Pacific Journal of Mathematics

STABILITY OF CAPILLARY HYPERSURFACES IN A EUCLIDEAN BALL

HAIZHONG LI AND CHANGWEI XIONG

Volume 297 No. 1 November 2018

PACIFIC JOURNAL OF MATHEMATICS
Vol. 297, No. 1, 2018
dx.doi.org/10.2140/pjm.2018.297.131

STABILITY OF CAPILLARY HYPERSURFACES IN A EUCLIDEAN BALL

HAIZHONG LI AND CHANGWEI XIONG

We study the stability of capillary hypersurfaces in a unit Euclidean ball. It is proved that if the center of mass of the domain enclosed by the immersed capillary hypersurface and the wetted part of the sphere is located at the origin, then the hypersurface is unstable. An immediate result is that all known examples except the totally geodesic ones and spherical caps are unstable. We also conjecture a precise delineation of the stable capillary hypersurfaces in unit Euclidean balls.

1. Introduction

Capillarity is an important physical phenomenon, which occurs when two different materials contact and do not mix. Given a container B with an incompressible liquid drop T in it, the interface of the liquid and the air is a capillary surface M. In absence of gravity, the interface M is of constant mean curvature and the contact angle of M to the boundary ∂B is constant. One should compare this setting with a soap bubble, where the surface has no boundary and constant mean curvature, or a soap film, having fixed boundary and constant mean curvature.

The literature for the study of capillarity is extensive. We refer to the book [Finn 1986], where the treatment of the theory is mainly in the nonparametric case and in the more general situation of presence of gravity. Also we mention [Finn 1999] for a more recent survey about this topic.

In this paper we are concerned with the special case that the container B is a unit Euclidean ball and no gravity is involved. We study the (weak) stability for capillary hypersurfaces. This problem has been discussed by Ros and Souam [1997], where they dealt with the surface case and obtained some topological and geometrical restrictions. For the hypersurface case with free boundary (the contact angle is $\pi/2$), Ros and Vergasta [1995] also proved some interesting results. Also see [Souam 1997] for relevant work in space forms. In addition, we would like to remark

The research of the authors was supported by NSFC No. 11831005 and No. 11671224. The authors would like to thank Prof. Ben Andrews for his continuous help.
MSC2010: 49Q10, 53A10.
Keywords: capillary hypersurface, instability, spherical boundary.

that the study of compact and constant (higher) mean curvature hypersurfaces in a Euclidean ball with free boundary is similar to that of closed and constant (higher) mean curvature hypersurfaces in a sphere in some sense. In that respect we refer the readers to [Barbosa et al. 1988; Alías et al. 2007; Cheng 2003; 2008].

In this paper we prove the following theorem.

Theorem 1.1. *Let $x : M^n \to \mathbb{R}^{n+1}$ be an immersed capillary hypersurface in a unit Euclidean ball B^{n+1} and Ω the wetted part of the boundary of the ball. Denote by T the domain enclosed by $x(M)$ and Ω. If the center of mass of T is at the origin, the capillary hypersurface M is unstable.*

Remark 1.2. When $n = 2$, our Theorem 1.1 was proved by P. I. Marinov in his Ph.D. thesis [2010]. It is worth pointing out that Marinov's argument depends crucially on conformal coordinates, which can not be extended to the higher dimensional case here.

Here since we assume M is immersed, $x(M)$ may have self-intersections. Thus we should be careful with the choice of T. When M is embedded, T is understood in the common sense. See Remark 2.1 below for more explanation.

We note the application of Theorem 1.1 to Delaunay hypersurfaces in particular. Recall that Delaunay hypersurfaces are the hypersurfaces of revolution with constant mean curvature. By Proposition 4.3 in [Hutchings et al. 2002], Delaunay hypersurfaces are classified as an unduloid, cylinder, nodoid, sphere, catenoid, or hyperplane. To guarantee the portion of a Delaunay hypersurface in a Euclidean ball is also capillary, it should have some symmetry. See Section 2 below for more details. In that case, we call it a Delaunay capillary hypersurface. From Theorem 1.1, then, we have the following corollary.

Corollary 1.3. *The only stable Delaunay capillary hypersurface M^n in a unit Euclidean ball B^{n+1} is a totally geodesic hypersurface or a spherical cap.*

Our approach for proving Theorem 1.1 is as follows. In the higher dimensional case, we find that we can construct a conformal Killing vector field $Y[\xi]$ for any fixed $\xi \in \mathbb{S}^n$ from the natural conformal transformation family on B^{n+1}. Using the normal part $\langle Y[\xi], N\rangle$ as the test function, we can define a symmetric bilinear form $Q(\xi_1, \xi_2)$ by following [Marinov 2010]. By summing Q over $(n+1)$ coordinate directions we find Q has at least one negative eigenvalue. This summation technique can be compared with J. Simons' work [1968]. At last, under the hypothesis of Theorem 1.1 we can derive the instability of the hypersurface. Our argument indicates that this conformal field is important and we can use it to conclude that the center of mass of minimal submanifolds with free boundary in a unit Euclidean ball is at the origin (see Proposition 4.2). We refer the readers to [Fraser and Schoen 2011; 2016] for the very recent work on the minimal submanifolds with free boundary.

Just like in the case $n = 2$ in [Marinov 2010], as an application of our argument, we give a new proof of the classical result due to Barbosa and do Carmo [1984] which states that the only closed stable immersed hypersurface of constant mean curvature in $\mathbb{R}^{n+1}$ is the round sphere.

The outline of this paper is as follows. In Section 2, after fixing some notation and definitions, we prove the stability of hyperplanes and spherical caps. Then we construct the crucial conformal vector field. We also review some known facts about the Delaunay hypersurfaces. In Section 3 we give the proof of Theorem 1.1. In the last section, we discuss some applications of our method.

2. Preliminaries

Notation and definitions. Let $x : M^n \to \mathbb{R}^{n+1}$ be an orientable immersed hypersurface in the unit Euclidean ball $B^{n+1} \subset \mathbb{R}^{n+1}$ with $x(\operatorname{int} M) \subset \operatorname{int} B^{n+1}$ and $x(\partial M) \subset \partial B^{n+1}$. Suppose $\Omega \subset \partial B^{n+1}$ such that $\partial\Omega = x(\partial M)$, and denote by $T \subset B^{n+1}$ the part of the ball satisfying $\partial T = x(M) \cup \Omega$.

Remark 2.1. If $x(M)$ has self-intersections, T may be viewed as the finite union of some domains T_i, $i = 1, \dots, m$, i.e., $T = \bigcup_{i=1}^m T_i$. Here the T_i may intersect with each other. If there is more than one choice for $\{T_i\}_{i=1}^m$, choose one and fix it. In the proof we will see that only the property $\partial T = x(M) \cup \Omega$ is needed. If there is no confusion, we write M for $x(M)$ and ∂M for $x(\partial M)$ for simplicity.

Let N be the unit normal of M pointing inwards to T and $\bar{N}$ the unit outward normal of ∂B^{n+1}. Denote by ν and $\bar{\nu}$ the conormals of ∂M in M and Ω, respectively. Let D be the connection of $\mathbb{R}^{n+1}$ and ∇ the connection of M. Then the second fundamental form of M in $\mathbb{R}^{n+1}$ is given by $\sigma(X_1, X_2) = \langle D_{X_1} X_2, N\rangle$ for all $X_1, X_2 \in T_p M$. When taking an orthonormal basis $\{e_i\}_{i=1}^n$ on TM, we also denote by h_{ij} the components $\sigma(e_i, e_j)$. So the mean curvature H of M is $H = \frac{1}{n}\sum_{i=1}^n h_{ii}$. And the second fundamental form of ∂B in $\mathbb{R}^{n+1}$ is given by $\Pi(Y_1, Y_2) = \langle D_{Y_1} Y_2, -\bar{N}\rangle$ for all $Y_1, Y_2 \in T_p(\partial B)$. Finally, let $\theta \in (0, \pi)$ be the angle between ν and $\bar{\nu}$. See Figure 1 for an illustration.

Following [Ros and Souam 1997], we discuss the variation of M.

Definition 2.2. An *admissible variation* of $x : M^n \to \mathbb{R}^{n+1}$ is a differentiable map $X : (-\varepsilon, \varepsilon) \times M \to \mathbb{R}^{n+1}$ such that $X_t : M^n \to \mathbb{R}^{n+1}$, $t \in (-\varepsilon, \varepsilon)$, given by $X_t(p) = X(t, p)$, $p \in M$, is an immersion satisfying $X_t(\operatorname{int} M) \subset \operatorname{int} B$ and $X_t(\partial M) \subset \partial B$ for all t, and $X_0 = x$.

Now for given $\theta \in (0, \pi)$, we define an energy functional

$$E(t) = |M(t)| - \cos\theta\,|\Omega(t)|, \tag{1}$$

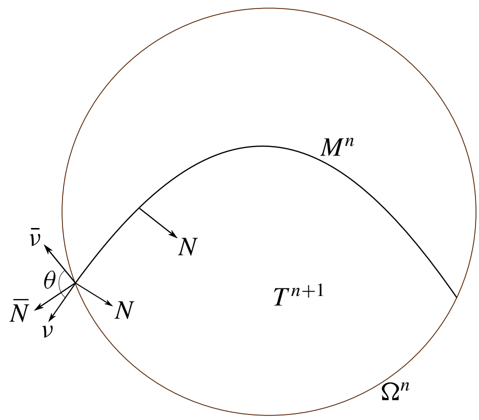

Figure 1. A typical illustration.

where $|\cdot|$ denotes the area function. The volume functional can be defined as

$$V(t) = \int_{[0,t]\times M} X^*\, dv,$$

where dv is the standard volume element of $\mathbb{R}^{n+1}$. Under these constraints, we define the following.

Definition 2.3. An immersed hypersurface $x : M^n \to \mathbb{R}^{n+1}$ is called capillary if $E'(0) = 0$ for any admissible volume-preserving variation of x.

Note that we have the formulas

$$E'(0) = -n \int_M Hf\, da + \int_{\partial M} \langle Y, \nu - \cos\theta\, \bar{\nu}\rangle\, ds, \tag{2}$$

$$V'(0) = \int_M f\, da, \tag{3}$$

where Y is the variational vector field $Y(p) = (\partial X/\partial t)(p)|_{t=0}$, f is its normal component $f = \langle Y, N\rangle$, and da and ds are the corresponding area elements.

From these formulas we see that M is capillary if and only if it has constant mean curvature and makes constant contact angle θ with ∂B. Furthermore, one can compute the second derivative of $E(t)$ at $t = 0$ with respect to an admissible volume-preserving variation to get

$$E''(0) = -\int_M \big(\Delta f + (|\sigma|^2 + \widetilde{\mathrm{Ric}}(N)) f\big) f\, da + \int_{\partial M} \Big(\frac{\partial f}{\partial \nu} - q f\Big) f\, ds \tag{4}$$

(see, e.g., the appendix of [Ros and Souam 1997]), where

$$f \in \mathcal{F} := \Big\{ f \in H^1(M) : \int_M f\, da = 0 \Big\},$$

$\widetilde{\mathrm{Ric}}(N)$ is the Ricci curvature of the ambient space and

$$q = \frac{1}{\sin\theta}\Pi(\bar{\nu}, \bar{\nu}) + \cot\theta\,\sigma(\nu, \nu). \tag{5}$$

In our setting, $\widetilde{\mathrm{Ric}}(N) = 0$ and $\Pi(\bar{\nu}, \bar{\nu}) = 1$.

Definition 2.4. A capillary hypersurface M is called (weakly) stable if $E''(0) \geq 0$ for all $f \in \mathcal{F}$.

In the sequel, we denote by $\partial^2 E(f)$ the quantity $E''(0)$ for a given function f.

Stable examples of capillary hypersurfaces. First we prove the stability of totally geodesic capillary hypersurfaces and spherical caps. The proof is similar to that of Proposition 1.1 in [Ros and Souam 1997]. We include it for completeness.

Proposition 2.5. *Let $B^{n+1} \subset \mathbb{R}^{n+1}$ be a unit Euclidean ball. Then totally geodesic capillary hypersurfaces and spherical caps are stable.*

Proof. First assume M is a totally geodesic capillary hypersurface, i.e., an n-dimensional ball $B^n(R)$ with radius R in B^{n+1}. Then the contact angle θ satisfies $\sin\theta = R$. By the definition of stability, we have to prove

$$\int_M |\nabla f|^2\,da \geq \frac{1}{R}\int_{\partial M} f^2\,ds \quad \forall f \in \mathcal{F}. \tag{6}$$

Consider now the $(n+1)$-dimensional ball B' of radius R having M as an equatorial totally geodesic hypersurface. Then by [Bokowski and Sperner 1979], M is area minimizing for the partitioning problem in B'. Thus M is stable in B', which is equivalent to the inequality (6).

Next assume M is a spherical cap in B^{n+1} with R being the radius of the sphere containing M and θ the contact angle. Consider the n-dimensional hyperplane P containing ∂M. Then M is a capillary hypersurface in a halfspace with a contact angle θ'. By [Gonzalez et al. 1980], M is stable in the halfspace, which means

$$\int_M \Big(|\nabla f|^2 - \frac{n}{R^2} f^2\Big)\,da \geq \frac{\cot\theta'}{R}\int_{\partial M} f^2\,ds \quad \forall f \in \mathcal{F}. \tag{7}$$

Elementary calculation leads to

$$\frac{1}{\sin\theta} + \frac{\cot\theta}{R} = \frac{\cot\theta'}{R}. \tag{8}$$

Now (7) and (8) together yield the stability of M in B^{n+1}. □

Conformal transformations on the Euclidean ball. Now we construct a conformal vector field. Fix a vector $a \in B^{n+1}$. Then

$$\varphi_a(x) = \frac{(1-|a|^2)x - (1 - 2\langle a, x\rangle + |x|^2)a}{1 - 2\langle a, x\rangle + |a|^2|x|^2} \tag{9}$$

defines a map from B^{n+1} to B^{n+1} and from $\mathbb{S}^n$ to $\mathbb{S}^n$ (see, e.g., Section 3.8 in [Schoen and Yau 1994]), since we have

$$1-|\varphi_a(x)|^2=\frac{(1-|a|^2)(1-|x|^2)}{1-2\langle a,x\rangle+|a|^2|x|^2}.$$

Moreover, φ_a is conformal. In fact, by a direct calculation we can check that

$$|d\varphi_a|^2=\left(\frac{1-|a|^2}{1-2\langle a,x\rangle+|a|^2|x|^2}\right)^2|dx|^2.$$

Note that $\varphi_a(a)=0$, $\varphi_a(0)=-a$, φ_a fixes two points $\pm a/|a|$ and φ_0 is an identity.

Next fix $\xi\in\mathbb{S}^n$. Let $a=t\xi$ with $-1<t<1$. Then

$$f_t(x)=\varphi_{t\xi}(x)=\frac{(1-t^2)x-(1-2t\langle\xi,x\rangle+|x|^2)t\xi}{1-2t\langle\xi,x\rangle+t^2|x|^2} \tag{10}$$

is a family of conformal transformations with parameter t. Thus f_t determines a conformal vector field $Y[\xi]$ as follows:

$$Y[\xi]=\frac{d}{dt}\bigg|_{t=0}f_t(x)=-(1+|x|^2)\xi+2\langle\xi,x\rangle x. \tag{11}$$

Note that $Y[\xi]$ is tangential along the sphere $\mathbb{S}^n$, since for all $x\in\mathbb{S}^n$,

$$\langle Y[\xi],x\rangle=-(1+|x|^2)\langle\xi,x\rangle+2\langle\xi,x\rangle|x|^2=0.$$

Delaunay hypersurfaces in Euclidean space. In this subsection, following [Hutchings et al. 2002], we review some facts about Delaunay hypersurfaces which are rotational and of constant mean curvature H. These hypersurfaces are the models we are concerned with in Theorem 1.1.

Let $M^n\subset\mathbb{R}^{n+1}$ be a hypersurface which is invariant under the action of the orthogonal group $O(n)$ fixing the x^1-axis. Assume M is generated by a curve Γ contained in the x^1x^2-plane. Then it suffices to determine the curve Γ.

Parametrize the curve $\Gamma=(x^1,x^2)$ by arc-length s. Denote by α the angle between the tangent to Γ and the positive x^1-direction and choose the normal vector $N=(\sin\alpha,-\cos\alpha)$. Then $(x^1,x^2;\alpha)$ satisfies the system of ordinary differential equations

$$\begin{cases}(x^1)'=\cos\alpha,\\ (x^2)'=\sin\alpha,\\ \alpha'=-nH+(n-1)\dfrac{\cos\alpha}{x^2}.\end{cases}$$

The first integral of this system is given by

$$(x^2)^{n-1}\cos\alpha-H(x^2)^n=F,$$

where the constant F is called the force of the curve Γ and it together with H will determine the curve as follows. (See Proposition 4.3 in [Hutchings et al. 2002].)

Proposition 2.6. *The curve Γ and the hypersurface M generated by Γ have the following several possible types.*

(a) *If $FH > 0$ then Γ is a periodic graph over the x^1-axis. It generates a periodic embedded unduloid, or a cylinder.*

(b) *If $FH < 0$ then Γ is a locally convex curve and M is a nodoid, which has self-intersections.*

(c) *If $F = 0$ and $H \neq 0$ then M is a sphere.*

(d) *If $H = 0$ and $F \neq 0$ we obtain a catenary which generates an embedded catenoid M with $F > 0$ if the normal points down and $F < 0$ if the normal points up.*

(e) *If $H = 0$ and $F = 0$ then Γ is a straight line orthogonal to the x^1-axis which generates a hyperplane.*

(f) *If M touches the x^1-axis, then it must be a sphere or a hyperplane.*

(g) *The curve Γ is determined, up to translation along the x^1-axis, by the pair (H, F).*

From this proposition, it is easy to see if M^n is the portion of an unduloid, cylinder, nodoid or a catenoid in a unit Euclidean ball B^{n+1} with revolution axis x^1, and moreover M is symmetric with respect to the hyperplane $\{x^1 = 0\}$, then M is a capillary hypersurface in B^{n+1}. In that case we call them Delaunay capillary hypersurfaces in B^{n+1}. Furthermore, the generalized body T enclosed by M and the wetted part of the sphere has the center of mass at the origin. So Theorem 1.1 is applicable.

3. Instability of capillary hypersurfaces

With the preparations above, we can define a "test function"

$$\phi[\xi] = \langle Y[\xi], N\rangle = \langle -(1+|x|^2)\xi + 2\langle \xi, x\rangle x, N\rangle. \tag{12}$$

We mention that we will also use the following expression of $\phi[\xi]$:

$$\phi[\xi] = \langle \xi, -(1+|x|^2)N + 2\langle x, N\rangle x\rangle. \tag{13}$$

Recall the second variational formula

$$\partial^2 E(\phi) = -\int_M L\phi \cdot \phi \, da + \int_{\partial M} (\phi_\nu - q\phi)\phi \, ds, \tag{14}$$

where $L = \Delta + |\sigma|^2$ and $q = \csc\theta + \cot\theta\sigma(\nu, \nu)$.

Now we can prove the following lemmas.

Lemma 3.1. *The vector ν is a principal direction for σ along ∂M. In particular, $D_\nu N = -\sigma(\nu, \nu)\nu$.*

Proof. It suffices to prove that $\sigma(\nu, X) = 0$ for all $X \in T_p(\partial M)$. In fact, we have

$$\begin{aligned}\sigma(\nu, X) &= \langle D_X \nu, N\rangle = \langle D_X(\cos\theta\bar{\nu} + \sin\theta\overline{N}), -\sin\theta\bar{\nu} + \cos\theta\overline{N}\rangle \\ &= \langle D_X\bar{\nu}, \overline{N}\rangle = -II(\bar{\nu}, X) = 0,\end{aligned}$$

where we used the facts that θ is constant, $\bar{\nu}$ and $\overline{N}$ are unit vectors, and ∂B is totally umbilical. Thus we complete the proof of Lemma 3.1. □

Lemma 3.2. *Along ∂M, we have*

$$\phi_\nu - q\phi = 0. \tag{15}$$

Proof. First, from (13) and Lemma 3.1 we have

$$\begin{aligned}\phi_\nu &= \big\langle \xi, -(1+|x|^2)N + 2\langle x, N\rangle x\big\rangle_\nu \\ &= \big\langle \xi, -2\langle x, \nu\rangle N + (1+|x|^2)\sigma(\nu, \nu)\nu - 2\langle x, \sigma(\nu, \nu)\nu\rangle x + 2\langle x, N\rangle \nu\big\rangle \\ &= 2\big\langle \xi, -\langle x, \nu\rangle N + \sigma(\nu, \nu)(\nu - \langle x, \nu\rangle x) + \langle x, N\rangle\nu\big\rangle,\end{aligned}$$

where in the third line we used $|x| = 1$ along ∂M.

Next, noticing that $x = \overline{N} = \cos\theta N + \sin\theta\nu$, we get

$$\begin{aligned}\phi_\nu &= 2\big\langle \xi, -\sin\theta N + \sigma(\nu, \nu)(\nu - \sin\theta(\cos\theta N + \sin\theta\nu)) + \cos\theta\nu\big\rangle \\ &= 2\big\langle \xi, (\sigma(\nu, \nu)\cos\theta + 1)(\cos\theta\nu - \sin\theta N)\big\rangle.\end{aligned}$$

On the other hand,

$$\begin{aligned}q\phi &= (\csc\theta + \cot\theta\sigma(\nu, \nu))\big\langle \xi, -(1+|x|^2)N + 2\langle x, N\rangle x\big\rangle \\ &= (\csc\theta + \cot\theta\sigma(\nu, \nu))2\langle \xi, -N + \cos\theta(\cos\theta N + \sin\theta\nu)\rangle \\ &= (1 + \cos\theta\sigma(\nu, \nu))2\langle \xi, -\sin\theta N + \cos\theta\nu\rangle,\end{aligned}$$

where again in the second line we used $|x| = 1$ along ∂M. Hence, we obtain

$$\phi_\nu - q\phi = 0.$$ □

The next lemma, which indicates the geometric meaning of Lemma 3.2, may have its own interest. Thus we also include it here.

Lemma 3.3. *Under the flow f_t, there holds*

$$\frac{d}{dt}\Big|_{t=0}\theta(t) = \phi_\nu - q\phi. \tag{16}$$

In particular, since f_t is conformal (angle preserving), $\phi_\nu - q\phi = 0$.

Proof. Following [Ros and Souam 1997], we denote by a "prime" the covariant derivative $(D/dt)|_{t=0}$. Also by the appendix of [Ros and Souam 1997], we have

$$\begin{aligned} \nu' &= \Big(\frac{\partial\phi}{\partial\nu} + \sigma(Y_0, \nu)\Big)N + \phi S_0(\nu) - \phi\sigma(\nu, \nu)\nu - S_1(Y_1) + \cot\theta\tilde{\nabla}\phi, \\ \bar{\nu}' &= -\Pi(Y, \bar{\nu})\bar{N} - S_2(Y_1) + (\csc\theta)\tilde{\nabla}\phi, \end{aligned}$$

where $\tilde{\nabla}$ denotes the gradient on ∂M, Y_0 (resp. Y_1) the tangent part of the variational vector field Y to M (resp. to ∂M), S_0 the shape operator of M in $\mathbb{R}^{n+1}$ with respect to N, and S_1 (resp. S_2) the shape operator of ∂M in M (resp. ∂B) with respect to ν (resp. $\bar{\nu}$).

Note that $\cos\theta(t) = \langle \nu, \bar{\nu}\rangle$, which implies

$$-\sin\theta \frac{d}{dt}\Big|_{t=0}\theta(t) = \langle \nu', \bar{\nu}\rangle + \langle \nu, \bar{\nu}'\rangle.$$

Taking into account that

$$\bar{\nu} = -\sin\theta N + \cos\theta\nu, \qquad \bar{N} = \cos\theta N + \sin\theta\nu,$$

we have

$$\begin{aligned} -\sin\theta \frac{d}{dt}\Big|_{t=0}\theta(t) &= \Big\langle \Big(\frac{\partial\phi}{\partial\nu} + \sigma(Y_0, \nu)\Big)N + \phi S_0(\nu) - \phi\sigma(\nu, \nu)\nu, -\sin\theta N + \cos\theta\nu \Big\rangle \\ &\qquad + \langle \nu, -\Pi(Y, \bar{\nu})(\cos\theta N + \sin\theta\nu)\rangle \\ &= -\sin\theta\Big(\frac{\partial\phi}{\partial\nu} + \sigma(Y_0, \nu)\Big) - \sin\theta\,\Pi(Y, \bar{\nu}), \end{aligned}$$

or

$$\frac{d}{dt}\Big|_{t=0}\theta(t) = \frac{\partial\phi}{\partial\nu} + \sigma(Y_0, \nu) + \Pi(Y, \bar{\nu}). \tag{17}$$

Again from the appendix of [Ros and Souam 1997], there hold

$$Y_0 = Y_1 - (\cot\theta)\phi\nu, \qquad Y = Y_1 - (\csc\theta)\phi\bar{\nu}, \qquad \sigma(Y_1, \nu) + \Pi(Y_1, \bar{\nu}) = 0.$$

Now plugging these equalities into (17), the lemma follows immediately. □

Lemma 3.4. *We have*

$$L\phi = -2n\langle \xi, N + Hx\rangle. \tag{18}$$

Proof. The proof is a direct calculation using the moving frame method. This method is very powerful in differential geometry. Take an orthonormal basis

$\{e_i : i = 1, \ldots, n\} \cup \{e_{n+1} = N\}$. Then we have the structure equations

$$dx = \sum_{i=1}^{n} \omega_i e_i,$$
$$de_i = \sum_{j=1}^{n} \omega_{ij} e_j + \sum_{j=1}^{n} h_{ij} \omega_j e_{n+1},$$
$$de_{n+1} = -\sum_{i,j=1}^{n} h_{ij} \omega_i e_j,$$

where ω_i is the dual form and ω_{ij} the connection form (see, e.g., [Cao and Li 2007]). Thus, we have

$$\begin{aligned}\Delta\phi &= \Delta\big\langle \xi, -(1+|x|^2)N + 2\langle x, N\rangle x\big\rangle \\ &= \Big\langle \xi, -\Big(\Delta|x|^2 \cdot N + 2\sum_{i=1}^{n} (|x|^2)_{,i} N_{,i} + (1+|x|^2)\Delta N\Big) \\ &\qquad + 2\Big(\Delta\langle x, N\rangle \cdot x + 2\sum_{i=1}^{n} \langle x, N\rangle_{,i} x_{,i} + \langle x, N\rangle \Delta x\Big)\Big\rangle.\end{aligned} \tag{19}$$

Note that

$$\Delta|x|^2 = 2nH\langle x, N\rangle + 2n,$$
$$\sum_{i=1}^{n} (|x|^2)_{,i} N_{,i} = -2\sum_{i,j=1}^{n} \langle x, e_i\rangle h_{ij} e_j.$$

Using the Codazzi equation to get $\sum_{i=1}^{n} h_{ij,i} = \sum_{i=1}^{n} h_{ii,j} = nH_{,j} = 0$, then we have

$$\Delta N = \sum_{i=1}^{n} N_{,ii} = -|\sigma|^2 N.$$

Moreover, we can get

$$\Delta\langle x, N\rangle = \langle \Delta x, N\rangle + 2\sum_{i=1}^{n} \langle x_{,i}, N_{,i}\rangle + \langle x, \Delta N\rangle = -nH - |\sigma|^2 \langle x, N\rangle,$$
$$\sum_{i=1}^{n} \langle x, N\rangle_{,i} x_{,i} = \sum_{i,j=1}^{n} \langle x, -h_{ij} e_j\rangle e_i = \sum_{i,j=1}^{n} -h_{ij} \langle x, e_j\rangle e_i.$$

Now substituting all these terms into (19) gives rise to

$$\Delta\phi = \langle \xi, -2n(N + Hx)\rangle - |\sigma|^2 \phi.$$

Therefore,

$$L\phi = \Delta\phi + |\sigma|^2 \phi = -2n\langle \xi, N + Hx\rangle. \qquad \square$$

Thus, we obtain

$$\partial^2 E(\phi) = -2n \int_M \langle \xi, N + Hx\rangle \cdot \big\langle \xi, (1+|x|^2)N - 2\langle x, N\rangle x\big\rangle \, da.$$

To analyze $\partial^2 E(\phi)$, we define a bilinear form

$$Q(\xi_1, \xi_2) = -2n \int_M \langle \xi_1, N + Hx\rangle \cdot \big\langle \xi_2, (1+|x|^2)N - 2\langle x, N\rangle x\big\rangle \, da \tag{20}$$

for all $\xi_1, \xi_2 \in \mathbb{S}^n$ (see [Marinov 2010] for the case $n = 2$). Denote by $\{\partial_A\}_{A=1}^{n+1}$ the standard coordinate vectors in $\mathbb{R}^{n+1}$. Then we have the following lemma.

Lemma 3.5. *Q has the following properties.*

(1) *Q is symmetric.*

(2) $\operatorname{tr} Q = \sum_{A=1}^{n+1} Q(\partial_A, \partial_A) \le 0$ *with equality if and only if* $|x| = \text{const}$ *on M.*

Proof. (1) First we prove Q is symmetric. Note that in fact Q is defined as

$$Q(\xi_1, \xi_2) = -\int_M L(\phi[\xi_1]) \cdot \phi[\xi_2] \, da.$$

Then Green's formula implies

$$Q(\xi_1, \xi_2) = -\int_M \phi[\xi_1] \cdot L(\phi[\xi_2]) \, da + \int_{\partial M} \big(\phi[\xi_1](\phi[\xi_2])_\nu - (\phi[\xi_1])_\nu \phi[\xi_2]\big) \, ds.$$

But Lemma 3.2 yields $(\phi[\xi_i])_\nu = q\phi[\xi_i]$, $i = 1, 2$. So the boundary term vanishes and then

$$Q(\xi_1, \xi_2) = Q(\xi_2, \xi_1).$$

(2) Next we calculate $\operatorname{tr} Q$:

$$\begin{aligned}
\operatorname{tr} Q &= \sum_{A=1}^{n+1} Q(\partial_A, \partial_A) \\
&= -2n \int_M \sum_{A=1}^{n+1} \langle \partial_A, N + Hx\rangle \cdot \big\langle \partial_A, (1+|x|^2)N - 2\langle x, N\rangle x\big\rangle \, da \\
&= -2n \int_M \big\langle N + Hx, (1+|x|^2)N - 2\langle x, N\rangle x\big\rangle \, da \\
&= -2n \int_M \big(H\langle x, N\rangle(1-|x|^2) + 1 + |x|^2 - 2\langle x, N\rangle^2\big) \, da \\
&\le -2n \int_M (H\langle x, N\rangle + 1)(1-|x|^2) \, da.
\end{aligned}$$

Also, we have $\Delta|x|^2 = 2n(H\langle x, N\rangle + 1)$. Consequently,

$$\begin{aligned}
\operatorname{tr} Q &\le -\int_M \Delta|x|^2 \cdot (1-|x|^2)\, da \\
&= \int_M \nabla|x|^2 \cdot \nabla(1-|x|^2)\, da - \int_{\partial M} \frac{\partial|x|^2}{\partial\nu}(1-|x|^2)\, ds \\
&= -\int_M |\nabla(|x|^2)|^2\, da \\
&\le 0,
\end{aligned}$$

where we have used $|x| = 1$ on ∂M to remove the boundary term. And it is easy to see $\operatorname{tr} Q = 0$ if and only if $|x| = \text{const}$.

This completes the proof of Lemma 3.5. □

Now if $|x| = \text{const}$ on M, since M has boundary $\partial M \subset \mathbb{S}^n$, we must have $|x| = 1$ on M. So $M \subset \mathbb{S}^n$. But that cannot happen because we assume that $\operatorname{int} M \subset \operatorname{int} B$. Thus by Lemma 3.5, Q has at least one negative eigenvalue. But on the other hand,

$$\operatorname{div}_{\mathbb{R}^{n+1}} Y[\xi] = \sum_{A=1}^{n+1} \langle D_{\partial_A} Y[\xi], \partial_A\rangle = 2(n+1)\langle \xi, x\rangle, \tag{21}$$

which by integration implies

$$\begin{aligned}
\int_M \phi\, da &= \int_M \langle Y[\xi], N\rangle\, da \\
&= -\int_T \operatorname{div}_{\mathbb{R}^{n+1}} Y[\xi]\, dv + \int_\Omega \langle Y[\xi], \bar{N}\rangle\, da \\
&= -2(n+1)\int_T \langle \xi, x\rangle\, dv.
\end{aligned} \tag{22}$$

So generally $\int_M \phi\, da \ne 0$. That means $\phi[\xi]$ is not a test function.

However, under the hypothesis of Theorem 1.1 that the center of mass of T is at the origin, we have $\int_M \phi\, da = -2(n+1)\int_T \langle \xi, x\rangle\, dv = 0$ for all $\xi \in \mathbb{S}^n$. So if we choose ξ as an eigenvector corresponding to the negative eigenvalue of Q, we have $\partial^2 E(\phi[\xi]) = Q(\xi, \xi) < 0$, which implies that M is unstable. This completes the proof of Theorem 1.1.

4. Other applications and a question

In this section we give several applications of the above argument and propose a conjecture on the topic.

Another criterion for instability. The following proposition is an immediate result.

Proposition 4.1. *If the bilinear form Q has two negative eigenvalues, then M is unstable.*

Proof. Assume Q is diagonalized such that ξ_1 and ξ_2 are the eigenvectors corresponding to the two negative eigenvalues. Then for real numbers c_1 and c_2 with $c_1^2+c_2^2\neq 0$,

$$Q(c_1\xi_1+c_2\xi_2, c_1\xi_1+c_2\xi_2)=c_1^2Q(\xi_1,\xi_1)+c_2^2Q(\xi_2,\xi_2)<0. \tag{23}$$

On the other hand,

$$\begin{aligned}\int_M \phi[c_1\xi_1+c_2\xi_2]\,da &= -2(n+1)\int_T\langle c_1\xi_1+c_2\xi_2, x\rangle\,dv\\ &= -2(n+1)\Big(c_1\int_T\langle\xi_1,x\rangle\,dv+c_2\int_T\langle\xi_2,x\rangle\,dv\Big).\end{aligned}$$

So we can always choose suitable c_1 and c_2 with $c_1^2+c_2^2\neq 0$ such that

$$\int_M \phi[c_1\xi_1+c_2\xi_2]\,da=0.$$

Then using $\phi[c_1\xi_1+c_2\xi_2]$ as a test function, from (23) we know M is unstable. □

The significance of the above proposition is as follows. For a given concrete capillary hypersurface M in B^{n+1}, the bilinear form Q is computable in principle. Then if Q has two negative eigenvalues, we can assert its instability. Also from this proposition we know that for hyperplanes and spherical caps Q has exactly one negative eigenvalue.

The center of mass of minimal submanifolds with free boundary. By free boundary we mean that M intersects ∂B^{n+1} orthogonally; that is, $\nu=x$ along ∂M. By analyzing the vector field $Y[\xi]$, we have the following proposition.

Proposition 4.2. *The center of mass of a minimal submanifold M^k with free boundary in a Euclidean ball is at the origin.*

Proof. Along M^k choose the orthonormal basis

$$\{e_i : i=1,\dots,k\}\cup\{e_\alpha : \alpha=k+1,\dots,n+1\}$$

such that $\{e_i \mid i=1,\dots,k\}\subset TM$. Then we have

$$\operatorname{div}_M Y[\xi]^T=\operatorname{div}_M\Big(Y[\xi]-\sum_\alpha\langle Y[\xi],e_\alpha\rangle e_\alpha\Big)=2k\langle\xi,x\rangle+\langle Y[\xi],k\vec{H}\rangle=2k\langle\xi,x\rangle.$$

By the divergence theorem, we have

$$2k\int_M\langle\xi,x\rangle\,da=\int_{\partial M}\langle Y[\xi]^T,\nu\rangle\,ds=\int_{\partial M}\langle Y[\xi],x\rangle\,ds=0,$$

where we have used the fact that $Y[\xi]$ is tangential to ∂B^{n+1}. □

This proposition shows that minimal submanifolds with free boundary have some symmetry. Comparing with it, we mention two other properties of M^k:

(1) The center of mass of the boundary ∂M is at the origin (a simple argument).

(2) The volume of M has a lower bound $|M^k| \geq |B^k|$, where B^k is a k-dimensional unit ball [Brendle 2012; Fraser and Schoen 2011; Ros and Vergasta 1995].

Stable immersed closed CMC hypersurfaces in $\mathbb{R}^{n+1}$**.** At last we give a new proof of a theorem by Barbosa and do Carmo by following Marinov's argument [2010] in the case $n = 2$.

Theorem 4.3 [Barbosa and do Carmo 1984; Marinov 2010]. *The only stable immersed closed hypersurface of constant mean curvature in* $\mathbb{R}^{n+1}$ *is the round sphere.*

Proof. By translation, assume the center of mass of a generalized body T enclosed by M is at the origin. So

$$\int_M \phi[\xi]\, da = 0$$

for all $\xi \in \mathbb{S}^n$. If M is the round sphere, we are done. Otherwise $|x| \neq \text{const}$. So by Lemma 3.5 the bilinear form Q has a negative eigenvalue. Choosing ξ as an eigenvector corresponding to the negative eigenvalue, we have

$$\partial^2 E(\phi[\xi]) = -\int_M L\phi[\xi] \cdot \phi[\xi]\, da = Q(\xi, \xi) < 0,$$

which shows that M is unstable. □

An open question. Since all the examples, i.e., the Delaunay capillary hypersurfaces, are known to be stable or unstable, we propose a conjecture as follows.

Conjecture 4.4. The only stable capillary hypersurface M^n ($n \geq 3$) in a unit Euclidean ball B^{n+1} is a totally geodesic hypersurface or a spherical cap.

There are some remarks on this conjecture.

(1) For $n \geq 2$, $H = 0$ and $\theta = \pi/2$, M must be totally geodesic [Ros and Vergasta 1995].

(2) For $n = 2$ and $\theta = \pi/2$, M is a totally geodesic disk, a spherical cap or a surface of genus 1 with embedded boundary having at most two connected components [Ros and Vergasta 1995].

(3) For $n = 2$ and $H = 0$, M is a totally geodesic disk or a surface of genus 1 with at most three connected boundary components [Ros and Souam 1997].

References

[Alías et al. 2007] L. J. Alías, A. Brasil, Jr., and O. Perdomo, "On the stability index of hypersurfaces with constant mean curvature in spheres", *Proc. Amer. Math. Soc.* **135**:11 (2007), 3685–3693. MR Zbl

[Barbosa and do Carmo 1984] J. L. Barbosa and M. do Carmo, "Stability of hypersurfaces with constant mean curvature", *Math. Z.* **185**:3 (1984), 339–353. MR Zbl

[Barbosa et al. 1988] J. L. Barbosa, M. do Carmo, and J. Eschenburg, "Stability of hypersurfaces of constant mean curvature in Riemannian manifolds", *Math. Z.* **197**:1 (1988), 123–138. MR Zbl

[Bokowski and Sperner 1979] J. Bokowski and E. Sperner, Jr., "Zerlegung konvexer Körper durch minimale Trennflächen", *J. Reine Angew. Math.* **311/312** (1979), 80–100. MR Zbl

[Brendle 2012] S. Brendle, "A sharp bound for the area of minimal surfaces in the unit ball", *Geom. Funct. Anal.* **22**:3 (2012), 621–626. MR Zbl

[Cao and Li 2007] L. Cao and H. Li, "r-minimal submanifolds in space forms", *Ann. Global Anal. Geom.* **32**:4 (2007), 311–341. MR Zbl

[Cheng 2003] Q.-M. Cheng, "Compact hypersurfaces in a unit sphere with infinite fundamental group", *Pacific J. Math.* **212**:1 (2003), 49–56. MR Zbl

[Cheng 2008] Q.-M. Cheng, "First eigenvalue of a Jacobi operator of hypersurfaces with a constant scalar curvature", *Proc. Amer. Math. Soc.* **136**:9 (2008), 3309–3318. MR Zbl

[Finn 1986] R. Finn, *Equilibrium capillary surfaces*, Grundlehren der Math. Wissenschaften **284**, Springer, 1986. MR Zbl

[Finn 1999] R. Finn, "Capillary surface interfaces", *Notices Amer. Math. Soc.* **46**:7 (1999), 770–781. MR Zbl

[Fraser and Schoen 2011] A. Fraser and R. Schoen, "The first Steklov eigenvalue, conformal geometry, and minimal surfaces", *Adv. Math.* **226**:5 (2011), 4011–4030. MR Zbl

[Fraser and Schoen 2016] A. Fraser and R. Schoen, "Sharp eigenvalue bounds and minimal surfaces in the ball", *Invent. Math.* **203**:3 (2016), 823–890. MR Zbl

[Gonzalez et al. 1980] E. Gonzalez, U. Massari, and I. Tamanini, "Existence and regularity for the problem of a pendent liquid drop", *Pacific J. Math.* **88**:2 (1980), 399–420. MR Zbl

[Hutchings et al. 2002] M. Hutchings, F. Morgan, M. Ritoré, and A. Ros, "Proof of the double bubble conjecture", *Ann. of Math.* (2) **155**:2 (2002), 459–489. MR Zbl

[Marinov 2010] P. I. Marinov, *Stability analysis of capillary surfaces with planar or spherical boundary in the absence of gravity*, Ph.D. thesis, University of Toledo, 2010, available at https://search.proquest.com/docview/849719123.

[Ros and Souam 1997] A. Ros and R. Souam, "On stability of capillary surfaces in a ball", *Pacific J. Math.* **178**:2 (1997), 345–361. MR Zbl

[Ros and Vergasta 1995] A. Ros and E. Vergasta, "Stability for hypersurfaces of constant mean curvature with free boundary", *Geom. Dedicata* **56**:1 (1995), 19–33. MR Zbl

[Schoen and Yau 1994] R. Schoen and S.-T. Yau, *Lectures on differential geometry*, Conf. Proc. Lecture Notes Geom. Topol. **1**, Int. Press, Cambridge, 1994. MR Zbl

[Simons 1968] J. Simons, "Minimal varieties in Riemannian manifolds", *Ann. of Math.* (2) **88** (1968), 62–105. MR Zbl

[Souam 1997] R. Souam, "On stability of stationary hypersurfaces for the partitioning problem for balls in space forms", *Math. Z.* **224**:2 (1997), 195–208. MR Zbl

Received July 5, 2016. Revised September 5, 2017.

HAIZHONG LI
DEPARTMENT OF MATHEMATICAL SCIENCES
TSINGHUA UNIVERSITY
BEIJING
CHINA
hli@math.tsinghua.edu.cn

CHANGWEI XIONG
MATHEMATICAL SCIENCES INSTITUTE
AUSTRALIAN NATIONAL UNIVERSITY
CANBERRA
AUSTRALIA
changwei.xiong@anu.edu.au

PACIFIC JOURNAL OF MATHEMATICS

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

msp.org/pjm

See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2018 is US $475/year for the electronic version, and $640/year for print and electronic.
Subscriptions, requests for back issues and changes of subscriber address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

PACIFIC JOURNAL OF MATHEMATICS

Volume 297 No. 1 November 2018

0030-8730(201811)297:1;1-R