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# A RESTRICTION ON THE ALEXANDER POLYNOMIALS OF $L$-SPACE KNOTS

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**Using an invariant defined by Rasmussen, we extend an argument given by Hedden and Watson which further restricts which Alexander polynomials can be realized by $L$-space knots.**

## 1. Introduction

In [Ozsváth and Szabó 2008], it is shown how the filtered chain homotopy type of the knot Floer complex $\mathrm{CFK}^-(K)$ can be used to compute the Heegaard Floer homology of $S^3_n(K)$, the rational homology sphere obtained by performing Dehn surgery along $K \subset S^3$ with slope $n$. In [Ozsváth and Szabó 2005], this relationship is used to investigate which knots admit lens space surgeries, using the fact that if $Y$ is a lens space, $Y$ has the "smallest possible" Heegaard Floer homology:

$$\operatorname{rank} \widehat{\mathrm{HF}}(Y) = |H_1(Y;\mathbb{Z})|. \tag{1}$$

More generally, a rational homology sphere which satisfies condition (1) is called an $L$-*space*. So, from a Heegaard–Floer perspective, a natural extension of the question "which knots admit lens space surgeries?" is "which knots admit $L$-space surgeries?".

Letting $A(x)$ denote the Alexander grading of a homogeneous element $x$ in $\mathrm{CFK}^-$, the following proposition is a straightforward consequence of [Ozsváth and Szabó 2005, Theorem 1.2]; see [Hom 2014, Remark 6.6].

**Proposition 1.1.** *Suppose $K \subset S^3$ is a knot on which some positive integral surgery yields an $L$-space. Then $\mathrm{CFK}^-(K)$ has a basis $\{x_{-k}, \dots, x_k\}$ with the following properties*:

- $A(x_i) = n_i$, *where* $n_{-k} < n_{-k+1} < \cdots < n_{k-1} < n_k$.
- $n_i = -n_{-i}$.
- *If* $i \equiv k \mod 2$, *then* $\partial(x_i) = 0$.
- *If* $i \equiv k+1 \mod 2$, *then* $\partial(x_i) = x_{i-1} + U^{n_{i+1}-n_i} x_{i+1}$. □

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Notice that $x_k$ is in the kernel of $\partial$ and not in the image. The complex $\widehat{\mathrm{CFK}}(K)$ is the quotient complex of $\mathrm{CFK}^-(K)$ obtained by setting $U=0$ (we refer the reader to [Ozsváth and Szabó 2004b] for details on knot Floer homology). It is a Heegaard Floer complex $\widehat{\mathrm{CF}}(S^3)$, with an additional filtration induced by $K$. After setting $U=0$, we still observe that

$$x_k \quad \text{generates} \quad H_*(\widehat{\mathrm{CFK}}(K), \partial) \cong \widehat{\mathrm{HF}}(S^3) \cong \mathbb{F}. \tag{2}$$

By convention then, $M(x_k)=0$ (where $M$ is the Maslov grading). Since $U$ decreases $M$ by 2, and $\partial$ decreases $M$ by 1, this determines the Maslov grading on all homogeneous elements of $\mathrm{CFK}^-(K)$.

Ozsváth and Szabó [2004b] also showed that the graded Euler characteristic of $\widehat{\mathrm{CFK}}(K)$ is the symmetrized Alexander polynomial of $K$,

$$\sum_i \chi(\widehat{\mathrm{CFK}}(K,i)) \cdot T^i = \Delta_K(T), \tag{3}$$

so a corollary to Proposition 1.1 is the following:

**Corollary 1.2** [Ozsváth and Szabó 2005, Corollary 1.3]. *If $K \subset S^3$ is a knot which admits an $L$-space surgery, then*

$$\Delta_K(T) = \sum_{i=-k}^{k} (-1)^{k+i}\, T^{n_i} \tag{4}$$

*for some sequence of integers $n_{-k} < n_{-k+1} < \cdots < n_{k-1} < n_k$ satisfying $n_{-i} = n_i$.*

**Remark 1.3.** Although Proposition 1.1 only applies to knots which have positive $L$-space surgeries, a knot $K$ has a negative $L$-space surgery if and only if its mirror image $\overline{K}$ has a positive $L$-space surgery. Corollary 1.2 then follows in this generality because $\Delta_K(T) = \Delta_{\overline{K}}(T)$.

In particular, all of the nonzero coefficients of $\Delta_K(T)$ are $\pm 1$. Note that

$$n_k = g(K) = |\tau(K)| = g_4(K), \tag{5}$$

where $g$ is the Seifert genus, $\tau$ is the Ozsváth–Szabó concordance invariant defined in [Ozsváth and Szabó 2003], and $g_4$ is the smooth four-genus of $K$. The first equality follows from the knot Floer homology detection of genus [Ozsváth and Szabó 2004a], the second follows from (2), and the third follows from the fact shown in [Ozsváth and Szabó 2003] that, for any knot $K$,

$$|\tau(K)| \le g_4(K) \le g(K).$$

This was the most general restriction on the Alexander polynomials of knots admitting $L$-space surgeries in the literature until Hedden and Watson showed the following proposition.

**Proposition 1.4** [Hedden and Watson 2018, Corollary 9]. *If $K \subset S^3$ is a knot which admits an $L$-space surgery, then $\Delta_K(T)$ is as described in Corollary 1.2, and further, $n_k - n_{k-1} = 1$.*

Originally stated by Hedden and Watson as a corollary to a more general restriction on knot Floer complexes, this particular result was already known by Rasmussen, as conveyed to the author in a private communication. The proof hinges on an invariant defined by Rasmussen, and a particular inequality which it satisfies. Roughly, if large $n$-surgery is done on an unknot and on a knot $K$, the differences in the $d$-invariants (defined in (8)) of the resulting manifolds are bounded above by numbers depending on $g_4(K)$. Proposition 1.4 is then proved by showing that if a complex has a basis as in Proposition 1.1 and $n_k - n_{k-1} > 1$, Rasmussen's inequality is violated, and therefore this complex cannot be the knot Floer complex of any knot.

Our aim here is to extend this argument. We will introduce Rasmussen's invariant and inequality in Section 2. In Section 3, we will show how to compute the invariant for $L$-space knots from their Alexander polynomials (that is, from the sequence of the $n_i$). We will then see that Rasmussen's inequality places further restrictions on the $n_i$, analogous to the restriction $n_k - n_{k-1} = 1$. As a result, it will be shown that certain symmetric Laurent polynomials satisfying Proposition 1.4 cannot be the Alexander polynomial of any $L$-space knot.

**Theorem 1.5.** *Suppose $K \subset S^3$ is a knot which admits an $L$-space surgery. Then its symmetrized Alexander polynomial can be written as*

$$\Delta_K(T) = \sum_{i=-k}^{k} (-1)^{k+i}\, T^{n_i},$$

*for some sequence of integers $n_{-k} < n_{-k+1} < \cdots < n_{k-1} < n_k$ satisfying the following:*

- *$n_i = -n_{-i}$,*
- *if we let $r_i = n_{k+2-2i} - n_{k+1-2i}$, then $r_1 = 1$, and for any $j \leq k$,*

$$\sum_{i=2}^{j} r_i \leq \sum_{i=k-j+2}^{k} r_i. \tag{6}$$

As we will explain in Section 3, the restriction is more easily stated in terms of a modified version of the Alexander polynomial,

$$\widetilde{\Delta}_K(T) := \frac{\Delta_K(T)}{1 - T^{-1}}.$$

It follows from Corollary 1.2 that when $K$ is a knot which admits an $L$-space surgery,

$$\widetilde{\Delta}_K(T) = \sum_{i=0}^{\infty} T^{a_i}$$

for some sequence of integers satisfying

- $a_0 = g(K)$,
- $a_{i+1} < a_i$,
- $a_i = -i$ for $i \geq g(K)$.

We can then rephrase Theorem 1.5 as follows.

**Theorem 1.6** (restatement of Theorem 1.5 in terms of $\widetilde{\Delta}$). *Suppose $K \subset S^3$ is a knot which admits an $L$-space surgery and $\{a_i\}$ is the decreasing sequence of integers such that*

$$\widetilde{\Delta}_K(T) = \sum_{i=0}^{\infty} T^{a_i}.$$

*Then, for all* $0 \leq i \leq g(K)$,

$$a_i \leq g(K) - 2i. \tag{7}$$

To see the preceding theorems as generalizations of Proposition 1.4, note that in the language of Theorem 1.5, Proposition 1.4 translates to the statement $r_1 = 1$; in the language of Theorem 1.6, it translates to $a_1 \leq g(K) - 2$.

As a concrete example, there does not exist a knot in $S^3$ which admits an $L$-space surgery and has Alexander polynomial

$$-1 + (T^2 + T^{-2}) - (T^3 + T^{-3}) + (T^4 + T^{-4}).$$

Correspondingly, there does not exist a knot in $S^3$ with knot Floer complex as shown in Figure 1. As demonstrated after the proof of Theorem 1.5, this settles the question of which polynomials belong to knots of genus 4 or less which admit $L$-space surgeries.

## 2. The invariant $h_m(K)$

A useful feature of Heegaard Floer theory is that its groups satisfy surgery exact triangles; for example, a long exact sequence between Heegaard Floer homology groups of manifolds which are 0-, $\infty$- and $n$-framed surgery along the same knot $K$ [Ozsváth and Szabó 2004c, Section 9]. Rasmussen [2003, Definition 7.1] defines an invariant $h_m(K)$ as the rank of a particular map in such a sequence (cf. [Frøyshov 2002], where an instanton-Floer invariant $h$ is introduced).

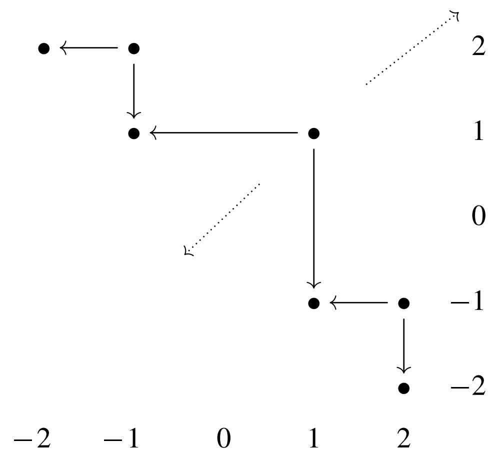


**Figure 1.** A bifiltered chain complex which cannot be the knot Floer complex of any knot in $S^3$.

Recall that if $(Y, \mathfrak{t})$ is a spin$^c$ rational homology sphere, Ozsváth and Szabó define the *$d$-invariant* of $(Y, \mathfrak{t})$ as

$$d(Y, \mathfrak{t}) = \min\{M(x) \mid x \in \operatorname{Im}(\pi_* : \mathrm{HF}^\infty(Y, \mathfrak{t}) \to \mathrm{HF}^+(Y, \mathfrak{t}))\}. \tag{8}$$

Rasmussen [2004, Section 2.2] shows that, in the case where $S^3_{-n}(K)$ is an $L$-space, the invariant $d(S^3_{-n}(K), \mathfrak{s}_m)$ is equal to twice $h_m(K)$, up to a shift which is independent of $K$. In particular, since $h_m(\text{unknot}) = 0$ for all $m$, we have

$$h_m(K) = \tfrac{1}{2}\big(d(S^3_{-n}(K), \mathfrak{s}_m) - d(S^3_{-n}(\text{unknot}), \mathfrak{s}_m)\big). \tag{9}$$

The key to obtaining restrictions on $L$-space knots is the following inequality, analogous to an inequality in instanton Floer homology proved by Frøyshov [2004].

**Proposition 2.1** [Rasmussen 2004, Theorem 2.3]. *Let $K$ be a knot in $S^3$ and let $g_4(K)$ be its slice genus. Then $h_m(K) = 0$ for $|m| > g_4(K)$, while for $|m| \leq g_4(K)$,*

$$h_m(K) \leq \left\lceil \tfrac{1}{2}(g_4(K) - |m|) \right\rceil. \tag{10}$$

Note that for a knot admitting an $L$-space surgery, due to (5), we can replace $g_4(K)$ with $g(K)$ and obtain

$$h_m(K) \leq \left\lceil \tfrac{1}{2}(g(K) - |m|) \right\rceil. \tag{11}$$

While the term $L$-space knot refers to a knot which admits an $L$-space surgery, different conventions are used regarding the restrictions on the slope of the $L$-space surgeries. Here it will be convenient to adopt the definition that an $L$-space knot in

$S^3$ is one with a *positive* $L$-space surgery. This is opposite Rasmussen's point of view [2004], but note that $K$ admits a positive $L$-space surgery if and only if its mirror image $\bar{K}$ admits a negative $L$-space surgery. Accordingly, we follow Hedden and Watson in defining

$$\bar{h}_m(K) := \tfrac{1}{2}\big(d(S^3_n(\text{unknot}), \mathfrak{s}_m) - d(S^3_n(K), \mathfrak{s}_m)\big) \tag{12}$$

and recall their observation that $\bar{h}_m(K) = h_m(\bar{K})$. Finally, we should note that $g(K) = g(\bar{K})$, so $\bar{h}_m$ satisfies the same inequality which $h_m$ does for knots admitting $L$-space surgeries: for $|m| \leq g(K)$,

$$\bar{h}_m(K) \leq \left\lceil \tfrac{1}{2}(g(K) - |m|) \right\rceil. \tag{13}$$

## 3. Values of $\bar{h}_m$ for $L$-space knots

Next, we recall how to compute $d$-invariants, and therefore $\bar{h}_m$, from $\mathrm{CFK}^-$. It was shown independently by Ozsváth and Szabó [2004b] and Rasmussen [2003] that for large $n$-surgery (that is, for $n \geq 2g(K) - 1$), the Heegaard Floer homology groups $\mathrm{HF}^-(S^3_n(K))$ are the homology groups of certain subcomplexes of $\mathrm{CFK}^-(K)$, up to a shift in Maslov grading which is independent of $K$. In particular, if we let $A_m$ denote the subcomplex consisting of elements with Alexander grading less than or equal to $m$, then

$$\mathrm{HF}^-(S^3_n(K), \mathfrak{s}_m) \cong H_*(A_m),$$

up to a shift in grading.[1] It follows that

$$d(S^3_n(K), \mathfrak{s}_m) = \max\{M(x) \mid x \text{ a nontorsion generator of } H_*(A_m)\} + c,$$

where $c$ is a constant which depends on $n$, but not on $K$. Therefore, the "shifted" $d$-invariant

$$\tilde{d}(K, m) := \max\{M(x) \mid x \text{ a nontorsion generator of } H_*(A_m)\} \tag{14}$$

is well-defined, and satisfies

$$d(S^3_n(\text{unknot}), \mathfrak{s}_m) - d(S^3_n(K), \mathfrak{s}_m) = \tilde{d}(\text{unknot}, m) - \tilde{d}(K, m) \tag{15}$$

for any sufficiently large $n$. For the unknot, we have the complex

$$\mathrm{CFK}^-(\text{unknot}) \cong \mathbb{F}[U],$$

where the generator has Maslov grading and Alexander grading equal to zero. Since multiplication by $U$ lowers the Alexander grading by 1 and the Maslov grading by 2,

$$\tilde{d}(\text{unknot}, m) = m - |m|.$$

[1] Here we are adopting the convention that $\mathrm{CF}^-$ and $\mathrm{CFK}^-$ contain the element 1 in $\mathbb{F}[U]$.

Therefore, we can rewrite inequality (13) using (15) and the above: if $K \subset S^3$ is an $L$-space knot, then for $|m| \leq g(K)$,

$$\begin{aligned} \bar{h}_m(K) = \tfrac{1}{2}(\tilde{d}(\text{unknot}, m) - \tilde{d}(K, m)) &\leq \left\lceil \tfrac{1}{2}(g(K) - |m|) \right\rceil, \\ -\tfrac{1}{2}\tilde{d}(K, m) &\leq \left\lceil \tfrac{1}{2}(g(K) - m) \right\rceil. \end{aligned} \tag{16}$$

With inequality (16) in hand, it remains to see how the values of $\tilde{d}$ are determined by the Alexander polynomial of an $L$-space knot.

Recall that the Alexander polynomial is the graded Euler characteristic of $\widehat{\text{CFK}}(K)$:

$$\sum_i \chi(\widehat{\text{CFK}}(K, i)) \cdot T^i = \Delta_K(T).$$

Further, $\text{CFK}^-$ is generated by the same set as $\widehat{\text{CFK}}$, over $\mathbb{F}[U]$ rather than $\mathbb{F}$. Since $U$ lowers the Alexander grading by 1 and preserves the parity of the Maslov grading,

$$\begin{aligned} \sum_i \chi(\text{CFK}^-(K, i)) \cdot T^i &= \sum_i \chi(\widehat{\text{CFK}}(K, i)) \cdot T^i \cdot (1 + T^{-1} + T^{-2} + \cdots) \\ &= \frac{\Delta_K(T)}{1 - T^{-1}} =: \widetilde{\Delta}_K(T). \end{aligned} \tag{17}$$

In other words, $\widetilde{\Delta}_K(T)$ is the graded Euler characteristic of $\text{CFK}^-(K)$.

**Remark 3.1.** If $K \subset S^3$ is a knot for which $\Delta_K(T)$ is of the type described in Corollary 1.2, then

$$\widetilde{\Delta}_K(T) = \sum_{i=0}^{\infty} T^{a_i},$$

where

- $a_0 = g(K)$,
- $a_{i+1} < a_i$, and
- $a_i = -i$ for all $i \geq g(K)$.

In [Krcatovich 2015], a reduced complex $\underline{\text{CFK}}^-$ was defined, and it was shown that for an $L$-space knot,

$$\underline{\text{CFK}}^-(K) \cong \mathbb{F}[U], \tag{18}$$

supported in Maslov grading zero [Krcatovich 2015, Corollary 4.2]. Roughly speaking, the complex $\text{CFK}^-$ has a filtration induced by $U$, and a filtration induced the knot; ignoring the knot filtration, one recovers $\text{CF}^-(S^3)$, whereas ignoring the $U$-filtration, one gets a "reduced" knot Floer complex. We refer the reader to [Krcatovich 2015] for a precise statement, and here simply remark that the structure of $\text{CFK}^-$ for an $L$-space knot, as described in Proposition 1.1, is what makes its reduced complex have a simple form.

Since the reduced complex is filtered chain homotopy equivalent to $\mathrm{CFK}^-(K)$ (with respect to the knot filtration), they have the same Euler characteristic. In particular, (18) says that every generator has even Maslov grading, so each contributes a positive term to the Euler characteristic. In other words, if

$$\widetilde{\Delta}_K(T) = \sum_{i=0}^{\infty} T^{a_i},$$

then $\underline{\mathrm{CFK}}^-(K)$ has one generator with Alexander grading $a_i$, for each $i \geq 0$. Since multiplication by $U$ is a filtered map (i.e., it never increases the Alexander grading), then necessarily

$$M(a_i) = -2i.$$

Figure 2 gives an illustration for the case of the (3, 4)-torus knot,[2] where

$$\Delta_{T_{3,4}}(T) = 1 - (T^2 + T^{-2}) + (T^3 + T^{-3}), \tag{19}$$

and therefore

$$\widetilde{\Delta}_{T_{3,4}}(T) = T^3 + 1 + T^{-1} + T^{-3} + T^{-4} + \cdots . \tag{20}$$

*Proof of Theorem 1.6.* First note that it is sufficient to prove the proposition for positive surgeries (see Remark 1.3).

So, let $K$ be an $L$-space knot in $S^3$, so that

$$\widetilde{\Delta}_K(T) = \sum_{i=0}^{\infty} T^{a_i}.$$

Then the reduced complex $\underline{\mathrm{CFK}}^-(K)$ consists of a single generator of Alexander grading $a_i$ and Maslov grading $-2i$, for each $i \geq 0$. Since the $a_i$ are strictly decreasing, it follows that $\tilde{d}$, as defined in (14), is given by

$$\tilde{d}(K, m) = -2 \min\{i \mid a_i \leq m\},$$

and therefore

$$\tilde{d}(K, a_i - 1) = -2(i + 1). \tag{21}$$

Substituting these values into inequality (16), we obtain

$$i + 1 \leq \left\lceil \tfrac{1}{2}(g(K) - (a_i - 1)) \right\rceil, \tag{22}$$

so

$$a_i \leq g(K) - 2i. \qquad \square$$

[2]It was shown by Moser [1971] that torus knots admit lens space (hence $L$-space) surgeries.

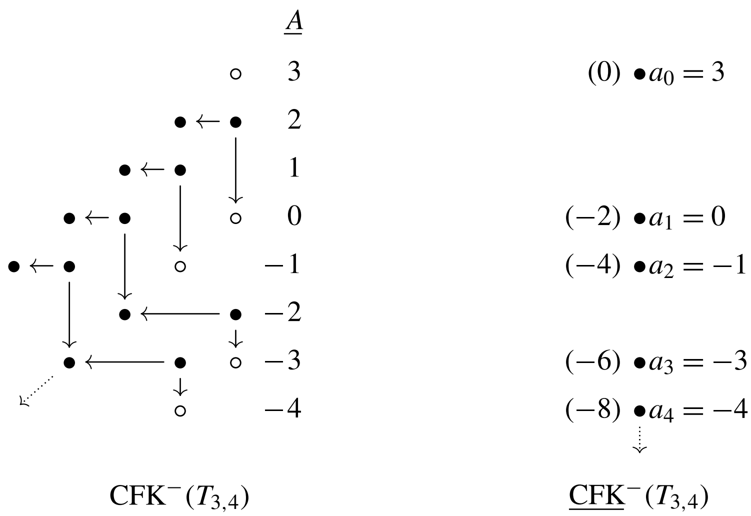


**Figure 2.** To the left is the knot Floer complex $\mathrm{CFK}^-$ for the (3,4)-torus knot, which, by Proposition 1.1, is determined by its Alexander polynomial. To the right is the reduced complex, which, for any $L$-space knot, is isomorphic to $\mathbb{F}[U]$, supported in Maslov grading zero. Note that the reduced complex on the right has a generator for each $\circ$ generator on the left, the bottom-most generator of each staircase summand.

*Proof of Theorem 1.5.* Let $K$ be an $L$-space knot, so that

$$\Delta_K(T) = \sum_{i=-k}^{k} (-1)^{k+i} T^{n_i}.$$

We have introduced the variables

$$\vec{r} = (r_1, \ldots, r_k)$$

as the "gaps" in the Alexander polynomial (the difference in the exponents of consecutive nonzero terms),

$$r_i = n_{k+2-2i} - n_{k+1-2i}.$$

While $\vec{r}$ records only every second gap, by the symmetry of $\Delta(T)$, this determines the polynomial uniquely. Diagramatically, notice that $\vec{r}$ is simply the list of horizontal lengths of a staircase summand of $\mathrm{CFK}^-$, in order from left to right. See Figure 3 for the example of the (4, 5)-torus knot, which has

$$\Delta_{T(4,5)}(T) = -1 + (T^2 + T^{-2}) - (T^5 + T^{-5}) + (T^6 + T^{-6}).$$

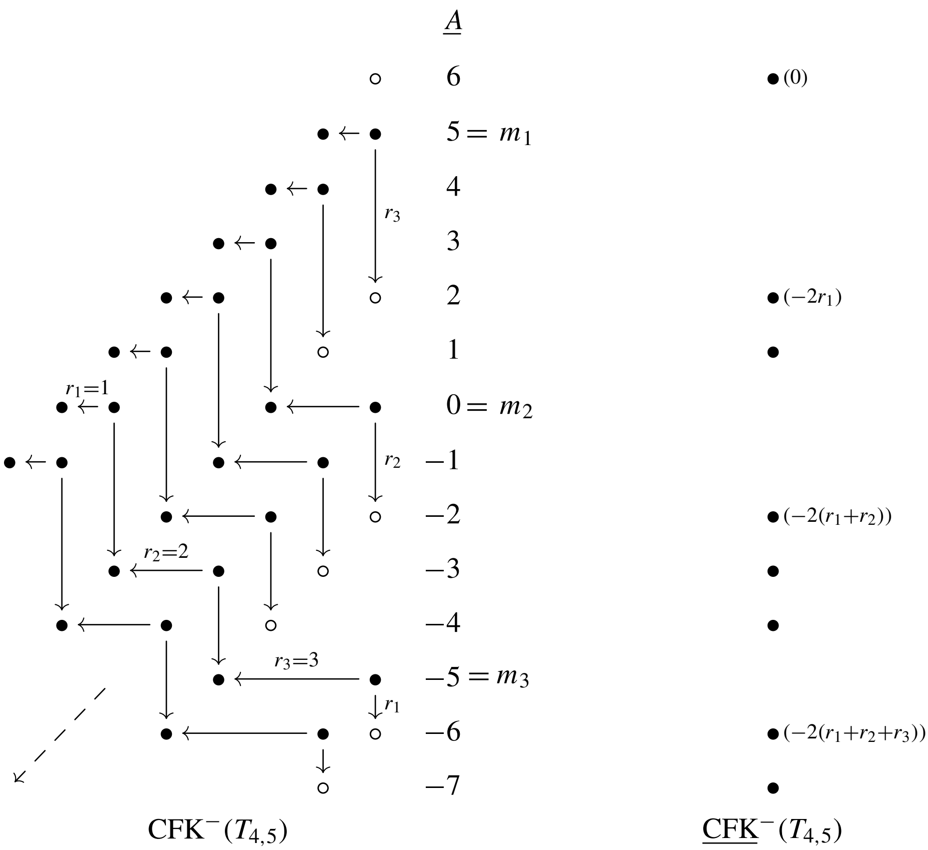


**Figure 3.** The complex $\mathrm{CFK}^-$ for the (4, 5)-torus knot, and its reduced form $\underline{\mathrm{CFK}}^-$. Note that the integers $r_1$, $r_2$ and $r_3$ are the horizontal lengths of each staircase, from left to right (and by symmetry, the vertical lengths, from bottom to top). This figure illustrates how the $m_j$ — the Alexander gradings at which the reduced complex "jumps" — are determined by the $r_i$, and further, how the values of $\tilde{d}(K, m_j)$, given in parentheses to the right, are determined by the $r_i$.

Next we observe how, given $\vec{r}$, to compute both sides of inequality (16) for any $m$, with Figure 3 as a guide. We focus on the values labeled $m_j$ in Figure 3; in other words, the values where we have the "jumps" in the reduced complex. More precisely, if we let

$$m_j = g(K) - \left( \sum_{i=1}^{j} r_i + \sum_{i=k-j+2}^{k} r_i \right),$$

then we have that

$$\tilde{d}(K, m_j) = -2\sum_{i=1}^{j} r_i.$$

Substituting these values into inequality (16) when $m = m_j$ gives

$$\sum_{i=1}^{j} r_i \le \left\lceil \frac{g(K) - \big(g(K) - \big(\sum_{i=1}^{j} r_i + \sum_{i=k-j+2}^{k} r_i\big)\big)}{2} \right\rceil. \tag{23}$$

The case $j = 1$ gives $r_1 \le \lceil \frac{1}{2} r_1 \rceil$, so, since each $r_i$ is a positive integer, $r_1$ must equal 1. Substituting this into (23) gives

$$\begin{aligned} 1 + \sum_{i=2}^{j} r_i &\le \left\lceil \frac{1 + \sum_{i=2}^{j} r_i + \sum_{i=k-j+2}^{k} r_i}{2} \right\rceil, \\ \sum_{i=2}^{j} r_i &\le \left\lceil \frac{-1 + \sum_{i=2}^{j} r_i + \sum_{i=k-j+2}^{k} r_i}{2} \right\rceil, \end{aligned}$$

from which it follows that

$$\sum_{i=2}^{j} r_i \le \sum_{i=k-j+2}^{k} r_i.$$

This is sufficient to prove the claim. We could similarly obtain inequalities by considering values of $m$ different from the $m_j$, but those would be no stronger, and therefore provide no more restrictions on $\vec{r}$. $\square$

As an example, consider a knot $K$ with

$$\Delta_K(T) = -1 + (T^2 + T^{-2}) - (T^3 + T^{-3}) + (T^4 + T^{-4}),$$

so that

$$\widetilde{\Delta}_K(T) = T^4 + T^2 + T + T^{-2} + T^{-4} + T^{-5} + \cdots.$$

This polynomial satisfies the restriction of Proposition 1.4, but if $K$ were an $L$-space knot, we would have $g(K) = 4$, and $a_2 = 1$. This violates inequality (7), so $K$ (and its mirror image) cannot admit an $L$-space surgery. Alternatively, this polynomial has gaps $\vec{r} = (1, 2, 1)$, and since $r_2 \not\le r_3$, this violates inequality (6).

In fact, this completely determines which Alexander polynomials are realized by $L$-space knots of genus less than or equal to 4. All other polynomials satisfying Proposition 1.4 are realized by known $L$-space knots. For knots of genus 5, Theorem 1.5 eliminates the polynomials corresponding to $\vec{r} = (1, 2, 1, 1)$ and $\vec{r} = (1, 3, 1)$, but there are still three more which are not realized by any $L$-space knot known to the author (corresponding to $(1, 1, 2, 1)$, $(1, 2, 2)$ and $(1, 4)$).

| $\vec{r}$ | $L$-space knot with corresponding $\Delta(T)$ |
|---|---|
| (1) | $T(2,3)$ |
| (1,1) | $T(2,5)$ |
| (1,1,1) | $T(2,7)$ |
| (1,2) | $T(3,4)$, (2, 3)-cable of $T(2,3)$ |
| (1,1,1,1) | $T(2,9)$ |
| (1,1,2) | $T(3,5)$ |
| (1,2,1) | excluded by Theorem 1.5 |
| (1,3) | (2, 5)-cable of $T(2,3)$ |

Finally, we should point out the relation between the sequence of the $a_i$ for an $L$-space knot and the gap function defined by Borodzik and Livingston [2014, Definition 2.6]; namely,

$$a_i = \min_m \{I_K(m) = i\} - g(K).$$

As pointed out to the author by Borodzik, their restrictions given in [Borodzik and Livingston 2016, Theorem 2.14] can be reinterpreted in terms of the $a_i$. Informally, if two $L$-space knots are related by a small number of crossing changes, they have similar Alexander polynomials. More precisely, if $a_i$ and $a_i'$ are the exponents of $\widetilde{\Delta}$ for two $L$-space knots which differ by changing $p$ positive crossings to negative crossings, then

$$|a_i - a_i'| \leq p \quad \text{for all } i.$$

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