}' \in X$ with $\|\mathrm{v}\| > r_1$ and $\|\mathrm{v} - \mathrm{v}'\| < 1$, if $s \in \Gamma_{\mathrm{w}}(k, -\mathrm{v})$*

for some $\mathrm{w} \in X$ *and* $s' \in \Gamma_{\mathrm{w}'}(k, \mathrm{v}')$ *for some* $\mathrm{w}' \in X$, *then* $s = s'$.

Proof. Suppose that $s \neq s'$. Then, fix $g \in C_0(S, X)$ such that

$$g(s) = -\mathrm{w}, \quad g(s') = -\mathrm{w}' \quad \text{and} \quad \|g\| = \max\{\|\mathrm{w}\|, \|\mathrm{w}'\|\}. \tag{4-1}$$

By applying the definitions of the sets $\Gamma_{\mathrm{w}}(k, -\mathrm{v})$ and $\Gamma_{\mathrm{w}'}(k, \mathrm{v}')$, respectively, to $T^{-1}g$, we get the inequalities

$$2\|\mathrm{w}\| = \|T(T^{-1}g)(s) - \mathrm{w}\| \leq M\omega(k, T^{-1}g, -\mathrm{v}) + 1, \tag{4-2}$$

and

$$2\|\mathrm{w}'\| = \|T(T^{-1}g)(s') - \mathrm{w}'\| \leq M\omega(k, T^{-1}g, \mathrm{v}') + 1. \tag{4-3}$$

Since $\|\mathrm{w}\| = \|\mathrm{v}\|/M - 1$, by (4-2) we obtain

$$\frac{2\|\mathrm{v}\|}{M} \leq M\omega(k, T^{-1}g, -\mathrm{v}) + 3, \tag{4-4}$$

and since $\|\mathrm{w}'\| = \|\mathrm{v}'\|/M - 1$ and $\|\mathrm{v} - \mathrm{v}'\| < 1$, according to (4-3) we have

$$\frac{2\|\mathrm{v}\|}{M} \leq M\omega(k, T^{-1}g, \mathrm{v}') + 3 + \frac{2}{M}, \tag{4-5}$$

and again by $\|\mathrm{v} - \mathrm{v}'\| < 1$, we see that

$$\begin{aligned}
\omega(k, T^{-1}g, \mathrm{v}') &= \max\{\|T^{-1}g\|, \|T^{-1}g(k) - \mathrm{v}'\|\} \\
&\leq \max\{\|T^{-1}g\|, \|T^{-1}g(k) - \mathrm{v}\|\} + 1 \\
&= \omega(k, T^{-1}g, \mathrm{v}) + 1.
\end{aligned}$$

Therefore, according to (4-5) we deduce that

$$\frac{2\|\mathrm{v}\|}{M} \leq M\omega(k, T^{-1}g, \mathrm{v}) + 3 + M + \frac{2}{M}. \tag{4-6}$$

Thus, putting $\Delta = 3 + M + 2/M$, it follows from (4-4) and (4-6) that

$$\frac{2\|\mathrm{v}\|}{M} \leq M \min\bigl\{\omega(k, T^{-1}g, -\mathrm{v}), \omega(k, T^{-1}g, \mathrm{v})\bigr\} + \Delta.$$

That is, $2\|\mathrm{v}\|/M$ is less than or equal to

$$M \min\bigl\{\max\{\|T^{-1}g\|, \|T^{-1}g(k) + \mathrm{v}\|\}, \max\{\|T^{-1}g\|, \|T^{-1}g(k) - \mathrm{v}\|\}\bigr\} + \Delta.$$

Then, by using the identity of real numbers a, b and c,

$$\min\{\max\{a, b\}, \max\{a, c\}\} = \max\{a, \min\{b, c\}\},$$

with

$$a = \|T^{-1}g\|, \quad b = \|T^{-1}g(k) + \mathrm{v}\| \quad \text{and} \quad c = \|T^{-1}g(k) - \mathrm{v}\|,$$

we have that

$$\frac{2\|\mathrm{v}\|}{M} \leq M \max\bigl\{\|T^{-1}g\|, \min\{\|T^{-1}g(k)+\mathrm{v}\|, \|T^{-1}g(k)-\mathrm{v}\|\}\bigr\} + \Delta.$$

Moreover, by applying Lemma 2.1 with $x = T^{-1}g(k)$ and $y = \mathrm{v}$ we conclude that

$$\frac{2\|\mathrm{v}\|}{M} \leq M \max\Bigl\{\|T^{-1}g\|, \frac{2}{S(X)} \max\{\|T^{-1}g(k)\|, \|\mathrm{v}\|\}\Bigr\} + \Delta. \tag{4-7}$$

On the other hand, putting $\Delta' = 2 - M$ and having in mind (4-1) we also infer that

$$\|T^{-1}g\| \leq M\|g\| + 1 = M \max\{\|\mathrm{w}\|, \|\mathrm{w}'\|\} + 1 \leq \|\mathrm{v}\| + \Delta'. \tag{4-8}$$

Therefore by (4-7) and (4-8) we conclude that

$$\frac{2\|\mathrm{v}\|}{M} \leq \frac{2M}{S(X)}(\|\mathrm{v}\| + \Delta') + \Delta,$$

that is,

$$2\|\mathrm{v}\|\Bigl(\frac{1}{M} - \frac{M}{S(X)}\Bigr) \leq \frac{2M\Delta'}{S(X)} + \Delta.$$

Since $M^2 < S(X)$, we have that $1/M - M/S(X) > 0$. Thus, there exists $r_1 \geq r_0$, depending only on M and $S(X)$, such that we have a contradiction for $\mathrm{v} \in X$ with $\|\mathrm{v}\| > r_1$. □

We consider r_1 given by Proposition 4.1 to be fixed. Since it depends only on M and $S(X)$, this same constant works for the sets $\Lambda_{\mathrm{v}}(s, \mathrm{w})$. The following consequence of the previous proposition will allow us to define, in the next section, a function $\varphi : K \to S$ which as we shall see in Section 7 will be a homeomorphism between K and S.

Corollary 4.2. *For all $k \in K$, $s, s' \in S$, and $\mathrm{v}, \mathrm{v}' \in X$, with $\|\mathrm{v}\| > r_1$, $\|\mathrm{v}'\| > r_1$ and $\|\mathrm{v} - \mathrm{v}'\| < 1$, if $s \in \Gamma_{\mathrm{w}}(k, \mathrm{v})$ and $s' \in \Gamma_{\mathrm{w}'}(k, \mathrm{v}')$ for some $\mathrm{w}, \mathrm{w}' \in X$, then $s = s'$.*

Proof. Since $\|-\mathrm{v}\| > r_1 \geq r_0$, by Proposition 3.1 there exists $\mathrm{w}'' \in X$ such that $\Gamma_{\mathrm{w}''}(k, -\mathrm{v}) \neq \varnothing$. Take $s'' \in \Gamma_{\mathrm{w}''}(k, -\mathrm{v})$.

Observe that since $s'' \in \Gamma_{\mathrm{w}''}(k, -\mathrm{v})$ and $s \in \Gamma_{\mathrm{w}}(k, \mathrm{v})$, it follows by Proposition 4.1 that $s'' = s$. Moreover, since $s'' \in \Gamma_{\mathrm{w}''}(k, -\mathrm{v})$ and $s' \in \Gamma_{\mathrm{w}'}(k, \mathrm{v}')$, again by Proposition 4.1 we infer that $s'' = s'$. Hence $s = s'$. □

5. The functions $\varphi : K \to S$ and $\psi : S \to K$

In this section, we will begin to construct a homeomorphism between K and S via the following proposition.

Proposition 5.1. *For all $k \in K$ there exists $s \in S$ such that for all $\mathrm{v} \in X$ with $\|\mathrm{v}\| > r_1$ and $\mathrm{w} \in X$, either $\Gamma_{\mathrm{w}}(k, \mathrm{v}) = \{s\}$ or $\Gamma_{\mathrm{w}}(k, \mathrm{v}) = \varnothing$.*

Proof. Take $k \in K$ and put $A = \{\mathrm{v} \in X : \|\mathrm{v}\| > r_1\}$. Hence, it suffices to prove that for any $\mathrm{v}, \mathrm{v}' \in A$, if $s \in \Gamma_{\mathrm{w}}(k, \mathrm{v})$ and $s' \in \Gamma_{\mathrm{w}'}(k, \mathrm{v}')$ for some $\mathrm{w}, \mathrm{w}' \in X$, then $s = s'$.

Suppose thus that $s \in \Gamma_{\mathrm{w}}(k, \mathrm{v})$ and $s' \in \Gamma_{\mathrm{w}'}(k, \mathrm{v}')$ for some $\mathrm{w}, \mathrm{w}' \in X$. We will distinguish two cases.

Case 1. X is of dimension at least 2. Therefore A is path-connected. So we may find points $\mathrm{u}_0, \ldots, \mathrm{u}_n$ in A such that $\mathrm{u}_0 = \mathrm{v}'$, $\mathrm{u}_n = \mathrm{v}$ and $\|\mathrm{u}_j - \mathrm{u}_{j-1}\| < 1$ for all $1 \le j \le n$. Put $s_0 = s'$ and $s_n = s$. Moreover, according to Proposition 3.1, for each $1 \le j \le n-1$, there exists $s_j \in S$ and $\mathrm{w}_j \in X$ such that $s_j \in \Gamma_{\mathrm{w}_j}(k, \mathrm{u}_j)$.

For each $1 \le j \le n$, since $\|\mathrm{u}_j - \mathrm{u}_{j-1}\| < 1$, Corollary 4.2 implies that $s_j = s_{j-1}$. By using this fact repeatedly, we conclude that $s' = s_1 = \cdots = s_{n-1} = s$.

Case 2. $X = \mathbb{R}$. In this case, fix w'' such that $\Gamma_{\mathrm{w}''}(k, -\mathrm{v}) \neq \varnothing$. Then, using Proposition 4.1 we have

$$\Gamma_{\mathrm{w}''}(k, -\mathrm{v}) = \Gamma_{\mathrm{w}}(k, \mathrm{v}) = \{s\}.$$

Since $A = (-\infty, -r_1) \cup (r_1, +\infty)$, there is a path in A connecting v' to either v or $-\mathrm{v}$. Then, proceeding as in Case 1 we conclude that $s' = s$. □

Thus, we are able to define the function $\varphi : K \to S$ where $\varphi(k)$ is the element s given by Proposition 5.1. By symmetry, we may also define a function $\psi : S \to K$ such that $\psi(s)$ is the element k given by the symmetric version of Proposition 5.1.

To show that in fact φ and ψ are continuous and $\psi^{-1} = \varphi$ we will still need to prove another property of the sets $\Gamma_{\mathrm{w}}(k, \mathrm{v})$.

6. Another decisive property of the sets $\Gamma_{\mathrm{w}}(k, \mathrm{v})$ when $M^2 < S(X)$

The next proposition will help us prove that functions φ and ψ defined in the previous section are homeomorphisms provided that we change the number r_1 in the statement of Proposition 5.1 by another convenient number greater than it. See Proposition 7.1.

Proposition 6.1. *There exists $r_2 > r_1$, depending only on M and $S(X)$, such that, for all $k \in K$ and $\mathrm{v} \in X$ with $\|\mathrm{v}\| > r_2$, if $s \in \Gamma_{\mathrm{w}}(k, \mathrm{v})$ for some $\mathrm{w} \in X$ and $\Lambda_{\mathrm{z}}(s, \mathrm{w}) \neq \varnothing$ for some $\mathrm{z} \in X$, then $\Lambda_{\mathrm{z}}(s, \mathrm{w}) = \{k\}$.*

Proof. Pick $k' \in \Lambda_{\mathrm{z}}(s, \mathrm{w})$ and we must show that $k' = k$. Suppose the contrary and fix $f \in C_0(K, X)$ such that

$$f(k) = \frac{\mathrm{v}}{2}, \quad f(k') = -\frac{\|\mathrm{v}\|}{2\|\mathrm{z}\|}\mathrm{z} \quad \text{and} \quad \|f\| = \frac{\|\mathrm{v}\|}{2}. \tag{6-1}$$

Thus,

$$\omega(k, f, \mathrm{v}) = \frac{\|\mathrm{v}\|}{2}.$$

Applying the definition of $\Gamma_{\mathrm{w}}(k, \mathrm{v})$ to f, we see that

$$\|Tf(s) - \mathrm{w}\| \le M\omega(k, f, \mathrm{v}) + 1 = \frac{M}{2}\|\mathrm{v}\| + 1.$$

Moreover, since

$$\|Tf\| \le M\|f\| + 1 = \frac{M}{2}\|\mathrm{v}\| + 1,$$

it follows that

$$\omega(s, Tf, \mathrm{w}) \le \frac{M}{2}\|\mathrm{v}\| + 1.$$

So, by applying the definition of $\Lambda_{\mathrm{z}}(s, \mathrm{w})$ to Tf, we have

$$\|f(k') - \mathrm{z}\| = \|T^{-1}(Tf)(k') - \mathrm{z}\| \le M\omega(s, Tf, \mathrm{w}) + 1 \le \frac{M^2}{2}\|\mathrm{v}\| + M + 1. \tag{6-2}$$

On the other hand, since $\|\mathrm{w}\| = \|\mathrm{v}\|/M - 1$ and $\|\mathrm{z}\| = \|\mathrm{w}\|/M - 1$ we obtain

$$\|\mathrm{z}\| = \Big(\frac{\|\mathrm{v}\|}{M} - 1\Big)\frac{1}{M} - 1 = \frac{\|\mathrm{v}\|}{M^2} - \frac{1}{M} - 1.$$

Furthermore, according to (6-1), $f(k')$ and z have opposite directions. Then

$$\|f(k') - \mathrm{z}\| = \|f(k')\| + \|\mathrm{z}\| = \frac{\|\mathrm{v}\|}{2} + \frac{\|\mathrm{v}\|}{M^2} - \frac{1}{M} - 1. \tag{6-3}$$

Therefore, putting $\Delta'' = M + 2 + 1/M$, by (6-2) and (6-3) we conclude that

$$\Big(\frac{1}{2} + \frac{1}{M^2} - \frac{M^2}{2}\Big)\|\mathrm{v}\| \le \Delta''. \tag{6-4}$$

Since $M^2 < S(X) \le 2$, it can be easily seen that

$$\frac{1}{2} + \frac{1}{M^2} - \frac{M^2}{2} > 0.$$

So, there exists $r_2 \ge r_1$ depending only on M and $S(X)$ such that the inequality (6-4) fails to be true for $\mathrm{v} \in X$ with $\|\mathrm{v}\| > r_2$, completing the proof of the proposition. □

As we did to r_0 and r_1, we may fix r_2 given by the Proposition 6.1, and it is clear that this constant also works for the for the sets $\Lambda_{\mathrm{v}}(s, \mathrm{w})$.

7. The topological spaces *K* and *S* are homeomorphic

Observe that the statements of Proposition 3.1, Corollary 4.2, Proposition 5.1 and Proposition 6.1 remain true if we change r_0 and r_1 to r_2. Consider thus φ and ψ defined as at the end of Section 5. To complete the proof of Theorem 1.1, we prove the following proposition.

Proposition 7.1. *The functions $\varphi : K \to S$ and $\psi : S \to K$ are continuous and $\psi = \varphi^{-1}$.*

Proof. First we will show that $\psi = \varphi^{-1}$. Fix $k \in K$. By the definition of $\varphi(k)$ there are v, w $\in X$ with $\|\mathrm{v}\| > (r_2 + 1)M$ such that

$$\varphi(k) \in \Gamma_{\mathrm{w}}(k, \mathrm{v}).$$

Thus, $\|\mathrm{w}\| > r_2$ and by Proposition 3.1 there exists $\mathrm{z} \in X$ satisfying $\Lambda_{\mathrm{z}}(\varphi(k), \mathrm{w}) \neq \varnothing$. Then, according to Proposition 6.1 we know that

$$\Lambda_{\mathrm{z}}(\varphi(k), \mathrm{w}) = \{k\}.$$

Therefore, it follows by the definition of ψ that $\psi(\varphi(k)) = k$. Hence $\psi \circ \varphi = \mathrm{Id}_K$. Analogously we deduce that $\varphi \circ \psi = \mathrm{Id}_S$.

We will now prove that φ is continuous. The proof that ψ is continuous is analogous. Observe that it suffices to prove that for each net $(k_j)_{j \in J}$ of K converging to $k \in K$, the net $(\varphi(k_j))_{j \in J}$ admits a subnet converging to $\varphi(k)$.

Assume then that $(k_j)_{j \in J}$ is a net of K converging to k. For all $j \in J$ take v_j and w_j such that $\|\mathrm{v}_j\| = c$, for some $c > r_2$, and

$$\varphi(k_j) \in \Gamma_{\mathrm{w}_j}(k_j, \mathrm{v}_j). \tag{7-1}$$

Since the nets $(\mathrm{v}_j)_{j \in J}$ and $(\mathrm{w}_j)_{j \in J}$ are contained in compact sets, we may assume that there are v, w $\in X$ such that $\mathrm{v}_j \to \mathrm{v}$ and $\mathrm{w}_j \to \mathrm{w}$.

For each $f \in C_0(K, X)$ we have

$$\omega(k_j, f, \mathrm{v}_j) \to \omega(k, f, \mathrm{v}), \tag{7-2}$$

and according to (7-1),

$$\|Tf(\varphi(k_j)) - \mathrm{w}_j\| \leq M\omega(k_j, f, \mathrm{v}_j) + 1, \quad \text{for all } j \in J. \tag{7-3}$$

Fix $f_1 \in C_0(K, X)$ satisfying $\|f_1\| = \|\mathrm{v}\|/2$ and $f_1(x) = \mathrm{v}/2$. Then (7-3) implies

$$\begin{aligned} \|Tf_1(\varphi(k_j))\| &\geq \|\mathrm{w}_j\| - \|Tf_1(\varphi(k_j)) - \mathrm{w}_j\| \\ &\geq \frac{c}{M} - M\omega(k_j, f_1, \mathrm{v}_j) - 2, \end{aligned}$$

for every $j \in J$. Notice that $\omega(k, f_1, \mathrm{v}) = \|\mathrm{v}\|/2 = c/2$, so by (7-2) we have

$$\liminf_{j \in J} \|Tf_1(\varphi(k_j))\| \geq \left(\frac{1}{M} - \frac{M}{2}\right)c - 2,$$

and since $c > r_2 \geq r_0$ and recalling (3-5), we obtain

$$\liminf_{j \in J} \|Tf_1(\varphi(k_j))\| > 0.$$

Since Tf_1 vanishes at infinity, this implies that $(\varphi(k_j))_{j \in J}$ admits a subnet converging to some $s \in S$, so we assume that $\varphi(k_j) \to s$. Hence, by (7-2) and (7-3),

$$\|Tf(s) - \mathrm{w}\| \leq M\omega(k, f, \mathrm{v}) + 1, \quad \text{for all } f \in C_0(K, X),$$

which means that $s \in \Gamma_{\mathrm{w}}(k, \mathrm{v}) = \{\varphi(k)\}$. Consequently $s = \varphi(k)$. ☐

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