Pacific Journal of Mathematics

INTERIOR GRADIENT ESTIMATES FOR WEAK SOLUTIONS OF QUASILINEAR p-LAPLACIAN TYPE EQUATIONS

TUOC PHAN

Volume 297 No. 1 November 2018

PACIFIC JOURNAL OF MATHEMATICS
Vol. 297, No. 1, 2018
dx.doi.org/10.2140/pjm.2018.297.195

INTERIOR GRADIENT ESTIMATES FOR WEAK SOLUTIONS OF QUASILINEAR p-LAPLACIAN TYPE EQUATIONS

TUOC PHAN

We study the interior weighted Sobolev regularity for weak solutions of the quasilinear equations of the form $\operatorname{div} A(x, u, \nabla u) = \operatorname{div} F$. The vector field A is allowed to be discontinuous in x, Hölder continuous in u and its growth in the gradient variable is like the p-Laplace operator with $1 < p < \infty$. We establish interior weighted $W^{1,q}$-regularity estimates for weak solutions to the equations for every $q > p$ assuming that the weak solutions are in the local John–Nirenberg BMO space. This paper therefore improves available results because it replaces the boundedness or continuity assumption on weak solutions by the borderline BMO one. Our regularity estimates also recover known results in which A is independent of the variable u. Our regularity theory complements the classical $C^{1,\alpha}$-regularity theory developed by many mathematicians including DiBenedetto and Tolksdorf for this general class of quasilinear elliptic equations.

1. Introduction

This paper establishes interior regularity estimates in weighted Sobolev spaces for weak solutions to the following general quasilinear p-Laplacian type equations:

$$\operatorname{div}[\boldsymbol{A}(x, u, \nabla u)] = \operatorname{div}[|\boldsymbol{F}|^{p-2}\boldsymbol{F}] \quad \text{in } B_{2R}, \tag{1-1}$$

where B_{2R} is the ball in $\mathbb{R}^n$ centered at the origin and with radius $2R$ for some $R > 0$, $\boldsymbol{F}$ is a given measurable vector field function, u is an unknown solution, and

$$\boldsymbol{A} = \boldsymbol{A}(x, z, \xi) : B_{2R} \times \mathbb{K} \times \mathbb{R}^n \to \mathbb{R}^n$$

is a given vector field. We assume that $\boldsymbol{A}(\,\cdot\,, z, \xi)$ is measurable in B_{2R} for every $(z, \xi) \in \mathbb{K} \times (\mathbb{R}^n \setminus \{0\})$, $\boldsymbol{A}(x, \cdot\,, \xi)$ Hölder continuous in $\mathbb{K}$ for a.e. $x \in B_{2R}$ and for all $\xi \in \mathbb{R}^n \setminus \{0\}$, and $\boldsymbol{A}(x, z, \cdot\,)$ differentiable in $\mathbb{R}^n \setminus \{0\}$ for each $z \in \mathbb{K}$ and for a.e. $x \in B_{2R}$. Here, $\mathbb{K}$ is an open interval in $\mathbb{R}$, which could be the same as $\mathbb{R}$. We

MSC2010: primary 35B65, 35J62, 35J70; secondary 35B45.
Keywords: quasilinear elliptic equations, quasilinear p-Laplacian type equations, Calderón–Zygmund regularity estimates, weighted Sobolev spaces.

assume in addition that there exist constants $\Lambda > 0$, $\alpha \in (0, 1]$, and $1 < p < \infty$ such that $\boldsymbol{A}$ satisfies the natural growth conditions

$$\langle \partial_\xi \boldsymbol{A}(x, z, \xi)\eta, \eta\rangle \geq \Lambda^{-1}|\xi|^{p-2}|\eta|^2 \quad \text{for a.e. } x \in B_{2R},\ \forall z \in \mathbb{K},\ \forall \xi, \eta \in \mathbb{R}^n \setminus \{0\}, \tag{1-2}$$

$$|\boldsymbol{A}(x, z, \xi)| + |\xi||\partial_\xi \boldsymbol{A}(x, z, \xi)| \leq \Lambda|\xi|^{p-1} \quad \text{for a.e. } x \in B_{2R},\ \forall z \in \mathbb{K},\ \forall \xi \in \mathbb{R}^n \setminus \{0\}, \tag{1-3}$$

$$|\boldsymbol{A}(x, z_1, \xi) - \boldsymbol{A}(x, z_2, \xi)| \leq \Lambda|\xi|^{p-1}|z_1 - z_2|^\alpha \quad \text{for a.e. } x \in B_{2R},\ \forall z_1, z_2 \in \mathbb{K},\ \forall \xi \in \mathbb{R}^n \setminus \{0\}. \tag{1-4}$$

Observe that under the conditions (1-2)–(1-4), the class of equations of the form (1-1) contains the well-known p-Laplace equations.

The focus of this paper is to investigate the regularity in weighted Sobolev spaces for weak solutions u of (1-1) when the nonlinearity of $\boldsymbol{A}$ depends on u as its variable. In this perspective, we would like to point out that, on the one hand, the $C^{1,\alpha}$-regularity theory for *bounded, weak solutions* of this class of equations has been investigated extensively, assuming some regularity of $\boldsymbol{A}$ in both x and z variables; see [DiBenedetto 1983; Evans 1982; Lewis 1983; Lieberman 1988; Gilbarg and Trudinger 1983; Ladyzhenskaya and Ural'tseva 1968; Malý and Ziemer 1997; Tolksdorf 1984; Ural'tseva 1968; Uhlenbeck 1977]. On the other hand, when $\boldsymbol{A}$ is discontinuous in x or $\boldsymbol{F}$ is not sufficiently regular, one does not expect those mentioned Schauder's type estimates for weak solutions of (1-1) to hold, and it is natural to search for L^q-estimates for the gradients instead; see [Gilbarg and Trudinger 1983; Ladyzhenskaya and Ural'tseva 1968; Maugeri et al. 2000; Krylov 2007; Malý and Ziemer 1997], for example. In this line of research, we note that in case $\boldsymbol{A} = \boldsymbol{A}_0$ for some $\boldsymbol{A}_0$ which is independent of the variable $z \in \mathbb{K}$, the equation (1-1) is reduced to

$$\operatorname{div}[\boldsymbol{A}_0(x, \nabla u)] = \operatorname{div}[|\boldsymbol{F}|^{p-2}\boldsymbol{F}] \quad \text{in } B_{2R}, \tag{1-5}$$

and the $W^{1,q}$-regularity estimates of Calderón–Zygmund type for weak solutions to the class of equations (1-5) has been studied by many authors; for example, see [Iwaniec 1983; DiBenedetto and Manfredi 1993; Byun and Wang 2012; Byun et al. 2007; Caffarelli and Peral 1998; Di Fazio 1996; Duzaar and Mingione 2010; 2011; Kinnunen and Zhou 1999; Maugeri et al. 2000; Mengesha and Phuc 2012; Dong and Kim 2010; Krylov 2007; 2008]. However, if $\boldsymbol{A}$ depends on the z-variable as in (1-1) and even with $\boldsymbol{F} = 0$, the $W^{1,q}$-regularity estimates become much more challenging, and not very well understood. This is due to the fact that the Calderón–Zygmund theory relies heavily on the scaling and dilation invariances of the considered class of equations; see [Wang 2003] for the geometric intuition of this fact. Since the

class of equations (1-5) is invariant under the scalings

$$u \mapsto u/\lambda \quad \text{and} \quad u(x) \mapsto \frac{u(rx)}{r} \qquad \text{for all positive numbers } r, \lambda, \tag{1-6}$$

the $W^{1,q}$-regularity of Calderón–Zygmund for weak solutions of (1-5) is therefore naturally expected. Meanwhile, the invariant homogeneity with respect to (1-6) is no longer available for (1-1). This fact presents a serious obstacle in obtaining $W^{1,q}$-estimates for the weak solutions of (1-1) as they do not generate enough estimates to carry out the proof by using existing methods.

In the recent work [Hoang et al. 2015; Nguyen and Phan 2016], the $W^{1,q}$-regularity estimates for weak solutions of (1-1) are addressed, and the $W^{1,q}$-regularity estimates are established assuming that the weak solutions are bounded. To overcome the loss of the homogeneity that we mentioned, we introduced in [Hoang et al. 2015; Nguyen and Phan 2016] some "double-scaling parameter" technique. Essentially, we study an enlarged class of "double parameter" equations of the type (1-1). Then, by a compactness argument, we successfully applied the perturbation method in [Caffarelli and Peral 1998] to tackle the problem. Careful analysis is required to ensure that all intermediate steps in the perturbation process are uniform with respect to the scaling parameters. See also [Byun et al. 2017; Phan 2017] for further implementation of this idea, and [Dong and Kim 2011] for some other related results in this line of research. In the papers [Hoang et al. 2015; Nguyen and Phan 2016; Byun et al. 2017], the a priori boundedness assumption on the weak solutions is essential to start the investigation of $W^{1,q}$-theory. This is because the approach uses the maximum principle for the unperturbed equations to implement the perturbation technique of [Caffarelli and Peral 1998]. We also would like to reference [Bögelein 2014], where the same $W^{1,p}$-theory for parabolic equations of type (1-1) is also achieved for continuous weak solutions.

A natural question arises from the mentioned work: *Is it necessary to assume that solutions are bounded, both for Sobolev regularity theory and Schauder's regularity*? In this paper, we give an answer to this question in the Sobolev regularity setting. In particular, we establish the $W^{1,q}$-regularity estimates for weak solutions of (1-1) by assuming that the solutions are in the BMO John–Nirenberg space, i.e., the borderline case. This is achieved in Theorem 1.1 below. Our paper therefore generalizes all results in [Bögelein 2014; Byun et al. 2017; Hoang et al. 2015; Nguyen and Phan 2016]. Moreover, this paper also simplifies many technical issues in [Hoang et al. 2015; Nguyen and Phan 2016], and gives a generic approach to unify and treat both classes of equations (1-1) and (1-5) at the same time. Unlike [Byun et al. 2017; Hoang et al. 2015; Nguyen and Phan 2016], we only use "one parameter" in the class of our equations. Precisely, we investigate the equation

$$\operatorname{div}[\boldsymbol{A}(x, \lambda u, \nabla u)] = \operatorname{div}[|\boldsymbol{F}|^{p-2}|\boldsymbol{F}] \quad \text{in } B_{2R}, \tag{1-7}$$

with the parameter $\lambda \geq 0$. The class of equations (1-7) is indeed the smallest one that is invariant with respect to the scalings and dilation (1-6) and that includes (1-1). When $\lambda = 0$, the equation (1-7) clearly becomes the equation (1-5). This paper therefore recovers known results such as [Iwaniec 1983; DiBenedetto and Manfredi 1993; Byun and Wang 2012; Byun et al. 2007; Caffarelli and Peral 1998; Di Fazio 1996; Duzaar and Mingione 2010; 2011; Kinnunen and Zhou 1999; Maugeri et al. 2000; Mengesha and Phuc 2012] regarding the interior regularity of weak solutions of (1-5).

From now on, the notation A_q with $q \geq 1$ stands for the class of Muckenhoupt weights, whose definition is recalled in Definition 2.3. Also, $B_R(y)$ is the ball in $\mathbb{R}^n$ with radius $R > 0$ and centered at $y \in \mathbb{R}^n$. For simplicity, we also write $B_R = B_R(0)$. Moreover, for some locally integrable function $f : U \to \mathbb{R}$ with some measurable set $U \subset \mathbb{R}^n$ and with $\rho_0 > 0$, the BMO seminorm of bounded mean oscillation of f is defined by

$$[\![f]\!]_{\mathrm{BMO}(U,\,\rho_0)} = \sup_{y\in U,\,0<\rho<\rho_0} \frac{1}{|B_\rho(y)|} \int_{B_\rho(y)\cap U} |f(x) - \bar{f}_{B_\rho(y)\cap U}|\,dx,$$
$$\text{where } \bar{f}_{B_\rho(y)\cap U} = \frac{1}{|B_\rho(y)|} \int_{B_\rho(y)\cap U} f(x)\,dx.$$

The main result of this paper is the following interior regularity estimates for weak solutions of (1-7) in weighted Lebesgue spaces.

Theorem 1.1. *Let $\Lambda > 0$, $M > 0$, $p, q > 1$, $\gamma \geq 1$, and $\alpha \in (0, 1]$. Then there exists a sufficiently small constant $\delta = \delta(p, q, n, \Lambda, M, \gamma, \alpha) > 0$ such that the following statement holds true. Assume that $\boldsymbol{A} : B_{2R} \times \mathbb{K} \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory map satisfying (1-2)–(1-4) and*

$$\begin{aligned}(1\text{-}8)\quad & [\![\boldsymbol{A}]\!]_{\mathrm{BMO}(B_R,\,R)} \\ &:= \sup_{0<\rho\leq R} \sup_{y\in B_R} \frac{1}{|B_\rho(y)|} \int_{B_\rho(y)} \left[\sup_{z\in\mathbb{K},\,\xi\in\mathbb{R}^n\setminus\{0\}} \frac{|\boldsymbol{A}(x,z,\xi) - \bar{\boldsymbol{A}}_{B_\rho(y)}(z,\xi)|}{|\xi|^{p-1}} \right] dx \\ &\leq \delta\end{aligned}$$

for some $R > 0$ and for some open interval $\mathbb{K} \subset \mathbb{R}$. Then for every $\boldsymbol{F} \in L^p(B_{2R}, \mathbb{R}^n)$, if u is a weak solution of

$$\operatorname{div}[\boldsymbol{A}(x, \lambda u, \nabla u)] = \operatorname{div}[|\boldsymbol{F}|^{p-2}\boldsymbol{F}] \quad \text{in } B_{2R}$$

with $[\![\lambda u]\!]_{\mathrm{BMO}(B_R,\,R)} \leq M$ for some $\lambda \geq 0$, the weighted regularity estimate

$$\int_{B_R} |\nabla u|^{pq}\omega(x)\,dx \leq C\left[\int_{B_{2R}} |\boldsymbol{F}|^{pq}\omega(x)\,dx + \omega(B_{2R})\left(\frac{1}{|B_{2R}|}\int_{B_{2R}} |\nabla u|^p\,dx\right)^q\right]$$

holds, as long as its right-hand side is finite, where $\omega \in A_q$ *with*

$$[\omega]_{A_q} \le \gamma, \qquad \bar{\boldsymbol{A}}_{B_\rho(y)}(z,\xi) := \fint_{B_\rho(y)} \boldsymbol{A}(x,z,\xi)\,dx,$$

and C *is a constant depending only on* q, p, n, Λ, α, M, $\mathbb{K}$, R, *and* γ.

We emphasize that the significant contribution in Theorem 1.1 is that it relaxes and do not requires the considered weak solutions to be bounded as in [Bögelein 2014; Byun et al. 2017; Hoang et al. 2015; Nguyen and Phan 2016]. This is completely new even for the case $\omega = 1$, in comparison to the known work that we already mentioned for both the Schauder and the Sobolev regularity theories regarding weak solutions of (1-1). Certainly, removing the boundedness assumption on solutions and replacing it by the condition that weak solutions are in BMO is valuable in the critical cases in which the L^∞-bound for solutions are not available; see [DiBenedetto and Manfredi 1993], for example. When $p = n$, our weak solutions are in $W^{1,n}$, and hence they are in BMO by the Sobolev embedding theorem. Therefore, in this case, our theorem is applicable directly, while results [Bögelein 2014; Byun et al. 2017; Hoang et al. 2015; Nguyen and Phan 2016] may not be. Note that M is not required to be small: our $[\![\lambda u]\!]_{\mathrm{BMO}(B_R,R)}$ is not necessarily small. When $\lambda = 0$, the condition $[\![\lambda u]\!]_{\mathrm{BMO}(B_R,R)} \le M$ is certainly held for every function u. Therefore, Theorem 1.1 recovers results in [Iwaniec 1983; Byun and Wang 2012; Byun et al. 2007; Caffarelli and Peral 1998; Di Fazio 1996; Duzaar and Mingione 2010; 2011; Kinnunen and Zhou 1999; Mengesha and Phuc 2012], in which the case that $\boldsymbol{A}$ is independent of $z \in \mathbb{K}$ is studied. This paper therefore unifies $W^{1,q}$-regularity estimates for both (1-1) and (1-5). We also would like to note that the fact that $\boldsymbol{A}$ is defined in $z \in \mathbb{K}$ only is important in many applications. A simple example is $\mathbb{K} = (0, \infty)$, meaning that (1-2)–(1-4) only hold for positive solutions u. In the study of cross-diffusion equations in [Hoang et al. 2015], $K = (0, M_0)$ for some $M_0 > 0$.

We remark that the smallness condition (1-8) on the mean oscillation of $\boldsymbol{A}$ with respect to the x-variable is necessary as there is a counterexample provided in [Meyers 1963] for linear equations. In this regard, we also would like to point out that in [Dong and Kim 2010], regularity estimates for weak solutions of equations with measurable coefficients that are small in partial BMO-seminorm are established.

This paper follows the perturbation approach of [Caffarelli and Peral 1998] and makes use of the Hardy–Littlewood maximal function; see also [Byun and Wang 2012; Byun et al. 2017; Nguyen and Phan 2016; Hoang et al. 2015; Phan 2017; Wang 2003]. One can also find in [Krylov 2007; 2008; Dong and Kim 2010; 2011] for a similar perturbation approach which uses the Fefferman–Stein sharp function. To overcome the loss of boundedness of solutions due to our assumption, instead of applying the maximum principle during the perturbation process as in prior work,

we directly derive and delicately use Hölder's regularity estimates for solutions of the corresponding homogeneous equations; see the estimates (3-4) and (3-15), for example. The well-known John–Nirenberg's theorem and reverse Hölder's inequality also play a very important role in our approach.

We now conclude this section by outlining the organization of this paper. Section 2 reviews some definitions and some known results needed in the paper. Intermediate steps in the approximation estimates required in the proof of Theorem 1.1 are established and proved in Section 3. Finally, Section 4 gives the proof of Theorem 1.1.

2. Definitions and preliminaries

Scaling invariances, and definitions of weak solutions. Let $\lambda' \geq 0$, and let us consider a function $u \in W^{1,p}_{\mathrm{loc}}(U)$ satisfying

$$\operatorname{div}[\boldsymbol{A}(x, \lambda' u, \nabla u)] = \operatorname{div}[|\boldsymbol{F}|^{p-2}\boldsymbol{F}] \quad \text{in } U,$$

in the sense of distribution, for some open bounded set $U \subset \mathbb{R}^n$. Then for some fixed $\lambda > 0$, the rescaled function

$$v(x) = \frac{u(x)}{\lambda} \quad \text{for } x \in U \tag{2-1}$$

solves the equation

$$\operatorname{div}[\hat{\boldsymbol{A}}(x, \hat{\lambda} v, \nabla v)] = \operatorname{div}[|\hat{\boldsymbol{F}}|^{p-2}\hat{\boldsymbol{F}}] \quad \text{in } U$$

in the distributional sense, where $\hat{\lambda} = \lambda\lambda' \geq 0$ and $\hat{\boldsymbol{A}} : U \times \mathbb{K} \times \mathbb{R}^n \to \mathbb{R}^n$ is defined by

$$\hat{\boldsymbol{A}}(x, z, \xi) = \frac{\boldsymbol{A}(x, z, \lambda\xi)}{\lambda^{p-1}} \quad \text{and} \quad \hat{\boldsymbol{F}}(x) = \frac{\boldsymbol{F}(x)}{\lambda^{p-1}}. \tag{2-2}$$

Remark 2.1. If $\boldsymbol{A} : U \times \mathbb{K} \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies the conditions (1-2)–(1-4) on $U \times \mathbb{K} \times \mathbb{R}^n$, then the rescaled vector field $\hat{\boldsymbol{A}} : U \times \mathbb{K} \times \mathbb{R}^n \to \mathbb{R}^n$ defined in (2-2) also satisfies the structural conditions (1-2)–(1-4) with the same constants Λ, p, and α. Moreover, $[\![\boldsymbol{A}]\!]_{\mathrm{BMO}(U,\rho_0)} = [\![\hat{\boldsymbol{A}}]\!]_{\mathrm{BMO}(U,\rho_0)}$ for any $\rho_0 > 0$.

In this paper, $C_0^\infty(U)$ is the set of all smooth compactly supported functions in U, $L^p(U, \mathbb{R}^n)$ with $1 \leq p < \infty$ is the Lebesgue space consisting of all measurable functions $f : U \to \mathbb{R}^n$ such that $|f|^p$ is integrable on U, and $W^{1,p}(U)$ is the standard Sobolev space on U. Moreover, $\langle \cdot, \cdot \rangle$ is the Euclidean inner product in $\mathbb{R}^n$. Let us now recall the definitions of weak solutions that we use throughout the paper.

Definition 2.2. Let $\mathbb{K} \subset \mathbb{R}$ be an interval, and let $\Lambda > 0$, $p > 1$, $\alpha \in (0, 1]$. Also, let $U \subset \mathbb{R}^n$ be an open bounded set in $\mathbb{R}^n$ with sufficiently smooth boundary ∂U, and let $\boldsymbol{A} : U \times \mathbb{K} \times \mathbb{R}^n \to \mathbb{R}^n$ satisfy conditions (1-2)–(1-4) on $U \times \mathbb{K} \times \mathbb{R}^n$.

(i) For every $\boldsymbol{F} \in L^p(U;\mathbb{R}^n)$ and $\lambda \geq 0$, a function $u \in W^{1,p}_{\mathrm{loc}}(U)$ is called a *weak solution* of

$$\operatorname{div}[\boldsymbol{A}(x,\lambda u,\nabla u)] = \operatorname{div}[|\boldsymbol{F}|^{p-2}\boldsymbol{F}] \quad \text{in } U$$

if $\lambda u(x) \in \mathbb{K}$ for a.e. $x \in U$, and

$$\int_U \langle \boldsymbol{A}(x,\lambda u,\nabla u), \nabla\varphi\rangle\, dx = \int_U \langle |\boldsymbol{F}|^{p-2}\boldsymbol{F}, \nabla\varphi\rangle\, dx \quad \forall \varphi \in C_0^\infty(U).$$

(ii) For every $\boldsymbol{F} \in L^p(U;\mathbb{R}^n)$, $g \in W^{1,p}(U)$, and $\lambda \geq 0$, a function $u \in W^{1,p}(U)$ is a *weak solution* of

$$\begin{cases} \operatorname{div}[\boldsymbol{A}(x,\lambda u,\nabla u)] = \operatorname{div}[|\boldsymbol{F}|^{p-2}\boldsymbol{F}] & \text{in } U, \\ u = g & \text{on } \partial U, \end{cases}$$

if $\lambda u(x) \in \mathbb{K}$ for a.e. $x \in U$, $u - g \in W_0^{1,p}(U)$, and

$$\int_U \langle \boldsymbol{A}(x,\lambda u,\nabla u), \nabla\varphi\rangle\, dx = \int_U \langle \boldsymbol{F}, \nabla\varphi\rangle\, dx \quad \forall \varphi \in C_0^\infty(U).$$

Muckenhoupt weights, weighted inequalities, and the crawling ink-spots lemma. This section recalls several analysis results and definitions that are needed in the paper. Firstly, we recall the definition of the A_p-Muckenhoupt class of weights introduced in [Muckenhoupt 1972].

Definition 2.3. Let $1 \leq p < \infty$. A nonnegative and locally integrable function $\omega : \mathbb{R}^n \to [0,\infty)$ is said to be in *the class A_p of Muckenhoupt weights* if

$$[\omega]_{A_p} := \sup_{\text{balls } B\subset\mathbb{R}^n} \left(\fint_B \omega(x)\,dx\right)\left(\fint_B \omega(x)^{1/(1-p)}\,dx\right)^{p-1} < \infty \quad \text{if } p > 1,$$

$$[\omega]_{A_1} := \sup_{\text{balls } B\subset\mathbb{R}^n} \left(\fint_B \omega(x)\,dx\right)\|\omega^{-1}\|_{L^\infty(B)} < \infty \quad \text{if } p = 1.$$

It turns out that the class of A_p-Muckenhoupt weights satisfies the reverse Hölder's inequality and the doubling properties. In particular, a measure of any A_p-weight is comparable with the Lebesgue measure in some sense. This is in fact a well-known result due to R. Coifman and C. Fefferman, and it is an important ingredient in the paper.

Lemma 2.4 [Coifman and Fefferman 1974]. *For $1 < p < \infty$, the following statements hold true*:

(i) *If $\mu \in A_p$, then for every ball $B \subset \mathbb{R}^n$ and every measurable set $E \subset B$,*

$$\mu(B) \leq [\mu]_{A_p}\left(\frac{|B|}{|E|}\right)^p \mu(E).$$

(ii) *If $\mu \in A_p$ with $[\mu]_{A_p} \le \gamma$ for some given $\gamma \ge 1$, then there are $C = C(\gamma, n)$ and $\beta = \beta(\gamma, n) > 0$ such that*

$$\mu(E) \le C\left(\frac{|E|}{|B|}\right)^{\beta} \mu(B)$$

for every ball $B \subset \mathbb{R}^n$ and every measurable set $E \subset B$.

Observe that in the above statement and in this paper, the notation

$$|U| = \int_U dx, \qquad \mu(U) = \int_U \mu(x)\, dx,$$

for every measurable set $U \subset \mathbb{R}^n$ is used.

Secondly, we state a standard result in measure theory.

Lemma 2.5. *Assume that $g \ge 0$ is a measurable function in a bounded subset $U \subset \mathbb{R}^n$. Let $\theta > 0$ and $N > 1$ be given constants. If μ is a weight function in $\mathbb{R}^n$, then for any $1 \le p < \infty$,*

$$g \in L^p(U, \mu) \Leftrightarrow S := \sum_{j \ge 1} N^{pj} \mu(\{x \in U : g(x) > \theta N^j\}) < \infty.$$

Moreover, there exists a constant $C > 0$ depending only on θ, N, and p such that

$$C^{-1} S \le \|g\|^p_{L^p(U,\mu)} \le C(\mu(U) + S),$$

where $L^p(U, \mu)$ is the weighted Lebesgue space with norm

$$\|g\|_{L^p(U,\mu)} = \left(\int_U |g(x)|^p \mu(x)\, dx\right)^{1/p}.$$

Thirdly, we discuss the Hardy–Littlewood maximal operator and its boundedness in weighted spaces. For a given locally integrable function $f : \mathbb{R}^n \to \mathbb{R}$, the Hardy–Littlewood maximal function is defined as

$$\mathcal{M} f(x) = \sup_{\rho > 0} \fint_{B_\rho(x)} |f(y)|\, dy \quad \text{for } x \in \mathbb{R}^n. \tag{2-3}$$

For a function f that is defined on a bounded domain U, we write

$$\mathcal{M}_U f(x) = \mathcal{M}(f \chi_U)(x),$$

where χ_U is the characteristic function of the set U. The following boundedness of the Hardy–Littlewood maximal operator $\mathcal{M} : L^q(\mathbb{R}^n, \omega) \to L^q(\mathbb{R}^n, \omega)$ is classical.

Lemma 2.6. *Let $\gamma \ge 1$ and $\omega \in A_q$ with $[\omega]_{A_q} \le \gamma$.*

(i) Strong (q, q): *Let $1 < q < \infty$. Then there exists a constant $C = C(\gamma, q, n)$ such that*

$$\|\mathcal{M}\|_{L^q(\mathbb{R}^n, \omega) \to L^q(\mathbb{R}^n, \omega)} \le C.$$

(ii) Weak (1, 1): *There exists a constant $C = C(n)$ such that for any $\lambda > 0$, we have*

$$\left|\{x \in \mathbb{R}^n : \mathcal{M}(f) > \lambda\}\right| \le \frac{C}{\lambda} \int_{\mathbb{R}^n} |f|\, dx.$$

Finally, we recall the following important lemma. This lemma is usually referred to as the "crawling ink-spots" lemma, and is originally due to N. V. Krylov and M. V. Safonov [Krylov and Safonov 1979; Safonov 1980].

Lemma 2.7 (crawling ink-spots). *Suppose that $\omega \in A_q$ with $[\omega]_{A_q} \le \gamma$ for some $1 < q < \infty$ and some $\gamma \ge 1$. Suppose also that $R > 0$ and that C, D are measurable sets satisfying $C \subset D \subset B_R$. Assume that there are $\rho_0 \in (0, R/2)$ and $0 < \epsilon < 1$ such that*

(i) *$\omega(C) < \epsilon\omega(B_{\rho_0}(y))$ for almost every $y \in B_R$, and*

(ii) *for all $x \in B_R$ and $\rho \in (0, \rho_0)$, if $\omega(C \cap B_\rho(x)) \ge \epsilon\omega(B_\rho(x))$, then*

$$B_\rho(x) \cap B_R \subset D.$$

Then

$$\omega(C) \le \epsilon_1 \omega(D) \quad \text{for } \epsilon_1 = \epsilon 20^{nq} \gamma^2.$$

Hölder regularity and self-improving regularity. We recall some classical regularity results. The first is about the interior Hölder regularity for weak solutions of homogeneous p-Laplacian type equations (1-5). This result is indeed a consequence of the well-known De Giorgi–Nash–Moser theory; see [Giusti 2003, Theorem 7.6; Ladyzhenskaya and Ural'tseva 1968, Theorem 1.1, p. 251].

Lemma 2.8. *Let $\Lambda > 0$, $p > 1$, and let $\mathbb{A}_0 : B_r \times \mathbb{R}^n \to \mathbb{R}^n$ be a Carathéodory map and satisfy (1-2)–(1-3) on $B_r \times \mathbb{R}^n$ with some $r > 0$. If $v \in W^{1,p}(B_r)$ is a weak solution of the equation*

$$\operatorname{div}[\mathbb{A}_0(x, \nabla v)] = 0 \quad \text{in } B_r,$$

then there is $C_0 > 0$ depending only on Λ, n, p such that

$$\|v\|_{L^\infty(B_{5r/6})} \le C_0 \left(\fint_{B_r} |v|^p\, dx \right)^{1/p}.$$

Moreover, there is a constant $\beta \in (0, 1)$ depending only on Λ, n, p, and $\|v\|_{L^\infty(B_{5r/6})}$ such that

$$|v(x) - v(y)| \le C_0 \|v\|_{L^\infty(B_{5r/6})} \left(\frac{|x-y|}{r} \right)^\beta \quad \forall x, y \in \overline{B}_{2r/3}.$$

We now recall a classical result on self-improving regularity estimates for weak solutions of p-Laplacian type equations. The following result is due to N. Meyers and A. Elcrat [1975, Theorem 1]; see also [DiBenedetto and Manfredi 1993] and, for the parabolic version, [Kinnunen and Lewis 2000].

Lemma 2.9. *Let $\Lambda > 0$, $p > 1$. Then there exists $p_0 = p_0(\Lambda, n, p) > p$ such that the following statement holds true. Suppose that $\mathbb{A}_0 : B_{2r} \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory map satisfying (1-2)–(1-3) on $B_{2r} \times \mathbb{R}^n$ with some $r > 0$. If $v \in W^{1,p}(B_{2r})$ is a weak solution of the equation*

$$\operatorname{div}[\mathbb{A}_0(x, \nabla v)] = 0 \quad \text{in } B_{2r},$$

then for every $p_1 \in [p, p_0]$, there exists a constant $C = C(\Lambda, p_1, p, n) > 0$ such that

$$\Big(\frac{1}{|B_r|}\int_{B_r} |\nabla v|^{p_1}\,dx\Big)^{1/p_1} \le C\Big(\frac{1}{|B_{2r}|}\int_{B_{2r}} |\nabla v|^{p}\,dx\Big)^{1/p}.$$

Some simple energy estimates. In this section we derive some elementary estimates which will be used frequently in the paper.

Lemma 2.10. *Let $\Lambda > 0$, $p > 1$, and let $U \subset \mathbb{R}^n$ be a bounded open set and $\mathbb{K}$ an interval in $\mathbb{R}$. Assume that $\boldsymbol{A} : U \times \mathbb{K} \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies (1-2)–(1-3) on $U \times \mathbb{K} \times \mathbb{R}^n$. Then for any functions $u, v \in W^{1,p}(U)$ and any nonnegative function $\phi \in C(\overline{U})$, the following hold*:

(i) *If $1 < p < 2$, then for every $\tau > 0$,*

$$\int_U |\nabla u - \nabla v|^p \phi\,dx \le \tau \int_U |\nabla u|^p \phi\,dx + C(\Lambda, p)\tau^{(p-2)/p}\int_U \langle \boldsymbol{A}(x, u, \nabla u) - \boldsymbol{A}(x, u, \nabla v), \nabla u - \nabla v\rangle \phi\,dx.$$

(ii) *If $p \ge 2$, then*

$$\int_U |\nabla u - \nabla v|^p \phi\,dx \le C(\Lambda, p)\int_U \langle \boldsymbol{A}(x, u, \nabla u) - \boldsymbol{A}(x, u, \nabla v), \nabla u - \nabla v\rangle \phi\,dx.$$

Proof. This lemma is well-known; see [Tolksdorf 1984, Lemma 1; Nguyen and Phan 2016, Lemma 3.1]. However, because it is important and also for completeness, we provide the proof. We first claim that from (1-2), the monotonicity property

$$\langle \boldsymbol{A}(x, z, \xi) - \boldsymbol{A}(x, z, \eta), \xi - \eta\rangle \ge \begin{cases} \gamma_0|\xi - \eta|^p & \text{if } p \ge 2,\\ \gamma_0(|\xi| + |\xi - \eta|)^{p-2}|\xi - \eta|^2 & \text{if } 1 < p < 2\end{cases} \tag{2-4}$$

of $\boldsymbol{A}$ holds true for all $(x, z) \in U \times \mathbb{K}$ and all $\xi, \eta \in \mathbb{R}^n \setminus \{0\}$, where $\gamma_0 = \gamma_0(\Lambda, p) > 0$ is a constant. To prove the claim, observe that for each $(x, z) \in U \times \mathbb{K}$ and each $\xi, \eta \in \mathbb{R}^n \setminus \{0\}$, we can write

$$\langle \boldsymbol{A}(x, z, \xi) - \boldsymbol{A}(x, z, \eta), \xi - \eta\rangle = \int_0^1 \big\langle \boldsymbol{A}_\xi(x, z, \xi + t(\eta - \xi))(\xi - \eta), \xi - \eta\big\rangle\,dt, \tag{2-5}$$

where $\boldsymbol{A}_\xi(x, z, \cdot)$ is the matrix of partial derivatives of $\boldsymbol{A}$ with respect to the third component variable in $\mathbb{R}^n \setminus \{0\}$ of $\boldsymbol{A}$. It follows from (1-2) that

$$\big\langle \boldsymbol{A}_\xi(x, z, \xi + t(\eta - \xi))(\xi - \eta), \xi - \eta \big\rangle \geq \Lambda^{-1} |\xi + t(\eta - \xi)|^{p-2} |\xi - \eta|^2. \tag{2-6}$$

Then, if $p \in (1, 2)$, we see that $|\xi + t(\eta - \xi)| \leq |\xi| + |\xi - \eta|$, and therefore,

$$\langle \boldsymbol{A}(x, z, \xi) - \boldsymbol{A}(x, z, \eta), \xi - \eta \rangle \geq \Lambda^{-1} (|\xi| + |\xi - \eta|)^{p-2} |\xi - \eta|^2.$$

Hence, the second estimate in (2-4) is proved. On the other hand, when $p \geq 2$, by (2-5)–(2-6), we see that

$$\langle \boldsymbol{A}(x, z, \xi) - \boldsymbol{A}(x, z, \eta), \xi - \eta \rangle \geq \Lambda^{-1} |\xi - \eta|^2 \int_0^{1/4} |\xi + t(\eta - \xi)|^{p-2}\, dt.$$

We may now assume without loss of generality that $|\xi - \eta| \neq 0$ and $|\eta| \leq |\xi|$. Let us define $t_0 = |\xi|/|\xi - \eta|$. Note that if $|\xi - \eta| \leq 2|\xi|$, then $t_0 \geq \frac{1}{2}$ and

$$|\xi + t(\eta - \xi)| \geq \big||\xi| - t|\xi - \eta|\big| = |t - t_0|\,|\xi - \eta| \geq \tfrac{1}{4}|\xi - \eta| \quad \forall t \in \big(0, \tfrac{1}{4}\big).$$

Otherwise, we have $|\eta| \leq |\xi| \leq \frac{1}{2}|\xi - \eta|$, and then

$$\begin{aligned} |\xi + t(\eta - \xi)| &= |(1 - t)(\xi - \eta) + \eta| \\ &\geq (1 - t)|\xi - \eta| - |\eta| \\ &\geq \tfrac{3}{4}|\xi - \eta| - \tfrac{1}{2}|\xi - \eta| = \tfrac{1}{4}|\xi - \eta| \quad \forall t \in \big(0, \tfrac{1}{4}\big). \end{aligned}$$

Hence, in conclusion, we have $|\xi + t(\eta - \xi)| \geq \frac{1}{4}|\xi - \eta|$ for all $t \in \big(0, \frac{1}{4}\big)$, and therefore,

$$\langle \boldsymbol{A}(x, z, \xi) - \boldsymbol{A}(x, z, \eta), \xi - \eta \rangle \geq \frac{1}{4^{p-1}\Lambda} |\xi - \eta|^p.$$

This proves the first estimate in (2-4) when $p \geq 2$, completing the proof of (2-4).

Finally, observe that from (2-4), (ii) becomes trivial. Therefore, it remains to prove (i) with $1 < p < 2$. In this case, for each $\xi, \eta \in \mathbb{R}^n \setminus \{0\}$ and each $\tau \in (0, 1)$, we can use Young's inequality to obtain

$$\begin{aligned} |\xi - \eta|^p &= (|\xi| + |\xi - \eta|)^{p(2-p)/2} (|\xi| + |\xi - \eta|)^{p(p-2)/2} |\xi - \eta|^p \\ &\leq \frac{\tau}{3^{-p}} (|\xi| + |\xi - \eta|)^p + C_p \tau^{(p-2)/p} (|\xi| + |\xi - \eta|)^{p-2} |\xi - \eta|^2. \end{aligned}$$

From this and (2-4), we infer that

$$\begin{aligned} |\xi - \eta|^p &\leq \tau |\xi|^p + C_p \tau^{(p-2)/p} (|\xi| + |\xi - \eta|)^{p-2} |\xi - \eta|^2 \\ &\leq \tau |\xi|^p + C(\Lambda, p) \tau^{(p-2)/p} \langle \boldsymbol{A}(x, z, \xi) - \boldsymbol{A}(x, z, \eta), \xi - \eta \rangle. \end{aligned}$$

Then (i) follows and the proof of Lemma 2.10 is complete. □

Lemma 2.11 (Caccioppoli type estimates). *Let $\Lambda > 0$, $p > 1$ be fixed. Then for every $r > 0$ and every $\boldsymbol{A}_0 : B_r \times \mathbb{R}^n$ satisfying (1-2)–(1-3) on $B_r \times \mathbb{R}^n$, if $v \in W^{1,p}(B_r)$ is a weak solution of*

$$\operatorname{div}[\boldsymbol{A}_0(x, \nabla v)] = 0 \quad \text{in } B_r$$

then it holds that

$$\int_{B_r} |\nabla v|^p \phi(x)^p \, dx \le C(\Lambda, p) \int_{B_r} |v-k|^p |\nabla \phi(x)|^p \, dx$$

for all $\phi \in C_0^1(B_r)$, $\phi \ge 0$, and for all $k \in \mathbb{R}$.

Proof. Since $(v-k)\phi \in W_0^{1,p}(B_r)$, we can use it as a test function. From this, together with Hölder's inequality and Young's inequality, we can infer that

$$\begin{aligned}
\int_{B_r(x_0)} &\langle \boldsymbol{A}_0(x, \nabla v) - \boldsymbol{A}_0(x, 0), \nabla v \rangle \phi^p \, dx \\
&= -p \int_{B_r(x_0)} \langle \boldsymbol{A}_0(x, \nabla v), \nabla \phi \rangle (v-k) \phi^{p-1} \, dx \\
&\le C(\Lambda, p) \int_{B_r(x_0)} |\nabla v|^{p-1} \phi^{p-1} |\nabla \phi| \, |v-k| \, dx \\
&\le \frac{1}{4} \int_{B_r(x_0)} |\nabla v|^p \phi^p(x) \, dx + C(\Lambda, p) \int_{B_r(x_0)} |v-k|^p |\nabla \phi|^p \, dx.
\end{aligned}$$

Now, by Lemma 2.10, it follows that

$$\begin{aligned}
\int_{B_r(x_0)} |\nabla v|^p \phi^p \, dx &\le \frac{1}{4} \int_{B_r(x_0)} |\nabla v|^p \phi^p \, dx \\
&\qquad + C(\Lambda, p) \int_{B_r(x_0)} \langle \boldsymbol{A}_0(x, \nabla v) - \boldsymbol{A}_0(x, 0), \nabla v \phi^p \rangle \, dx \\
&\le \frac{1}{2} \int_{B_r(x_0)} |\nabla v|^p \phi^p \, dx + C(\Lambda, p) \int_{B_r(x_0)} |v-k|^p |\nabla \phi|^p \, dx.
\end{aligned}$$

Therefore,

$$\int_{B_r(x_0)} |\nabla v|^p \phi(x)^p \, dx \le C(\Lambda, p) \int_{B_r(x_0)} |v-k|^p |\nabla \phi(x)|^p \, dx,$$

as desired. □

A known approximation estimate. We recall a known approximation estimate established in [Byun and Wang 2012; Byun et al. 2007] and many other papers for the solutions of equations of the type (1-5) in which the vector field $\boldsymbol{A}_0$ is independent of the variable $z \in \mathbb{K}$. This approximation estimate will be used in an intermediate step for the proof of Theorem 1.1.

Lemma 2.12. *Let $\Lambda > 0$, $p > 1$ be fixed. Then for every $\epsilon \in (0, 1)$, there exists a sufficiently small number $\delta_0 = \delta_0(\epsilon, \Lambda, n, p) \in (0, \epsilon)$ such that the following holds. Assume that $\boldsymbol{A}_0 : B_{2R} \times \mathbb{R}^n \to \mathbb{R}^n$ is such that (1-2)–(1-3) hold, and*

$$\sup_{\substack{\xi \in \mathbb{R}^n \\ \xi \neq 0}} \sup_{\substack{x \in B_{2R} \\ 0 < \rho < R}} \frac{1}{|B_\rho(x)|} \int_{B_\rho(x)} \frac{|\boldsymbol{A}_0(y, \xi) - \overline{\boldsymbol{A}}_{0, B_\rho(x)}|}{|\xi|^{p-1}}\, dy \le \delta_0.$$

Then for every $x_0 \in B_R$ and $r \in (0, R/2)$, and for $\boldsymbol{G} \in L^p(B_{2R}, \mathbb{R}^n)$, if w is a weak solution in $W^{1,p}(B_{2r}(x_0))$ of

$$\operatorname{div}[\boldsymbol{A}_0(x, \nabla w)] = \operatorname{div}[|\boldsymbol{G}|^{p-2}\boldsymbol{G}] \quad \text{in } B_{2r}(x_0)$$

satisfying

$$\frac{1}{|B_{2r}(x_0)|} \int_{B_{2r}(x_0)} |\nabla w|^p\, dx \le 1,$$

and if

$$\frac{1}{|B_{2r}(x_0)|} \int_{B_{2r}(x_0)} |\boldsymbol{G}|^p\, dx \le \delta_0^p,$$

then there is $h \in W^{1,p}(B_{7r/4}(x_0))$ such that

$$\frac{1}{|B_{7r/4}(x_0)|} \int_{B_{7r/4}(x_0)} |\nabla w - \nabla h|^p\, dx \le \epsilon^p, \quad \|\nabla h\|_{L^\infty(B_{3r/2}(x_0))} \le C(\Lambda, n, p).$$

3. Interior approximation estimates

In this section, let $\boldsymbol{A} : B_{2R} \times \mathbb{K} \times \mathbb{R}^n \to \mathbb{R}^n$ satisfy (1-2)–(1-4) on $B_{2R} \times \mathbb{K} \times \mathbb{R}^n$ for some $R > 0$ and some open interval $\mathbb{K} \subset \mathbb{R}$. We study the weak solutions $u \in W^{1,p}(B_{2R})$ of the scaling parameter equation

$$\operatorname{div}[\boldsymbol{A}(x, \lambda u, \nabla u)] = \operatorname{div}[|\boldsymbol{F}|^{p-2}\boldsymbol{F}] \quad \text{in } B_{2R}, \tag{3-1}$$

with the parameter $\lambda \ge 0$. Our goal in this section is to provide necessary estimates for proving Theorem 1.1. Our approach is based on the perturbation technique introduced in [Caffarelli and Peral 1998] together with the "scaling parameter" technique introduced in [Hoang et al. 2015; Nguyen and Phan 2016]. The approach is also influenced by recent developments [Bögelein 2014; Byun and Wang 2012; Byun et al. 2007; 2017; Phan 2017]. In our first step, we fix u in $\boldsymbol{A}$ and then approximate the solution u of (3-1) by a solution of the corresponding homogeneous equations with the fixed u coefficient, as in [Bögelein 2014; Byun et al. 2017].

Lemma 3.1. *Let $\Lambda, M > 0$, $p > 1$ be fixed and $\kappa \in (0, 1]$. Then, for every small $\epsilon \in (0, 1)$, there exists a sufficiently small number $\delta_1 = \delta_1(\epsilon, \Lambda, n, p, \kappa) \in (0, \epsilon)$ such that the following holds. Assume that $\boldsymbol{A} : B_{2R} \times \mathbb{K} \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies (1-2)–(1-4) with some $\mathbb{K} \subset \mathbb{R}$ and some $R > 0$, and that $\boldsymbol{F} \in L^p(B_{2R}, \mathbb{R}^n)$ satisfies*

$$\fint_{B_r(x_0)} |\boldsymbol{F}|^p \, dx \le \delta_1^p$$

for some $x_0 \in B_R$ *and* $r \in (0, R)$. *Suppose also that* $u \in W^{1,p}(B_{2R})$ *is a weak solution of* (3-1) *satisfying*

$$\fint_{B_r(x_0)} |\nabla u|^p \, dx \le 1 \quad \textit{and} \quad \lambda \left(\fint_{B_r(x_0)} |u - \bar{u}_{B_r(x_0)}|^p \right)^{1/p} \le M$$

for some $\lambda \ge 0$. *Then*

$$\fint_{B_r(x_0)} |\nabla u - \nabla v|^p \, dx \le \epsilon^p \kappa^n, \tag{3-2}$$

where $v \in W^{1,p}(B_r)$ *is the weak solution of*

$$\begin{cases} \operatorname{div}[\boldsymbol{A}(x, \lambda u, \nabla v)] = 0 & \textit{in } B_r(x_0), \\ v = u - \bar{u}_{B_r(x_0)} & \textit{on } \partial B_r(x_0). \end{cases} \tag{3-3}$$

Moreover, it also holds that

$$\lambda \left(\fint_{B_r(x_0)} |v|^p \, dx \right)^{1/p} \le C(n, p)[M + \lambda r \epsilon \kappa^{n/p}]. \tag{3-4}$$

Proof. Note that for $\widetilde{\boldsymbol{A}}_0(x, \xi) := \boldsymbol{A}(x, \lambda u(x), \xi)$, we see that $\widetilde{\boldsymbol{A}}_0$ is independent of the variable $z \in \mathbb{K}$, and it satisfies the assumptions (1-2)–(1-3). The equation (3-3) is written as

$$\begin{cases} \operatorname{div}[\widetilde{\boldsymbol{A}}_0(x, \nabla v)] = 0 & \text{in } B_r(x_0), \\ v = u - \bar{u}_{B_r(x_0)} & \text{on } \partial B_r(x_0), \end{cases} \tag{3-5}$$

and we note that the existence of the weak solution v of (3-5) follows from the standard theory in calculus of variation. Therefore, it remains to prove the estimates (3-2) and (3-4). Since $v - [u - \bar{u}_{B_r(x_0)}] \in W_0^{1,p}(B_r(x_0))$, we can take it as a test function for (3-3); we obtain

$$\int_{B_r(x_0)} \langle \boldsymbol{A}(x, \lambda u, \nabla v), \nabla u - \nabla v \rangle \, dx = 0.$$

Similarly, we can use $v - [u - \bar{u}_{B_r(x_0)}]$ as a test function for the equation for (3-1) to see that

$$\int_{B_r(x_0)} \langle \boldsymbol{A}(x, \lambda u, \nabla u), \nabla u - \nabla v \rangle \, dx = \int_{B_r(x_0)} \langle |\boldsymbol{F}|^{p-2} \boldsymbol{F}, \nabla u - \nabla v \rangle \, dx.$$

It then follows from these two identities that

$$\begin{aligned} \int_{B_r(x_0)} \big\langle \boldsymbol{A}(x, \lambda u, \nabla u) - \boldsymbol{A}(x, \lambda u, \nabla v), \nabla v - \nabla u \big\rangle \, dx \\ = \int_{B_r(x_0)} \langle |\boldsymbol{F}|^{p-2} \boldsymbol{F}, \nabla u - \nabla v \rangle \, dx. \end{aligned} \tag{3-6}$$

We only consider the case $1 < p < 2$, because the case $p \geq 2$ is similar, and simpler. It follows from Lemma 2.10(i), Remark 2.1, and (3-6) that for each $\tau \in (0, 1)$,

$$\begin{aligned}
&\int_{B_r(x_0)} |\nabla u - \nabla v|^p \, dx \\
&\quad \leq \tau \int_{B_r(x_0)} |\nabla u|^p \, dx \\
&\qquad\qquad + C(\Lambda, \tau, p) \int_{B_r(x_0)} \big\langle \boldsymbol{A}(x, \lambda u, \nabla u) - \boldsymbol{A}(x, \lambda u, \nabla v), \nabla v - \nabla u \big\rangle \, dx \\
&\quad \leq \tau \int_{B_r(x_0)} |\nabla u|^p \, dx + C(\Lambda, \tau, p) \int_{B_r(x_0)} \big| \langle |\boldsymbol{F}|^{p-2} \boldsymbol{F}, \nabla u - \nabla v \rangle \big| \, dx \\
&\quad \leq \tau \int_{B_r(x_0)} |\nabla u|^p \, dx + \frac{1}{2} \int_{B_r(x_0)} |\nabla u - \nabla v|^p \, dx + C(\Lambda, \tau, p) \int_{B_r(x_0)} |\boldsymbol{F}|^p \, dx,
\end{aligned}$$

where in the last step, we have used Hölder's inequality and Young's inequality. Hence, by canceling similar terms, we obtain

$$\fint_{B_r(x_0)} |\nabla u - \nabla v|^p \, dx \leq 2\tau \fint_{B_r(x_0)} |\nabla u|^p \, dx + C(\Lambda, \tau, p) \fint_{B_r(x_0)} |\boldsymbol{F}|^p \, dx.$$

Now, choosing $\tau = \epsilon^p \kappa^n / 4$ and $\delta_1 = \delta_1(\epsilon, \Lambda, n, p, \kappa) \in (0, \epsilon)$ sufficiently small so that $C(\Lambda, \tau, p)\delta^p < \epsilon^p \kappa^n / 2$, the estimate (3-2) follows. It remains to prove (3-4). By Poincaré's inequality, we see that

$$\begin{aligned}
&\left(\fint_{B_r(x_0)} |v|^p \, dx \right)^{1/p} \\
&\quad \leq C(p) \left[\left(\fint_{B_r(x_0)} \big| v - [u - \bar{u}_{B_r(x_0)}] \big|^p \, dx \right)^{1/p} + \left(\fint_{B_r(x_0)} |u - \bar{u}_{B_r(x_0)}|^p \, dx \right)^{1/p} \right] \\
&\quad \leq C(n, p) \left[r \left(\fint_{B_r(x_0)} |\nabla v - \nabla u|^p \, dx \right)^{1/p} + \left(\fint_{B_r(x_0)} |u - \bar{u}_{B_r(x_0)}|^p \, dx \right)^{1/p} \right].
\end{aligned}$$

From this and since $\kappa \in (0, 1)$, it follows that

$$\lambda \left(\fint_{B_r(x_0)} |v|^p \, dx \right)^{1/p} \leq C(n, p)[M + r\lambda\epsilon\kappa^{n/p}],$$

as desired. □

Next, we approximate the solution u by the solution w of the following equation, whose principal part is a vector field that is independent of w and has small oscillation with respect to the x-variable:

$$\begin{cases} \operatorname{div}[\boldsymbol{A}(x, \lambda \bar{u}_{B_{\kappa r}(x_0)}, \nabla w)] = 0 & \text{in } B_{\kappa r}(x_0), \\ w = v & \text{on } \partial B_{\kappa r}(x_0), \end{cases} \tag{3-7}$$

where v is the weak solution of (3-3) and $\kappa \in \left(0, \frac{1}{3}\right)$ is sufficiently small to be determined. Our next result is in the same fashion as Lemma 3.1.

Lemma 3.2. *Let $\Lambda, M > 0$, $p > 1$, and $\alpha \in (0, 1]$ be fixed, and let $\epsilon \in (0, 1)$. There exist positive, sufficiently small numbers $\kappa = \kappa(\epsilon, \Lambda, M, p, n, \alpha) \in \left(0, \frac{1}{3}\right)$ and $\delta_2 = \delta_2(\epsilon, \Lambda, M, n, \alpha, p) \in (0, \epsilon)$ such that the following holds. Assume that $\boldsymbol{A} : B_{2R} \times \mathbb{K} \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies (1-2)–(1-4) with some $R > 0$ and some open interval $\mathbb{K} \subset \mathbb{R}$, and assume that $\boldsymbol{F} \in L^p(B_{2R}, \mathbb{R}^n)$ and*

$$\fint_{B_r(x_0)} |\boldsymbol{F}|^p \, dx \le \delta_2^p$$

for some $x_0 \in B_R$ and $r \in (0, R/2)$. Then, for every $\lambda > 0$, if $u \in W^{1,p}(B_{2R})$ is a weak solution of (3-1) satisfying

$$\fint_{B_{2\kappa r}(x_0)} |\nabla u|^p \, dx \le 1, \quad \fint_{B_r(x_0)} |\nabla u|^p \, dx \le 1, \quad \text{and} \quad [\![\lambda u]\!]_{\mathrm{BMO}(B_R, R)} \le M,$$

then it holds that

$$\left(\fint_{B_{\kappa r}(x_0)} |\nabla u - \nabla w|^p \, dx\right)^{1/p} \le \epsilon \quad \text{and} \quad \left(\fint_{B_{\kappa r}} |\nabla w|^p \, dx\right)^{1/p} \le C_0(n, p), \tag{3-8}$$

where w is the weak solution of (3-7).

Proof. For a given sufficiently small $\epsilon > 0$, let $\epsilon' \in (0, \epsilon/2)$ and $\kappa \in \left(0, \frac{1}{3}\right)$, both sufficiently small and depending on $\epsilon, \Lambda, M, n, \alpha, p$, which will be determined. Then, let $\delta_2 = \delta_1(\epsilon', \Lambda, n, p, \kappa) > 0$, where δ_1 is defined as in Lemma 3.1. Let v be the solution of (3-3). By using Lemma 3.1, we see that

$$\begin{aligned} &\fint_{B_r(x_0)} |\nabla u - \nabla v|^p \, dx \le (\epsilon')^p \kappa^n, \\ &\lambda \left(\fint_{B_r(x_0)} |v|^p \, dx\right)^{1/p} \le C(n, p)[r\epsilon'\lambda\kappa^{n/p} + M]. \end{aligned} \tag{3-9}$$

Observe also that the first inequality in (3-9), the assumption in the lemma, and the fact that both ϵ and κ are small imply that

$$\begin{aligned} \left(\fint_{B_{2\kappa r}(x_0)} |\nabla v|^p \, dx\right)^{1/p} &\le \left(\fint_{B_{2\kappa r}(x_0)} |\nabla u - \nabla v|^p \, dx\right)^{1/p} + \left(\fint_{B_{2\kappa r}(x_0)} |\nabla u|^p \, dx\right)^{1/p} \\ &\le \left(\frac{1}{2^n \kappa^n} \fint_{B_r(x_0)} |\nabla u - \nabla v|^p \, dx\right)^{1/p} + \left(\fint_{B_{2\kappa r}(x_0)} |\nabla u|^p \, dx\right)^{1/p} \\ &\le \frac{\epsilon'}{2^{n/p}} + 1 \le 2. \end{aligned} \tag{3-10}$$

On the other hand, from the Caccioppoli type estimate in Lemma 2.11, (3-9), and $\kappa \in \left(0, \frac{1}{3}\right)$, we also see that

$$\left(\frac{1}{|B_{2\kappa r}(x_0)|}\int_{B_{2\kappa r}(x_0)}|\nabla v|^p\,dx\right)^{1/p}\le\frac{C(\Lambda,n,p)}{(1-2\kappa)r\kappa^{n/p}}\left(\frac{1}{|B_r(x_0)|}\int_{B_r(x_0)}|v|^p\,dx\right)^{1/p}\le C(\Lambda,n,p)\big[\epsilon'+M(\lambda\kappa^{n/p}r)^{-1}\big]. \tag{3-11}$$

Now, let w be the weak solution of (3-7). As in the proof of Lemma 3.1, the existence of w is assured. Therefore, it remains to prove the estimate (3-8). Taking $w-v\in W_0^{1,p}(B_{\kappa r}(x_0))$ as a test function for (3-7) and (3-3), we obtain

$$\int_{B_{\kappa r}(x_0)}\langle \boldsymbol{A}(x,\lambda u,\nabla v),\nabla w-\nabla v\rangle\,dx=\int_{B_{\kappa r}(x_0)}\langle \boldsymbol{A}(x,\lambda\bar{u}_{B_{\kappa r}(x_0)},\nabla w),\nabla w-\nabla v\rangle\,dx=0. \tag{3-12}$$

Again, we only need to consider the case $1<p<2$, as $p\ge 2$ can be done similarly using (ii) of Lemma 2.10. From now on, for simplicity, we write $\hat{u}=u-\bar{u}_{B_{\kappa r}(x_0)}$. We can use Lemma 2.10(i), the condition (1-4), and (3-12) to obtain, with some $\tau>0$ sufficiently small to be determined,

$$\begin{aligned}
&\int_{B_{\kappa r}(x_0)}|\nabla v-\nabla w|^p\,dx\\
&\quad\le\tau\int_{B_{\kappa r}(x_0)}|\nabla v|^p\,dx+\Bigg(C(\Lambda,p)\tau^{(p-2)/p}\\
&\qquad\times\int_{B_{\kappa r}(x_0)}\big\langle \boldsymbol{A}(x,\lambda\bar{u}_{B_{\kappa r}(x_0)},\nabla v)-\boldsymbol{A}(x,\lambda\bar{u}_{B_{\kappa r}(x_0)},\nabla w),\nabla v-\nabla w\big\rangle\,dx\Bigg)\\
&\quad\le\tau\int_{B_{\kappa r}(x_0)}|\nabla v|^p\,dx+\Bigg(C(\Lambda,p)\tau^{(p-2)/p}\\
&\qquad\times\int_{B_{\kappa r}(x_0)}\big\langle \boldsymbol{A}(x,\lambda\bar{u}_{B_{\kappa r}(x_0)},\nabla v)-\boldsymbol{A}(x,\lambda u,\nabla v),\nabla v-\nabla w\big\rangle\,dx\Bigg)\\
&\quad\le\tau\int_{B_{\kappa r}(x_0)}|\nabla v|^p\,dx+C(\Lambda,p)\tau^{(p-2)/p}\int_{B_{\kappa r}(x_0)}|\lambda\hat{u}|^{\alpha}|\nabla v|^{p-1}|\nabla v-\nabla w|\,dx\\
&\quad\le\frac{1}{2}\int_{B_{\kappa r}(x_0)}|\nabla v-\nabla w|^p\,dx+\tau\int_{B_{\kappa r}(x_0)}|\nabla v|^p\,dx\\
&\qquad+C(\Lambda,p)\tau^{(p-2)/(p-1)}\int_{B_{\kappa r}(x_0)}|\lambda\hat{u}|^{\alpha p/(p-1)}|\nabla v|^p\,dx,
\end{aligned}$$

where in the last step, we have used the Hölder's inequality and Young's inequality. Hence, by canceling similar terms, we obtain

$$\begin{aligned}
&\frac{1}{|B_{\kappa r}(x_0)|}\int_{B_{\kappa r}(x_0)}|\nabla v-\nabla w|^p\,dx\\
&\quad\le\frac{2\tau}{|B_{\kappa r}(x_0)|}\int_{B_{\kappa r}(x_0)}|\nabla v|^p\,dx+\frac{C(\Lambda,p)\tau^{(p-2)/(p-1)}}{|B_{\kappa r}(x_0)|}\int_{B_{\kappa r}(x_0)}|\lambda\hat{u}|^{\alpha p/(p-1)}|\nabla v|^p\,dx.
\end{aligned}$$

For q_1 greater than but sufficiently close to p and depending only on Λ, p, we write

$$q_1 = \frac{\alpha p p_1}{(p-1)(p_1-p)} > p.$$

Using Hölder's inequality, the self-improving regularity estimate (i.e., Lemma 2.9), and (3-10), we then obtain

$$\begin{aligned}
&\frac{1}{|B_{\kappa r}(x_0)|}\int_{B_{\kappa r}(x_0)} |\nabla v - \nabla w|^p\,dx \\
&\quad\le \frac{2\tau}{|B_{\kappa r}(x_0)|}\int_{B_{\kappa r}(x_0)} |\nabla v|^p\,dx + \Bigg[C(\Lambda, p)\tau^{(p-2)/(p-1)} \\
&\qquad\times\left(\frac{1}{|B_{\kappa r}(x_0)|}\int_{B_{\kappa r}(x_0)} |\lambda\hat{u}|^{q_1}\right)^{(p_1-p)/p_1}\left(\frac{1}{|B_{\kappa r}(x_0)|}\int_{B_{\kappa r}(x_0)} |\nabla v|^{p_1}\,dx\right)^{p/p_1}\Bigg] \\
&\quad\le C(\Lambda, n, p)\Bigg[2\tau + \tau^{(p-2)/(p-1)}\left(\frac{1}{|B_{\kappa r}(x_0)|}\int_{B_{\kappa r}(x_0)} |\lambda\hat{u}|^{q_1}\,dx\right)^{(p_1-p)/p_1}\Bigg] \\
&\qquad\qquad\times\left(\frac{1}{|B_{2\kappa r}(x_0)|}\int_{B_{2\kappa r}(x_0)} |\nabla v|^p\,dx\right).
\end{aligned}$$

Now, from the well-known John–Nirenberg theorem, we further write

$$\begin{aligned}
&\frac{1}{|B_{\kappa r}(x_0)|}\int_{B_{\kappa r}(x_0)} |\lambda\hat{u}|^{q_1}\,dx \\
&\quad= \frac{1}{|B_{\kappa r}(x_0)|}\int_{B_{\kappa r}(x_0)} |\lambda\hat{u}|^{p/2}|\lambda\hat{u}|^{q_1-p/2}\,dx \\
&\quad\le \left(\frac{1}{|B_{\kappa r}(x_0)|}\int_{B_{\kappa r}(x_0)} |\lambda\hat{u}|^{p}\,dx\right)^{1/2}\left(\frac{1}{|B_{\kappa r}(x_0)|}\int_{B_{\kappa r}(x_0)} |\lambda\hat{u}|^{2q_1-p}\,dx\right)^{1/2} \\
&\quad\le C(n, \alpha, p)[\![\lambda u]\!]_{\mathrm{BMO}(B_R,R)}^{q_1-p/2}\left(\frac{1}{|B_{\kappa r}(x_0)|}\int_{B_{\kappa r}(x_0)} |\lambda\hat{u}|^{p}\,dx\right)^{1/2} \\
&\quad= C(n, M, \alpha, p)\left(\frac{1}{|B_{\kappa r}(x_0)|}\int_{B_{\kappa r}(x_0)} |\lambda\hat{u}|^{p}\,dx\right)^{1/2}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\frac{1}{|B_{\kappa r}(x_0)|}\int_{B_{\kappa r}(x_0)} |\nabla v - \nabla w|^p\,dx \\
&\quad\le C(\Lambda, n, \alpha, p)\Bigg[2\tau + \tau^{(p-2)/(p-1)}\left(\frac{1}{|B_{\kappa r}(x_0)|}\int_{B_{\kappa r}(x_0)} |\lambda\hat{u}|^{p}\,dx\right)^{(p_1-p)/2p_1}\Bigg] \\
&\qquad\qquad\times\left(\frac{1}{|B_{2\kappa r}(x_0)|}\int_{B_{2\kappa r}(x_0)} |\nabla v|^p\,dx\right).
\end{aligned} \tag{3-13}$$

From (3-11) and $[\![\lambda u]\!]_{\mathrm{BMO}(B_{2R},R)} \le M$, we can take $\tau = \frac{1}{2}$ in (3-13) to obtain in particular

$$\begin{aligned}\left(\frac{1}{|B_{\kappa r}(x_0)|}\int_{B_{\kappa r}(x_0)} |\nabla v - \nabla w|^p\,dx\right)^{1/p} &\\ &\le C(\Lambda, M, n, \alpha, p)\left(\frac{1}{|B_{2\kappa r}(x_0)|}\int_{B_{2\kappa r}(x_0)} |\nabla v|^p\,dx\right)^{1/p}\\ &\le C_1(\Lambda, M, n, p)\big[\epsilon' + M(r\kappa^{n/p}\lambda)^{-1}\big].\end{aligned}$$

Hence, if $\dfrac{\epsilon\kappa^{n/p}\lambda r}{4MC_1(\Lambda, M, n, p)} \ge 1$, we choose ϵ' sufficiently small so that

$$C_1(\Lambda, n, p)\epsilon' < \frac{\epsilon}{4}.$$

Then

$$\left(\frac{1}{|B_{\kappa r}(x_0)|}\int_{B_{\kappa r}(x_0)} |\nabla v - \nabla w|^p\,dx\right)^{1/p} \le \frac{\epsilon}{2}.$$

From this, the first estimate in (3-9), and the triangle inequality, the first estimate of (3-8) follows. Therefore, it remains to consider the case

$$\lambda\kappa^{n/p} r\epsilon \le 4MC_1(\Lambda, M, n, p). \tag{3-14}$$

In this case, we first note that from our choice that $\epsilon' \le \epsilon$, we particularly have

$$\lambda\kappa^{n/p}\epsilon' r \le C(\Lambda, M, n, p).$$

Then, it follows from the second estimate in (3-9) that

$$\lambda\left(\frac{1}{|B_r(x_0)|}\int_{B_r(x_0)} |v|^p\,dx\right)^{1/p} \le C(\Lambda, M, n, p).$$

On the other hand, from (3-3), and the scaling invariances discussed at the beginning of Section 2, we observe that $\tilde{v}(x) = \lambda v(x - x_0)$ is a weak solution of

$$\operatorname{div}[\hat{\boldsymbol{A}}_0(x, \nabla\tilde{v})] = 0 \quad \text{in } B_r,$$

where $\hat{\boldsymbol{A}}_0(x, \xi) = \lambda^{p-1}\boldsymbol{A}(x - x_0, \lambda u(x - x_0), \lambda^{-1}\xi)$ for all $x \in B_r$, $\xi \in \mathbb{R}^n$. From this and Remark 2.1, we can apply Hölder's regularity theory in Lemma 2.8 for the solution $\tilde{v}$ to find that there is $\beta \in (0, 1)$ depending only on Λ, M, n, p such that

$$\begin{aligned}&\lambda\|v\|_{L^\infty(B_{5r/6}(x_0))} \le C(\Lambda, M, n, p),\\ &\lambda|v(x) - v(y)| \le C(\Lambda, M, p, n)\kappa^\beta \quad \forall x, y \in \bar{B}_{\kappa r}(x_0).\end{aligned} \tag{3-15}$$

The estimate (3-15), (3-10), and (3-13) imply that

$$\begin{aligned}(3\text{-}16)\quad &\frac{1}{|B_{\kappa r}(x_0)|}\int_{B_{\kappa r}(x_0)}|\nabla v-\nabla w|^p\,dx\\ &\le C(\Lambda,M,n,\alpha,p)\\ &\quad\times\Big[2\tau+\tau^{(p-2)/(p-1)}\Big(\frac{1}{|B_{\kappa r}(x_0)|}\int_{B_{\kappa r}(x_0)}|\lambda\hat{u}|^p\,dx\Big)^{(p_1-p)/2p_1}\Big].\end{aligned}$$

On the other hand, for $v'=v+\bar{u}_{B_{\kappa r}}$, we can write

$$\begin{aligned}&\frac{1}{|B_{\kappa r}(x_0)|}\int_{B\kappa r(x_0)}|\lambda\hat{u}|^p\,dx\\ &\le C(p)\Big[\frac{1}{|B_{\kappa r}(x_0)|}\int_{B_{\kappa r}(x_0)}|\lambda(u-v')|^p\,dx+\frac{1}{|B_{\kappa r}(x_0)|}\int_{B_{\kappa r}(x_0)}|\lambda(v'-\bar{v}'_{B_{\kappa r}(x_0)})|^p\,dx\\ &\qquad+\frac{1}{|B_{\kappa r}(x_0)|}\int_{B_{\kappa r}(x_0)}|\lambda(\bar{u}_{B_{\kappa r}(x_0)}-\bar{v}'_{B_{\kappa r}(x_0)})|^p\,dx\Big]\\ &\le C(n,p)\Big[\frac{1}{\kappa^n|B_r(x_0)|}\int_{B_r(x_0)}|\lambda(u-v')|^p\,dx\\ &\qquad+\frac{1}{|B_{\kappa r}(x_0)|}\int_{B_{\kappa r}(x_0)}|\lambda(v-\bar{v}_{B_{\kappa r}(x_0)})|^p\,dx\Big].\end{aligned}$$

Since $u-v'\in W_0^{1,2}(B_r(x_0))$, we can use Poincaré's inequality for the first term in the right-hand side of the last estimate to obtain

$$\begin{aligned}&\Big(\frac{1}{|B_{\kappa r}(x_0)|}\int_{B_{\kappa r}(x_0)}|\lambda\hat{u}|^p\,dx\Big)^{1/p}\\ &\qquad\le C(\Lambda,n,p)\Big[\frac{\lambda r}{\kappa^{n/p}}\Big(\frac{1}{|B_r(x_0)|}\int_{B_r(x_0)}|\nabla u-\nabla v|^p\,dx\Big)^{1/p}\\ &\qquad\qquad+\lambda\sup_{x,y\in\bar{B}_{\kappa r}(x_0)}|v(x)-v(y)|\Big].\end{aligned}$$

From this estimate, (3-9), and (3-15), we infer that

$$\Big(\frac{1}{|B_{\kappa r}(x_0)|}\int_{B_{\kappa r}(x_0)}|\lambda\hat{u}|^p\,dx\Big)^{1/p}\le C(\Lambda,p,n)\big[\lambda r\epsilon'+\kappa^\beta\big].$$

From this, we can control the estimate in (3-16) as

$$\begin{aligned}&\frac{1}{|B_{\kappa r}(x_0)|}\int_{B_{\kappa r}(x_0)}|\nabla v-\nabla w|^p\,dx\\ &\qquad\le C(\Lambda,M,n,\alpha,p)\big[2\tau+\tau^{(p-2)/(p-1)}(\lambda r\epsilon'+\kappa^\beta)^{p(p_1-p)/2p_1}\big].\end{aligned}$$

Then, combining this last estimate with (3-14), we obtain

$$\frac{1}{|B_{\kappa r}(x_0)|}\int_{B_{\kappa r}(x_0)}|\nabla v-\nabla w|^p\,dx \le C_2(\Lambda, M, \alpha, p, n)\left[\tau+\tau^{(p-2)/(p-1)}\left(\frac{\epsilon'}{\epsilon\kappa^{n/p}}+\kappa^\beta\right)^{p(p_1-p)/2p_1}\right].$$

We firstly choose $\tau > 0$ so that

$$C_2(\Lambda, M, n, \alpha, p)\tau = \frac{1}{2}\left(\frac{\epsilon}{2}\right)^p.$$

Next, we choose κ sufficiently small depending only on Λ, n, α, p, and ϵ so that

$$\kappa^\beta \le \frac{1}{2}\left[\frac{(\epsilon/2)^p}{4C_2(\Lambda, M, p, \alpha, n)\tau^{(p-2)/(p-1)}}\right]^{2p_1/p(p_1-p)},$$

and finally we choose $\epsilon' \in (0, \epsilon/2)$ sufficiently small so that

$$\epsilon' \le \frac{\kappa^{n/p}\epsilon}{2}\left[\frac{(\epsilon/2)^p}{4C_2(\Lambda, M, p, \alpha, n)\tau^{(p-2)/(p-1)}}\right]^{2p_1/p(p_1-p)}.$$

From these choices, it follows that

$$\left(\frac{1}{|B_{\kappa r}(x_0)|}\int_{B_{\kappa r}(x_0)}|\nabla v-\nabla w|^p\,dx\right)^{1/p} \le \frac{\epsilon}{2}.$$

The first estimate (3-8) then holds thanks to this estimate, the first estimate in (3-9), and the triangle inequality.

Finally, to complete the proof, it remains to verify the second estimate of (3-8). By using the triangle inequality, the assumption of the lemma, and the fact that $\epsilon \in (0, 1)$, we see that

$$\begin{aligned}\left(\fint_{B_{\kappa r}(x_0)}|\nabla w|^p\,dx\right)^{1/p} &\le \left(\fint_{B_{\kappa r}(x_0)}|\nabla w-\nabla u|^p\,dx\right)^{1/p}+\left(\fint_{B_{\kappa r}(x_0)}|\nabla u|^p\,dx\right)^{1/p}\\ &\le \epsilon+\left(2^n\fint_{B_{2\kappa r}(x_0)}|\nabla u|^p\,dx\right)^{1/p}\\ &\le \epsilon+2^{n/p}\le 1+2^{n/p}=C_0(n, p).\end{aligned}$$

The proof is therefore complete. □

Summarizing these efforts, we can state and prove the main result of the section.

Proposition 3.3. *Let $\Lambda > 0$, $p > 1$, and $\alpha \in (0, 1]$ be fixed. Then, for every $\epsilon \in (0, 1)$, there exist sufficiently small numbers $\kappa = \kappa(\epsilon, \Lambda, M, p, n, \alpha) \in \left(0, \frac{1}{2}\right]$ and $\delta = \delta(\epsilon, \Lambda, M, \alpha, n, p) \in (0, \epsilon)$ such that the following holds. Assume that $\boldsymbol{A} : B_{2R}\times\mathbb{K}\times\mathbb{R}^n \to \mathbb{R}^n$ such that (1-2)–(1-4) and (1-8) hold for some $R > 0$ and*

some open interval $\mathbb{K} \subset \mathbb{R}$, *and assume that*

$$\fint_{B_{2r}(x_0)} |\boldsymbol{F}|^p \, dx \le \delta^p$$

for some $x_0 \in \overline{B}_R$ *and some* $r \in (0, R/2)$. *Then for every* $\lambda \ge 0$, *if* $u \in W^{1,p}(B_{2R})$ *is a weak solution of* (3-1) *satisfying*

$$\fint_{B_{4\kappa r}(x_0)} |\nabla u|^p \, dx \le 1, \qquad \fint_{B_{2r}(x_0)} |\nabla u|^p \, dx \le 1, \quad \text{and} \quad [\![\lambda u]\!]_{\mathrm{BMO}(B_R,R)} \le M,$$

then there is $h \in W^{1,p}(B_{7\kappa r/2}(x_0))$ *such that*

$$\fint_{B_{7\kappa r/2}(x_0)} |\nabla u - \nabla h|^p \, dx \le \epsilon^p, \qquad \|\nabla h\|_{L^\infty(B_{3\kappa r}(x_0))} \le C(\Lambda, n, p). \tag{3-17}$$

Proof. For given ϵ, let

$$\delta = \min\{\delta_0(\epsilon/[2C_0(n,p)], \Lambda, n, p), \delta_2(\epsilon/2, \Lambda, M, \alpha, p)\},$$

where δ_0 is defined in Lemma 2.12, δ_2 is defined in Lemma 3.2, and $C_0(n,p) > 1$ is a constant defined in (3-8). We now prove our Lemma 3.2 with this choice of δ, κ. Note that since both numbers δ_0, δ_2 are independent of λ, so are δ, κ. If $\lambda = 0$, then our proposition follows directly from Lemma 2.12 with $\boldsymbol{G}$ replaced by $\boldsymbol{F}$ and for $\kappa = \frac{1}{2}$. Also, when $\lambda > 0$, let κ be a number defined as in Lemma 3.2. Then our proposition follows directly by applying Lemma 3.2 with r replaced by $2r$, Lemma 2.12 with $\boldsymbol{G} = 0$ and r replaced by $2\kappa r$, and the triangle inequality. □

4. Level set estimates and proof of Theorem 1.1

Level set estimates. Recall that the Hardy–Littlewood maximal function $\mathcal{M}(f)$ is defined in (2-3), and $\mathcal{M}_U(f) = \mathcal{M}(f\chi_U)$ for an open set U and its characteristic function χ_U. Our first result of this subsection is the following important lemma on the density of the level sets of a solution u of (3-1).

Lemma 4.1. *Let* Λ, M *be positive numbers,* $p, \gamma > 1, \alpha \in (0, 1]$, *and let* $\epsilon > 0$ *be sufficiently small. Then there exist a sufficiently large number* $N = N(\Lambda, n, p) \ge 1$ *and two positive sufficiently small numbers* $\kappa = \kappa(\epsilon, \Lambda, M, p, n, \gamma, \alpha) \in \left(0, \frac{1}{2}\right]$ *and* $\delta = \delta(\epsilon, \Lambda, M, p, n, \gamma, \alpha) \in (0, \epsilon)$ *such that the following statement holds. Suppose that* $\boldsymbol{A} : B_{2R} \times \mathbb{K} \times \mathbb{R}^n \to \mathbb{R}^n$ *such that* (1-2)–(1-4) *and* (1-8) *hold for some* $R > 0$ *and some open interval* $\mathbb{K} \subset \mathbb{R}$. *Suppose also that* $u \in W^{1,p}(B_{2R})$ *is a weak solution of* (3-1) *satisfying* $[\![\lambda u]\!]_{\mathrm{BMO}(B_R,R)} \le M$ *with some* $\lambda \ge 0$. *If* $y \in B_R$ *and* $\rho \in (0, \kappa_0)$ *such that*

$$B_\rho(y) \cap \left\{B_R : \mathcal{M}_{B_{2R}}(|\nabla u|^p) \le 1\right\} \cap \left\{B_R : \mathcal{M}_{B_{2R}}(|\boldsymbol{F}|^p) \le \delta^p\right\} \ne \varnothing$$

for $\kappa_0 = \min\{1, R\}\kappa/6$, *then*

$$\omega\big(\{x \in B_R : \mathcal{M}_{B_{2R}}(|\nabla u|^p) > N\} \cap B_\rho(y)\big) \le \epsilon\omega(B_\rho(y)) \tag{4-1}$$

for $\omega \in A_q$ *with* $[\omega]_{A_q} \le \gamma$ *and* $q > 1$.

Proof. The proof is standard using Proposition 3.3. However, as Proposition 3.3 is stated differently compared to the other similar approximation estimates in the literature, details of the proof of this lemma are required. For a given $\epsilon > 0$, let $\epsilon' > 0$ be a positive number to be determined depending only on ϵ, Λ, n, p, and γ. Then let $\kappa = \kappa(\epsilon', \Lambda, M, p, n, \alpha)$ and $\delta = \delta(\epsilon', \Lambda, M, p, n, \alpha)$ be the numbers defined in Proposition 3.3. We prove the lemma with this choice of δ, κ. By the assumption, we can find

$$x_0 \in B_\rho(y) \cap \{B_R : \mathcal{M}_{B_{2R}}(|\nabla u|^p) \le 1\} \cap \{B_R : \mathcal{M}_{B_{2R}}(|\boldsymbol{F}|^p) \le \delta^p\}. \tag{4-2}$$

Let $r = \kappa^{-1}\rho \in (0, R/6)$. Since $\rho \in (0, \kappa_0)$ and κ is sufficiently small, we have $B_{4r}(y) \subset B_{5r}(x_0) \subset B_{2R}$. From this and (4-2), it follows that

$$\fint_{B_{4r}(y)} |\nabla u|^p\,dx \le \frac{|B_{5r}(x_0)|}{|B_{4r}(y)|} \fint_{B_{5r}(x_0)} |\nabla u|^p\,dx \le \Big(\frac{5}{4}\Big)^n,$$
$$\fint_{B_{4r}(y)} |\boldsymbol{F}|^p\,dx \le \frac{|B_{5r}(x_0)|}{|B_{4r}(y)|} \fint_{B_{5r}(x_0)} |\boldsymbol{F}|^p\,dx \le \Big(\frac{5}{4}\Big)^n \delta^p.$$

Moreover, we also have $B_{8\rho}(y) \subset B_{9\rho}(x_0) \subset B_{2R}$ and therefore

$$\fint_{B_{8\kappa r}(y)} |\nabla u|^p\,dx = \fint_{B_{8\rho}(y)} |\nabla u|^p\,dx \le \frac{|B_{9\rho}(x_0)|}{|B_{8\rho}(y)|} \fint_{B_{9\rho}(x_0)} |\nabla u|^p\,dx \le \Big(\frac{9}{8}\Big)^n.$$

Hence, all conditions in Proposition 3.3 are satisfied with some suitable scaling. From this and our choice of κ, δ, we can apply Proposition 3.3 to find a function $h \in W^{1,p}(B_{7\rho/2}(y))$ satisfying

$$\fint_{B_{7\rho/2}(y)} |\nabla u - \nabla h|^p\,dx \le (\epsilon')^p \Big(\frac{3}{2}\Big)^n, \qquad \|\nabla h\|_{L^\infty(B_{3\rho}(y))} \le C_*(\Lambda, n, p),$$

where in the above estimates, we have used the fact that $\kappa r = \rho$. Let us now denote

$$N = \max\{2^p C_*^p, 2^n\}.$$

We prove (4-1) with this choice of N. To this end, we firstly prove that

$$\{x \in B_\rho(y) : \mathcal{M}_{B_{7\rho/2}(y)}(|\nabla u - \nabla h|^p)(x) \le C_*^p\} \subset \{x \in B_\rho(y) : \mathcal{M}_{B_{2R}}(|\nabla u|^p)(x) \le N\}. \tag{4-3}$$

To prove this statement, let x be a point in the set on the left side of (4-3). We verify that

$$\mathcal{M}_{B_{2R}}(|\nabla u|^p)(x) \le N. \tag{4-4}$$

Let $\rho' > 0$ be any number. If $\rho' < 2\rho$, then $B_{\rho'}(x) \subset B_{3\rho}(y) \subset B_{2R}$, and it follows that

$$\begin{aligned}\Big(\fint_{B_{\rho'}(x)} |\nabla u(z)|^p\,dz\Big)^{1/p} &\le \Big(\fint_{B_{\rho'}(x)} |\nabla u(z) - \nabla h(z)|^p\,dz\Big)^{1/p} + \Big(\fint_{B_{\rho'}(x)} |\nabla h(z)|^p\,dz\Big)^{1/p}\\ &\le \big(\mathcal{M}_{B_{7\rho/2}(y)}(|\nabla u - \nabla h|^p)(x)\big)^{1/p} + \|\nabla h\|_{L^\infty(B_{3\rho}(y))} \le 2C_* \le N^{1/p}.\end{aligned}$$

On the other hand, if $\rho' \ge 2\rho$, we note that $B_{\rho'}(x) \subset B_{2\rho'}(x_0)$, and it follows from this and (4-2) that

$$\begin{aligned}\frac{1}{|B_{\rho'}(x)|}\int_{B_{\rho'}(x)\cap B_{2R}} |\nabla u(z)|^p\,dz &\le \frac{|B_{2\rho'}(x_0)|}{|B_{\rho'}(x)|}\frac{1}{|B_{2\rho'}(x_0)|}\int_{B_{2\rho'}(x_0)\cap B_{2R}} |\nabla u(z)|^p\,dz \le 2^n \le N.\end{aligned}$$

Hence, (4-4) is verified and therefore (4-3) is proved. Observe that (4-3) is in fact equivalent to

$$\begin{aligned}&\big\{x \in B_\rho(y) : \mathcal{M}_{B_{2R}}(|\nabla u|^p)(x) > N\big\}\\ &\qquad\subset E := \big\{x \in B_\rho(y) : \mathcal{M}_{B_{7\rho/2}(y)}(|\nabla u - \nabla h|^p)(x) > C_*^p\big\}.\end{aligned} \tag{4-5}$$

On the other hand, from the weak type (1, 1) estimate of the Hardy–Littlewood maximal function (see Lemma 2.6), it is true that

$$\frac{|E|}{|B_\rho(y)|} \le \frac{C(n)}{C_*^p}\fint_{B_{7\rho/2}(y)} |\nabla u - \nabla h|^p\,dz \le C_1(\Lambda, n, p)(\epsilon')^p.$$

From this and the doubling property of A_q-weights as in (ii) of Lemma 2.4, it follows that

$$\frac{\omega(E)}{\omega(B_\rho(y))} \le C(n,\gamma)\Big(\frac{|E|}{|B_\rho(y)|}\Big)^\beta \le C'(\Lambda, n, p, \gamma)(\epsilon')^{p\beta}$$

for some $\beta = \beta(\gamma, n) > 0$. Therefore, by choosing ϵ' depending on $\epsilon, \Lambda, n, p, \gamma$ such that

$$C'(\Lambda, n, p, \gamma)(\epsilon')^{p\beta} = \epsilon,$$

we obtain

$$\omega(E) \le \epsilon\omega(B_\rho(y)).$$

From this estimate and the definition of E in (4-5), the estimate (4-1) follows and the proof is complete. □

The following level set estimate is a direct corollary of Lemma 4.1 and Lemma 2.7, which is also the main result of this subsection.

Lemma 4.2. *Let Λ, M be positive numbers, $p, \gamma > 1$, $\alpha \in (0, 1]$, and let $\epsilon > 0$ be sufficiently small. Then there exist a sufficiently large number $N = N(\Lambda, n, p) \geq 1$ and a sufficiently small number $\delta = \delta(\epsilon, \Lambda, M, p, n, \alpha) \in (0, \epsilon)$ such that the following statement holds. Assume that $\boldsymbol{A} : B_{2R} \times \mathbb{K} \times \mathbb{R}^n \to \mathbb{R}^n$ is such that (1-2)–(1-4) and (1-8) hold for some $R > 0$ and some open interval $\mathbb{K} \subset \mathbb{R}$. Suppose also that for any $\lambda \geq 0$, if $u \in W^{1,p}(B_{2R})$ is a weak solution of (3-1) satisfying*

$$\begin{aligned} &[\![\lambda u]\!]_{\mathrm{BMO}(B_R, R)} \leq M, \\ &\omega(\{B_R : \mathcal{M}_{B_{2R}}(|\nabla u|^p) > N\}) \leq \epsilon\omega(B_{\kappa_0}(y)) \quad \forall y \in \bar{B}_R, \end{aligned} \tag{4-6}$$

for some $\omega \in A_q$ with $q > 1$ and $[\omega]_{A_q} \leq \gamma$, then

$$\begin{aligned} &\omega\big(\big\{B_R : \mathcal{M}_{B_{2R}}(|\nabla u|^p) > N\big\}\big) \\ &\quad \leq \epsilon_1\big[\omega\big(\big\{B_R : \mathcal{M}_{B_{2R}}(|\nabla u|^p) > 1\big\}\big) + \omega\big(\big\{B_R : \mathcal{M}_{B_{2R}}(|\boldsymbol{F}|^p) > \delta^p\big\}\big)\big], \end{aligned} \tag{4-7}$$

with ϵ_1 as defined in Lemma 2.7 and κ_0 as defined in Lemma 4.1.

Proof. Let N, κ_0, δ be defined as in Lemma 4.1. We apply Lemma 2.7 with

$$C = \big\{x \in B_R : \mathcal{M}_{B_{2R}}(|\nabla u|^p)(x) > N\big\}$$

and

$$D = \big\{x \in B_R : \mathcal{M}_{B_{2R}}(|\nabla u|^p)(x) > 1\big\} \cup \big\{x \in B_R : \mathcal{M}_{B_{2R}}(|\boldsymbol{F}|^p)(x) > \delta^p\big\}.$$

Observe that by the second condition in (4-6), (i) of Lemma 2.7 is satisfied. On the other hand, by Lemma 4.1, (ii) of Lemma 2.7 also holds true. Therefore, both conditions of Lemma 2.7 are valid, and (4-7) follows directly from Lemma 2.7. □

Proof of the interior $W^{1,q}$-regularity estimates. From Lemma 4.2 and an iterating procedure, we obtain the following lemma:

Lemma 4.3. *Let Λ, M, p, α, ϵ, N, δ, κ, κ_0 and $\boldsymbol{A}$, R be as in Lemma 4.2. Then, for any $\lambda \geq 0$, if $u \in W^{1,p}(B_{2R})$ is a weak solution of (3-1) satisfying*

$$[\![\lambda u]\!]_{\mathrm{BMO}(B_R, R)} \leq M \quad \text{and} \quad \omega\big(\big\{B_R : \mathcal{M}_{B_{2R}}(|\nabla u|^p) > N\big\}\big) \leq \epsilon\omega(B_{\kappa_0}(y)) \quad \forall y \in \bar{B}_R$$

for some $\omega \in A_q$ with $q > 1$ and $[\omega]_{A_q} \leq \gamma$, then with ϵ_1 defined as in Lemma 2.7, and for any $k \in \mathbb{N}$, the following estimate holds:

$$\begin{aligned} &\omega\big(\big\{B_R : \mathcal{M}_{B_{2R}}(|\nabla u|^p) > N^k\big\}\big) \\ &\quad \leq \epsilon_1^k\omega\big(\big\{B_R : \mathcal{M}_{B_{2R}}(|\nabla u|^p) > 1\big\}\big) + \sum_{i=1}^{k} \epsilon_1^i\omega\big(\big\{B_R : \mathcal{M}_{B_{2R}}(|\boldsymbol{F}|^p) > \delta^p N^{k-i}\big\}\big). \end{aligned} \tag{4-8}$$

Proof. The proof is based on induction on $k \in \mathbb{N}$, and an iteration of Lemma 4.2. See, for example, [Phan 2017, Lemma 4.10]. □

We now can complete the proof of Theorem 1.1.

Proof of Theorem 1.1. The proof now is quite standard. However, we include it here for completeness, and for the transparency regarding the role of the scaling parameter λ. Let $N = N(\Lambda, p, n)$ be defined as in Lemma 4.3. For $q > 1$, we choose $\epsilon > 0$ sufficiently small and depending only on Λ, n, p, q, and γ such that

$$\epsilon_1 N^q = \frac{1}{2},$$

where ϵ_1 is defined as in Lemma 4.3. With this ϵ, we can now choose

$$\delta = \delta(\epsilon, \Lambda, M, p, q, n, \alpha), \quad \kappa = \kappa(\epsilon, \Lambda, M, p, q, n, \gamma, \alpha), \quad \kappa_0 = \min\{1, R\}\kappa/6$$

as determined by Lemma 4.3. Assume that the assumptions of Theorem 1.1 hold with this choice of δ. For $\lambda \geq 0$, let us assume that u is a weak solution of (3-1) satisfying $[\![\lambda u]\!]_{\mathrm{BMO}(B_R)} \leq M$, and let

$$E = E(\lambda, N) = \big\{B_R : \mathcal{M}_{B_{2R}}(|\nabla u|^p) > N\big\}. \tag{4-9}$$

We now prove the estimate in Theorem 1.1 with the additional assumption that

$$\omega(E) \leq \epsilon\omega(B_{\kappa_0}(y)) \quad \forall y \in \overline{B}_R. \tag{4-10}$$

Let us now consider the sum

$$S = \sum_{k=1}^{\infty} N^{qk}\omega\big(\big\{B_R : \mathcal{M}_{B_{2R}}(|\nabla u|^p) > N^k\big\}\big).$$

From (4-10), we can apply Lemma 4.3 to obtain

$$S \leq \sum_{k=1}^{\infty} N^{kq} \sum_{i=1}^{k} \epsilon_1^i \omega\big(\big\{B_R : \mathcal{M}_{B_{2R}}(|\boldsymbol{F}|^p) > \delta^p N^{k-i}\big\}\big) + \sum_{k=1}^{\infty} (N^q\epsilon_1)^k \omega\big(\big\{B_R : \mathcal{M}_{B_{2R}}(|\nabla u|^p) > 1\big\}\big).$$

By Fubini's theorem, the above estimate can be rewritten as

$$S \leq \sum_{j=1}^{\infty} (N^q\epsilon_1)^j \sum_{k=j}^{\infty} N^{q(k-j)} \omega\big(\big\{B_R : \mathcal{M}_{B_{2R}}(|\boldsymbol{F}|^p) > \delta^p N^{k-j}\big\}\big) + \sum_{k=1}^{\infty} (N^q\epsilon_1)^k \omega\big(\big\{B_R : \mathcal{M}_{B_{2R}}(|\nabla u|^p) > 1\big\}\big). \tag{4-11}$$

Observe that

$$\omega\big(\big\{B_R : \mathcal{M}_{B_{2R}}(|\nabla u|^p) > 1\big\}\big) \leq \omega(B_R).$$

From this, the choice of ϵ, Lemma 2.5, and (4-11) it follows that

$$S \le C\big[\|\mathcal{M}_{B_{2R}}(|\boldsymbol{F}|^p)\|^q_{L^q(B_R,\omega)} + \omega(B_R)\big].$$

Applying Lemma 2.5 again, we infer that

$$\|\mathcal{M}_{B_{2R}}(|\nabla u|^p)\|^q_{L^q(B_R,\omega)} \le C\big[\|\mathcal{M}_{B_{2R}}(|\boldsymbol{F}|^p)\|^q_{L^q(B_{2R},\omega)} + \omega(B_R)\big].$$

Also, by Lebesgue's differentiation theorem, it is true that

$$|\nabla u(x)|^p \le \mathcal{M}_{B_{2R}}(|\nabla u|^p)(x) \quad \text{a.e. } x \in B_R.$$

Hence,

$$\|\nabla u\|^{pq}_{L^{pq}(B_R,\omega)} \le C\big[\|\mathcal{M}_{B_{2R}}(|\boldsymbol{F}|^p)\|^q_{L^q(B_R,\omega)} + \omega(B_R)\big].$$

From this and Lemma 2.6, it follows that

$$\|\nabla u\|_{L^{pq}(B_R,\omega)} \le C\big[\|\boldsymbol{F}\|_{L^{pq}(B_{2R},\omega)} + \omega(B_R)^{1/q}\big]. \tag{4-12}$$

Summarizing the efforts, we conclude that (4-12) holds true as long as u is a weak solution of (3-1) for $\lambda \ge 0$ and (4-10) holds.

It now remains to remove the additional assumption (4-10). To this end, assume all assumptions in Theorem 1.1 hold, and let u be a weak solution of (3-1) with some $\lambda \ge 0$. Let $\mu > 0$ sufficiently large to be determined, and let $\lambda' = \lambda\mu \ge 0$, $u_\mu = u/\mu$, and $\boldsymbol{F}_\mu = \boldsymbol{F}/\mu$. We note that u_μ is a weak solution of

$$\operatorname{div}[\hat{\boldsymbol{A}}(x, \lambda' u_\mu, \nabla u_\mu)] = \operatorname{div}[|\boldsymbol{F}_\mu|^{p-2}\boldsymbol{F}_\mu] \quad \text{in } B_{2R}, \tag{4-13}$$

where

$$\hat{\boldsymbol{A}}(x, z, \xi) = \frac{\boldsymbol{A}(x, z, \mu\xi)}{\mu^{p-1}}.$$

Note that by Remark 2.1, $\hat{\boldsymbol{A}}$ satisfies (1-2)–(1-4) with the same constants Λ, p, α. Moreover, $\hat{\boldsymbol{A}}$ also satisfies (1-8). We then denote

$$E_\mu = \big\{B_R : \mathcal{M}_{B_{2R}}(|\nabla u_\mu|^p) > N\big\}$$

and assume that

$$K_0 = \left(\frac{1}{|B_{2R}|}\int_{B_{2R}} |\nabla u|^p\,dx\right)^{1/p} > 0. \tag{4-14}$$

We claim that we can choose $\mu = CK_0$ with some sufficiently large constant C depending only on Λ, M, p, q, n, and R/κ_0 such that

$$\omega(E_M) \le \epsilon\omega(B_{\kappa_0}(y)) \quad \forall y \in \overline{B}_R. \tag{4-15}$$

If this holds, we can apply (4-12) for u_μ, which is a weak solution of (4-13), to obtain

$$\|\nabla u_\mu\|_{L^{pq}(B_R,\omega)} \le C\big[\|\boldsymbol{F}_\mu\|_{L^{pq}(B_{2R},\omega)} + \omega(B_R)^{1/q}\big].$$

Then, by multiplying this equality with μ, we obtain

$$\|\nabla u\|_{L^{pq}(B_R,\omega)} \le C\big[\|\boldsymbol{F}\|_{L^{pq}(B_{2R},\omega)} + \omega(B_R)^{1/q} K_0\big].$$

The proof of Theorem 1.1 is therefore complete if we can prove (4-15). To this end, using the doubling property of $\omega \in A_q$ as in (i) of Lemma 2.4, we have

$$\frac{\omega(E_\mu)}{\omega(B_{\kappa_0}(y))} = \frac{\omega(E_\mu)}{\omega(B_{2R})}\frac{\omega(B_{2R})}{\omega(B_{\kappa_0}(y))} \le \gamma \frac{\omega(E_\mu)}{\omega(B_{2R})}\left(\frac{2R}{\kappa_0}\right)^{nq}.$$

From this, and using (ii) of Lemma 2.4 again, we can find $\beta = \beta(\gamma, n) > 0$ such that

$$\frac{\omega(E_\mu)}{\omega(B_{\kappa_0}(y))} \le C(\gamma, n)\left(\frac{2R}{\kappa_0}\right)^{nq}\left(\frac{|E_\mu|}{|B_{2R}|}\right)^{\beta/p}. \tag{4-16}$$

Now, by the definition of E_μ and the weak type $(1, 1)$ estimate for the maximal function, we see that

$$\begin{aligned}\frac{|E_\mu|}{|B_{2R}|} &= \big|\{B_R : \mathcal{M}_{B_{2R}}(|\nabla u|^p) > N\mu^p\}\big| \,/\, |B_{2R}| \\ &= \frac{C(n, p)}{N\mu^p}\frac{1}{|B_{2R}|}\int_{B_{2R}} |\nabla u|^p\, dx \le \frac{C(p, n)K_0^p}{N\mu^p},\end{aligned}$$

where K_0 is defined in (4-14). From this estimate and (4-16), it follows that

$$\frac{\omega(E_\mu)}{\omega(B_{\kappa_0}(y))} \le C^*(\Lambda, \gamma, p, n)\left(\frac{2R}{\kappa_0}\right)^{nq}\left(\frac{K_0}{\mu}\right)^{\beta}.$$

Now we choose μ such that

$$\mu = K_0\left[\epsilon^{-1}C^*(\Lambda, \gamma, p, n)\left(\frac{2R}{\kappa_0}\right)^{nq}\right]^{1/\beta}.$$

Then it follows that

$$\omega(E_\mu) \le \epsilon\omega(B_{\kappa_0}(y)) \quad \forall y \in \overline{B}_R.$$

This proves (4-15) and completes the proof of Theorem 1.1. □

Acknowledgements

T. Phan's research is supported by the Simons Foundation, grant #354889. The author would like to thank the anonymous referees for valuable comments and suggestions, which significantly improved the presentation of the paper.

References

[Bögelein 2014] V. Bögelein, "Global Calderón–Zygmund theory for nonlinear parabolic systems", *Calc. Var. Partial Differential Equations* **51**:3-4 (2014), 555–596. MR Zbl

[Byun and Wang 2012] S.-S. Byun and L. Wang, "Nonlinear gradient estimates for elliptic equations of general type", *Calc. Var. Partial Differential Equations* **45**:3-4 (2012), 403–419. MR Zbl

[Byun et al. 2007] S.-S. Byun, L. Wang, and S. Zhou, "Nonlinear elliptic equations with BMO coefficients in Reifenberg domains", *J. Funct. Anal.* **250**:1 (2007), 167–196. MR Zbl

[Byun et al. 2017] S.-S. Byun, D. K. Palagachev, and P. Shin, "Global Sobolev regularity for general elliptic equations of p-Laplacian type", preprint, 2017. arXiv

[Caffarelli and Peral 1998] L. A. Caffarelli and I. Peral, "On $W^{1,p}$ estimates for elliptic equations in divergence form", *Comm. Pure Appl. Math.* **51**:1 (1998), 1–21. MR Zbl

[Coifman and Fefferman 1974] R. R. Coifman and C. Fefferman, "Weighted norm inequalities for maximal functions and singular integrals", *Studia Math.* **51** (1974), 241–250. MR Zbl

[Di Fazio 1996] G. Di Fazio, "L^p estimates for divergence form elliptic equations with discontinuous coefficients", *Boll. Un. Mat. Ital. A* (7) **10**:2 (1996), 409–420. MR Zbl

[DiBenedetto 1983] E. DiBenedetto, "$C^{1+\alpha}$ local regularity of weak solutions of degenerate elliptic equations", *Nonlinear Anal.* **7**:8 (1983), 827–850. MR Zbl

[DiBenedetto and Manfredi 1993] E. DiBenedetto and J. Manfredi, "On the higher integrability of the gradient of weak solutions of certain degenerate elliptic systems", *Amer. J. Math.* **115**:5 (1993), 1107–1134. MR Zbl

[Dong and Kim 2010] H. Dong and D. Kim, "Elliptic equations in divergence form with partially BMO coefficients", *Arch. Ration. Mech. Anal.* **196**:1 (2010), 25–70. MR Zbl

[Dong and Kim 2011] H. Dong and D. Kim, "Global regularity of weak solutions to quasilinear elliptic and parabolic equations with controlled growth", *Comm. Partial Differential Equations* **36**:10 (2011), 1750–1777. MR Zbl

[Duzaar and Mingione 2010] F. Duzaar and G. Mingione, "Gradient estimates via linear and nonlinear potentials", *J. Funct. Anal.* **259**:11 (2010), 2961–2998. MR Zbl

[Duzaar and Mingione 2011] F. Duzaar and G. Mingione, "Gradient estimates via non-linear potentials", *Amer. J. Math.* **133**:4 (2011), 1093–1149. MR Zbl

[Evans 1982] L. C. Evans, "A new proof of local $C^{1,\alpha}$ regularity for solutions of certain degenerate elliptic PDE", *J. Differential Equations* **45**:3 (1982), 356–373. MR Zbl

[Gilbarg and Trudinger 1983] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, 2nd ed., Grundlehren der Math. Wissenschaften **224**, Springer, 1983. MR Zbl

[Giusti 2003] E. Giusti, *Direct methods in the calculus of variations*, World Sci., River Edge, NJ, 2003. MR Zbl

[Hoang et al. 2015] L. T. Hoang, T. V. Nguyen, and T. V. Phan, "Gradient estimates and global existence of smooth solutions to a cross-diffusion system", *SIAM J. Math. Anal.* **47**:3 (2015), 2122–2177. MR Zbl

[Iwaniec 1983] T. Iwaniec, "Projections onto gradient fields and L^p-estimates for degenerated elliptic operators", *Studia Math.* **75**:3 (1983), 293–312. MR Zbl

[Kinnunen and Lewis 2000] J. Kinnunen and J. L. Lewis, "Higher integrability for parabolic systems of p-Laplacian type", *Duke Math. J.* **102**:2 (2000), 253–271. MR Zbl

[Kinnunen and Zhou 1999] J. Kinnunen and S. Zhou, "A local estimate for nonlinear equations with discontinuous coefficients", *Comm. Partial Differential Equations* **24**:11-12 (1999), 2043–2068. MR Zbl

[Krylov 2007] N. V. Krylov, "Parabolic and elliptic equations with VMO coefficients", *Comm. Partial Differential Equations* **32**:1-3 (2007), 453–475. MR Zbl

[Krylov 2008] N. V. Krylov, *Lectures on elliptic and parabolic equations in Sobolev spaces*, Graduate Studies in Math. **96**, Amer. Math. Soc., Providence, RI, 2008. MR Zbl

[Krylov and Safonov 1979] N. V. Krylov and M. V. Safonov, "An estimate for the probability of a diffusion process hitting a set of positive measure", *Dokl. Akad. Nauk SSSR* **245**:1 (1979), 18–20. In Russian; translation in *Soviet Math. Dokl.* **20** (1979), 253–256. MR Zbl

[Ladyzhenskaya and Ural'tseva 1968] O. A. Ladyzhenskaya and N. N. Ural'tseva, *Linear and quasilinear elliptic equations*, Academic Press, New York, 1968. MR Zbl

[Lewis 1983] J. L. Lewis, "Regularity of the derivatives of solutions to certain degenerate elliptic equations", *Indiana Univ. Math. J.* **32**:6 (1983), 849–858. MR Zbl

[Lieberman 1988] G. M. Lieberman, "Boundary regularity for solutions of degenerate elliptic equations", *Nonlinear Anal.* **12**:11 (1988), 1203–1219. MR Zbl

[Malý and Ziemer 1997] J. Malý and W. P. Ziemer, *Fine regularity of solutions of elliptic partial differential equations*, Math. Surveys and Monographs **51**, Amer. Math. Soc., Providence, RI, 1997. MR Zbl

[Maugeri et al. 2000] A. Maugeri, D. K. Palagachev, and L. G. Softova, *Elliptic and parabolic equations with discontinuous coefficients*, Math. Research **109**, Wiley, Berlin, 2000. MR Zbl

[Mengesha and Phuc 2012] T. Mengesha and N. C. Phuc, "Global estimates for quasilinear elliptic equations on Reifenberg flat domains", *Arch. Ration. Mech. Anal.* **203**:1 (2012), 189–216. MR Zbl

[Meyers 1963] N. G. Meyers, "An L^p-estimate for the gradient of solutions of second order elliptic divergence equations", *Ann. Scuola Norm. Sup. Pisa* (3) **17** (1963), 189–206. MR Zbl

[Meyers and Elcrat 1975] N. G. Meyers and A. Elcrat, "Some results on regularity for solutions of non-linear elliptic systems and quasi-regular functions", *Duke Math. J.* **42** (1975), 121–136. MR Zbl

[Muckenhoupt 1972] B. Muckenhoupt, "Weighted norm inequalities for the Hardy maximal function", *Trans. Amer. Math. Soc.* **165** (1972), 207–226. MR Zbl

[Nguyen and Phan 2016] T. Nguyen and T. Phan, "Interior gradient estimates for quasilinear elliptic equations", *Calc. Var. Partial Differential Equations* **55**:3 (2016), art. id. 59. MR Zbl

[Phan 2017] T. Phan, "Weighted Calderón–Zygmund estimates for weak solutions of quasi-linear degenerate elliptic equations", submitted, 2017. arXiv

[Safonov 1980] M. V. Safonov, "Harnack's inequality for elliptic equations and Hölder property of their solutions", *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov.* **96** (1980), 272–287. In Russian; translated in *J. Soviet Math.* **21**:5 (1983), 851–863. MR Zbl

[Tolksdorf 1984] P. Tolksdorf, "Regularity for a more general class of quasilinear elliptic equations", *J. Differential Equations* **51**:1 (1984), 126–150. MR Zbl

[Uhlenbeck 1977] K. Uhlenbeck, "Regularity for a class of non-linear elliptic systems", *Acta Math.* **138**:3-4 (1977), 219–240. MR Zbl

[Ural'tseva 1968] N. N. Ural'tseva, "Degenerate quasilinear elliptic systems", *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov.* **7** (1968), 184–222. In Russian. MR Zbl

[Wang 2003] L. H. Wang, "A geometric approach to the Calderón–Zygmund estimates", *Acta Math. Sin.* (*Engl. Ser.*) **19**:2 (2003), 381–396. MR Zbl

Received May 5, 2017. Revised January 14, 2018.

TUOC PHAN
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF TENNESSEE
KNOXVILLE, TN
UNITED STATES
phan@math.utk.edu

PACIFIC JOURNAL OF MATHEMATICS

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

msp.org/pjm

See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2018 is US $475/year for the electronic version, and $640/year for print and electronic.
Subscriptions, requests for back issues and changes of subscriber address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

PACIFIC JOURNAL OF MATHEMATICS

Volume 297 No. 1 November 2018