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ON LEGENDRE CURVES IN NORMED PLANES

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Legendre curves are smooth plane curves which may have singular points, but still have a well defined smooth normal (and corresponding tangent) vector field. Because of the existence of singular points, the usual curvature concept for regular curves cannot be extended to these curves. However, Fukunaga and Takahashi defined and studied functions that play the role of curvature functions of a Legendre curve, and whose ratio extends the curvature notion in the usual sense. In the same direction, our paper is devoted to the extension of the concept of circular curvature from regular to Legendre curves, but additionally referring not only to the Euclidean plane. For the first time we will extend the concept of Legendre curves to normed planes. Generalizing in such a way the results of the mentioned authors, we define new functions that play the role of circular curvature of Legendre curves, and tackle questions concerning existence, uniqueness, and invariance under isometries for them. Using these functions, we study evolutes, involutes, and pedal curves of Legendre curves for normed planes, and the notion of contact between such curves is correspondingly extended, too. We also provide new ways to calculate the Maslov index of a front in terms of our new curvature functions. It becomes clear that an inner product is not necessary in developing the theory of Legendre curves. More precisely, only a fixed norm and the associated orthogonality (of Birkhoff type) are necessary.

1. Introduction

The concept of *curvature* of regular curves in the Euclidean plane can be extended to normed planes in several ways (see [Balestro et al. 2018] for an exposition of the topic, and [Martini and Wu 2014] refers, more generally, to classical curve theory in such planes). One of the curvature types obtained by these extensions, namely the *circular curvature*, can be regarded as the inverse of the radius of a second-order contact circle at the respective point of the curve. Therefore it turns out that the

investigation of the differential geometry of these curves from the viewpoint of singularity theory is also due to this context (see [Izumiya et al. 2016] and [Balestro et al. 2018, Section 9]). In the Euclidean subcase, the concept of curvature can be carried over to certain curves containing singular points. This was done in [Fukunaga and Takahashi 2013], and the aim of the present paper is to investigate this framework more generally for normed planes, using the concept of circular curvature, and also extending the usual inner product orthogonality to *Birkhoff orthogonality*.

Let us say a few words about the motivations for studying Legendre curves in two-dimensional Banach spaces. Minkowski geometry (i.e., the geometry of finite-dimensional real Banach spaces; see [Thompson 1996]) is more than 100 years old and can be seen as a starting field and a "special case" of Banach space theory (see, e.g., [Johnson and Lindenstrauss 2001; 2003]), meaning the restriction to the geometric view on the finite-dimensional situation. But it can also be considered as a "subcase" of Finsler geometry (see [Matsumoto 1986] and [Bao et al. 2000]) meaning here the local situation in tangent spaces. Regarding methods and tools, Minkowski geometry is also closely related to classical convexity (excellently presented in [Schneider 2013]), and additionally many of its outcomes generalize results from convexity. But the field of relative differential geometry (see the survey [Barthel and Kern 1994] and, as a nice example for classical results in this direction, [Heil 1970]) is also conceptually related. Thus, there should be lively connections between Minkowski geometry and the fields of functional analysis, differential geometry, and classical convexity. The existence of such connections is obvious for functional analysis and convexity, e.g., by various articles in [Johnson and Lindenstrauss 2001; 2003] and by [Thompson 1996]. But in the case of (even classical) differential geometry and Minkowski geometry, not many articles exist which combine these fields. Even now, some classical theory of curves in Minkowski planes is not really developed (this situation was already discussed in [Martini and Wu 2014]). Thus, we started to write conceptual papers in this direction, to develop some systematized tools for investigating curves and surfaces in normed planes and spaces from the viewpoint of differential geometry (see [Balestro et al. 2018; 2017a; 2017b; 2017c]). In particular, a comprehensive study of all curvature types of curves in normed planes is given in [Balestro et al. 2018], and this systematization automatically led to (curvature concepts for) Legendre curves in Minkowski planes. These curves have interesting applications, e.g., referring to contact manifolds or to fronts, and we hope that such related notions can also be successfully extended to Minkowski geometry. Similar investigations will follow, even generalized for gauges (i.e., generalized Minkowski spaces whose unit ball is still a convex body, but no longer symmetric with respect to the origin). For example, results on respective generalizations of types of multifocal curves (such as multifocal Cassini curves, or multifocal ellipses) with applications in location science and other fields can be found in [Jahn et al. 2016].

We start with some basic definitions. A *normed* (or *Minkowski*) *plane* $(X, \|\cdot\|)$ is a two-dimensional real vector space X endowed with a norm $\|\cdot\| : X \to \mathbb{R}$, whose *unit ball* is the set $B := \{x \in X : \|x\| \leq 1\}$, namely a compact convex set centered at the origin o which is an interior point of B. The boundary $S := \{x \in X : \|x\| = 1\}$ of B is called the *unit circle*, and all homothetic copies of B and S will be called *Minkowski balls* and *Minkowski circles*, respectively. We will always assume that the plane is *smooth*, which means that S is a smooth curve, and also *strictly convex*, meaning that S does not contain straight line segments. In a normed plane $(X, \|\cdot\|)$ we define an orthogonality relation by stating that two vectors $x, y \in X$ are *Birkhoff orthogonal* (denoted by $x \dashv_B y$) whenever $\|x + ty\| \geq \|x\|$ for each $t \in \mathbb{R}$. Geometrically this means that if $x \dashv_B y$ and $x \neq 0$, then the Minkowski circle centered at the origin which passes through x is supported by a line in the direction of y. Useful references with respect to Minkowski geometry (i.e., the geometry of finite-dimensional real Banach spaces) are [Thompson 1996; Martini et al. 2001; Martini and Swanepoel 2004]; for orthogonality types in Minkowski spaces we refer to [Alonso et al. 2012].

One should notice that Birkhoff orthogonality is not necessarily a symmetric relation. Actually, we may endow the plane with a new associated norm which reverses the orthogonality relation. To do so, we fix a nondegenerate symplectic bilinear form $[\,\cdot\,,\cdot\,] : X \times X \to \mathbb{R}$ (which is unique up to rescaling) and define the associated *antinorm* to be

$$\|x\|_a = \sup\{|[x, y]| : y \in S\}, \quad x \in X.$$

It is easily seen that $\|\cdot\|_a$ is a norm on X, and that it reverses Birkhoff orthogonality. Moreover, the *unit anticircle* (i.e., the unit circle of the antinorm) solves the isoperimetric problem in the original Minkowski plane (see [Busemann 1947]). The planes where Birkhoff orthogonality is symmetric are called *Radon planes*, and their unit circles are called *Radon curves*. In this case, we clearly have that the unit circle and the unit anticircle are homothets, and we will always assume that the fixed symplectic bilinear form is rescaled in such way that they coincide. A comprehensive exposition on this topic is [Martini and Swanepoel 2006].

A smooth curve $\gamma : J \to X$ is said to be *regular* if $\gamma'(t) \neq 0$ for every $t \in J$. If a curve is not regular, then a point, where the derivative vanishes, is called a *singular point* of γ. The *length* of a curve $\gamma : [a, b] \to X$ is defined as usual in terms of the norm by

$$l(\gamma) := \sup_P \sum_{j=1}^{n} \|\gamma(t_j) - \gamma(t_{j-1})\|,$$

where the supremum is taken over all partitions of $P = \{a = t_0, \ldots, t_n = b\}$ of $[a, b]$. It is clear that we can define here the standard arc-length parametrization, and that if s is an arc-length parameter in γ, then $\|\gamma'(s)\| = 1$. We now define the *circular curvature* for a regular curve $\gamma : [0, l(\gamma)] \to X$ parametrized by arc-length (for the

sake of simplicity). To do so, let $\varphi(u) : [0, l(S)] \to S$ be a parametrization of the unit circle by arc-length. Let $u(s) : [0, l(\gamma)] \to [0, l(S)]$ be the function such that $\dot{\gamma}(s) = \frac{d\varphi}{du}(u(s))$. Then the *circular curvature* of γ at $\gamma(s)$ is defined as

$$k(s) := \dot{u}(s).$$

We define the *left normal field* of γ to be the unit vector field $\eta : [0, l(\gamma)] \to S$ such that $\eta(s) \dashv_B \dot{\gamma}(s)$ and $[\eta(s), \dot{\gamma}(s)] > 0$ for each $s \in [0, l(\gamma)]$. Writing $\dot{\gamma}(s) = \frac{d\varphi}{du}(u(s))$, we have that the left normal field is given by $\eta(s) = \varphi(u(s))$. Therefore, we have the Frenet-type formula

$$\dot{\eta}(s) = \dot{u}(s)\frac{d\varphi}{du}(u(s)) = k(s)\dot{\gamma}(s).$$

The *center of curvature* of γ at $\gamma(s)$ is the point $c(s) := \gamma(s) - k(s)^{-1}\eta(s)$, and we call the number $\rho(s) := k(s)^{-1}$ the *curvature radius* of γ at $\gamma(s)$. The circle centered in $c(s)$ and having radius $\rho(s)$ is the *osculating circle* of γ at $\gamma(s)$. It is easily seen that this circle has second-order contact with γ at $\gamma(s)$. From the viewpoint of singularity theory, the *distance squared function* of γ to a point $p \in X$ is the function $D_p(s) := \|\gamma(s) - p\|^2$. We can obtain the centers of curvature of a given curve as follows.

Proposition 1.1. *Let $\gamma : [0, l(\gamma)] \to X$ be a smooth and regular curve parametrized by arc length. Then the function $D_p(s) = \|\gamma(s) - p\|^2$ is such that $\dot{D}_p(s_0) = \ddot{D}_p(s_0) = 0$ if and only if p is the center of curvature of γ at $\gamma(s_0)$.*

Proof. See [Balestro et al. 2018, Proposition 9.1]. $\square$

   Throughout the text, we will call the circular curvature simply *curvature*, and the left normal field will be referred to as the *normal field*.

2. Curvature of curves with singularities

The main objective of this paper is to extend and study the concept of curvature for curves in normed planes which have certain types of "well-behaving" singularities. Roughly speaking, in certain situations a curve can have a singularity, but we are still able to derive a natural tangent direction corresponding to the respective curve point. For example, let $\gamma(t) : I \to X$ be a curve, and assume that γ has a (unique, for the sake of simplicity) isolated singularity at $t_0 \in I$ (that is, $\gamma'(t)$ does not vanish in a punctured neighborhood of t_0). If both limits

$$\lim_{t \to t_0^{\pm}} \frac{\gamma'(t)}{\|\gamma'(t)\|}$$

exist and are equal up to the sign, then we can naturally define a field of tangent (or normal) directions through the entire γ. This kind of singularity appears, for example, in *evolutes* of regular curves (see [Balestro et al. 2018, Section 9]).

In singularity theory, submanifolds with singularities but well-defined tangent spaces are usually called *frontals* (see [Ishikawa 2016]). If the ambient space is two-dimensional, then these submanifolds are precisely the curves which have well-defined tangent fields, even if they contain singularities. Such curves were studied in [Fukunaga and Takahashi 2013; 2014; 2015; 2016]. Heuristically speaking, the existence of a well-defined tangent field has no relation to the metric of the plane. Therefore, we can reobtain the definitions posed by the mentioned authors, but now regarding the usual tools and machinery of planar Minkowski geometry.

We define a *Legendre curve* to be a smooth map $(\gamma, \eta) : I \to X \times S$ such that $\eta(t) \dashv_B \gamma'(t)$ for every $t \in I$. If a Legendre curve is an *immersion* (i.e., if the derivatives of γ and η do not vanish at the same time), then we call it a *Legendre immersion*. A curve $\gamma : I \to X$ is said to be a *frontal* if there exists a smooth map $\eta : I \to S$ such that (γ, η) is a Legendre curve. Finally, we say that γ is a *front* if there exists a smooth map $\eta : I \to S$ such that (γ, η) is a Legendre immersion.

Since we are dealing with smooth and strictly convex normed planes, it follows that Birkhoff orthogonality is unique on both sides. Define the map $b : X \setminus \{o\} \to S$ (where o again denotes the origin of the plane) which associates to each $v \in X \setminus \{o\}$ the unique vector $b(v) \in S$ such that $v \dashv_B b(v)$ and $[v, b(v)] > 0$. A Legendre curve is defined heuristically by guaranteeing the existence of a *normal field* to γ, instead of a tangent field. But now we simply use the map b to define a "tangent field". We just have to define, for a Legendre curve $(\gamma, \eta) : I \to X \times S$, the vector field $\xi(t) := b(\eta(t))$. Of course, $\xi(t)$ points in the direction of $\gamma'(t)$. Then there exists a smooth function $\alpha : I \to \mathbb{R}$ such that

$$(2\text{-}1) \qquad \gamma'(t) = \alpha(t)\xi(t), \quad t \in I.$$

Also, since $\eta'(t)$ supports the unit circle at $\eta(t)$, it follows that there exists a smooth function $\kappa : I \to \mathbb{R}$ such that

$$(2\text{-}2) \qquad \eta'(t) = \kappa(t)\xi(t), \quad t \in I.$$

We call the pair (α, κ) the *curvature* of the Legendre curve (γ, η) with respect to the parameter t. This terminology makes sense since it is easy to see that the curvature of a Legendre curve depends on its parametrization. To justify why this pair of functions represents an analogous concept of curvature for Legendre curves, we will show that it yields the usual (circular) curvature of a regular curve.

Lemma 2.1. *Let $\gamma : I \to X$ be a regular curve in a normed plane. Clearly, if $\eta : I \to S$ is its normal vector field, then (γ, η) is a Legendre curve. Therefore, its circular curvature $k : I \to \mathbb{R}$ is given by*

$$k(t) = \frac{\kappa(t)}{\alpha(t)},$$

where κ and α are defined as above.

Proof. For the Legendre curve (γ, η) we have the equalities (2-1) and (2-2). Notice that, since γ is regular, the function α does not vanish. Hence we may write

$$\eta'(t) = \frac{\kappa(t)}{\alpha(t)}\gamma'(t).$$

On the other hand, let s be an arc-length parameter in γ and, as usual, let $\varphi(u)$ be an arc-length parametrization of the unit circle. We denote the derivative with respect to s by a superscribed dot, and write

$$\dot{t}(s)\gamma'(t) = \dot{\gamma}(s) = \frac{d\varphi}{du}(u(s)),$$

where $u(s)$ is as in the definition of circular curvature. We have that $k(s) = \dot{u}(s)$ and $\eta(s) = \varphi(u(s))$. Differentiating this last equality, we get

$$\dot{t}(s)\eta'(t) = \dot{u}(s)\frac{d\varphi}{du}(u(s)) = \dot{u}(s)\dot{\gamma}(s) = \dot{u}(s)\dot{t}(s)\gamma'(t),$$

and since $\dot{t}(s)$ does not vanish, it follows that $\eta'(t) = \dot{u}(s)\gamma'(t) = k(t)\gamma'(t)$. This gives the desired equality. □

Remark 2.2. When working in the Euclidean plane, one gets a second Frenet-type formula by differentiating the field $\xi(t)$, and the same curvature function $\kappa(t)$ is obtained (see [Fukunaga and Takahashi 2013]). This is not the case here. The first problem that appears is that the derivative of $\xi(t)$ does not necessarily point in the direction of $\eta(t)$. We can overcome this problem by restricting ourselves to Radon planes. However, even in this small class of norms we do not reobtain the same curvature function. Indeed, since $\xi(t) = b(\eta(t))$, we have

$$\xi'(t) = Db_{\eta(t)}(\eta'(t)) = \kappa(t)Db_{\eta(t)}(\xi(t)),$$

where Db denotes the usual differential of the map $b : X \setminus \{o\} \to S$, which is no longer a (linear) rotation. It turns out that, since the considered plane is Radon, the vector $Db_{\eta(t)}(\xi(t))$ is a positive multiple of the vector $-\eta(t)$, but it is not necessarily unit. If we define the map $\rho : S \to \mathbb{R}$ by $\rho(v) = \|Db_v(b(v))\|$, then we may write

$$(2\text{-}3) \qquad\qquad \xi'(t) = -\kappa(t)\rho(\eta(t))\eta(t).$$

The function ρ is constant, however, if and only if the plane is Euclidean (see [Balestro and Shonoda 2018] for a proof). If we return to the general case, we clearly have

$$(2\text{-}4) \qquad\qquad \xi'(t) = -\kappa(t)\rho(\eta(t))b(\xi(t)),$$

and this equality will be used in Section 5, where the function ρ will appear in the curvature pair of the evolute of a front.

We give an example to illustrate the concepts we have just introduced.

Example 2.3. Let $\|\cdot\|$ be the usual $l_p - l_q$ norm in $\mathbb{R}^2$, for some $1 < p, q < \infty$ such that $1/p + 1/q = 1$ (that is, we endow the first and third quadrants with the l_p norm, and the second and fourth with the l_q norm). The parametrized curve $\gamma : \mathbb{R} \to \mathbb{R}^2$ given by $\gamma(t) = (t^3, t^2)$ has a cusp singularity at $t = 0$, but it is a Legendre curve. Indeed, the tangent vector at $t = 0$ is simply the vector $(0, 1)$. For $t > 0$, the tangent vector to γ points in a direction of the first quadrant, and supports the (oriented) unit circle in the fourth quadrant. Equality (2-1) becomes

$$\gamma'(t) = (3t^2, 2t) = [(3t^2)^p + (2t)^p]^{1/p} \cdot \frac{(3t^2, 2t)}{[(3t^2)^p + (2t)^p]^{1/p}},$$

from where $\alpha(t) = [(3t^2)^p + (2t)^p]^{1/p}$. To obtain (2-2) we start by recalling that an $l_p - l_q$ norm is a Radon norm, for which Birkhoff orthogonality is symmetric; see [Martini and Swanepoel 2006]. Thus, to obtain the direction of η we just have to differentiate the unit tangent vector (since γ' rotates counterclockwise, the derivative has the same orientation as η). To that end, assume that the first portion of the unit circle in the first quadrant is parametrized by $s \mapsto (s^{1/p}, (1-s)^{1/p})$, where $s = s(t)$, and write

$$\frac{3t^2}{2t} = \frac{s^{1/p}}{(1-s)^{1/p}}.$$

We get $s(t) = (3t^2)^p / ((3t^2)^p + (2t)^p)$. Then $\eta(t)$ is the unit vector in the direction of $(s^{-1/q}, -(1-s)^{-1/q})$, that is:

$$\eta(t) = \frac{(s^{-1/q}, -(1-s)^{-1/q})}{\left(\frac{1}{s} + \frac{1}{1-s}\right)^{1/q}} = ((1-s)^{1/q}, -s^{1/q}),$$

where we recall that we normalized with respect to the norm l_q, since η is a vector of the fourth quadrant. Finally,

$$\eta'(t) = -\frac{s'(t)}{q}((1-s)^{-1/p}, s^{-1/p}) = \frac{-s'(t)}{q(1-s)^{1/p}s^{1/p}}(s^{1/p}, (1-s)^{1/p}),$$

and since $(s^{1/p}, (1-s)^{1/p})$ is the unit tangent vector to γ, we get

$$\kappa(t) = -\frac{s'(t)}{q(1-s(t))^{1/p}s(t)^{1/p}}.$$

For $t < 0$ we proceed similarly, and the value of $\kappa(0)$ can be obtained by taking limits.

At this point, we have seen that we can extend the definitions, which are common for the Euclidean subcase, in a way that everything still makes sense and has analogous behavior. However, a simple question arises: if a certain fixed curve

in the plane is a Legendre curve (immersion) with respect to a fixed norm, is it then necessarily a Legendre curve (immersion) with respect to any other (smooth and strictly convex) norm? The answer is positive, and we can briefly explain the argument. Let S_1 and S_2 be unit circles with respect to two different norms, and denote the respective Birkhoff orthogonality relations by $\dashv_B^1$ and $\dashv_B^2$. Consider the map $T : S_1 \to S_2$ which associates each $v \in S_1$ to the unique $T(v) \in S_2$ such that $T(v) \dashv_B^2 b_1(v)$ and $[T(v), b_1(v)] > 0$, where b_1 is the usual map b of the geometry given by S_1. The map T is clearly smooth, and if (γ, η) is a Legendre curve (immersion) with respect to the norm of S_1, then $(\gamma, T(\eta))$ is a Legendre curve (immersion) with respect to the norm of S_2. The details are left to the reader.

3. Existence, uniqueness, and invariance under isometries

This section is concerned with natural questions regarding the generalized objects that we have defined. We start by asking whether or not there exists a corresponding Legendre curve whose curvature is given by certain fixed smooth functions $\kappa, \alpha : I \to \mathbb{R}$. For simplicity, throughout this section we assume that $I = [0, c]$.

Theorem 3.1 (existence theorem). *Let* $(\alpha, \kappa) : I \to \mathbb{R}^2$ *be a smooth function. Then there exists a Legendre curve* $(\gamma, \eta) : I \to X \times S$ *whose curvature is* (α, κ).

Proof. First, define the function $u : I \to \mathbb{R}$ by

$$u(t) = \int_0^t \kappa(s)\,ds, \quad t \in I.$$

Now, define $\eta : I \to S$ by $\eta(t) = \varphi(u(t))$, and $\gamma : I \to X$ by

$$\gamma(t) = \int_0^t \alpha(s)b(\eta(s))\,ds, \quad t \in I.$$

We claim that the pair (γ, η) is a Legendre curve with curvature (α, κ). To verify this, we differentiate γ to obtain $\gamma'(t) = \alpha(t)b(\eta(t))$. Notice that $\eta(t) \dashv_B \gamma'(t)$. Therefore, in view of the previous notation, we indeed have $\xi(t) = b(\eta(t))$, and consequently (2-1) holds. Now, differentiating η yields

$$\eta'(t) = u'(t)\frac{d\varphi}{du}(u(t)) = \kappa(t)\xi(t),$$

since $\varphi(u)$ is an arc-length parametrization of the unit circle. $\qquad\square$

Of course, the next natural question is whether or not such a Legendre curve is uniquely determined if we fix initial conditions $\gamma(0) \in X$ and $\eta(0) \in S$. We give now a positive answer to this question using the standard theory of ordinary differential equations.

Theorem 3.2 (uniqueness theorem). *Let* $(\alpha, \kappa) : I \to \mathbb{R}^2$ *be a smooth function and fix* $(p, v) \in X \times S$. *Then there exists a unique Legendre curve* $(\gamma, \eta) : I \to X \times S$ *whose curvature is* (α, κ) *and such that* $\gamma(0) = p$ *and* $\eta(0) = v$.

Proof. From the construction in the previous theorem, it is clear that to determine a vector field $\eta : I \to S$ such that $\eta'(t) = \kappa(t)b(\eta(t))$ with initial condition $\eta(0) = v$ is equivalent to finding a function $u : I \to \mathbb{R}$ that solves the initial value problem

$$\begin{cases} u'(t) = \kappa(t), & t \in I, \\ u(0) = u_0, \end{cases}$$

where $u_0 \in \mathbb{R}$ is such that $v = \varphi(u_0)$. Uniqueness of such a function is guaranteed by the standard theory of ordinary differential equations (see, for instance, [Coddington and Levinson 1955]).

Now the tangent vector field $\gamma'(t) = \alpha(t)\xi(t)$ is completely determined (where $\xi(t) = b(\eta(t))$, as usual). Since it is clear that smooth curves with the same tangent vector field must be equal up to translation, the proof is complete. $\square$

As a consequence of the uniqueness theorem, we have a characterization of the Minkowski circle. See [Fukunaga and Takahashi 2015, Proposition 2.12] for the Euclidean version of this characterization.

Proposition 3.3. *A Legendre curve* $(\gamma, \eta) : I \to X \times S$ *is contained in a Minkowski circle if and only if there exists a constant* $c \in \mathbb{R}$ *such that* $\alpha(t) = c\kappa(t)$ *for all* $t \in I$.

Proof. If γ is contained in a Minkowski circle of radius c, then the circular curvature equals $1/c$ (see [Balestro et al. 2018, Theorem 6.1]). Therefore, from Lemma 2.1 it follows that $\alpha(t) = c\kappa(t)$ for every $t \in I$. The converse follows immediately from the uniqueness theorem. $\square$

Let $(\gamma, \eta) : I \to X \times S$ be a Legendre curve with curvature (α, κ), and let $T : X \to X$ be an *isometry* of the plane, i.e., a norm-preserving map. An isometry is called *orientation-preserving* if the sign of the fixed determinant form remains invariant under its action. Since Birkhoff orthogonality is defined in terms of distances, it is clear that $(T\gamma, T\eta)$ is still a Legendre curve, and hence it has a curvature function (κ_T, α_T).

Theorem 3.4 (invariance under isometries). *The curvature of a Legendre curve is invariant under an orientation-preserving isometry of the plane.*

Proof. Using the same notation as above, we have to prove that $\kappa = \kappa_T$ and $\alpha = \alpha_T$. Recall that an isometry of a normed plane must be linear up to translation, and then we may consider it as linear, for the sake of simplicity (see [Balestro et al. 2018]). Hence, from (2-1) and (2-2) we have the equalities

$$(T\gamma)'(t) = T\gamma'(t) = \alpha(t)T\xi(t) \quad \text{and} \quad (T\eta)'(t) = T\eta'(t) = \kappa(t)T\xi(t).$$

Therefore, in order to prove that $\kappa = \kappa_T$ and $\alpha = \alpha_T$ it suffices to show that $T\xi(t) = b(T\eta(t))$, where we recall that $\xi(t) = b(\eta(t))$. But this comes immediately, since $T\eta(t) \dashv_B Tb(\eta(t))$, $\|Tb(\eta(t))\| = \|b(\eta(t))\| = 1$, and T is orientation-preserving. □

Remark 3.5. Clearly, if the considered isometry is orientation-reversing, then we have $\kappa_T = -\kappa$ and $\alpha_T = -\alpha$.

4. Ordinary cusps of closed fronts

A singularity $t_0 \in I$ of a smooth curve $\gamma : I \to X$ is said to be an *ordinary cusp* if $\gamma''(t_0)$ and $\gamma'''(t_0)$ are linearly independent vectors. Our next statement shows that we can describe an ordinary cusp of a front in terms of the curvature functions of an associated Legendre immersion.

Lemma 4.1. *Let* $\gamma : I \to X$ *be a front, and let* $\eta : I \to S$ *be a smooth vector field such that* (γ, η) *is a Legendre immersion. A point* $t_0 \in I$ *is an ordinary cusp if and only if* $\alpha'(t_0) \neq 0$, *where* $\alpha : I \to \mathbb{R}$ *is defined as in* (2-1).

Proof. This comes from the two-fold straightforward differentiation of (2-1). In a singular point $t_0 \in I$ we have

$$\gamma''(t_0) = \alpha'(t_0)\xi(t_0) \quad \text{and} \quad \gamma'''(t_0) = 2\alpha'(t_0)\xi'(t_0) + \alpha''(t_0)\xi(t_0).$$

Since (γ, η) is an immersion, we have $\eta'(t_0) \neq 0$. Thus, $\xi'(t_0) = Db_{\eta(t_0)}(\eta'(t_0)) \neq 0$ and $\xi(t_0) \dashv_B \xi'(t_0)$. The desired result follows. □

It is clear that the definition of an ordinary cusp does not involve any metric or orthogonality concept fixed in the plane. Indeed, one just needs differentiation of a curve to define an ordinary cusp. The previous lemma, despite being easy and intuitive, shows us that, using the curvature defined by the Minkowski metric, one can characterize an ordinary cusp of a Legendre immersion in the same way as we would in the standard Euclidean metric.

Continuing in this direction, we proceed to formalize the idea that *the orientation changes when we pass through an ordinary cusp*, and we will do this by using only the machinery defined here. Indeed, since $\text{sgn}[\eta(t), \gamma'(t)] = \text{sgn}(\alpha(t))$, it follows that the orientation of the basis $\{\eta(t), \gamma'(t)\}$ changes. By the last lemma, the sign of α changes at a point $t_0 \in I$ if and only if $\gamma(t_0)$ is an ordinary cusp, and then it follows that the orientation of the basis $\{\eta(t), \gamma'(t)\}$ (well-defined in a punctured neighborhood of t_0) changes when, and only when, we pass through an ordinary cusp. As a consequence we reobtain the following well-known result.

Proposition 4.2. *Let* $\gamma : S^1 \to X$ *be a closed front, where* S^1 *is the usual circle* $\mathbb{R}/\mathbb{Z}$. *Then* γ *has an even number of ordinary cusps.*
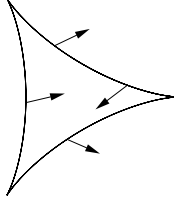
Figure 1. A front must have an even number of ordinary cusps.

Proof. The intuitive idea here is that we must pass through an even number of ordinary cusps so that the sign of $[\eta(t), \gamma'(t)]$ is not inverted when we return to the initial point (see Figure 1). We will formalize this.

Let $\eta : S^1 \to S$ be a normal vector field such that (γ, η) is a Legendre immersion. We identify S^1 with the interval $[0, 1]$ and, up to a translation in the parameter, assume that $\gamma(0) = \gamma(1)$ is a regular point. Let $\{t_1 < t_2 < \cdots < t_m\}$ be the set of all ordinary cusps of γ, and assume that m is odd. It is clear that the sign of $[\eta(t), \gamma'(t)]$ is constant in each interval (t_{j-1}, t_j), and also before t_1 and after t_m. Since there is no ordinary cusp in the interval $(t_m, 1 + t_1)$, it also follows that m is even. Otherwise, the sign of $[\eta(t), \gamma'(t)]$ would be distinct in $(t_m, 1]$ and $[0, t_1)$. $\square$

Remark 4.3. In view of Lemma 4.1, an ordinary cusp is a zero-crossing of α, and the converse is also true. Since γ is closed, we have that α is a periodic smooth function, and then we must have an even number of zero-crossings. This (a little less geometric) argument also works for proving Proposition 4.2.

Our next task is to obtain the *Maslov index* (or *zigzag number*) of a closed front using the generalized curvature of a Legendre immersion (for the Euclidean case this was done in [Fukunaga and Takahashi 2013]). By [Saji et al. 2009] we are inspired to formulate the following

Definition 4.4. Let $t_0 \in I$ be an ordinary cusp of a front $\gamma : I \to X$ with associated normal field η. Then, if $[\eta(t_0), \eta'(t_0)] > 0$, we say that t_0 is a *zig*, and if $[\eta(t_0), \eta'(t_0)] < 0$, we say that t_0 is a *zag*.

Notice that we always have $[\eta(t_0), \eta'(t_0)] \neq 0$ on an ordinary cusp (γ is a front). Geometrically, by this definition we can distinguish whether the normal field rotates counterclockwise or clockwise in the neighborhood of an ordinary cusp, and this is equivalent to the definition given in [Saji et al. 2009]. As one may expect, whether an ordinary cusp is a zig or a zag does not depend on the metric (and consequently not on the orthogonality relation) fixed in the plane.

Proposition 4.5. *Let $\gamma : I \to X$ be a front, and let $t_0 \in I$ be an ordinary cusp. Then we have one of the following statements.*

(a) *For every $\varepsilon > 0$ there exist $t_1, t_2 \in (t_0 - \varepsilon, t_0 + \varepsilon)$ such that $t_1 < t_0 < t_2$ and $[\gamma'(t_1), \gamma'(t_2)] < 0$, or*

(b) *for every $\varepsilon > 0$ there exist $t_1, t_2 \in (t_0 - \varepsilon, t_0 + \varepsilon)$ such that $t_1 < t_0 < t_2$ and*
    $[\gamma'(t_1), \gamma'(t_2)] > 0$.

In the first case, the cusp is a zig. In the second one, we have a zag.

Proof. Assume that (γ, η) is a Legendre immersion, and let α be as in (2-1). Since t_0 is an ordinary cusp, it follows that for small $\varepsilon > 0$ we have that α has constant and distinct signs in each of the lateral neighborhoods $(t_0 - \varepsilon, t_0)$ and $(t_0, t_0 + \varepsilon)$ of t_0. Therefore, for any $t_1, t_2 \in (t_0 - \varepsilon, t_0 + \varepsilon)$ with $t_1 < t_0 < t_2$ it holds that $\alpha(t_1)\alpha(t_2) < 0$. Now we write

$$[\gamma'(t_1), \gamma'(t_2)] = \alpha(t_1)\alpha(t_2)[\xi(t_1), \xi(t_2)].$$

Hence, in $(t_0 - \varepsilon, t_0 + \varepsilon)$ the sign of $[\gamma'(t_1), \gamma'(t_2)]$ for $t_1 < t_0 < t_2$ depends only on the sign of $[\xi(t_1), \xi(t_2)]$. On the other hand, since (γ, η) is an immersion, we have that $\eta'(t_0) \neq 0$. Then, taking a smaller $\varepsilon > 0$ if necessary, we may assume that η is injective when restricted to the interval $(t_0 - \varepsilon, t_0 + \varepsilon)$. Consequently, ξ is also injective in $(t_0 - \varepsilon, t_0 + \varepsilon)$, and the sign of $[\xi(t_1), \xi(t_2)]$ for $t_1, t_2 \in (t_0 - \varepsilon, t_0 + \varepsilon)$ with $t_1 < t_2$ only depends on how η walks through the unit circle in this interval (clockwise or counterclockwise). $\qquad\square$

A zero-crossing of the curvature function κ of a Legendre immersion is called an *inflection point*. One can have a better understanding of the classification of ordinary cusps by noticing that two consecutive ordinary cusps of a frontal have different types if and only if there is an odd number of inflection points between them. Indeed, this follows from the fact that $\mathrm{sgn}[\eta(t), \eta'(t)] = \mathrm{sgn}(\kappa(t)[\eta(t), b(\eta(t))]) = \mathrm{sgn}(\kappa(t))$ and from the continuity of κ.

Let $\gamma : S^1 \to X$ be a closed front, and let $C_\gamma := \{t_1, \ldots, t_m\}$ be the set of its (ordered) ordinary cusps. Attribute the letter a to a zig, and b to a zag, and form the word $w_\gamma := t_1 t_2 \cdots t_m$. Since m is even, it follows that the identification of w_γ in the free product $\mathbb{Z}_2 * \mathbb{Z}_2$ (considering the reduction $a^2 = b^2 = 1$) must be of the form $(ab)^k$ or $(ba)^k$. The number k is called the *Maslov index* (or *zigzag number*) of γ, and it will be denoted by $z(\gamma)$.

We will follow [Saji et al. 2009] to obtain the Maslov index in terms of the curvature pair of a Legendre immersion, but now the considered curvature pair is given by Birkhoff orthogonality instead of Euclidean orthogonality. First, let $\mathrm{P}^1(\mathbb{R})$ be the real projective line, and let $[x : y]$, defined as y/x, be coordinates on it. The curvature pair of a Legendre immersion can then be regarded as the smooth map $k_\gamma : S^1 \to \mathrm{P}^1(\mathbb{R})$ given by

$$k_\gamma(t) = [\alpha(t) : \kappa(t)],$$

where α and κ are, as usual, given as in (2-1) and (2-2). If we identify canonically the projective line with the one-dimensional circle (see Figure 2), then we can naturally define the *rotation number* of k_γ as its absolute number of (complete)
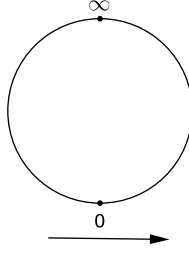
Figure 2. Identification $P^1(\mathbb{R}) \simeq S^1$, with orientation (positive ratios are on the right-hand side).

turns over the circle, counted with sign depending on the orientation. We say that a front is *generic* if all of its singular points are ordinary cusps and all of its self-intersections are *double points*, which means that if $t_0 \neq t_1$ and $\gamma(t_0) = \gamma(t_1)$, then $\eta(t_0)$ and $\eta(t_1)$ are linearly independent vectors.

Theorem 4.6. *Let $\gamma : S^1 \to X$ be a generic closed front. Then the zigzag number of γ equals the rotation number of k_γ.*

Proof. Following [Saji et al. 2009], the strategy of the proof is to count the number of times that k_γ passes through the point $[0 : 1]$ ($= \infty$) two consecutive times with the same orientation. First, notice that $k_\gamma(t) = [0 : 1]$ if and only if t is an ordinary cusp. Now observe that the sign of $\kappa(t)/\alpha(t)$ in a punctured neighborhood of a cusp is the same as the sign of $\alpha\kappa$. Therefore, we can decide whether we have a clockwise or a counterclockwise ∞-crossing at a singularity $t_0 \in S^1$ looking to the sign of $(\alpha\kappa)'(t_0) = \alpha'(t_0)\kappa(t_0)$. Namely, if $\alpha'(t_0)\kappa(t_0) < 0$, the ∞-crossing is counterclockwise, and if $\alpha'(t_0)\kappa(t_0) > 0$, then it is clockwise.

Now let $t_0, t_1 \in S^1$ be two consecutive singularities. It is clear that $\alpha'(t_0)$ and $\alpha'(t_1)$ have opposite signs. Therefore, if we have two consecutive zigs, or two consecutive zags, then the associated consecutive ∞-crossings have the same orientation, and consequently play no role in the rotation number. On the other hand, a zig followed by a zag (or vice-versa) yields a complete (positive or negative) turn over $P^1(\mathbb{R})$. This shows what we had to prove. $\qquad\square$

Remark 4.7. Since the choice of a normal field to turn a front into a Legendre immersion is not unique, and the associated curvature pair is not invariant under a reparametrization of the front, a comment is due. The classification of *all* ordinary cusps will change if we replace the field η by $-\eta$, and hence the Maslov index remains the same. Also, it is easily seen that a reparametrization of the front yields a new curvature pair where both previous curvature functions are multiplied by the same function. Therefore, the map $k_\gamma : S^1 \to P^1(\mathbb{R})$ defined previously is invariant under a reparametrization of the front, and so is its rotation number.

We shall describe now another way to obtain the Maslov index of a closed front. In view of Theorem 4.6, what changes is that we count the rotation number of k_γ by regarding zero-crossings instead of ∞-crossings. Geometrically, instead of using types of singular points, we classify inflection points (recall that, for us, an inflection point is a zero-crossing of the curvature function κ). As the reader will notice, this approach has the advantage of avoiding the use of reductions in free products.

Let, as usual, $(\gamma, \eta) : S^1 \to X \times S$ be a generic closed front with associated curvature pair (α, κ). We say that an inflection point $t_0 \in S^1$ is a *flip* if t_0 is a zero-crossing from negative to positive of $\alpha(t)\kappa(t)$, and a *flop* if t_0 is a zero-crossing from positive to negative of $\alpha(t)\kappa(t)$. Notice that every inflection point is a flip or a flop, since $\alpha(t_0)$ does not vanish (γ is a front) and t_0 is a zero-crossing of κ.

Theorem 4.8. *The Maslov index of a generic closed front γ is half the absolute value of the difference between its number of flips and flops. In other words,*

$$z(\gamma) = \tfrac{1}{2}|\# \text{flip} - \# \text{flop}|,$$

where $\#$flip and $\#$flop denote the number of inflection points for each respective type.

Proof. Before proving the theorem, it is interesting to capture the combinatorial flavor of the problem. Notice first that a flip corresponds to a counterclockwise zero-crossing of k_γ in $\mathrm{P}^1(\mathbb{R})$, and a flop corresponds to a clockwise zero-crossing. Between two consecutive singularities, we have two possibilities:

(1) The number of inflection points is even. In this case we have the same number of flips and flops, since α does not change its sign between two consecutive zeros.

(2) The number of inflection points is odd. In this case we have $|\# \text{flip} - \# \text{flop}| = 1$ between these singularities.

Moreover, successive zero-crossings of κ are always alternate, and then we have two consecutive flips (or flops) when there is a singular point between two consecutive inflection points. The reader is invited to draw some concrete examples, to better capture the ideas. We will give an analytic proof, however. As noticed in [Fukunaga and Takahashi 2013], the zigzag number is half the absolute value of the degree of the map $k_\gamma : S^1 \to \mathrm{P}^1(\mathbb{R})$. Since S^1 is path connected, we can calculate the degree of k_γ by counting the points of the set $k_\gamma^{-1}([1:0])$ where the derivative is orientation-preserving/reversing. In other words, the degree of k_γ is the difference between the numbers of counterclockwise and clockwise zero-crossings in $\mathrm{P}^1(\mathbb{R})$. Since each counterclockwise zero-crossing corresponds to a flip, and each clockwise zero-crossing corresponds to a flop, we have

$$z(\gamma) = \tfrac{1}{2}\deg(k_\gamma) = \tfrac{1}{2}|\# \text{flip} - \# \text{flop}|,$$

as we aimed to prove.   □

5. Evolutes and involutes of fronts

Let $\gamma : I \to X$ be a smooth regular curve whose circular curvature k does not vanish. Then the *evolute* of γ is the curve $e_\gamma : I \to X$ defined as

$$e_\gamma(t) = \gamma(t) - r(t)\eta(t),$$

where $r(t) := k(t)^{-1}$ is the curvature radius of γ at $t \in I$ and $\eta(t)$ is the left normal vector to γ at $t \in I$ (both defined as in our introduction). A *parallel* of γ is a curve of the type

(5-1) $$\gamma_d(t) = \gamma(t) + d\eta(t),$$

for some fixed $d \in \mathbb{R}$. As in the Euclidean case, the singular points of the parallels of γ sweep out the evolute of γ (see [Balestro et al. 2018, Section 9]). Based on this characterization, we will follow [Fukunaga and Takahashi 2014] to define the evolute of a front in a Minkowski plane.

First, let $(\gamma, \eta) : I \to X$ be a Legendre immersion. Then, using the normal field η we can define a *parallel of the front* γ exactly by (5-1).

Lemma 5.1. *A parallel of a front* $\gamma : I \to X$ *is also a front.*

Proof. Let (γ, η) be a Legendre immersion. We shall see that (γ_d, η) is a Legendre immersion. From (2-1) and (2-2) we have $\gamma'_d(t) = \gamma'(t) + d\eta'(t) = (\alpha(t) + d\kappa(t))\xi(t)$. Therefore, $\eta(t) \dashv_B \gamma'_d(t)$ for each $t \in I$. It only remains to prove that (γ_d, η) is an immersion. For this, just write down the equations

(5-2) $$\gamma'_d(t) = (\alpha(t) + d\kappa(t))\xi(t) \quad \text{and} \quad \eta'(t) = \kappa(t)\xi(t),$$

and observe that γ'_d and η' vanish simultaneously if and only if α and κ vanish simultaneously. But this would contradict the hypothesis that (γ, η) is a Legendre immersion. □

Example 5.2. Let $\mathbb{R}^2$ be endowed with the usual l_p norm, where $1 < p < +\infty$, and let $\gamma(t) = (t, t^2/2)$ be defined for $t > 0$. To determine the parallels of γ we have to determine the normal field $\eta(t)$. For that, it suffices to find the point where the tangent direction $\gamma'(t) = (1, t)$ supports the unit circle (in the fourth quadrant, according to the orientation of γ). We consider the parametrization of the portion of the unit circle in the fourth quadrant given by $s \mapsto (s^{1/p}, -(1-s)^{1/p})$, and differentiating we obtain a vector that has to point in the direction of $\gamma'(t) = (1, t)$. Hence

$$\frac{(1-s)^{1/q}}{s^{1/q}} = \frac{1}{t},$$

from which we get $s = 1/(1 + t^{-q})$. Therefore,

$$\eta(t) = \left(\frac{1}{(1 + t^{-q})^{1/p}}, \frac{-t^{-q/p}}{(1 + t^{-q})^{1/p}} \right).$$
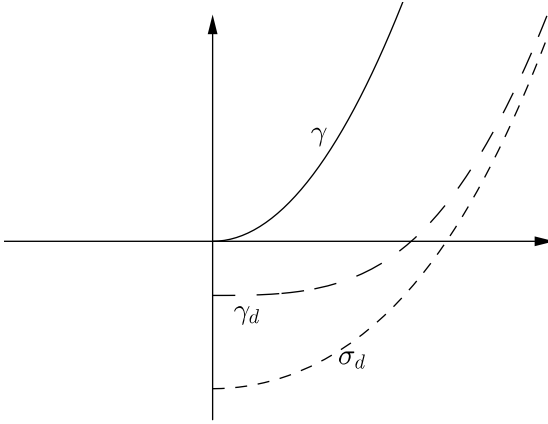
Figure 3. Parallels of γ.

The expression for the parallels of γ follows. In Figure 3 we illustrate a parallel with respect to the l_p norm (denoted by γ_d), and also a parallel in the usual Euclidean norm (denoted by σ_d). Both are constructed with the same value of $d > 0$.

From now on we will *always* assume that $\gamma : I \to X$ is a front, and that the pair (γ, η) is an associated Legendre immersion whose curvature pair (α, κ) is such that κ does not vanish. Then, we define the *evolute* of γ to be

$$(5\text{-}3) \qquad e_\gamma(t) = \gamma(t) - \frac{\alpha(t)}{\kappa(t)}\eta(t), \quad t \in I.$$

Notice that this definition makes sense (as an extension of the usual evolute of a regular curve) in view of Lemma 2.1. Also, observe that a front and its evolute intersect in (and only in) singular points of the front. Further, as we have mentioned, the evolute of a front is the set of singular points of the parallels of this front. We will now prove this.

Proposition 5.3. *The set of points of the evolute of a front γ is precisely the set of singular points of the parallels of γ.*

Proof. For each $t \in I$, the point $e_\gamma(t)$ belongs to the parallel given by $d = -\frac{\alpha(t)}{\kappa(t)}$. Since $\gamma_d'(t) = (\alpha(t) + d\kappa(t))\xi(t)$, it follows that γ_d is singular at that point. On the other hand, a singular point of a parallel γ_d must be given by some $t \in I$ such that $d = -\frac{\alpha(t)}{\kappa(t)}$. $\qquad \square$

We will verify that the evolute of a front is also a front, whose curvature can be obtained in terms of (α, κ). To do so, from now on we consider the map b defined only for unit vectors. We do this because the restriction $b|_S : S \to S$ is bijective, and hence invertible. Let (γ, η) be a Legendre immersion with associated curvature function given by (α, κ), and let e_γ be its evolute. We will write $\nu(t) = -b^{-1}(\eta(t))$. Thus, we have the following.

Theorem 5.4. *The pair (e_γ, v) is a Legendre immersion with associated curvature given by the equalities*

$$e_\gamma'(t) = -\frac{d}{dt}\left(\frac{\alpha(t)}{\kappa(t)}\right)\eta(t) \quad and \quad v'(t) = \beta(t)\eta(t).$$

Since $\eta(t) = -b(v(t))$, it follows that the curvature pair of (e_γ, v) is given by $\left(\frac{d}{dt}\left(\frac{\alpha}{\kappa}\right), -\beta\right)$. Moreover, the function $-\beta(t)$ is given by

$$-\beta(t) = \frac{\kappa(t)}{\rho(t)},$$

where $\rho(t) := \rho(v(t))$ is as defined in [Remark 2.2](#).

Proof. The first equation comes easily by differentiating (5-3). For the second, first notice that since v is a unit vector field, it follows that $v(t) \dashv_B v'(t)$, and therefore $v'(t)$ is parallel to $\eta(t)$. Notice that already this characterizes the pair (e_γ, v) as a Legendre curve. Before showing that this is indeed an immersion, we will prove the expression for $-\beta$. Differentiating v yields

$$v'(t) = -Db_{\eta(t)}^{-1}(\eta'(t)) = -\kappa(t)Db_{\eta(t)}^{-1}(b(\eta(t))) = -\kappa(t)\bar\rho(t)\eta(t),$$

for some function $\bar\rho$. We will prove then that $\bar\rho(t) = \rho(v(t))^{-1}$. To do this, let $v = \eta(t)$ and write down the equalities

$$Db_{b^{-1}(v)}(v) = \rho(b^{-1}(v))b(v) \quad and \quad Db_v^{-1}(b(v)) = \bar\rho(v)v.$$

Therefore,

$$\bar\rho(v)v = Db_v^{-1}(b(v)) = \frac{1}{\rho(b^{-1}(v))}Db_v^{-1}(Db_{b^{-1}(v)}(v)) = \frac{1}{\rho(b^{-1}(v))}v,$$

and since $\rho(b^{-1}(v)) = \rho(-v(t)) = \rho(v(t))$, the desired result follows. Now, since by our hypothesis the function κ does not vanish, it follows that (e_γ, v) is, in fact, a Legendre immersion. $\qquad\square$

Remark 5.5. Unlike in the Euclidean subcase, here the function $\rho(t)$ appears. This function carries, somehow, the "distortion" of the unit circle of the considered norm with respect to the Euclidean unit circle. Such functions appear even in Radon planes.

An evolute of a regular curve in the Euclidean plane is the envelope of its normal lines, and the same holds for the evolute of a regular curve in a Minkowski plane, when we replace inner product orthogonality by Birkhoff orthogonality (see [Craizer 2014]). From (5-3) and [Theorem 5.4](#) it follows that the tangent line of the evolute e_γ of a Legendre immersion (γ, η) at $t \in I$ is precisely the normal line of γ at $\gamma(t)$. Therefore, the evolute of a Legendre immersion can be regarded as the envelope of the normal line field of the immersion.

The family of normal lines of a Legendre immersion (γ, η) is the zero set of the function $F : I \times X \to \mathbb{R}$ given by $F(t, v) = [\gamma(t) - v, \eta(t)]$. Indeed, for each fixed $t \in I$ the zero set of $F_t(v) := F(t, v)$ is the normal line of γ at $\gamma(t)$. Therefore, we could expect that the points, for which both F and its derivative with respect to t vanish, describe the evolute of the Legendre immersion (see [Bruce and Giblin 1981]). This is indeed true, as we shall see next.

Proposition 5.6. *For the function $F : I \times X \to \mathbb{R}$ defined above we have*

$$F(t, v) = \frac{\partial F}{\partial t}(t, v) = 0 \ \text{if and only if} \ v = \gamma(t) - \frac{\alpha(t)}{\kappa(t)}\eta(t).$$

Therefore, the envelope of the normal line field of a Legendre immersion is precisely its evolute.

Proof. It is clear that $F(t, v) = 0$ if and only if $\gamma(t) - v = \lambda\eta(t)$ for some $\lambda \in \mathbb{R}$. Differentiating, we have

$$\frac{\partial F}{\partial t}(t, v) = [\gamma'(t), \eta(t)] + [\gamma(t) - v, \eta'(t)] = \alpha(t)[\xi(t), \eta(t)] + \kappa(t)[\gamma(t) - v, \xi(t)].$$

Hence, $F(t, v) = \frac{\partial F}{\partial t}(t, v) = 0$ if and only if $(\alpha(t) - \lambda\kappa(t))[\xi(t), \eta(t)] = 0$ and $\gamma(t) - v = \lambda\eta(t)$. Since $[\xi(t), \eta(t)]$ does not vanish, the desired result follows. $\square$

The involute of a regular curve γ is a curve whose evolute is γ (see [Balestro et al. 2018] and [Craizer 2014]). We can easily extend this definition to our new context, in a manner similar to that for the Euclidean subcase in [Fukunaga and Takahashi 2016]. An *involute* of a Legendre immersion $(\gamma, \eta) : [0, c] \to X \times S$ whose curvature κ does not vanish is a Legendre immersion whose evolute is (γ, η).

Theorem 5.7. *Let $(\gamma, \eta) : [0, c] \to X \times S$ be a Legendre immersion with curvature pair (α, κ), and assume that κ does not vanish. For any $d \in \mathbb{R}$, the map*

$$(\sigma, \xi) : [0, c] \to X \times S,$$

where $\xi(t) = b(\eta(t))$, as usual, and

$$(5\text{-}4) \qquad \sigma(t) = \gamma(0) - \int_0^t \left(\int_0^s \alpha(\tau) \, d\tau \right) \xi'(s) \, ds + d\xi(t)$$

is a Legendre immersion with curvature pair

$$(5\text{-}5) \qquad \left(\kappa(t)\rho(\eta(t)) \left(-d + \int_0^t \alpha(\tau) \, d\tau \right), -\kappa(t)\rho(\eta(t)) \right) : [0, c] \to \mathbb{R}^2,$$

and with ρ defined as in (2-4). Moreover, (σ, ξ) is an involute of (γ, η).

Proof. First, differentiation of σ yields

$$\sigma'(t) = \left(d - \int_0^t \alpha(\tau) \, d\tau \right) \xi'(t) = \kappa(t)\rho(\eta(t)) \left(-d + \int_0^t \alpha(\tau) \, d\tau \right) b(\xi(t)),$$

where the last equality comes from (2-4). Notice that $\xi(t) \dashv \sigma'(t)$ for each $t \in [0, c]$, and hence the pair (σ, ξ) is a Legendre curve. The derivative of the normal field ξ is given by (2-4), and then the curvature pair of (σ, ξ) is precisely the one given in (5-5). Since $\kappa(t)\rho(\eta(t))$ does not vanish, it follows that (σ, ξ) is indeed an immersion.

It remains to show that the evolute of (σ, ξ) is (γ, η). From the definition, the evolute of σ is the curve

$$e_\sigma(t) = \sigma(t) + \left(-d + \int_0^t \alpha(\tau)\,d\tau\right)\xi(t), \quad t \in [0, c].$$

Note that $e_\sigma(0) = \gamma(0)$, so it suffices to show that e_σ and γ have the same derivative. A simple calculation gives $e_\sigma'(t) = \alpha(t)\xi(t) = \gamma'(t)$, which concludes the proof. $\square$

Observe that a front has a family of involutes (with parameter $d \in \mathbb{R}$), which is a family of parallel curves. In view of Proposition 5.3, a front can be characterized as the set of singular points of these involutes. This remark is the reason for our slightly different approach in comparison with [Fukunaga and Takahashi 2016]. Also one would expect that this happens since any of the parallels has the same normal vector field, and therefore yields the same envelope (which is γ, in view of Proposition 5.6).

6. Singular points and vertices of Legendre immersions

A point where the derivative of the curvature of a regular curve vanishes is usually called a *vertex*. We shall extend this definition to fronts in normed planes, in the same way as in [Fukunaga and Takahashi 2014] for the Euclidean subcase. Let $(\gamma, \eta) : [0, c] \to X \times S$ be a Legendre immersion with curvature pair (α, κ), and assume that κ does not vanish. We say that $t_0 \in [0, c]$ is a *vertex* of the front γ (or of the associated Legendre immersion) if

$$\frac{d}{dt}\left(\frac{\alpha}{\kappa}\right)(t_0) = 0.$$

Notice that, as in the regular case, a vertex of a front corresponds to a singular point of its evolute (and that the converse also holds). A vertex which is a regular point of γ is said to be a *regular vertex*. As one would suspect, we can reobtain the vertex in terms of the function F which describes the normal line field of the front (see Proposition 5.6).

Lemma 6.1. *Let $(\gamma, \eta) : I \to X \times S$ be a Legendre immersion, and let $F : I \times X \to \mathbb{R}$ be defined as $F(t, v) = [\gamma(t) - v, \eta(t)]$. Therefore, $t_0 \in I$ is a vertex of γ if and only if*

$$\frac{\partial^2 F}{\partial t^2}(e_\gamma(t_0), t_0) = 0.$$

Proof. A simple calculation gives

$$\frac{\partial^2 F}{\partial t^2}(t, v) = [\gamma''(t), \eta(t)] + [\gamma(t) - v, \eta''(t)].$$

Hence, in a point $(e_\gamma(t), t)$ we have

$$\frac{\partial^2 F}{\partial t^2}(e_\gamma(t), t) = [\gamma''(t), \eta(t)] + \left[\frac{\alpha(t)}{\kappa(t)}\eta(t), \eta''(t)\right] = [\eta(t), \xi(t)]\left(\frac{\alpha(t)}{\kappa(t)}\kappa'(t) - \alpha'(t)\right),$$

and it is clear that the latter vanishes at $t_0 \in I$ if and only if $\frac{d}{dt}\left(\frac{\alpha}{\kappa}\right)(t_0) = 0$. □

The easy observation that the function α/κ must have a local extremum strictly between two consecutive singular points leads to the following version of the *four vertex theorem*.

Proposition 6.2. *Either of the following conditions is sufficient for a closed front* $\gamma : S^1 \to X$ *to have at least four vertices*:

(a) γ *has at least four singular points.*

(b) γ *has at least two singular points which are not ordinary cusps.*

Proof. For (a), notice that if γ has at least four singular points, then α/κ has at least four local extrema, each of them corresponding to a vertex. For (b), just notice that a singular point which is not an ordinary cusp is, in particular, a vertex. Indeed, the derivative

$$\frac{d}{dt}\left(\frac{\alpha}{\kappa}\right)(t) = \frac{\alpha'(t)\kappa(t) - \alpha(t)\kappa'(t)}{\kappa(t)^2}$$

vanishes whenever $\alpha(t) = \alpha'(t) = 0$ (and this happens in a singular point which is not an ordinary cusp, see Lemma 4.1). In addition to these vertices, the existence of two regular vertices (guaranteed by the two singular points) finishes the proof. □

It is easy to see that a singular point of a Legendre curve in a Minkowski plane is still a singular point if we change the considered norm. Moreover, an ordinary cusp remains an ordinary cusp. However, a vertex of a Legendre curve may not be a vertex of it if we change the norm of the plane. Indeed, every point of a circle is a vertex (the circular curvature is constant); this is no longer the case when we change the norm.

But somehow we can still relate numbers of singular points to numbers of vertices. Since there is at least one vertex strictly between two consecutive singular points of a front, we have that the number of vertices of a closed front is greater than or equal to its number of singular points. Therefore, if $\Sigma(\gamma)$ and $V(\gamma)$ denote the set of singular points and the set of vertices of γ, respectively, and σ is an involute of γ, then we have

$$\#\Sigma(\sigma) \leq \#\Sigma(\gamma) \leq \#V(\gamma),$$

where the first inequality comes from the observation that the vertices of σ correspond to the singular points of γ, since γ is the evolute of σ. This observation is proved for the Euclidean subcase in [Fukunaga and Takahashi 2016], and what we wanted to show is that it only depends on the fact that there always exists at least one vertex between two consecutive singular points of a Legendre curve.

7. Contact between Legendre curves

The concept of *contact* between regular plane curves intends, intuitively, to describe how "similar" two curves are in a neighborhood of a point. In [Fukunaga and Takahashi 2013, Section 3] this notion is extended to Legendre curves as follows: given $k \in \mathbb{N}$, two Legendre curves $(\gamma_1, \eta_1) : I_1 \to X \times S$ and $(\gamma_2, \eta_2) : J \to X \times S$ are said to have *k-th order contact* at $t = t_0$, $u = u_0$ if

$$\frac{d^j}{dt^j}(\gamma_1, \eta_1)(t_0) = \frac{d^j}{du^j}(\gamma_2, \eta_2)(u_0) \quad \text{for } j = 0, \dots, k-1$$

and

$$\frac{d^k}{dt^k}(\gamma_1, \eta_1)(t_0) \neq \frac{d^k}{du^k}(\gamma_2, \eta_2)(u_0).$$

If only the first condition holds, then we say that the curves have *at least k-th order contact* at $t = t_0$ and $u = u_0$. In the mentioned paper, this was defined exactly in the same way, but considering that the normal vector field of each Legendre curve is the one given by the Euclidean orthogonality. We shall see that if two Legendre curves have k-th order contact for a given fixed norm, then they have k-th order contact for any norm.

Proposition 7.1. *Let (γ, η) and $(\bar{\gamma}, \bar{\eta})$ be Legendre curves which have k-th order contact at $t = t_0$ and $u = u_0$. Therefore, changing the norm of the plane, the new Legendre curves (derived in the same sense as discussed in the last paragraph of Section 2) still have k-th order contact at $t = t_0$ and $u = u_0$.*

Proof. Assume that $\| \cdot \|_1$ and $\| \cdot \|_2$ are smooth and strictly convex norms in the plane with unit circles S_1 and S_2, respectively. Denote by $h : S_1 \to S_2$ the map introduced in the last paragraph of Section 2, which takes each vector $v \in S_1$ to the vector $h(v) \in S_2$ such that S_2 is supported at $h(v)$ by the same direction which supports S_1 at v.

Let (γ, η) be a Legendre curve in the norm $\| \cdot \|_1$. Then $(\gamma, h \circ \eta)$ is a Legendre curve in the norm $\| \cdot \|_2$. Writing $v = h \circ \eta$, the strategy is to prove that, for any $m \in \mathbb{N}$, the m-th derivative of v at $t = t_0$ only depends on h and $\eta^{(j)}(t_0)$ for $j = 0, \dots, m$. For this reason, we note

$$v^{(m)}(t_0) = D^m h_{(\eta(t_0), \eta'(t_0), \dots, \eta^{(m-1)}(t_0))}(\eta^{(m)}(t_0)),$$

where $D^m h_{(\eta(t_0), \eta'(t_0), \dots, \eta^{(m-1)}(t_0))}$ is the usual m-th derivative of the map h, which is a linear map defined over $T_{\eta^{(m-1)}(t_0)}(\cdots (T_{\eta'(t_0)}(T_{\eta(t_0)} S_1)))$. Since this derivative

clearly depends only on h and $\eta^{(j)}(t_0)$ for $j = 0, \ldots, m$, it follows that $\nu^{(m)}(t_0)$ also only depends on it.

Hence, if a change of the norm carries over the normal fields η and $\bar{\eta}$ to ν and $\bar{\nu}$, respectively, then we have that $\nu^{(j)}(t_0) = \bar{\nu}^{(j)}(u_0)$ for every $j = 0, \ldots, k$ if and only if the same happens for η and $\bar{\eta}$ (the "only if" part comes from h being an immersion for all $v \in S_1$). $\qquad\square$

It is a well known fact that the contact between two regular curves can be characterized by means of their curvatures. In [Fukunaga and Takahashi 2013, Theorem 3.1] this is extended to Legendre curves using the developed curvature functions. We shall verify that we can obtain an analogous result (in one of the directions, only) when we are not working in the Euclidean subcase.

Theorem 7.2. *Let $(\gamma_1, \eta_1) : I_1 \to X \times S$ and $(\gamma_2, \eta_2) : I_2 \to X \times S$ be Legendre curves with curvature pairs (α_1, κ_1) and (α_2, κ_2), respectively. If these curves have at least k-th order contact at $t = t_0$ and $u = u_0$, then*

$$\frac{d^j}{dt^j}(\alpha_1, \kappa_1)(t_0) = \frac{d^j}{du^j}(\alpha_2, \kappa_2)(u_0), \quad \text{for} \quad j = 0, \ldots, k-1.$$

However, the converse may not be true (even up to isometry) if the norm is not Euclidean.

Proof. The proof is essentially the same as in the mentioned theorem in [Fukunaga and Takahashi 2013]. Differentiating (2-1) and (2-2), we have the equalities

$$\gamma^{(k)}(t) = \sum_{j=0}^{k} \binom{k}{j} \alpha^{(j)}(t) \xi^{(k-j)}(t) \quad \text{and} \quad \eta^{(k)}(t) = \sum_{j=0}^{k} \binom{k}{j} \kappa^{(j)}(t) \xi^{(k-j)}(t).$$

If $k = 1$, then we have $\alpha_1'(t_0)\xi_1(t_0) = \alpha_2'(u_0)\xi_2(u_0)$ and $\kappa_1(t_0)\xi_1(t_0) = \kappa_2(u_0)\xi_2(u_0)$. Since $\eta_1(t_0) = \eta_2(u_0)$, it follows that $\xi_1(t_0) = \xi_2(u_0)$, and so we have $\alpha_1(t_0) = \alpha_2(u_0)$ and $\kappa_1(t_0) = \kappa_2(t_0)$. Regarding higher order contact, one just has to proceed inductively by using the previous differentiation formulas.

We illustrate the fact that the converse does not necessarily hold if the norm is not Euclidean with a constructive example. Take two disjoint arcs γ_1 and γ_2 in the unit circle which do not overlap under an isometry (the existence of such arcs is guaranteed by [Balestro et al. 2018, Proposition 7.1]). Assume that these arcs are parametrized by arc-length and choose parameters t_0 and u_0 such that the supporting directions to $\gamma_1(t_0)$ and $\gamma_2(u_0)$ are distinct. These arcs, together with their respective normal vector fields (η_1 and η_2, say), are Legendre curves whose curvatures and derivatives of curvatures coincide. However, there is no isometry carrying $\gamma_1(t_0)$ to $\gamma_2(u_0)$ and $\eta_1(t_0)$ to $\eta_2(u_0)$, and hence these curves do not have contact of any order up to isometry. $\qquad\square$

8. Pedal curves of frontals

A *pedal curve* of a regular curve γ is usually defined to be the locus of the orthogonal projections of a fixed point p to the tangent lines of γ. The existence of a tangent field allows us to carry over this definition to frontals in a straightforward manner.

Definition 8.1. Let $(\gamma, \eta) : I \to X \times S$ be a Legendre curve, and let $\xi(t) = b(\eta(t))$, as usual. Fix a point $p \in X$. The *pedal curve* of the frontal γ with respect to p is the curve $\gamma_p : I \to X$ which associates to each $t \in I$ the unique point $\gamma_p(t)$ of the line $s \mapsto \gamma(t) + s\xi(t)$ such that $\gamma_p(t) - p \dashv_B \xi(t)$ (see Figure 4). In other words, $\gamma_p(t)$ is the intersection of the parallel to $\eta(t)$ drawn through p with the tangent line of γ at $\gamma(t)$.

It is useful, however, to have a formula for the pedal curve which we can work with. To this end, fix $t \in I$ and let $\alpha, \beta \in \mathbb{R}$ be constants such that $\gamma_p(t) = \gamma(t) + \alpha\xi(t)$ and $p - \gamma(t) = \beta\eta(t)$. From the vectorial sum $\alpha\xi(t) + \beta\eta(t) = p - \gamma(t)$ we have

$$\alpha[\xi(t), \eta(t)] = [p - \gamma(t), \eta(t)].$$

Since $\eta(t) \dashv_B \xi(t)$ and the basis $\{\eta(t), \xi(t)\}$ is positively oriented, the above equality reads $\alpha\|\xi(t)\|_a = [\gamma(t) - p, \eta(t)]$. Hence,

$$(8\text{-}1) \qquad \gamma_p(t) = \gamma(t) + [\gamma(t) - p, \eta(t)]\xi_a(t),$$

where $\xi_a(t) = \xi(t)/\|\xi(t)\|_a$ is the vector ξ normalized in the antinorm.

Notice that from our geometric definition it follows that a pedal curve of a frontal with respect to a given point does not depend on the parametrization of the frontal. We give an illustrated example of a pedal curve of a regular curve.

Example 8.2. Consider the space $\mathbb{R}^2$ endowed with the usual l_p norm for some $1 < p < +\infty$, and let $q \in \mathbb{R}$ be such that $1/p + 1/q = 1$. A simple calculation shows that the right pedal curve of the unit circle with respect to the point $(0, 1)$ is obtained by joining the curve

$$\sigma(t) = \begin{cases} (t^{1/p} - t^{1/p}(1-t)^{1/q}, t + (1-t)^{1/p}), & t \in [0, 1], \\ ((2-t)^{1/p} + (2-t)^{1/p}(t-1)^{1/q}, 2 - t - (t-1)^{1/p}), & t \in [1, 2], \end{cases}$$

with its reflection through the y-axis. Figure 5 illustrates the case $p = 3$.
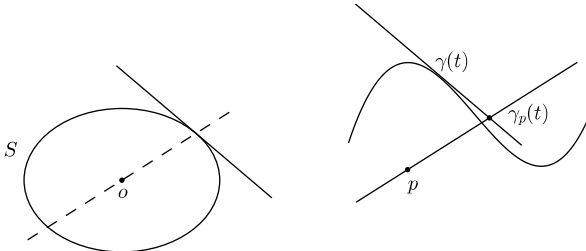
Figure 4. Constructing a pedal curve.
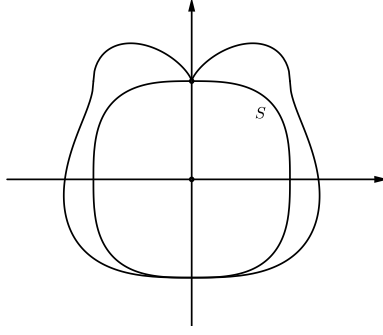
Figure 5. The unit circle of l_3 and its pedal curve with respect to the point $(0, 1)$.

As an interesting property of pedal curves, we will prove that a frontal can be regarded as the envelope of a certain family of lines defined by (any) one of its pedal curves.

Proposition 8.3. *Let γ_p be a pedal curve of a frontal γ. Then γ is the envelope of the family of lines*

$$\{l_t : s \mapsto \gamma_p(t) + sb(\gamma_p(t) - p)\}_{t \in I},$$

where l_{t_0} is defined by taking limits if $p = \gamma_p(t_0)$ for some $t_0 \in I$. As in Proposition 5.6, if $F : I \times X \to \mathbb{R}$ is given by $F(t, v) = [\gamma_p(t) - v, b(\gamma_p(t) - p)]$, then

$$F(t, v) = \frac{\partial F}{\partial t}(t, v) = 0 \quad \text{if and only if } v = \gamma(t).$$

In particular, any frontal is a pedal curve of some curve in the plane.

Proof. We may assume, without loss of generality, that locally $b(\gamma_p(t) - p) = \xi(t)$. Differentiating F and applying (8-1) yields

$$\frac{\partial F}{\partial t}(t, v) = [\gamma_p'(t), \xi(t)] + [\gamma_p(t) - v, \xi'(t)]$$

$$= [\gamma(t) - p, \eta(t)] \cdot [\xi_a'(t), \xi(t)] + [\gamma_p(t) - v, \xi'(t)].$$

Assume that $F(t, v) = \frac{\partial F}{\partial t}(t, v) = 0$. From $F(t, v) = 0$ we have that $\gamma_p(t) - v = \alpha \xi(t)$ for some $\alpha \in \mathbb{R}$. Due to the other equality and to the above calculation, we get

$$0 = [\gamma(t) - p, \eta(t)] \cdot [\xi_a'(t), \xi(t)] + \alpha[\xi(t), \xi'(t)].$$

From the definition of $\xi_a(t)$ it follows that

$$[\xi_a'(t), \xi(t)] = \frac{[\xi(t), \xi'(t)]}{[\xi(t), \eta(t)]},$$

and substituting this into the previous equality yields immediately the equality $\alpha[\xi(t), \eta(t)] = [p - \gamma(t), \eta(t)]$. Therefore, from (8-1) we have $v = \gamma(t)$. The converse is straightforward. $\square$

As usual, we assume that $\gamma : I \to X$ is a frontal with associated normal field η whose curvature is (α, κ), and we also assume that ξ and ξ_a are defined as before. Notice that differentiation of (8-1) yields

$$\gamma_p'(t) = \gamma'(t) + [\gamma'(t), \eta(t)]\xi_a(t) + [\gamma(t) - p, \eta'(t)]\xi_a(t) + [\gamma(t) - p, \eta(t)]\xi_a'(t)$$
$$= [\gamma(t) - p, \eta'(t)]\xi_a(t) + [\gamma(t) - p, \eta(t)]\xi_a'(t),$$

where the second equality is justified since $[\gamma'(t), \eta(t)]\xi_a(t) = -\gamma'(t)$. From the definition of ξ_a and from (2-4), the above equality may be written as

$$(8\text{-}2) \quad \gamma_p' = \frac{\kappa}{[\eta, \xi]}\left(\left([\gamma - p, \xi] + \rho\frac{[\gamma - p, \eta] \cdot [\eta, b(\xi)]}{[\eta, \xi]}\right)\xi - \rho[\gamma - p, \eta]b(\xi)\right),$$

where we are omitting the variable t for the sake of having a clearer notation. Also we are denoting $\rho = \rho(t) = \rho(\eta(t))$. Notice that if t_0 is a point where κ vanishes, then t_0 is a singular point of the pedal curve γ_p. Also, if p is a point of γ, then it is also a singular point of γ_p. Finally, the only remaining possibility of γ_p having a singular point would be if

$$[\gamma(t) - p, \xi(t)] + \rho(t)\frac{[\gamma(t) - p, \eta(t)] \cdot [\eta(t), b(\xi(t))]}{[\eta(t), \xi(t)]} = 0$$

and

$$[\gamma(t) - p, \eta(t)] = 0,$$

but if the second equality holds, and $p \notin \gamma(I)$, then the first equality does not hold. Indeed, if $\gamma(t) - p \neq 0$, then either $[\gamma(t) - p, \xi(t)] \neq 0$ or $[\gamma(t) - p, \eta(t)] \neq 0$. If p is a point of γ, then γ_p is not necessarily a frontal (a counterexample is given by Example 8.2, in view of Proposition 4.2). However, based on this observation we can prove that if $p \notin \gamma(I)$, then γ_p is a frontal.

Theorem 8.4. *Let $(\gamma, \eta) : I \to X \times S$ be a Legendre curve with curvature (α, κ). If $p \in X \setminus \gamma(I)$, then the pedal curve γ_p is a frontal. Moreover, the singular points of γ_p correspond exactly to the points where κ vanishes.*

Proof. The equality (8-2) can be written as

$$\gamma_p'(t) = \frac{\kappa(t)}{[\eta(t), \xi(t)]}\zeta(t),$$

where $\zeta(t)$ is the nonvanishing vector field

$$\zeta = \left([\gamma - p, \xi] + \rho\frac{[\gamma - p, \eta] \cdot [\eta, b(\xi)]}{[\eta, \xi]}\right)\xi - \rho[\gamma - p, \eta]b(\xi).$$

Here again we omitted, for the sake of simplicity, the parameter. Therefore, abusing the notation and setting $\nu(t) = b^{-1}(\zeta(t))$, we have that (γ_p, ν) is a Legendre curve (with tangent field given by $\zeta/\|\zeta\|$). Also, since ζ does not vanish, it follows that $t \in I$ is a singular point of the pedal curve γ_p if and only if $\kappa(t) = 0$. $\square$

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REMARKS ON CRITICAL METRICS OF
THE SCALAR CURVATURE AND VOLUME FUNCTIONALS
ON COMPACT MANIFOLDS WITH BOUNDARY

HALYSON BALTAZAR AND ERNANI RIBEIRO, JR.

**We provide a general Bochner type formula which enables us to prove
some rigidity results for *V*-static spaces. In particular, we show that
an *n*-dimensional positive static triple with connected boundary and
positive scalar curvature must be isometric to the standard hemisphere,
provided that the metric has zero radial Weyl curvature and satisfies a
suitable pinching condition. Moreover, we classify *V*-static spaces with
nonnegative sectional curvature.**

1. Introduction

Let (M^n, g) be a connected Riemannian manifold. Following the terminology used
by Miao and Tam [2009] as well as Corvino, Eichmair and Miao [Corvino et al.
2013], we say that g is a *V-static metric* if there is a smooth function f on M^n and
a constant κ satisfying the *V*-static equation

$$\text{(1-1)} \qquad \mathfrak{L}_g^*(f) = -(\Delta f)g + \text{Hess } f - f \text{ Ric} = \kappa g,$$

where $\mathfrak{L}_g^*$ is the formal L^2-adjoint of the linearization of the scalar curvature
operator $\mathfrak{L}_g$, which plays an important role in problems related to prescribing the
scalar curvature function. Here, Ric, Δ and Hess stand, respectively, for the Ricci
tensor, the Laplacian operator and the Hessian form on M^n. Such a function f is
called *V-static potential*.

It is well known that *V*-static metrics are important in understanding the interplay
between volume and scalar curvature. They arise from the modified problem of
finding stationary points for the volume functional on the space of metrics whose
scalar curvature is equal to a given constant (see [Corvino et al. 2013; Miao and
Tam 2009; 2011; Yuan 2016]). In general, the scalar curvature is not sufficient for
controlling the volume. However, Miao and Tam [2012] proved a rigidity result for
the upper hemisphere with respect to nondecreasing scalar curvature and volume.

Corvino et al. [2013] were able to show that when the metric g does not admit a nontrivial solution to (1-1), then one can achieve simultaneously a prescribed perturbation of the scalar curvature that is compactly supported in a bounded domain Ω and a prescribed perturbation of the volume by a small deformation of the metric in $\overline{\Omega}$. We highlight that a Riemannian metric g for which there exists a nontrivial function f satisfying (1-1) must have constant scalar curvature R (see [Corvino et al. 2013, Proposition 2.1; Miao and Tam 2009, Theorem 7]).

The case where $\kappa \neq 0$ in (1-1) and the potential function f vanishes on the boundary was studied by Miao and Tam [2009]. In this approach, a *Miao–Tam critical metric* is a 3-tuple (M^n, g, f), where (M^n, g) is a compact Riemannian manifold of dimension at least 3 with a smooth boundary ∂M and $f : M^n \to \mathbb{R}$ is a smooth function such that $f^{-1}(0) = \partial M$ satisfying the overdetermined-elliptic system

(1-2) $$\mathfrak{L}_g^*(f) = -(\Delta f)g + \operatorname{Hess} f - f \operatorname{Ric} = g.$$

Miao and Tam [2009] showed that these critical metrics arise as critical points of the volume functional on M^n when restricted to the class of metrics g with prescribed constant scalar curvature such that $g_{|T\partial M} = h$ for a prescribed Riemannian metric h on the boundary. Some explicit examples of Miao–Tam critical metrics are in the form of warped products and those examples include the spatial Schwarzschild metrics and AdS–Schwarzschild metrics restricted to certain domains containing their horizon and bounded by two spherically symmetric spheres (see Corollaries 3.1 and 3.2 in [Miao and Tam 2011]). For more details see, for instance, [Baltazar and Ribeiro 2017; Barros et al. 2015; Batista et al. 2017; Corvino et al. 2013; Miao and Tam 2009; 2011; Yuan 2016].

We also remark that (1-1) can be seen as a generalization of the *static equation* $\mathfrak{L}_g^*(f) = 0$ (see [Ambrozio 2017; Corvino 2000]), namely, $\kappa = 0$ in (1-1). We remember that a *positive static triple* is a triple (M^n, g, f) consisting of a connected n-dimensional smooth manifold M with boundary ∂M (possibly empty), a complete Riemannian metric g on M and a nontrivial static potential $f \in C^\infty(M)$ that is nonnegative, vanishes precisely on ∂M and satisfies the static equation

(1-3) $$\mathfrak{L}_g^*(f) = -(\Delta f)g + \operatorname{Hess} f - f \operatorname{Ric} = 0.$$

For the sake of completeness, it is very important to recall the following classical example of a positive static triple with nonempty boundary.

Example 1.1. An example of positive static triple with connected nonempty boundary is given by choosing $(\mathbb{S}_+^n(r), g)$ to be the open upper n-hemisphere $\mathbb{S}_+^n(r)$ of radius r in $\mathbb{R}^{n+1}$ endowed with the Euclidean metric g. Hence, $\partial M = \mathbb{S}^{n-1}(r)$ is the equator and the corresponding height function f on $\mathbb{S}_+^n(r)$ is positive, vanishes along $\partial M = \mathbb{S}^{n-1}(r)$ and satisfies (1-3).

It has been conjectured in 1984 that the only static vacuum spacetime with positive cosmological constant and connected event horizon is the de Sitter space of radius r. This conjecture is the so-called *cosmic no-hair conjecture* and it was formulated by Boucher, Gibbons and Horowitz in [Boucher et al. 1984]. It is closely related to the Fischer–Marsden conjecture (see [Shen 1997]). It should be emphasized that there are positive static triples with double boundary, such as the *Nariai space*. Hence, connectedness of the boundary is essential for Conjecture 1.2 to be true.

Conjecture 1.2 [Boucher et al. 1984, cosmic no-hair conjecture]. *Example 1.1 is the only possible n-dimensional positive static triple with single-horizon (connected) and positive scalar curvature.*

In the last decades some partial answers to Conjecture 1.2 were achieved. For instance, if (M^n, g) is Einstein it suffices to apply the Obata type theorem due to Reilly [1977] (see also [Obata 1962]) to conclude that Conjecture 1.2 is true. Moreover, Kobayashi [1982] and Lafontaine [1983] proved independently that such a conjecture is also true under the conformally flat condition.

For what follows, we recall that the Bach tensor on a Riemannian manifold (M^n, g), $n \geq 4$, is defined in terms of the components of the Weyl tensor W_{ikjl} as

(1-4) $$B_{ij} = \frac{1}{n-3} \nabla^k \nabla^l W_{ikjl} + \frac{1}{n-2} R_{kl} W_i{}^k{}_j{}^l,$$

while for $n = 3$ it is given by

(1-5) $$B_{ij} = \nabla^k C_{kij},$$

where C_{ijk} stands for the Cotton tensor. We say that (M^n, g) is Bach-flat when $B_{ij} = 0$.

Qing and Yuan [2013] obtained a classification result for static spaces under Bach-flat assumption. In particular, it is not hard to see that the method used by Qing and Yuan implies that such a conjecture is also true under Bach-flat assumption (see Theorem 1.3 below). Gibbons, Hartnoll and Pope [Gibbons et al. 2003] constructed counterexamples to the cosmic no-hair conjecture in the cases $4 \leq n \leq 8$. However, it remains interesting to show under which conditions such a conjecture remains true. For more details on this subject and further partial answers see, for instance, [Ambrozio 2017; Boucher et al. 1984; Chruściel 2011; Hijazi et al. 2015; Shen 1997]. Next, let us recall the following useful classification.

Theorem 1.3 [Kobayashi 1982; Lafontaine 1983; Qing and Yuan 2013]. *Let (M^n, g, f) be an n-dimensional positive static triple with scalar curvature $R = n(n-1)$. Suppose that (M^n, g) is Bach-flat, then (M^n, g, f) is covered by a static triple equivalent to one of the following static triples*:

(1) *The standard hemisphere with canonical metric*

$$(\mathbb{S}^n_+, g_{\mathbb{S}^{n-1}}, f = x_{n+1}).$$

(2) *The standard cylinder over* $\mathbb{S}^{n-1}$ *with the product metric*

$$\left(M = \left[0, \tfrac{\pi}{\sqrt{n}}\right] \times \mathbb{S}^{n-1}, \; g = dt^2 + \tfrac{n-2}{n} g_{\mathbb{S}^{n-1}}, \; f(t) = \sin(\sqrt{n}t)\right).$$

(3) *For some constant* $m \in (0, \sqrt{(n-2)^{n-2}/n^n})$ *we consider the Schwarzschild space defined by*

$$\left(M = [r_1, r_2] \times \mathbb{S}^{n-1}, \; g = \frac{1}{1 - 2mt^{2-n} - t^2} dt^2 + t^2 g_{\mathbb{S}^{n-1}}, \; f(t) = \sqrt{1 - 2mt^{2-n} - t^2}\right),$$

where $r_1 < r_2$ *are the positive zeroes of* f.

Ambrozio [2017] obtained interesting classification results for static three-dimensional manifolds with positive scalar curvature. To do so, he proved a Bochner type formula for three-dimensional positive static triples involving the traceless Ricci tensor and the Cotton tensor. A similar Bochner type formula was obtained by Batista et al. [2017] for three-dimensional Riemannian manifolds satisfying (1-2). Those formulae may be used to rule out some possible new examples. In this article, we extend such Bochner type formulae for a more general class of metrics and arbitrary dimension $n > 2$. To be precise, we have established the following result.

Theorem 1.4. *Let* (M^n, g, f, κ) *be a connected, smooth Riemannian manifold and* f *is a smooth function on* M^n *satisfying the V-static* (1-1). *Then we have*

$$(1\text{-}6) \quad \tfrac{1}{2} \operatorname{div}(f \nabla |\mathrm{Ric}|^2) = \left(\tfrac{n-2}{n-1} |C_{ijk}|^2 + |\nabla \operatorname{Ric}|^2\right) f + \tfrac{n\kappa}{n-1} |\mathring{\mathrm{Ric}}|^2$$

$$+ \left(\tfrac{2}{n-1} R |\mathring{\mathrm{Ric}}|^2 + \tfrac{2n}{n-2} \operatorname{tr}(\mathring{\mathrm{Ric}}^3)\right) f - \tfrac{n-2}{n-1} W_{ijkl} \nabla_l f C_{ijk} - 2 f W_{ijkl} R_{ik} R_{jl},$$

where C *stands for the Cotton tensor,* W *is the Weyl tensor and* $\mathring{\mathrm{Ric}}$ *is the traceless Ricci tensor.*

Remembering that three-dimensional Riemannian manifolds have vanishing Weyl tensor, it is easy to see that Theorem 1.4 is a generalization, for any dimension, of Theorem 3 in [Batista et al. 2017] as well as Proposition 12 in [Ambrozio 2017].

Before presenting a couple of applications of the above formula it is fundamental to remember that a Riemannian manifold (M^n, g) has *zero radial Weyl curvature* when

$$(1\text{-}7) \qquad\qquad W(\cdot, \cdot, \cdot, \nabla f) = 0,$$

for a suitable potential function f on M^n. This class of manifolds includes the case of locally conformally flat manifolds. Moreover, this condition has been used to classify gradient Ricci solitons as well as quasi-Einstein manifolds (see [Catino

2012; He et al. 2012; Petersen and Wylie 2010]). Here, we shall use this condition to obtain the following corollary.

Corollary 1.5. *Let (M^n, g, f) be a compact, oriented, connected Miao–Tam critical metric with positive scalar curvature and nonnegative potential function f. Suppose that*

- *M^n has zero radial Weyl curvature and*
- *$|\mathring{\text{Ric}}|^2 \leq \frac{1}{n(n-1)} R^2$.*

Then M^n must be isometric to a geodesic ball in $\mathbb{S}^n$.

It is not difficult to see that the above result generalizes Corollary 1 in [Batista et al. 2017]. Next, we get the following result for static spaces.

Corollary 1.6. *Let (M^n, g, f) be a compact, oriented, connected positive static triple with positive scalar curvature. Suppose that*

- *M^n has zero radial Weyl curvature and*
- *$|\mathring{\text{Ric}}|^2 \leq \frac{1}{n(n-1)} R^2$.*

Then one of the following assertions holds:

(1) *M^n is equivalent to the standard hemisphere of $\mathbb{S}^n$; or*

(2) *$|\mathring{\text{Ric}}|^2 = \frac{1}{n(n-1)} R^2$ and (M^n, g, f) is covered by a static triple that is equivalent to the standard cylinder.*

Remark 1.7. It is worthwhile to remark that Corollary 1.6 can be seen as a partial answer to Conjecture 1.2.

Remark 1.8. We also point out that four-dimensional V-static spaces with zero radial Weyl curvature must be locally conformally flat. To prove this claim it suffices to apply the same arguments used in the initial part of the proof of Theorem 2 in [Barros et al. 2015].

In order to proceed, we recall that a classical lemma due to Berger guarantees that any two symmetric tensors T on a Riemannian manifold (M^n, g) with nonnegative sectional curvature must satisfy

(1-8) $$(\nabla_i \nabla_j T_{ik} - \nabla_j \nabla_i T_{ik}) T_{jk} \geq 0.$$

In fact, we have

$$(\nabla_i \nabla_j T_{ik} - \nabla_j \nabla_i T_{ik}) T_{jk} = \sum_{i<j} R_{ijij} (\lambda_i - \lambda_j)^2,$$

where the λ_i's are the eigenvalues of tensor T (see Lemma 4.1 in [Cao 2007]). Here, we shall use these data jointly with Theorem 1.4 to deduce a rigidity result for three-dimensional Miao–Tam critical metrics with nonnegative sectional curvature

(see also Proposition 4.2 in Section 4 for a version in arbitrary dimension). More precisely, we have established the following result.

Theorem 1.9. *Let (M^3, g, f) be a three-dimensional compact, oriented, connected Miao–Tam critical metric with smooth boundary ∂M and nonnegative sectional curvature, with f assumed to be nonnegative. Then M^3 is isometric to a geodesic ball in a simply connected space form $\mathbb{R}^3$ or $\mathbb{S}^3$.*

Finally, we get the following result for positive static triples.

Theorem 1.10. *Let (M^n, g, f) be a positive static triple with nonnegative sectional curvature, zero radial Weyl curvature and scalar curvature $R = n(n-1)$. Then, up to a finite quotient, M^n is isometric to either the standard hemisphere $\mathbb{S}^n_+$ or the standard cylinder over $\mathbb{S}^{n-1}$ with the product metric described in Theorem 1.3.*

2. Preliminaries

In this section we shall present some preliminaries which will be useful for the establishment of the desired results. Firstly, we remember that a V-static space is a Riemannian manifold (M^n, g) with a nontrivial solution (f, κ) satisfying the overdetermined-elliptic system

$$-(\Delta f)g + \text{Hess} f - f \, \text{Ric} = \kappa g,$$

where κ is a constant. As usual, we rewrite in the tensorial language as

$$(2\text{-}1) \qquad -(\Delta f)g_{ij} + \nabla_i \nabla_j f - f R_{ij} = \kappa g_{ij}.$$

Tracing (2-1) we deduce that f also satisfies the equation

$$(2\text{-}2) \qquad \Delta f + \frac{R}{n-1} f + \frac{n\kappa}{n-1} = 0.$$

Moreover, by using (2-2) it is not difficult to check that

$$(2\text{-}3) \qquad f \, \mathring{\text{Ric}} = \mathring{\text{Hess}} f,$$

where $\mathring{T}$ stands for the traceless part of T.

Before proceeding we recall two special tensors in the study of curvature for a Riemannian manifold (M^n, g), $n \geq 3$. The first one is the Weyl tensor W which is defined by the decomposition formula

$$(2\text{-}4) \quad R_{ijkl} = W_{ijkl} + \frac{1}{n-2}(R_{ik}g_{jl} + R_{jl}g_{ik} - R_{il}g_{jk} - R_{jk}g_{il})$$
$$- \frac{R}{(n-1)(n-2)}(g_{jl}g_{ik} - g_{il}g_{jk}),$$

where R_{ijkl} stands for the Riemann curvature operator Rm, whereas the second one is the Cotton tensor C given by

$$(2\text{-}5) \qquad C_{ijk} = \nabla_i R_{jk} - \nabla_j R_{ik} - \frac{1}{2(n-1)}(\nabla_i R g_{jk} - j R g_{ik}).$$

Note that C_{ijk} is skew-symmetric in the first two indices and trace-free in any two indices. These two above tensors satisfy

$$(2\text{-}6) \qquad C_{ijk} = -\frac{n-2}{n-3}\nabla_l W_{ijkl},$$

provided $n \geq 4$.

For our purpose we also remember that as a consequence of the Bianchi identity

$$(2\text{-}7) \qquad (\mathrm{div}\, Rm)_{jkl} = \nabla_k R_{jl} - \nabla_l R_{jk}.$$

Moreover, from commutation formulas (Ricci identities), for any Riemannian manifold (M^n, g) we have

$$(2\text{-}8) \qquad \nabla_i \nabla_j R_{kl} - \nabla_j \nabla_i R_{kl} = R_{ijks} R_{sl} + R_{ijls} R_{ks};$$

for more details, see [Chow et al. 2007; Viaclovsky 2011].

To conclude this section, we shall present the following lemma for V-static spaces, which was previously obtained in [Barros et al. 2015] for Miao–Tam critical metrics.

Lemma 2.1. *Let (M^n, g) be a connected, smooth Riemannian manifold and f be a smooth function on M^n satisfying* (1-1). *Then we have*:

$$f(\nabla_i R_{jk} - \nabla_j R_{ik}) = R_{ijkl}\nabla_l f + \frac{R}{n-1}(\nabla_i f g_{jk} - \nabla_j f g_{ik}) - (\nabla_i f R_{jk} - \nabla_j f R_{ik}).$$

Proof. The proof is standard, and it follows the same steps of Lemma 1 in [Barros et al. 2015]. For the sake of completeness we shall sketch it here. Firstly, since g is parallel we may use (2-1) to infer

$$(2\text{-}9) \qquad (\nabla_i f) R_{jk} + f \nabla_i R_{jk} = \nabla_i \nabla_j \nabla_k f - (\nabla_i \Delta f) g_{jk}.$$

Next, since M^n has constant scalar curvature we have from (2-2) that

$$\nabla_i \Delta f = -\frac{R}{n-1}\nabla_i f,$$

which substituted into (2-9) gives

$$(2\text{-}10) \qquad f \nabla_i R_{jk} = -(\nabla_i f) R_{jk} + \nabla_i \nabla_j \nabla_k f + \frac{R}{n-1}\nabla_i f g_{jk}.$$

Finally, we apply the Ricci identity to arrive at

$$f(\nabla_i R_{jk} - \nabla_j R_{ik}) = R_{ijkl}\nabla_l f + \frac{R}{n-1}(\nabla_i f g_{jk} - \nabla_j f g_{ik}) - (\nabla_i f R_{jk} - \nabla_j f R_{ik}). \qquad \square$$

3. A Bochner type formula and applications

In this section we shall provide a general Bochner type formula, which enables us to prove some rigidity results for V-static spaces. To do so, we shall obtain some identities involving the Cotton tensor and Weyl tensor on Riemannian manifolds satisfying the V-static equation. Following the notation employed in [Barros et al. 2015], we can use (2-4) jointly with Lemma 2.1 to obtain

$$(3\text{-}1) \qquad\qquad f C_{ijk} = T_{ijk} + W_{ijkl} \nabla_l f,$$

where the auxiliary tensor T_{ijk} is defined as

$$(3\text{-}2) \quad T_{ijk} = \frac{n-1}{n-2}(R_{ik}\nabla_j f - R_{jk}\nabla_i f) + \frac{1}{n-2}(g_{ik} R_{js}\nabla_s f - g_{jk} R_{is}\nabla_s f)$$
$$- \frac{R}{n-2}(g_{ik}\nabla_j f - g_{jk}\nabla_i f).$$

In the sequel, we obtain a divergent formula for any Riemannian manifold (M^n, g) with constant scalar curvature.

Lemma 3.1. *Let (M^n, g) be a connected Riemannian manifold with constant scalar curvature and $f : M \to \mathbb{R}$ be a smooth function defined on M. Then we have*

$$\operatorname{div}(f\nabla|\mathrm{Ric}|^2) = -f|C_{ijk}|^2 + 2f|\nabla\mathrm{Ric}|^2 + \langle\nabla f, \nabla|\mathrm{Ric}|^2\rangle + \frac{2n}{n-2}f R_{ij} R_{ik} R_{jk}$$
$$- \frac{4n-2}{(n-1)(n-2)}f R|\mathring{\mathrm{Ric}}|^2 - \frac{2}{n(n-2)}f R^3$$
$$+ 2\nabla_i(f C_{ijk} R_{jk}) + 2C_{ijk}\nabla_j f R_{ik} - 2f W_{ijkl} R_{ik} R_{jl}.$$

Proof. Firstly, since M^n has constant scalar curvature we immediately get

$$f|C_{ijk}|^2 = f|\nabla_i R_{jk} - \nabla_j R_{ik}|^2 = 2f|\nabla\mathrm{Ric}|^2 - 2f\nabla_i R_{jk}\nabla_j R_{ik}.$$

On the other hand, easily one verifies that

$$\nabla_j(f\nabla_i R_{jk} R_{ik}) = \nabla_j f\nabla_i R_{jk} R_{ik} + f\nabla_j\nabla_i R_{jk} R_{ik} + f\nabla_i R_{jk}\nabla_j R_{ik}.$$

Hence, it follows that

$$f|C_{ijk}|^2 = 2f|\nabla\mathrm{Ric}|^2 - 2\nabla_j(f\nabla_i R_{jk} R_{ik}) + 2\nabla_j f\nabla_i R_{jk} R_{ik} + 2f\nabla_j\nabla_i R_{jk} R_{ik}.$$

Next, from the commutation formula for second covariant derivative of the Ricci curvature (see (2-8)) combined with (2-5), we deduce

$$(3\text{-}3) \quad f|C_{ijk}|^2 = 2f|\nabla\mathrm{Ric}|^2 + 2\nabla_j f(C_{ijk} + \nabla_j R_{ik})R_{ik}$$
$$+ 2f(R_{ij} R_{ik} R_{jl} - R_{ijkl} R_{ik} R_{jl}) - 2\nabla_j(f\nabla_i R_{jk} R_{ik})$$
$$= 2f|\nabla\mathrm{Ric}|^2 + 2C_{ijk}\nabla_j f R_{ik} + \langle\nabla f, \nabla|\mathrm{Ric}|^2\rangle$$
$$+ 2f(R_{ij} R_{ik} R_{jk} - R_{ijkl} R_{ik} R_{jl}) - 2\nabla_j(f\nabla_i R_{jk} R_{ik}).$$

Now, substituting (2-4) into (3-3) we achieve

$$
\begin{aligned}
f|C_{ijk}|^2 = {}& 2f|\nabla\operatorname{Ric}|^2 + 2C_{ijk}\nabla_j f R_{ik} + \langle\nabla f, \nabla|\operatorname{Ric}|^2\rangle + 2f R_{ij}R_{ik}R_{jk} \\
& - 2f W_{ijkl}R_{ik}R_{jl} - \frac{2f}{n-2}(2R|\operatorname{Ric}|^2 - 2R_{ij}R_{ik}R_{jk}) \\
& \qquad\qquad + \frac{2Rf}{(n-1)(n-2)}(R^2 - |\operatorname{Ric}|^2) - 2\nabla_j(f\nabla_i R_{jk}R_{ik}) \\
= {}& 2f|\nabla\operatorname{Ric}|^2 + 2C_{ijk}\nabla_j f R_{ik} + \langle\nabla f, \nabla|\operatorname{Ric}|^2\rangle + \frac{2n}{n-2}f R_{ij}R_{ik}R_{jk} \\
& - 2f W_{ijkl}R_{ik}R_{jl} - \frac{(4n-2)}{(n-1)(n-2)}f R|\operatorname{Ric}|^2 \\
& \qquad\qquad + \frac{2}{(n-1)(n-2)}f R^3 - 2\nabla_j(f\nabla_i R_{jk}R_{ik}),
\end{aligned}
$$

which can be rewritten as

$$
\begin{aligned}
f|C_{ijk}|^2 = {}& 2f|\nabla\operatorname{Ric}|^2 + 2C_{ijk}\nabla_j f R_{ik} + \langle\nabla f, \nabla|\operatorname{Ric}|^2\rangle + \frac{2n}{n-2}f R_{ij}R_{ik}R_{jk} \\
& - 2f W_{ijkl}R_{ik}R_{jl} - \frac{(4n-2)}{(n-1)(n-2)}f R|\mathring{\operatorname{Ric}}|^2 \\
& \qquad\qquad - \frac{2}{n(n-2)}f R^3 - 2\nabla_j(f\nabla_i R_{jk}R_{ik}) \\
= {}& 2f|\nabla\operatorname{Ric}|^2 + 2C_{ijk}\nabla_j f R_{ik} + \langle\nabla f, \nabla|\operatorname{Ric}|^2\rangle + \frac{2n}{n-2}f R_{ij}R_{ik}R_{jk} \\
& - 2f W_{ijkl}R_{ik}R_{jl} - \frac{(4n-2)}{(n-1)(n-2)}f R|\mathring{\operatorname{Ric}}|^2 \\
& \qquad\qquad - \frac{2}{n(n-2)}f R^3 + 2\nabla_i(f C_{ijk}R_{jk}) - \operatorname{div}(f\nabla|\operatorname{Ric}|^2),
\end{aligned}
$$

where we used (2-5) to justify the second equality. So, the proof is completed. $\square$

Proceeding, we shall deduce another divergent formula, which plays a crucial role in the proof of Theorem 1.4.

Lemma 3.2. *Let (M^n, g, f, κ) be a V-static space. Then we have*

$$
\begin{aligned}
\tfrac{1}{2}\operatorname{div}(f\nabla|\operatorname{Ric}|^2) \\
= -f|C_{ijk}|^2 + f|\nabla\operatorname{Ric}|^2 + \langle\nabla f, \nabla|\operatorname{Ric}|^2\rangle - \frac{n\kappa}{n-1}|\mathring{\operatorname{Ric}}|^2 + 2\nabla_i(f C_{ijk}R_{jk}).
\end{aligned}
$$

Proof. To start with, we use Lemma 2.1 together with (2-5) to infer

$$
\begin{aligned}
\nabla_i(\nabla_j f R_{ik}R_{jk} + R_{ijkl}\nabla_l f R_{jk}) \\
= \nabla_i(\nabla_j f R_{ik}R_{jk}) + \nabla_i\Big[f C_{ijk}R_{jk} - \frac{R}{n-1}(\nabla_i f R - \nabla_j f R_{ji}) \\
+ (|\operatorname{Ric}|^2\nabla_i f - \nabla_j f R_{ik}R_{jk})\Big].
\end{aligned}
$$

Rearranging the terms we immediately deduce

$$\nabla_i(\nabla_j f\, R_{ik} R_{jk} + R_{ijkl}\nabla_l f\, R_{jk})$$
$$= \nabla_i(f C_{ijk} R_{jk}) + \nabla_i\left[-\frac{R^2}{n-1}\nabla_i f + \frac{R}{n-1}R_{ji}\nabla_j f + |\mathrm{Ric}|^2\nabla_i f\right],$$

and remembering that (M^n, g) has constant scalar curvature we use the twice-contracted second Bianchi identity $(2\,\mathrm{div}\,\mathrm{Ric} = \nabla R = 0)$ to get

$$(3\text{-}4)\quad \nabla_i(\nabla_j f\, R_{ik} R_{jk} + R_{ijkl}\nabla_l f\, R_{jk})$$
$$= \nabla_i(f C_{ijk} R_{jk}) - \frac{R^2}{n-1}\Delta f + \frac{R}{n-1}\nabla_i\nabla_j f\, R_{ji} + |\mathrm{Ric}|^2\Delta f + \langle\nabla f, \nabla|\mathrm{Ric}|^2\rangle.$$

Therefore, substituting (2-1) and (2-2) into (3-4) we obtain

$$\nabla_i(\nabla_j f\, R_{ik} R_{jk} + R_{ijkl}\nabla_l f\, R_{jk})$$
$$= \nabla_i(f C_{ijk} R_{jk}) + \langle\nabla f, \nabla|\mathrm{Ric}|^2\rangle - \frac{R^2}{n-1}\Delta f$$
$$+ \frac{R}{n-1}(f R_{ij} + (\Delta f + \kappa)g_{ij})R_{ji} + \Delta f |\mathrm{Ric}|^2$$
$$= \nabla_i(f C_{ijk} R_{jk}) + \langle\nabla f, \nabla|\mathrm{Ric}|^2\rangle + \frac{R}{n-1}f|\mathrm{Ric}|^2 + \frac{R^2\kappa}{n-1} + \frac{-Rf - n\kappa}{n-1}|\mathrm{Ric}|^2.$$

From this, it follows that

$$(3\text{-}5)\quad \nabla_i(\nabla_j f\, R_{ik} R_{jk} + R_{ijkl}\nabla_l f\, R_{jk})$$
$$= \nabla_i(f C_{ijk} R_{jk}) + \langle\nabla f, \nabla|\mathrm{Ric}|^2\rangle - \frac{n\kappa}{n-1}|\mathring{\mathrm{Ric}}|^2.$$

At the same time, notice that

$$\nabla_i(\nabla_j f\, R_{ik} R_{jk} + R_{ijkl}\nabla_l f\, R_{jk})$$
$$= \nabla_i\nabla_j f\, R_{ik} R_{jk} + \nabla_j f\, R_{ik}\nabla_i R_{jk} + \nabla_i R_{ijkl}\nabla_l f\, R_{jk}$$
$$+ R_{ijkl}\nabla_i\nabla_l f\, R_{jk} + R_{ijkl}\nabla_l f\,\nabla_i R_{jk}.$$

Hence, it follows from Lemma 2.1 and (2-1) that

$$\nabla_i(\nabla_j f\, R_{ik} R_{jk} + R_{ijkl}\nabla_l f\, R_{jk})$$
$$= f(R_{ij} R_{ik} R_{jk} - R_{ijkl} R_{ik} R_{jl}) + \nabla_j f\, R_{ik}\nabla_i R_{jk} + C_{ijk}\nabla_j f\, R_{ik}$$
$$+ f C_{ijk}\nabla_i R_{jk} + (\nabla_i f\, R_{jk} - \nabla_j f\, R_{ik})\nabla_i R_{jk}$$
$$= f(R_{ij} R_{ik} R_{jk} - R_{ijkl} R_{ik} R_{jl}) + C_{ijk}\nabla_j f\, R_{ik}$$
$$+ f C_{ijk}\nabla_i R_{jk} + \tfrac{1}{2}\langle\nabla f, \nabla|\mathrm{Ric}|^2\rangle.$$

Proceeding, we use that the Cotton tensor is skew-symmetric in the first two indices and (3-3) to infer

(3-6) $\quad \nabla_i(\nabla_j f R_{ik} R_{jk} + R_{ijkl}\nabla_l f R_{jk})$
$$= f|C_{ijk}|^2 - f|\nabla \text{Ric}|^2 + \nabla_j(f\nabla_i R_{jk}R_{ik})$$
$$= f|C_{ijk}|^2 - f|\nabla \text{Ric}|^2 - \nabla_i(fC_{ijk}R_{jk}) + \tfrac{1}{2}\text{div}(f\nabla|\text{Ric}|^2).$$

Finally, it suffices to compare (3-5) and (3-6) to get the desired result. □

Proof of Theorem 1.4. A simple computation using (3-1) as well as (3-2) allows us to deduce
$$f|C_{ijk}|^2 = \frac{2(n-1)}{(n-2)}R_{ik}\nabla_j f C_{ijk} + W_{ijkl}\nabla_l f C_{ijk},$$

where we have used that C_{ijk} is skew-symmetric in the first two indices and trace-free in any two indices. Whence, substituting this data into Lemma 3.1 we obtain

(3-7) $\quad \text{div}(f\nabla|\text{Ric}|^2)$
$$= 2f|\nabla \text{Ric}|^2 + \langle\nabla f, \nabla|\text{Ric}|^2\rangle + \frac{2n}{(n-2)}f R_{ij}R_{ik}R_{jk}$$
$$- \frac{1}{(n-1)}f|C_{ijk}|^2 - \frac{4n-2}{(n-1)(n-2)}f R|\mathring{\text{Ric}}|^2 - \frac{2}{n(n-2)}f R^3$$
$$+ 2\nabla_i(fC_{ijk}R_{jk}) - \frac{(n-2)}{(n-1)}W_{ijkl}\nabla_l f C_{ijk} - 2f W_{ijkl}R_{ik}R_{jl}.$$

Now, comparing the expression obtained in (3-7) with Lemma 3.2 we arrive at

(3-8) $\quad \tfrac{1}{2}\text{div}(f\nabla|\text{Ric}|^2)$
$$= \left(\frac{n-2}{n-1}|C_{ijk}|^2 + |\nabla \text{Ric}|^2\right)f + \frac{n\kappa}{n-1}|\mathring{\text{Ric}}|^2 + \frac{2n}{n-2}f R_{ij}R_{ik}R_{jk}$$
$$- \frac{4n-2}{(n-1)(n-2)}f R|\mathring{\text{Ric}}|^2 - \frac{2}{n(n-2)}f R^3$$
$$- \frac{n-2}{n-1}W_{ijkl}\nabla_l f C_{ijk} - 2f W_{ijkl}R_{ik}R_{jl}.$$

On the other hand, recalling that $\mathring{R}_{ij} = R_{ij} - R^2/ng$, it is easy to check that
$$f R_{ij}R_{ik}R_{jk} = f\mathring{R}_{ij}\mathring{R}_{jk}\mathring{R}_{ik} + \frac{3}{n}f R|\mathring{\text{Ric}}|^2 + \frac{f R^3}{n^2}.$$

This substituted into (3-8) gives

$\tfrac{1}{2}\text{div}(f\nabla|\text{Ric}|^2)$
$$= \left(\frac{n-2}{n-1}|C_{ijk}|^2 + |\nabla \text{Ric}|^2\right)f + \frac{n\kappa}{n-1}|\mathring{\text{Ric}}|^2 + \left(\frac{2}{n-1}R|\mathring{\text{Ric}}|^2 + \frac{2n}{n-2}\text{tr}(\mathring{\text{Ric}}^3)\right)f$$
$$- \frac{n-2}{n-1}W_{ijkl}\nabla_l f C_{ijk} - 2f W_{ijkl}R_{ik}R_{jl},$$

which finishes the proof of the theorem. □

Proof of Corollaries 1.5 and 1.6. In order to prove Corollaries 1.5 and 1.6, we recall that the Cotton tensor and the divergence of the Weyl tensor are related as follows:

$$(3\text{-}9) \qquad\qquad C_{ijk} = -\frac{n-2}{n-3}\nabla_l W_{ijkl}.$$

Notice also that the zero radial Weyl curvature condition, namely, $W_{ijkl}\nabla_l f = 0$, jointly with (3-9) and (2-1) yields

$$\begin{aligned}
0 &= \nabla_i(W_{ijkl}\nabla_k f\, R_{jl})\\
&= \nabla_i W_{ijkl}\nabla_k f\, R_{jl} + W_{ijkl}\nabla_i\nabla_k f\, R_{jl}\\
&= \frac{n-3}{n-2}C_{klj}\nabla_k f\, R_{jl} + f\, W_{ijkl}R_{ik}R_{jl}.
\end{aligned}$$

By using that the Cotton tensor is skew-symmetric in the two first indices we obtain

$$f\, W_{ijkl}R_{ik}R_{jl} = \frac{n-3}{2(n-2)}C_{ijk}(\nabla_j f\, R_{ik} - \nabla_i f\, R_{ik}),$$

which can be succinctly rewritten as

$$f\, W_{ijkl}R_{ik}R_{jl} = \frac{n-3}{2(n-1)}C_{ijk}T_{ijk}.$$

From this, it follows from (3-1) that

$$(3\text{-}10) \qquad\qquad f\, W_{ijkl}R_{ik}R_{jl} = \frac{n-3}{2(n-1)}f\,|C_{ijk}|^2.$$

Now, comparing (3-10) with Theorem 1.4 we achieve

$$\frac{1}{2}\operatorname{div}(f\nabla|\mathrm{Ric}|^2) = \Big(\frac{n-2}{n-1}|C_{ijk}|^2 + |\nabla\mathrm{Ric}|^2\Big)f + \frac{n\kappa}{n-1}|\mathring{\mathrm{Ric}}|^2 \\
+ \Big(\frac{2}{n-1}R|\mathring{\mathrm{Ric}}|^2 + \frac{2n}{n-2}\operatorname{tr}(\mathring{\mathrm{Ric}}^3)\Big)f - \frac{n-3}{n-1}f\,|C_{ijk}|^2,$$

so that

$$(3\text{-}11)\quad \frac{1}{2}\operatorname{div}(f\nabla|\mathrm{Ric}|^2) = \Big(\frac{1}{n-1}|C_{ijk}|^2 + |\nabla\mathrm{Ric}|^2\Big)f + \frac{n\kappa}{n-1}|\mathring{\mathrm{Ric}}|^2 \\
+ \Big(\frac{2}{n-1}R|\mathring{\mathrm{Ric}}|^2 + \frac{2n}{n-2}\operatorname{tr}(\mathring{\mathrm{Ric}}^3)\Big)f.$$

Before proceeding it is important to remember that the classical Okumura's lemma [1974, Lemma 2.1] guarantees

$$(3\text{-}12) \qquad\qquad \operatorname{tr}(\mathring{\mathrm{Ric}}^3) \geq -\frac{n-2}{\sqrt{n(n-1)}}|\mathring{\mathrm{Ric}}|^3.$$

Therefore, upon integrating (3-11) over M we use (3-12) to arrive at

$$(3\text{-}13) \quad 0 \geq \int_M \left(\frac{1}{n-1} |C_{ijk}|^2 + |\nabla \operatorname{Ric}|^2 \right) f \, dM_g + \frac{n\kappa}{n-1} \int_M |\mathring{\operatorname{Ric}}|^2 \, dM_g$$

$$+ \int_M \frac{2n}{\sqrt{n(n-1)}} |\mathring{\operatorname{Ric}}|^2 \left(\frac{R}{\sqrt{n(n-1)}} - |\mathring{\operatorname{Ric}}| \right) f \, dM_g.$$

We now suppose that $\kappa = 1$, that is, (M^n, g) is a Miao–Tam critical metric, and we may use our assumption with (3-13) to conclude that $|\mathring{\operatorname{Ric}}|^2 = 0$ and this forces M^n to be Einstein. So, it suffices to apply Theorem 1.1 in [Miao and Tam 2011] to conclude that (M^n, g) is isometric to a geodesic ball in $\mathbb{S}^n$ and this concludes the proof of Corollary 1.5.

From now on we assume that $\kappa = 0$, that is, (M^n, g) is a static space. In this case, our assumption substituted into (3-13) guarantees that either $|\mathring{\operatorname{Ric}}|^2 = 0$ or $|\mathring{\operatorname{Ric}}|^2 = R^2/(n(n-1))$. In the first case, we conclude that (M^n, g) is an Einstein manifold. Then, it suffices to apply Lemma 3 in [Reilly 1977] to conclude that M^n is isometric to a hemisphere of $\mathbb{S}^n$. In the second one, notice that M^n must have a vanishing Cotton tensor and parallel Ricci curvature. From this, we can use (1-4) to infer

$$(n-2)B_{ij} = \nabla_k C_{kij} + W_{ikjl} R_{kl} = W_{ikjl} R_{kl},$$

and consequently, by using the static equation, we deduce

$$(n-2)f B_{ij} = W_{ikjl} \nabla_k \nabla_l f = \nabla_k (W_{ijkl} \nabla_l f) - \nabla_k W_{ikjl} \nabla_l f.$$

Hence, our assumption on Weyl curvature tensor jointly with (3-9) yields

$$(n-2)f B_{ij} = -\nabla_k W_{jlik} \nabla_l f = \frac{n-3}{n-2} C_{jli} \nabla_l f = 0.$$

From here it follows that (M^n, g) is Bach-flat. Hence, the result follows from Corollary 4.4 (see also Theorem 1 in [Ambrozio 2017] for $n = 3$). This is what we wanted to prove. $\qquad\square$

4. Critical metrics with nonnegative sectional curvature

In the last decades there have been a lot of interesting results concerning the geometry of manifolds with nonnegative sectional curvature. In this context, as it was previously mentioned, any two symmetric tensors T on a Riemannian manifold (M^n, g) with nonnegative sectional curvature must satisfy

$$(4\text{-}1) \qquad\qquad (\nabla_i \nabla_j T_{ik} - \nabla_j \nabla_i T_{ik}) T_{jk} \geq 0.$$

In fact, we have

$$(\nabla_i \nabla_j T_{ik} - \nabla_j \nabla_i T_{ik}) T_{jk} = \sum_{i<j} R_{ijij} (\lambda_i - \lambda_j)^2,$$

where the λ_i are the eigenvalues of tensor T (see Lemma 4.1 in [Cao 2007]). In particular, choosing $T = \text{Ric}$ we immediately get

$$(\nabla_i \nabla_j R_{ik} - \nabla_j \nabla_i R_{ik}) R_{jk} \geq 0.$$

This combined with (2-8) yields

$$(4\text{-}2) \qquad\qquad R_{ij} R_{jk} R_{ik} - R_{ijkl} R_{jl} R_{ik} \geq 0.$$

In the sequel, we shall deduce a useful expression for $R_{ij} R_{jk} R_{ik} - R_{ijkl} R_{jl} R_{ik}$ on any Riemannian manifold.

Lemma 4.1. *Let (M^n, g) be a Riemannian manifold. Then we have*

$$R_{ij} R_{jk} R_{ik} - R_{ijkl} R_{jl} R_{ik} = \frac{1}{n-1} R|\mathring{\text{Ric}}|^2 + \frac{n}{n-2} \text{tr}(\mathring{\text{Ric}}^3) - W_{ijkl} R_{ik} R_{jl}.$$

Proof. By using the definition of the Riemann tensor (2-4) we obtain

$$R_{ij} R_{jk} R_{ik} - R_{ijkl} R_{jl} R_{ik}$$
$$= \frac{n}{n-2} R_{ij} R_{jk} R_{ik} - W_{ijkl} R_{jl} R_{ik} - \frac{(2n-1)}{(n-1)(n-2)} R|\text{Ric}|^2 + \frac{R^3}{(n-1)(n-2)}$$

so that

$$(4\text{-}3) \quad R_{ij} R_{jk} R_{ik} - R_{ijkl} R_{jl} R_{ik}$$
$$= \frac{n}{n-2} R_{ij} R_{jk} R_{ik} - W_{ijkl} R_{jl} R_{ik} - \frac{(2n-1)}{(n-1)(n-2)} R|\mathring{\text{Ric}}|^2 - \frac{1}{n(n-2)} R^3.$$

On the other hand, we already know that

$$R_{ij} R_{ik} R_{jk} = \mathring{R}_{ij} \mathring{R}_{jk} \mathring{R}_{ik} + \frac{3}{n} R|\mathring{\text{Ric}}|^2 + \frac{R^3}{n^2}.$$

This substituted into (4-3) gives the desired result. $\square$

Since three-dimensional Riemannian manifolds have vanishing Weyl tensor, the proof of Theorem 1.9 follows as an immediate consequence of the following slightly stronger result.

Proposition 4.2. *Let (M^n, g, f) be a compact, oriented, connected Miao–Tam critical metric with smooth boundary ∂M and nonnegative sectional curvature, f, which is also assumed to be nonnegative. Suppose that M^n has zero radial Weyl curvature, then M^n is isometric to a geodesic ball in a simply connected space form $\mathbb{R}^n$ or $\mathbb{S}^n$.*

Proof. To begin with, we multiply by f the expression obtained in Lemma 4.1 and then we use Theorem 1.4 to infer

$$\tfrac{1}{2} \text{div}(f \nabla |\text{Ric}|^2) = \left(\frac{n-2}{n-1} |C_{ijk}|^2 + |\nabla \text{Ric}|^2 \right) f + \frac{n}{n-1} |\mathring{\text{Ric}}|^2$$
$$+ 2(R_{ij} R_{jk} R_{ik} - R_{ijkl} R_{jl} R_{ik}) f - \frac{n-2}{n-1} W_{ijkl} \nabla_l f C_{ijk}$$

and since M^n has zero radial Weyl curvature we get

$$\tfrac{1}{2}\operatorname{div}(f\nabla|\mathrm{Ric}|^2) = \left(\frac{n-2}{n-1}|C_{ijk}|^2 + |\nabla\mathrm{Ric}|^2\right)f + \frac{n}{n-1}|\mathring{\mathrm{Ric}}|^2$$
$$+ 2(R_{ij}R_{jk}R_{ik} - R_{ijkl}R_{jl}R_{ik})f.$$

Finally, upon integrating the above expression over M^n we use (4-2) to conclude that $\mathring{\mathrm{Ric}} = 0$ and then $(M^n g)$ is Einstein. Hence, we apply Theorem 1.1 in [Miao and Tam 2011] to conclude that (M^n, g) is isometric to a geodesic ball in $\mathbb{R}^n$ or $\mathbb{S}^n$. This finishes the proof of the proposition. □

Proceeding, we shall prove Theorem 1.10, which was announced in Section 1.

Theorem 4.3. *Let (M^n, g, f) be a positive static triple with nonnegative sectional curvature, zero radial Weyl curvature and scalar curvature $R = n(n-1)$. Then up to a finite quotient M^n is isometric to either the standard hemisphere $\mathbb{S}^n_+$ or the standard cylinder over $\mathbb{S}^{n-1}$ with the product metric described in Theorem 1.3.*

Proof. The proof looks like the one from the previous theorem. In fact, substituting Lemma 4.1 into Theorem 1.4 we arrive at

$$\tfrac{1}{2}\operatorname{div}(f\nabla|\mathrm{Ric}|^2) = \left(\frac{1}{n-1}|C_{ijk}|^2 + |\nabla\mathrm{Ric}|^2\right)f + 2(R_{ij}R_{jk}R_{ik} - R_{ijkl}R_{jl}R_{ik})f.$$

We integrate the above expression over M^n and then use (4-2) to conclude that (M^n, g) must have vanishing Cotton tensor and parallel Ricci curvature. Finally, it suffices to repeat the same arguments applied in the final steps of the proof of Corollary 1.6. So, the proof is completed. □

As an immediate consequence of the previous theorem we get the following result.

Corollary 4.4. *Let (M^3, g, f) be a three-dimensional positive static triple with nonnegative sectional curvature and normalized scalar curvature $R = 6$. Then, up to a finite quotient, M^3 is isometric to either the standard hemisphere $\mathbb{S}^3_+$ or the standard cylinder over $\mathbb{S}^2$ with the product metric described in Theorem 1.3.*

We point out that, by a different approach, Ambrozio [2017] was able to show that a three-dimensional compact positive static triple with scalar curvature 6 and nonnegative Ricci curvature must be equivalent to the standard hemisphere or be covered by the standard cylinder.

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CHERLIN'S CONJECTURE FOR SPORADIC SIMPLE GROUPS

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We prove Cherlin's conjecture, concerning binary primitive permutation groups, for those groups with socle isomorphic to a sporadic simple group.

1. Introduction

In this paper we consider the following conjecture which was given first in [Cherlin 2000]:

Conjecture 1.1. *A finite primitive binary permutation group must be one of* :

(1) *A symmetric group Sym(n) acting naturally on n elements.*

(2) *A cyclic group of prime order acting regularly on itself.*

(3) *An affine orthogonal group $V \cdot O(V)$ with V a vector space over a finite field equipped with an anisotropic quadratic form acting on itself by translation, with complement the full orthogonal group $O(V)$.*

Thanks to work of Cherlin himself [2016], and of Wiscons [2016], Conjecture 1.1 has been reduced to a statement about almost simple groups. In particular, to prove Conjecture 1.1 it would be sufficient to prove the following statement.

Conjecture 1.2. *If G is a binary almost simple primitive permutation group on the set Ω, then $G = \mathrm{Sym}(\Omega)$.*

In this paper, we prove this conjecture for almost simple groups with sporadic socle. Formally, our main result is the following:

Theorem 1.3. *Let G be an almost simple primitive permutation group with socle isomorphic to a sporadic simple group. Then G is not binary.*

Note that we include the group $^2F_4(2)'$ in the list of sporadic groups — this group is sometimes considered "the 27-th sporadic group" — so Theorem 1.3 applies to this group too.

The terminology of Theorem 1.3 and the preceding conjectures is all fairly standard in the world of group theory, with the possible exception of the word

"binary". Roughly speaking an action is binary if the induced action on ℓ-tuples can be deduced from the induced action on pairs (for any integer $\ell > 2$); a formal definition of a "binary permutation group" is given in Section 2.

1A. *Context and methods.* We will not spend much time here trying to motivate the study of binary permutation groups. As will be clear on reading the definition of binary in Section 2, this notion is a particular instance of the more general concept of "arity" or "relational complexity". These notions, which we define below in group theoretic terms, can also be formulated from a model theoretic point of view where they are best understood as properties of "relational structures". These connections, which run very deep, are explored at length in [Cherlin 2000], to which we refer the interested reader.

Theorem 1.3 settles Conjecture 1.2 for one of the families given by the classification of finite simple groups. It is the third recent result in this direction: Conjecture 1.2 has also been settled for groups with alternating socle [Gill and Spiga 2016], and for groups with socle a rank 1 group of Lie type [Gill et al. 2017]. Work is ongoing for the groups that remain (groups with socle a group of Lie type of rank at least 2) [Gill et al. ≥ 2018].

Our proof of Theorem 1.3 builds on ideas developed in [Gill and Spiga 2016] and [Gill et al. 2017], in particular the notion of a "strongly nonbinary action". In addition to this known approach, we also make use of a number of new lemmas — we mention, in particular, Lemma 2.7, which connects the "binariness" of an action to a bound on the number of orbits in the induced action on ℓ-tuples. These lemmas are gathered together in Section 2.

In addition to these new lemmas, though, this paper is very focused on adapting known facts about binary actions to create computational tests that can be applied using a computer algebra package like GAP or Magma. This process of developing tests is explained in great detail in Section 3.

In the final two sections we describe the outcome of these computations. In Section 4 we are able to give a proof of Theorem 1.3 for all of the sporadic groups barring the Monster. In Section 5 we give a proof of Theorem 1.3 for the Monster. The sheer size of the Monster means that some of the computational procedures that we exploit for the other groups are no longer available to us, and so our methods need to be refined to deal with this special case.

2. Definitions and lemmas

Throughout this section G is a finite group acting (not necessarily faithfully) on a set Ω of cardinality t. Given a subset Λ of Ω, we write

$$G_\Lambda := \{g \in G \mid \lambda^g \in \Lambda, \text{ for all } \lambda \in \Lambda\}$$

for the set-wise stabilizer of Λ, $G_{(\Lambda)} := \{g \in G \mid \lambda^g = \lambda, \text{ for all } \lambda \in \Lambda\}$ for the

pointwise stabilizer of Λ, and G^Λ for the permutation group induced on Λ by the action of G_Λ. In particular, $G^\Lambda \cong G_\Lambda / G_{(\Lambda)}$.

Given a positive integer r, the group G is called *r-subtuple complete* with respect to the pair of *n*-tuples $I, J \in \Omega^n$, if it contains elements that map every subtuple of length r in I to the corresponding subtuple in J, i.e.,

for every $\{k_1, k_2, \ldots, k_r\} \subseteq \{1, \ldots, n\}$,

$$\text{there exists } h \in G \text{ with } I_{k_i}^h = J_{k_i}, \text{ for every } i \in \{1, \ldots, r\}.$$

Here I_k denotes the k-th element of tuple I and I_k^g denotes the image of I_k under the action of g. Note that *n*-subtuple completeness simply requires the existence of an element of G mapping I to J.

Definition 2.1. The action of G is said to be of *arity* r if, for all $n \in \mathbb{N}$ with $n \geq r$ and for all *n*-tuples $I, J \in \Omega^n$, r-subtuple completeness (with respect to I and J) implies *n*-subtuple completeness (with respect to I and J). Note that in the literature the concept of "arity" is also known by the name "relational complexity".

When the action of G has arity 2, we say that the action of G is *binary*. If G is given to us as a permutation group, then we say that G is a *binary permutation group*.

A pair (I, J) of *n*-tuples of Ω is called a *nonbinary witness for the action of G on Ω* if G is 2-subtuple complete with respect to I and J, but not *n*-subtuple complete, that is, I and J are not G-conjugate. To show that the action of G on Ω is nonbinary it is sufficient to find a nonbinary witness (I, J).

We now recall some useful definitions introduced in [Gill et al. 2017]. We say that the action of G on Ω is *strongly nonbinary* if there exists a nonbinary witness (I, J) such that

- I and J are t-tuples where $|\Omega| = t$;
- the entries of I and J comprise all the elements of Ω.

We give a standard example, taken from [Gill et al. 2017], showing how strongly nonbinary actions can arise.

Example 2.2. Let G be a subgroup of $\mathrm{Sym}(\Omega)$, let $g_1, g_2, \ldots, g_r$ be elements of G, and let $\tau, \eta_1, \ldots, \eta_r$ be elements of $\mathrm{Sym}(\Omega)$ with

$$g_1 = \tau \eta_1, \ g_2 = \tau \eta_2, \ \ldots, \ g_r = \tau \eta_r.$$

Suppose that, for every $i \in \{1, \ldots, r\}$, the support of τ is disjoint from the support of η_i; moreover, suppose that, for each $\omega \in \Omega$, there exists $i \in \{1, \ldots, r\}$ (which may depend upon ω) with $\omega^{\eta_i} = \omega$. Suppose, in addition, $\tau \notin G$. Now, writing $\Omega = \{\omega_1, \ldots, \omega_t\}$, observe that

$$((\omega_1, \omega_2, \ldots, \omega_t), (\omega_1^\tau, \omega_2^\tau, \ldots, \omega_t^\tau))$$

is a nonbinary witness. Thus the action of G on Ω is strongly nonbinary.

The following lemma, taken from [Gill et al. 2017], shows a crucial property of the notion of strongly nonbinary action: it allows one to argue "inductively" on set-stabilizers (see also Lemma 2.8).

Lemma 2.3. *Let Ω be a G-set and let $\Lambda \subseteq \Omega$. If G^Λ is strongly nonbinary, then G is not binary in its action on Ω.*

Proof. Write $\Lambda := \{\lambda_1, \ldots, \lambda_\ell\}$ and assume that G^Λ is strongly nonbinary. Then there exists $\sigma \in \mathrm{Sym}(\ell)$ with $I := (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ and $J := (\lambda_{1^\sigma}, \lambda_{2^\sigma}, \ldots, \lambda_{\ell^\sigma})$ a nonbinary witness for the action of G_Λ on Λ. Now, observe that (I, J) is also a nonbinary witness for the action of G on Ω because any (putative) element g of G mapping I to J fixes Λ set-wise and hence $g \in G_\Lambda$. $\qquad\square$

Next we need an observation, made first in [Gill et al. 2017], that the existence of a strongly nonbinary witness is related to the classic concept of 2-*closure* introduced by Wielandt [1964]: given a permutation group G on Ω, the 2-closure of G is the set

$$G^{(2)} := \{\sigma \in \mathrm{Sym}(\Omega) \mid \text{ for all } (\omega_1, \omega_2) \in \Omega \times \Omega,$$
$$\text{there exists } g_{\omega_1\omega_2} \in G \text{ with } \omega_1^\sigma = \omega_1^{g_{\omega_1\omega_2}}, \omega_2^\sigma = \omega_2^{g_{\omega_1\omega_2}}\},$$

that is, $G^{(2)}$ is the largest subgroup of $\mathrm{Sym}(\Omega)$ having the same orbitals as G. The group G is said to be 2-*closed* if and only if $G = G^{(2)}$.

Lemma 2.4. *Let G be a permutation group on Ω. Then G is strongly nonbinary if and only if G is not 2-closed.*

Proof. Write $\Omega := \{\omega_1, \ldots, \omega_t\}$. If G is not 2-closed, then there exists $\sigma \in G^{(2)} \setminus G$. Set $I := (\omega_1, \ldots, \omega_t)$ and $J := I^\sigma = (\omega_1^\sigma, \ldots, \omega_t^\sigma)$; observe that I and J are 2-subtuple complete (because $\sigma \in G^{(2)}$) and are not G-conjugate (because $\sigma \notin G$). Thus (I, J) is a strongly nonbinary witness. The converse is similar. $\qquad\square$

Our next two lemmas make use of Lemma 2.3 and Example 2.2 to yield easy criteria for showing that a permutation group is not binary.

Lemma 2.5. *Let G be a transitive permutation group on Ω, let $\alpha \in \Omega$ and let p be a prime with p dividing both $|\Omega|$ and $|G_\alpha|$ and with p^2 not dividing $|G_\alpha|$. Suppose that G contains an elementary abelian p-subgroup $V = \langle g, h \rangle$ with $g \in G_\alpha$, with h and gh conjugate to g via G. Then G is not binary.*

Proof. Let $g \in G_\alpha$ and let $h \in g^G$ with $\langle g, h \rangle$ an elementary abelian p-subgroup of G of order p^2 with gh also conjugate to g via G. In particular, $h = g^x$, for some $x \in G$. Write $\alpha_0 := \alpha$ and $\alpha_p := \alpha^x$.

Since $g \in G_{\alpha_0}$ and $h \in G_{\alpha_p}$ commute, $\alpha_0^{h^i}$ is fixed by g and $\alpha_p^{g^i}$ is fixed by h, for every i. Write $\alpha_i := \alpha_0^{h^i}$ and $\alpha_{p+i} := \alpha_p^{g^i}$, for every $i \in \{0, \ldots, p-1\}$. Moreover, g acts as a p-cycle on $\{\alpha_p, \ldots, \alpha_{2p-1}\}$ and h acts as a p-cycle on $\{\alpha_0, \ldots, \alpha_{p-1}\}$.

Since gh is conjugate to g via an element of G, there exists $y \in G$ with $gh = g^y$. Write $\alpha_{2p} = \alpha^y$. Observe that gh fixes $(\alpha_{2p})^{g^{-i}} = \alpha_{2p}^{h^i}$ for every i. Write

$\alpha_{2p+i} := \alpha_{2p}^{g^i}$, for every $i \in \{0, \ldots, p-1\}$. Thus g and h act as inverse p-cycles on $\{\alpha_{2p}, \ldots, \alpha_{3p-1}\}$.

Write $\Lambda := \{\alpha_0, \ldots, \alpha_{3p-1}\}$. We have

$$g^\Lambda = (\alpha_p, \ldots, \alpha_{2p-1})(\alpha_{3p-1}, \ldots, \alpha_{2p}),$$
$$h^\Lambda = (\alpha_0, \ldots, \alpha_{p-1})(\alpha_{2p}, \ldots, \alpha_{3p-1}),$$
$$(gh)^\Lambda = (\alpha_0, \ldots, \alpha_{p-1})(\alpha_p, \ldots, \alpha_{2p-1}).$$

If G^Λ is strongly nonbinary, then G is not binary by Lemma 2.3. Assume that G^Λ is not strongly nonbinary. Then, in view of Example 2.2, there exists $f \in G$ with $f^\Lambda = (\alpha_p, \ldots, \alpha_{2p-1})$. This is a contradiction, because by hypothesis $|G_\alpha|$ is not divisible by p^2 but $\langle g, f \rangle$ has order divisible by p^2 and fixes $\alpha_0 = \alpha$. $\square$

Lemma 2.6. *Let G be a permutation group on Ω and suppose that g and h are elements of G of order p where p is a prime such that g, h and gh^{-1} are all G-conjugate. Suppose that $V = \langle g, h \rangle$ is elementary abelian of order p^2. Suppose, finally, that G does not contain any elements of order p that fix more points of Ω than g. If $|\mathrm{Fix}(V)| < |\mathrm{Fix}(g)|$, then G is not binary.*

We remark that there are well-known formulae that we can use to calculate $\mathrm{Fix}(V)$ and $|\mathrm{Fix}(g)|$ when G is transitive (see for instance [Liebeck and Saxl 1991, Lemma 2.5]). Suppose that M is the stabilizer of a point in Ω; then we have

$$(2\text{-}1) \quad |\mathrm{Fix}_\Omega(g)| = \frac{|\Omega| \cdot |M \cap g^G|}{|g^G|}, \quad |\mathrm{Fix}_\Omega(V)| = \frac{|\Omega| \cdot |\{V^g \mid g \in G, V^g \le M\}|}{|V^G|}.$$

Proof. We let

$$\Lambda := \mathrm{Fix}(g) \cup \mathrm{Fix}(h) \cup \mathrm{Fix}(gh^{-1}).$$

Observe, first, that Λ, $\mathrm{Fix}(g)$, $\mathrm{Fix}(h)$ and $\mathrm{Fix}(gh^{-1})$ are g-invariant and h-invariant. Observe, second, that

$$\mathrm{Fix}(g) \cap \mathrm{Fix}(h) = \mathrm{Fix}(g) \cap \mathrm{Fix}(gh^{-1}) = \mathrm{Fix}(h) \cap \mathrm{Fix}(gh^{-1}) = \mathrm{Fix}(V).$$

Write τ_1 for the permutation induced by g on $\mathrm{Fix}(gh^{-1})$, τ_2 for the permutation induced by g on $\mathrm{Fix}(h)$, and τ_3 for the permutation induced by h on $\mathrm{Fix}(g)$ (observe that τ_i's are non trivial as gh^{-1}, h and g are conjugate). Since $|\mathrm{Fix}(V)| < |\mathrm{Fix}(g)|$, we conclude that τ_1, τ_2 and τ_3 are disjoint nontrivial permutations. What is more, g induces the permutation $\tau_1\tau_2$ on Λ, while h induces the permutation $\tau_1\tau_3$ on Λ.

In view of Example 2.2, G^Λ is strongly nonbinary provided there is no element $f \in G_\Lambda$ that induces the permutation τ_1. Arguing by contradiction, if such an element f exists, then f has order divisible by p and $f^{o(f)/p}$ is a p-element fixing more points than g, which is a contradiction. Thus G^Λ is strongly nonbinary and G is not binary by Lemma 2.3. $\square$

For the rest of this section we assume that G is transitive. Given $\ell \in \mathbb{N} \setminus \{0\}$, we denote by $\Omega^{(\ell)}$ the subset of the Cartesian product Ω^ℓ consisting of the ℓ-tuples $(\omega_1, \ldots, \omega_\ell)$ with $\omega_i \neq \omega_j$, for every two distinct elements $i, j \in \{1, \ldots, \ell\}$. We denote by $r_\ell(G)$ the number of orbits of G on $\Omega^{(\ell)}$.

Let $\pi : G \to \mathbb{N}$ be the permutation character of G, that is, $\pi(g) = \text{fix}_\Omega(g)$ where $\text{fix}_\Omega(g)$ is the cardinality of the fixed point set $\text{Fix}_\Omega(g) := \{\omega \in \Omega \mid \omega^g = \omega\}$ of g. From the orbit counting lemma, we have

$$r_\ell(G) = \frac{1}{|G|} \sum_{g \in G} \text{fix}_\Omega(g)(\text{fix}_\Omega(g) - 1) \cdots (\text{fix}_\Omega - (\ell - 1))$$

$$= \langle \pi(\pi - 1) \cdots (\pi - (\ell - 1)), 1 \rangle_G,$$

where 1 is the principal character of G and $\langle \cdot, \cdot \rangle_G$ is the natural Hermitian product on the space of $\mathbb{C}$-class functions of G.

Lemma 2.7. *If G is transitive and binary, then $r_\ell(G) \leq r_2(G)^{\ell(\ell-1)/2}$ for each $\ell \in \mathbb{N}$.*

Note that this lemma is, in effect, an immediate consequence of the fact that, for a binary action, the orbits on pairs "determine" orbits on ℓ-tuples. Thus, to uniquely determine the orbit of a particular ℓ-tuple, it is enough to specify the orbits of all $\binom{\ell}{2}$ pairs making up the ℓ-tuple.

Proof. We write $r_2 := r_2(G)$ and $r_\ell := r_\ell(G)$ and we assume that $r_\ell > r_2^{(\ell-1)\ell/2}$ for some $\ell \in \mathbb{N}$. Clearly, $\ell > 2$.

Let
$$(\omega_{1,1}, \ldots, \omega_{1,\ell}), \ldots, (\omega_{r_\ell,1}, \ldots, \omega_{r_\ell,\ell})$$
be a family of representatives for the G-orbits on $\Omega^{(\ell)}$. From the pigeon-hole principle, at least r_ℓ/r_2 of these elements have the first two coordinates in the same G-orbit. Formally, there exists $\kappa \in \mathbb{N}$ with $\kappa \geq r_\ell/r_2$ and a subset $\{i_1, \ldots, i_\kappa\}$ of $\{1, \ldots, r_\ell\}$ of cardinality κ such that the κ pairs

$$(\omega_{i_1,1}, \omega_{i_1,2}), \ldots, (\omega_{i_\kappa,1}, \omega_{i_\kappa,2})$$

are in the same G-orbit. By considering all possible pairs of coordinates, this argument can be easily generalized. Indeed, from the pigeon-hole principle, there exists κ with $\kappa \geq r_\ell/r_2^{(\ell-1)\ell/2} > 1$ and a subset $\{i_1, \ldots, i_\kappa\}$ of $\{1, \ldots, r_\ell\}$ of cardinality κ such that, for each $1 \leq u < v \leq \ell$, the κ pairs

$$(\omega_{i_1,u}, \omega_{i_1,v}), \ldots, (\omega_{i_\kappa,u}, \omega_{i_\kappa,v})$$

are in the same G-orbit. In other words, the ℓ-tuples

$$(\omega_{i_1,1}, \ldots, \omega_{i_1,\ell}), \ldots, (\omega_{i_\kappa,1}, \ldots, \omega_{i_\kappa,\ell})$$

are 2-subtuple complete. Since G is binary, these ℓ-tuples must be in the same G-orbit, contradicting $\kappa > 1$. $\qquad\square$

Observe that when $r_2(G) = 1$, that is, G is 2-transitive, Lemma 2.7 yields $r_\ell(G) = 1$ for every $\ell \in \{2, \ldots |\Omega|\}$. Therefore $G = \mathrm{Sym}(\Omega)$ is the only 2-transitive binary group.

Lemma 2.8. *Let G be transitive, let α be a point of Ω and let $\Lambda \subseteq \Omega$ be a G_α-orbit. If G is binary, then G_α^Λ is binary. In particular, if $g \in G$ and the action of G_α on the right cosets of $G_\alpha \cap G_\alpha^g$ in G_α is not binary, then G is not binary.*

Proof. Assume that G is binary. Let $\ell \in \mathbb{N}$ and let $I := (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ and $J := (\lambda_1', \lambda_2', \ldots, \lambda_\ell')$ be two tuples in Λ^ℓ that are 2-subtuple complete for the action of G_α on Λ. Clearly, $I_0 := (\alpha, \lambda_1, \lambda_2, \ldots, \lambda_\ell)$ and $J_0 := (\alpha, \lambda_1', \lambda_2', \ldots, \lambda_\ell')$ are 2-subtuple complete for the action of G on Ω. As G is binary, I_0 and J_0 are in the same G-orbit; hence I and J are in the same G_α-orbit. From this we deduce that G_α^Λ is binary.

Suppose now that $g \in G$ and that the action of G_α on the right cosets of $G_\alpha \cap G_\alpha^g$ in G_α is not binary. Set $\beta := \alpha^g$ and $\Lambda := \beta^{G_\alpha}$. Now Λ is a G_α-orbit contained in $\Omega \setminus \{\alpha\}$ and the action of G_α on Λ is permutation isomorphic to the action of G_α on the right cosets of $G_\alpha \cap G_\beta = G_\alpha \cap G_\alpha^g$ in G_α. Therefore, G_α^Λ is not binary and hence G is not binary. $\qquad\square$

3. On computation

In this section we explain how to make use of the lemmas given in the previous section in a computational setting. The computational problem we are faced with is as follows: given a transitive action of a group G on a set Ω, we wish to show that the action is nonbinary; in some cases we will require more, namely that the action is strongly nonbinary. If the set Ω is small enough, then we can often exhibit G as a permutation group in the computer algebra package Magma and compute explicitly; when Ω gets too large, then this may be infeasible and we may know only the isomorphism type of G and the isomorphism type of a point-stabilizer.

3A. *Test 1: using Lemma 2.7.* In some cases, Lemma 2.7 is very efficient for dealing with some primitive actions of almost simple groups G with socle a sporadic simple group. In particular, whenever the permutation character of G is available in GAP or in Magma, we can simply check directly the inequality in Lemma 2.7. For instance, using this method it is easy to verify that each faithful primitive action of M_{11} is nonbinary.

For practical purposes, it is worth mentioning that apart from

- the Monster,
- the action of the Baby Monster on the cosets of a maximal subgroup of type $(2^2 \times F_4(2)) : 2$,

each permutation character of each primitive permutation representation of an almost simple group with socle a sporadic simple group is available in GAP via the package

"The GAP Character Table Library". Therefore, for the proof of Theorem 1.3, we can quickly and easily use Lemma 2.7 except for the Monster. To give a rough idea of the time to perform this test, in the Baby Monster (except for the action on the cosets on a maximal subgroup of type $(2^2 \times F_4(2)) : 2$), it takes less than two minutes to perform this checking. (The permutation character of the Baby Monster G on the cosets of a maximal subgroup M of type $(2^2 \times F_4(2)) : 2$ is missing from the GAP library because the conjugacy fusion of some of the elements of M in G remains a mystery: this information is vital for computing the permutation character.)

For reasons that will be more clear later, for the proof of Theorem 1.3, we need to prove the nonbinariness of permutation groups $G \leq \mathrm{Sym}(\Omega)$ that are not necessarily almost simple, let alone having socle a sporadic simple group. When $|\Omega|$ is relatively small (for practical purposes, here relatively small means at most 10^9), we can afford to compute the permutation character and check the inequality in Lemma 2.7.

3B. *Test 2*: *using Lemma 2.4.* By connecting the notion of strong-nonbinariness to 2-closure, Lemma 2.4 yields an immediate computational dividend: there are built-in routines in GAP and Magma to compute the 2-closure of a permutation group.

Thus if Ω is small enough, say $|\Omega| \leq 10^6$, then we can easily check whether or not the group G is 2-closed. Thus we can ascertain whether or not G is strongly nonbinary.

3C. *Test 3*: *a direct analysis.* The next test we discuss is feasible once again provided $|\Omega| \leq 10^6$. It simply tests whether or not 2-subtuple-completeness implies 3-subtuple completeness, and the procedure is as follows:

We fix $\alpha \in \Omega$, we compute the orbits of G_α on $\Omega \setminus \{\alpha\}$ and we select a set of representatives $\mathcal{O}$ for these orbits. Then, for each $\beta \in \mathcal{O}$, we compute the orbits of $G_\alpha \cap G_\beta$ on $\Omega \setminus \{\alpha, \beta\}$ and we select a set of representatives $\mathcal{O}_\beta$. Then, for each $\gamma \in \mathcal{O}_\beta$, we compute $\gamma^{G_\alpha} \cap \gamma^{G_\beta}$. Finally, for each $\gamma' \in \gamma^{G_\alpha} \cap \gamma^{G_\beta}$, we test whether the two triples (α, β, γ) and (α, β, γ') are G-conjugate. If the answer is "no", then G is not binary because by construction (α, β, γ) and (α, β, γ') are 2-subtuple complete. In particular, in this circumstance, we can break all the "for loops" and deduce that G is not binary.

If the answer is "yes", for every β, γ, γ', then we cannot deduce that G is binary, but we can keep track of these cases for a deeper analysis. We observe that, if the answer is "yes", for every β, γ, γ', then 2-subtuple completeness implies 3-subtuple completeness.

3D. *Test 4*: *using Lemma 2.8.* The next test is particularly useful in cases where Ω is very large, since its computational complexity is independent of $|\Omega|$. Let us

suppose that G and its subgroup M are stored in a library as abstract groups (or as matrix groups or as permutation groups). When $|G : M|$ is too large, it is impractical (and sometimes impossible) to construct G as a permutation group on the coset space $\Omega := G/M$ with point stabilizer M. However, using Lemma 2.8, we can still prove that G acting on Ω is nonbinary: all we need is $g \in G$ such that the action of M on $M \cap M^g$ is nonbinary. Now, for carefully chosen g, $|M : M \cap M^g|$ might be much smaller than $|G : M|$ and we can use one of the previous tests to ascertain whether or not M in its action on $M/(M \cap M^g)$ is binary.

3E. *Test 5*: *a new lemma*. Our final test requires an extra lemma which we include here, rather than in the earlier section, as its computational aspect is somehow inherent in its very statement.

Lemma 3.1. *Let G be a primitive group on a set Ω, let α be a point of Ω, let M be the stabilizer of α in G and let d be an integer with $d \geq 2$. Suppose $M \neq 1$ and, for each transitive action of M on a set Λ such that*

(1) *$|\Lambda| > 1$,*

(2) *every composition factor of M is isomorphic to some section of M^Λ,*

(3) *either $M_{(\Lambda)} = 1$ or, given $\lambda \in \Lambda$, the stabilizer M_λ has a normal subgroup N with $N \neq M_{(\Lambda)}$ and $N \cong M_{(\Lambda)}$, and*

(4) *M is binary in its action on Λ,*

we have that d divides $|\Lambda|$. Then either d divides $|\Omega| - 1$ or G is not binary.

Proof. Suppose that G is binary. Since $\{\beta \in \Omega \mid \beta^m = \beta, \text{ for all } m \in M\}$ is a block of imprimitivity for G and since G is primitive, we obtain that either M fixes each point of Ω or α is the only point fixed by M. The former possibility is excluded because $M \neq 1$ by hypothesis. Therefore α is the only point fixed by M. Let $\Lambda \subseteq \Omega \setminus \{\alpha\}$ be an M-orbit. Thus $|\Lambda| > 1$ and (1) holds. Since G is a primitive group on Ω, from [Dixon and Mortimer 1996, Theorem 3.2C], we obtain that every composition factor of M is isomorphic to some section of M^Λ and hence (2) holds. From Lemma 2.8, the action of M on Λ is binary and hence (4) holds. Let now $\lambda \in \Lambda$ and consider the orbital graph $\Gamma := (\alpha, \lambda)^G$. Observe that Γ is connected because G is primitive. Let $g \in G$ with $\alpha^g = \lambda$. Clearly, Λ is the set of out-neighbors of α in Γ and $\Lambda' := \Lambda^g$ is the set of out-neighbors of $\alpha^g = \lambda$ in Γ. Set $N := (G_\lambda)_{(\Lambda')}$. Clearly, $(G_\alpha)_{(\Lambda)} = M_{(\Lambda)}$ and $(G_{\alpha^g})_{(\Lambda^g)} = (G_\lambda)_{(\Lambda')} = N$ are isomorphic because they are G-conjugate via the element g. Moreover, $M_{(\Lambda)} = (G_\alpha)_{(\lambda^{G_\alpha})}$ is normalized by G_α and, similarly, N is normalized by G_λ; therefore they are both normalized by

$$G_\alpha \cap G_\lambda = M \cap G_\lambda = M_\lambda.$$

If $M_{(\Lambda)}$ and N are equal, an easy connectedness argument yields that $M_{(\Lambda)} = 1$. Therefore (3) also holds.

Since the four hypotheses in the statement of this lemma hold for the action of $M = G_\alpha$ on its G_α-orbit Λ, we get d divides $|\Lambda|$. Since this argument does not depend on the G_α-orbit $\Lambda \subseteq \Omega \setminus \{\alpha\}$, we obtain that $\Omega \setminus \{\alpha\}$ has cardinality divisible by d. Thus $|\Omega| - 1$ is divisible by d.   □

When it comes to implementing Lemma 3.1 on a computer, it is important to observe that we do *not* need to construct the embedding of $M = G_\alpha$ in G; indeed we do not need the group G stored in our computer at all. Instead we need only the index $|G : M| = |\Omega|$ and the abstract group M (given as a group of matrices, or as a permutation group, or as a finitely presented group).

Given $|\Omega|$ and M, we may choose a prime p (typically $p = 2$) with p not dividing $|\Omega| - 1$ and we construct all the transitive permutation representations of degree greater than 1 and relatively prime to p of M satisfying (1), (2) and (3). If none of these permutation representations is binary (and we can use any of Tests 1 to 4 to test this), we infer that every transitive permutation representation of M of degree greater than 1 satisfying (1), (2), (3) and (4) has degree divisible by p. Now, from Lemma 3.1, we get that G in its action on the set M of right cosets of M in G is not binary because p does not divide $|\Omega| - 1$.

We give an explicit example to show how easily Lemma 3.1 can be applied. The Monster G has a maximal subgroup M isomorphic to $\mathrm{PGL}_2(19)$. The index of M in G is

$$118131202455338139749482442245864145761075200000000 \sim 10^{50}$$

and we can easily observe that this number is even. After implementing Lemma 3.1 on a computer, it takes the blink of an eye to prove that each permutation representation of M of degree at least 1 and odd is nonbinary. Thus G acting on the cosets of M is nonbinary. Observe that besides $|G : M|$ and the isomorphism class of M, no information about G is needed.

4. The non-Monster groups

The centerpiece of this section is Table 1; it summarizes the results of applying the tests described in the previous section to all almost simple groups with a sporadic socle, barring the Monster.

Table 1 consists of two columns: the first column lists all of the almost simple groups G with socle a sporadic simple group (recall that we include Tits group $^2F_4(2)'$ in the list of sporadic groups). In the second column, we list all pairs $(M, \circ)$, where M is a maximal subgroup of G with the property that the action of G on the set G/M of right cosets of M in G satisfies Lemma 2.7 (in other words, the action is not excluded by Test 1, and hence is a potentially binary action). We use the ATLAS [Conway et al. 1985] notation for the group M.

Now the symbol ∘ is either ∞ or a prime p or "?". We write ∘ = ∞ if we have proved the nonbinariness of G in its action on G/M using Tests 2 or 3; we write ∘ = p if we have proved the nonbinariness of G in its action on G/M using Test 5 applied to the prime p; and we write ∘ =? if both methods have failed. The symbol "−" in the second column means that each primitive permutation representation of G is not binary via Lemma 2.7 (Test 1).

We have made use of the fact that full information on the maximal subgroups for each group in the first column of Table 1 is available: these are all stored in GAP or in Magma. To be more precise, in each case, either the maximal subgroup M is stored providing a generating set (written as words in the standard generators for G), or when such information is not available (for instance, for some of the maximal subgroups of Fi_{23}), the group M is explicitly described (for instance, as a p-local subgroup) and hence also in this case it is possible to construct M with a computer.

We are now able to prove Theorem 1.3 for all groups bar the Monster.

Proposition 4.1. *Let G be an almost simple primitive group with socle a sporadic simple group. If G is binary*, *then G is the Monster group.*

Proof. In view of Table 1, it suffices to consider the case that G is either Co_3, or Ru, or B: these are the only groups having a "?" symbol in one of their rows. We first assume that G is either Co_3 or Ru; here, in view of Table 1 the group G is acting on the cosets of $M = A_4 \times S_5$ when $G = \mathrm{Co}_3$, or $M = 5 : 4 \times A_5$ when $G = \mathrm{Ru}$. Given a Sylow 2-subgroup P of M, in both cases it is easy to verify with Magma that there exists $g \in \mathbf{N}_G(P)$ with $M \cap M^g = P$. When $G = \mathrm{Co}_3$, P is of type $2 \times 2 \times D_4$ and, when $G = \mathrm{Ru}$, P is of type $4 \times 2 \times 2$. Another computation shows that the actions of $A_4 \times S_5$ on the cosets of $2 \times 2 \times D_4$, and of $5 : 4 \times A_4$ on the cosets of $4 \times 2 \times 2$ are not binary. Therefore, G in its action on the cosets of M is not binary by Lemma 2.8.

Finally assume that G is the Baby Monster B. In view of Table 1, G is acting on the cosets of M where M is of one of the following types:

$$(2^2 \times F_4(2)) : 2, \quad 3^{1+8}.2^{1+6}.U_4(2).2, \quad (3^2 : D_8 \times U_4(3).2^2).2, \quad 3^2.3^3.3^6.(S_4 \times 2S_4).$$

Let Ω be the set of right cosets of M in G and let $\alpha \in \Omega$ with $G_\alpha = M$ (that is, α is the coset M). We go through the four remaining cases one at a time.

Case 1: $M \cong (3^2 : D_8 \times U_4(3).2^2).2$. Observe that a Sylow 7-subgroup of G has order $7^2 = 49$, that G has a unique conjugacy class of elements of order 7, and that $|M|$ and $|G : M|$ are both divisible by 7. Then Lemma 2.5 implies that G is not binary.

Case 2: $M \cong 3^{1+8}.2^{1+6}.U_4(2).2$. The group G has two conjugacy classes of elements of order 5, with the ATLAS notation, of type 5A and of type 5B. By

group	outcome of tests
M_{11}	$-$
M_{12}	$-$
$M_{12}.2$	$-$
M_{22}	$-$
$M_{22}.2$	$-$
M_{23}	$-$
M_{24}	$(L_2(7), \infty)$
J_1	$(D_6 \times D_{10}, \infty)$, $(7:6, \infty)$
J_2	(A_5, ∞)
$J_2.2$	(S_5, ∞)
J_3	$-$
$J_3.2$	$(19:18, 3)$
J_4	$(M_{22}:2, 2)$, $(11_+^{1+2}:(5 \times 2S_4), 2)$, $(L_2(32):5, 11)$, $(L_2(23):2, 2)$, $(U_3(3), 2)$, $(29:28, 2)$, $(43:14, 7)$, $(37:12, 2)$
$^2F_4(2)'$	$-$
$^2F_4(2)$	$(M, 2)$ where M has order 156
Suz	$(A_7, 2)$, $(L_2(25), 2)$
Suz.2	$(S_7, 7)$
McL	$-$
McL.2	$-$
HS	$(M_{22}, 2)$
HS.2	$(M_{22}:2, 2)$
Co_3	$(A_4 \times S_5, ?)$
Co_2	$(5_+^{1+2}:4S_4, 2)$
Co_1	$(A_9 \times S_3, 3)$, $((A_7 \times L_2(7)):2, 2)$, $((D_{10} \times (A_5 \times A_5).2).2, 2)$, $(5_+^{1+2}:GL_2(5), 2)$, $(5^3:(4 \times A_5).2, 2)$, $(5^2:4A_4, 2)$, $(7^2:(3 \times 2A_4), 2)$
He	$(5^2:4A_4, 2)$
He.2	$-$
Fi_{22}	$-$
$Fi_{22}.2$	$-$
Fi_{23}	$(L_2(23), 2)$
Fi_{24}'	$((A_5 \times A_9):2, 3)$, $(A_6 \times L_2(8):3, 2)$, $(7:6 \times A_7, 7)$, $(U_3(3).2, 2)$ $(U_3(3).2, 2)$, $(L_2(13).2, 2)$, $(L_2(13).2, 2)$, $(29:14, 7)$
Fi_{24}	$(S_5 \times S_9, 3)$, $(S_6 \times L_2(8):3, 2)$, $(7:6 \times S_7, 7)$, $(7_+^{1+2}:(6 \times S_3).2, 2)$, $(29:28, 7)$
Ru	$(L_2(13):2, 2)$, $(5:4 \times A_5, ?)$, $(A_6.2^2, 2)$, $(5_+^{1+2}:[2^5], 2)$, $(3.A_6.2^2, 2)$
O'N	$(A_7, 2)$, $(A_7, 2)$
O'N.2	$(31:30, 5)$, $(L_2(7):2, 2)$, $(A_6:2_2, 2)$
Ly	$(67:22, 11)$, $(37:18, 3)$
Th	$(3^5:2S_6, 2)$, $(5_+^{1+2}:4S_4, 2)$, $(5^2:GL_2(5), 2)$, $(7^2:(3 \times 2S_4), 2)$, $(L_2(19).2, 2)$, $(L_3(3), 2)$, $(M_{10} = A_6.2_3, 2)$, $(31:15, 4)$, $(S_5, 5)$
HN	$(3_+^{1+4}:4A_5, 2)$
HN.2	$-$
B	$((2^2 \times F_4(2)):2, ?)$, $(3^{1+8}.2^{1+6}.U_4(2).2, ?)$, $((3^2:D_8 \times U_4(3).2^2).2, ?)$, $(5:4 \times HS:2, 2)$, $(3^2.3^3.3^6.(S_4 \times 2S_4), ?)$, $(S_4 \times ^2F_4(2), 2)$, $(S_5 \times (M_{22}:2), 2)$, $((S_6 \times (L_3(4):2)).2, 2)$, $(5^3:L_3(5), 2)$, $(5^{1+4}.2^{1+4}.A_5.4, 2)$, $((S_6 \times S_6).4, 2)$, $((5^2:4S_4) \times S_5, 2)$, $(L_2(49).2, 2)$, $(L_2(31), 2)$, $(M_{11}, 2)$, $(L_3(3), 2)$, $(L_2(17):2, 2)$, $(L_2(11):2, 2)$, $(47:23, 23)$

Table 1. Disposing of some of the sporadic simple groups.

computing the permutation character of G via the package the GAP character table library, we see that an element of type 5A fixes no point and that an element of type 5B fixes 25000 points. Observe that $|M|$ is divisible by 5, but not by $5^2 = 25$. Moreover, using the ATLAS [Conway et al. 1985], we see that G contains an elementary abelian 5-group V of order 5^3 generated by three elements of type 5B; moreover, the normalizer of V is a maximal subgroup of G of type $5^3 : L_3(5)$. In particular, each nonidentity 5-element of V is of type 5B, because $L_3(5)$ acts transitively on the nonzero vectors of 5^3. Since $|M|$ is not divisible by 25, we conclude that $\mathrm{Fix}(V)$ is empty. Now we apply Lemma 2.6 to L, a subgroup of M of order 25 such that $|\mathrm{Fix}(L)| < |\mathrm{Fix}(g)|$. We conclude that G is not binary.

Case 3: $M \cong 3^2.3^3.3^6.(S_4 \times 2S_4)$. From the ATLAS [Conway et al. 1985], we see that $M = \mathbf{N}_G(V)$, where V is an elementary abelian 3-subgroup of order 3^2 with $V \setminus \{1\}$ consisting only of elements of type 3B. For the proof of this case, we refer to [Wilson 1987] and [Wilson 1999]. According to Wilson [1987, Section 3], there exist three G-conjugacy classes of elementary abelian 3-subgroups of G of order 3^2 consisting only of elements of type 3B, denoted in [Wilson 1987] as having type (a) or (b) or (c). Moreover, from [Wilson 1987, Proposition 3.1], we see that only for the elementary abelian 3-groups of type (a) the normalizers are maximal subgroups of G and of shape $3^2.3^3.3^6.(S_4 \times 2S_4)$. Thus V is of type (a). Let V_1, V_2, V_3 be representatives for the conjugacy classes of elementary 3-subgroups of G of order 3^2 and consisting only of elements of type 3B. We may assume that $V_1 = V$. From [Wilson 1987] or [Wilson 1999], for $W \in \{V_1, V_2, V_3\}$, $\mathbf{N}_G(W)$ has shape $3^2.3^3.3^6.(S_4 \times 2S_4)$, $(3^2 \times 3^{1+4}).(2^2 \times 2A_4).2$, and $(3^2 \times 3^{1+4}).(2 \times 2S_4)$; in [Wilson 1987; 1999], these cases are referred to as type (a), type (b) and type (c), respectively.

Next, we consider a maximal subgroup K of G isomorphic to $\mathrm{PSL}_3(3)$. From [Wilson 1999] (pages 9 and 10 and the discussion therein on the interaction between K and the types (a), (b) and (c)), we infer that K contains a conjugate of V. In particular, replacing K by a suitable G-conjugate, we may assume that $V \leq K$, and more specifically,

$$M \cap K = \mathbf{N}_K(V).$$

Take $\Lambda := \alpha^K$ and observe that Λ is a K-orbit on Ω and that the stabilizer of the point α in K is $\mathbf{N}_K(V)$. Moreover, since K is maximal in G, we get $G_\Lambda = K$. We claim that $G^\Lambda = K^\Lambda$ is strongly nonbinary, from which it follows that G is not binary by Lemma 2.3. Observe that the action of K on Λ is permutation isomorphic to the action of K on the set of right cosets of $\mathbf{N}_K(V)$ in K.

Now, we consider the abstract group $K_0 = \mathrm{PSL}_3(3)$, we consider an elementary abelian 3-subgroup V_0 of order 9 of K_0, we compute $N_0 := \mathbf{N}_{K_0}(V_0)$ and we consider the action of K_0 on the set Λ_0 of right cosets of N_0 in K_0. A straightforward

computation shows that K_0 is not 2-closed in this action, and hence K_0 in its action on Λ_0 is strongly nonbinary by Lemma 2.4.

Case 4: $M \cong (2^2 \times F_4(2)) : 2$. Here we cannot invoke the GAP character table library to understand whether $F_4(2)$ contains elements of type 5A or 5B, because the fusion of M in G is (in some cases) still unknown. As we mentioned above, G has two conjugacy classes of elements of order 5, denoted by 5A and 5B; what is more the group $F_4(2)$ contains a unique conjugacy class of elements of order 5. Observe that the centralizers in G have elements of type 5A and 5B which have orders 44352000 and 6000000, respectively. Now, the centralizer in $F_4(2)$ of an element of order 5 has cardinality 3600. Since 3600 does not divide 6000000, we get that M contains only elements of type 5A; in particular elements of type 5B do not fix any element of Ω.

Using (2-1), we conclude that if g is an element of order 5 in M, then

$$|\mathrm{Fix}_\Omega(g)| = \frac{|G|}{|M|} \frac{|M : \mathbf{C}_M(g)|}{|G : \mathbf{C}_G(g)|} = \frac{|\mathbf{C}_G(g)|}{|\mathbf{C}_M(g)|} = \frac{44352000}{3600 \times 4 \times 2} = 1540.$$

Now let V be a Sylow 5-subgroup of M and observe that V has order 5^2 and $V \setminus \{1\}$ consists only of elements of type 5A. Referring to [Wilson 1987, Section 6], we see that G contains only one conjugacy class of elementary abelian groups of order 25 for which the nontrivial elements are all of type 5A. Thus V is a representative of this G-conjugacy class. Now, Theorem 6.4 in [Wilson 1987] yields $N_G(V) \cong 5^2 : 4S_4 \times S_5$. Appealing to (2-1) again, we conclude that

$$|\mathrm{Fix}_\Omega(V)| = \frac{|G|}{|M|} \frac{|M : \mathbf{N}_M(V)|}{|G : \mathbf{N}_G(V)|} = \frac{|\mathbf{N}_G(V)|}{|\mathbf{N}_M(V)|} = \frac{28800}{19200} = 15.$$

Now Lemma 2.6 implies that G is not binary. □

5. The Monster

We prove Theorem 1.3 for the Monster. Our proof will break down into several parts, and to ensure we cover all possibilities we make use of a recent account of the classification of the maximal subgroups of the sporadic simple groups in [Wilson 2017].

From [Wilson 2017, Section 3.6], we see that the classification of the maximal subgroups of the Monster G is complete except for a few small open cases. In particular, if M is a maximal subgroup of G, then either

(**a**) M is in [Wilson 2017, Section 4], or

(**b**) M is almost simple with socle isomorphic to $L_2(8)$, $L_2(13)$, $L_2(16)$, $U_3(4)$ or $U_3(8)$.

From here on G will always denote the Monster group, and M will be a maximal subgroup of G. We consider the action of G on cosets of M.

maximal subgroup	prime	maximal subgroup	prime
$2.B$	11	$(D_{10} \times \mathrm{HS}).2$	7
$2^{1+24}.\mathrm{Co}_1$	11	$(3^2 : 2 \times O_8^+(3)).S_4$	7
$3.\mathrm{Fi}_{24}$	11	$3^{2+5+10}.(M_{11} \times 2S_4)$	11
$2^2.{}^2E_6(2).S_3$	11	$5^{1+6} : 2J_2 : 4$	7
$2^{10+16}.O_{10}^+(2)$	7	$(A_5 \times A_{12}) : 2$	7
$2^{2+11+22}.(M_{24} \times S_3)$	7	$(A_5 \times U_3(8) : 3_1) : 2$	7
$3^{1+12}.2\mathrm{Suz}.2$	7	$(L_3(2) \times S_4(4) : 2).2$	7
$2^{5+10+20}.(S_3 \times L_5(2))$	7	$(5^2 : [2^4] \times U_5(5)).S_3$	7
$2^{3+6+12+18}.(L_3(2) \times 3S_6)$	7	$7^{1+4} : (3 \times 2S_7)$	5
$3^8.O_8^-(3).2_3$	7	$L_2(16).2$	5

Table 2. Primitive actions of the Monster for Lemma 5.2.

5A. *The almost simple subgroups in (b).* We begin by applying Test 5 to those groups in category (***b***). Provided that such a group M is not isomorphic to $L_2(16).2$, we find that, by applying Test 5 with the prime 2 or 3, we can immediately show that G in its action on G/M is not binary.

The group $M = L_2(16).2$ is exceptional here: for each prime p dividing $|M|$, there exists a permutation representation of M of degree coprime to p satisfying the four conditions in Lemma 3.1; hence we cannot apply Test 5. We defer the treatment of $L_2(16).2$ to Section 5B below.

From here on we will consider those groups in category (***a***), as well as the deferred group $L_2(16).2$.

5B. *Constructing a strongly nonbinary subset.* For our next step, we will apply Lemma 2.5 to the remaining group, $L_2(16).2$, from category (***b***) and to the groups from category (***a***). We start with a technical lemma; this is then followed by the statement that we need, Lemma 5.2.

Lemma 5.1. *Let G be the Monster, let $p \in \{5, 7, 11\}$ and let $x \in G$ with $o(x) = p$. Then there exists $g \in G$ with $\langle x, x^g \rangle$ elementary abelian of order p^2 and with xx^g conjugate to x via an element of G.*

Proof. When $p = 11$, there is nothing to prove: G has a unique conjugacy class of elements of order 11 and a Sylow 11-subgroup of G is elementary abelian of order 11^2.

When $p \in \{5, 7\}$, it is enough to read [Conway et al. 1985, page 234]: G contains two conjugacy classes of elements of order p. Moreover, G contains two elementary abelian p-subgroups V and V' both of order p^2, with V generated by two elements of type pA and with V' generated by two elements of type pB. Moreover, $\mathbf{N}_G(V)$ and $\mathbf{N}_G(V')$ act transitively on the nonidentity elements of V and of V', respectively.

This lemma can also be easily deduced from [Wilson 1988]. $\qquad\square$

maximal subgroup	prime	maximal subgroup	prime
$(7 : 3 \times \mathrm{He}) : 2$	2	$(7^2 : (3 \times 2A_4) \times L_2(7)).2$	2
$(A_6 \times A_6 \times A_6).(2 \times S_4)$	2	$(13 : 6 \times L_3(3)).2$	2
$(5^2 : [2^4] \times U_3(5)).S_3$	2	$13^{1+2} : (3 \times 4S_4)$	2
$(L_2(11) \times M_{12}) : 2$	2	$L_2(71)$	2
$(A_7 \times (A_5 \times A_5) : 2^2) : 2$ [1]	2	$L_2(59)$	5
$5^4 : (3 \times 2L_2(25)) : 2$	2	$11^2 : (5 \times 2A_5)$	2
$7^{2+1+2} : \mathrm{GL}_2(7)$	2	$L_2(41)$	2
$M_{11} \times A_6.2^2$	2	$L_2(29) : 2$	2
$(S_5 \times S_5 \times S_5) : S_3$	3	$7^2 : \mathrm{SL}_2(7)$	2
$(L_2(11) \times L_2(11)) : 4$	2	$L_2(19) : 2$	2
$13^2 : (2L_2(13).4)$	2	$41 : 40$	2

Table 3. Primitive actions of the Monster for Lemma 5.3.

Lemma 5.2. *Let G be the Monster and let M be a maximal subgroup of G. If M is as in the maximal subgroup columns of Table 2, then the action of G on the right cosets of M in G is not binary.*

Note that the final line of the table is the remaining group from category (***b***), hence, once this lemma is disposed of, we only deal with groups from category (***a***).

Proof. If suffices to compare $|G : M|$ with $|M|$ and apply Lemmas 2.5 and 5.1. For simplicity we highlight in Table 2 the prime p that we use to apply Lemma 2.5. □

5C. *Using Test 5.* We next apply Test 5 to the remaining maximal subgroups of G. The statement that we need is the following.

Lemma 5.3. *Let G be the Monster and let M be a maximal subgroup of G. If M is as in maximal subgroup columns of Table 3, then the action of G on the right cosets of M in G is not binary.*

Proof. Table 3 lists precisely those remaining maximal subgroups that can be excluded using Test 5, together with the prime p that has been used. □

5D. *The remainder.* By ruling out the groups listed in Tables 2 and 3, we are left with precisely five subgroups on Wilson's list [2017]. We now deal with these one at a time and, in so doing, we complete the proof of Theorem 1.3. The remaining groups are as follows: $S_3 \times \mathrm{Th}$, $3^{3+2+6+6} : (L_3(3) \times SD_{16})$, $(7 : 3 \times \mathrm{He}) : 2$, $5^{3+3}.(2 \times L_3(5))$, $5^{2+2+4} : (S_3 \times \mathrm{GL}_2(5))$.

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[1]This action turned out to be rather problematic. Each transitive action of odd degree satisfying the conditions (1), (2), (3) in Lemma 3.1 is not binary. However, for some of these actions to witness the nonbinariness we had to resort to 4-tuples, which was particularly time consuming.

Case 1: $M \cong S_3 \times \mathrm{Th}$. Here we refer to [Wilson 1988, Section 2]. There are three conjugacy classes of elements of order 3 in the Monster G, of type 3A, 3B and 3C, and the normalizers of the cyclic subgroups generated by the elements of type 3C are maximal subgroups of G conjugate to M. Choose x, an element of type 3C with $M = \mathbf{N}_G(\langle x \rangle)$. We write $M := H \times K$, where $H \cong S_3$ and $K \cong \mathrm{Th}$. From the first two lines of the proof of Proposition 2.1 of [Wilson 1988], for every $y \in K$ of order 3, xy is an element of type 3C. From the subgroup structure of the Thompson group Th, the group K contains an element y of order 3 with $\mathbf{N}_K(\langle y \rangle)$ of shape $(3 \times G_2(3)) : 2$ and maximal in K. Since x and xy are in the same G-conjugacy class, there exists $g \in G$ with $x^g = xy$. Moreover, an easy computation inside the direct product $M = H \times K$ yields that $M \cap M^g = \mathbf{N}_G(\langle x \rangle) \cap \mathbf{N}_G(\langle xy \rangle) = \mathbf{N}_M(\langle xy \rangle) \cong (\langle x \rangle \times \mathbf{C}_K(y)) : 2$ has shape $(3 \times 3 \times G_2(3)) : 2$. This shows that the action of M on the right cosets of $M \cap M^g$ is permutation isomorphic to the primitive action of Th on the right cosets of $(3 \times G_2(3)) : 2$. In other words, G has a suborbit inducing a primitive action of the sporadic Thompson group. From Proposition 4.1, this action is not binary, and hence the action of G on the right cosets of M is not binary by Lemma 2.8.

Case 2: $M \cong 3^{3+2+6+6} : (L_3(3) \times SD_{16})$. Arguing as in the previous case, we note that M contains only elements of type 13A and no elements of type 13B. Let Q be a 13-Sylow subgroup of M and let P be a 13-Sylow subgroup of G with $Q \leq P$. Observe that P is an extraspecial group of exponent 13 of order 13^3 and that Q has order 13. Replacing P by a suitable G-conjugate we may also assume that $Q \neq \mathbf{Z}(P)$. (Observe that to guarantee that we may actually assume that $Q \neq \mathbf{Z}(P)$ we need to use [Wilson 1988, page 15], which describes how the 13-elements of type A and B are partitioned in P. Indeed, not all 13-elements of type B are in $\mathbf{Z}(P)$ and hence, if accidentally $Q = \mathbf{Z}(P)$, we may replace Q with a suitable conjugate.)

Let $\alpha \in \Omega$ with $G_\alpha = M$ and set $\Lambda := \alpha^P$. From the previous paragraph, P acts faithfully on the set Λ and $|\Lambda| = 13^2$. Now the permutation group P in its action on Λ is not 2-closed; indeed the 2-closure of P in its action on Λ is of order 13^{14}, it is a Sylow 13-subgroup of $\mathrm{Sym}(\Lambda)$ (this follows from an easy computation or directly from [Dobson and Witte 2002]). Since P embeds into G^Λ, the 2-closure of G^Λ contains the 2-closure of P, but since 13^{14} does not divide the order of $|G|$, G^Λ is not 2-closed. Lemmas 2.3 and 2.4 imply that the action is not binary.

Case 3: $M \cong (7 : 3 \times \mathrm{He}) : 2$. Observe that He has a unique conjugacy class of elements of order 5 and that its Sylow 5-subgroups are elementary abelian of order 5^2. Thus, we let $V := \langle g, h \rangle$ be an elementary abelian 5-subgroup of M and we note that g, h and gh are M-conjugate and hence G-conjugate. The group G has two conjugacy classes of elements of order 5, denoted 5A and 5B. We claim that M contains only elements of type 5A. Indeed, a computation inside the Held group He reveals that $\mathbf{C}_M(g)$ contains an element of order $7 \times 3 \times 5 = 105$ and hence G

contains an element x of order 105 with $x^{21} = g$ being an element of order 5. By considering the power information on the conjugacy classes of G, we see that g belongs to the conjugacy class of type 5A. Since all 5-elements are conjugate in M, we get that M contains only 5-elements of type 5A.

We now calculate the number of fixed points of g and of V on Ω, making use of (2-1). Using the information on the conjugacy classes of He and G we deduce

$$|\mathrm{Fix}_\Omega(g)| = \frac{|G|}{|M|} \frac{|M : \mathbf{C}_M(g)|}{|G : \mathbf{C}_G(g)|} = \frac{|\mathbf{C}_G(g)|}{|\mathbf{C}_M(g)|} = \frac{1365154560000000}{12600} = 108345600000.$$

Next, since V is a Sylow 5-subgroup of M, we deduce that $|\mathbf{N}_M(V)| = 50400$ using the structure of the Held group. Moreover, from [Wilson 1988, Section 9], we get that the normalizer of an elementary abelian 5-subgroup of the Monster consisting only of elements of type 5A is maximal in G and is of the form $(5^2 : 4 \cdot 2^2 \times U_3(5)) : S_3$. In particular, $|\mathbf{N}_G(V)| = 302400000$. Thus

$$|\mathrm{Fix}_\Omega(V)| = \frac{|G|}{|M|} \frac{|M : \mathbf{N}_M(V)|}{|G : \mathbf{N}_G(V)|} = \frac{|\mathbf{N}_G(V)|}{|\mathbf{N}_M(V)|} = \frac{302400000}{50400} = 6000.$$

Now Lemma 2.6 implies that G is not binary.

Case 4: $M \cong 5^{3+3}.(2 \times L_3(5))$. Let P be a Sylow 31-subgroup of M and observe that P is also a Sylow 31-subgroup of G. Recall that G has a maximal subgroup $K := C \times D$, where $C \cong S_3$ and $D \cong \mathrm{Th}$ (as usual Th denotes the sporadic Thompson group). Now, by considering the subgroup structure of Th, we see that D contains a maximal subgroup isomorphic to $2^5.L_5(2)$ and hence D contains a Frobenius subgroup F isomorphic to $2^5 : 31$. Replacing F by a suitable conjugate we may assume that $P \leq F$.

Comparing the subgroup structure of M and of F, we deduce $M \cap F = P$. Consider $\Lambda := \alpha^F$. By construction, as $M = G_\alpha$, we get $|\Lambda| = 32$ and F acts as a 2-transitive Frobenius group of degree 32 on Λ. Since the 2-closure of a 2-transitive group of degree 32 is $\mathrm{Sym}(32)$ and since G has no sections isomorphic to $\mathrm{Sym}(32)$, we deduce from Lemma 2.4 that G^Λ is strongly nonbinary. Therefore G is not binary by Lemma 2.3.

Case 5: $M \cong 5^{2+2+4} : (S_3 \times \mathrm{GL}_2(5))$. For this last case we invoke again the help of a computer-aided computation based on Lemma 3.1, but applied in a slightly different way than what we have described in Test 5. (We thank Tim Dokchitser for hosting the computations required for dealing with this case.) Observe that $|\Omega| - 1$ is divisible by 5, but not by 5^2.

With Magma we construct all the transitive permutation representations on a set Λ of degree greater than 1 and with $|\Lambda|$ not divisible by 5^2 of M. (Considering that a Sylow 5-subgroup of M has index 576, this computation does require some time but it is feasible.) Next, with a case-by-case analysis we see that none of these

permutation representations satisfies (1), (2), (3) and (4). Therefore, every transitive permutation representation of M of degree greater than 1 satisfying (1), (2), (3) and (4) has degree divisible by 25. Now, from Lemma 3.1 applied with $d := 25$, we get that G in its action on the set M of right cosets of M in G is not binary because 25 does not divide $|\Omega| - 1$.

Acknowledgments

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A CHARACTERIZATION OF ROUND SPHERES
IN SPACE FORMS

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Let $\mathbb{Q}_c^{n+1}$ be the complete simply connected $(n+1)$-dimensional space form of curvature c. We obtain a new characterization of geodesic spheres in $\mathbb{Q}_c^{n+1}$ in terms of the higher order mean curvatures. In particular, we prove that the geodesic sphere is the only complete bounded immersed hypersurface in $\mathbb{Q}_c^{n+1}$, $c \leq 0$, with constant mean curvature and constant scalar curvature. The proof relies on the well known Omori–Yau maximum principle, a formula of Walter for the Laplacian of the r-th mean curvature of a hypersurface in a space form, and a classical inequality of Gårding for hyperbolic polynomials.

1. Introduction

A question of interest in differential geometry is whether the geodesic sphere is the only compact oriented hypersurface in the $(n + 1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$ with constant r-th mean curvature H_r, for some $r = 1, \ldots, n$. Here H_1, H_2, and H_n are the mean curvature, the scalar curvature, and the Gauss–Kronecker curvature, respectively; see the definitions in Section 2. When $r = 1$ this question is the well known Hopf conjecture, and when $r = 2$ it is a problem proposed by Yau [1982, Problem 31, p. 677].

As proved by Alexandrov [1958] for $r = 1$, and by Ros [1988; 1987] for any r (see also [Montiel and Ros 1991] and the appendix by Korevaar in [Ros 1988]), the above question has an affirmative answer for embedded hypersurfaces. In the immersed case, the question has a negative answer when $r = 1$ — see the examples of nonspherical compact hypersurfaces with constant mean curvature in the Euclidean space constructed by Wente [1986] and Hsiang, Teng and Yu [Hsiang et al. 1983] — and an affirmative answer when $r = n$ (by a theorem of Hadamard). The problem is still unsolved for $1 < r < n$. For partial answers when $r = 2$ (Yau's problem), see [Cheng 2002; Li 1996; Okayasu 2005].

Because of the difficulty of the above question, it is natural to attempt to obtain the rigidity of the sphere in $\mathbb{R}^{n+1}$ under geometric conditions stronger than requiring that H_r be constant for some r. In this regard, Gardner [1970] proved that if a compact oriented hypersurface M^n in $\mathbb{R}^{n+1}$ has two consecutive mean curvatures H_r and H_{r+1} constant, for some $r = 1, \ldots, n-1$, then it is a geodesic sphere. This result was extended to compact hypersurfaces in any space form by Bivens [1983]. For improvements on Bivens' result, see [Koh 1998; Wang 2014].

Cheng and Wan [1994] proved that a complete hypersurface M^3 with constant scalar curvature R and constant mean curvature $H \neq 0$ in $\mathbb{R}^4$ is a generalized cylinder $\mathbb{S}^k(a) \times \mathbb{R}^{3-k}$, for some $k = 1, 2, 3$ and some $a > 0$; see [Núñez 2017] for results of this nature in higher dimensions. From this result one obtains the following improvement, when $n = 3$ and $r = 1$, to the theorem of Gardner referred to above: *The geodesic spheres are the only complete bounded hypersurfaces in $\mathbb{R}^4$ with constant scalar curvature and constant mean curvature* (cf. Corollary 1.2).

Our main result (Theorem 1.1) provides a new characterization of geodesic spheres in space forms. There are many results of this nature in the literature, most of which assure that a compact hypersurface that satisfies certain geometric conditions is a geodesic sphere. What makes the characterization provided by Theorem 1.1 special is that the geometric conditions in it are imposed on a *complete* hypersurface (that is bounded when $c \leq 0$, and contained in a spherical cap when $c > 0$), and not on a *compact* one.

In the theorem below and throughout this work, $\mathbb{Q}_c^{n+1}$ stands for the $(n+1)$-dimensional complete simply connected space of constant sectional curvature c.

Theorem 1.1. *Let M^n be a complete Riemannian manifold with scalar curvature R bounded from below, and let $f : M^n \to \mathbb{Q}_c^{n+1}$ be an isometric immersion. In the case $c \leq 0$, assume that $f(M^n)$ is bounded, and in the case $c > 0$, that $f(M^n)$ lies inside a geodesic ball of radius $\rho < \pi/2\sqrt{c}$. If the mean curvature H is constant and, for some $r = 2, \ldots, n$, the r-th mean curvature H_r is constant, then $f(M^n)$ is a geodesic sphere of $\mathbb{Q}_c^{n+1}$.*

The following results follow immediately from the above theorem. Notice that the hypothesis in Theorem 1.1 that the scalar curvature of M^n is bounded from below is superfluous when $r = 2$.

Corollary 1.2. *Let $f : M^n \to \mathbb{Q}_c^{n+1}$ be an isometric immersion of a complete Riemannian manifold M^n in $\mathbb{Q}_c^{n+1}$. In the case $c \leq 0$, assume that $f(M^n)$ is bounded, and in the case $c > 0$, that $f(M^n)$ lies inside a geodesic ball of radius $\rho < \pi/2\sqrt{c}$. If the mean curvature H and the scalar curvature R are constant, then $f(M^n)$ is a geodesic sphere of $\mathbb{Q}_c^{n+1}$.*

Corollary 1.3. *Let $f : M^n \to \mathbb{Q}_c^{n+1}$ be an isometric immersion of a compact Riemannian manifold M^n in $\mathbb{Q}_c^{n+1}$. In the case $c > 0$, assume that $f(M)$ is contained*

in an open hemisphere of $\mathbb{S}_c^{n+1}$. *If the mean curvature H is constant and, for some $r = 2, \ldots, n$, the r-th mean curvature H_r is constant, then $f(M^n)$ is a geodesic sphere of* $\mathbb{Q}_c^{n+1}$.

Remark 1.4. The examples of [Wente 1986; Hsiang et al. 1983] referred to in the second paragraph of this section show that the hypothesis that H_r is constant for some r, $2 \leq r \leq n$, can not be removed from Theorem 1.1. It is surely a difficult question to know whether the theorem holds without the assumption that H is constant (cf. Yau's problem mentioned in the beginning of this section). We do not know whether Theorem 1.1 (for $r \geq 3$) holds without the hypothesis that the scalar curvature of M is bounded below.

The proof of Theorem 1.1 relies on the well known Omori–Yau maximum principle [Cheng and Yau 1975; Omori 1967; Yau 1975], a formula of Walter [1985] for the Laplacian of the r-th mean curvature of a hypersurface in a space form, and a classical inequality of Gårding [1959] for hyperbolic polynomials.

2. Preliminaries

Given an isometric immersion $f : M^n \to N^{n+k}$ of an n-dimensional Riemannian manifold M^n into an $(n + k)$-dimensional Riemannian manifold N^{n+k}, denote by $\sigma : TM \times TM \to TM^\perp$ the (vector valued) second fundamental form of f, and by A_ξ the shape operator of the immersion with respect to a (locally defined) unit normal vector field ξ. From the Gauss formula one obtains, for all smooth vector fields X and Y,

$$(2\text{-}1) \qquad \langle A_\xi X, Y \rangle = \langle \sigma(X, Y), \xi \rangle.$$

In the particular case that M and N are orientable and $k = 1$, one may choose a global unit normal vector field ξ and so define a (symmetric) 2-tensor field h on M by $h(X, Y) = \langle \sigma(X, Y), \xi \rangle$. Then, by (2-1),

$$h(X, Y) = \langle AX, Y \rangle, \quad X, Y \in \mathfrak{X}(M),$$

where $A = A_\xi$ is the shape operator of the immersion with respect to ξ. If we assume further that N^{n+1} has constant sectional curvature, it follows from the symmetry of h and the Codazzi equation that the covariant derivative ∇h of h is symmetric. From now on we denote by h_{ij} and h_{ijk} the components of h and ∇h, respectively, in a local orthonormal frame field $\{e_1, \ldots, e_n\}$, i.e.,

$$h_{ij} = h(e_i, e_j), \qquad h_{ijk} = \nabla h(e_i, e_j, e_k).$$

Given an isometric immersion $f : M^n \to N^{n+1}$, denote by $\lambda_1, \ldots, \lambda_n$ the principal curvatures of M^n with respect to a global unit normal vector field ξ (i.e., the eigenvalues of the shape operator $A = A_\xi$). It is well known that if we label the

principal curvatures at each point by the condition $\lambda_1 \leq \cdots \leq \lambda_n$, then the resulting functions $\lambda_i : M \to \mathbb{R}$, $i = 1, \ldots, n$, are continuous.

The r-th mean curvature H_r, $1 \leq r \leq n$, of M^n is defined by

$$(2\text{-}2) \qquad \binom{n}{r} H_r = \sum_{i_1 < \cdots < i_r} \lambda_{i_1} \cdots \lambda_{i_r}.$$

Notice that H_1 is the mean curvature H $(= \frac{1}{n} \operatorname{tr} A$, where $\operatorname{tr} A$ is the trace of A) and $H_n = \lambda_1 \lambda_2 \cdots \lambda_n$ is the Gauss–Kronecker curvature of the immersion. In the particular case that N^{n+1} has constant sectional curvature, the function H_2 is up to a constant the (normalized) scalar curvature R of M^n. In fact, if N^{n+1} has constant sectional curvature c and if $\{e_1, \ldots, e_n\}$ is an orthonormal basis for the tangent space at a given point of M^n such that $A e_i = \lambda_i e_i$, $i = 1, \ldots, n$, then the sectional curvature $K(e_i, e_j)$ of the plane spanned by e_i and e_j is given by

$$K(e_i, e_j) = c + \lambda_i \lambda_j$$

by the Gauss equation, and so

$$R = \frac{1}{\binom{n}{2}} \sum_{i<j} K(e_i, e_j) = \frac{1}{\binom{n}{2}} \sum_{i<j} (c + \lambda_i \lambda_j) = c + H_2.$$

The squared norm $|A|^2$ of the shape operator A is defined as the trace of A^2. It is easy to see that

$$|A|^2 = \sum_i \lambda_i^2.$$

From (2-2) and the last two equalities we obtain the following useful relation involving the mean curvature H, the norm $|A|$ of the shape operator A, and the normalized scalar curvature R:

$$(2\text{-}3) \qquad n^2 H^2 = \left(\sum_{i=1}^n \lambda_i \right)^2 = \sum_{i=1}^n \lambda_i^2 + \sum_{i \neq j} \lambda_i \lambda_j = |A|^2 + n(n-1)(R-c).$$

In terms of the r-th symmetric function $\sigma_r : \mathbb{R}^n \to \mathbb{R}$,

$$(2\text{-}4) \qquad \sigma_r(x_1, \ldots, x_n) = \sum_{i_1 < \cdots < i_r} x_{i_1} \cdots x_{i_r},$$

the equality (2-2) can be rewritten as

$$\binom{n}{r} H_r = \sigma_r \circ \vec{\lambda},$$

where $\vec{\lambda} = (\lambda_1, \ldots, \lambda_n)$ is the principal curvature vector of the immersion. In order to unify the notation, we define $H_0 = 1 = \sigma_0$ and $H_r = 0 = \sigma_r$ for all $r \geq n+1$.

As one might expect, the knowledge of the properties of the symmetric functions is very important to the study of the higher order mean curvatures of a hypersurface. In order to state a property of the symmetric functions that is relevant to us, we summarize below some of the results of the classical article [Gårding 1959] on hyperbolic polynomials; see also [Caffarelli et al. 1985, p. 268; Fontenele and Silva 2001, p. 217].

Let $P : \mathbb{R}^n \to \mathbb{R}$ be a homogenous polynomial of degree m and let $a = (a_1, \ldots, a_n)$ be a fixed vector of $\mathbb{R}^n$. We say that P is hyperbolic with respect to the vector a, or in short, that P is a-hyperbolic, if for every $x \in \mathbb{R}^n$ the polynomial in s, $P(sa + x)$, has m real roots. Denote by Γ_P the connected component of the set $\{P \neq 0\}$ that contains a. Gårding [1959] proved that Γ_P is an open convex cone, with vertex at the origin, and that the homogenous polynomial of degree $m - 1$ defined by

$$Q(x) = \frac{d}{ds}\Big|_{s=0} P(sa + x) = \sum_{j=1}^{n} a_j \frac{\partial P}{\partial x_j}(x)$$

is also a-hyperbolic. Moreover, $\Gamma_P \subset \Gamma_Q$.

As can easily be seen, the n-th symmetric function σ_n is hyperbolic with respect to the vector $a = (1, \ldots, 1)$. Applying the results of the previous paragraph to σ_n, and observing that

$$\sigma_r(x) = \frac{1}{(n-r)!} \frac{d^{n-r}}{ds^{n-r}}\Big|_{s=0} \sigma_n(sa + x), \quad r = 1, \ldots, n-1,$$

one concludes that σ_r, $1 \leq r \leq n$, is hyperbolic with respect to $a = (1, \ldots, 1)$, and that $\Gamma_1 \supset \Gamma_2 \supset \cdots \supset \Gamma_n$, where $\Gamma_r := \Gamma_{\sigma_r}$.

Gårding [1959] established an inequality for hyperbolic polynomials involving their completely polarized forms. A particular case of this inequality, from which the general case is derived, says that

$$\frac{1}{m} \sum_{k=1}^{n} y_k \frac{\partial P}{\partial x_k}(x) \geq P(y)^{1/m} P(x)^{1-1/m}, \quad \forall x, y \in \Gamma_P.$$

As observed in [Caffarelli et al. 1985, p. 269], the above inequality is equivalent to the assertion that $P^{1/m}$ is a concave function on Γ_P. In particular, we have the following result, which plays an important role in the proof of Theorem 1.1.

Proposition 2.1. *For each $r = 1, 2, \ldots, n$, the function $\sigma_r^{1/r}$ is concave on Γ_r.*

3. The Laplacian of the r-th mean curvature

The symmetric functions σ_r, $1 \leq r \leq n$, defined by (2-4), arise naturally from the identity

$$\prod_{s=1}^{n}(x_s+t) = \sum_{r=0}^{n}\sigma_r(x)t^{n-r},$$

which is valid for all $x = (x_1,\ldots,x_n) \in \mathbb{R}^n$ and $t \in \mathbb{R}$. Differentiating this identity with respect to x_j, one obtains

$$\prod_{s\neq j}(x_s+t) = \sum_{r=0}^{n}\frac{\partial \sigma_r}{\partial x_j}(x)t^{n-r}, \quad j = 1,\ldots,n.$$

Differentiating the above equality with respect to x_i for $i \neq j$ yields

$$\prod_{s\neq i,j}(x_s+t) = \sum_{r=0}^{n}\frac{\partial^2 \sigma_r}{\partial x_i \partial x_j}(x)t^{n-r}, \quad i \neq j.$$

Hence,

(3-1)
$$\frac{\partial^2 \sigma_r}{\partial x_i \partial x_j}(x) = \begin{cases} \sigma_{r-2}(\widehat{x_i},\widehat{x_j}), & i \neq j, \\ 0, & i = j, \end{cases}$$

where $\sigma_{r-2}(\widehat{x_i},\widehat{x_j}) = \sigma_{r-2}(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_{j-1},x_{j+1},\ldots,x_n)$.

Walter [1985] established a formula for the Laplacian of the r-th mean curvature of a hypersurface in a space of constant sectional curvature. For the convenience of the reader, we state that formula below. Recall that the Laplacian Δu of a C^2-function u defined on a Riemannian manifold $(M, \langle \cdot, \cdot \rangle)$ is the trace of the 2-tensor field Hess u, called the Hessian of u, defined by Hess $u(X,Y) = \langle \nabla_X \nabla u, Y \rangle$, for all $X, Y \in \mathfrak{X}(M)$.

Proposition 3.1. *Let M^n be an orientable hypersurface of an orientable Riemannian manifold N_c^{n+1} of constant sectional curvature c. Then for every $r = 1,\ldots,n$ and every $p \in M^n$,*

$$\binom{n}{r}\Delta H_r = n\sum_{j}\frac{\partial \sigma_r}{\partial x_j}(\vec{\lambda})\,\mathrm{Hess}\,H(e_j,e_j) - \sum_{i<j}\frac{\partial^2 \sigma_r}{\partial x_i \partial x_j}(\vec{\lambda})(\lambda_i-\lambda_j)^2 K_{ij}$$
$$+ \sum_{i,j,k}\frac{\partial^2 \sigma_r}{\partial x_i \partial x_j}(\vec{\lambda})(h_{iik}h_{jjk} - h_{ijk}^2),$$

where $\lambda_1,\ldots,\lambda_n$ are the principal curvatures of M^n at p, $\vec{\lambda} = (\lambda_1,\ldots,\lambda_n)$, $\{e_1,\ldots,e_n\}$ is an orthonormal basis of T_pM that diagonalizes the shape operator A, and K_{ij} is the sectional curvature of M^n in the plane spanned by $\{e_i,e_j\}$.

4. Complete and bounded hypersurfaces

In the proof of Theorem 1.1, besides Propositions 2.1 and 3.1, we use the following result.

Proposition 4.1. *Let M^n be a complete Riemannian manifold with sectional curvature K bounded from below and $f : M^n \to \mathbb{Q}_c^{n+k}$ an isometric immersion of M^n into the $(n+k)$-dimensional complete simply connected space $\mathbb{Q}_c^{n+k}$ of constant sectional curvature c. In the case $c \leq 0$, assume that $f(M^n)$ is bounded, and in the case $c > 0$, that $f(M^n)$ lies inside a geodesic ball of radius $\rho < \pi/2\sqrt{c}$. Then there exist $p \in M$ and a unit vector $\xi_0 \in (f_* T_p M)^\perp$ such that for any unit vector $v \in T_p M$,*

$$(4\text{-}1) \qquad \langle A_{\xi_0} v, v \rangle > \begin{cases} 0, & c \geq 0, \\ \sqrt{-c}, & c < 0. \end{cases}$$

We believe that the above proposition is known, but since we were unable to find a reference for it in the literature, we prove it below. The main ingredient in this proof is the following well known maximum principle due to Omori and Yau [Cheng and Yau 1975; Omori 1967; Yau 1975]; see [Fontenele and Xavier 2011, Theorem 3.4] for a conceptual refinement of this principle.

Omori–Yau maximum principle. *Let M^n be a complete Riemannian manifold with sectional curvature or Ricci curvature bounded from below, and let $f : M \to \mathbb{R}$ be a C^2-function bounded from above. Then for every $\varepsilon > 0$, there exists $x_\varepsilon \in M$ such that*

$$f(x_\varepsilon) > \sup f - \varepsilon, \quad \|\nabla f(x_\varepsilon)\| < \varepsilon, \quad \operatorname{Hess} f(x_\varepsilon)(v, v) < \varepsilon \|v\|^2 \quad \forall v \in T_{x_\varepsilon} M - \{0\}$$

or

$$f(x_\varepsilon) > \sup f - \varepsilon, \quad \|\nabla f(x_\varepsilon)\| < \varepsilon, \quad \Delta f(x_\varepsilon) < \varepsilon,$$

respectively.

The following lemma, which is also used in the proof of Proposition 4.1, expresses the gradient and Hessian of the restriction of a function to a submanifold in terms of the space gradient and Hessian; see [Dajczer 1990, p. 46] for a proof. In its statement, we use the symbol ∇ for the gradient of any function involved.

Lemma 4.2. *Let $f : M^n \to N^{n+k}$ be an isometric immersion of a Riemannian manifold M^n into a Riemannian manifold N^{n+k}, and let $g : N \to \mathbb{R}$ be a function of class C^2. Then for all $p \in M$ and $v, w \in T_p M$, one has*

$$f_*(\nabla(g \circ f)(p)) = \big[\nabla g(f(p))\big]^\top,$$

$$(4\text{-}2) \qquad \operatorname{Hess}(g \circ f)_p(v, w) = \operatorname{Hess} g_{f(p)}(f_* v, f_* w) + \big\langle \nabla g(f(p)), \sigma_p(v, w) \big\rangle,$$

where σ is the second fundamental form of the immersion, f_* is the differential of f and "$\top$" means orthogonal projection onto $f_*(T_p M)$.

Proof of Proposition 4.1. By hypothesis, $f(M)$ is contained in some closed ball $\bar{B}_\rho(q_o)$ of center q_o and radius ρ, with $\rho < \pi/2\sqrt{c}$ if $c > 0$. Let $r(\cdot) = d(\cdot, q_0)$ be the distance function from the point q_0 in $\mathbb{Q}_c^{n+k}$ and let $g = r \circ f$. Since g is

bounded from above (for $f(M) \subset \bar{B}_\rho(q_0)$) and the sectional curvatures of M are bounded from below, the Omori–Yau maximum principle assures us that, for every $\varepsilon > 0$, there exist $x_\varepsilon \in M$ such that

$$g(x_\varepsilon) > \sup g - \varepsilon, \quad \|\nabla g(x_\varepsilon)\| < \varepsilon, \quad \operatorname{Hess} g_{x_\varepsilon}(v, v) < \varepsilon \|v\|^2, \quad \forall v \in T_{x_\varepsilon} M.$$

From the last two inequalities and Lemma 4.2, we obtain

$$(4\text{-}3) \qquad\qquad \varepsilon > \|\nabla g(x_\varepsilon)\| = \|\nabla r(f(x_\varepsilon))^\top\|$$

and, for every $v \in T_{x_\varepsilon} M$,

$$(4\text{-}4) \quad \varepsilon \|v\|^2 > \operatorname{Hess} g_{x_\varepsilon}(v, v) = \operatorname{Hess} r_{f(x_\varepsilon)}(f_* v, f_* v) + \langle \sigma_{x_\varepsilon}(v, v), \nabla r(f(x_\varepsilon)) \rangle,$$

where the superscript "$\top$" indicates orthogonal projection on $f_*(T_{x_\varepsilon} M)$.

For every $v \in T_{x_\varepsilon} M$, write

$$(4\text{-}5) \qquad\qquad f_* v = v_1 + v_2,$$

where v_1 and v_2 are the components of $f_* v$ that are parallel and orthogonal, respectively, to $\nabla r(f(x_\varepsilon))$. Recalling that $\bar{\nabla}_{\nabla r} \nabla r = 0$, where $\bar{\nabla}$ is the Riemannian connection of $\mathbb{Q}_c^{n+k}$, one has

$$(4\text{-}6) \qquad \operatorname{Hess} r_{f(x_\varepsilon)}(f_* v, f_* v) = \operatorname{Hess} r_{f(x_\varepsilon)}(v_1 + v_2, v_1 + v_2)$$
$$= \operatorname{Hess} r_{f(x_\varepsilon)}(v_2, v_2).$$

Note that v_2 is tangent to the geodesic sphere S of $\mathbb{Q}_c^{n+k}$ centered at q_0 that contains $f(x_\epsilon)$. Applying (4-2) for the inclusion $\iota : S \to \mathbb{Q}_c^{n+k}$ and $g = r$, one obtains

$$(4\text{-}7) \qquad\qquad \operatorname{Hess} r_{f(x_\varepsilon)}(v_2, v_2) = \langle B v_2, v_2 \rangle,$$

where B is the shape operator of S with respect to $-\nabla r$. Since the principal curvatures of a geodesic sphere of radius t in $\mathbb{Q}_c^{n+k}$ are constant and given by

$$(4\text{-}8) \qquad \mu_c(t) = \begin{cases} \sqrt{c} \cot(\sqrt{c}\, t), & c > 0,\ 0 < t < \pi/\sqrt{c}, \\ 1/t, & c = 0,\ t > 0, \\ \sqrt{-c} \coth(\sqrt{-c}\, t), & c < 0,\ t > 0, \end{cases}$$

it follows from (4-6) and (4-7) that

$$(4\text{-}9) \qquad\qquad \operatorname{Hess} r_{f(x_\varepsilon)}(f_* v, f_* v) = \mu_c(r(f(x_\varepsilon))) \|v_2\|^2.$$

As $\|\nabla r\| \equiv 1$, by (4-5) one has $v_1 = \langle f_* v, \nabla r(f(x_\varepsilon)) \rangle \nabla r(f(x_\varepsilon))$. Then, by (4-3),

$$\|v_1\| = \left| \langle f_* v, \nabla r(f(x_\varepsilon))^\top \rangle \right| \leq \|f_* v\| \|\nabla r(f(x_\varepsilon))^\top\| < \varepsilon \|v\|.$$

From (4-5) and the above inequality, we obtain

$$(4\text{-}10) \qquad \|v_2\|^2 = \|f_* v\|^2 - \|v_1\|^2 = \|v\|^2 - \|v_1\|^2 > (1 - \varepsilon^2)\|v\|^2.$$

Hence, by (4-4), (4-9), and (4-10),

$$\varepsilon\|v\|^2 > \mu_c(r(f(x_\varepsilon)))(1-\varepsilon^2)\|v\|^2 + \langle\sigma_{x_\varepsilon}(v,v),\nabla r(f(x_\varepsilon))\rangle.$$

Since μ_c is decreasing and $r(f(x_\varepsilon)) \leq \rho$, it follows that

$$\varepsilon\|v\|^2 > \mu_c(\rho)(1-\varepsilon^2)\|v\|^2 + \langle\sigma_{x_\varepsilon}(v,v),\nabla r(f(x_\varepsilon))\rangle$$
$$= \mu_c(\rho)(1-\varepsilon^2)\|v\|^2 + \langle\sigma_{x_\varepsilon}(v,v),\nabla r(f(x_\varepsilon))^\perp\rangle,$$

where $\nabla r(f(x_\varepsilon))^\perp$ is the component of $\nabla r(f(x_\varepsilon))$ that is orthogonal to $f_*(T_{x_\varepsilon}M)$. Setting $\xi_\varepsilon = -\nabla r(f(x_\varepsilon))^\perp/\|\nabla r(f(x_\varepsilon))^\perp\|$, it follows from (2-1) and the above inequality that

$$(4\text{-}11) \qquad \langle A_{\xi_\varepsilon}v,v\rangle = \langle\sigma_{x_\varepsilon}(v,v),\xi_\varepsilon\rangle > \frac{\mu_c(\rho)(1-\varepsilon^2)-\varepsilon}{\|\nabla r(f(x_\varepsilon))^\perp\|}$$

for all $v \in T_{x_\varepsilon}M$, $\|v\| = 1$. Since, by (4-3), the term on the right-hand side of (4-11) tends to $\mu_c(\rho)$ when $\varepsilon \to 0$, and, by (4-8), $\mu_c(\rho) > 0$ for $c \geq 0$ and $\mu_c(\rho) > \sqrt{-c}$ for $c < 0$, (4-1) is fulfilled choosing $p = x_\varepsilon$ and $\xi_0 = \xi_\varepsilon$, where ε is any positive number sufficiently small. $\qquad\square$

5. Proof of Theorem 1.1

Since H is constant and R is bounded from below, from (2-3) one obtains that $|A|^2$ is bounded, and so that the sectional curvatures of M^n are bounded from below. Then, by Proposition 4.1, there exist a point $p \in M$ and a unit vector $\xi_0 \in (f_*T_pM)^\perp$ such that

$$(5\text{-}1) \qquad \langle A_{\xi_0}v,v\rangle > \alpha_c\|v\|^2, \qquad v \in T_pM,$$

where

$$\alpha_c = \begin{cases} 0, & c \geq 0, \\ \sqrt{-c}, & c < 0. \end{cases}$$

Choosing the unit normal vector field ξ such that $\xi(p) = \xi_0$, by (5-1) the principal curvatures of M at p satisfy

$$(5\text{-}2) \qquad \lambda_i(p) > \alpha_c \geq 0, \quad i = 1,\ldots,n.$$

By Proposition 3.1, as H and H_r are constant one has

$$(5\text{-}3) \qquad \sum_{i<j}\frac{\partial^2\sigma_r}{\partial x_i\partial x_j}(\vec{\lambda})(\lambda_i-\lambda_j)^2 K_{ij} = \sum_{i,j,k}\frac{\partial^2\sigma_r}{\partial x_i\partial x_j}(\vec{\lambda})(h_{iik}h_{jjk}-h_{ijk}^2),$$

where $\vec{\lambda} = (\lambda_1,\ldots,\lambda_n)$. From (5-2) one obtains that $H_r > 0$ and that $\vec{\lambda}(p)$ belongs to the Gårding's cone Γ_r (see Section 2). Then, since M is connected,

$$\vec{\lambda}(q) \in \Gamma_r, \quad \forall q \in M.$$

By Proposition 2.1, $W_r = \sigma_r^{1/r}$ is a concave function on Γ_r. Thus,

$$(5\text{-}4) \qquad \sum_{i,j} y_i y_j \frac{\partial^2 W_r}{\partial x_i \partial x_j}(x) \le 0$$

for all $x \in \Gamma_r$ and $(y_1, \ldots, y_n) \in \mathbb{R}^n$. A simple computation shows that

$$\frac{\partial^2 W_r}{\partial x_i \partial x_j} = \frac{1}{r}\sigma_r^{(1/r)-2}\left(\frac{1-r}{r}\frac{\partial \sigma_r}{\partial x_i}\frac{\partial \sigma_r}{\partial x_j} + \sigma_r \frac{\partial^2 \sigma_r}{\partial x_i \partial x_j}\right).$$

Using the above equality in (5-4), we conclude that

$$(5\text{-}5) \qquad \sigma_r(x) \sum_{i,j} y_i y_j \frac{\partial^2 \sigma_r}{\partial x_i \partial x_j}(x) \le \frac{r-1}{r} \sum_{i,j} y_i y_j \frac{\partial \sigma_r}{\partial x_i}(x)\frac{\partial \sigma_r}{\partial x_j}(x)$$

$$= \frac{r-1}{r}\left(\sum_j y_j \frac{\partial \sigma_r}{\partial x_j}(x)\right)^2,$$

for all $x \in \Gamma_r$ and $(y_1, \ldots, y_n) \in \mathbb{R}^n$. Taking $x = \vec{\lambda}$ and $y_i = h_{iik}$, $i = 1, \ldots, n$, in (5-5), one obtains

$$(5\text{-}6) \qquad \binom{n}{r} H_r \sum_{i,j} h_{iik} h_{jjk} \frac{\partial^2 \sigma_r}{\partial x_i \partial x_j}(\vec{\lambda}) \le \frac{r-1}{r}\left(\sum_j h_{jjk}\frac{\partial \sigma_r}{\partial x_j}(\vec{\lambda})\right)^2, \quad \forall k.$$

We claim that in a basis that diagonalizes A,

$$(5\text{-}7) \qquad \sum_j h_{jjk}\frac{\partial \sigma_r}{\partial x_j}(\vec{\lambda}) = \binom{n}{r} e_k(H_r).$$

The claim can be proved using the formula $\binom{n}{r} e_k(H_r) = \operatorname{tr}(P_{r-1}\nabla_{e_k} A)$ [Rosenberg 1993, p. 225], where P_{r-1} is the $(r-1)$-th Newton tensor associated with the shape operator A of M.

Since H_r is a positive constant, from (5-6) and (5-7) one obtains

$$\sum_{i,j} h_{iik} h_{jjk} \frac{\partial^2 \sigma_r}{\partial x_i \partial x_j}(\vec{\lambda}) \le 0, \quad k = 1, \ldots, n.$$

Using this information in (5-3), we conclude that the inequality

$$\sum_{i<j} \frac{\partial^2 \sigma_r}{\partial x_i \partial x_j}(\vec{\lambda})(\lambda_i - \lambda_j)^2 K_{ij} \le -\sum_{i,j,k} h_{ijk}^2 \frac{\partial^2 \sigma_r}{\partial x_i \partial x_j}(\vec{\lambda})$$

holds at every point of M. Since, by (3-1) and (5-2),

$$(5\text{-}8) \qquad \frac{\partial^2 \sigma_r}{\partial x_i \partial x_j}(\vec{\lambda}(p)) = \begin{cases} \sigma_{r-2}(\widehat{\lambda}_i(p), \widehat{\lambda}_j(p)) > 0, & i \ne j, \\ 0, & i = j, \end{cases}$$

it follows that

$$(5\text{-}9) \qquad \sum_{i<j} \frac{\partial^2 \sigma_r}{\partial x_i \partial x_j}(\vec{\lambda}(p))(\lambda_i(p) - \lambda_j(p))^2 K_{ij}(p) \leq 0.$$

Since, by (5-2) and the Gauss equation,

$$K_{ij}(p) = c + \lambda_i(p)\lambda_j(p) > c + \alpha_c^2 \geq 0, \quad i \neq j,$$

it follows from (5-8) and (5-9) that $\lambda_1(p) = \cdots = \lambda_n(p) = H$.

Let U be the set of umbilic points of M. Let B be the set

$$B = \{p \in U : \lambda_i(p) > \alpha_c \text{ for all } i = 1, \ldots, n\} \subset U.$$

By the argument above, B is nonempty and open. Assuming $B \neq U$, we can find a point $q \in \partial B \subset U$. By continuity of the principal curvatures, they are all constant and bigger than α_c at q, and hence $B = U$. Since U is closed, open, and nonempty, M is totally umbilical. To finish, it is well known that the only complete totally umbilical hypersurfaces in a space form are the geodesic spheres [Spivak 1975, pp. 75–79].

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A NON-STRICTLY PSEUDOCONVEX DOMAIN FOR WHICH THE SQUEEZING FUNCTION TENDS TO 1 TOWARDS THE BOUNDARY

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In recent work by Zimmer it was proved that if $\Omega \subset \mathbb{C}^n$ is a bounded convex domain with C^∞-smooth boundary, then Ω is strictly pseudoconvex provided that the squeezing function approaches 1 as one approaches the boundary. We show that this result fails if Ω is only assumed to be C^2-smooth.

1. Introduction

We recall the definition of the squeezing function $S_\Omega(z)$ on a bounded domain $\Omega \subset \mathbb{C}^n$. If $z \in \Omega$, and $f_z : \Omega \to \mathbb{B}^n$ is an embedding with $f_z(z) = 0$, we set

$$(1\text{-}1) \qquad S_{\Omega,f_z}(z) := \sup\{r > 0 : B_r(0) \subset f_z(\Omega)\},$$

and then

$$(1\text{-}2) \qquad S_\Omega(z) := \sup_{f_z}\{S_{\Omega,f_z}(z)\}.$$

A guiding question is the following: which complex analytic properties of Ω are encoded by the behaviour of S_Ω? For instance, if S_Ω is bounded away from 0, then Ω is necessarily pseudoconvex, and the Kobayashi–, Carathéodory–, Bergman– and the Kähler–Einstein metrics are complete, and they are pairwise quasi-isometric; see [Liu, Sun and Yau 2004; Yeung 2009]. Recently, Zimmer [2018b] proved that if

$$(1\text{-}3) \qquad \lim_{z \to b\Omega} S_\Omega(z) = 1$$

for a C^∞-smooth, bounded convex domain, then the domain Ω is necessarily strictly

pseudoconvex.[1] In this short note we will show that this does not hold for C^2-smooth domains.

Theorem 1.1. *There exists a bounded convex C^2-smooth domain $\Omega \subset \mathbb{C}^n$ which is not strongly pseudoconvex, but*

$$(1\text{-}4) \qquad \qquad \lim_{z \to b\Omega} S_\Omega(z) = 1,$$

where $S_\Omega(z)$ denotes the squeezing function on Ω.

For further results about the squeezing function the reader may also consult the references [Diederich, Fornæss and Wold 2016; Deng, Guan and Zhang 2012; 2016; Fornæss and Rong 2016; Fornæss and Wold 2015; Kim and Zhang 2016; Liu, Sun and Yau 2004; Yeung 2009; Zimmer 2018b]. In the last section we will post some open problems.

2. The construction

The construction in $\mathbb{R}^n$ and curvature estimates. We start by describing a construction of a convex domain Ω in $\mathbb{R}^n$ with a single non-strictly convex point. Afterwards we will explain how to make the construction give the conclusion of Theorem 1.1 for each $n = 2m$, when we make the identification with $\mathbb{C}^m$.

Let $x = (x_1, \ldots, x_n)$ denote the coordinates on $\mathbb{R}^n$. For any $k \in \mathbb{N}$ we let B_k denote the ball

$$(2\text{-}1) \qquad \qquad B_k := \{x \in \mathbb{R}^n : x_1^2 + \cdots + x_{n-1}^2 + (x_n - k)^2 < k^2\}.$$

On some fixed neighbourhood of the origin, each boundary bB_k may be written as a graph of a function

$$(2\text{-}2) \quad x_n = \psi_k(x') = \psi_k(x_1, \ldots, x_{n-1}) = k - \sqrt{k^2 - \|x'\|^2} = \frac{1}{2k}\|x'\|^2 + O(\|x\|^3).$$

Fix a smooth cut-off function $\chi(x') = \chi(|x'|)$ with compact support in $\{|x'| < 1\}$ which is one near the origin. We will create a new limit-graphing function $f(x')$ by subsequently gluing the functions ψ_k and ψ_{k+1} by setting

$$(2\text{-}3) \qquad \qquad g_k(x') = \psi_k(x') + \chi\left(\frac{x'}{\epsilon_k}\right)(\psi_{k+1}(x') - \psi_k(x')),$$

where the sequence ϵ_k will converge rapidly to zero, and the boundary of our domain Ω will be defined (locally) as the graph Σ of the function f defined as follows: Start by setting $f_k := \psi_k$ for some $k \in \mathbb{N}$. Then define f_{k+1} inductively by

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[1]Added in proof: Zimmer [2018a] has subsequently improved his results to convex domains with $C^{2,\alpha}$-boundary.

setting $f_{k+1} = f_k$ for $\|x'\| \geq \epsilon_k$ and then $f_{k+1} = g_k$ for $\|x'\| < \epsilon_k$. Finally we set $f = \lim_{k \to \infty} f_k$.

To show that the limit function f is C^2-smooth (if the ϵ_k's converge rapidly to 0), we need to show that the sequence $\{f_k\}$ is a Cauchy sequence with respect to the C^2-norm, i.e., we need to estimate the derivatives

$$(2\text{-}4) \qquad \sigma_{ij}^k(x') := \frac{\partial^2}{\partial x_i \partial x_j} \left(\chi \left(\frac{x'}{\epsilon_k} \right) (\psi_{k+1}(x') - \psi_k(x')) \right).$$

Note first that

$$(2\text{-}5) \qquad \psi_{k+1}(x') - \psi_k(x') = \frac{-1}{2k(k+1)} \|x'\|^2 + O(\|x'\|^3).$$

We see that

$$|\sigma_{ij}^k(x')| = \left(\frac{1}{\epsilon_k^2} O(\|x'\|^2) + \frac{1}{\epsilon_k} O(\|x'\|) \right) \frac{1}{2k(k+1)} + \frac{1}{\epsilon_k^2} O(\|x'\|^3) + \frac{1}{\epsilon_k} O(\|x'\|^2),$$

and so for $\|x'\| < \epsilon_k$ we have

$$(2\text{-}6) \qquad |\sigma_{ij}^k(x')| \leq C \cdot \frac{1}{2k(k+1)} + O(\epsilon_k),$$

where the constants are independent of any particular choice of ϵ_k. So if ϵ_k is small enough we see that $|\sigma_{ij}^k|$ is of order of magnitude $1/k^2$, which shows that $\{f_k\}$ will be a Cauchy sequence.

To ensure that Ω is convex we will need to estimate the curvature of Σ, and estimates of the curvature of the partial graphs $\Sigma_k = \{x, g_k(x)\}$ will be necessary to prove Theorem 1.1. Informally our goal is to show the following: *There exist $N, m \in \mathbb{N}, N > m$, such that if $k \geq N$ and if ϵ_k is sufficiently small (depending on k), then Σ_k curves, at every point and in all directions, more than bB_{k+m} and less than bB_{k-m}.*

We make this more precise. The surface Σ_k has a defining function $\rho_k(x) = g_k(x') - x_n$. If v_p is a tangent vector to Σ_k at $p = (x', g_k(x))$, the curvature of Σ_k in the direction of v_p is defined as

$$(2\text{-}7) \qquad \kappa_p^{\Sigma_k}(v_p) := \frac{H\rho_k(p)(v_p)}{\|\nabla \rho_k(p)\| \|v_p\|^2},$$

where $\nabla \rho_k$ is the gradient, and $H\rho_k$ is the Hessian of ρ_k (which is equal to the Hessian of g_k). The curvature (2-7) depends only on the direction of v_p, and the curvature of bB_k is $\frac{1}{k}$ at all points and in all directions. The precise statement of our goal stated above is this:

Lemma 2.1. *Let ψ_k and χ be defined as above for $k \in \mathbb{N}$. There exist $N, m \in \mathbb{N}$, $N > m$, such that if each ϵ_k is sufficiently small (depending on k), and $k \geq N$, then*

$$(2\text{-}8) \qquad \frac{1}{k+m} \leq \kappa_p^{\Sigma_k}(v_p) \leq \frac{1}{k-m},$$

for all v_p tangent to Σ_k.

It is now easy to see that if $\epsilon_k \searrow 0$ sufficiently fast, then Ω is convex, and strictly convex away from the origin. If we let Ω_k denote the domain whose boundary near the origin is given by the graph of f_k, we see that Ω_k is strictly convex, the Hessian being positive definite everywhere. Moreover, $\Omega = \cup_k \Omega_k$, and so Ω is convex.

Proof of Lemma 2.1. When we estimate the curvature we may assume that the functions g_k are simply

$$(2\text{-}9) \qquad g_k(x') = \psi_k(x') - \chi\left(\frac{x'}{\epsilon_k}\right)\left(\frac{1}{2k(k+1)}\right)|x'|^2 =: \psi_k(x') + \sigma_k(x'),$$

since the higher order terms missing in this expression of g_k can be made insignificant by choosing ϵ_k small enough. Because of the $|x'|^2$ term it is easy to see that

$$(2\text{-}10) \qquad dg_k(x') = d\psi_k(x') + \triangle_k(x')$$

and

$$(2\text{-}11) \qquad Hg_k(x') = H\psi_k(x') + h_k(x'),$$

where the coefficients in both $\triangle_k$ and h_k are of order of magnitude $1/k^2$ independently of k and of the choice of a small ϵ_k.

Fix a point x' and a vector $v \in \mathbb{R}^{n-1}$ with $\|v\| = 1$. Then a tangent vector v_p at the point $(x', g_k(x'))$ is given by

$$(2\text{-}12) \qquad v_p = (v, dg_k(x')(v)) = (v, d\psi_k(x')(v) + \triangle_k(x')(v)).$$

Estimating the curvature we see that

$$\kappa_p^{\Sigma_k}(v_p) = \frac{(H\psi_k(x') + h_k(x'))(v_p)}{\|\nabla \rho_k(p)\| \|v_p\|^2}$$

$$= \frac{(H\psi_k(x'))((v, d\psi_k(x')v) + (0', \triangle_k(x')(v)))}{\|-e_n + \nabla\psi_k(p) + \nabla\sigma_k(x')\| \|(v, d\psi_k(x')(v)) + (0', \triangle_k(x'))\|^2} + O\left(\frac{1}{k^2}\right)$$

$$= \frac{(H\psi_k(x'))((v, d\psi_k(x')v))}{\|-e_n + \nabla\psi_k(x')\| \left(1 + O\left(\frac{1}{k^2}\right)\right) \|(v, d\psi_k(x')(v))\|^2 \left(1 + O\left(\frac{1}{k^2}\right)\right)^2} + O\left(\frac{1}{k^2}\right)$$

$$= \frac{(H\psi_k(x'))((v, d\psi_k(x')v))}{\|-e_n + \nabla\psi_k(x')\| \|(v, d\psi_k(x')(v))\|^2} + O\left(\frac{1}{k^2}\right) = \frac{1}{k} + O\left(\frac{1}{k^2}\right),$$

where the term $\frac{1}{k}$ comes from the fact that the expression above is the formula for the curvature of a ball of radius k. From this it is straightforward to deduce the existence of an m such that the lemma holds.   □

The squeezing function on Ω. We will now explain why the squeezing function goes to 1 uniformly as we approach $b\Omega$ provided that the ϵ_k's decrease sufficiently fast. Let N, m be as in Lemma 2.1, and start by setting $f_k = \psi_k$ for some $k > N$.

Fix some small $\delta_k > 0$. By Lemma 2.1, if ϵ_k is small enough, we can for each $p = (x', x_n) \in b\Omega_k$, with $\|x'\| < \delta_k$, find a ball B of radius $k + m$ containing Ω_k such that $p \in bB$. By the same lemma we can for each such p also find a local piece of a ball of radius $k - m$ touching p from the inside of Ω_k, and the size of the local ball is uniform. So using Lemma 3.1 we may find a $t_k > 0$ small enough such that

$$(2\text{-}13) \qquad\qquad S_{\Omega_k}(x', x_n) \geq 1 - \frac{m}{(k+m)}$$

if $x_n \leq t_k$.

Next, again by Lemma 2.1, we find a $\delta_{k+1} < \delta_k$ such that if ϵ_{k+1} is small enough, then for each $p = (x', x_n) \in b\Omega_{k+1}$ with $\|x'\| < \delta_{k+1}$, we may oscillate with balls of radius $k + 1 - m$ and $k + 1 + m$ respectively. So there is a $t_{k+1} < t_k$ such that

$$(2\text{-}14) \qquad\qquad S_{\Omega_{k+1}}(x', x_n) \geq 1 - \frac{m}{(k+1+m)}$$

if $x_n \leq t_{k+1}$. Furthermore, by further decreasing ϵ_{k+1}, we can keep the estimate (2-13) with Ω_k replaced by Ω_{k+1}. The reason is the following. First of all, by [Fornæss and Wold 2015], there exists a constant C_k such that

$$(2\text{-}15) \qquad\qquad S_{\Omega_k}(z) \geq 1 - C_k \cdot \mathrm{dist}(z, b\Omega_k),$$

and near any compact $K \subset b\Omega_k$ away from 0, this estimate is not going to be disturbed by a small perturbation of $b\Omega_k$ near the point 0; the estimate is obtained by using oscillating balls at points of K whose boundaries will stay bounded away from 0. Furthermore, on any compact subset of Ω_k we have that $S_{\Omega_{k+1}} \to S_{\Omega_k}$ as $\epsilon_{k+1} \to 0$.

Continuing in this fashion, we obtain a decreasing sequence $0 < t_j < t_{j+1}$, $j = k, k + 1, \ldots$, and an increasing sequence of domains Ω_j, such that for each j we have

$$(2\text{-}16) \qquad\qquad S_{\Omega_j}(x', x_n) \geq 1 - \frac{m}{(k+i+m)}$$

for $t_{k+i} \leq x_n \leq t_{k+i-1}$, for $i \leq j$. The result now follows from Lemma 3.2.

3. Lemmata

Let $0 < s < 1/2$, $0 < d < r < 1$, and set $B_s = B(s, 1 - s)$, the ball of radius $1 - s$ centred at $(s, 0')$. Furthermore, we set

$$(3\text{-}1) \qquad B_{s,d} = B_s \cap \{(z_1, z') \in \mathbb{B}^n : \mathcal{Re}(z_1) > d\}.$$

Lemma 3.1. *If $B_{s,d} \subset \Omega \subset \mathbb{B}^n$, and if $r > 1 - \frac{sd}{4}$, then $S_\Omega(r, 0) > 1 - s$.*

Proof. Set $\mu = 1 - s$ and $\eta = \frac{d}{2}$, and then

$$(3\text{-}2) \qquad B_\eta^\mu = \left\{(z_1, z') \in \mathbb{C}^n : |z_1 - (1 - \eta)|^2 + \frac{\eta}{\mu}|z'|^2 < \eta^2\right\}.$$

Then certainly $\mathcal{Re}(z_1) > d$ on B_η^μ, and we also have that $B_\eta^\mu \subset B_s$. To see the latter, we translate the two balls sending $(1, 0')$ to the origin, where they are defined by

$$(3\text{-}3) \qquad \widetilde{B}_s = \{(z_1, z') : 2\mu\,\mathcal{Re}(z_1) + |z|^2 < 0\},$$

and

$$(3\text{-}4) \qquad \widetilde{B}_\eta^\mu = \left\{(z_1, z') : 2\eta\,\mathcal{Re}(z_1) + |z_1|^2 + \frac{\eta}{\mu}|z'|^2 < 0\right\}.$$

Also,

$$2\eta\mathcal{Re}(z_1) + |z_1|^2 + \frac{\eta}{\mu}|z'|^2 < 0 \Rightarrow 2\eta\,\mathcal{Re}(z_1) + \frac{\eta}{\mu}|z_1|^2 + \frac{\eta}{\mu}|z'|^2 < 0$$

$$\Leftrightarrow 2\mu\mathcal{Re}(z_1) + |z|^2 < 0.$$

According to Lemma 3.5 in [Fornæss and Wold 2015] we have

$$(3\text{-}5) \qquad S_\Omega(r, 0) \geq \sqrt{\mu}\,\sqrt{1 - 2(1 - r)\frac{1}{\eta}} = \sqrt{(1 - s)\left(1 - \frac{4(1 - r)}{d}\right)},$$

from which the lemma follows easily. $\qquad\square$

Lemma 3.2. *Let $\Omega_j \subset \Omega_{j+1}$ for $j \in \mathbb{N}$, set $\Omega = \cup_j \Omega_j$, and assume that Ω is bounded. Let $z \in \Omega$, and assume that $S_{\Omega_j}(z) > 1 - \delta$ for all j large enough so that $z \in \Omega_j$. Then $S_\Omega(z) \geq 1 - \delta$.*

Proof. Let $f_j : \Omega_j \to \mathbb{B}^n$ be an embedding such that $f_j(z) = 0$ and $B_{1-\delta}(0) \subset f_j(\Omega_j)$. By passing to a subsequence we may assume that $f_j \to f : \Omega \to \mathbb{B}^n$ uniformly on compact sets, with $f(z) = 0$. Setting $g_j = f_j^{-1} : B_{1-\delta}(0) \to \Omega$ we may also assume that $g_j \to g$ uniformly on compact sets. Then $f|_{g(B_{1-\delta}(0))} = g^{-1}$, from which the result follows. $\qquad\square$

4. Some open problems

Problem 4.1. Does Zimmer's result hold for pseudoconvex domains of class C^∞?

Problem 4.2. How much smoothness is needed for Zimmer's result hold for pseudoconvex domains?

Problem 4.3. Let $\Omega \subset \mathbb{C}^2$ be a bounded pseudoconvex domain of class C^∞. Is $S_\Omega(z)$ bounded away from zero?

In light of the result of [Deng, Guan and Zhang 2016], the answer to the last question is affirmative for bounded strictly pseudoconvex domains of class $\mathbb{C}^2$ in all dimensions. For strictly convex domains in $\mathbb{C}^n$, this was proved in [Yeung 2009]. Furthermore, it has been shown in [Kim and Zhang 2016] that the same holds for bounded convex domains without any further regularity assumptions, and by [Nikolov and Andreev 2017], it even holds for bounded $\mathbb{C}$-convex domains in general. On the other hand, by [Fornæss and Rong 2016], the answer is negative in general for $n \geq 3$.

Quantifying the asymptotic behaviour of the squeezing function, we showed in [Fornæss and Wold 2015] that

(i) $S_\Omega(z) \geq 1 - C \operatorname{dist}(z, b\Omega)$, and

(ii) $S_\Omega(z) \geq 1 - C \sqrt{\operatorname{dist}(z, b\Omega)}$,

for strongly pseudoconvex domains of class C^4 and C^3 respectively. In [Diederich, Fornæss and Wold 2016] we showed that if the squeezing function approaches 1 essentially faster than in (i), then Ω is biholomorphic to the unit ball.

Problem 4.4. What is the optimal estimate for the squeezing function for strictly pseudoconvex domains of class C^k with $k < 4$?

Let $\phi : \mathbb{B}^2 \to \mathbb{C}^2$ be defined as

$$\phi(z_1, z_2) := (z_1, -z_2 \log(z_1 - 1)).$$

Then $\Omega := \phi(\mathbb{B}^2)$ is of class C^1, and $(1, 0)$ is a non-strictly pseudoconvex boundary point of Ω. So S_Ω being identically equal to 1 does not even imply strict pseudoconvexity in the case of C^1-smooth boundaries.

Problem 4.5. Let $\phi : \mathbb{B}^n \to \Omega$ be a biholomorphism, and assume that Ω is a bounded C^2-smooth domain. Is Ω strictly pseudoconvex?

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AN AMIR–CAMBERN THEOREM FOR QUASI-ISOMETRIES OF $C_0(K, X)$ SPACES

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Let X be a finite-dimensional Banach space. We prove that if K and S are locally compact Hausdorff spaces and there exists a bijective map $T : C_0(K, X) \to C_0(S, X)$ such that

$$\frac{1}{M} \|f - g\| - L \leq \|T(f) - T(g)\| \leq M \|f - g\| + L,$$

for every $f, g \in C_0(K, X)$ then K and S are homeomorphic, whenever $L \geq 0$ and $1 \leq M^2 < S(X)$, where $S(X)$ denotes the Schäffer constant of X.

This nonlinear vector-valued extension of the Amir–Cambern theorem via quasi-isometries T with large M was previously unknown even for the classical ℓ_p^n spaces, $1 < p < \infty$, $p \neq 2$ and $n \geq 2$.

1. Introduction

If K is a locally compact Hausdorff space and X is a Banach space, we denote by $C_0(K, X)$ the Banach space of continuous functions vanishing at infinity on K, taking values in X, and provided with the usual supremum norm. If X is the scalar field ($\mathbb{R}$ or $\mathbb{C}$) we will denote this space by $C_0(K)$. In the case where K is a compact Hausdorff space we write $C(K, X)$ instead of $C_0(K, X)$.

The well-known Banach–Stone theorem states that if K and S are locally compact Hausdorff spaces, then the existence of a linear isometry T from $C_0(K)$ onto $C_0(S)$ implies that K and S are homeomorphic [Banach 1932; Behrends 1979; Stone 1937]. Amir [1965] and Cambern [1967] independently generalized this theorem by proving that if $C_0(K)$ and $C_0(S)$ are isomorphic under a linear isomorphism T satisfying $\|T\| \|T^{-1}\| < 2$, then K and S must also be homeomorphic. The constant 2 is the best possible for the formulation of this result [Cambern 1970; Cohen 1975].

Various authors, beginning with Jerison [1950], have considered the problem of determining geometric properties of X which allow generalizations of these theorems to the $C_0(K, X)$ spaces; see for instance [Cidral et al. 2015].

In the present paper we strengthen the Amir–Cambern theorem by showing that the conclusion holds if the requirement that T be a linear isomorphism with $\|T\|\|T^{-1}\| < 2$ is replaced by the requirement that T be a *bijective coarse (M, L)-quasi-isometry* on $C_0(K, X)$ spaces for finite-dimensional spaces X with $L \geq 0$ and M satisfying $1 \leq M^2 < S(X)$, where $S(X)$ is the following parameter introduced by Schäffer [Gao and Lau 1990; Schäffer 1976] for Banach spaces X:

$$S(X) = \inf\big\{\max\{\|x + y\|, \|x - y\|\} : \|x\| = 1 \text{ and } \|y\| = 1\big\}.$$

Recall that a Banach space X is called uniformly nonsquare [James 1964, Definition 1.1] if there exists $0 < \delta < 1$ such that for any $x, y \in X$, with $\|x\| = 1$ and $\|y\| = 1$, we have

$$\min\{\|x + y\|, \|x - y\|\} \leq 2(1 - \delta).$$

Then, X is uniformly nonsquare if and only if $S(X) > 1$ [Kato et al. 2001, Proposition 1]. Moreover, $S(\mathbb{R}) = 2$, $S(\mathbb{C}) = \sqrt{2}$, and $1 \leq S(X) \leq \sqrt{2}$ for every Banach space with dimension greater than or equal to 2 [Gao and Lau 1990, Theorem 2.5]. If X is a Hilbert space with dimension at least 2 then $S(X) = \sqrt{2}$, but this equality does not characterize the Hilbert spaces X; see for instance [Komuro et al. 2016, p.1].

A bijective map $T : C_0(K, X) \to C_0(S, X)$ is said to be a coarse (M, L)-quasi-isometry or simply an (M, L)-quasi-isometry if for some constants $M \geq 1$ and $L \geq 0$ the inequalities

$$\frac{1}{M}\|f - g\| - L \leq \|T(f) - T(g)\| \leq M\|f - g\| + L$$

are satisfied for all $f, g \in C_0(K, X)$. This notion includes some important concepts used in the nonlinear classification of Banach spaces [Benyamini and Lindenstrauss 2000; Godefroy et al. 2014; Górak 2011; Kalton 2008].

Thus, the main aim of this work is to prove the following nonlinear vector-valued extension of the Amir–Cambern theorem via quasi-isometries.

Theorem 1.1. *Let X be a finite-dimensional Banach space with $S(X) > 1$. Suppose that K and S are locally compact Hausdorff spaces and there exists a bijective (M, L)-quasi-isometry T from functions $C_0(K, X)$ onto $C_0(S, X)$ satisfying*

$$M^2 < S(X),$$

then K and S are homeomorphic.

The starting point of our research toward proving Theorem 1.1 was the fact that, for the particular case where X is a *finite-dimensional strictly convex space* [Clarkson 1936] and $M < 1 + \epsilon_0$ for some $\epsilon_0 > 0$, the theorem was proved by Jarosz [1989, Theorem 4]. However, even in the case where $X = \mathbb{R}$, the arguments presented in the

proof of [Jarosz 1989, Theorem 1] require ϵ_0 to be very small, namely $\epsilon_0 < 10^{-30}$. In addition, if X has dimension at least 2, ϵ_0 depends on the modulus of convexity of X and nothing is established about it beyond its existence.

This result of Jarosz naturally leads us to the following problem.

Problem 1.2. When can the Amir–Cambern theorem be extended for $C_0(K, X)$ spaces to (M, L)-quasi-isometries with $M > 1$?

Theorem 1.1 states that every finite-dimensional uniformly nonsquare space and, in particular, every finite-dimensional strictly convex space provides a positive solution to the above problem for a range M depending on a geometrical property of X. Notice also that in the special case where $X = \ell_p^n$ (the real n-dimensional l_p space, $1 < p < \infty$ and $n \geq 2$), the following immediate corollary of Theorem 1.1 was only known when $p = 2$ [Galego and Da Silva 2018, Main Theorem].

Corollary 1.3. *Let* $1 < p < \infty$ *and* $n \geq 2$. *Suppose that* K *and* S *are locally compact Hausdorff spaces and* T *is a bijective* (M, L)-*quasi-isometry from* $C_0(K, \ell_p^n)$ *onto* $C_0(S, \ell_p^n)$ *satisfying*
$$M^2 < \min\{2^{1/p}, 2^{1-1/p}\},$$
then K *and* S *are homeomorphic.*

Proof. It suffices to recall that by [Gao and Lau 1990, Theorem 3.1], for every $1 < p < \infty$ and $n \geq 2$, we know that
$$S(\ell_p^n) = \min\{2^{1/p}, 2^{1-1/p}\}. \qquad \square$$

The case $X = \mathbb{R}$ of Theorem 1.1 was proved in [Galego and Porto da Silva 2016, Main Theorem]. On the other hand, Theorem 1.1 does not apply to $X = \ell_\infty^n$, $n \geq 2$, the real n-dimensional l_∞ space, because in this case $S(X) = 1$ and moreover by a well-known result of Sundaresan [1973, p.22] there are nonhomeomorphic compact Hausdorff spaces K and S such that $C(K, X)$ is isometric with $C(S, X)$.

Notice that, in view of Problem 1.2 and in connection with Theorem 1.1, the following question arises naturally.

Problem 1.4. Suppose that X is a Banach space such that there exists $c > 1$ satisfying the following property: for any locally compact Hausdorff spaces K and S and bijective (M, L)-quasi-isometry T from $C_0(K, X)$ onto $C_0(S, X)$ with
$$M^2 < c,$$
it follows that K and S are homeomorphic.
Then:

(1) Is it true that $S(X) > 1$?

(2) Is $c \leq S(X)$?

(3) Does it follow that X is a finite-dimensional space?

2. An inequality involving the Schäffer constant

We begin the proof of Theorem 1.1 by establishing an inequality related to the Schäffer constant that will be very useful later. The constant

$$(2\text{-}1) \qquad J(X) = \sup\{\min\{\|x + y\|, \|x - y\|\} : \|x\| = 1 \text{ and } \|y\| = 1\},$$

is called the nonsquare or James constant of X.

If X is a real Banach space of finite dimension at least 2, then according to [Casini 1986, Proposition 2.1] or [Gao and Lau 1990, Theorem 2.5]

$$(2\text{-}2) \qquad\qquad\qquad J(X)S(X) = 2.$$

This fact also holds if X is a complex Banach space, for if $X_{\mathbb{R}}$ is its natural real Banach space structure, we have that $J(X) = J(X_{\mathbb{R}})$ and $S(X) = S(X_{\mathbb{R}})$. So, from now on, we shall not distinguish the scalar field of X.

Lemma 2.1. *Let X be a Banach space and $x, y \in X$. Then*

$$\min\{\|x + y\|, \|x - y\|\} \le \frac{2}{S(X)} \max\{\|x\|, \|y\|\}.$$

Proof. First assume that $X = \mathbb{R}$. Then $S(X) = 2$ and for each pair $x, y \in \mathbb{R}$,

$$\min\{|x + y|, |x - y|\} = \big||x| - |y|\big| \le \max\{|x|, |y|\},$$

so we are done.

Suppose now that the dimension of X is at least 2 and fix $x, y \in X$. Since $S(X) \le 2$, the lemma follows trivially in the case that $x = 0$ or $y = 0$. Thus, we assume that $x, y \ne 0$ and put $\hat{x} = x/\|x\|$, $\hat{y} = y/\|y\|$. We may also assume that

$$(2\text{-}3) \qquad \|x\| = \max\{\|x\|, \|y\|\} \quad \text{and} \quad \|\hat{x} + \hat{y}\| = \min\{\|\hat{x} + \hat{y}\|, \|\hat{x} - \hat{y}\|\}.$$

Next, by (2-1) and (2-2) we infer that

$$\frac{2}{S(X)} = J(X) = \sup\{\min\{\|u + v\|, \|u - v\|\} : \|u\| = 1 \text{ and } \|v\| = 1\}.$$

Then, by (2-3) it follows that

$$\|\hat{x} + \hat{y}\| \le \frac{2}{S(X)},$$

and putting $\alpha = \|y\|/\|x\| \in (0, 1]$, we note that

$$\frac{\|x + y\|}{\|x\|} = \|\hat{x} + \alpha\hat{y}\| \le (1 - \alpha)\|\hat{x}\| + \alpha\|\hat{x} + \hat{y}\| \le \frac{2}{S(X)},$$

and we are also done. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

3. Special sets associated to (M, L)-quasi-isometries

In a recent study of (M, L)-quasi-isometries between the spaces $C_0(K)$ and $C_0(S)$ [Galego and Porto da Silva 2016], subsets $\Gamma_w(k, v)$ and $\Gamma_v(s, w)$ of S and K, respectively, where $k \in K$, $s \in S$ and v and w are suitable elements of $\mathbb{R}$, were introduced. In this section, we introduce the definitions and a property (Proposition 3.1) of an analogous class of these sets for v, w $\in X$ instead of $\mathbb{R}$.

From now on, as in the hypotheses of Theorem 1.1, we fix a finite-dimensional Banach space X with $S(X) > 1$ and a bijective coarse (M, L)-quasi-isometry

$$T : C_0(K, X) \to C_0(S, X)$$

satisfying $M^2 < S(X)$ and $L \geq 0$. One can easily see that for any $\alpha > 0$, the map $f \mapsto \alpha T(f/\alpha)$ is a bijective coarse $(M, \alpha L)$-quasi-isometry, so it is possible to change the value of L as we wish. Then, we may suppose that $L = 1$. Moreover, notice that the map $f \mapsto T(f) - T(0)$ is a bijective coarse quasi-isometry with the same constants (M, L) of T, with the additional property that it maps 0 to 0. For that reason we may suppose that $T(0) = 0$. Finally, notice that the map T^{-1} does not necessarily have the same constants (M, L) of T; in fact we can only guarantee that it is a bijective coarse (M, ML)-quasi-isometry. Thus, we may actually suppose that $L = 1/M$, and this will ensure that both the maps T and T^{-1} are bijective coarse $(M, 1)$-quasi-isometries.

Let H be a locally compact Hausdorff space, $k \in H$, $f \in C_0(H, X)$ and v $\in X$. Following [Galego and Porto da Silva 2016, Definition 2.2] we set

$$\omega(k, f, v) = \max\{\|f\|, \|f(k) - v\|\}.$$

Let v, w $\in X$ satisfy $\|v\| \geq M$ and $\|w\| = \|v\|/M - 1$. Following [Galego and Porto da Silva 2016, Definition 3.1], we set

$$\Gamma_w(k, v) = \Big\{s \in S : \|Tf(s) - w\| \leq M\omega(k, f, v) + 1, \ \text{for all } f \in C_0(K, X)\Big\}.$$

Similarly, for $s \in S$, w, z $\in X$ with $\|w\| \geq M$ and $\|z\| = \|w\|/M - 1$, we also set

$$\Lambda_z(s, w) = \Big\{k \in K : \|T^{-1}g(k) - z\| \leq M\omega(s, g, w) + 1, \ \text{for all } g \in C_0(S, X)\Big\}.$$

Since it is required in the definition of the sets $\Gamma_w(k, v)$ and $\Lambda_z(s, w)$ that $\|v\| \geq M$ and $\|w\| = \|v\|/M - 1$ and, respectively, $\|w\| \geq M$ and $\|z\| = \|w\|/M - 1$, these restrictions on the norms of the parameters will be implicit in every usage of these sets.

It is important to have in mind that, since both T and T^{-1} are bijective coarse $(M, 1)$-quasi-isometries, for any result involving the sets $\Gamma_w(k, v)$, a similar result holds for the sets $\Lambda_z(s, w)$. We will use the same result label to refer to either case.

The proof of the following proposition is essentially the same as the proof of [Galego and Porto da Silva 2016, Proposition 3.2].

Proposition 3.1. *There exists $r_0 > 0$, depending only on M, such that, for all $k \in K$ and $v \in X$ with $\|v\| \geq r_0$, there exists $w \in X$ such that $\Gamma_w(k, v) \neq \varnothing$.*

Proof. Let $\mathcal{V}_k$ denote the set of open neighborhoods of k. For each $U \in \mathcal{V}_k$ we fix $f_U \in C_0(K)$ such that the image of f_U is contained in $[0, 1]$, $f_U(k) = 1$ and $f_U|_{K \setminus U} \equiv 0$. We consider the net $(v \cdot f_U)_{U \in \mathcal{V}_k}$.

Claim. $\limsup \|f - v \cdot f_U\| \leq \omega(k, f, v)$,   for all $f \in C_0(K, X)$.

Given $\varepsilon > 0$, fix $U_\varepsilon \in \mathcal{V}_k$ such that

$$\|f(u) - f(k)\| < \varepsilon, \quad \text{for all } u \in U_\varepsilon.$$

Pick $U \in \mathcal{V}_k$ such that $U \subset U_\varepsilon$, and we shall evaluate $\|f - v \cdot f_U\|$. If $u \in U$, then

$$\|f(u) - v \cdot f_U(u)\| \leq \|f(k) - v \cdot f_U(u)\| + \varepsilon.$$

Notice that $v \cdot f_U$ has its image contained in the segment $\{\alpha v : \alpha \in [0, 1]\}$, and then

$$\|f(k) - v \cdot f_U(u)\| \leq \max\{\|f(k)\|, \|f(k) - v\|\}.$$

Therefore,

$$(3\text{-}1) \qquad \|f(u) - v \cdot f_U(u)\| \leq \max\{\|f(k)\|, \|f(k) - v\|\} + \varepsilon, \quad \text{for all } u \in U.$$

On the other hand, if $u \in K \setminus U$, then $f_U(u) = 0$, and consequently,

$$(3\text{-}2) \qquad \|f(u) - v \cdot f_U(u)\| = \|f(u)\| \leq \|f\|, \quad \text{for all } u \in K \setminus U.$$

By (3-1) and (3-2), we conclude that

$$\|f - v \cdot f_U\| \leq \max\{\|f\|, \|f(k) - v\|\} + \varepsilon,$$

and the claim is proved.

For each U, we fix $s_U \in S$ such that

$$\|T(v \cdot f_U)(s_U)\| = \|T(v \cdot f_U)\|.$$

Since $\|v \cdot f_U\| = \|v\|$ and $T(0) = 0$, we have that

$$\|v\|/M - 1 \leq \|T(v \cdot f_U)(s_U)\| \leq M\|v\| + 1.$$

Then, the net $(T(v \cdot f_U)(s_U))_{U \in \mathcal{V}_k} \subset X$ is bounded and since X is finite-dimensional, we may assume that $T(v \cdot f_U)(s_U) \to w_0$, for some $w_0 \in X$. Moreover, we have

$$(3\text{-}3) \qquad\qquad\qquad \|w_0\| \geq \|v\|/M - 1.$$

The vector w_0 will be later used to define w.

Now, let us see that $(s_U)_{U \in \mathcal{V}_k}$ admits a convergent subnet. It follows by the claim that

$$(3\text{-}4) \quad \limsup \|Tf(s_U) - T(v \cdot f_U)(s_U)\| \leq \limsup \|Tf - T(v \cdot f_U)\|$$
$$\leq M \limsup \|f - v \cdot f_U\| + 1$$
$$\leq M\omega(k, f, v) + 1, \quad \text{for all } f \in C_0(K, X).$$

Fix $f_0 \in C_0(K, X)$ such that $\|f_0\| = \|v\|/2$ and $f_0(k) = v/2$. Then $\omega(k, f_0, v) = \|v\|/2$ and, by (3-3) and (3-4), we have

$$(3\text{-}5) \quad \liminf \|Tf_0(s_U)\|$$
$$\geq \liminf \|T(v \cdot f_U)(s_U)\| - \limsup \|Tf_0(s_U) - T(v \cdot f_U)(s_U)\|$$
$$\geq \|w_0\| - (M\omega(k, f_0, v) + 1)$$
$$\geq \|v\|/M - 1 - (M\|v\|/2 + 1)$$
$$= \|v\|(1/M - M/2) - 2.$$

Since $M^2 < S(X) \leq 2$, we have that $1/M - M/2 > 0$, and then, there exists r_0 depending only on M such that for $\|v\| > r_0$, we have

$$\liminf \|Tf_0(s_U)\| > 0.$$

Since Tf_0 vanishes at infinity, it follows that $(s_U)_{U \in \mathcal{V}_k}$ admits a convergent subnet, so we may assume that $s_U \to s$. By (3-4), we derive that

$$(3\text{-}6) \qquad \|Tf(s) - w_0\| \leq M\omega(k, f, v) + 1, \quad \text{for all } f \in C_0(K, X).$$

Define $w = \alpha_0 w_0$, with $\alpha_0 = (\|v\|/M - 1)/\|w_0\|$. We have that $\|w\| = \|v\|/M - 1$ and, by (3-3), $\alpha_0 \leq 1$.

We will conclude the proof by showing that (3-6) is also satisfied for w instead of w_0. Given $f \in C_0(K, X)$, notice that

$$\|Tf(s)\| \leq \|Tf\| \leq M\|f\| + 1 \leq M\omega(k, f, v) + 1,$$

then, by (3-6),

$$\|Tf(s) - w\| \leq \alpha_0\|Tf(s) - w_0\| + (1 - \alpha_0)\|Tf(s)\| \leq M\omega(k, f, v) + 1. \quad \square$$

From now on, we consider r_0 given by Proposition 3.1 to be fixed. Since r_0 depends only on M, this same constant works for the sets $\Lambda_v(s, w)$.

4. The special sets $\Gamma_w(k, v)$ when $M^2 < S(X)$

In this section we state a fundamental proposition concerning the special sets $\Gamma_w(k, v)$ associated to the $(M, 1)$-quasi-isometry T that we are considering.

Proposition 4.1. *There exists $r_1 > r_0$, depending only on M and $S(X)$, such that, for all $k \in K$, $v \in X$ and $v' \in X$ with $\|v\| > r_1$ and $\|v - v'\| < 1$, if $s \in \Gamma_w(k, -v)$*

for some $w \in X$ *and* $s' \in \Gamma_{w'}(k, v')$ *for some* $w' \in X$, *then* $s = s'$.

Proof. Suppose that $s \neq s'$. Then, fix $g \in C_0(S, X)$ such that

(4-1) $g(s) = -w, \quad g(s') = -w' \quad \text{and} \quad \|g\| = \max\{\|w\|, \|w'\|\}.$

By applying the definitions of the sets $\Gamma_w(k, -v)$ and $\Gamma_{w'}(k, v')$, respectively, to $T^{-1}g$, we get the inequalities

(4-2) $2\|w\| = \|T(T^{-1}g)(s) - w\| \leq M\omega(k, T^{-1}g, -v) + 1,$

and

(4-3) $2\|w'\| = \|T(T^{-1}g)(s') - w'\| \leq M\omega(k, T^{-1}g, v') + 1.$

Since $\|w\| = \|v\|/M - 1$, by (4-2) we obtain

(4-4) $\dfrac{2\|v\|}{M} \leq M\omega(k, T^{-1}g, -v) + 3,$

and since $\|w'\| = \|v'\|/M - 1$ and $\|v - v'\| < 1$, according to (4-3) we have

(4-5) $\dfrac{2\|v\|}{M} \leq M\omega(k, T^{-1}g, v') + 3 + \dfrac{2}{M},$

and again by $\|v - v'\| < 1$, we see that

$$\omega(k, T^{-1}g, v') = \max\{\|T^{-1}g\|, \|T^{-1}g(k) - v'\|\}$$
$$\leq \max\{\|T^{-1}g\|, \|T^{-1}g(k) - v\|\} + 1$$
$$= \omega(k, T^{-1}g, v) + 1.$$

Therefore, according to (4-5) we deduce that

(4-6) $\dfrac{2\|v\|}{M} \leq M\omega(k, T^{-1}g, v) + 3 + M + \dfrac{2}{M}.$

Thus, putting $\Delta = 3 + M + 2/M$, it follows from (4-4) and (4-6) that

$$\dfrac{2\|v\|}{M} \leq M \min\{\omega(k, T^{-1}g, -v), \omega(k, T^{-1}g, v)\} + \Delta.$$

That is, $2\|v\|/M$ is less than or equal to

$M \min\{\max\{\|T^{-1}g\|, \|T^{-1}g(k) + v\|\}, \max\{\|T^{-1}g\|, \|T^{-1}g(k) - v\|\}\} + \Delta.$

Then, by using the identity of real numbers a, b and c,

$$\min\{\max\{a, b\}, \max\{a, c\}\} = \max\{a, \min\{b, c\}\},$$

with

$$a = \|T^{-1}g\|, \quad b = \|T^{-1}g(k) + v\| \quad \text{and} \quad c = \|T^{-1}g(k) - v\|,$$

we have that

$$\frac{2\|v\|}{M} \leq M \max\left\{\|T^{-1}g\|, \min\{\|T^{-1}g(k) + v\|, \|T^{-1}g(k) - v\|\}\right\} + \Delta.$$

Moreover, by applying Lemma 2.1 with $x = T^{-1}g(k)$ and $y = v$ we conclude that

$$(4\text{-}7) \qquad \frac{2\|v\|}{M} \leq M \max\left\{\|T^{-1}g\|, \frac{2}{S(X)} \max\{\|T^{-1}g(k)\|, \|v\|\}\right\} + \Delta.$$

On the other hand, putting $\Delta' = 2 - M$ and having in mind (4-1) we also infer that

$$(4\text{-}8) \qquad \|T^{-1}g\| \leq M\|g\| + 1 = M \max\{\|w\|, \|w'\|\} + 1 \leq \|v\| + \Delta'.$$

Therefore by (4-7) and (4-8) we conclude that

$$\frac{2\|v\|}{M} \leq \frac{2M}{S(X)}(\|v\| + \Delta') + \Delta,$$

that is,

$$2\|v\|\left(\frac{1}{M} - \frac{M}{S(X)}\right) \leq \frac{2M\Delta'}{S(X)} + \Delta.$$

Since $M^2 < S(X)$, we have that $1/M - M/S(X) > 0$. Thus, there exists $r_1 \geq r_0$, depending only on M and $S(X)$, such that we have a contradiction for $v \in X$ with $\|v\| > r_1$. $\qquad\square$

We consider r_1 given by Proposition 4.1 to be fixed. Since it depends only on M and $S(X)$, this same constant works for the sets $\Lambda_v(s, w)$. The following consequence of the previous proposition will allow us to define, in the next section, a function $\varphi : K \to S$ which as we shall see in Section 7 will be a homeomorphism between K and S.

Corollary 4.2. *For all $k \in K$, $s, s' \in S$, and $v, v' \in X$, with $\|v\| > r_1$, $\|v'\| > r_1$ and $\|v - v'\| < 1$, if $s \in \Gamma_w(k, v)$ and $s' \in \Gamma_{w'}(k, v')$ for some $w, w' \in X$, then $s = s'$.*

Proof. Since $\|-v\| > r_1 \geq r_0$, by Proposition 3.1 there exists $w'' \in X$ such that $\Gamma_{w''}(k, -v) \neq \varnothing$. Take $s'' \in \Gamma_{w''}(k, -v)$.

Observe that since $s'' \in \Gamma_{w''}(k, -v)$ and $s \in \Gamma_w(k, v)$, it follows by Proposition 4.1 that $s'' = s$. Moreover, since $s'' \in \Gamma_{w''}(k, -v)$ and $s' \in \Gamma_{w'}(k, v')$, again by Proposition 4.1 we infer that $s'' = s'$. Hence $s = s'$. $\qquad\square$

5. The functions $\varphi : K \to S$ and $\psi : S \to K$

In this section, we will begin to construct a homeomorphism between K and S via the following proposition.

Proposition 5.1. *For all $k \in K$ there exists $s \in S$ such that for all $v \in X$ with $\|v\| > r_1$ and $w \in X$, either $\Gamma_w(k, v) = \{s\}$ or $\Gamma_w(k, v) = \varnothing$.*

Proof. Take $k \in K$ and put $A = \{v \in X : \|v\| > r_1\}$. Hence, it suffices to prove that for any v, v′ $\in A$, if $s \in \Gamma_w(k, v)$ and $s' \in \Gamma_{w'}(k, v')$ for some w, w′ $\in X$, then $s = s'$.

Suppose thus that $s \in \Gamma_w(k, v)$ and $s' \in \Gamma_{w'}(k, v')$ for some w, w′ $\in X$. We will distinguish two cases.

Case 1. X is of dimension at least 2. Therefore A is path-connected. So we may find points $u_0, \ldots, u_n$ in A such that $u_0 = v'$, $u_n = v$ and $\|u_j - u_{j-1}\| < 1$ for all $1 \le j \le n$. Put $s_0 = s'$ and $s_n = s$. Moreover, according to Proposition 3.1, for each $1 \le j \le n - 1$, there exists $s_j \in S$ and $w_j \in X$ such that $s_j \in \Gamma_{w_j}(k, u_j)$.

For each $1 \le j \le n$, since $\|u_j - u_{j-1}\| < 1$, Corollary 4.2 implies that $s_j = s_{j-1}$. By using this fact repeatedly, we conclude that $s' = s_1 = \cdots = s_{n-1} = s$.

Case 2. $X = \mathbb{R}$. In this case, fix w″ such that $\Gamma_{w''}(k, -v) \ne \varnothing$. Then, using Proposition 4.1 we have

$$\Gamma_{w''}(k, -v) = \Gamma_w(k, v) = \{s\}.$$

Since $A = (-\infty, -r_1) \cup (r_1, +\infty)$, there is a path in A connecting v′ to either v or $-v$. Then, proceeding as in Case 1 we conclude that $s' = s$. □

Thus, we are able to define the function $\varphi : K \to S$ where $\varphi(k)$ is the element s given by Proposition 5.1. By symmetry, we may also define a function $\psi : S \to K$ such that $\psi(s)$ is the element k given by the symmetric version of Proposition 5.1.

To show that in fact φ and ψ are continuous and $\psi^{-1} = \varphi$ we will still need to prove another property of the sets $\Gamma_w(k, v)$.

6. Another decisive property of the sets $\Gamma_w(k, v)$ when $M^2 < S(X)$

The next proposition will help us prove that functions φ and ψ defined in the previous section are homeomorphisms provided that we change the number r_1 in the statement of Proposition 5.1 by another convenient number greater than it. See Proposition 7.1.

Proposition 6.1. *There exists $r_2 > r_1$, depending only on M and $S(X)$, such that, for all $k \in K$ and v $\in X$ with $\|v\| > r_2$, if $s \in \Gamma_w(k, v)$ for some w $\in X$ and $\Lambda_z(s, w) \ne \varnothing$ for some z $\in X$, then $\Lambda_z(s, w) = \{k\}$.*

Proof. Pick $k' \in \Lambda_z(s, w)$ and we must show that $k' = k$. Suppose the contrary and fix $f \in C_0(K, X)$ such that

$$(6\text{-}1) \qquad f(k) = \frac{v}{2}, \quad f(k') = -\frac{\|v\|}{2\|z\|}z \quad \text{and} \quad \|f\| = \frac{\|v\|}{2}.$$

Thus,

$$\omega(k, f, v) = \frac{\|v\|}{2}.$$

Applying the definition of $\Gamma_w(k, v)$ to f, we see that

$$\|Tf(s) - w\| \leq M\omega(k, f, v) + 1 = \frac{M}{2}\|v\| + 1.$$

Moreover, since

$$\|Tf\| \leq M\|f\| + 1 = \frac{M}{2}\|v\| + 1,$$

it follows that

$$\omega(s, Tf, w) \leq \frac{M}{2}\|v\| + 1.$$

So, by applying the definition of $\Lambda_z(s, w)$ to Tf, we have

$$(6\text{-}2) \quad \|f(k') - z\| = \|T^{-1}(Tf)(k') - z\| \leq M\omega(s, Tf, w) + 1 \leq \frac{M^2}{2}\|v\| + M + 1.$$

On the other hand, since $\|w\| = \|v\|/M - 1$ and $\|z\| = \|w\|/M - 1$ we obtain

$$\|z\| = \left(\frac{\|v\|}{M} - 1\right)\frac{1}{M} - 1 = \frac{\|v\|}{M^2} - \frac{1}{M} - 1.$$

Furthermore, according to (6-1), $f(k')$ and z have opposite directions. Then

$$(6\text{-}3) \qquad \|f(k') - z\| = \|f(k')\| + \|z\| = \frac{\|v\|}{2} + \frac{\|v\|}{M^2} - \frac{1}{M} - 1.$$

Therefore, putting $\Delta'' = M + 2 + 1/M$, by (6-2) and (6-3) we conclude that

$$(6\text{-}4) \qquad \left(\frac{1}{2} + \frac{1}{M^2} - \frac{M^2}{2}\right)\|v\| \leq \Delta''.$$

Since $M^2 < S(X) \leq 2$, it can be easily seen that

$$\frac{1}{2} + \frac{1}{M^2} - \frac{M^2}{2} > 0.$$

So, there exists $r_2 \geq r_1$ depending only on M and $S(X)$ such that the inequality (6-4) fails to be true for $v \in X$ with $\|v\| > r_2$, completing the proof of the proposition. $\square$

As we did to r_0 and r_1, we may fix r_2 given by the Proposition 6.1, and it is clear that this constant also works for the for the sets $\Lambda_v(s, w)$.

7. The topological spaces K and S are homeomorphic

Observe that the statements of Proposition 3.1, Corollary 4.2, Proposition 5.1 and Proposition 6.1 remain true if we change r_0 and r_1 to r_2. Consider thus φ and ψ defined as at the end of Section 5. To complete the proof of Theorem 1.1, we prove the following proposition.

Proposition 7.1. *The functions $\varphi : K \to S$ and $\psi : S \to K$ are continuous and $\psi = \varphi^{-1}$.*

Proof. First we will show that $\psi = \varphi^{-1}$. Fix $k \in K$. By the definition of $\varphi(k)$ there are v, w $\in X$ with $\|v\| > (r_2 + 1)M$ such that

$$\varphi(k) \in \Gamma_w(k, v).$$

Thus, $\|w\| > r_2$ and by Proposition 3.1 there exists $z \in X$ satisfying $\Lambda_z(\varphi(k), w) \neq \varnothing$. Then, according to Proposition 6.1 we know that

$$\Lambda_z(\varphi(k), w) = \{k\}.$$

Therefore, it follows by the definition of ψ that $\psi(\varphi(k)) = k$. Hence $\psi \circ \varphi = \mathrm{Id}_K$. Analogously we deduce that $\varphi \circ \psi = \mathrm{Id}_S$.

We will now prove that φ is continuous. The proof that ψ is continuous is analogous. Observe that it suffices to prove that for each net $(k_j)_{j \in J}$ of K converging to $k \in K$, the net $(\varphi(k_j))_{j \in J}$ admits a subnet converging to $\varphi(k)$.

Assume then that $(k_j)_{j \in J}$ is a net of K converging to k. For all $j \in J$ take v_j and w_j such that $\|v_j\| = c$, for some $c > r_2$, and

$$(7\text{-}1) \qquad\qquad \varphi(k_j) \in \Gamma_{w_j}(k_j, v_j).$$

Since the nets $(v_j)_{j \in J}$ and $(w_j)_{j \in J}$ are contained in compact sets, we may assume that there are v, w $\in X$ such that $v_j \to v$ and $w_j \to w$.

For each $f \in C_0(K, X)$ we have

$$(7\text{-}2) \qquad\qquad \omega(k_j, f, v_j) \to \omega(k, f, v),$$

and according to (7-1),

$$(7\text{-}3) \qquad \|Tf(\varphi(k_j)) - w_j\| \leq M\omega(k_j, f, v_j) + 1, \quad \text{for all } j \in J.$$

Fix $f_1 \in C_0(K, X)$ satisfying $\|f_1\| = \|v\|/2$ and $f_1(x) = v/2$. Then (7-3) implies

$$\|Tf_1(\varphi(k_j))\| \geq \|w_j\| - \|Tf_1(\varphi(k_j)) - w_j\|$$
$$\geq \frac{c}{M} - M\omega(k_j, f_1, v_j) - 2,$$

for every $j \in J$. Notice that $\omega(k, f_1, v) = \|v\|/2 = c/2$, so by (7-2) we have

$$\liminf_{j \in J} \|Tf_1(\varphi(k_j))\| \geq \left(\frac{1}{M} - \frac{M}{2}\right)c - 2,$$

and since $c > r_2 \geq r_0$ and recalling (3-5), we obtain

$$\liminf_{j \in J} \|Tf_1(\varphi(k_j))\| > 0.$$

Since Tf_1 vanishes at infinity, this implies that $(\varphi(k_j))_{j \in J}$ admits a subnet converging to some $s \in S$, so we assume that $\varphi(k_j) \to s$. Hence, by (7-2) and (7-3),

$$\|Tf(s) - w\| \leq M\omega(k, f, v) + 1, \quad \text{for all } f \in C_0(K, X),$$

which means that $s \in \Gamma_w(k, v) = \{\varphi(k)\}$. Consequently $s = \varphi(k)$. $\square$

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WEAK AMENABILITY OF LIE GROUPS MADE DISCRETE

SØREN KNUDBY

We completely characterize connected Lie groups all of whose countable subgroups are weakly amenable. We also provide a characterization of connected semisimple Lie groups that are weakly amenable. Finally, we show that a connected Lie group is weakly amenable if the group is weakly amenable as a discrete group.

1. Statement of the results

Weak amenability for locally compact groups was introduced in [Cowling and Haagerup 1989]. The property has proven useful as a tool in operator algebras going back to Haagerup's result on the free groups [1978], results on lattices on simple Lie groups and their group von Neumann algebras [Cowling and Haagerup 1989; Haagerup 2016], and more recently in several results on Cartan rigidity in the theory of von Neumann algebras (see, e.g., [Popa and Vaes 2014a; 2014b]). Due to its many applications in operator algebras, the study of weak amenability, especially for discrete groups, is important.

A locally compact group G is weakly amenable if the constant function 1 on G can be approximated uniformly on compact subsets by compactly supported Herz–Schur multipliers, uniformly bounded in norm (see Section 2 for details). The optimal uniform norm bound is the *Cowling–Haagerup constant* (or the *weak amenability constant*), denoted here $\Lambda(G)$.

Weak amenability has been quite well studied, especially in the setting of connected Lie groups. The combined results of [Cowling 1983; Cowling and Haagerup 1989; de Cannière and Haagerup 1985; Dorofaeff 1993; 1996; Haagerup 2016; Hansen 1990] provides a characterization of weak amenability for simple Lie groups, which we record in Theorem 1.1. For partial results in the nonsimple case, we refer to [Cowling et al. 2005; Knudby 2016].

Theorem 1.1. *A connected simple Lie group G is weakly amenable if and only if the real rank of G is zero or one. In that case, the weak amenability constant is*

$$(1.2) \qquad \Lambda(G) = \begin{cases} 1 & \text{when } G \text{ has real rank zero,} \\ 1 & \text{when } G \approx \text{SO}(1, n), \ n \geq 2, \\ 1 & \text{when } G \approx \text{SU}(1, n), \ n \geq 2, \\ 2n - 1 & \text{when } G \approx \text{Sp}(1, n), \ n \geq 2, \\ 21 & \text{when } G \approx \text{F}_{4(-20)}. \end{cases}$$

Above, $G \approx H$ means that G is locally isomorphic to H.

In Section 3, we observe how the classification of simple Lie groups that are weakly amenable can be extended to include all semisimple Lie groups. Since it is not known in general if weak amenability of connected Lie groups is preserved under local isomorphism, it is not entirely obvious how to deduce the semisimple case from the simple case. We prove the following:

Theorem 3.1. *Let G be a connected semisimple Lie group. Then G is locally isomorphic to a direct product $S_1 \times \cdots \times S_n$ of connected simple Lie groups $S_1, \ldots, S_n$, and G is weakly amenable if and only if each S_i is weakly amenable. In fact,*

$$\Lambda(G) = \prod_{i=1}^{n} \Lambda(S_i).$$

Combining Theorems 3.1 and 1.1 one can then compute the value $\Lambda(G)$ for any connected semisimple Lie group G. Our proof of Theorem 3.1 relies on an inequality proved by Cowling for discrete groups; see (2.6). In order to apply Cowling's inequality, we pass to lattices by using Haagerup's result (2.5) that this does not change the Cowling–Haagerup constant. The same trick is also used in our proof of Theorem 5.4.

Theorem 3.1 was previously known under the additional assumption that the semisimple Lie group had finite center or finite fundamental group. Indeed, under this additional assumption, Theorem 3.1 then follows from Theorem 1.1 and an application of the well-known permanence properties (2.1) and (2.3) below. The assumption of finite center or finite fundamental group can be considered as extreme cases, and Theorem 3.1 then settles the intermediate cases in which the center and the fundamental group are both infinite.

In a similar spirit, the characterization of weak amenability for connected Lie groups in general has been done in the cases where the Levi factor has finite center [Cowling et al. 2005] or the Lie group has trivial fundamental group [Knudby 2016]; the case of a finite fundamental group then follows from (2.1). Some intermediate cases remain open.

For a locally compact group G, we let G_d denote the same group G equipped with the discrete topology. There has previously been some interest in studying the relationship between properties of G and G_d; see, e.g., [Bédos 1994; Bekka and Valette 1993; de Cornulier 2006; Knudby and Li 2015]. For instance, it is known that if G_d is an amenable group, then G is an amenable group. The analogous question about weak amenability is open:

Question 1 [Knudby and Li 2015]. If G_d is weakly amenable, is G weakly amenable?

It is a fact that a discrete group is weakly amenable if and only if all of its countable subgroups are weakly amenable (see Lemma 2.7). It thus makes no difference if one studies weak amenability of G_d or of all countable subgroups of G. Note that countable subgroups of G are always viewed with the discrete topology which might differ from the subspace topology coming from G.

Our main result is the following characterization of connected Lie groups all of whose countable subgroups are weakly amenable.

Theorem 4.11. *Let G be a connected Lie group, and let G_d denote the group G equipped with the discrete topology. The following are equivalent.*

(1) *G is locally isomorphic to $R \times \mathrm{SO}(3)^a \times \mathrm{SL}(2, \mathbb{R})^b \times \mathrm{SL}(2, \mathbb{C})^c$, for a solvable connected Lie group R and integers a, b, c.*

(2) *G_d is weakly amenable with constant 1.*

(3) *G_d is weakly amenable.*

(4) *Every countable subgroup of G is weakly amenable with constant 1.*

(5) *Every countable subgroup of G is weakly amenable.*

In [Knudby and Li 2015], Theorem 4.11 was proved in the special case where G is a simple Lie group. In order to remove the assumption of simplicity, one needs to deal with certain semidirect products, some of which were dealt with in [Knudby 2016]. In Section 4 we obtain nonweak amenability results for the remaining semidirect products (see Proposition 4.9) and thus obtain Theorem 4.11.

Our proof of Theorem 4.11 relies in part on the methods of de Cornulier [2006], where he proved that (1) in Theorem 4.11 is equivalent to:

(6) *G_d has the Haagerup property.*

It was conjectured by Cowling (see [Cherix et al. 2001, p. 7]) that a locally compact group G satisfies $\Lambda(G) = 1$ if and only if G has the Haagerup property. Although this is now known to be false in this generality (see [Ozawa and Popa 2010, Remark 2.13; de Cornulier et al. 2008, Corollary 2]), Theorem 4.11 together with de Cornulier's result [2006, Theorem 1.14] establishes Cowling's conjecture for connected Lie groups made discrete.

As another application of Theorem 4.11, we are able to settle Question 1 in the case of connected Lie groups. In the last section, we establish the following.

Corollary 5.5. *Let G be a connected Lie group. If G_{d} is weakly amenable, then G is weakly amenable. In this case, $\Lambda(G_{\mathrm{d}}) = \Lambda(G) = 1$.*

We remark that our proof of Corollary 5.5 relies on the classification obtained in Theorem 4.11. It would be preferable to have a direct proof avoiding the classification.

2. Preliminaries

2A. *Weak amenability.* Let G be a locally compact group. A *Herz–Schur multiplier* is a complex function φ on G of the form $\varphi(y^{-1}x) = \langle P(x), Q(y) \rangle$, where $P, Q : G \to \mathcal{H}$ are bounded continuous functions from G to a Hilbert space $\mathcal{H}$ and $x, y \in G$. Note that φ is continuous and bounded by $\|P\|_{\infty}\|Q\|_{\infty}$. The Herz–Schur norm of φ is defined as

$$\|\varphi\|_{B_2} = \inf\{\|P\|_{\infty}\|Q\|_{\infty}\},$$

where the infimum is taken over all P, Q as above. With this norm and pointwise operations, the Herz–Schur multipliers form a unital Banach algebra.

The group G is *weakly amenable* if there is a net (φ_i) of compactly supported Herz–Schur multipliers converging to 1 uniformly on compact subsets of G and satisfying $\sup_i \|\varphi_i\|_{B_2} \leq C$ for some $C \geq 1$. The *weak amenability constant* $\Lambda(G)$ is the infimum of those $C \geq 1$ for which such a net exists, with the understanding that $\Lambda(G) = \infty$ if G is not weakly amenable. There are several equivalent definitions of weak amenability in the literature; see, e.g., [Cowling and Haagerup 1989, Proposition 1.1]. Weak amenability of groups should however not be confused with weak amenability of Banach algebras.

Weak amenability is preserved under several group constructions. We list here the known results needed later on and refer to [Brown and Ozawa 2008, Section 12.3; Cowling 1989, Section III; Cowling and Haagerup 1989, Section 1; Haagerup 2016; Jolissaint 2015] for proofs. When K is a compact normal subgroup of G,

$$(2.1) \qquad \Lambda(G) = \Lambda(G/K).$$

If $(G_i)_{i \in I}$ is a directed collection of open subgroups in G then

$$(2.2) \qquad \Lambda\left(\bigcup_{i \in I} G_i\right) = \sup_{i \in I} \Lambda(G_i).$$

For two locally compact groups G and H,

$$(2.3) \qquad \Lambda(G \times H) = \Lambda(G)\Lambda(H).$$

If G has a closed normal subgroup N such that the quotient G/N is amenable then

$$(2.4) \qquad \Lambda(N) = \Lambda(G).$$

We remark that (2.4) is stated in [Jolissaint 2015] only for second countable groups, but it is not difficult to deduce the general statement from this and the Kakutani–Kodaira theorem [Hewitt and Ross 1979, Theorem 8.7] using (2.1) and (2.2).

Recall that a lattice Γ in a locally compact group G is a discrete subgroup such that the homogeneous space G/Γ admits a G-invariant probability measure, where G acts on G/Γ by left translation. If Γ is a lattice in a second countable, locally compact group G, then

$$(2.5) \qquad \Lambda(\Gamma) = \Lambda(G).$$

When Z is a central subgroup of a discrete group G then

$$(2.6) \qquad \Lambda(G) \leq \Lambda(G/Z).$$

A remark on (2.6) is in order. Much work related to weak amenability for connected Lie groups would be significantly easier if (2.6) holds true for nondiscrete groups G as well. For instance, [Hansen 1990] would then have been an immediate consequence of earlier work such as [Cowling 1983; Cowling and Haagerup 1989], and our Theorem 3.1 would also be an immediate consequence of earlier work. It would even be relatively easy to complete the characterization of weak amenability for connected Lie groups. Needless to say, we have not been able to generalize (2.6) to the nondiscrete case so far. Sometimes, one can reduce the general case to the discrete case and then apply (2.6). In the present paper, this is done using lattices as is most explicitly seen in the proof of Theorem 3.1, but also in Theorem 5.4.

Lemma 2.7. *Let G be a discrete group. Then G is weakly amenable if and only if every countable subgroup of G is weakly amenable.*

Proof. Clearly, weak amenability of G implies that every subgroup of G is weakly amenable. Assume conversely that G is not weakly amenable. We claim that G contains a countable subgroup which is not weakly amenable. Since G is the directed union of all its countable subgroups, it follows from (2.2) that there is a sequence, $G_1, G_2, \ldots$, of countable subgroups of G such that $\Lambda(G_n) \geq n$. Let G_∞ be the subgroup of G generated by $G_1, G_2, \ldots$. Then G_∞ is a countable subgroup of G and G_∞ is not weakly amenable. This completes the proof. $\square$

2B. *Structure of Lie groups.* Loosely speaking, two Lie groups are locally isomorphic if they admit homeomorphic neighborhoods of the identity on which the group laws (here only partially defined) are identical. Equivalently, two Lie groups are

locally isomorphic if and only if their Lie algebras are isomorphic; see [Helgason 1978, Theorem II.1.11].

A connected Lie group G has a simply connected covering $\widetilde{G}$ which is a Lie group locally isomorphic to G in such a way that the covering map is a group homomorphism. The kernel of the covering homomorphism is a discrete central subgroup of $\widetilde{G}$. Conversely, any connected Lie group locally isomorphic to G is a quotient of $\widetilde{G}$ by a discrete central subgroup. For a discrete subgroup N of the center $Z(\widetilde{G})$ of $\widetilde{G}$, then the center of the quotient $\widetilde{G}/N$ is precisely the quotient of the center $Z(\widetilde{G})/N$. See, e.g., [Chevalley 1946, Chapter II; Knapp 2002, Section I.11] for details.

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. Then G admits a Levi decomposition $G = RS$. Here, R is the solvable closed connected Lie subgroup of G associated with the solvable radical of $\mathfrak{g}$. The group S is a semisimple connected Lie subgroup of G associated with a (semisimple) Levi subalgebra $\mathfrak{s}$ of $\mathfrak{g}$. We refer to Section 3.18 of [Varadarajan 1974] and especially Theorem 3.18.13 therein for details. The semisimple Lie algebra $\mathfrak{s}$ splits as a direct sum $\mathfrak{s} = \mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_n$ of simple Lie algebras (for some $n \geq 0$), and if S_i denotes the connected Lie subgroup of G associated with the Lie subalgebra $\mathfrak{s}_i$, then S is locally isomorphic to the direct product $S_1 \times \cdots \times S_n$.

3. Weak amenability of semisimple Lie groups

The computation below of $\Lambda(G)$ for all semisimple Lie groups G basically relies on three facts: the existence of lattices in semisimple Lie groups, the permanence results stated in Section 2, and most importantly that $\Lambda(G)$ is known for all simple Lie groups.

Theorem 3.1. *Let G be a connected semisimple Lie group. Then G is locally isomorphic to a direct product $S_1 \times \cdots \times S_n$ of connected simple Lie groups $S_1, \ldots, S_n$, and G is weakly amenable if and only if each S_i is weakly amenable. In fact,*

$$\Lambda(G) = \prod_{i=1}^{n} \Lambda(S_i).$$

Proof. Let Z denote the center of G, $\widetilde{G}$ the universal cover of G, and $\overline{G} = G/Z$. By semisimplicity, Z is discrete. The Lie algebra $\mathfrak{g}$ of G is a direct sum $\mathfrak{g} = \mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_n$ of simple Lie algebras. Let $\widetilde{S}_i$ and $\overline{S}_i$ denote the analytic subgroups of $\widetilde{G}$ and $\overline{G}$ corresponding to $\mathfrak{s}_i$, respectively. Then we have the following direct product decompositions

$$\widetilde{G} = \prod_{i=1}^{n} \widetilde{S}_i \quad \text{and} \quad \overline{G} = \prod_{i=1}^{n} \overline{S}_i.$$

Let $\overline{\Gamma}$ be a lattice in $\overline{G}$ (a lattice exists by [Raghunathan 1972, Theorem 14.1]).

Consider the covering homomorphisms

$$\widetilde{G} \to G \quad \text{and} \quad G \to \overline{G},$$

and let Γ be the lift of $\overline{\Gamma}$ to G, and let $\widetilde{\Gamma}$ be the lift of Γ to $\widetilde{G}$. Then $\Gamma \le G$ is a lattice, and $\widetilde{\Gamma} \le \widetilde{G}$ is a lattice. Using (2.5), (2.3), and (2.6) we obtain

$$\Lambda(G) = \Lambda(\Gamma) \le \Lambda(\overline{\Gamma}) = \Lambda(\overline{G}) = \prod_{i=1}^{n} \Lambda(\overline{S}_i),$$

$$\Lambda(G) = \Lambda(\Gamma) \ge \Lambda(\widetilde{\Gamma}) = \Lambda(\widetilde{G}) = \prod_{i=1}^{n} \Lambda(\widetilde{S}_i).$$

By Theorem 1.1, we have $\Lambda(\overline{S}_i) = \Lambda(\widetilde{S}_i)$ for every i, concluding the proof. $\square$

4. Weak amenability of Lie groups made discrete

When G is a Lie group we denote by G_{d} the group G equipped with the discrete topology. We recall a result needed in the proof of Theorem 4.11:

Theorem 4.1 [Knudby and Li 2015, Theorem 1.10]. *For a connected simple Lie group S, the following are equivalent.*

- *S is locally isomorphic to* $\mathrm{SO}(3)$, $\mathrm{SL}(2, \mathbb{R})$, *or* $\mathrm{SL}(2, \mathbb{C})$.
- *S_{d} is weakly amenable.*
- *S_{d} is weakly amenable with constant* 1.

In order to generalize Theorem 4.1 to nonsimple Lie groups we need to consider certain semidirect products which we now describe. A main ingredient to prove nonweak amenability of these semidirect products is this:

Theorem 4.2 [Knudby 2016, Theorem 5]. *Let $H \curvearrowright N$ be an action by automorphisms of a discrete group H on a discrete group N, and let $G = N \rtimes H$ be the corresponding semidirect product group. Let N_0 be a proper subgroup of N. Suppose*

(1) *H is not amenable;*

(2) *N is amenable;*

(3) *N_0 is H-invariant;*

(4) *for every $x \in N \setminus N_0$, the stabilizer of x in H is amenable.*

Then G is not weakly amenable.

The semidirect products of interest also appear in [de Cornulier 2006] to which we refer the reader for further details.

The irreducible real representations of $SL(2, \mathbb{R})$ and $SU(2)$ are well known. We describe them below.

For each natural number $n \geq 1$, the group $SL(2, \mathbb{R})$ has a unique irreducible real representation V_n of dimension n; see [Lang 1975, p. 107]. It may be realized as the natural action of $SL(2, \mathbb{R})$ on the homogeneous polynomials in two real variables of degree $n - 1$.

Similarly, the group $SU(2)$ acts on the homogeneous polynomials in two complex variables of degree $n - 1$. When $n = 2m$ is even, this representation is still irreducible as a real representation V_{4m} of dimension $4m$. When $n = 2m + 1$ is odd, the representation is the complexification of an irreducible real representation V_{2m+1} of dimension $2m + 1$. The representations V_{2m+1} and V_{4m} make up all the irreducible real representations of $SU(2)$. We refer to [Bröcker and tom Dieck 1985; Itzkowitz et al. 1991] for details.

Let S be $SL(2, \mathbb{R})$ or $SU(2)$, and let $\mathfrak{s}$ be the Lie algebra of S. If V is a real irreducible representation of S, then V also carries the derived representation of $\mathfrak{s}$. Let $\mathrm{Alt}_{\mathfrak{s}}(V)$ denote the real vector space of alternating bilinear forms φ on V that are $\mathfrak{s}$-invariant, that is, bilinear forms $\varphi : V \times V \to \mathbb{R}$ satisfying

$$\varphi(x, x) = 0 \quad \text{and} \quad \varphi(s.x, y) + \varphi(x, s.y) = 0 \qquad \text{for all } s \in \mathfrak{s}, \ x, y \in V.$$

The Lie group $H(V)$ is defined as $V \times \mathrm{Alt}_{\mathfrak{s}}(V)^*$ with group multiplication given by

$$(x, z)(x', z') = (x + x', z + z' + e_{x,x'}), \qquad x, x' \in V, \ z, z' \in \mathrm{Alt}_{\mathfrak{s}}(V)^*,$$

where $e_{x,x'} \in \mathrm{Alt}_{\mathfrak{s}}(V)^*$ is the evaluation functional defined by $e_{x,x'}(\varphi) = \varphi(x, x')$. The group S acts on $H(V)$ by $s.(x, z) = (s.x, z)$. When $Z \subseteq \mathrm{Alt}_{\mathfrak{s}}(V)^*$ is a subspace, we obtain a quotient group $H(V)/Z$, and the action of S on $H(V)$ descends to an action on $H(V)/Z$. In this way we obtain the semidirect product

$$H(V)/Z \rtimes S.$$

Lemma 4.3. *If G is a proper, real algebraic subgroup of $SL(2, \mathbb{R})$ or $SU(2)$, then G_{d} is amenable.*

Proof. Let S be $SL(2, \mathbb{R})$ or $SU(2)$, and let $\mathfrak{s}$ be the Lie algebra of S. The group G has only finitely many components (in the usual Hausdorff topology); see [Whitney 1957, Theorem 3; Platonov and Rapinchuk 1994, Theorem 3.6]. It is therefore enough to show that the identity component G_0 of G is amenable as a discrete group.

Since G_0 is a connected, proper, closed subgroup of S, its Lie algebra $\mathfrak{g}$ is a proper Lie subalgebra of $\mathfrak{s}$. The dimension of $\mathfrak{g}$ is therefore at most two, and $\mathfrak{g}$ must be a solvable Lie algebra. So G_0 is a solvable group. In particular, G_0 is amenable in the discrete topology. $\square$

In what follows below, we have to exclude the trivial irreducible representation of S. We thus assume from now on that dim $V \geq 2$.

Lemma 4.4. *If* dim $V \geq 2$ *and if* $(x, z) \in H(V)/Z$ *and* $x \neq 0$, *then the stabilizer of* (x, z) *in* S *is amenable in the discrete topology.*

Proof. The stabilizer of (x, z) in S coincides with the stabilizer of x in S. Since $x \neq 0$, and S acts irreducibly on V, the stabilizer of x in S is a proper subgroup. It follows from the explicit description of the action of S on V as the action on the homogeneous polynomials in two variables that the stabilizer is moreover a real algebraic subgroup. Hence, Lemma 4.3 shows that the stabilizer is amenable in the discrete topology. □

Proposition 4.5. *If* dim $V \geq 2$, *the group* $H(V)/Z \rtimes S$ *is not weakly amenable in the discrete topology.*

Proof. We intend to apply Theorem 4.2 with $H = S_{\mathrm{d}}$, $N = (H(V)/Z)_{\mathrm{d}}$ and $N_0 = (\mathrm{Alt}_{\mathfrak{s}}(V)^*/Z)_{\mathrm{d}}$. Clearly, H is not amenable, N is amenable, and N_0 is invariant under H (in fact, H acts trivially on N_0). It remains to check that every element of $N \setminus N_0$ has amenable stabilizer. This is Lemma 4.4. □

In the case $S = \mathrm{SL}(2, \mathbb{R})$, the group $H(V)/Z \rtimes S$ is not simply connected, since $\mathrm{SL}(2, \mathbb{R})$ is not simply connected. Let $\widetilde{S} = \widetilde{\mathrm{SL}}(2, \mathbb{R})$ denote the universal covering group of $\mathrm{SL}(2, \mathbb{R})$. The covering homomorphism $\widetilde{S} \to S$ has kernel isomorphic to the group of integers $\mathbb{Z}$. The group $\widetilde{S}$ acts on $H(V)/Z$ through the action of S. Since the stabilizer of $(x, z) \in H(V)/Z$ in $\widetilde{S}$ is an extension by $\mathbb{Z}$ of the stabilizer in S, and since amenability is preserved by extensions, the following is immediate from Lemma 4.4.

Lemma 4.6. *If* dim $V \geq 2$ *and if* $(x, z) \in H(V)/Z$ *and* $x \neq 0$, *then the stabilizer of* (x, z) *in* $\widetilde{S}$ *is amenable in the discrete topology.*

Applying Theorem 4.2 with $H = \widetilde{S}_{\mathrm{d}}$, $N = (H(V)/Z)_{\mathrm{d}}$ and $N_0 = (\mathrm{Alt}_{\mathfrak{s}}(V)^*/Z)_{\mathrm{d}}$, we obtain the following:

Proposition 4.7. *If* dim $V \geq 2$, *the group* $H(V)/Z \rtimes \widetilde{S}$ *is not weakly amenable in the discrete topology.*

Proposition 4.8. *If* dim $V \geq 2$ *and if* G *is a connected Lie group locally isomorphic to* $H(V)/Z \rtimes S$, *then* G_{d} *is not weakly amenable.*

Proof. Let $\widetilde{G}$ be the universal cover of G. Then $G = \widetilde{G}/D$ for some discrete central subgroup D of $\widetilde{G}$. By (2.6), it is enough to prove that $\widetilde{G}_{\mathrm{d}}$ is not weakly amenable, and hence we may (and will) assume that G is simply connected.

If $S = \mathrm{SU}(2)$, the group $H(V)/Z \rtimes S$ is simply connected, so $\widetilde{G} = H(V)/Z \rtimes S$ and we apply Proposition 4.5. If $S = \mathrm{SL}(2, \mathbb{R})$ and $\widetilde{S} = \widetilde{\mathrm{SL}}(2, \mathbb{R})$, then $H(V)/Z \rtimes \widetilde{S}$ is simply connected, so $\widetilde{G} = H(V)/Z \rtimes \widetilde{S}$ and we apply Proposition 4.7. □

For now, let $S = \mathrm{SL}(2, \mathbb{R})$ and $V = V_n$. If $n = 2m + 1$ is odd, the space $\mathrm{Alt}_{\mathfrak{s}}(V)^*$ is trivial and

$$H(V) \rtimes S = \mathbb{R}^{2m+1} \rtimes \mathrm{SL}(2, \mathbb{R}).$$

If $n = 2m$ is even, the space $\mathrm{Alt}_{\mathfrak{s}}(V)^*$ is one-dimensional and $H(V)$ is the $2m+1$-dimensional real Heisenberg group H_{2m+1}. If $Z = \mathrm{Alt}_{\mathfrak{s}}(V)^*$, then

$$H(V) \rtimes S = H_{2m+1} \rtimes \mathrm{SL}(2, \mathbb{R}), \qquad H(V)/Z \rtimes S = \mathbb{R}^{2m} \rtimes \mathrm{SL}(2, \mathbb{R}).$$

When $S = \mathrm{SU}(2)$ and $V = V_{2m+1}$, the space $\mathrm{Alt}_{\mathfrak{s}}(V)^*$ is trivial, and with the notation of de Cornulier [2006] we have

$$H(V) \rtimes S = D_{2m+1}^{\mathbb{R}} \rtimes \mathrm{SU}(2).$$

When $S = \mathrm{SU}(2)$ and $V = V_{4m}$, the space $\mathrm{Alt}_{\mathfrak{s}}(V)^*$ is three-dimensional. If $Z_i \subseteq \mathrm{Alt}_{\mathfrak{s}}(V)^*$ is a subspace of dimension $3 - i$, then with the notation of de Cornulier we have

$$H(V)/Z_i \rtimes S = HU_{4m}^i \rtimes \mathrm{SU}(2).$$

Using the perhaps more illuminating description of the groups $H(V)/Z \rtimes S$ just given, Proposition 4.8 translates as:

Proposition 4.9. *Let G be a connected Lie group locally isomorphic to one of the following groups*:

- $D_{2n+1}^{\mathbb{R}} \rtimes \mathrm{SU}(2)$ *for some $i = 0, 1, 2, 3$ and some $n \geq 1$.*
- $HU_{4n}^i \rtimes \mathrm{SU}(2)$ *for some $i = 0, 1, 2, 3$ and some $n \geq 1$.*
- $\mathbb{R}^n \rtimes \mathrm{SL}(2, \mathbb{R})$ *for some $n \geq 2$.*
- $H_{2n+1} \rtimes \mathrm{SL}(2, \mathbb{R})$ *for some $n \geq 1$.*

Then G_{d} is not weakly amenable.

Proposition 4.10. *Let G be a connected Lie group, and let $G = RS$ be a Levi decomposition* (see Section 2B), *where R is the solvable radical and S is a semisimple Levi factor. If $[R, S] \neq 1$, then G_{d} is not weakly amenable.*

Proof. This follows basically from structure theory of Lie algebras together with Proposition 4.9. Indeed, given $[R, S] \neq 1$ it follows from Propositions 3.4 and 3.8 of [de Cornulier 2006] that G contains a connected Lie subgroup H locally isomorphic to one of the groups listed in Proposition 4.9. Since H_{d} is not weakly amenable, G_{d} is not weakly amenable. $\qquad\square$

Theorem 4.11. *Let G be a connected Lie group, and let G_{d} denote the group G equipped with the discrete topology. The following are equivalent.*

(1) *G is locally isomorphic to $R \times \mathrm{SO}(3)^a \times \mathrm{SL}(2, \mathbb{R})^b \times \mathrm{SL}(2, \mathbb{C})^c$, for a solvable connected Lie group R and integers a, b, c.*

(2) G_d *is weakly amenable with constant* 1.

(3) G_d *is weakly amenable.*

(4) *Every countable subgroup of G is weakly amenable with constant* 1.

(5) *Every countable subgroup of G is weakly amenable.*

Proof. Let $G = RS$ be a Levi decomposition of G (see Section 2B).

$(1) \Rightarrow (2)$: If S is a semisimple Levi factor in G, then by assumption S is normal in G, and the group G_d/S_d is solvable, since it is a quotient of the solvable group R_d. By (2.4), it is enough to show that S_d is weakly amenable with constant 1.

Using (2.6), we may assume that the center of S is trivial. Then S is a direct product of factors $SO(3)$, $PSL(2, \mathbb{R})$, and $PSL(2, \mathbb{C})$. An application of (2.3) and Theorem 4.1 shows that S_d is weakly amenable with constant 1.

$(2) \Rightarrow (4)$: This is clear.

$(4) \Rightarrow (5)$: This is clear.

$(5) \Rightarrow (3)$: This is Lemma 2.7.

$(3) \Rightarrow (1)$: Suppose G does not satisfy (1). If $[R, S] \neq 1$, then Proposition 4.10 shows that G_d is not weakly amenable. Otherwise $[R, S] = 1$ and S contains a simple Lie subgroup not locally isomorphic to $SO(3)$, $SL(2, \mathbb{R})$, or $SL(2, \mathbb{C})$. It then follows from Theorem 4.1 that S_d is not weakly amenable, and hence G_d is not weakly amenable either. □

5. Weak amenability of Lie groups with and without topology

As a consequence of Theorem 4.11, we can answer in the affirmative (part of) Question 1.8 of [Knudby and Li 2015] — the case of connected Lie groups. Indeed, we show below that, for a connected Lie group G, if G_d is weakly amenable then G too is weakly amenable. We first establish a few lemmas.

Lemma 5.1. *Let m, n be nonnegative integers, and let $D \subseteq \mathbb{R}^m \times \mathbb{Z}^n$ be a discrete subgroup. There is a discrete subgroup $D' \subseteq \mathbb{R}^m \times \mathbb{Z}^n$ such that $D \subseteq D'$ and D' is cocompact in $\mathbb{R}^m \times \mathbb{Z}^n$.*

Proof. Our proof is an application of the characterization of compactly generated, locally compact abelian groups; see [Hewitt and Ross 1979, Theorem 9.8]. As $\mathbb{R}^m \times \mathbb{Z}^n$ is compactly generated, so is the quotient $(\mathbb{R}^m \times \mathbb{Z}^n)/D$. Therefore the quotient is of the form $\mathbb{R}^a \times \mathbb{Z}^b \times C$, where a and b are integers and C is a compact abelian group. Clearly, $\mathbb{R}^a \times \mathbb{Z}^b \times C$ has a cocompact discrete subgroup, $\mathbb{Z}^a \times \mathbb{Z}^b \times \{0\}$, and its preimage in $\mathbb{R}^m \times \mathbb{Z}^n$ is a discrete, cocompact subgroup which contains D. □

Our next lemma establishes the existence of lattices in certain Lie groups. There are well-known results of Malcev about existence of lattices in nilpotent Lie groups and of Borel about existence of lattices in semisimple Lie groups; see [Raghunathan 1972, Theorems 2.12 and 14.1]. However, we are interested in some intermediate cases such as the following example, which we have included to give the reader an intuition about the succeeding proof.

Example 5.2. Fix an irrational number θ. Let H be the universal covering group of $SL(2, \mathbb{R})$. Its center is infinite cyclic, and we let z denote a generator of the center of H. Consider the group $D = \{(-m - n\theta, z^m, z^n) \mid m, n \in \mathbb{Z}\}$ which is central in $\mathbb{R} \times H \times H$, and let G be the quotient group $G = (\mathbb{R} \times H \times H)/D$. We will describe a lattice in G.

The group $SL(2, \mathbb{R})$ admits a lattice F isomorphic to the free group on two generators. By freeness, F lifts to a subgroup $\widetilde{F}$ of H. Then $(\mathbb{Z} \times \widetilde{F} \times \widetilde{F})D$ is a lattice in $\mathbb{R} \times H \times H$, and it obviously contains D, so it factors down to a lattice in $(\mathbb{R} \times H \times H\times)/D$.

Lemma 5.3. *A connected Lie group locally isomorphic to*

$$\mathbb{R}^m \times SL(2, \mathbb{R})^n,$$

where m and n are nonnegative integers, contains a lattice.

Proof. We first introduce some notation. For any Lie group L, let $Z(L)$ denote the center of L. We use 1 to denote the neutral element (or 0 for the group $\mathbb{R}$). Let H be the universal covering group of $SL(2, \mathbb{R})$. Its center $Z(H)$ is infinite cyclic.

Set $\widetilde{G} = \mathbb{R}^m \times H^n$. Then $\widetilde{G}$ is a simply connected and connected Lie group, and any connected Lie group G locally isomorphic to $\mathbb{R}^m \times SL(2, \mathbb{R})^n$ is of the form $G = \widetilde{G}/D$ for some discrete central subgroup D of $\widetilde{G}$. Let $\pi : \widetilde{G} \to G$ be the quotient homomorphism $\pi(x) = xD$.

Suppose $D \subseteq D'$ for some other discrete central subgroup D' in $\widetilde{G}$ and that $\widetilde{G}/D'$ contains a lattice. Then the preimage under $\widetilde{G}/D \to \widetilde{G}/D'$ of any lattice in $\widetilde{G}/D'$ is a lattice in $\widetilde{G}/D$. The center of $\widetilde{G}$ is $Z(\widetilde{G}) = \mathbb{R}^m \times Z(H)^n \simeq \mathbb{R}^m \times \mathbb{Z}^n$, so by Lemma 5.1 we may without loss of generality suppose that D is discrete and cocompact in $Z(\widetilde{G})$.

The quotient $H/Z(H)$ is $PSL(2, \mathbb{R})$, and it is well known that $PSL(2, \mathbb{R})$ has a lattice F isomorphic to a free group on two generators; see, e.g., [Bekka et al. 2008, Example B.2.5(iv)]. By the universal property of free groups, there is a subgroup $\widetilde{F} \subseteq H$ such that the quotient map $H \to PSL(2, \mathbb{R})$ maps $\widetilde{F}$ bijectively onto F. The preimage of F in H is $\widetilde{F}Z(H)$, which is a lattice in H. Also, $\widetilde{F} \cap Z(H) = \{1\}$.

Consider the subgroup $\Gamma = \{0\}^m \times \widetilde{F}^n$ in $\widetilde{G}$. We will show that $\pi(\Gamma) = \Gamma D/D$ is a lattice in G.

The group ΓD is discrete in $\widetilde{G}$: Since it is a countable subgroup, it is enough to see that ΓD is closed in $\widetilde{G}$. Now, ΓD is clearly closed in $\Gamma Z(\widetilde{G})$, which is closed in $\widetilde{G}$ since $\Gamma Z(\widetilde{G}) = \mathbb{R}^m \times (\widetilde{F} Z(H))^n$.

The group $\pi(\Gamma)$ is discrete in G: As π is an open map, $\pi(W)$ is an open set in G and $\pi(\Gamma) \cap \pi(W) = \{1\}$. Indeed, if $w \in W$ and $\gamma \in \Gamma$ satisfy $\pi(w) = \pi(\gamma)$, then it follows that $w \in \Gamma D$ so $w = 1$. Thus, $\pi(\Gamma)$ is discrete in G.

The group $\pi(\Gamma)$ has finite covolume in G: Let $\psi : G \to G/Z(G)$ be the quotient homomorphism. As D is discrete and $\widetilde{G}$ is connected, $Z(G) = Z(\widetilde{G})/D$. We thus have isomorphisms

$$G/Z(G) \simeq \widetilde{G}/Z(\widetilde{G}) \simeq \mathrm{PSL}(2, \mathbb{R})^n,$$

and under these isomorphisms $\psi\pi(\Gamma) = F^n$. As F is a lattice in $\mathrm{PSL}(2, \mathbb{R})$, there is a Borel set (even a Borel fundamental domain) $\Omega \subseteq G/Z(G)$ of finite measure such that $\Omega(\psi\pi(\Gamma)) = G/Z(G)$; see [Bekka et al. 2008, Proposition B.2.4]. By outer regularity, we may assume that Ω is, in addition, open (but no longer a fundamental domain). The inverse image $\psi^{-1}(\Omega) \subseteq G$ is then also open and $\psi^{-1}(\Omega)\pi(\Gamma) = G$. As $\psi^{-1}(\Omega)$ is open, its characteristic function is lower semicontinuous, and it follows from Weil's integration formula for lower semicontinuous functions (see [Reiter and Stegeman 2000, (3.3.13)]) that the Haar measure of $\psi^{-1}(\Omega)$ is the Haar measure of Ω multiplied by the Haar measure of $Z(G)$. As D is cocompact in $Z(\widetilde{G})$, the center $Z(G) = Z(\widetilde{G})/D$ is compact. Therefore $Z(G)$ has finite Haar measure, and in conclusion $\psi^{-1}(\Omega)$ has finite Haar measure.

By [Bekka et al. 2008, Proposition B.2.4] it follows that $\pi(\Gamma)$ is a lattice in G, and this completes the proof. $\qquad\square$

Theorem 5.4. *Let G be a connected Lie group locally isomorphic to*

$$G \approx R \times \mathrm{SO}(3)^a \times \mathrm{SL}(2, \mathbb{R})^b \times \mathrm{SL}(2, \mathbb{C})^c,$$

for a solvable connected Lie group R and integers a, b, c. Then G is weakly amenable with constant 1, i.e., $\Lambda(G) = 1$.

Proof. The strategy of the proof is to reduce the problem to the case where G is locally isomorphic to the group appearing in Lemma 5.3. This is done in several steps. We first show how to get rid of the factors $\mathrm{SO}(3)$ and $\mathrm{SL}(2, \mathbb{C})$. Then we show how to reduce the radical to an abelian group.

Let $\mathfrak{g}$ be the Lie algebra of G. With $\mathfrak{r}$ the solvable radical in $\mathfrak{g}$, we have (recall $\mathfrak{so}(3) = \mathfrak{su}(2)$)

$$\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{su}(2)^a \oplus \mathfrak{sl}(2, \mathbb{R})^b \oplus \mathfrak{sl}(2, \mathbb{C})^c.$$

Set $\mathfrak{s}_0 = \mathfrak{su}(2)^a \oplus \mathfrak{sl}(2, \mathbb{C})^c \subseteq \mathfrak{g}$ and let S_0 be the connected semisimple Lie subgroup of G associated with $\mathfrak{s}_0$. Note that the center $Z(S_0)$ of S_0 is finite, since the simply connected group $\mathrm{SU}(2)^a \times \mathrm{SL}(2, \mathbb{C})^c$ has finite center.

Set $\mathfrak{h} = \mathfrak{r} \oplus \mathfrak{sl}(2, \mathbb{R})^b$ so that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{s}_0$, and let H be the connected Lie subgroup of G associated with $\mathfrak{h}$. As $[\mathfrak{h}, \mathfrak{s}_0] = 0$, the subgroups H and S_0 commute, and the multiplication map $\varphi : H \times S_0 \to G$ is a homomorphism. The image $\varphi(H \times S_0)$ is a connected Lie subgroup of G containing both H and S_0. It follows that φ is surjective and $G \simeq (H \times S_0)/\ker \varphi$.

The kernel $\ker \varphi$ is precisely

$$\ker \varphi = \{(h, h^{-1}) \mid h \in H \cap S_0\}.$$

Since H and S_0 commute, the group $H \cap S_0$ is central in S_0 and hence finite. Then $\ker \varphi$ is also a finite group. By (2.1) and (2.3), we have

$$\Lambda(G) = \Lambda(H \times S_0) = \Lambda(H)\Lambda(S_0).$$

Note that $\Lambda(S_0) = 1$ by Theorem 3.1, since by Theorem 1.1 both SU(2) and SL(2, $\mathbb{C}$) are weakly amenable with constant 1 (recall SL(2, $\mathbb{C}$) $\approx$ SO(1, 3)).

We have thus reduced the problem to the case where $G = H$ is locally isomorphic to $R \times \mathrm{SL}(2, \mathbb{R})^b$. Let $G = RS$ be a Levi decomposition of G. Then the closure $\bar{S}$ of S in G is a closed connected normal subgroup of G whose Lie algebra is a subalgebra of $\mathfrak{g}$, and the quotient $G/\bar{S}$ is solvable. By (2.4), it suffices to prove that $\bar{S}$ is weakly amenable with constant 1. We may thus suppose that S is a dense connected Lie subgroup of G.

When S is dense, a theorem of Mostow [1950, §6] shows that G is of the form $G = SC$ where C is a connected Lie subgroup of the center of G and in the closure of the center of S. It follows that the solvable radical is abelian and G is locally isomorphic to $\mathbb{R}^n \times \mathrm{SL}(2, \mathbb{R})^b$ for some integer n.

If G is simply connected, then $G = \mathbb{R}^n \times \widetilde{\mathrm{SL}}(2, \mathbb{R})^b$, where $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ denotes the universal covering group of SL(2, $\mathbb{R}$). That $\Lambda(G) = 1$ follows from [Hansen 1990], using the equality $\widetilde{\mathrm{SL}}(2, \mathbb{R}) = \widetilde{\mathrm{SU}}(1, 1)$ and the product formula (2.3); see also Theorem 1.1.

The general case can then be deduced from the simply connected case as follows. Let $\widetilde{G}$ be the universal covering group of G. By Lemma 5.3, there is a lattice Γ in G. Let $\widetilde{\Gamma}$ be the preimage of Γ in $\widetilde{G}$ under the covering homomorphism $\widetilde{G} \to G$. Then $\widetilde{\Gamma}$ is a lattice in $\widetilde{G}$, and $\widetilde{\Gamma}$ is a central extension of Γ. By (2.5) and (2.6) we have

$$\Lambda(G) = \Lambda(\Gamma) \le \Lambda(\widetilde{\Gamma}) = \Lambda(\widetilde{G}) = 1.$$

This shows that $\Lambda(G) = 1$, and the proof is complete. $\qquad\square$

Corollary 5.5. *Let G be a connected Lie group. If G_{d} is weakly amenable, then G is weakly amenable. In this case, $\Lambda(G_{\mathrm{d}}) = \Lambda(G) = 1$.*

Proof. This is immediate from Theorems 4.11 and 5.4. $\qquad\square$

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A RESTRICTION ON THE ALEXANDER POLYNOMIALS
OF *L*-SPACE KNOTS

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**Using an invariant defined by Rasmussen, we extend an argument given by
Hedden and Watson which further restricts which Alexander polynomials
can be realized by *L*-space knots.**

1. Introduction

In [Ozsváth and Szabó 2008], it is shown how the filtered chain homotopy type
of the knot Floer complex $\mathrm{CFK}^-(K)$ can be used to compute the Heegaard Floer
homology of $S^3_n(K)$, the rational homology sphere obtained by performing Dehn
surgery along $K \subset S^3$ with slope n. In [Ozsváth and Szabó 2005], this relationship
is used to investigate which knots admit lens space surgeries, using the fact that if
Y is a lens space, Y has the "smallest possible" Heegaard Floer homology:

$$(1) \qquad \qquad \mathrm{rank}\, \widehat{\mathrm{HF}}(Y) = |H_1(Y; \mathbb{Z})|.$$

More generally, a rational homology sphere which satisfies condition (1) is called
an *L-space*. So, from a Heegaard–Floer perspective, a natural extension of the
question "which knots admit lens space surgeries?" is "which knots admit *L*-space
surgeries?".

Letting $A(x)$ denote the Alexander grading of a homogeneous element x in CFK^-,
the following proposition is a straightforward consequence of [Ozsváth and Szabó
2005, Theorem 1.2]; see [Hom 2014, Remark 6.6].

Proposition 1.1. *Suppose $K \subset S^3$ is a knot on which some positive integral surgery
yields an L-space. Then $\mathrm{CFK}^-(K)$ has a basis $\{x_{-k}, \ldots, x_k\}$ with the following
properties*:

- $A(x_i) = n_i$, *where* $n_{-k} < n_{-k+1} < \cdots < n_{k-1} < n_k$.
- $n_i = -n_{-i}$.
- *If* $i \equiv k \mod 2$, *then* $\partial(x_i) = 0$.
- *If* $i \equiv k+1 \mod 2$, *then* $\partial(x_i) = x_{i-1} + U^{n_{i+1}-n_i} x_{i+1}$. □

Notice that x_k is in the kernel of ∂ and not in the image. The complex $\widehat{\mathrm{CFK}}(K)$ is the quotient complex of $\mathrm{CFK}^-(K)$ obtained by setting $U = 0$ (we refer the reader to [Ozsváth and Szabó 2004b] for details on knot Floer homology). It is a Heegaard Floer complex $\widehat{\mathrm{CF}}(S^3)$, with an additional filtration induced by K. After setting $U = 0$, we still observe that

$$(2) \qquad x_k \quad \text{generates} \quad H_*(\widehat{\mathrm{CFK}}(K), \partial) \cong \widehat{\mathrm{HF}}(S^3) \cong \mathbb{F}.$$

By convention then, $M(x_k) = 0$ (where M is the Maslov grading). Since U decreases M by 2, and ∂ decreases M by 1, this determines the Maslov grading on all homogeneous elements of $\mathrm{CFK}^-(K)$.

Ozsváth and Szabó [2004b] also showed that the graded Euler characteristic of $\widehat{\mathrm{CFK}}(K)$ is the symmetrized Alexander polynomial of K,

$$(3) \qquad \sum_i \chi(\widehat{\mathrm{CFK}}(K, i)) \cdot T^i = \Delta_K(T),$$

so a corollary to Proposition 1.1 is the following:

Corollary 1.2 [Ozsváth and Szabó 2005, Corollary 1.3]. *If $K \subset S^3$ is a knot which admits an L-space surgery, then*

$$(4) \qquad \Delta_K(T) = \sum_{i=-k}^{k} (-1)^{k+i} \, T^{n_i}$$

for some sequence of integers $n_{-k} < n_{-k+1} < \cdots < n_{k-1} < n_k$ satisfying $n_{-i} = n_i$.

Remark 1.3. Although Proposition 1.1 only applies to knots which have positive L-space surgeries, a knot K has a negative L-space surgery if and only if its mirror image $\overline{K}$ has a positive L-space surgery. Corollary 1.2 then follows in this generality because $\Delta_K(T) = \Delta_{\overline{K}}(T)$.

In particular, all of the nonzero coefficients of $\Delta_K(T)$ are ± 1. Note that

$$(5) \qquad n_k = g(K) = |\tau(K)| = g_4(K),$$

where g is the Seifert genus, τ is the Ozsváth–Szabó concordance invariant defined in [Ozsváth and Szabó 2003], and g_4 is the smooth four-genus of K. The first equality follows from the knot Floer homology detection of genus [Ozsváth and Szabó 2004a], the second follows from (2), and the third follows from the fact shown in [Ozsváth and Szabó 2003] that, for any knot K,

$$|\tau(K)| \le g_4(K) \le g(K).$$

This was the most general restriction on the Alexander polynomials of knots admitting L-space surgeries in the literature until Hedden and Watson showed the following proposition.

Proposition 1.4 [Hedden and Watson 2018, Corollary 9]. *If $K \subset S^3$ is a knot which admits an L-space surgery, then $\Delta_K(T)$ is as described in Corollary 1.2, and further, $n_k - n_{k-1} = 1$.*

Originally stated by Hedden and Watson as a corollary to a more general restriction on knot Floer complexes, this particular result was already known by Rasmussen, as conveyed to the author in a private communication. The proof hinges on an invariant defined by Rasmussen, and a particular inequality which it satisfies. Roughly, if large n-surgery is done on an unknot and on a knot K, the differences in the d-invariants (defined in (8)) of the resulting manifolds are bounded above by numbers depending on $g_4(K)$. Proposition 1.4 is then proved by showing that if a complex has a basis as in Proposition 1.1 and $n_k - n_{k-1} > 1$, Rasmussen's inequality is violated, and therefore this complex cannot be the knot Floer complex of any knot.

Our aim here is to extend this argument. We will introduce Rasmussen's invariant and inequality in Section 2. In Section 3, we will show how to compute the invariant for L-space knots from their Alexander polynomials (that is, from the sequence of the n_i). We will then see that Rasmussen's inequality places further restrictions on the n_i, analogous to the restriction $n_k - n_{k-1} = 1$. As a result, it will be shown that certain symmetric Laurent polynomials satisfying Proposition 1.4 cannot be the Alexander polynomial of any L-space knot.

Theorem 1.5. *Suppose $K \subset S^3$ is a knot which admits an L-space surgery. Then its symmetrized Alexander polynomial can be written as*

$$\Delta_K(T) = \sum_{i=-k}^{k} (-1)^{k+i} \, T^{n_i},$$

for some sequence of integers $n_{-k} < n_{-k+1} < \cdots < n_{k-1} < n_k$ satisfying the following:

- $n_i = -n_{-i}$,
- *if we let $r_i = n_{k+2-2i} - n_{k+1-2i}$, then $r_1 = 1$, and for any $j \leq k$,*

$$(6) \qquad \sum_{i=2}^{j} r_i \leq \sum_{i=k-j+2}^{k} r_i.$$

As we will explain in Section 3, the restriction is more easily stated in terms of a modified version of the Alexander polynomial,

$$\widetilde{\Delta}_K(T) := \frac{\Delta_K(T)}{1 - T^{-1}}.$$

It follows from Corollary 1.2 that when K is a knot which admits an L-space surgery,

$$\widetilde{\Delta}_K(T) = \sum_{i=0}^{\infty} T^{a_i}$$

for some sequence of integers satisfying

- $a_0 = g(K)$,

- $a_{i+1} < a_i$,

- $a_i = -i$ for $i \geq g(K)$.

We can then rephrase Theorem 1.5 as follows.

Theorem 1.6 (restatement of Theorem 1.5 in terms of $\widetilde{\Delta}$). *Suppose $K \subset S^3$ is a knot which admits an L-space surgery and $\{a_i\}$ is the decreasing sequence of integers such that*

$$\widetilde{\Delta}_K(T) = \sum_{i=0}^{\infty} T^{a_i}.$$

Then, for all $0 \leq i \leq g(K)$,

(7) $a_i \leq g(K) - 2i.$

To see the preceding theorems as generalizations of Proposition 1.4, note that in the language of Theorem 1.5, Proposition 1.4 translates to the statement $r_1 = 1$; in the language of Theorem 1.6, it translates to $a_1 \leq g(K) - 2$.

As a concrete example, there does not exist a knot in S^3 which admits an L-space surgery and has Alexander polynomial

$$-1 + (T^2 + T^{-2}) - (T^3 + T^{-3}) + (T^4 + T^{-4}).$$

Correspondingly, there does not exist a knot in S^3 with knot Floer complex as shown in Figure 1. As demonstrated after the proof of Theorem 1.5, this settles the question of which polynomials belong to knots of genus 4 or less which admit L-space surgeries.

2. The invariant $h_m(K)$

A useful feature of Heegaard Floer theory is that its groups satisfy surgery exact triangles; for example, a long exact sequence between Heegaard Floer homology groups of manifolds which are 0-, ∞- and n-framed surgery along the same knot K [Ozsváth and Szabó 2004c, Section 9]. Rasmussen [2003, Definition 7.1] defines an invariant $h_m(K)$ as the rank of a particular map in such a sequence (cf. [Frøyshov 2002], where an instanton-Floer invariant h is introduced).
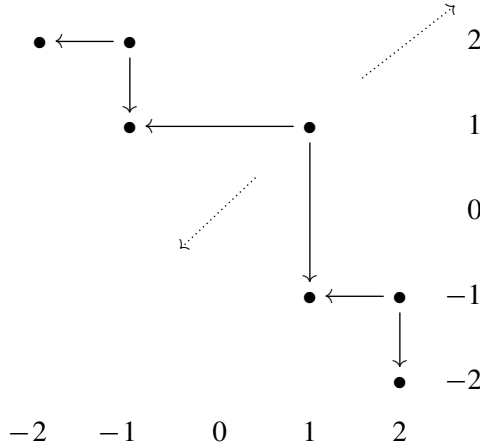
Figure 1. A bifiltered chain complex which cannot be the knot Floer complex of any knot in S^3.

Recall that if $(Y, \mathfrak{t})$ is a spin^c rational homology sphere, Ozsváth and Szabó define the *d-invariant* of $(Y, \mathfrak{t})$ as

$$(8) \qquad d(Y, \mathfrak{t}) = \min\{M(x) \mid x \in \mathrm{Im}(\pi_* : \mathrm{HF}^\infty(Y, \mathfrak{t}) \to \mathrm{HF}^+(Y, \mathfrak{t}))\}.$$

Rasmussen [2004, Section 2.2] shows that, in the case where $S^3_{-n}(K)$ is an L-space, the invariant $d(S^3_{-n}(K), \mathfrak{s}_m)$ is equal to twice $h_m(K)$, up to a shift which is independent of K. In particular, since $h_m(\text{unknot}) = 0$ for all m, we have

$$(9) \qquad h_m(K) = \tfrac{1}{2}\big(d(S^3_{-n}(K), \mathfrak{s}_m) - d(S^3_{-n}(\text{unknot}), \mathfrak{s}_m)\big).$$

The key to obtaining restrictions on L-space knots is the following inequality, analogous to an inequality in instanton Floer homology proved by Frøyshov [2004].

Proposition 2.1 [Rasmussen 2004, Theorem 2.3]. *Let K be a knot in S^3 and let $g_4(K)$ be its slice genus. Then $h_m(K) = 0$ for $|m| > g_4(K)$, while for $|m| \le g_4(K)$,*

$$(10) \qquad h_m(K) \le \left\lceil \tfrac{1}{2}(g_4(K) - |m|) \right\rceil.$$

Note that for a knot admitting an L-space surgery, due to (5), we can replace $g_4(K)$ with $g(K)$ and obtain

$$(11) \qquad h_m(K) \le \left\lceil \tfrac{1}{2}(g(K) - |m|) \right\rceil.$$

While the term L-space knot refers to a knot which admits an L-space surgery, different conventions are used regarding the restrictions on the slope of the L-space surgeries. Here it will be convenient to adopt the definition that an L-space knot in

S^3 is one with a *positive* L-space surgery. This is opposite Rasmussen's point of view [2004], but note that K admits a positive L-space surgery if and only if its mirror image $\bar{K}$ admits a negative L-space surgery. Accordingly, we follow Hedden and Watson in defining

$$(12) \qquad \bar{h}_m(K) := \tfrac{1}{2}\big(d(S_n^3(\text{unknot}), \mathfrak{s}_m) - d(S_n^3(K), \mathfrak{s}_m)\big)$$

and recall their observation that $\bar{h}_m(K) = h_m(\bar{K})$. Finally, we should note that $g(K) = g(\bar{K})$, so $\bar{h}_m$ satisfies the same inequality which h_m does for knots admitting L-space surgeries: for $|m| \leq g(K)$,

$$(13) \qquad \bar{h}_m(K) \leq \left\lceil \tfrac{1}{2}(g(K) - |m|) \right\rceil.$$

3. Values of $\bar{h}_m$ for L-space knots

Next, we recall how to compute d-invariants, and therefore $\bar{h}_m$, from CFK^-. It was shown independently by Ozsváth and Szabó [2004b] and Rasmussen [2003] that for large n-surgery (that is, for $n \geq 2g(K) - 1$), the Heegaard Floer homology groups $\text{HF}^-(S_n^3(K))$ are the homology groups of certain subcomplexes of $\text{CFK}^-(K)$, up to a shift in Maslov grading which is independent of K. In particular, if we let A_m denote the subcomplex consisting of elements with Alexander grading less than or equal to m, then

$$\text{HF}^-(S_n^3(K), \mathfrak{s}_m) \cong H_*(A_m),$$

up to a shift in grading.[1] It follows that

$$d(S_n^3(K), \mathfrak{s}_m) = \max\{M(x) \mid x \text{ a nontorsion generator of } H_*(A_m)\} + c,$$

where c is a constant which depends on n, but not on K. Therefore, the "shifted" d-invariant

$$(14) \qquad \tilde{d}(K, m) := \max\{M(x) \mid x \text{ a nontorsion generator of } H_*(A_m)\}$$

is well-defined, and satisfies

$$(15) \qquad d(S_n^3(\text{unknot}), \mathfrak{s}_m) - d(S_n^3(K), \mathfrak{s}_m) = \tilde{d}(\text{unknot}, m) - \tilde{d}(K, m)$$

for any sufficiently large n. For the unknot, we have the complex

$$\text{CFK}^-(\text{unknot}) \cong \mathbb{F}[U],$$

where the generator has Maslov grading and Alexander grading equal to zero. Since multiplication by U lowers the Alexander grading by 1 and the Maslov grading by 2,

$$\tilde{d}(\text{unknot}, m) = m - |m|.$$

---

[1]Here we are adopting the convention that CF^- and CFK^- contain the element 1 in $\mathbb{F}[U]$.

Therefore, we can rewrite inequality (13) using (15) and the above: if $K \subset S^3$ is an L-space knot, then for $|m| \leq g(K)$,

(16)
$$\bar{h}_m(K) = \tfrac{1}{2}(\tilde{d}(\text{unknot}, m) - \tilde{d}(K, m)) \leq \left\lceil \tfrac{1}{2}(g(K) - |m|) \right\rceil,$$
$$-\tfrac{1}{2}\tilde{d}(K, m) \leq \left\lceil \tfrac{1}{2}(g(K) - m) \right\rceil.$$

With inequality (16) in hand, it remains to see how the values of $\tilde{d}$ are determined by the Alexander polynomial of an L-space knot.

Recall that the Alexander polynomial is the graded Euler characteristic of $\widehat{\text{CFK}}(K)$:

$$\sum_i \chi(\widehat{\text{CFK}}(K, i)) \cdot T^i = \Delta_K(T).$$

Further, CFK^- is generated by the same set as $\widehat{\text{CFK}}$, over $\mathbb{F}[U]$ rather than $\mathbb{F}$. Since U lowers the Alexander grading by 1 and preserves the parity of the Maslov grading,

(17)
$$\sum_i \chi(\text{CFK}^-(K, i)) \cdot T^i = \sum_i \chi(\widehat{\text{CFK}}(K, i)) \cdot T^i \cdot (1 + T^{-1} + T^{-2} + \cdots)$$
$$= \frac{\Delta_K(T)}{1 - T^{-1}} =: \widetilde{\Delta}_K(T).$$

In other words, $\widetilde{\Delta}_K(T)$ is the graded Euler characteristic of $\text{CFK}^-(K)$.

Remark 3.1. If $K \subset S^3$ is a knot for which $\Delta_K(T)$ is of the type described in Corollary 1.2, then

$$\widetilde{\Delta}_K(T) = \sum_{i=0}^{\infty} T^{a_i},$$

where

- $a_0 = g(K)$,
- $a_{i+1} < a_i$, and
- $a_i = -i$ for all $i \geq g(K)$.

In [Krcatovich 2015], a reduced complex $\underline{\text{CFK}}^-$ was defined, and it was shown that for an L-space knot,

(18)
$$\underline{\text{CFK}}^-(K) \cong \mathbb{F}[U],$$

supported in Maslov grading zero [Krcatovich 2015, Corollary 4.2]. Roughly speaking, the complex CFK^- has a filtration induced by U, and a filtration induced the knot; ignoring the knot filtration, one recovers $\text{CF}^-(S^3)$, whereas ignoring the U-filtration, one gets a "reduced" knot Floer complex. We refer the reader to [Krcatovich 2015] for a precise statement, and here simply remark that the structure of CFK^- for an L-space knot, as described in Proposition 1.1, is what makes its reduced complex have a simple form.

Since the reduced complex is filtered chain homotopy equivalent to $CFK^-(K)$ (with respect to the knot filtration), they have the same Euler characteristic. In particular, (18) says that every generator has even Maslov grading, so each contributes a positive term to the Euler characteristic. In other words, if

$$\widetilde{\Delta}_K(T) = \sum_{i=0}^{\infty} T^{a_i},$$

then $\underline{CFK}^-(K)$ has one generator with Alexander grading a_i, for each $i \geq 0$. Since multiplication by U is a filtered map (i.e., it never increases the Alexander grading), then necessarily

$$M(a_i) = -2i.$$

Figure 2 gives an illustration for the case of the $(3, 4)$-torus knot,[2] where

$$(19) \qquad\qquad \Delta_{T_{3,4}}(T) = 1 - (T^2 + T^{-2}) + (T^3 + T^{-3}),$$

and therefore

$$(20) \qquad\qquad \widetilde{\Delta}_{T_{3,4}}(T) = T^3 + 1 + T^{-1} + T^{-3} + T^{-4} + \cdots.$$

Proof of Theorem 1.6. First note that it is sufficient to prove the proposition for positive surgeries (see Remark 1.3).

So, let K be an L-space knot in S^3, so that

$$\widetilde{\Delta}_K(T) = \sum_{i=0}^{\infty} T^{a_i}.$$

Then the reduced complex $\underline{CFK}^-(K)$ consists of a single generator of Alexander grading a_i and Maslov grading $-2i$, for each $i \geq 0$. Since the a_i are strictly decreasing, it follows that $\tilde{d}$, as defined in (14), is given by

$$\tilde{d}(K, m) = -2 \min\{i \mid a_i \leq m\},$$

and therefore

$$(21) \qquad\qquad \tilde{d}(K, a_i - 1) = -2(i + 1).$$

Substituting these values into inequality (16), we obtain

$$(22) \qquad\qquad i + 1 \leq \left\lceil \tfrac{1}{2}(g(K) - (a_i - 1)) \right\rceil,$$

so

$$a_i \leq g(K) - 2i. \qquad\qquad\qquad \square$$

---

[2]It was shown by Moser [1971] that torus knots admit lens space (hence L-space) surgeries.

$\underline{A}$

○	3	$(0) \ \bullet a_0 = 3$
● ← ●	2	
● ← ●	1	
● ← ● ○	0	$(-2) \ \bullet a_1 = 0$
● ← ● ○	-1	$(-4) \ \bullet a_2 = -1$
● ← ●	-2	
● ← ● ○	-3	$(-6) \ \bullet a_3 = -3$
○	-4	$(-8) \ \bullet a_4 = -4$

$\mathrm{CFK}^-(T_{3,4})$ $\underline{\mathrm{CFK}^-}(T_{3,4})$
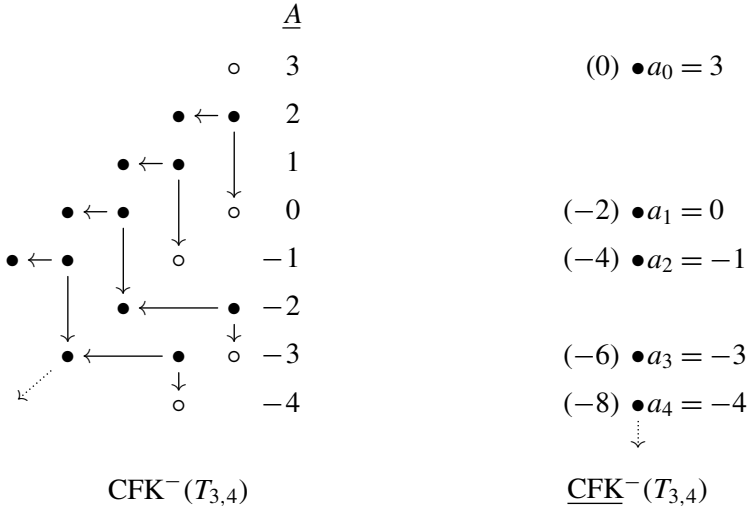
Figure 2. To the left is the knot Floer complex CFK^- for the $(3,4)$-torus knot, which, by Proposition 1.1, is determined by its Alexander polynomial. To the right is the reduced complex, which, for any L-space knot, is isomorphic to $\mathbb{F}[U]$, supported in Maslov grading zero. Note that the reduced complex on the right has a generator for each ○ generator on the left, the bottom-most generator of each staircase summand.

Proof of Theorem 1.5. Let K be an L-space knot, so that

$$\Delta_K(T) = \sum_{i=-k}^{k} (-1)^{k+i} T^{n_i}.$$

We have introduced the variables

$$\vec{r} = (r_1, \ldots, r_k)$$

as the "gaps" in the Alexander polynomial (the difference in the exponents of consecutive nonzero terms),

$$r_i = n_{k+2-2i} - n_{k+1-2i}.$$

While $\vec{r}$ records only every second gap, by the symmetry of $\Delta(T)$, this determines the polynomial uniquely. Diagramatically, notice that $\vec{r}$ is simply the list of horizontal lengths of a staircase summand of CFK^-, in order from left to right. See Figure 3 for the example of the $(4, 5)$-torus knot, which has

$$\Delta_{T(4,5)}(T) = -1 + (T^2 + T^{-2}) - (T^5 + T^{-5}) + (T^6 + T^{-6}).$$

$\underline{A}$

6       $\bullet (0)$

$5 = m_1$

4

r_3

3

2       $\bullet (-2r_1)$

1       $\bullet$

$0 = m_2$

r_2 -1

-2 $\bullet (-2(r_1+r_2))$

$r_1=1$

-3 $\bullet$

$r_2=2$

-4 $\bullet$

$r_3=3$

$-5 = m_3$

$\downarrow r_1$

-6 $\bullet (-2(r_1+r_2+r_3))$

-7 $\bullet$

$\mathrm{CFK}^-(T_{4,5})$ $\underline{\mathrm{CFK}}^-(T_{4,5})$
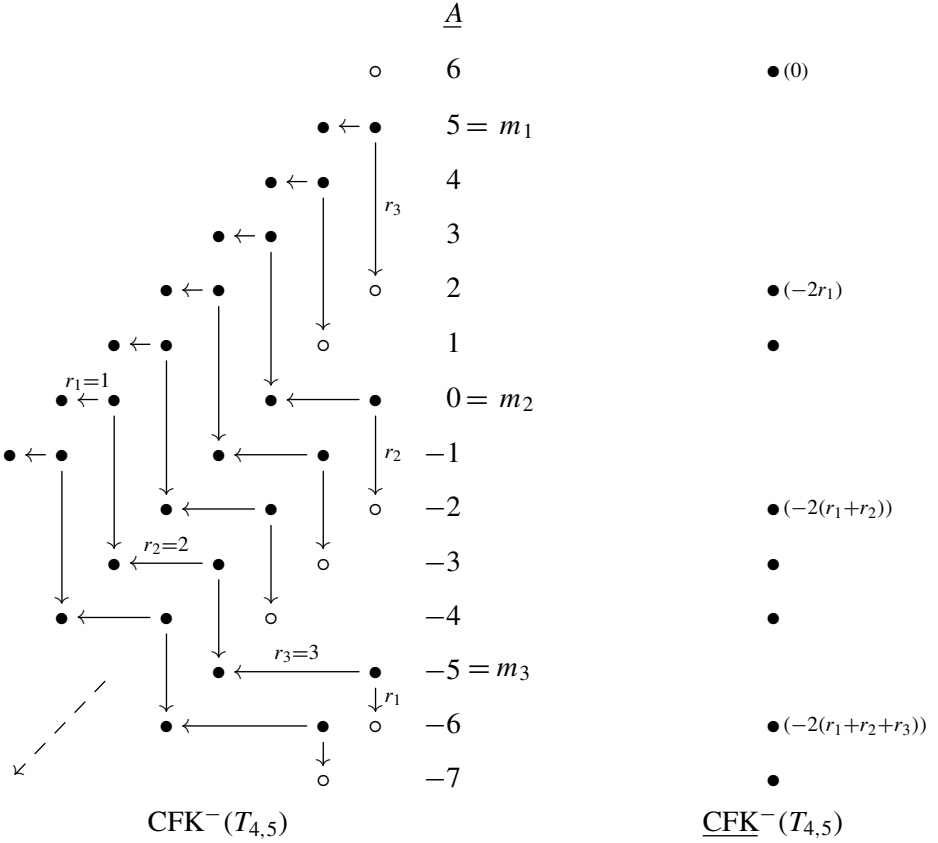
Figure 3. The complex CFK^- for the $(4, 5)$-torus knot, and its reduced form $\underline{\mathrm{CFK}}^-$. Note that the integers r_1, r_2 and r_3 are the horizontal lengths of each staircase, from left to right (and by symmetry, the vertical lengths, from bottom to top). This figure illustrates how the m_j — the Alexander gradings at which the reduced complex "jumps" — are determined by the r_i, and further, how the values of $\tilde{d}(K, m_j)$, given in parentheses to the right, are determined by the r_i.

Next we observe how, given $\vec{r}$, to compute both sides of inequality (16) for any m, with Figure 3 as a guide. We focus on the values labeled m_j in Figure 3; in other words, the values where we have the "jumps" in the reduced complex. More precisely, if we let

$$m_j = g(K) - \left(\sum_{i=1}^{j} r_i + \sum_{i=k-j+2}^{k} r_i \right),$$

then we have that

$$\tilde{d}(K, m_j) = -2 \sum_{i=1}^{j} r_i.$$

Substituting these values into inequality (16) when $m = m_j$ gives

$$(23) \qquad \sum_{i=1}^{j} r_i \le \left\lceil \frac{g(K) - \left(g(K) - \left(\sum_{i=1}^{j} r_i + \sum_{i=k-j+2}^{k} r_i\right)\right)}{2} \right\rceil.$$

The case $j = 1$ gives $r_1 \le \left\lceil \frac{1}{2} r_1 \right\rceil$, so, since each r_i is a positive integer, r_1 must equal 1. Substituting this into (23) gives

$$1 + \sum_{i=2}^{j} r_i \le \left\lceil \frac{1 + \sum_{i=2}^{j} r_i + \sum_{i=k-j+2}^{k} r_i}{2} \right\rceil,$$

$$\sum_{i=2}^{j} r_i \le \left\lceil \frac{-1 + \sum_{i=2}^{j} r_i + \sum_{i=k-j+2}^{k} r_i}{2} \right\rceil,$$

from which it follows that

$$\sum_{i=2}^{j} r_i \le \sum_{i=k-j+2}^{k} r_i.$$

This is sufficient to prove the claim. We could similarly obtain inequalities by considering values of m different from the m_j, but those would be no stronger, and therefore provide no more restrictions on $\vec{r}$. □

As an example, consider a knot K with

$$\Delta_K(T) = -1 + (T^2 + T^{-2}) - (T^3 + T^{-3}) + (T^4 + T^{-4}),$$

so that

$$\widetilde{\Delta}_K(T) = T^4 + T^2 + T + T^{-2} + T^{-4} + T^{-5} + \cdots.$$

This polynomial satisfies the restriction of Proposition 1.4, but if K were an L-space knot, we would have $g(K) = 4$, and $a_2 = 1$. This violates inequality (7), so K (and its mirror image) cannot admit an L-space surgery. Alternatively, this polynomial has gaps $\vec{r} = (1, 2, 1)$, and since $r_2 \not\le r_3$, this violates inequality (6).

In fact, this completely determines which Alexander polynomials are realized by L-space knots of genus less than or equal to 4. All other polynomials satisfying Proposition 1.4 are realized by known L-space knots. For knots of genus 5, Theorem 1.5 eliminates the polynomials corresponding to $\vec{r} = (1, 2, 1, 1)$ and $\vec{r} = (1, 3, 1)$, but there are still three more which are not realized by any L-space knot known to the author (corresponding to $(1, 1, 2, 1)$, $(1, 2, 2)$ and $(1, 4)$).

$\vec{r}$	L-space knot with corresponding $\Delta(T)$
(1)	$T(2,3)$
(1,1)	$T(2,5)$
(1,1,1)	$T(2,7)$
(1,2)	$T(3,4)$, $(2,3)$-cable of $T(2,3)$
(1,1,1,1)	$T(2,9)$
(1,1,2)	$T(3,5)$
(1,2,1)	excluded by Theorem 1.5
(1,3)	$(2,5)$-cable of $T(2,3)$

Finally, we should point out the relation between the sequence of the a_i for an L-space knot and the gap function defined by Borodzik and Livingston [2014, Definition 2.6]; namely,

$$a_i = \min_m \{I_K(m) = i\} - g(K).$$

As pointed out to the author by Borodzik, their restrictions given in [Borodzik and Livingston 2016, Theorem 2.14] can be reinterpreted in terms of the a_i. Informally, if two L-space knots are related by a small number of crossing changes, they have similar Alexander polynomials. More precisely, if a_i and a_i' are the exponents of $\widetilde{\Delta}$ for two L-space knots which differ by changing p positive crossings to negative crossings, then

$$|a_i - a_i'| \le p \quad \text{for all } i.$$

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STABILITY OF CAPILLARY HYPERSURFACES IN A EUCLIDEAN BALL

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We study the stability of capillary hypersurfaces in a unit Euclidean ball. It is proved that if the center of mass of the domain enclosed by the immersed capillary hypersurface and the wetted part of the sphere is located at the origin, then the hypersurface is unstable. An immediate result is that all known examples except the totally geodesic ones and spherical caps are unstable. We also conjecture a precise delineation of the stable capillary hypersurfaces in unit Euclidean balls.

1. Introduction

Capillarity is an important physical phenomenon, which occurs when two different materials contact and do not mix. Given a container B with an incompressible liquid drop T in it, the interface of the liquid and the air is a capillary surface M. In absence of gravity, the interface M is of constant mean curvature and the contact angle of M to the boundary ∂B is constant. One should compare this setting with a soap bubble, where the surface has no boundary and constant mean curvature, or a soap film, having fixed boundary and constant mean curvature.

The literature for the study of capillarity is extensive. We refer to the book [Finn 1986], where the treatment of the theory is mainly in the nonparametric case and in the more general situation of presence of gravity. Also we mention [Finn 1999] for a more recent survey about this topic.

In this paper we are concerned with the special case that the container B is a unit Euclidean ball and no gravity is involved. We study the (weak) stability for capillary hypersurfaces. This problem has been discussed by Ros and Souam [1997], where they dealt with the surface case and obtained some topological and geometrical restrictions. For the hypersurface case with free boundary (the contact angle is $\pi/2$), Ros and Vergasta [1995] also proved some interesting results. Also see [Souam 1997] for relevant work in space forms. In addition, we would like to remark

that the study of compact and constant (higher) mean curvature hypersurfaces in a Euclidean ball with free boundary is similar to that of closed and constant (higher) mean curvature hypersurfaces in a sphere in some sense. In that respect we refer the readers to [Barbosa et al. 1988; Alías et al. 2007; Cheng 2003; 2008].

In this paper we prove the following theorem.

Theorem 1.1. *Let $x : M^n \to \mathbb{R}^{n+1}$ be an immersed capillary hypersurface in a unit Euclidean ball B^{n+1} and Ω the wetted part of the boundary of the ball. Denote by T the domain enclosed by $x(M)$ and Ω. If the center of mass of T is at the origin, the capillary hypersurface M is unstable.*

Remark 1.2. When $n = 2$, our Theorem 1.1 was proved by P. I. Marinov in his Ph.D. thesis [2010]. It is worth pointing out that Marinov's argument depends crucially on conformal coordinates, which can not be extended to the higher dimensional case here.

Here since we assume M is immersed, $x(M)$ may have self-intersections. Thus we should be careful with the choice of T. When M is embedded, T is understood in the common sense. See Remark 2.1 below for more explanation.

We note the application of Theorem 1.1 to Delaunay hypersurfaces in particular. Recall that Delaunay hypersurfaces are the hypersurfaces of revolution with constant mean curvature. By Proposition 4.3 in [Hutchings et al. 2002], Delaunay hypersurfaces are classified as an unduloid, cylinder, nodoid, sphere, catenoid, or hyperplane. To guarantee the portion of a Delaunay hypersurface in a Euclidean ball is also capillary, it should have some symmetry. See Section 2 below for more details. In that case, we call it a Delaunay capillary hypersurface. From Theorem 1.1, then, we have the following corollary.

Corollary 1.3. *The only stable Delaunay capillary hypersurface M^n in a unit Euclidean ball B^{n+1} is a totally geodesic hypersurface or a spherical cap.*

Our approach for proving Theorem 1.1 is as follows. In the higher dimensional case, we find that we can construct a conformal Killing vector field $Y[\xi]$ for any fixed $\xi \in \mathbb{S}^n$ from the natural conformal transformation family on B^{n+1}. Using the normal part $\langle Y[\xi], N \rangle$ as the test function, we can define a symmetric bilinear form $Q(\xi_1, \xi_2)$ by following [Marinov 2010]. By summing Q over $(n + 1)$ coordinate directions we find Q has at least one negative eigenvalue. This summation technique can be compared with J. Simons' work [1968]. At last, under the hypothesis of Theorem 1.1 we can derive the instability of the hypersurface. Our argument indicates that this conformal field is important and we can use it to conclude that the center of mass of minimal submanifolds with free boundary in a unit Euclidean ball is at the origin (see Proposition 4.2). We refer the readers to [Fraser and Schoen 2011; 2016] for the very recent work on the minimal submanifolds with free boundary.

Just like in the case $n = 2$ in [Marinov 2010], as an application of our argument, we give a new proof of the classical result due to Barbosa and do Carmo [1984] which states that the only closed stable immersed hypersurface of constant mean curvature in $\mathbb{R}^{n+1}$ is the round sphere.

The outline of this paper is as follows. In Section 2, after fixing some notation and definitions, we prove the stability of hyperplanes and spherical caps. Then we construct the crucial conformal vector field. We also review some known facts about the Delaunay hypersurfaces. In Section 3 we give the proof of Theorem 1.1. In the last section, we discuss some applications of our method.

2. Preliminaries

Notation and definitions. Let $x : M^n \to \mathbb{R}^{n+1}$ be an orientable immersed hypersurface in the unit Euclidean ball $B^{n+1} \subset \mathbb{R}^{n+1}$ with $x(\mathrm{int}\, M) \subset \mathrm{int}\, B^{n+1}$ and $x(\partial M) \subset \partial B^{n+1}$. Suppose $\Omega \subset \partial B^{n+1}$ such that $\partial \Omega = x(\partial M)$, and denote by $T \subset B^{n+1}$ the part of the ball satisfying $\partial T = x(M) \cup \Omega$.

Remark 2.1. If $x(M)$ has self-intersections, T may be viewed as the finite union of some domains T_i, $i = 1, \ldots, m$, i.e., $T = \bigcup_{i=1}^{m} T_i$. Here the T_i may intersect with each other. If there is more than one choice for $\{T_i\}_{i=1}^{m}$, choose one and fix it. In the proof we will see that only the property $\partial T = x(M) \cup \Omega$ is needed. If there is no confusion, we write M for $x(M)$ and ∂M for $x(\partial M)$ for simplicity.

Let N be the unit normal of M pointing inwards to T and $\overline{N}$ the unit outward normal of ∂B^{n+1}. Denote by ν and $\bar{\nu}$ the conormals of ∂M in M and Ω, respectively. Let D be the connection of $\mathbb{R}^{n+1}$ and ∇ the connection of M. Then the second fundamental form of M in $\mathbb{R}^{n+1}$ is given by $\sigma(X_1, X_2) = \langle D_{X_1} X_2, N \rangle$ for all $X_1, X_2 \in T_p M$. When taking an orthonormal basis $\{e_i\}_{i=1}^{n}$ on TM, we also denote by h_{ij} the components $\sigma(e_i, e_j)$. So the mean curvature H of M is $H = \frac{1}{n} \sum_{i=1}^{n} h_{ii}$. And the second fundamental form of ∂B in $\mathbb{R}^{n+1}$ is given by $\Pi(Y_1, Y_2) = \langle D_{Y_1} Y_2, -\overline{N} \rangle$ for all $Y_1, Y_2 \in T_p(\partial B)$. Finally, let $\theta \in (0, \pi)$ be the angle between ν and $\bar{\nu}$. See Figure 1 for an illustration.

Following [Ros and Souam 1997], we discuss the variation of M.

Definition 2.2. An *admissible variation* of $x : M^n \to \mathbb{R}^{n+1}$ is a differentiable map $X : (-\varepsilon, \varepsilon) \times M \to \mathbb{R}^{n+1}$ such that $X_t : M^n \to \mathbb{R}^{n+1}$, $t \in (-\varepsilon, \varepsilon)$, given by $X_t(p) = X(t, p)$, $p \in M$, is an immersion satisfying $X_t(\mathrm{int}\, M) \subset \mathrm{int}\, B$ and $X_t(\partial M) \subset \partial B$ for all t, and $X_0 = x$.

Now for given $\theta \in (0, \pi)$, we define an energy functional

$$(1) \qquad\qquad E(t) = |M(t)| - \cos\theta \, |\Omega(t)|,$$
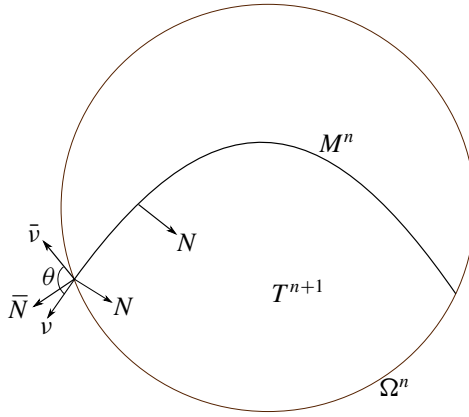
Figure 1. A typical illustration.

where $|\cdot|$ denotes the area function. The volume functional can be defined as

$$V(t) = \int_{[0,t] \times M} X^* \, dv,$$

where dv is the standard volume element of $\mathbb{R}^{n+1}$. Under these constraints, we define the following.

Definition 2.3. An immersed hypersurface $x : M^n \to \mathbb{R}^{n+1}$ is called capillary if $E'(0) = 0$ for any admissible volume-preserving variation of x.

Note that we have the formulas

$$(2) \qquad E'(0) = -n \int_M Hf \, da + \int_{\partial M} \langle Y, v - \cos\theta \, \bar{v} \rangle \, ds,$$

$$(3) \qquad V'(0) = \int_M f \, da,$$

where Y is the variational vector field $Y(p) = (\partial X/\partial t)(p)|_{t=0}$, f is its normal component $f = \langle Y, N \rangle$, and da and ds are the corresponding area elements.

From these formulas we see that M is capillary if and only if it has constant mean curvature and makes constant contact angle θ with ∂B. Furthermore, one can compute the second derivative of $E(t)$ at $t = 0$ with respect to an admissible volume-preserving variation to get

$$(4) \qquad E''(0) = -\int_M \big(\Delta f + (|\sigma|^2 + \widetilde{\mathrm{Ric}}(N)) f\big) f \, da + \int_{\partial M} \left(\frac{\partial f}{\partial v} - qf\right) f \, ds$$

(see, e.g., the appendix of [Ros and Souam 1997]), where

$$f \in \mathcal{F} := \left\{ f \in H^1(M) : \int_M f \, da = 0 \right\},$$

$\widetilde{\mathrm{Ric}}(N)$ is the Ricci curvature of the ambient space and

$$(5) \qquad q = \frac{1}{\sin\theta}\Pi(\bar{\nu}, \bar{\nu}) + \cot\theta\,\sigma(\nu, \nu).$$

In our setting, $\widetilde{\mathrm{Ric}}(N) = 0$ and $\Pi(\bar{\nu}, \bar{\nu}) = 1$.

Definition 2.4. A capillary hypersurface M is called (weakly) stable if $E''(0) \geq 0$ for all $f \in \mathcal{F}$.

In the sequel, we denote by $\partial^2 E(f)$ the quantity $E''(0)$ for a given function f.

Stable examples of capillary hypersurfaces. First we prove the stability of totally geodesic capillary hypersurfaces and spherical caps. The proof is similar to that of Proposition 1.1 in [Ros and Souam 1997]. We include it for completeness.

Proposition 2.5. *Let $B^{n+1} \subset \mathbb{R}^{n+1}$ be a unit Euclidean ball. Then totally geodesic capillary hypersurfaces and spherical caps are stable.*

Proof. First assume M is a totally geodesic capillary hypersurface, i.e., an n-dimensional ball $B^n(R)$ with radius R in B^{n+1}. Then the contact angle θ satisfies $\sin\theta = R$. By the definition of stability, we have to prove

$$(6) \qquad \int_M |\nabla f|^2 \, da \geq \frac{1}{R}\int_{\partial M} f^2 \, ds \quad \forall f \in \mathcal{F}.$$

Consider now the $(n+1)$-dimensional ball B' of radius R having M as an equatorial totally geodesic hypersurface. Then by [Bokowski and Sperner 1979], M is area minimizing for the partitioning problem in B'. Thus M is stable in B', which is equivalent to the inequality (6).

Next assume M is a spherical cap in B^{n+1} with R being the radius of the sphere containing M and θ the contact angle. Consider the n-dimensional hyperplane P containing ∂M. Then M is a capillary hypersurface in a halfspace with a contact angle θ'. By [Gonzalez et al. 1980], M is stable in the halfspace, which means

$$(7) \qquad \int_M \left(|\nabla f|^2 - \frac{n}{R^2}f^2\right) da \geq \frac{\cot\theta'}{R}\int_{\partial M} f^2 \, ds \quad \forall f \in \mathcal{F}.$$

Elementary calculation leads to

$$(8) \qquad \frac{1}{\sin\theta} + \frac{\cot\theta}{R} = \frac{\cot\theta'}{R}.$$

Now (7) and (8) together yield the stability of M in B^{n+1}. $\square$

Conformal transformations on the Euclidean ball. Now we construct a conformal vector field. Fix a vector $a \in B^{n+1}$. Then

$$(9) \qquad \varphi_a(x) = \frac{(1 - |a|^2)x - (1 - 2\langle a, x\rangle + |x|^2)a}{1 - 2\langle a, x\rangle + |a|^2|x|^2}$$

defines a map from B^{n+1} to B^{n+1} and from $\mathbb{S}^n$ to $\mathbb{S}^n$ (see, e.g., Section 3.8 in [Schoen and Yau 1994]), since we have

$$1 - |\varphi_a(x)|^2 = \frac{(1 - |a|^2)(1 - |x|^2)}{1 - 2\langle a, x\rangle + |a|^2|x|^2}.$$

Moreover, φ_a is conformal. In fact, by a direct calculation we can check that

$$|d\varphi_a|^2 = \left(\frac{1 - |a|^2}{1 - 2\langle a, x\rangle + |a|^2|x|^2}\right)^2 |dx|^2.$$

Note that $\varphi_a(a) = 0$, $\varphi_a(0) = -a$, φ_a fixes two points $\pm a/|a|$ and φ_0 is an identity.

Next fix $\xi \in \mathbb{S}^n$. Let $a = t\xi$ with $-1 < t < 1$. Then

$$(10) \qquad f_t(x) = \varphi_{t\xi}(x) = \frac{(1 - t^2)x - (1 - 2t\langle\xi, x\rangle + |x|^2)t\xi}{1 - 2t\langle\xi, x\rangle + t^2|x|^2}$$

is a family of conformal transformations with parameter t. Thus f_t determines a conformal vector field $Y[\xi]$ as follows:

$$(11) \qquad Y[\xi] = \frac{d}{dt}\Big|_{t=0} f_t(x) = -(1 + |x|^2)\xi + 2\langle\xi, x\rangle x.$$

Note that $Y[\xi]$ is tangential along the sphere $\mathbb{S}^n$, since for all $x \in \mathbb{S}^n$,

$$\langle Y[\xi], x\rangle = -(1 + |x|^2)\langle\xi, x\rangle + 2\langle\xi, x\rangle|x|^2 = 0.$$

Delaunay hypersurfaces in Euclidean space. In this subsection, following [Hutchings et al. 2002], we review some facts about Delaunay hypersurfaces which are rotational and of constant mean curvature H. These hypersurfaces are the models we are concerned with in Theorem 1.1.

Let $M^n \subset \mathbb{R}^{n+1}$ be a hypersurface which is invariant under the action of the orthogonal group $O(n)$ fixing the x^1-axis. Assume M is generated by a curve Γ contained in the x^1x^2-plane. Then it suffices to determine the curve Γ.

Parametrize the curve $\Gamma = (x^1, x^2)$ by arc-length s. Denote by α the angle between the tangent to Γ and the positive x^1-direction and choose the normal vector $N = (\sin\alpha, -\cos\alpha)$. Then $(x^1, x^2; \alpha)$ satisfies the system of ordinary differential equations

$$\begin{cases} (x^1)' = \cos\alpha, \\ (x^2)' = \sin\alpha, \\ \alpha' = -nH + (n-1)\dfrac{\cos\alpha}{x^2}. \end{cases}$$

The first integral of this system is given by

$$(x^2)^{n-1}\cos\alpha - H(x^2)^n = F,$$

where the constant F is called the force of the curve Γ and it together with H will determine the curve as follows. (See Proposition 4.3 in [Hutchings et al. 2002].)

Proposition 2.6. *The curve Γ and the hypersurface M generated by Γ have the following several possible types.*

(a) *If $FH > 0$ then Γ is a periodic graph over the x^1-axis. It generates a periodic embedded unduloid, or a cylinder.*

(b) *If $FH < 0$ then Γ is a locally convex curve and M is a nodoid, which has self-intersections.*

(c) *If $F = 0$ and $H \neq 0$ then M is a sphere.*

(d) *If $H = 0$ and $F \neq 0$ we obtain a catenary which generates an embedded catenoid M with $F > 0$ if the normal points down and $F < 0$ if the normal points up.*

(e) *If $H = 0$ and $F = 0$ then Γ is a straight line orthogonal to the x^1-axis which generates a hyperplane.*

(f) *If M touches the x^1-axis, then it must be a sphere or a hyperplane.*

(g) *The curve Γ is determined, up to translation along the x^1-axis, by the pair (H, F).*

From this proposition, it is easy to see if M^n is the portion of an unduloid, cylinder, nodoid or a catenoid in a unit Euclidean ball B^{n+1} with revolution axis x^1, and moreover M is symmetric with respect to the hyperplane $\{x^1 = 0\}$, then M is a capillary hypersurface in B^{n+1}. In that case we call them Delaunay capillary hypersurfaces in B^{n+1}. Furthermore, the generalized body T enclosed by M and the wetted part of the sphere has the center of mass at the origin. So Theorem 1.1 is applicable.

3. Instability of capillary hypersurfaces

With the preparations above, we can define a "test function"

$$(12) \qquad \phi[\xi] = \langle Y[\xi], N \rangle = \langle -(1 + |x|^2)\xi + 2\langle \xi, x \rangle x, N \rangle.$$

We mention that we will also use the following expression of $\phi[\xi]$:

$$(13) \qquad \phi[\xi] = \langle \xi, -(1 + |x|^2)N + 2\langle x, N \rangle x \rangle.$$

Recall the second variational formula

$$(14) \qquad \partial^2 E(\phi) = -\int_M L\phi \cdot \phi \, da + \int_{\partial M} (\phi_\nu - q\phi)\phi \, ds,$$

where $L = \Delta + |\sigma|^2$ and $q = \csc\theta + \cot\theta\sigma(\nu, \nu)$.

Now we can prove the following lemmas.

Lemma 3.1. *The vector v is a principal direction for σ along ∂M. In particular, $D_v N = -\sigma(v, v)v$.*

Proof. It suffices to prove that $\sigma(v, X) = 0$ for all $X \in T_p(\partial M)$. In fact, we have

$$\sigma(v, X) = \langle D_X v, N \rangle = \langle D_X(\cos\theta\,\bar{v} + \sin\theta\,\bar{N}), -\sin\theta\,\bar{v} + \cos\theta\,\bar{N}\rangle$$
$$= \langle D_X \bar{v}, \bar{N}\rangle = -II(\bar{v}, X) = 0,$$

where we used the facts that θ is constant, $\bar{v}$ and $\bar{N}$ are unit vectors, and ∂B is totally umbilical. Thus we complete the proof of Lemma 3.1.   □

Lemma 3.2. *Along ∂M, we have*

(15) $$\phi_v - q\phi = 0.$$

Proof. First, from (13) and Lemma 3.1 we have

$$\phi_v = \langle \xi, -(1 + |x|^2)N + 2\langle x, N\rangle x\rangle_v$$
$$= \langle \xi, -2\langle x, v\rangle N + (1 + |x|^2)\sigma(v, v)v - 2\langle x, \sigma(v, v)v\rangle x + 2\langle x, N\rangle v\rangle$$
$$= 2\langle \xi, -\langle x, v\rangle N + \sigma(v, v)(v - \langle x, v\rangle x) + \langle x, N\rangle v\rangle,$$

where in the third line we used $|x| = 1$ along ∂M.

Next, noticing that $x = \bar{N} = \cos\theta N + \sin\theta v$, we get

$$\phi_v = 2\langle \xi, -\sin\theta N + \sigma(v, v)(v - \sin\theta(\cos\theta N + \sin\theta v)) + \cos\theta v\rangle$$
$$= 2\langle \xi, (\sigma(v, v)\cos\theta + 1)(\cos\theta v - \sin\theta N)\rangle.$$

On the other hand,

$$q\phi = (\csc\theta + \cot\theta\sigma(v, v))\langle \xi, -(1 + |x|^2)N + 2\langle x, N\rangle x\rangle$$
$$= (\csc\theta + \cot\theta\sigma(v, v))2\langle \xi, -N + \cos\theta(\cos\theta N + \sin\theta v)\rangle$$
$$= (1 + \cos\theta\sigma(v, v))2\langle \xi, -\sin\theta N + \cos\theta v\rangle,$$

where again in the second line we used $|x| = 1$ along ∂M. Hence, we obtain

$$\phi_v - q\phi = 0.   \qquad \square$$

The next lemma, which indicates the geometric meaning of Lemma 3.2, may have its own interest. Thus we also include it here.

Lemma 3.3. *Under the flow f_t, there holds*

(16) $$\frac{d}{dt}\Big|_{t=0} \theta(t) = \phi_v - q\phi.$$

In particular, since f_t is conformal (angle preserving), $\phi_v - q\phi = 0$.

Proof. Following [Ros and Souam 1997], we denote by a "prime" the covariant derivative $(D/dt)|_{t=0}$. Also by the appendix of [Ros and Souam 1997], we have

$$v' = \left(\frac{\partial \phi}{\partial v} + \sigma(Y_0, v)\right)N + \phi S_0(v) - \phi \sigma(v, v)v - S_1(Y_1) + \cot \theta \tilde{\nabla} \phi,$$

$$\bar{v}' = -\Pi(Y, \bar{v})\bar{N} - S_2(Y_1) + (\csc \theta)\tilde{\nabla}\phi,$$

where $\tilde{\nabla}$ denotes the gradient on ∂M, Y_0 (resp. Y_1) the tangent part of the variational vector field Y to M (resp. to ∂M), S_0 the shape operator of M in $\mathbb{R}^{n+1}$ with respect to N, and S_1 (resp. S_2) the shape operator of ∂M in M (resp. ∂B) with respect to v (resp. $\bar{v}$).

Note that $\cos \theta(t) = \langle v, \bar{v} \rangle$, which implies

$$-\sin \theta \frac{d}{dt}\Big|_{t=0} \theta(t) = \langle v', \bar{v} \rangle + \langle v, \bar{v}' \rangle.$$

Taking into account that

$$\bar{v} = -\sin \theta N + \cos \theta v, \qquad \bar{N} = \cos \theta N + \sin \theta v,$$

we have

$$-\sin \theta \frac{d}{dt}\Big|_{t=0} \theta(t) = \left\langle \left(\frac{\partial \phi}{\partial v} + \sigma(Y_0, v)\right)N + \phi S_0(v) - \phi \sigma(v, v)v, -\sin \theta N + \cos \theta v \right\rangle$$
$$+ \langle v, -\Pi(Y, \bar{v})(\cos \theta N + \sin \theta v) \rangle$$
$$= -\sin \theta \left(\frac{\partial \phi}{\partial v} + \sigma(Y_0, v)\right) - \sin \theta \Pi(Y, \bar{v}),$$

or

$$(17) \qquad \frac{d}{dt}\Big|_{t=0} \theta(t) = \frac{\partial \phi}{\partial v} + \sigma(Y_0, v) + \Pi(Y, \bar{v}).$$

Again from the appendix of [Ros and Souam 1997], there hold

$$Y_0 = Y_1 - (\cot \theta)\phi v, \qquad Y = Y_1 - (\csc \theta)\phi \bar{v}, \qquad \sigma(Y_1, v) + \Pi(Y_1, \bar{v}) = 0.$$

Now plugging these equalities into (17), the lemma follows immediately. $\square$

Lemma 3.4. *We have*

$$(18) \qquad L\phi = -2n\langle \xi, N + Hx \rangle.$$

Proof. The proof is a direct calculation using the moving frame method. This method is very powerful in differential geometry. Take an orthonormal basis

$\{e_i : i = 1, \ldots, n\} \cup \{e_{n+1} = N\}$. Then we have the structure equations

$$dx = \sum_{i=1}^{n} \omega_i e_i,$$

$$de_i = \sum_{j=1}^{n} \omega_{ij} e_j + \sum_{j=1}^{n} h_{ij} \omega_j e_{n+1},$$

$$de_{n+1} = - \sum_{i,j=1}^{n} h_{ij} \omega_i e_j,$$

where ω_i is the dual form and ω_{ij} the connection form (see, e.g., [Cao and Li 2007]). Thus, we have

$$(19) \quad \Delta\phi = \Delta\langle\xi, -(1+|x|^2)N + 2\langle x, N\rangle x\rangle$$

$$= \Big\langle \xi, -\Big(\Delta|x|^2 \cdot N + 2\sum_{i=1}^{n}(|x|^2)_{,i} N_{,i} + (1+|x|^2)\Delta N\Big)$$

$$+ 2\Big(\Delta\langle x, N\rangle \cdot x + 2\sum_{i=1}^{n}\langle x, N\rangle_{,i} x_{,i} + \langle x, N\rangle\Delta x\Big)\Big\rangle.$$

Note that

$$\Delta|x|^2 = 2nH\langle x, N\rangle + 2n,$$

$$\sum_{i=1}^{n}(|x|^2)_{,i} N_{,i} = -2\sum_{i,j=1}^{n}\langle x, e_i\rangle h_{ij} e_j.$$

Using the Codazzi equation to get $\sum_{i=1}^{n} h_{ij,i} = \sum_{i=1}^{n} h_{ii,j} = nH_{,j} = 0$, then we have

$$\Delta N = \sum_{i=1}^{n} N_{,ii} = -|\sigma|^2 N.$$

Moreover, we can get

$$\Delta\langle x, N\rangle = \langle\Delta x, N\rangle + 2\sum_{i=1}^{n}\langle x_{,i}, N_{,i}\rangle + \langle x, \Delta N\rangle = -nH - |\sigma|^2\langle x, N\rangle,$$

$$\sum_{i=1}^{n}\langle x, N\rangle_{,i} x_{,i} = \sum_{i,j=1}^{n}\langle x, -h_{ij} e_j\rangle e_i = \sum_{i,j=1}^{n} -h_{ij}\langle x, e_j\rangle e_i.$$

Now substituting all these terms into (19) gives rise to

$$\Delta\phi = \langle\xi, -2n(N + Hx)\rangle - |\sigma|^2\phi.$$

Therefore,

$$L\phi = \Delta\phi + |\sigma|^2\phi = -2n\langle\xi, N + Hx\rangle. \qquad \square$$

Thus, we obtain

$$\partial^2 E(\phi) = -2n \int_M \langle \xi, N + Hx \rangle \cdot \langle \xi, (1 + |x|^2)N - 2\langle x, N \rangle x \rangle da.$$

To analyze $\partial^2 E(\phi)$, we define a bilinear form

(20) $$Q(\xi_1, \xi_2) = -2n \int_M \langle \xi_1, N + Hx \rangle \cdot \langle \xi_2, (1 + |x|^2)N - 2\langle x, N \rangle x \rangle da$$

for all $\xi_1, \xi_2 \in \mathbb{S}^n$ (see [Marinov 2010] for the case $n = 2$). Denote by $\{\partial_A\}_{A=1}^{n+1}$ the standard coordinate vectors in $\mathbb{R}^{n+1}$. Then we have the following lemma.

Lemma 3.5. *Q has the following properties.*

(1) *Q is symmetric.*

(2) $\operatorname{tr} Q = \sum_{A=1}^{n+1} Q(\partial_A, \partial_A) \leq 0$ *with equality if and only if* $|x| = $ const *on M.*

Proof. (1) First we prove Q is symmetric. Note that in fact Q is defined as

$$Q(\xi_1, \xi_2) = -\int_M L(\phi[\xi_1]) \cdot \phi[\xi_2] \, da.$$

Then Green's formula implies

$$Q(\xi_1, \xi_2) = -\int_M \phi[\xi_1] \cdot L(\phi[\xi_2]) \, da + \int_{\partial M} \left(\phi[\xi_1](\phi[\xi_2])_v - (\phi[\xi_1])_v \phi[\xi_2]\right) ds.$$

But Lemma 3.2 yields $(\phi[\xi_i])_v = q\phi[\xi_i]$, $i = 1, 2$. So the boundary term vanishes and then

$$Q(\xi_1, \xi_2) = Q(\xi_2, \xi_1).$$

(2) Next we calculate $\operatorname{tr} Q$:

$$\operatorname{tr} Q = \sum_{A=1}^{n+1} Q(\partial_A, \partial_A)$$

$$= -2n \int_M \sum_{A=1}^{n+1} \langle \partial_A, N + Hx \rangle \cdot \langle \partial_A, (1 + |x|^2)N - 2\langle x, N \rangle x \rangle da$$

$$= -2n \int_M \langle N + Hx, (1 + |x|^2)N - 2\langle x, N \rangle x \rangle da$$

$$= -2n \int_M \left(H\langle x, N \rangle(1 - |x|^2) + 1 + |x|^2 - 2\langle x, N \rangle^2\right) da$$

$$\leq -2n \int_M (H\langle x, N \rangle + 1)(1 - |x|^2) \, da.$$

Also, we have $\Delta |x|^2 = 2n(H\langle x, N\rangle + 1)$. Consequently,

$$
\begin{aligned}
\operatorname{tr} Q &\leq - \int_M \Delta |x|^2 \cdot (1 - |x|^2)\, da \\
&= \int_M \nabla |x|^2 \cdot \nabla (1 - |x|^2)\, da - \int_{\partial M} \frac{\partial |x|^2}{\partial \nu} (1 - |x|^2)\, ds \\
&= - \int_M |\nabla(|x|^2)|^2\, da \\
&\leq 0,
\end{aligned}
$$

where we have used $|x| = 1$ on ∂M to remove the boundary term. And it is easy to see $\operatorname{tr} Q = 0$ if and only if $|x| = \text{const}$.

This completes the proof of Lemma 3.5. $\qquad\qquad\square$

Now if $|x| = \text{const}$ on M, since M has boundary $\partial M \subset \mathbb{S}^n$, we must have $|x| = 1$ on M. So $M \subset \mathbb{S}^n$. But that cannot happen because we assume that $\operatorname{int} M \subset \operatorname{int} B$. Thus by Lemma 3.5, Q has at least one negative eigenvalue. But on the other hand,

$$
(21) \qquad \operatorname{div}_{\mathbb{R}^{n+1}} Y[\xi] = \sum_{A=1}^{n+1} \langle D_{\partial_A} Y[\xi], \partial_A \rangle = 2(n+1)\langle \xi, x\rangle,
$$

which by integration implies

$$
\begin{aligned}
(22) \qquad \int_M \phi\, da &= \int_M \langle Y[\xi], N\rangle\, da \\
&= - \int_T \operatorname{div}_{\mathbb{R}^{n+1}} Y[\xi]\, dv + \int_\Omega \langle Y[\xi], \bar{N}\rangle\, da \\
&= -2(n+1) \int_T \langle \xi, x\rangle\, dv.
\end{aligned}
$$

So generally $\int_M \phi\, da \neq 0$. That means $\phi[\xi]$ is not a test function.

However, under the hypothesis of Theorem 1.1 that the center of mass of T is at the origin, we have $\int_M \phi\, da = -2(n+1)\int_T \langle \xi, x\rangle\, dv = 0$ for all $\xi \in \mathbb{S}^n$. So if we choose ξ as an eigenvector corresponding to the negative eigenvalue of Q, we have $\partial^2 E(\phi[\xi]) = Q(\xi, \xi) < 0$, which implies that M is unstable. This completes the proof of Theorem 1.1.

4. Other applications and a question

In this section we give several applications of the above argument and propose a conjecture on the topic.

Another criterion for instability. The following proposition is an immediate result.

Proposition 4.1. *If the bilinear form Q has two negative eigenvalues, then M is unstable.*

Proof. Assume Q is diagonalized such that ξ_1 and ξ_2 are the eigenvectors corresponding to the two negative eigenvalues. Then for real numbers c_1 and c_2 with $c_1^2 + c_2^2 \neq 0$,

$$(23) \qquad Q(c_1\xi_1 + c_2\xi_2, c_1\xi_1 + c_2\xi_2) = c_1^2 Q(\xi_1, \xi_1) + c_2^2 Q(\xi_2, \xi_2) < 0.$$

On the other hand,

$$\int_M \phi[c_1\xi_1 + c_2\xi_2]\, da = -2(n+1)\int_T \langle c_1\xi_1 + c_2\xi_2, x\rangle\, dv$$
$$= -2(n+1)\left(c_1\int_T \langle \xi_1, x\rangle\, dv + c_2\int_T \langle \xi_2, x\rangle\, dv\right).$$

So we can always choose suitable c_1 and c_2 with $c_1^2 + c_2^2 \neq 0$ such that

$$\int_M \phi[c_1\xi_1 + c_2\xi_2]\, da = 0.$$

Then using $\phi[c_1\xi_1 + c_2\xi_2]$ as a test function, from (23) we know M is unstable. $\square$

The significance of the above proposition is as follows. For a given concrete capillary hypersurface M in B^{n+1}, the bilinear form Q is computable in principle. Then if Q has two negative eigenvalues, we can assert its instability. Also from this proposition we know that for hyperplanes and spherical caps Q has exactly one negative eigenvalue.

The center of mass of minimal submanifolds with free boundary. By free boundary we mean that M intersects ∂B^{n+1} orthogonally; that is, $\nu = x$ along ∂M. By analyzing the vector field $Y[\xi]$, we have the following proposition.

Proposition 4.2. *The center of mass of a minimal submanifold M^k with free boundary in a Euclidean ball is at the origin.*

Proof. Along M^k choose the orthonormal basis

$$\{e_i : i = 1, \ldots, k\} \cup \{e_\alpha : \alpha = k+1, \ldots, n+1\}$$

such that $\{e_i \mid i = 1, \ldots, k\} \subset TM$. Then we have

$$\mathrm{div}_M Y[\xi]^T = \mathrm{div}_M\left(Y[\xi] - \sum_\alpha \langle Y[\xi], e_\alpha\rangle e_\alpha\right) = 2k\langle \xi, x\rangle + \langle Y[\xi], k\vec{H}\rangle = 2k\langle \xi, x\rangle.$$

By the divergence theorem, we have

$$2k\int_M \langle \xi, x\rangle\, da = \int_{\partial M} \langle Y[\xi]^T, \nu\rangle\, ds = \int_{\partial M} \langle Y[\xi], x\rangle\, ds = 0,$$

where we have used the fact that $Y[\xi]$ is tangential to ∂B^{n+1}. $\square$

This proposition shows that minimal submanifolds with free boundary have some symmetry. Comparing with it, we mention two other properties of M^k:

(1) The center of mass of the boundary ∂M is at the origin (a simple argument).

(2) The volume of M has a lower bound $|M^k| \geq |B^k|$, where B^k is a k-dimensional unit ball [Brendle 2012; Fraser and Schoen 2011; Ros and Vergasta 1995].

Stable immersed closed CMC hypersurfaces in $\mathbb{R}^{n+1}$. At last we give a new proof of a theorem by Barbosa and do Carmo by following Marinov's argument [2010] in the case $n = 2$.

Theorem 4.3 [Barbosa and do Carmo 1984; Marinov 2010]. *The only stable immersed closed hypersurface of constant mean curvature in $\mathbb{R}^{n+1}$ is the round sphere.*

Proof. By translation, assume the center of mass of a generalized body T enclosed by M is at the origin. So

$$\int_M \phi[\xi] \, da = 0$$

for all $\xi \in \mathbb{S}^n$. If M is the round sphere, we are done. Otherwise $|x| \neq$ const. So by Lemma 3.5 the bilinear form Q has a negative eigenvalue. Choosing ξ as an eigenvector corresponding to the negative eigenvalue, we have

$$\partial^2 E(\phi[\xi]) = -\int_M L\phi[\xi] \cdot \phi[\xi] \, da = Q(\xi, \xi) < 0,$$

which shows that M is unstable. □

An open question. Since all the examples, i.e., the Delaunay capillary hypersurfaces, are known to be stable or unstable, we propose a conjecture as follows.

Conjecture 4.4. The only stable capillary hypersurface M^n ($n \geq 3$) in a unit Euclidean ball B^{n+1} is a totally geodesic hypersurface or a spherical cap.

There are some remarks on this conjecture.

(1) For $n \geq 2$, $H = 0$ and $\theta = \pi/2$, M must be totally geodesic [Ros and Vergasta 1995].

(2) For $n = 2$ and $\theta = \pi/2$, M is a totally geodesic disk, a spherical cap or a surface of genus 1 with embedded boundary having at most two connected components [Ros and Vergasta 1995].

(3) For $n = 2$ and $H = 0$, M is a totally geodesic disk or a surface of genus 1 with at most three connected boundary components [Ros and Souam 1997].

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NON-MINIMALITY OF CERTAIN
IRREGULAR COHERENT PREMINIMAL AFFINIZATIONS

ADRIANO MOURA AND FERNANDA PEREIRA

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra of type D or E and λ be a dominant integral weight whose support bounds the subdiagram of type D_4. We study certain quantum affinizations of the simple $\mathfrak{g}$-module of highest weight λ which we term preminimal affinizations of order 2 (this is the maximal order for such λ). This class can be split in two: the coherent and the incoherent affinizations. If λ is regular, Chari and Pressley proved that the associated minimal affinizations belong to one of the three equivalent classes of coherent preminimal affinizations. In this paper we show that, if λ is irregular, the coherent preminimal affinizations are not minimal under certain hypotheses. Since these hypotheses are always satisfied if $\mathfrak{g}$ is of type D_4, this completes the classification of minimal affinizations for type D_4 by giving a negative answer to a conjecture of Chari and Pressley stating that the coherent and the incoherent affinizations were equivalent in type D_4 (this corrects the opposite claim made by the first author in a previous publication).

1. Introduction

This is the second paper of a series based on a project aiming at describing the classification of the Drinfeld polynomials of the irregular minimal affinizations of type D. The theory of minimal affinizations, initiated in [Chari 1995; Chari and Pressley 1996a], is an object of intensive study due to its rich structure and connections to other areas such as mathematical physics and combinatorics [Chari and Hernandez 2010; Zhang et al. 2016; Hernandez 2007; 2010; Hernandez and Leclerc 2016; Li and Qiao 2017; Naoi 2013; 2014]. We refer to the first two paragraphs of the first paper of the series [Moura and Pereira 2017] for an account of the status of this classification problem when this project started. The main result of the present paper (Theorem 2.4.4) is one of the crucial steps towards the final classification: it will provide one of the tools we shall use to compare certain affinizations or to show that they are not comparable. Moreover, together with the results of [Chari and Pressley 1996b], Theorem 2.4.4 completes the classification

in type D_4 by showing that the corresponding elements of two different families of affinizations described in [Chari and Pressley 1996b] are not equivalent, contrary to what was conjectured there. In fact, these elements are comparable and one is strictly larger than the other. Hence, only one of them, which we term an incoherent preminimal affinization of order 2 here, is actually a minimal affinization. The corresponding coherent affinization (the one that follows the pattern of the minimal affinizations in the regular case) is actually not minimal. The study made here goes beyond type D and proves that, under certain conditions, the coherent affinizations are also not minimal in type E. Since the proof of this result is rather lengthy and there are several parts which are interesting in their own right, we deem it appropriate to write a paper focusing exclusively on this result.

We now give a not so formal description of the main results of the paper. Let $U_q(\tilde{\mathfrak{g}})$ denote the quantum affine algebra associated to a simply laced finite type Kac–Moody algebra $\mathfrak{g}$. If $\boldsymbol{\omega}$ is a Drinfeld polynomial with classical weight λ, then the associated irreducible $U_q(\tilde{\mathfrak{g}})$-module $V_q(\boldsymbol{\omega})$, when regarded as module for $U_q(\mathfrak{g})$, decomposes as a direct sum of the form

$$V_q(\boldsymbol{\omega}) \cong V_q(\lambda) \oplus \bigoplus_\mu V_q(\mu)^{\oplus m_\mu},$$

where the sum is over all dominant integral weights μ such that $\mu < \lambda$ in the usual partial order on weights, m_μ is a nonnegative integer, and $V_q(\mu)$ denotes the irreducible $U_q(\mathfrak{g})$-module of highest weight μ. Any module satisfying this kind of decomposition is said to be an affinization of $V_q(\lambda)$ and an isomorphism as $U_q(\mathfrak{g})$-modules define an equivalence relation on the class of affinizations of $V_q(\lambda)$. Moreover, the partial order on weights induces a partial order on the set of equivalence classes of affinizations which obviously admit minimal elements, termed minimal affinizations [Chari 1995]. If $\mathfrak{g}$ is of type A, since quantum analogues of evaluation maps exist [Jimbo 1986], $V_q(\boldsymbol{\omega})$ is minimal if and only if $m_\mu = 0$ for all $\mu < \lambda$. In that case, the roots of each polynomial $\boldsymbol{\omega}_i(u)$ form what is called a q-string, where $i \in I$ and I is an index set for the nodes of the Dynkin diagram of $\mathfrak{g}$. Moreover, if we denote by a_i the center of the q-string associated to node i and let i_1 be the first node in supp(λ), the support of λ, there exists a strictly monotonic function f defined on supp(λ) such that

$$a_i = a_{i_1} q^{f(i)}.$$

If f is increasing, we say $V_q(\boldsymbol{\omega})$ is an increasing minimal affinization. Otherwise, we say it is decreasing. Although the increasing and decreasing minimal affinizations are equivalent, the understanding of the combinations of increasing and decreasing patterns for diagram subalgebras of type A is a key point for describing the minimal affinizations outside type A.

More precisely, it was proved in [Chari and Pressley 1996a] that, if $V_q(\boldsymbol{\omega})$ is a minimal affinization and $J \subseteq I$ corresponds to a connected subdiagram of type A

which remains connected after removing the trivalent node i_*, then the associated J-tuple of polynomials corresponds to a minimal affinization of type A. We say that $\boldsymbol{\omega}$ is preminimal if it satisfies this property. Thus, if $V_q(\boldsymbol{\omega})$ is minimal, $\boldsymbol{\omega}$ is preminimal. If i is an extremal node, let I_i be the maximal subdiagram of type A not containing i. We say that $\boldsymbol{\omega}$ is i-minimal if the associated I_i-tuple of polynomials corresponds to a minimal affinization of type A. The order of minimality of a preminimal $\boldsymbol{\omega}$ is defined as the cardinality of the set of extremal nodes i such that $\boldsymbol{\omega}$ is i-minimal. Hence, the order can be 0, 1, 2, 3. It follows from the results of [Chari and Pressley 1996a] that, if $\mathrm{supp}(\lambda) \subseteq I_i$ for some extremal node i, then $V_q(\boldsymbol{\omega})$ is a minimal affinization if and only if $\boldsymbol{\omega}$ is preminimal of order 3. On the other hand, if $\mathrm{supp}(\lambda)$ bounds the diagram of type D_4, the order of any preminimal $\boldsymbol{\omega}$ is at most 2 and, if $i_* \in \mathrm{supp}(\lambda)$, $V_q(\boldsymbol{\omega})$ is a minimal affinization if and only if $\boldsymbol{\omega}$ is preminimal of order 2. Note that, in this case, there are three equivalence classes of minimal affinizations, one for each extremal node, the node i for which i-minimality fails.

The purpose of this paper is to describe a few properties of the preminimal affinizations of order 2 when $i_* \notin \mathrm{supp}(\lambda)$ and $\mathrm{supp}(\lambda)$ bounds the subdiagram of type D_4. For type D_4, it follows from [Chari and Pressley 1996b] that, if $V_q(\boldsymbol{\omega})$ is a minimal affinization, then $\boldsymbol{\omega}$ has order 1 or 2. It was clear from [Chari and Pressley 1996b] that not all Drinfeld polynomials of order 1 correspond to minimal affinizations. However, the conjecture mentioned in the first paragraph can be rephrased as "all preminimal Drinfeld polynomials of order 2 correspond to minimal affinizations". The preminimal Drinfeld polynomials of order 2 can be encoded by the following pictures:
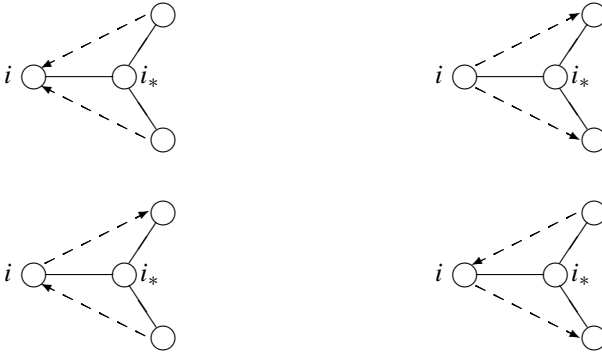
The arrows point in the direction that the function f decreases and i is the node for which i-minimality fails. Drinfeld polynomials satisfying either of the pictures in each line give rise to equivalent affinizations. Note that, if $i_* \in \mathrm{supp}(\lambda)$, Drinfeld polynomials satisfying the diagrams in the second line do not exist. We say that the ones satisfying the first line are coherent (because the arrows agree) and the ones satisfying the second line are incoherent. The notion of coherent and incoherent preminimal Drinfeld polynomials of order 2 can be extended for rank higher than 4,

including type E. Note that the incoherent ones do not exist when supp(λ) intersects more than once the connected subdiagram having i and i_* as extremal nodes. Otherwise, we conjecture that the coherent ones are not minimal affinizations and prove that this is indeed the case in type D as well as under certain conditions on supp(λ) when $\mathfrak{g}$ is of type E. In fact, the proof consists of showing that the coherent affinizations are strictly larger than their incoherent counterpart when the incoherent ones exist.

Part of the proof of Theorem 2.4.4 is based on a result about the multiplicity of $V_q(\nu)$ as a summand of the coherent and incoherent affinizations where ν is a specific dominant weight. The precise statement is in Proposition 2.4.6, which can be proved in greater generality than Theorem 2.4.4. The proof of Proposition 2.4.6 is a combination of arguments in the context of graded limits as well as in the context of qcharacters. Generators and relations for the graded limits of the coherent affinizations were described in [Moura 2010] for type D_4. This result has been extended for type D_n, $n > 4$, in [Naoi 2014] in the case $i_* \in$ supp(λ). However, as far as we can tell, the argument of [Naoi 2014] also works when $i_* \notin$ supp(λ). Inspired by Proposition 2.4.6, we define in Section 2.5 a quotient of the "coherent graded limits" and prove that it projects onto the corresponding "incoherent graded limits" under the hypotheses of Theorem 2.4.4. It is tempting to conjecture that these projections are actually isomorphisms (see Remark 2.5.5 for further comments).

Beside Proposition 2.4.6, by considering diagram subalgebras, the proof of Theorem 2.4.4 also relies on proving that certain tensor products of minimal affinizations in types A and D are irreducible. From type A we need tensor products of a general minimal affinization with a Kirillov–Reshetikhin module supported on an extremal node. This was exactly the topic of our first paper in this series, [Moura and Pereira 2017], where we described a necessary and sufficient condition for the irreducibility of such tensor products. In fact, this criterion for irreducibility is half of the main result of [Moura and Pereira 2017]. The other half will be crucial in the proof of the final classification as it also provides a tool to compare certain affinizations or to show that they are not comparable. From type D, we use a sufficient condition for the irreducibility of tensor products of two Kirillov–Reshetikhin modules associated to distinct extremal nodes proved in [Chari 2002]. For the proof of the final classification of minimal affinizations in type D we will need sharper results which will appear in [Pereira $\geq$ 2018] (see also [Pereira 2014]).

The paper is organized as follows. In Section 2, we fix the basic notation and review the background needed to state the main results. Further technical background is reviewed in Section 3. In Section 3.1, we compute the dimension of certain weight spaces of certain $\mathfrak{g}$-modules in terms of a "modified" Kostant partition function (see Proposition 3.1.1 and (3-1-3)). Such dimensions play a crucial

role in the proof of Proposition 2.4.6. In Section 3.2, in addition to reviewing
some known facts about diagram subalgebras, we prove Lemma 3.2.4 which is
an important technical ingredient to be used in the proof of Theorem 2.4.4. The
few facts about qcharacters that we need are reviewed in Section 3.3 and the
aforementioned criteria for irreducibility of certain tensor products is reviewed in
Section 3.4. Section 3.5 contains the necessary technical background on graded
limits. Section 4 is entirely dedicated to the proof of the main results: Theorem 2.4.4
and Proposition 2.4.6. In Section 4.1, we deduce some facts about qcharac-
ters and tensor products of Kirillov–Reshetikhin modules. Upper bounds for
certain outer multiplicities are obtained in Section 4.2 by studying graded lim-
its. The main technical obstacles for proving Theorem 2.4.4 in greater gen-
erality when $\mathfrak{g}$ is of type E arise from Lemma 4.2.3. Although the extra hy-
potheses are necessary for the validity of that lemma, they may not be neces-
sary for the validity of Theorem 2.4.4. However, the proof with the techniques
employed here, would require a much more intricate analysis (see the last para-
graph of Section 2.4 for more precise comments). The heart of the proof of
Proposition 2.4.6 is in Sections 4.3 and 4.4, where we study irreducible factors
of "incoherent" and "coherent" tensor products of Kirillov–Reshetikhin modules
associated to extremal nodes of the Dynkin diagram. Theorem 2.4.4 is finally
proved in Section 4.5.

2. The main results

Throughout the paper, let $\mathbb{C}$ and $\mathbb{Z}$ denote the sets of complex numbers and integers,
respectively. Let also $\mathbb{Z}_{\geq m}$, $\mathbb{Z}_{<m}$, etc. denote the obvious subsets of $\mathbb{Z}$. Given a ring
$\mathbb{A}$, the underlying multiplicative group of units is denoted by $\mathbb{A}^{\times}$. The symbol $\cong$
means "isomorphic to". We shall use the symbol $\diamond$ to mark the end of remarks,
examples, and statements of results whose proofs are postponed. The symbol $\square$
will mark the end of proofs as well as of statements whose proofs are omitted.

2.1. _Classical and quantum algebras._ Let I be the set of vertices of a finite-type
simply laced indecomposable Dynkin diagram and let $\mathfrak{g}$ be the associated simple
Lie algebra over $\mathbb{C}$ with a fixed Cartan subalgebra $\mathfrak{h}$. Fix a set of positive roots R^{+}
and let $\mathfrak{g}_{\pm\alpha}$, $\alpha \in R^{+}$, and $\mathfrak{g} = \mathfrak{n}^{-} \oplus \mathfrak{h} \oplus \mathfrak{n}^{+}$ be the associated root spaces and
triangular decomposition. The simple roots will be denoted by α_i, the fundamental
weights by ω_i, $i \in I$, Q, P, Q^{+}, P^{+} will denote the root and weight lattices with
corresponding positive cones, respectively. Let also $h_{\alpha} \in \mathfrak{h}$ be the coroot associated
to $\alpha \in R^{+}$. If $\alpha = \alpha_i$ is simple, we often simplify notation and write h_i. We denote
by $x_{\alpha}^{\pm}$ any element spanning the root space $\mathfrak{g}_{\pm\alpha}$. In particular, we shall write

$$(2\text{-}1\text{-}1) \qquad\qquad [x_{\alpha}^{-}, x_{\beta}^{-}] = x_{\alpha+\beta}^{-}.$$

Let $C = (c_{ij})_{i,j \in I}$ be the Cartan matrix of $\mathfrak{g}$, i.e., $c_{i,j} = \alpha_j(h_i)$.

By abuse of language, we will refer to any subset J of I as a subdiagram of the Dynkin diagram of $\mathfrak{g}$. Given $J \subseteq I$, we let $\bar{J}$ be the minimal connected subdiagram of I containing J and let ∂J be the subset of J consisting of nodes connected to at most one other node of J and

$$\mathring{J} = J \setminus \partial J.$$

For $i, j \in I$, set

$$[i, j] = \overline{\{i, j\}}, \qquad (i, j] = [i, j] \setminus \{i\}, \qquad [i, j) = [i, j] \setminus \{j\}.$$

Define also the distance between i, j as

$$d(i, j) = \#[i, j].$$

For a subdiagram $J \subseteq I$, we let $\mathfrak{g}_J$ be the subalgebra of $\mathfrak{g}$ generated by the corresponding simple root vectors, $\mathfrak{h}_J = \mathfrak{h} \cap \mathfrak{g}_J$ and so on. Also, let Q_J be the subgroup of Q generated by α_j, $j \in J$, and $R_J^+ = R^+ \cap Q_J$. Set

$$(2\text{-}1\text{-}2) \qquad\qquad \vartheta_J = \sum_{j \in J} \alpha_j$$

which is an element of R_J if J is connected. When $J = I$ we may simply write ϑ. Given $\lambda \in P$, let λ_J denote the restriction of λ to $\mathfrak{h}_J^*$ and let $\lambda^J \in P$ be such that $\lambda^J(h_j) = \lambda(h_j)$ if $j \in J$ and $\lambda^J(h_j) = 0$ otherwise. The support of $\mu \in P$ is defined by

$$\operatorname{supp}(\mu) = \{i \in I : \mu(h_i) \neq 0\}.$$

Given $\eta = \sum_{i \in I} s_i \alpha_i \in Q$, set

$$\operatorname{rsupp}(\eta) = \{i \in I : s_i \neq 0\}, \quad \operatorname{ht}_i(\eta) = s_i, \ i \in I, \quad \text{and} \quad \operatorname{ht}(\eta) = \sum_{i \in I} s_i.$$

For a Lie algebra $\mathfrak{a}$ over $\mathbb{C}$, let $\tilde{\mathfrak{a}} = \mathfrak{a} \otimes \mathbb{C}[t, t^{-1}]$ be its loop algebras and identify $\mathfrak{a}$ with the subalgebra $\mathfrak{a} \otimes 1$. Then, $\tilde{\mathfrak{g}} = \tilde{\mathfrak{n}}^- \oplus \tilde{\mathfrak{h}} \oplus \tilde{\mathfrak{n}}^+$ and $\tilde{\mathfrak{h}}$ is an abelian subalgebra. We denote by $\mathfrak{a}[t]$ the subalgebra determined by $\mathfrak{a} \otimes \mathbb{C}[t]$. Let also $\mathfrak{a}[t]_+ = \mathfrak{a} \otimes t\mathbb{C}[t]$. The elements $x_\alpha^\pm \otimes t^r$, $\alpha \in R^+$, $r \in \mathbb{Z}$, will be denoted by $x_{\alpha,r}^\pm$ and similarly we define $h_{\alpha,r}$. Given $a \in \mathbb{C}$, let τ_a be the Lie algebra automorphism of $\mathfrak{a}[t]$ defined by

$$(2\text{-}1\text{-}3) \qquad \tau_a(x \otimes f(t)) = x \otimes f(t - a) \quad \text{for every } x \in \mathfrak{a}, \ f(t) \in \mathbb{C}[t].$$

Let $\mathbb{F}$ be an algebraic closure of $\mathbb{C}(q)$, the ring of rational functions on an indeterminate q, and let $U_q(\mathfrak{g})$ and $U_q(\tilde{\mathfrak{g}})$ be the associated Drinfeld–Jimbo quantum groups over $\mathbb{F}$. We use the notation as in [Moura 2010, Section 1.2]. In particular, the Drinfeld loop-like generators of $U_q(\tilde{\mathfrak{g}})$ are denoted by $x_{i,r}^\pm$, $h_{i,s}$, $k_i^{\pm 1}$, $i \in I$, $r, s \in \mathbb{Z}$, $s \neq 0$. Also, $U_q(\mathfrak{g})$ is the subalgebra of $U_q(\tilde{\mathfrak{g}})$ generated by $x_i^\pm = x_{i,0}^\pm$, $k_i^{\pm 1}$, $i \in I$, and the subalgebras $U_q(\mathfrak{n}^\pm)$, $U_q(\mathfrak{h})$, $U_q(\tilde{\mathfrak{n}}^\pm)$, $U_q(\tilde{\mathfrak{h}})$, $U_q(\mathfrak{g}_J)$, $U_q(\tilde{\mathfrak{g}}_J)$, where $J \subseteq I$, are defined in the expected way.

The ℓ-weight lattice of $U_q(\tilde{\mathfrak{g}})$ is the multiplicative group $\mathcal{P}_q$ of n-tuples of rational functions $\boldsymbol{\mu} = (\boldsymbol{\mu}_i(u))_{i \in I}$ with values in $\mathbb{F}$ such that $\boldsymbol{\mu}_i(0) = 1$ for all $i \in I$. The elements of the submonoid $\mathcal{P}_q^+$ of $\mathcal{P}_q$ consisting of n-tuples of polynomials will be referred to as dominant ℓ-weights or Drinfeld polynomials. Given $a \in \mathbb{F}^\times$ and $\mu \in P$, let $\boldsymbol{\omega}_{\mu,a} \in \mathcal{P}_q$ be the element whose i-th rational function is

$$(1 - au)^{\mu(h_i)}, \quad i \in I.$$

In the case that $\mu = \omega_i$ for some i, we often simplify notation and write $\boldsymbol{\omega}_{i,a}$. Note that $\mathcal{P}_q$ is the (multiplicative) free abelian group on the set $\{\boldsymbol{\omega}_{i,a} : i \in I, a \in \mathbb{F}^\times\}$, let $\mathcal{P}$ denote the subgroup generated by $\{\boldsymbol{\omega}_{i,a} : i \in I, a \in \mathbb{C}^\times\}$, and $\mathcal{P}^+ = \mathcal{P}_q^+ \cap \mathcal{P}$. If

$$(2\text{-}1\text{-}4) \qquad \boldsymbol{\mu} = \prod_{(i,a) \in I \times \mathbb{F}^\times} \boldsymbol{\omega}_{i,a}^{p_{i,a}}$$

we shall say that $\boldsymbol{\omega}_{i,a}$ (respectively, $\boldsymbol{\omega}_{i,a}^{-1}$) appears in $\boldsymbol{\mu}$ if $p_{i,a} > 0$ (respectively, $p_{i,a} < 0$). Let $\mathrm{wt} : \mathcal{P}_q \to P$ be the group homomorphism determined by setting $\mathrm{wt}(\boldsymbol{\omega}_{i,a}) = \omega_i$. We have an injective map $\boldsymbol{\Psi} : \mathcal{P}_q \to (U_q(\tilde{\mathfrak{h}}))^*$, $\boldsymbol{\omega} \mapsto \boldsymbol{\Psi}_{\boldsymbol{\omega}}$, (see [Moura 2010, Section 1.8]) and, hence, we identify $\mathcal{P}_q$ with its image in $(U_q(\tilde{\mathfrak{h}}))^*$ under $\boldsymbol{\Psi}$. Similarly, there is an injective map $\mathcal{P} \to \tilde{\mathfrak{h}}^*$. Following [Chari and Moura 2005], given $i \in I$, $a \in \mathbb{F}^\times$, $m \in \mathbb{Z}_{\geq 0}$, define

$$\boldsymbol{\omega}_{i,a,m} = \prod_{j=0}^{m-1} \boldsymbol{\omega}_{i,aq^{m-1-2j}} \quad \text{and} \quad \boldsymbol{\alpha}_{i,a} = \boldsymbol{\omega}_{i,aq,2} \prod_{j \neq i} \boldsymbol{\omega}_{j,aq,-c_{j,i}}^{-1}.$$

For $\boldsymbol{\omega} \in \mathcal{P}_q$, let $\boldsymbol{\omega}_J$ be the associated J-tuple of rational functions and let $\mathcal{P}_J = \{\boldsymbol{\omega}_J : \boldsymbol{\omega} \in \mathcal{P}_q\}$. Similarly define $\mathcal{P}_J^+$. Notice that $\boldsymbol{\omega}_J$ can be regarded as an element of the ℓ-weight lattice of $U_q(\tilde{\mathfrak{g}}_J)$. Let $\pi_J : \mathcal{P}_q \to \mathcal{P}_J$ denote the map $\boldsymbol{\omega} \mapsto \boldsymbol{\omega}_J$. If $J = \{j\}$ is a singleton, we write π_j instead of π_J. An ℓ-weight $\boldsymbol{\omega}$ is said to be J-dominant if $\boldsymbol{\omega}_J \in \mathcal{P}_J^+$. Let also $\mathcal{Q}_J \subset \mathcal{P}_J$ (respectively, $\mathcal{Q}_J^+$) be the subgroup (submonoid) generated by $\pi_J(\boldsymbol{\alpha}_{j,a})$, $j \in J$, $a \in \mathbb{C}^\times$. When no confusion arises, we shall simply write $\boldsymbol{\alpha}_{j,a}$ for its image in $\mathcal{P}_J$ under π_J. Let

$$\iota_J : \mathbb{Z}[\mathcal{Q}_J] \to \mathbb{Z}[\mathcal{Q}_q],$$

be the ring homomorphism such that $\iota_J(\boldsymbol{\alpha}_{j,a}) = \boldsymbol{\alpha}_{j,a}$ for all $j \in J$, $a \in \mathbb{C}^\times$. We shall often abuse notation and identify $\mathcal{Q}_J$ with its image under ι_J. In particular, given $\boldsymbol{\mu} \in \mathcal{P}_q$, we set

$$\boldsymbol{\mu} \mathcal{Q}_J = \{\boldsymbol{\mu}\boldsymbol{\alpha} : \boldsymbol{\alpha} \in \iota_J(\mathcal{Q}_J)\}.$$

It will also be useful to introduce the element $\boldsymbol{\omega}^J \in \mathcal{P}_q$ defined by

$$(\boldsymbol{\omega}^J)_j(u) = \boldsymbol{\omega}_j(u) \quad \text{if } j \in J \quad \text{and} \quad (\boldsymbol{\omega}^J)_j(u) = 1 \text{ otherwise.}$$

2.2. Finite-dimensional representations. We let $\mathcal{C}$ denote the category of finite-dimensional $\mathfrak{g}$-modules and denote by $V(\lambda)$ an irreducible $\mathfrak{g}$-module of highest weight $\lambda \in P^+$. The character of a $\mathfrak{g}$-module V will be denoted by $\mathrm{ch}(V)$. We think of $\mathrm{ch}(V)$ as an element of the group ring $\mathbb{Z}[P]$. Let also $\mathcal{C}_q$ be the category of all finite-dimensional type 1 modules of $U_q(\mathfrak{g})$. Thus, a finite-dimensional $U_q(\mathfrak{g})$-module V is in $\mathcal{C}_q$ if $V = \bigoplus_{\mu \in P} V_\mu$ where

$$V_\mu = \{v \in V : k_i v = q^{\mu(h_i)} v \ \text{ for all } i \in I\}.$$

The character of V, also denoted by $\mathrm{ch}(V)$, is defined in the obvious way. The following theorem summarizes the basic facts about $\mathcal{C}_q$.

Theorem 2.2.1. *Let V be an object of $\mathcal{C}_q$. Then*:

(a) $\dim V_\mu = \dim V_{w\mu}$ *for all $w \in \mathcal{W}$.*

(b) *V is completely reducible.*

(c) *For each $\lambda \in P^+$ the $U_q(\mathfrak{g})$-module $V_q(\lambda)$ generated by a vector v satisfying*

$$x_i^+ v = 0, \qquad k_i v = q^{\lambda(h_i)} v, \qquad (x_i^-)^{\lambda(h_i)+1} v = 0, \quad \text{for all } i \in I,$$

is irreducible and finite-dimensional. If $V \in \mathcal{C}_q$ is irreducible, then V is isomorphic to $V_q(\lambda)$ for some $\lambda \in P^+$.

(d) *For all $\lambda \in P^+$, $\mathrm{ch}(V_q(\lambda)) = \mathrm{ch}(V(\lambda))$.* □

If $J \subseteq I$, we shall denote by $V_q(\lambda_J)$ the simple $U_q(\mathfrak{g}_J)$-module of highest weight λ_J. Since $\mathcal{C}_q$ is semisimple, it is easy to see that, if $\lambda \in P^+$ and $v \in V_q(\lambda)_\lambda$ is nonzero, then $U_q(\mathfrak{g}_J)v \cong V_q(\lambda_J)$.

Let $\widetilde{\mathcal{C}}_q$ be the category of all finite-dimensional ℓ-weight modules of $U_q(\widetilde{\mathfrak{g}})$. Thus, a finite-dimensional $U_q(\widetilde{\mathfrak{g}})$-module V is in $\widetilde{\mathcal{C}}_q$ if

$$V = \bigoplus_{\boldsymbol{\omega} \in \mathcal{P}_q} V_{\boldsymbol{\omega}},$$

where

$$v \in V_{\boldsymbol{\omega}} \ \Leftrightarrow \ \text{there exists } k \gg 0 \text{ such that } (\eta - \boldsymbol{\Psi}_{\boldsymbol{\omega}}(\eta))^k v = 0 \text{ for all } \eta \in U_q(\widetilde{\mathfrak{h}}).$$

$V_{\boldsymbol{\omega}}$ is called the ℓ-weight space of V associated to $\boldsymbol{\omega}$. A nonzero vector $v \in V_{\boldsymbol{\omega}}$ is said to be a highest-ℓ-weight vector if

$$\eta v = \boldsymbol{\Psi}_{\boldsymbol{\omega}}(\eta) v \ \text{ for every } \eta \in U_q(\widetilde{\mathfrak{h}}) \quad \text{and} \quad x_{i,r}^+ v = 0 \ \text{ for all } i \in I, \, r \in \mathbb{Z}.$$

V is said to be a highest-ℓ-weight module if it is generated by a highest-ℓ-weight vector. Note that if $V \in \widetilde{\mathcal{C}}_q$, then $V \in \mathcal{C}_q$ and

$$V_\lambda = \bigoplus_{\boldsymbol{\omega}:\mathrm{wt}(\boldsymbol{\omega})=\lambda} V_{\boldsymbol{\omega}}.$$

Given $\boldsymbol{\omega} \in \mathcal{P}_q^+$, the local Weyl module $W_q(\boldsymbol{\omega})$ is the $U_q(\tilde{\mathfrak{g}})$-module generated by a vector w satisfying the defining relations

$$x_{i,r}^+ w = 0, \quad x w = \boldsymbol{\Psi}_{\boldsymbol{\omega}}(x) w, \quad (x_i^-)^{\mathrm{wt}(\boldsymbol{\omega})+1} w = 0$$

for all $i \in I$, $r \in \mathbb{Z}$, $x \in U_q(\tilde{\mathfrak{h}})$. It was proved in [Chari and Pressley 2001] that $W_q(\boldsymbol{\omega}) \in \widetilde{\mathcal{C}}_q$ for every $\boldsymbol{\omega} \in \mathcal{P}_q^+$ and every finite-dimensional highest-ℓ-weight module of highest ℓ-weight $\boldsymbol{\omega}$ is a quotient of $W_q(\boldsymbol{\omega})$. Standard arguments show that $W_q(\boldsymbol{\omega})$ has a unique irreducible quotient, denoted by $V_q(\boldsymbol{\omega})$. In particular, we get the following classification of the simple objects of $\widetilde{\mathcal{C}}_q$.

Theorem 2.2.2. *If V is a simple object of $\widetilde{\mathcal{C}}_q$, then V is isomorphic to $V_q(\boldsymbol{\omega})$ for some $\boldsymbol{\omega} \in \mathcal{P}_q^+$.*   □

Analogous results hold for the category $\mathcal{C}$ of finite-dimensional $\tilde{\mathfrak{g}}$-modules. In particular, given $\boldsymbol{\omega} \in \mathcal{P}^+$, we let $W(\boldsymbol{\omega})$ and $V(\boldsymbol{\omega})$ denote the corresponding local Weyl module and irreducible module, respectively. The following is a corollary of the proof that $W(\boldsymbol{\omega})$ is finite-dimensional.

Proposition 2.2.3. *Suppose V is a highest-ℓ-weight module for $\tilde{\mathfrak{g}}$ and let v be a highest-ℓ-weight vector. Then, $V = U(\mathfrak{n}^-[t])v$.*   □

2.3. Minimal affinizations. Since $\mathcal{C}_q$ is semisimple, for any object $V \in \widetilde{\mathcal{C}}_q$ we have an isomorphism of $U_q(\mathfrak{g})$-modules

$$V \cong \bigoplus_{\mu \in P^+} V_q(\mu)^{\oplus m_\mu(V)}$$

for some $m_\mu(V) \in \mathbb{Z}_{\geq 0}$. We shall refer to the number $m_\mu(V)$ as the multiplicity of $V_q(\mu)$ in V.

Given $\lambda \in P^+$, $V \in \widetilde{\mathcal{C}}_q$ is said to be an affinization of $V_q(\lambda)$ if

$$(2\text{-}3\text{-}1) \qquad m_\lambda(V) = 1 \quad \text{and} \quad m_\mu(V) \neq 0 \quad \Rightarrow \quad \mu \leq \lambda.$$

Two affinizations of $V_q(\lambda)$ are said to be equivalent if they are isomorphic as $U_q(\mathfrak{g})$-modules. The partial order on P^+ induces a natural partial order on the set of (equivalence classes of) affinizations of $V_q(\lambda)$. Namely, if V and W are affinizations of $V_q(\lambda)$, say that $V \leq W$ if, for each $\mu \in P^+$, one of the following conditions holds:

(i) $m_\mu(V) \leq m_\mu(W)$.

(ii) If $m_\mu(V) > m_\mu(W)$, there exists $\nu > \mu$ such that $m_\nu(V) < m_\nu(W)$.

A minimal element of this partial order is said to be a minimal affinization [Chari 1995]. Clearly, a minimal affinization of $V_q(\lambda)$ must be irreducible as a $U_q(\tilde{\mathfrak{g}})$-module and, hence, is of the form $V_q(\boldsymbol{\omega})$ for some $\boldsymbol{\omega} \in \mathcal{P}_q^+$ such that $\mathrm{wt}(\boldsymbol{\omega}) = \lambda$. More generally, any quotient of $W_q(\boldsymbol{\omega})$ is an affinization of $V_q(\lambda)$ provided $\mathrm{wt}(\boldsymbol{\omega}) = \lambda$.

Suppose $\mathfrak{g}$ is of type A_n. It follows from [Jimbo 1986] that $V_q(\lambda)$ has a unique equivalence class of minimal affinizations and $V_q(\omega)$ is a minimal affinization if, and only if, it is an irreducible $U_q(\mathfrak{g})$-module. To describe the elements of $\mathcal{P}_q^+$ with this property, identify I with $\{1, 2, \ldots, n\}$ in such a way that $c_{i,i+1} = -1$ for all $1 \le i < n$ and ω_1 is the highest weight of the standard representation of $\mathfrak{g}$. Given $i, j \in I$, $i \le j$, set

$$_i|\lambda|_j = \sum_{k=i}^{j} \lambda(h_k).$$

If $i = 1$, we may write $|\lambda|_j$ instead of $_1|\lambda|_j$ and similarly if $j = n$. For $i > j$, we set $_i|\lambda|_j = 0$ and, for $i \le j$, define

$$p_{i,j}(\lambda) = p_{j,i}(\lambda) = {}_{i+1}|\lambda|_j + {}_i|\lambda|_{j-1} + j - i.$$

In particular, $p_{i,j}(\lambda) = 0$ if $i = j$ and

$$p_{i,j}(\lambda) = \lambda(h_i) + \lambda(h_j) + 2\,_{i+1}|\lambda|_{j-1} + j - i \quad \text{if} \quad i < j.$$

Theorem 2.3.1 [Chari and Pressley 1994b]. *$V_q(\omega)$ is a minimal affinization of $V_q(\lambda)$ if and only if there exist $a_i \in \mathbb{F}^\times$, $i \in I$, and $\epsilon = \pm 1$ such that*

$$(2\text{-}3\text{-}2) \qquad \omega = \prod_{i \in I} \omega_{i,a_i,\lambda(h_i)} \quad \text{with} \quad \frac{a_i}{a_j} = q^{\epsilon p_{i,j}(\lambda)} \quad \text{for all } i < j. \qquad \square$$

Notice that (2-3-2) is equivalent to saying that there exist $a \in \mathbb{F}^\times$ and $\epsilon = \pm 1$ such that

$$(2\text{-}3\text{-}3) \qquad \omega = \prod_{i \in I} \omega_{i,a_i,\lambda(h_i)} \quad \text{with} \quad a_i = aq^{\epsilon p_{i,n}(\lambda)} \quad \text{for all } i \in I.$$

If $\#\operatorname{supp}(\lambda) > 1$, the pair (a, ϵ) in (2-3-3) is unique. In that case, if ω satisfies (2-3-2) with $\epsilon = 1$, we say that $V_q(\omega)$ is a decreasing minimal affinization. Otherwise, we say $V_q(\omega)$ is an increasing minimal affinization. If $\#\operatorname{supp}(\lambda) = 1$, ω can be represented in the form (2-3-3) by two choices of pairs (a, ϵ), one for each value of ϵ. We do not fix a preferred presentation in that case. We consider ω to be simultaneously increasing and decreasing if $\#\operatorname{supp}(\lambda) \le 1$.

Assume now that $\mathfrak{g}$ is of type D or E, let i_* be the trivalent node, $\omega \in \mathcal{P}_q^+$ and $\lambda = \operatorname{wt}(\omega)$. We will say that ω is preminimal if $V_q(\omega_J)$ is a minimal affinization for any connected subdiagram J of type A such that $J \setminus \{i_*\}$ is connected. It was proved in [Chari and Pressley 1996a, Proposition 4.2] that, if $V_q(\omega)$ is a minimal affinization, then ω is preminimal. Henceforth, we assume ω is preminimal. It will be proved in Lemma 3.2.4 that

$$(2\text{-}3\text{-}4) \quad m_\mu(V_q(\omega)) > 0 \quad \Rightarrow \quad J_\mu \text{ is connected, where } J_\mu = \operatorname{rsupp}(\lambda - \mu).$$

Given $i \in \partial I$, let

$$I_i = \overline{\partial I \setminus \{i\}}.$$

Thus, I_i is the maximal connected subdiagram of type A which does not contain i, $i_* \in I_i$, and $I_i \setminus \{i_*\}$ is disconnected. We will say that $\boldsymbol{\omega}$ is i-minimal if $V_q(\boldsymbol{\omega}_{I_i})$ is a minimal affinization. Define the minimality order of $\boldsymbol{\omega}$ as

$$\mathrm{mo}(\boldsymbol{\omega}) = \#\{i \in \partial I : V_q(\boldsymbol{\omega}) \text{ is } i\text{-minimal}\}.$$

If $\boldsymbol{\omega}$ is preminimal of minimality order k, we shall simply say $\boldsymbol{\omega}$ is preminimal of order k. The minimality order of $V_q(\boldsymbol{\omega})$ is set to be $\mathrm{mo}(\boldsymbol{\omega})$. Note that, if $\mathrm{mo}(\boldsymbol{\omega}) = 3$, then $V_q(\boldsymbol{\omega}_J)$ is a minimal affinization for every connected subdiagram J of type A. One easily checks using Theorem 2.3.1 that

$$(2\text{-}3\text{-}5) \qquad \mathrm{mo}(\boldsymbol{\omega}) = 3 \quad \Rightarrow \quad \overline{\mathrm{supp}}(\lambda) \text{ is of type } A \quad \Rightarrow \quad \mathrm{mo}(\boldsymbol{\omega}) \geq 2.$$

The next theorem was proved in [Chari and Pressley 1996a; 1996b].

Theorem 2.3.2. (a) *If $\overline{\mathrm{supp}}(\lambda)$ is of type A, then $V_q(\boldsymbol{\omega})$ is a minimal affinization if and only if $\boldsymbol{\omega}$ is preminimal of order 3. In particular, $V_q(\lambda)$ has a unique equivalence class of minimal affinizations.*

  (b) *If $\overline{\mathrm{supp}}(\lambda)$ is not of type A and $\lambda(h_{i_*}) \neq 0$, then $V_q(\boldsymbol{\omega})$ is a minimal affinization if and only if $\boldsymbol{\omega}$ is preminimal of order 2. In particular, $V_q(\lambda)$ has three equivalence classes of minimal affinizations.*

  (c) *If $\mathfrak{g}$ is of type D_4, $\overline{\mathrm{supp}}(\lambda)$ is not of type A, $\lambda(h_{i_*}) = 0$, and $V_q(\boldsymbol{\omega})$ is a minimal affinization, the order of $\boldsymbol{\omega}$ is either 2 or 1. Moreover, the number of equivalence classes of minimal affinizations of order 1 grows unboundedly with λ.* □

Outside type A, a minimal affinization is typically not irreducible as a $U_q(\mathfrak{g})$-module even under the assumption of part (a) of Theorem 2.3.2. If λ satisfies the hypothesis of either part (a) or (b) of Theorem 2.3.2 it is said to be regular. Otherwise it is said to be irregular. If $\mathfrak{g}$ is of type D and λ is regular, the characters of the minimal affinizations were computed in [Naoi 2013; 2014] in terms of Demazure operators.

2.4. *Coherent and incoherent affinizations.* Assume $\mathfrak{g}$ is of type D or E. For a connected subdiagram $J \subseteq I$ of type A, we shall say that a total ordering $<$ on J is monotonic if, for all $i, k \in J$, we have

$$c_{i,k} = -1 \text{ and } i < k \quad \Rightarrow \quad \{j \in J : i < j < k\} = \varnothing.$$

For each such subdiagram there are exactly two choices of monotonic orderings and the maximum and minimum of a monotonic ordering belong to ∂J. Each monotonic ordering on J with the corresponding order-preserving identification of J with $\{1, \ldots, n\}$, $n = \#J$, induces an isomorphism of algebras

$$U_q(\tilde{\mathfrak{g}}_J) \cong U_q(\tilde{\mathfrak{sl}}_{n+1}).$$

Then, given $\boldsymbol{\omega} \in \mathcal{P}_q^+$ such that $V_q(\boldsymbol{\omega}_J)$ is a minimal affinization, we shall say that $\boldsymbol{\omega}_J$ is increasing or decreasing, with respect to the given ordering, if $V_q(\boldsymbol{\omega}_J)$ is an increasing or a decreasing minimal affinization for $U_q(\tilde{\mathfrak{sl}}_{n+1})$ after pulling back the action by the above isomorphism. If J' is another connected subdiagram of type A, we shall say that a choice of monotonic orderings on J and J' is coherent if they coincide on $J \cap J'$. In that case, we shall say that J and J' are coherently ordered. Evidently, given any two intersecting such diagrams, there exists at least one choice of coherent orderings. The following lemma is also easily established.

Lemma 2.4.1. *Let J, J' be coherently ordered connected subdiagrams of type A. Suppose $\boldsymbol{\omega} \in \mathcal{P}_q^+$ is such that both $V_q(\boldsymbol{\omega}_J)$ and $V_q(\boldsymbol{\omega}_{J'})$ are minimal affinizations and that*

$$\#(\mathrm{supp}(\mathrm{wt}(\boldsymbol{\omega})) \cap J \cap J') > 1.$$

Then, $\boldsymbol{\omega}_J$ is increasing if, and only if, $\boldsymbol{\omega}_{J'}$ is increasing. □

Note that the assumption on the cardinality is essential in the above lemma.

Suppose $\boldsymbol{\omega} \in \mathcal{P}_q^+$ is preminimal of order 2, let $k \in \partial I$ be the node such that $\boldsymbol{\omega}$ is not k-minimal, and choose coherent monotonic orderings on $I_l, l \in \partial I \setminus \{k\}$. We shall say that $\boldsymbol{\omega}$ is coherent if $V_q(\boldsymbol{\omega}_{I_l})$, $l \neq k$, are either both increasing or both decreasing minimal affinizations. Otherwise, we say that $\boldsymbol{\omega}$ is incoherent. Note that the property of being coherent is intrinsic to $\boldsymbol{\omega}$, i.e., it does not depend on the choice of coherent monotonic orderings on I_l, $l \in \partial I_k$. Moreover, it follows from the previous lemma that

$$(2\text{-}4\text{-}1) \qquad \boldsymbol{\omega} \text{ incoherent} \quad \Rightarrow \quad \#(\mathrm{supp}(\boldsymbol{\omega}) \cap [k, i_*]) \leq 1.$$

We can graphically represent coherent Drinfeld polynomials by the pictures

$$(2\text{-}4\text{-}2)$$
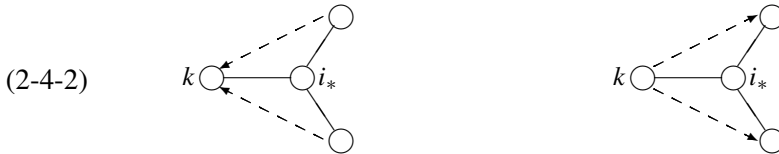
where the first means it is decreasing towards k and the second means it is decreasing away from k. Similarly, incoherent Drinfeld polynomials can be represented by

$$(2\text{-}4\text{-}3)$$
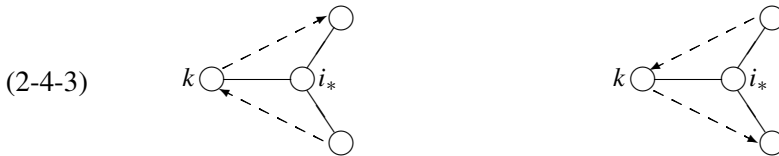
These pictures are inspired by those in the main theorem of [Chari and Pressley 1996b]. More involved pictures appear in the main theorem of [Pereira 2014].

Conjecture 2.4.2. *Let $\omega \in \mathcal{P}_q^+$ be preminimal of order 2 and let $k \in \partial I$ be the node such that ω is not k-minimal. If $\#(\mathrm{supp}(\mathrm{wt}(\omega)) \cap [i_*, k]) \le 1$ and ω is coherent, $V_q(\omega)$ is not a minimal affinization.*   ◇

Remark 2.4.3. Note that, if $\mathrm{supp}(\mathrm{wt}(\omega)) \subseteq I_k$, this conjecture follows from part (a) of Theorem 2.3.2, since the minimal affinizations have minimality order 3. In that case, note that, if $\#(\mathrm{supp}(\mathrm{wt}(\omega)) \cap I_l) \ge 2$ for $l \in \partial I_k$, a graphic representation of the Drinfeld polynomial of the minimal affinizations follows the picture (2-4-3). In other words, we can informally say that the minimal affinizations are incoherent, even though the notion is not defined when the minimality order is 3.

It follows from the above paragraph that it remains to prove the conjecture when $\#(\mathrm{supp}(\mathrm{wt}(\omega)) \cap [i_*, k]) = 1$ and $i_* \notin \mathrm{supp}(\mathrm{wt}(\omega))$. If $\#(\mathrm{supp}(\mathrm{wt}(\omega)) \cap [i_*, k]) > 1$, the conclusion of the conjecture is false by part (b) of Theorem 2.3.2. In fact, in the context of Theorem 2.3.2(b), $V_q(\omega)$ is a minimal affinization if and only if ω is coherent. We shall see in a forthcoming publication that the conclusion of the conjecture remains false if $\#(\mathrm{supp}(\mathrm{wt}(\omega)) \cap [i_*, k]) > 1$ and $i_* \notin \mathrm{supp}(\mathrm{wt}(\omega))$ (see also [Pereira 2014]). Note that this situation is realizable only if $\#I > 4$.   ◇

We will prove that the conclusion of Conjecture 2.4.2 holds under certain extra hypotheses if $\mathfrak{g}$ is of type E. To state them, we introduce the following notation. Given $\lambda \in P^+$ and $i \in \partial I$, if $\mathrm{supp}(\lambda) \cap (i_*, i] \ne \varnothing$, let $i_\lambda \in \mathrm{supp}(\lambda) \cap (i_*, i]$ be the element which is closest to i_*. Otherwise, set $i_\lambda = i$. Set also

$$I^\lambda = \overline{\{i_\lambda : i \in \partial I\}} \quad \text{and} \quad I_i^\lambda = I_i \cap I^\lambda \quad \text{for } i \in \partial I.$$

Note $i_\lambda \in I_j^\lambda$ if, and only if, $j \ne i$. The main result of this paper is:

Theorem 2.4.4. *Let $\omega \in \mathcal{P}_q^+$ be preminimal of order 2 and let $k \in \partial I$ be the node such that ω is not k-minimal. Set $\lambda = \mathrm{wt}(\omega)$ and assume $\#(\mathrm{supp}(\lambda) \cap [i_*, k]) = 1$, $i_* \notin \mathrm{supp}(\lambda)$, and ω is coherent. Then, $V_q(\omega)$ is not a minimal affinization provided either one of the following hypothesis holds*:

(i) $\mathfrak{g}$ *is of type D.*

(ii) I^λ *is of type D_4 and $d(k, i_*) > 1$.*

(iii) $\mathfrak{g}$ *is of type E_6 and $\mathrm{supp}(\lambda) = \partial I$.*

More precisely, $V_q(\omega) > V_q(\varpi)$ for any $\varpi \in \mathcal{P}_q^+$ which is preminimal of order 2, not k-minimal, incoherent, and such that $\mathrm{wt}(\varpi) = \lambda$.   ◇

Remark 2.4.5. This completes the classification of minimal affinizations for $\mathfrak{g}$ of type D_4. Namely, it was proved in [Chari and Pressley 1996b] that, if $V_q(\omega)$ is an irregular minimal affinization, then it must belong to three explicitly described families of preminimal affinizations (each family contains more than one equivalence class of affinizations). One of these families consists of preminimal affinizations

of order 1. This is family (c) in the notation of Chari and Pressley, and they show
that the elements belonging to this class are minimal affinizations. Moreover, they
are not comparable to any element of the other two families which are formed
by the preminimal affinizations of order 2: the coherent and incoherent families;
(a) and (b) in the notation of Chari and Pressley. They conjectured [1996b] that
a given coherent preminimal affinization was equivalent to its incoherent coun-
terpart. This would imply that all members of all three families were minimal
affinizations, thus completing the classification. Theorem 2.4.4 shows that the
coherent preminimal affinizations listed in [Chari and Pressley 1996b] are actually
not minimal affinizations. In fact, the proof of Theorem 2.4.4 will show that a given
coherent preminimal affinization is strictly larger than its incoherent counterpart
in the partial order of affinizations. Thus, all elements of the incoherent family
are minimal affinizations. The classification of irregular minimal affinizations for
type D_4 can then be summarized as: the three equivalent classes of incoherent
preminimal affinizations of order 2 together with the preminimal affinizations of
order 1 listed in family (c) of the main theorem of [Chari and Pressley 1996b].
See also Remark 4.2.4 for comments related to the structure of these affinizations
including an explanation of the erroneous announcement about the correctness of
the conjecture from [Chari and Pressley 1996b] made in [Moura 2010].   ◇

The proof of Theorem 2.4.4, given in Section 4.5, relies on tensor product results
from [Chari 2002; Moura and Pereira 2017], which will be reviewed in Section 3.4,
and on the computation of certain outer multiplicities for preminimal affinizations
satisfying

$$(2\text{-}4\text{-}4) \qquad\qquad \mathrm{mo}(\boldsymbol{\omega}_{I^\lambda}) = 2 \quad \text{and} \quad \lambda(h_{i_*}) = 0,$$

where $\lambda = \mathrm{wt}(\boldsymbol{\omega})$, which we now explain. Thus, let $\boldsymbol{\omega} \in \mathcal{P}^+$ be preminimal satisfy-
ing (2-4-4), let $k \in \partial I$ be such that

$$(2\text{-}4\text{-}5) \qquad\qquad \boldsymbol{\omega}_{I^\lambda} \text{ is not } k_\lambda\text{-minimal,}$$

and set

$$V = V_q(\boldsymbol{\omega}).$$

One easily checks that $\mathrm{supp}(\lambda)$ intersects both connected components of $I_k \setminus \{i_*\}$.
Notice however that we are allowing the possibility

$$\mathrm{supp}(\lambda) \cap [k, i_*] = \varnothing,$$

in which case $k_\lambda = k$. Recall (2-1-2) and set

$$(2\text{-}4\text{-}6) \qquad\qquad \nu = \lambda - \vartheta_{I^\lambda} \quad \text{and} \quad \nu_l = \lambda - \vartheta_{I_l^\lambda} \quad \text{for } l \in \partial I.$$

The following proposition will be crucial in the proof of Theorem 2.4.4.

Proposition 2.4.6. *In the above notation, we have*:

(a) *Let $l \in \partial I$. Then, $m_{\nu_l}(V) = \delta_{l,k}$ and $m_{\lambda - \alpha_{l_\lambda}}(V) = 0$ if $l_\lambda \in \mathrm{supp}(\lambda)$.*

(b) *If $\mu \in P^+$ satisfies $v < \mu < \lambda$ and $m_\mu(V) > 0$, then $\mu = v_k$.*

(c) *If ω_{I^λ} is coherent and $k_\lambda \in \mathrm{supp}(\lambda)$, $m_v(V) = 1$.*

(d) *If ω_{I^λ} is incoherent and $k_\lambda \in \mathrm{supp}(\lambda)$, $m_v(V) = 0$.*   ◇

Since ω_{I^λ} is l-minimal for $l \neq k$ and I_l is of type A, the equality $m_{v_l}(V) = 0$ is immediate from the well-known Lemma 3.2.1 below which also implies $m_{\lambda - \alpha_{l_\lambda}}(V) = 0$ for all $l \in \partial I$ such that $l_\lambda \in \mathrm{supp}(\lambda)$. The remaining statement of part (a) (the equality $m_{v_k}(V) = 1$) will be proved in Section 4.1.

Using part (a), part (b) is then easily proved as follows. The condition $\mu > v$, together with (2-3-4), implies that $\mu = \lambda - \vartheta_J$ for some connected subdiagram J properly contained in I^λ. One easily checks that, for such J, we have

(2-4-7) $\lambda - \vartheta_J \in P^+ \iff J = [l_\lambda, m_\lambda]$ with $l, m \in \partial I$, $l_\lambda, m_\lambda \in \mathrm{supp}(\lambda)$.

Hence, $\mu = v_l$ for some $l \in \partial I$ or $\mu = \lambda - \alpha_{l_\lambda}$ with $l_\lambda \in \mathrm{supp}(\lambda)$ and part (a) implies $\mu = v_k$.

Note that parts (a) and (b) of Proposition 2.4.6 imply that

$$m_v(V) = \dim(V_v) - \dim(V_q(\lambda)_v) - \dim(V_q(v_k)_v).$$

Hence, proving parts (c) and (d) is equivalent to proving that

(2-4-8) $\dim(V_v) = \dim(V_q(\lambda)_v) + \dim(V_q(v_k)_v) + \xi,$

where $\xi = 1$ for part (c) and $\xi = 0$ for (d). The proof of (2-4-8) will be given in Sections 4.3 and 4.4 using qcharacter theory.

Part of the hypotheses on Theorem 2.4.4 is explained by the following lemma. The remaining hypotheses are needed so that we remain within the cases covered by Theorem 3.4.1 presently and may be dropped once a more general version of that result is obtained.

Lemma 2.4.7. *Let V be as in Proposition 2.4.6 and assume one of the following:*

(i) $\mathfrak{g}$ *is of type D.*

(ii) I_k^λ *is of type A_3.*

(iii) $\mathfrak{g}$ *is of type E_6 and $\mathrm{supp}(\lambda) = \partial I$.*

If $\mu \in P^+$ satisfies $m_\mu(V) > 0$ and $\mu \neq \lambda$, then $\mu \leq v_k$.   ◇

The conclusion of this lemma is false outside these hypotheses. However, although we use it in a strong manner in the proof of Theorem 2.4.4, these hypotheses may not be necessary conditions for the validity of Conjecture 2.4.2. In fact, we believe the approach we use here can be carried out in broader generality and the first step is to replace this lemma by a characterization of the maximal elements of the set $\{\mu \in P^+ : \mu < \lambda, m_\mu(V) > 0\}$. We will address this in a future work.

In the case that hypothesis (ii) holds, Lemma 2.4.7 is a simple consequence of Lemma 3.2.4. More generally, it will be proved as a consequence of Lemma 4.2.3.

2.5. *Graded limits.* Let $\lambda \in P^+$, and suppose $\boldsymbol{\omega} \in \mathcal{P}_q^+$ is of the form

$$(2\text{-}5\text{-}1) \qquad \boldsymbol{\omega} = \prod_{i \in I} \boldsymbol{\omega}_{i,a_i,\lambda(h_i)} \quad \text{for some } a_i \in \mathbb{F}^{\times}.$$

Suppose further that $a_i/a_j \in q^{\mathbb{Z}}$ for all $i, j \in I$ (which is the case if $V_q(\boldsymbol{\omega})$ is a minimal affinization). In that case, there exists a $\mathfrak{g}[t]$-module $L(\boldsymbol{\omega})$, referred to as the graded limit of $V_q(\boldsymbol{\omega})$, satisfying

$$(2\text{-}5\text{-}2) \qquad \operatorname{ch}(L(\boldsymbol{\omega})) = \operatorname{ch}(V_q(\boldsymbol{\omega})).$$

The construction of $L(\boldsymbol{\omega})$ and the related literature will be revised in Section 3.5.

The graded local Weyl module of highest weight λ is the $\mathfrak{g}[t]$-module $W(\lambda)$ generated by a vector w satisfying the defining relations

$$\mathfrak{n}^+[t]w = \mathfrak{h}[t]_+ w = 0, \quad hw = \lambda(h)w, \quad (x_i^-)^{\lambda(h_i)+1}w = 0$$

for all $h \in \mathfrak{h}$ and $i \in I$. It is known that $W(\lambda)$ is finite-dimensional and any finite-dimensional graded $\mathfrak{g}[t]$-module generated by a highest-weight vector of weight λ is a quotient of $W(\lambda)$ (see [Chari et al. 2010]).

Assume $\mathfrak{g}$ is of type D or E and, for $k \in \partial I$, let $M_k(\lambda)$ be the quotient of $W(\lambda)$ by the submodule generated by

$$(2\text{-}5\text{-}3) \qquad x_{\vartheta_J,1}^- w \quad \text{with } J \subseteq I_i, \text{ for all } i \in \partial I_k.$$

The following lemma will be proved in Section 3.5.

Lemma 2.5.1. *Let $\boldsymbol{\omega} \in \mathcal{P}_q^+$ and suppose $k \in \partial I$ is such that $V_q(\boldsymbol{\omega}_{I_i})$ is a minimal affinization for $i \in \partial I_k$. Then, $L(\boldsymbol{\omega})$ is a quotient of $M_k(\lambda)$. In particular,*

$$m_\mu(V_q(\boldsymbol{\omega})) \le m_\mu(M_k(\lambda)) \quad \text{for all } \mu \in P^+. \qquad \diamond$$

Conjecture 2.5.2. *Let $\boldsymbol{\omega} \in \mathcal{P}_q^+$ be preminimal of order 2 and let $k \in \partial I$ be such that $\boldsymbol{\omega}$ is not k-minimal. If $\boldsymbol{\omega}$ is coherent and $\operatorname{supp}(\lambda) \cap [k, i_*] \ne \varnothing$, then $L(\boldsymbol{\omega}) \cong M_k(\lambda)$. $\diamond$*

This is a partial rephrasing of a conjecture from [Moura 2010] which was proved therein for $\mathfrak{g}$ of type D_4 and was proved in [Naoi 2013; 2014] in the case that $\mathfrak{g}$ is of type D, λ is regular, and $V_q(\boldsymbol{\omega})$ is a minimal affinization. The proof for type D_4 given in [Moura 2010] depends only on the hypothesis that $\boldsymbol{\omega}$ is coherent, regardless of whether λ is irregular or whether $V_q(\boldsymbol{\omega})$ is a minimal affinization and, as far as we can tell, the same should be true for the proof given in [Naoi 2013; 2014]. In particular, these proofs also provide formulas for computing the graded character of $L(\boldsymbol{\omega})$. Parts (a) and (c) of Proposition 2.4.6 as well as Lemma 2.4.7 in the case

where $\mathfrak{g}$ is of type D and ω is coherent can then be deduced from such computations. However, one needs much less information about the graded character to prove these statements. Namely, after Lemma 2.5.1, it suffices to prove

$$(2\text{-}5\text{-}4) \qquad m_\mu(M_k(\lambda)) \neq 0 \quad \Rightarrow \quad \mu \leq \nu_k,$$

$$(2\text{-}5\text{-}5) \quad \mu \in \{\nu_k, \nu\}, \ k_\lambda \in \text{supp}(\lambda) \quad \Rightarrow \quad m_\mu(M_k(\lambda)) = 1 \ \text{ and } \ m_\mu(L(\omega)) \neq 0.$$

We will actually prove the following slightly stronger result.

For a graded vector space V, let $V[s]$ be the s-th graded piece. If V is a graded $\mathfrak{g}[t]$-module, then $V[s]$ is a $\mathfrak{g}$-submodule of V for every s and we set

$$m_\mu^s(V) = m_\mu(V[s]), \quad \mu \in P^+.$$

Proposition 2.5.3. *Let $\omega \in \mathcal{P}_q^+$ be preminimal of order 2 and let $k \in \partial I$ be such that ω is not k-minimal. We have*:

(a) $m_{\nu_k}^s(M_k(\lambda)) = \delta_{s,1}$ *and, if $k_\lambda \in \text{supp}(\lambda)$, $m_\nu^s(M_k(\lambda)) = \delta_{s,1}$.*

(b) $m_{\nu_k}(L(\omega)) \neq 0$ *and, if ω is coherent and $k_\lambda \in \text{supp}(\lambda)$, then $m_\nu(L(\omega)) \neq 0$.*

(c) *Under the hypotheses of Lemma 2.4.7, if $\mu \in P^+$ satisfies $m_\mu^s(M_k(\lambda)) > 0$ for some $s \in \mathbb{Z}$ and $\mu < \lambda$ then $\mu \leq \nu_k$.*   ◇

The first statements in parts (a) and (b) as well part (c) will be proved in Section 4.2. The second equality in part (a) is a consequence of the second statement of part (b) together with Lemma 2.5.1 and (4-2-6). The second statement of part (b) is a consequence of Proposition 2.4.6(c) which will be proved in Section 4.4 (see also Remark 4.2.4).

It follows from Proposition 2.5.3 that

$$(2\text{-}5\text{-}6) \qquad M_k(\lambda) \cong_{\mathfrak{g}} V(\lambda) \oplus V(\nu_k) \oplus N \oplus \bigoplus_{\mu < \nu_k, \, \mu \nleq \nu} V(\mu)^{\oplus m_\mu(M_k(\lambda))},$$

where

$$N \cong \begin{cases} V(\nu) & \text{if } k_\lambda \in \text{supp}(\lambda), \\ 0 & \text{otherwise.} \end{cases}$$

Moreover,

$$V(\nu_k) \oplus N \subseteq M_k(\lambda)[1].$$

Let $N_k(\lambda)$ be the quotient of $M_k(\lambda)$ by the $\mathfrak{g}[t]$-submodule generated by N. In light of the above results, part (d) of Proposition 2.4.6 becomes equivalent to the following lemma.

Lemma 2.5.4. *Assume ω is incoherent and not k-minimal. Then, under the hypothesis of Proposition 2.4.6, $L(\omega)$ is a quotient of $N_k(\lambda)$.*   ◇

Remark 2.5.5. The theory of $\mathfrak{g}$-stable Demazure modules plays a prominent role in the study of graded limits of minimal affinizations. In [Li and Naoi 2016; Naoi 2013; 2014], it has been proved that the graded limits of minimal affinizations $\mathfrak{g}$ of classical type or G_2 with regular highest weight are generalized Demazure modules. It appears to us that this is no longer the case for the incoherent minimal affinizations as the simplest case does not appear to be even a Chari–Venkatesh module. Understanding the structure of the module $N_k(\lambda)$, which is most likely isomorphic to the graded limit of the incoherent minimal affinizations, from the point of view of Demazure theory is certainly a topic that must be investigated. We shall come back to this in the future. ◇

3. Technical background

In this section we review the technical background we shall need for proving Proposition 2.4.6 and Theorem 2.4.4.

3.1. *On the dimensions of certain weight spaces.* Let $p : Q \to \mathbb{Z}$ be Kostant's partition function. In other words, $p(\eta)$ is the number of ways of writing η as a sum of positive roots or, equivalently,

$$p(\eta) = \#\mathcal{P}_\eta,$$

where

$$\mathcal{P}_\eta = \left\{ \xi : R^+ \to \mathbb{Z}_{\geq 0} : \eta = \sum_{\alpha \in R^+} \xi(\alpha)\,\alpha \right\}.$$

In particular, $p(\alpha_i) = 1$ for all simple roots and $\eta \in Q \setminus Q^+ \Rightarrow p(\eta) = 0$. The Poincaré–Birkhoff–Witt (PBW) theorem implies $\dim(U(\mathfrak{n}^+)_\eta) = p(\eta)$. In particular, for $\lambda \in \mathfrak{h}^*$ and $M(\lambda)$ the Verma module of highest-weight λ, we have

$$\dim(M(\lambda)_{\lambda-\eta}) = p(\eta).$$

In the proof of (2-4-8), we will use a similarly flavored formula which applies to $V(\lambda)$, $\lambda \in P^+$, for certain $\eta \in Q^+$.[1] Thus, consider

$$\mathcal{P}_\eta^\lambda = \{\xi \in \mathcal{P}_\eta : \alpha \in \mathrm{supp}(\xi) \Rightarrow \mathrm{rsupp}(\alpha) \cap \mathrm{supp}(\lambda) \neq \varnothing\}$$

where

$$\mathrm{supp}(\xi) = \{\alpha \in R^+ : \xi(\alpha) \neq 0\}.$$

Let v be a highest-weight vector for $V(\lambda)$ and recall that, for all subdiagram $J \subseteq I$,

$$(3\text{-}1\text{-}1) \qquad J \cap \mathrm{supp}(\lambda) = \varnothing \ \text{ and } \ \mathrm{rsupp}(\alpha) \subseteq J \quad \Rightarrow \quad x_\alpha^- v = 0.$$

---

[1]Although we are assuming throughout the text that $\mathfrak{g}$ is simply laced, Section 3.1 is valid in complete generality with no need of modifications in the text.

A straightforward application of the PBW theorem then gives

$$(3\text{-}1\text{-}2) \qquad \dim(V(\lambda)_{\lambda-\eta}) \leq \#\mathcal{P}_\eta^\lambda \quad \text{for all } \lambda \in P^+, \eta \in Q.$$

Proposition 3.1.1. *If $J \subseteq I$ is connected and $\lambda \in P^+$ satisfies $\mathrm{supp}(\lambda) \cap J \subseteq \partial J$, then*

$$\dim(V(\lambda)_{\lambda-\vartheta_J}) = \#\mathcal{P}_{\vartheta_J}^\lambda.$$

Proof. Since $\dim(V(\lambda)_{\lambda-\eta}) = \dim(V(\lambda_J)_{\lambda_J-\eta_J})$ if $\eta \in Q_J$, we may assume $J = I$. It is well known that we have an isomorphism of $\mathfrak{n}^-$-modules

$$V(\lambda) \cong U(\mathfrak{n}^-)/U_\lambda \quad \text{with} \quad U_\lambda = \sum_{i \in I} U(\mathfrak{n}^-)(x_i^-)^{\lambda(h_i)+1}.$$

Setting

$$\mathfrak{n}_\lambda^- = \bigoplus_{\alpha \in R^+ : \, \mathrm{rsupp}(\alpha) \cap \mathrm{supp}(\lambda) \neq \varnothing} \mathfrak{g}_{-\alpha},$$

it follows from the PBW theorem that we have an isomorphism of vector spaces

$$V(\lambda)_{\lambda-\vartheta} \cong (U(\mathfrak{n}_\lambda^-)/U_\lambda')_{-\vartheta} \quad \text{with} \quad U_\lambda' = \sum_{i \in I : \, \lambda(h_i) \neq 0} U(\mathfrak{n}_\lambda^-)(x_i^-)^{\lambda(h_i)+1}.$$

Since $(U_\lambda')_{-\vartheta} = 0$ and $\dim(U(\mathfrak{n}_\lambda^-)_{-\vartheta}) = \#\mathcal{P}_\vartheta^\lambda$, the proposition follows. □

Let us make explicit all possible values of $\#\mathcal{P}_{\vartheta_J}^\lambda$. As in the proof of the proposition, to simplify notation, we assume $J = I$ and, hence, $\mathrm{supp}(\lambda) \subseteq \partial I$. In that case,

$$(3\text{-}1\text{-}3) \qquad \#\mathcal{P}_\vartheta^\lambda = \begin{cases} 1 & \text{if } \#\mathrm{supp}(\lambda) = 1, \\ \#\overline{\mathrm{supp}}(\lambda) & \text{if } \#\mathrm{supp}(\lambda) = 2, \\ 3(n-2)+1 & \text{if } \#\mathrm{supp}(\lambda) = 3 \text{ and } \mathfrak{g} \text{ is of type } D_n, \\ 4(n-2)-2 & \text{if } \#\mathrm{supp}(\lambda) = 3 \text{ and } \mathfrak{g} \text{ is of type } E_n. \end{cases}$$

To prove this, we will explicitly describe the elements of $\mathcal{P}_\vartheta^\lambda$. Notice that

$$\xi \in \mathcal{P}_\vartheta^\lambda \quad \Rightarrow \quad \#\mathrm{supp}(\xi) \leq \#\mathrm{supp}(\lambda) \text{ and } \xi(\alpha) \leq 1 \text{ for all } \alpha \in R^+.$$

Therefore, in order to describe ξ, it suffices to describe its support. If $\#\mathrm{supp}(\lambda) = 1$, the unique element $\xi \in \mathcal{P}_\vartheta^\lambda$ is characterized by $\mathrm{supp}(\xi) = \{\vartheta\}$. If $\mathrm{supp}(\lambda) = \{k, l\}$ with $k \neq l$, then, for each $i \in [k, l]$ let ξ_i be the element whose support is

$$\{\vartheta_{[k,i]}, \vartheta_{(i,l]}\} \setminus \{0\}.$$

One easily checks that $\mathcal{P}_\vartheta^\lambda = \{\xi_i : i \in [k, l]\}$, which proves (3-1-3) in this case. Finally, assume $\#\mathrm{supp}(\lambda) = 3$ and write $\partial I = \{k, l, m\}$ such that $\{m\}$ is a connected component of $I \setminus \{i_*\}$ and $\#[l, i_*] \leq \#[k, i_*]$. In particular, $I_m = I \setminus \{m\} = [k, l]$

and, for type D, l and m are spin nodes. For any connected subdiagram $I' \subseteq I$ and $i \in I'$, set

$$\mathscr{P}(I') = \{J \subseteq I' : J \text{ is connected}\},$$
$$\mathscr{P}_i(I') = \{J \in \mathscr{P}(I') : i \in J\},$$
$$\mathscr{P}_i^o(I') = \mathscr{P}_i(I') \cup \{\varnothing\}.$$

In the case that $I' = I$ we may simply write $\mathscr{P}$ and $\mathscr{P}_i$. Note that

(3-1-4) $\#\mathscr{P}_k(I_m) = n - 1$ and $\#\mathscr{P}_k^o([k, i_*)) = \begin{cases} n - 2 & \text{for type } D, \\ n - 3 & \text{for type } E. \end{cases}$

Given $J \in \mathscr{P}_k(I_m)$, let ξ_J be determined by

$$\text{supp}(\xi_J) = \{\alpha_m, \vartheta_J, \vartheta_{I_m \setminus J}\} \setminus \{0\}.$$

Given $J \in \mathscr{P}_k^o([k, i_*))$, let ξ_J' and ξ_J'' be determined by

$$\text{supp}(\xi_J') = \{\vartheta_J, \vartheta_{I \setminus [k,m]}, \vartheta_{[k,m] \setminus J}\} \setminus \{0\} \quad \text{and} \quad \text{supp}(\xi_J'') = \{\vartheta_J, \vartheta_{I \setminus J}\} \setminus \{0\}.$$

One easily checks that the elements ξ_J, $\xi_{J'}'$, $\xi_{J'}''$, $J \in \mathscr{P}_k(I_m)$, $J' \in \mathscr{P}_k^o([k, i_*))$ are all distinct. Moreover, if $\mathfrak{g}$ is of type D, then

$$\mathcal{P}_\vartheta^\lambda = \{\xi_J, \xi_{J'}', \xi_{J'}'' : J \in \mathscr{P}_k(I_m), J' \in \mathscr{P}_k^o([k, i_*))\},$$

which proves (3-1-3). Consider also ξ_J''', $J \in \mathscr{P}_k^o([k, i_*))$, determined by

$$\text{supp}(\xi_J''') = \{\vartheta_J, \alpha_l, \vartheta_{I \setminus (J \cup \{l\})}\} \setminus \{0\}.$$

If $\mathfrak{g}$ is of type D, we have $\xi_J' = \xi_J'''$ for all $J \in \mathscr{P}_k^o([k, i_*))$. However, for type E, these are actually new elements and one easily checks that

$$\mathcal{P}_\vartheta^\lambda = \{\xi_J, \xi_{J'}', \xi_{J'}'', \xi_{J'}''' : J \in \mathscr{P}_k(I_m), J' \in \mathscr{P}_k^o([k, i_*))\},$$

completing the proof of (3-1-3).

It will be useful to compare $\dim(V(\lambda)_{\lambda - \vartheta})$ with $\dim(W_{\lambda - \vartheta})$ where

(3-1-5) $$W = \bigotimes_{i \in \partial I} V(\lambda_i), \qquad \lambda_i = \lambda(h_i)\omega_i,$$

and we keep assuming $\text{supp}(\lambda) \subseteq \partial I$. We will see that

(3-1-6) $\dim(W_{\lambda - \vartheta}) = \dim(V(\lambda)_{\lambda - \vartheta}) + m$ with $m = \begin{cases} 0 & \text{if } \#\text{supp}(\lambda) = 1, \\ 1 & \text{if } \#\text{supp}(\lambda) = 2, \\ n + 1 & \text{if } \#\text{supp}(\lambda) = 3. \end{cases}$

Let $\mathscr{J}^\lambda$ be the set of families $J = (J_i)_{i \in \text{supp}(\lambda)}$ of disjoint connected subdiagrams of I satisfying

$$i \notin J_i \quad \Leftrightarrow \quad J_i = \varnothing$$

and, given $\eta \in Q$, set

$$\mathscr{I}_\eta^\lambda = \left\{ J \in \mathscr{I}^\lambda : \eta = \sum_{i \in \mathrm{supp}(\lambda)} \vartheta_{J_i} \right\}.$$

One easily sees that

(3-1-7) $\mathrm{ht}_i(\eta) \le 1$ for all $i \in I$ $\Rightarrow$ $\dim(W_{\lambda-\eta}) = \# \mathscr{I}_\eta^\lambda.$

Consider the map $\Psi : \mathscr{I}_\vartheta^\lambda \to \mathcal{P}_\vartheta^\lambda$ determined by

$$\mathrm{supp}(\Psi(J)) = \{\vartheta_{J_i} : i \in \partial I\} \setminus \{0\},$$

which is clearly surjective. We claim that, for all $\xi \in \mathcal{P}_\vartheta^\lambda$, we have

(3-1-8) $\# \Psi^{-1}(\xi) = \Phi(\xi) + 1$ where $\Phi(\xi) = \# \mathrm{supp}(\lambda) - \# \mathrm{supp}(\xi).$

Assuming this, we complete the proof of (3-1-6) as follows. If $\# \mathrm{supp}(\lambda) = 1$, $\Phi(\xi) = 0$ for all $\xi \in \mathcal{P}_\vartheta^\lambda$. In other words, Ψ is bijective and (3-1-6) follows from Proposition 3.1.1 and (3-1-7). If $\# \mathrm{supp}(\lambda) = 2$, there is a unique $\xi \in \mathcal{P}_\vartheta^\lambda$ such that $\Phi(\xi) \ne 0$: the one whose support is $\{\vartheta\}$. Therefore, $\# \mathscr{I}_\vartheta^\lambda = 1 + \# \mathcal{P}_\vartheta^\lambda$ and (3-1-6) follows. Finally, if $\# \mathrm{supp}(\lambda) = 3$, we have to count the sets

$$\{\xi \in \mathcal{P}_\vartheta^\lambda : \Phi(\xi) = 1\} \quad \text{and} \quad \{\xi \in \mathcal{P}_\vartheta^\lambda : \Phi(\xi) = 2\}.$$

The second set has exactly one element: the one whose support is $\{\vartheta\}$. Therefore, we are left to show that the first set has $n - 1$ elements. But indeed, ξ belongs to that set if, and only if, there exists $i \in I \setminus \{i_*\}$ such that

$$\mathrm{supp}(\xi) = \{\vartheta_{[i,\partial i]}, \vartheta_{I \setminus [i,\partial i]}\},$$

where ∂i is the element of ∂I lying in the same connected component of $I \setminus \{i_*\}$ as i.

It remains to prove (3-1-8). Fix $J \in \Psi^{-1}(\xi)$. If $\Phi(\xi) = 0$, then $J_i \ne \varnothing$ for all $i \in \mathrm{supp}(\lambda)$ and the claim is clear. If $\Phi(\xi) = 1$, then there exist $k, l \in \mathrm{supp}(\lambda)$ such that $k \in J_l$ and, hence, $J_k = \varnothing$. One easily checks that the unique other element of $\Psi^{-1}(\xi)$ is the one obtained from J by switching J_k and J_l. Finally, if $\Phi(\xi) = 2$, we must have $\# \mathrm{supp}(\lambda) = 3$ and there exists unique $k \in \mathrm{supp}(\lambda)$ such that $J_k \ne \varnothing$. In particular, $\mathrm{supp}(\lambda) \subseteq J_k$ and the other two elements of $\Psi^{-1}(\xi)$ are obtained from J by moving J_k to any of the other two positions. This completes the proof of (3-1-6).

Finally, we deduce some information about the outer multiplicities in W. Namely, write

$$W \cong \bigoplus_{\mu \in P^+} V(\mu)^{\oplus m_\mu(W)}.$$

Proposition 3.1.2. *Let $\mu \in P^+$ be such that $\mathrm{ht}_i(\lambda - \mu) \leq 1$ for all $i \in I$. Then, $m_\mu(W) \neq 0$ if and only if $\mu = \lambda - \vartheta_J$ with $J = \overline{S}$ for some $S \subseteq \mathrm{supp}(\lambda), \#S \neq 1$. In that case, $m_\mu(W) = 1$ if $\#S < 3$ and $m_\mu(W) = 2$ if $\#S = 3$.*

Proof. Set $J = \mathrm{rsupp}(\lambda - \mu)$. If $J \cap \mathrm{supp}(\lambda) = \varnothing$, then for all $J' \subseteq J$,

$$\dim(V(\lambda_i)_{\lambda_i - \vartheta_{J'}}) = 0, \quad \text{for all } i \in \partial I,$$

and, hence, $\dim(W_\mu) = 0$. If $\#J \cap \mathrm{supp}(\lambda) = 1, i \in \partial I$, and $J' \subseteq J$, then $\dim(V(\lambda_i)_{\lambda_i - \vartheta_{J'}}) \leq 1$ with equality holding if, and only if, $i \in J'$. In particular, $\dim(W_\mu) = \dim(V(\lambda)_\mu)$ and, hence, $m_\mu(W) = 0$. Similarly, we conclude that, if each connected component of J intersects $\mathrm{supp}(\lambda)$ in at most one node, then $m_\mu(W) = 0$.

Let $k, l \in \mathrm{supp}(\lambda), k \neq l$. If $J = [k, l]$, then $\dim(W_\mu) = \dim(V(\lambda)_\mu) + 1$ by (3-1-6) and, hence, $m_\mu(W) = 1$. This proves the proposition if I has no trivalent node and we can assume $\mathfrak{g}$ is of type D or E. Since $\mathfrak{g}$ is simply laced and one easily sees that if there exists $j \in \partial J$ such that $j \notin \mathrm{supp}(\lambda)$, then $\mu = \lambda - \vartheta_J \notin P^+$. Hence, we can assume $\#\mathrm{supp}(\lambda) = 3$ and $J = I$.

It follows from the cases already considered that

$$m_{\lambda - \vartheta_{[i,j]}}(W) = 1 \quad \text{for all } i, j \in \mathrm{supp}(\lambda), i \neq j.$$

Let $k \in \partial I$. Writing $\nu = \lambda - \vartheta$ and using (3-1-3) with $I' = (i_*, k]$ in place of I and $(\lambda - \vartheta_{I_k})^{I'}$ in place of λ, we see that

(3-1-9) $$\dim(V(\lambda - \vartheta_{I_k})_\nu) = d(i_*, k).$$

One easily checks that

(3-1-10) $$\sum_{k \in \partial I} d(k, i_*) = n - 1.$$

Combining this with (3-1-6) we get

$$\dim(W_\nu) - \dim(V(\lambda)_\nu) - \sum_{k \in \partial I} \dim(V(\lambda - \vartheta_{I_k})_\nu) = 2.$$

Since no other irreducible factor of W has ν as weight, we conclude $m_\nu(W) = 2$. $\square$

3.2. *Reduction to diagram subalgebras.* We now collect several useful technical results related to the action of diagram subalgebras.

Lemma 3.2.1 [Chari and Pressley 1996a, Lemma 2.4]. *Suppose $\varnothing \neq J \subseteq I$ defines a connected subdiagram of the Dynkin diagram of $\mathfrak{g}$, let V be a highest-ℓ-weight module with highest-ℓ-weight $\boldsymbol{\omega} \in \mathcal{P}^+$, $\lambda = \mathrm{wt}(\boldsymbol{\omega})$, $v \in V_\lambda \setminus \{0\}$, and $V_J = U_q(\tilde{\mathfrak{g}}_J)v$. Then, $m_\mu(V) = m_{\mu_J}(V_J)$ for all $\mu \in \lambda - Q_J^+$.* $\square$

Keeping the notation of Lemma 3.2.1, notice that if V is irreducible, then $V_J \cong V_q(\omega_J)$. Hence,

$$(3\text{-}2\text{-}1) \qquad v \in \omega Q_J \implies \dim(V_q(\omega)_v) = \dim(V_q(\omega_J)_{v_J}).$$

The next lemma is an easy consequence of [Chari and Pressley 1996a, Lemma 2.6].

Lemma 3.2.2. *Let $i_0 \in I$ be such that*

$$I = J_1 \sqcup \{i_0\} \sqcup J_2 \quad (disjoint\ union)$$

where J_1 is of type A, $J_2 \sqcup \{i_0\}$ is connected and $c_{jk} = 0$ for all $j \in J_1$, $k \in J_2$. Let $\omega \in \mathcal{P}^+$, $\lambda = \mathrm{wt}(\omega)$, and suppose $V_q(\omega_{J_1})$ is a minimal affinization of $V_q(\lambda_{J_1})$. Let also

$$\mu = \lambda - \sum_{j \in I \setminus \{i_0\}} s_j \alpha_j \quad with\ s_j \in \mathbb{Z}_{\geq 0}\ for\ all\ j \in I \setminus i_0.$$

If $m_\mu(V_q(\omega)) > 0$, then $s_j = 0$ for all $j \in J_1$. $\qquad\qquad\square$

Proposition 3.2.3 [Chari and Pressley 1996a, Proposition 3.3]. *Suppose $\mathfrak{g}$ is of type A and let $\omega \in \mathcal{P}^+$, $\lambda = \mathrm{wt}(\omega)$ be such that*

(i) *$V_q(\omega)$ is not a minimal affinization of $V_q(\lambda)$, and*

(ii) *$V_q(\omega_{I \setminus \{i\}})$ is a minimal affinization of $V_q(\lambda_{I \setminus \{i\}})$ for any $i \in \partial I$.*

Then, $m_{\lambda - \vartheta}(V_q(\omega)) > 0$. $\qquad\qquad\square$

Lemma 3.2.4. *Suppose $\mathfrak{g}$ is of type D or E, let $\omega \in \mathcal{P}_q^+$ be preminimal, $\lambda = \mathrm{wt}(\omega)$, and $V = V_q(\omega)$. Let $\mu \in P^+$ be such that*

$$(3\text{-}2\text{-}2) \qquad\qquad \mu < \lambda \quad and \quad m_\mu(V) \neq 0.$$

Then, $J_\mu := \mathrm{rsupp}(\lambda - \mu)$ is connected and $m_\mu(V) = m_{\mu_{J_\mu}}(V_q(\omega_{J_\mu}))$. Moreover, for each $k \in \partial I$ we have:

(a) *If $m \notin J_\mu$ for some $m \in [i_*, k]$, then $[m, k] \cap J_\mu = \varnothing$. In particular, $i_* \in J_\mu$.*

(b) *There exists unique $j \in [i_*, k]$ such that $(j, k] \cap J_\mu = \varnothing$ and $[j, i_*] \subseteq J_\mu$.*

(c) *If ω is k-minimal, then $j \neq i_*$.*

Proof. Assuming parts (a) and (b), the first two claims of the lemma can be proved as follows. Let j_k be defined as in (b) for each $k \in \partial I$. It is clear from (a) and (b) that $J_\mu = \overline{\{j_k : k \in \partial I\}}$, showing that it is connected. The second claim of the lemma is then immediate from Lemma 3.2.1.

The first claim in part (a) follows from an application of Lemma 3.2.2 with $i_0 = m$, $J_1 = (m, k]$, and $J_2 = I \setminus [m, k]$. For the second, note that, since $i_* \in [i_*, m]$ for all $m \in \partial I$, if we had $i_* \notin J_\mu$, it would follow that $J_\mu = \varnothing$, contradicting the first assumption in (3-2-2). For part (b), let j be the element of $J_\mu \cap [i_*, k]$ which is closest to k. Then, part (a) implies that j satisfies the desired properties.

To prove (c), note that, if $j = i_*$, we would have $(i_*, k] \cap J_\mu = \varnothing$ and, hence, $\mu \in \lambda - Q_{I_k}^+$. Since I_k is of type A and $\mu < \lambda$, we would have

$$m_{\mu_{J_k}}(V_q(\boldsymbol{\omega}_{I_k})) = 0.$$

On the other hand, Lemma 3.2.1 would imply that

$$m_\mu(V) = m_{\mu_{J_k}}(V_q(\boldsymbol{\omega}_{I_k})),$$

contradicting the second assumption from (3-2-2). □

We can now give a proof of Lemma 2.4.7 under the assumption that hypothesis (ii) is satisfied. Recalling the notation there, we have

$$V_q(\boldsymbol{\omega}_{I_m^\lambda}) \text{ is a minimal affinization for } m \neq k.$$

Defining j_m, $m \in \partial I$, as in Lemma 3.2.4, it follows that $j_m \neq i_*$ for $m \in \partial I_k$. Hypothesis (ii) implies that $I_k^\lambda \subseteq J_\mu$ and, hence, $\mu \leq \nu_k$.

3.3. qCharacters. Let $\mathbb{Z}[\mathcal{P}]$ be the integral group ring over $\mathcal{P}$. Given $\chi \in \mathbb{Z}[\mathcal{P}]$, say

$$\chi = \sum_{\boldsymbol{\mu} \in \mathcal{P}} \chi(\boldsymbol{\mu}) \, \boldsymbol{\mu},$$

we identify it with the function $\mathcal{P} \to \mathbb{Z}$, $\boldsymbol{\mu} \to \chi(\boldsymbol{\mu})$. Conversely, any function $\mathcal{P} \to \mathbb{Z}$ with finite support can be identified with an element of $\mathbb{Z}[\mathcal{P}]$. The qcharacter of $V \in \widetilde{\mathcal{C}}_q$ is the element qch(V) corresponding to the function

$$\boldsymbol{\mu} \mapsto \dim(V_{\boldsymbol{\mu}}).$$

We set

$$\mathrm{wt}_\ell(V) = \{\boldsymbol{\mu} \in \mathcal{P}_q : V_{\boldsymbol{\mu}} \neq 0\} \quad \text{and} \quad \mathrm{wt}_\ell(V_\mu) = \{\boldsymbol{\mu} \in \mathrm{wt}_\ell(V) : \mathrm{wt}(\boldsymbol{\mu}) = \mu\},$$

for all $\mu \in P$.

The Frenkel–Mukhin algorithm [2001] is one of the main tools for computing qcharacters of simple objects of $\widehat{\mathcal{C}}_q$, although it is not applicable to any such object. From the basic theory leading to the algorithm, we will only need the following result here (a proof can also be found in [Chari and Moura 2005]).

Lemma 3.3.1. *Let $V \in \widehat{\mathcal{C}}_q$, $i \in I$, and $\boldsymbol{\varpi} \in \mathcal{P}_q$. Suppose there exists $v \in V_{\boldsymbol{\varpi}} \setminus \{0\}$ satisfying $x_{i,r}^+ v = 0$ for all $r \in \mathbb{Z}$ and that $\boldsymbol{\varpi}^{\{i\}} = \omega_{i,a,m}$ for some $a \in \mathbb{F}^\times$, $m > 0$. Then,*

$$(3\text{-}3\text{-}1) \qquad \qquad \boldsymbol{\varpi} \, \boldsymbol{\alpha}_{i,aq^{m-1}}^{-1} \in \mathrm{wt}_\ell(V). \qquad \qquad \square$$

3.4. *Tensor products.* The algebra $U_q(\tilde{\mathfrak{g}})$ is a Hopf algebra. We now review the facts about tensor products of objects from $\widehat{\mathcal{C}}_q$ that we need.

It is well known that the tensor product of weight vectors is a weight vector and, hence, if $V, W \in \mathcal{C}_q$, we have $\mathrm{ch}(V \otimes W) = \mathrm{ch}(V)\mathrm{ch}(W)$. Although the tensor product of ℓ-weight vectors is not an ℓ-weight vector in general, it was proved in [Frenkel and Reshetikhin 1999] (see also [Chari and Moura 2005]) that we still have

$$(3\text{-}4\text{-}1) \qquad \mathrm{qch}(V \otimes W) = \mathrm{qch}(V)\mathrm{qch}(W) \quad \text{for every } V, W \in \widetilde{\mathcal{C}}_q.$$

It turns out that tensor products of nontrivial simple objects from $\widetilde{\mathcal{C}}_q$ may be simple as well. For the proof of Theorem 2.4.4, we will need some sufficient criteria for the irreducibility of certain tensor products of minimal affinizations which we now recall. The following is the first half of main result of [Moura and Pereira 2017].

Theorem 3.4.1. *Let $\mathfrak{g}$ be of type A_n, $\lambda \in P^+ \setminus \{0\}$, and consider*

$$\boldsymbol{\pi} = \prod_{i \leq j} \boldsymbol{\omega}_{i, aq^{-p_{i,j}(\lambda)}, \lambda(h_i)} \quad \text{and} \quad \boldsymbol{\pi}' = \boldsymbol{\omega}_{n, b, \eta}$$

for some $a, b \in \mathbb{F}^\times$ and $\eta \in \mathbb{Z}_{>0}$ where $j = \max\{i \in I : i \in \mathrm{supp}(\lambda)\}$. Then, $V_q(\boldsymbol{\pi}) \otimes V_q(\boldsymbol{\pi}')$ is reducible if and only if there exist $s \in \mathbb{Z}$, $j' \in \mathrm{supp}(\lambda)$, and $\eta' \in \mathbb{Z}_{>0}$ such that $b = aq^s$ and either one of the following options hold:

(i) $\eta' \leq \min\{\lambda(h_{j'}), \eta\}$ *and* $s + \eta + n - j' + 2 = -p_{j',j}(\lambda) - \lambda(h_{j'}) + 2\eta'$.

(ii) $\eta' \leq \min\{|\lambda|, \eta\}$ *and* $\lambda(h_j) + n - j + 2 = s - \eta + 2\eta'$. □

Remark 3.4.2. Note that $V_q(\boldsymbol{\pi})$, with $\boldsymbol{\pi}$ as in Theorem 3.4.1, is an increasing minimal affinization. Similar results for decreasing minimal affinizations as well as for tensor products with Kirillov–Reshetikhin (KR) modules associated to the first node can be obtained from Theorem 3.4.1 by means of duality arguments. The precise statements can be found in [Moura and Pereira 2017, Corollary 4.2.2]. The second half of Theorem 3.4.1 states that when such tensor products are reducible, they are length-two modules and the Drinfeld polynomial of the irreducible factor with lower highest-weight is explicitly described. ◇

We will also need a criterion that guarantees the irreducibility of tensor products of KR modules associated to nodes in ∂I when $\mathfrak{g}$ is of type D. To deduce it, we begin by recalling some facts about duality (a slightly more complete review was given in [Moura and Pereira 2017, Section 4.1]). For any two finite-dimensional $U_q(\tilde{\mathfrak{g}})$-modules V and W, we have

$$(3\text{-}4\text{-}2) \qquad\qquad (V \otimes W)^* \cong W^* \otimes V^*.$$

Also, given $\boldsymbol{\omega} \in \mathcal{P}^+$, we have

(3-4-3) $V_q(\boldsymbol{\omega})^* \cong V_q(\boldsymbol{\omega}^*)$ where $\boldsymbol{\omega}_i^*(u) = \boldsymbol{\omega}_{w_0 \cdot i}(q^{-h^\vee} u)$.

Here, $h^\vee$ is the dual Coxeter number of $\mathfrak{g}$, w_0 is the longest element of $\mathcal{W}$ and $w_0 \cdot i = j$ if and only if $w_0 \omega_i = -\omega_j$. The following lemma is well known and easily established.

Lemma 3.4.3. *Suppose V is an object from $\widetilde{\mathcal{C}}_q$. Then, V is simple if and only if both V and V^* are highest-ℓ-weight modules.* $\square$

The following is a rewriting of part of [Chari 2002, Corollary 6.2].

Proposition 3.4.4. *Suppose $\mathfrak{g}$ is of type D_n, let $i, j \in \partial I$ be distinct, $m_i, m_j \in \mathbb{Z}_{>0}$, $a_i, a_j \in \mathbb{F}^\times$, and let $V = V_q(\boldsymbol{\omega}_{i,a_i,m_i}) \otimes V_q(\boldsymbol{\omega}_{j,a_j,m_j})$ and $m = \min\{m_i, m_j\}$. The following are sufficient conditions for V to be a highest-ℓ-weight module:*

(a) $a_j/a_i \neq q^{m_i+m_j+2(2s-p)}$ *for all $1 \leq p \leq m$, $1 \leq s \leq \lfloor (n-1)/2 \rfloor$ if both i and j are spin nodes.*

(b) $a_j/a_i \neq q^{m_i+m_j+n-2p}$ *for all $1 \leq p \leq m$ if either i or j is not a spin node.* $\square$

Remark 3.4.5. There is a typo in [Chari 2002, Corollary 6.2] regarding part (a) of the above proposition. Namely, the range for the parameter s is claimed to be $0 \leq s \leq \lfloor (n-1)/2 \rfloor$. The absence of the possibility $s = 0$ is crucial for our purposes. We have rechecked the computations related to the proof of [Chari 2002, Corollary 6.2] and have established that indeed $s = 0$ can be removed from the range. Note that this correction is compatible with part (b) of the proposition in the sense that, in type D_4, since all elements of ∂I "are spin nodes", part (a) should "coincide" with (b). If $s = 0$ were allowed, the number of obstructions coming from (a) would be twice as many as from part (b). With this correction, parts (a) and (b) coincide in all elements of ∂I for type D_4. $\diamond$

Recall that, if $\mathfrak{g}$ is of type D, then

(3-4-4) $w_0 \cdot i = i$ if i is not a spin node.

In particular, if i is a spin node, so is $w_0 \cdot i$. Then, combining the last proposition with (3-4-2), (3-4-3), and Lemma 3.4.3, one easily establishes:

Corollary 3.4.6. *Suppose $\mathfrak{g}$ is of type D_n, let $i, j \in \partial I$ be distinct, $m_i, m_j \in \mathbb{Z}_{>0}$, $a_i, a_j \in \mathbb{F}^\times$, $V = V_q(\boldsymbol{\omega}_{i,a_i,m_i}) \otimes V_q(\boldsymbol{\omega}_{j,a_j,m_j})$ and $m = \min\{m_i, m_j\}$. The following are sufficient conditions for V to be irreducible.*

(a) $(a_j/a_i)^{\pm 1} \neq q^{m_i+m_j+2(2s-p)}$ *for all $1 \leq p \leq m$, $1 \leq s \leq \lfloor (n-1)/2 \rfloor$ if both i and j are spin nodes.*

(b) $(a_j/a_i)^{\pm 1} \neq q^{m_i+m_j+n-2p}$ *for all $1 \leq p \leq m$ if either i or j is not a spin node.* $\square$

Remark 3.4.7. Using the combinatorics of qcharacters in terms of tableaux, a necessary and sufficient condition in the context of part (a) of the above corollary was obtained in [Pereira 2014]. Moreover, in the case that V is reducible, an explicit description of the Drinfeld polynomial of its irreducible factor whose highest weight is the second highest was also obtained. Comments about the difference between the sufficient condition given by Corollary 3.4.6 and the necessary and sufficient condition obtained in [Pereira 2014] will appear in [Pereira $\geq$ 2018]. For the moment, it suffices to say that $s = 0$ (see previous remark) indeed corresponds to an irreducible tensor product according to [Pereira 2014]. ◇

3.5. *Classical and graded limits.* Let $\mathbb{A} = \mathbb{C}[q, q^{-1}] \subseteq \mathbb{F}$ and let $U_{\mathbb{A}}(\tilde{\mathfrak{g}})$ be the $\mathbb{A}$-subalgebra of $U_q(\tilde{\mathfrak{g}})$ generated by the elements $(x_{i,r}^{\pm})^{(k)}, k_i^{\pm 1}$ for $i \in I, r \in \mathbb{Z}$, and $k \in \mathbb{Z}_{\geq 0}$ where $(x_{i,r}^{\pm})^{(k)} = (x_{i,r}^{\pm})^k/([k]!)$. Define $U_{\mathbb{A}}(\mathfrak{g})$ similarly and notice that $U_{\mathbb{A}}(\mathfrak{g}) = U_{\mathbb{A}}(\tilde{\mathfrak{g}}) \cap U_q(\mathfrak{g})$. For the proof of the next proposition, see [Chari 2001, Lemma 2.1] and the locally cited references.

Proposition 3.5.1. *We have $U_q(\tilde{\mathfrak{g}}) = \mathbb{F} \otimes_{\mathbb{A}} U_{\mathbb{A}}(\tilde{\mathfrak{g}})$ and $U_q(\mathfrak{g}) = \mathbb{F} \otimes_{\mathbb{A}} U_{\mathbb{A}}(\mathfrak{g})$.* ◇

Regard $\mathbb{C}$ as an $\mathbb{A}$-module by letting q act as 1 and set

$$(3\text{-}5\text{-}1) \qquad \overline{U_q(\tilde{\mathfrak{g}})} = \mathbb{C} \otimes_{\mathbb{A}} U_{\mathbb{A}}(\tilde{\mathfrak{g}}) \quad \text{and} \quad \overline{U_q(\mathfrak{g})} = \mathbb{C} \otimes_{\mathbb{A}} U_{\mathbb{A}}(\mathfrak{g}).$$

Denote by $\overline{\eta}$ the image of $\eta \in U_{\mathbb{A}}(\tilde{\mathfrak{g}})$ in $\overline{U_q(\tilde{\mathfrak{g}})}$. The proof of the next proposition can be found in [Chari and Pressley 1994a, Proposition 9.2.3] and [Lusztig 1993].

Proposition 3.5.2. *$U(\tilde{\mathfrak{g}})$ is isomorphic to the quotient of $\overline{U_q(\tilde{\mathfrak{g}})}$ by the ideal generated by $\overline{k}_i - 1$, $i \in I$. In particular, the category of $\overline{U_q(\tilde{\mathfrak{g}})}$-modules on which k_i act as the identity operator for all $i \in I$ is equivalent to the category of all $\tilde{\mathfrak{g}}$-modules.* ◇

Denote by $\mathcal{P}_{\mathbb{A}}^+$ the subset of $\mathcal{P}_q$ consisting of n-tuples of polynomials with coefficients in $\mathbb{A}$. Let also $\mathcal{P}_{\mathbb{A}}^{\times}$ be the subset of $\mathcal{P}_{\mathbb{A}}^+$ consisting of n-tuples of polynomials whose leading terms are in $\mathbb{C}q^{\mathbb{Z}} \setminus \{0\} = \mathbb{A}^{\times}$. Given $\boldsymbol{\omega} \in \mathcal{P}_{\mathbb{A}}^+$, let $\overline{\boldsymbol{\omega}}$ be the element of $\mathcal{P}_q^+$ obtained from $\boldsymbol{\omega}$ by evaluating q at 1. Given a $U_{\mathbb{A}}(\tilde{\mathfrak{g}})$-submodule L of a $U_q(\tilde{\mathfrak{g}})$-module V, define

$$(3\text{-}5\text{-}2) \qquad\qquad \bar{L} = \mathbb{C} \otimes_{\mathbb{A}} L.$$

Then, $\bar{L}$ is a $\tilde{\mathfrak{g}}$-module by Proposition 3.5.2. The next theorem was proved in [Chari and Pressley 2001].

Theorem 3.5.3. *Let $\boldsymbol{\omega} \in \mathcal{P}_{\mathbb{A}}^{\times}$, $V \in \tilde{\mathcal{C}}_q$ be a highest-ℓ-weight module of highest ℓ-weight $\boldsymbol{\omega}$, $v \in V_{\boldsymbol{\omega}} \setminus \{0\}$, and $L = U_{\mathbb{A}}(\tilde{\mathfrak{g}})v$. Then, $\bar{L}$ is a highest-ℓ-weight module for $\tilde{\mathfrak{g}}$ with highest-ℓ-weight $\overline{\boldsymbol{\omega}}$ and $\mathrm{ch}(\bar{L}) = \mathrm{ch}(V)$.* ◇

Given $\boldsymbol{\omega} \in \mathcal{P}_{\mathbb{A}}^{\times}$, we denote by $\overline{V_q(\boldsymbol{\omega})}$ the $\tilde{\mathfrak{g}}$-module $\bar{L}$ with L as in the above theorem.

Assume $\overline{\boldsymbol{\omega}} = \boldsymbol{\omega}_{\lambda,a}$ for some $a \in \mathbb{C}^{\times}$ and let $\bar{v}$ be a nonzero vector in $\overline{V_q(\boldsymbol{\omega})}_{\lambda}$. It follows from Theorem 3.5.3 that

$$(h \otimes f(t))\bar{v} = f(a)h\bar{v} \quad \text{for all } h \in \mathfrak{h}, \ f(t) \in \mathbb{C}[t, t^{-1}].$$

Moreover, it follows from the proof of [Moura 2010, Proposition 3.13] that, if J is a connected subdiagram of type A such that $V_q(\boldsymbol{\omega}_J)$ is a minimal affinization, then

$$(3\text{-}5\text{-}3) \qquad x_{\alpha,r}^{-}\bar{v} = a^r x_{\alpha}^{-}\bar{v} \quad \text{for all } \alpha \in R_J^{+}, \ r \geq 0.$$

We shall regard $\overline{V_q(\boldsymbol{\omega})}$ as a $\mathfrak{g}[t]$-module which is generated by $\bar{v}$ by Proposition 2.2.3. Denote by $L(\boldsymbol{\omega})$ the $\mathfrak{g}[t]$-module obtained from $\overline{V_q(\boldsymbol{\omega})}$ by pulling-back the action by the automorphism τ_a defined in (2-1-3) and let $v \in L(\boldsymbol{\omega})_{\lambda} \setminus \{0\}$. It follows from the above considerations that $L(\boldsymbol{\omega}) = U(\mathfrak{g}[t])v$, $\mathfrak{n}^{+}[t]v = \mathfrak{h}[t]_{+}v = 0$, $hv = \lambda(h)v$ for all $h \in \mathfrak{h}$. Hence, $L(\boldsymbol{\omega})$ is a quotient of $W(\lambda)$. Moreover, Theorem 3.5.3 and part (d) of Theorem 2.2.1 imply (2-5-2). Also, by (3-5-3), if J is a connected subdiagram of type A such that $V_q(\boldsymbol{\omega}_J)$ is a minimal affinization,

$$(3\text{-}5\text{-}4) \qquad x_{\alpha,r}^{-}v = 0 \quad \text{for all} \quad \alpha \in R_J^{+}, \ r > 0,$$

which easily implies Lemma 2.5.1.

4. Proofs

4.1. *On characters and tensor products of KR-modules.* We will need some information on qcharacters and tensor products of Kirillov–Reshetikhin modules. Thus, let $\boldsymbol{\omega} = \boldsymbol{\omega}_{i,a,m}$ for some $i \in I$, $a \in \mathbb{F}$, $m > 0$ and let $v \in L(\boldsymbol{\omega})_{m\omega_i} \setminus \{0\}$. We begin with the following well-known fact:

$$(4\text{-}1\text{-}1) \qquad \mu \in P^{+}, \ \mu < m\omega_i, \ m_{\mu}(L(\boldsymbol{\omega})) \neq 0 \quad \Rightarrow \quad \text{ht}_i(\lambda - \mu) > 1.$$

Since, for all connected subdiagrams J, we have $\text{ht}_i(\vartheta_J) \leq 1$, this implies

$$(4\text{-}1\text{-}2) \qquad \dim(V_q(\boldsymbol{\omega})_{m\omega_i - \vartheta_J}) = \dim(V(m\omega_i)_{m\omega_i - \vartheta_J}) \leq 1,$$

where the inequality follows from Proposition 3.1.1 which also implies that

$$(4\text{-}1\text{-}3) \qquad \dim(V_q(\boldsymbol{\omega})_{m\omega_i - \vartheta_J}) = 1 \quad \Leftrightarrow \quad i \in J.$$

Let M be a tensor product of KR-modules associated to distinct nodes, say $V_q(\boldsymbol{\omega}_{i,a_i,m_i})$, $i \in I$, and set $\lambda = \sum_i m_i \omega_i$. Let also $W = \otimes_{i \in I} V(m_i \omega_i)$. It follows from the above discussion that

$$(4\text{-}1\text{-}4) \qquad \dim(W_{\lambda - \vartheta_J}) = \dim(M_{\lambda - \vartheta_J}) \quad \text{for all } J \subseteq I.$$

In particular, Proposition 3.1.2 applies to M in place of W. Therefore, if V, ν_k, and ν are as in Proposition 2.4.6, it follows that

$$(4\text{-}1\text{-}5) \qquad\qquad m_{\nu_k}(V) \leq 1 \quad \text{and} \quad m_\nu(V) \leq 2.$$

Proposition 3.2.3 then implies that $m_{\nu_k}(V) = 1$, thus completing the proof of part (a) of Proposition 2.4.6. A proof of (4-1-1) will be reviewed along the way when we perform some estimates using graded limits in Section 4.2. These estimates will also imply that we actually have

$$(4\text{-}1\text{-}6) \qquad\qquad m_\nu(V) \leq 1,$$

an improvement of (4-1-5) which will be crucial in our approach for proving the last two parts of Proposition 2.4.6.

Since most of the literature on qcharacters uses the Y-notation of [Frenkel and Reshetikhin 1999], we shall write all arguments within the context of qcharacters using that notation as well. Given $a \in \mathbb{F}^\times$, $i \in I$, $r \in \mathbb{Z}$, $m \in \mathbb{Z}_{\geq 0}$, set

$$(4\text{-}1\text{-}7) \qquad\qquad Y_{i,r,m} = \omega_{i,aq^{m+r-1},m}.$$

Let $\mathcal{P}_\mathbb{Z}$ be the submonoid of $\mathcal{P}_q$ generated by $Y_{i,r} := Y_{i,r,1}$, $i \in I$, $r \in \mathbb{Z}$. Following [Frenkel and Mukhin 2001], given $\omega \in \mathcal{P}_\mathbb{Z} \setminus \{\mathbf{1}\}$, define

$$(4\text{-}1\text{-}8) \qquad r(\omega) := \max\{r \in \mathbb{Z} : Y_{i,r}^{\pm 1} \text{ appears in } \omega \text{ for some } i \in I\}.$$

Then, ω is said to be right negative if $Y_{i,r(\omega)}$ does not appear in ω for all $i \in I$. Clearly, the product of right negative ℓ-weights is a right negative ℓ-weight and a dominant ℓ-weight is not right negative.

Given a connected subdiagram $J \subseteq I$ define

$$J^+ = \{i \in I \setminus J : c_{i,j} < 0 \text{ for some } j \in J\}.$$

If $l \in J$, define also

$$(4\text{-}1\text{-}9) \qquad\qquad \partial_l J = \begin{cases} l & \text{if } J = \{l\}, \\ (\partial J) \setminus \{l\} & \text{otherwise,} \end{cases}$$

and,

$$(4\text{-}1\text{-}10) \quad Y_{l,r,m}(J) =$$

$$Y_{l,r,m-1}\left(\prod_{i \in J^+} Y_{i,r+2(m-1)+d(i,l)}\right)\left(\prod_{i \in \partial_l J} Y_{i,r+2m+d(i,l)}\right)^{-1} \times (Y_{i_*,r+2m+d(i_*,l)})^\epsilon$$

for all $r, m \in \mathbb{Z}$, $m > 0$, where

$$\epsilon = 0 \text{ if } J \text{ is of type } A \text{ and } \epsilon = 1 \text{ otherwise.}$$

In particular,

$$Y_{l,r,m}(J) \text{ is right negative} \quad \text{and} \quad r(Y_{l,r,m}(J)) = r + 2m + d_{l,J},$$

where $d_{l,J} = \max\{d(l,j) : j \in J\}$. Set also

$$(4\text{-}1\text{-}11) \qquad\qquad Y_{l,r,m}(J) = Y_{l,r,m} \quad \text{if } l \notin J.$$

Lemma 4.1.1. *Let $V = V_q(Y_{l,r,m})$ for some $l \in I$, $r, m \in \mathbb{Z}$, $m > 0$, and $\lambda = m\omega_l$. For every connected subdiagram $J \subseteq I$ containing l, $\dim(V_{\lambda - \vartheta_J}) = 1$ and $V_{\lambda - \vartheta_J} = V_{Y_{l,r,m}(J)}$.*

Proof. After (4-1-2) and (4-1-3), it suffices to show that

$$(4\text{-}1\text{-}12) \qquad\qquad Y_{l,r,m}(J) \in \mathrm{wt}_\ell(V)$$

for every connected subdiagram J which is either empty or contains l. This is obvious if $J = \varnothing$. Otherwise, let $i \in \partial_l J$, $J' = J \setminus \{i\}$, and assume, by induction hypothesis on $\#J$, that $Y_{l,r,m}(J') \in \mathrm{wt}_\ell(V)$. Note also that $\lambda - \vartheta_{J'} + \alpha_i$ is not a weight of V and, hence, if $v \in V_{Y_{l,r,m}(J')}$, we have

$$x_{i,r}^+ v = 0 \quad \text{for all } r \in \mathbb{Z}.$$

It follows that the hypotheses of Lemma 3.3.1 are satisfied and one easily checks that $Y_{l,r,m}(J)$ is obtained from $Y_{l,r,m}(J')$ using (3-3-1), thus proving (4-1-12). $\square$

Lemma 4.1.2. *Let $i, j \in I$ and $\omega = Y_{i,r_i,m_i} Y_{j,r_j,m_j} \varpi$ for some $r_i, r_j, m_i, m_j \in \mathbb{Z}$, $m_i, m_j > 0$, and $\varpi \in \mathcal{P}^+$ such that $[i,j] \cap \mathrm{supp}(\mathrm{wt}(\varpi)) = \varnothing$. Let also $\lambda = \mathrm{wt}(\omega)$ and assume $r_i \le r_j$. Then,*

$$\dim(V_q(\omega)_{\lambda - \vartheta_{[i,j]}}) = \begin{cases} d(i,j) + 1 & \text{if } r_j - r_i = 2m_i + d(i,j), \\ d(i,j) + 2 & \text{otherwise.} \end{cases}$$

Proof. By Lemma 3.2.1, we may assume $\{i,j\} = \partial I$ and, hence, $\varpi = 1$ and $\mathfrak{g}$ is of type A. If $r_j - r_i = 2m_i + d(i,j)$, then $V_q(\omega)$ is a minimal affinization. Hence, it is isomorphic to $V(\lambda)$ as $U_q(\mathfrak{g})$-module and we are done by Proposition 3.1.1 and (3-1-3). Otherwise, $V_q(\omega)$ is not a minimal affinization and Proposition 3.2.3 implies

$$\dim(V_q(\omega)_{\lambda - \vartheta}) \ge \dim(V(\lambda)_{\lambda - \vartheta}) + 1 = d(i,j) + 2.$$

The opposite inequality is immediate from Proposition 3.1.2. $\square$

Remark 4.1.3. It is actually not difficult to prove the following improvement of the previous lemma:

$$\varpi Y_{i,r_i,m_i}([i,j] \setminus J) Y_{i,r_2,m_2}(J) \in \mathrm{wt}_\ell(V_q(\omega))$$

for all connected subdiagrams J containing j. Moreover, if $r_j - r_i \ne 2m_i + d(i,j)$,

the same holds for $J = \varnothing$. Thus, these are exactly the elements of $\text{wt}_\ell(V_q(\boldsymbol{\omega})_{\lambda - \vartheta_{[i,j]}})$ all with multiplicity 1. $\diamond$

4.2. *Computations with graded limits.*

We now compute some upper bounds for outer multiplicities in graded limits which will lead to proofs of Lemma 2.4.7, (4-1-6), and most of Proposition 2.5.3.

Given $a \in \mathbb{Z}_{>0}$ and $i \in I$, set

$$R_a = \{\alpha \in R^+ : \max\{\text{ht}_i(\alpha) : i \in I\} = a\}$$

and note that R^+ is the disjoint union of the sets R_a. Fix $k \in \partial I$, recall the definition of $M_k(\lambda)$ from (2-5-3), and let

$$v \in M_k(\lambda)_\lambda \setminus \{0\}.$$

We obviously have

(4-2-1) $\qquad\qquad x_{\alpha,s}^- v = 0 \quad \Rightarrow \quad x_{\alpha,r}^- v = 0 \quad \text{for all } r \geq s.$

Lemma 4.2.1. *Let $\alpha \in R_1$. Then, $x_{\alpha,2}^- v = 0$. Moreover, if there exists $l \in \partial I_k$ such that either $l_\lambda \notin \text{rsupp}(\alpha)$ or $l_\lambda = l \notin \text{supp}(\lambda)$, then $x_{\alpha,1}^- v = 0$.* $\diamond$

Proof. Let $\partial I_k = \{l, m\}$. Then, if $\alpha \in R_1$, we have $\alpha = \beta + \gamma$ with $\text{rsupp}(\beta) \subseteq I_l$ and $\text{rsupp}(\gamma) \subseteq I_m$. By definition of $M_k(\lambda)$,

$$x_{\beta,1}^- v = x_{\gamma,1}^- v = 0$$

and, hence,

$$x_{\alpha,2}^- v = [x_{\beta,1}^-, x_{\gamma,1}^-] v = 0.$$

For the second statement, note there exists a connected subdiagram $J \subseteq [l_\lambda, i_*)$ such that

$$\text{rsupp}(\alpha) \subseteq I_l \cup J, \quad \text{supp}(\lambda) \cap J = \varnothing, \quad \text{and} \quad \alpha = \beta + \gamma \quad \text{for some}$$
$$\beta \in R_J^+ \cup \{0\} \quad \text{and} \quad \gamma \in R_{I_l}^+ \cup \{0\}.$$

If either $\beta = 0$ or $\gamma = 0$, there is nothing else to do. Otherwise, since $J \cap \text{supp}(\lambda) = \varnothing$, $x_\beta^- v = 0$ and it follows that

(4-2-2) $\qquad\qquad\qquad\qquad x_{\alpha,1}^- v = [x_\beta^-, x_{\gamma,1}^-] v = 0.$ $\qquad\square$

Consider the basis $B := \{x_{\alpha,r}^- : \alpha \in R^+, r \geq 0\}$ of $\mathfrak{n}^-[t]$. Given a subset $S \subseteq B$ and a choice of total order on S, we denote by $U(S)$ the subspace of $U(\mathfrak{n}^-[t])$ spanned by PBW monomials formed from elements of S. Let

(4-2-3) $\quad R_1^\lambda = \{\alpha \in R_1 : I_k^\lambda \subseteq \text{rsupp}(\alpha)\}, \quad R_{>1} = R^+ \setminus R_1,$

(4-2-4) $\quad S_1^\lambda = \{x_{\alpha,1}^- : \alpha \in R_1^\lambda\}, \qquad\qquad S_{>1} = \{x_{\alpha,r}^- : \alpha \in R_{>1}, r > 0\}.$

Note, by inspecting the root systems, that $\vartheta_{I_k^\lambda}$ is a minimal element of $R_{>1} \cup R_1^\lambda$.

Lemma 4.2.1 implies that

$$(4\text{-}2\text{-}5) \qquad M_k(\lambda) = U(\mathfrak{n}^-)U(S_{>1})U(S_1^\lambda)v.$$

Standard arguments (cf. [Moura 2010, Lemma 2.3]) and the aforementioned minimality of $\vartheta_{I_k^\lambda}$ imply

$$m_{\nu_k}^s(M_k(\lambda)) \leq \delta_{s,1}.$$

Then, if $\boldsymbol{\omega} \in \mathcal{P}_q^+$ is l-minimal for $l \in \partial I_k$ and $\mathrm{wt}(\boldsymbol{\omega}) = \lambda$, it follows from Lemmas 2.5.1 and 3.2.1 that

$$m_{\nu_k}(V_q(\boldsymbol{\omega})) \leq 1.$$

This recovers the first part of (4-1-5) and, furthermore, if $\boldsymbol{\omega}$ is not k-minimal, Proposition 3.2.3 implies the first statements of parts (a) and (b) of Proposition 2.5.3. Moreover, the only PBW monomial $x \in U(S_{>1})U(S_1^\lambda)$ such that xv has weight $\nu = \lambda - \vartheta_{I^\lambda}$ is clearly $x = x_{\vartheta_{I^\lambda},1}^-$. Therefore,

$$(4\text{-}2\text{-}6) \qquad m_\nu^s(M_k(\lambda)) \leq \delta_{s,1}.$$

In particular, if V is as in Proposition 2.4.6,

$$(4\text{-}2\text{-}7) \quad \dim(V(\lambda)_\nu) + \dim(V(\nu_k)_\nu) \leq \dim(V_\nu) \leq \dim(V(\lambda)_\nu) + \dim(V(\nu_k)_\nu) + 1,$$

where the first inequality follows from parts (a) and (b) of Proposition 2.4.6, and the second follows from (4-2-6). Regarding the proof of Proposition 2.5.3, it remains to prove the second statement of part (b) as well as part (c). The remainder of this subsection is dedicated to the latter.

Remark 4.2.2. If V is a KR-module associated to the node $i \in I$ and v is the image of its highest-weight vector in the graded limit, a similar argument to the above proves

$$x_{\alpha,1}^- v = 0 \quad \text{if } \mathrm{ht}_i(\alpha) \leq 1$$

which implies (4-1-1) (cf. [Chari 2001; Chari and Moura 2006; Hatayama et al. 1999; Moura 2010; Moura and Pereira 2011]).   ◇

For the remainder of this subsection, assume the hypotheses of Lemma 2.4.7.

Lemma 4.2.3. *Let $\alpha \in R^+$. If $l \in \partial I_k$ exists such that $l_\lambda \notin \mathrm{rsupp}(\alpha)$, $x_{\alpha,1}^- v = 0$.* ◇

Proof. If I_k^λ is of type A_3, it follows that $\mathrm{rsupp}(\alpha) \subseteq I_l$ and there is nothing to do. Hence, we can assume either hypothesis (i) or (iii) of Lemma 2.4.7 is satisfied. In particular, $n > 4$ and, if $\mathfrak{g}$ is of type D, k is a spin node while l is not a spin node.

Assume $\alpha \in R_a$. Since the case $a = 1$ was proved in Lemma 4.2.1, we also assume $a \geq 2$. If $\mathfrak{g}$ is of type D, we have $\alpha = \beta + \gamma$ with $\beta, \gamma \in R^+$ such that $\mathrm{rsupp}(\beta) \subseteq [i_*, l]$, $\mathrm{rsupp}(\beta) \cap \mathrm{supp}(\lambda) = \varnothing$, and $\gamma \in R_1$. Since $l_\lambda \notin \mathrm{rsupp}(\gamma)$, the case $a = 1$ implies that $x_{\gamma,1}^- v = 0$ and we are done, using (4-2-2) once more. We

are left with the case that hypothesis (iii) of Proposition 2.4.6 is satisfied, i.e., $\mathfrak{g}$ is of type E_6 and $\operatorname{supp}(\lambda) = \partial I$. Recall that, for type E_6, we have

$$R_a \neq \varnothing \quad \Leftrightarrow \quad a \leq 3 \quad \text{and} \quad a = 3 \quad \Rightarrow \quad \operatorname{rsupp}(\alpha) = I.$$

Hence, we must have $a = 2$. An inspection of the root system shows that $\alpha = \beta + \gamma$ with $\beta \in (m, i_*]$ for some $m \in \partial I_k$ and $\gamma \in R_1$ such that $l_\lambda \notin \operatorname{rsupp}(\gamma)$. In particular, $\operatorname{rsupp}(\beta) \cap \operatorname{supp}(\lambda) = \varnothing$ and (4-2-2) completes the proof as before.   □

Let

(4-2-8) $R^\lambda = \{\alpha \in R^+ : I_k^\lambda \subseteq \operatorname{rsupp}(\alpha)\}$ and $S_{>1}^\lambda = \{x_{\alpha,r}^- : \alpha \in R^\lambda \setminus R_1, r > 0\}.$

This time we obviously have

(4-2-9)
$$\vartheta_{I_k^\lambda} = \min R^\lambda$$

while Lemma 4.2.3 implies that

(4-2-10)
$$M_k(\lambda) = U(\mathfrak{n}^-) U(S_{>1}^\lambda) U(S_1^\lambda) v.$$

Another application of [Moura 2010, Lemma 2.3] proves

$$\mu \in P^+, \quad \mu < \lambda, \quad m_\mu(M_k(\lambda)) \neq 0 \quad \Rightarrow \quad \mu \leq \nu_k.$$

This, together with Lemmas 2.5.1 and 3.2.1, implies Lemma 2.4.7 and part (c) of Proposition 2.5.3.

Remark 4.2.4. One can proceed with the methods used in the proof of [Moura 2010, (5-10)] to prove that, if $\boldsymbol{\omega}_{I^\lambda}$ is coherent, then

$$m_{\lambda - s\vartheta_{I_k^\lambda}}(V_q(\boldsymbol{\omega})) = m_{\lambda - s\vartheta_{I_k^\lambda}}^s(M_k(\lambda)) = 1$$

for all $1 \leq s \leq m = \min\{\lambda(h_i) : i \in I_k^\lambda\}$ and $m_\nu(V_q(\boldsymbol{\omega})) = m_\nu^1(M_k(\lambda)) = 1$. This would complete the proof of Proposition 2.5.3 as well as of part (c) of Proposition 2.4.6. However, since the proof of part (d) of Proposition 2.4.6 along the same lines is still unclear to us (see Remark 2.5.5), we will not proceed in that direction here. Instead, we will give proofs for both cases within the same spirit using qcharacters.

We recall that [Moura 2010, (5-10)] is a formula for all outer multiplicities of $L(\boldsymbol{\omega})$ for type D_4 whose proof implies the validity of Conjecture 2.5.2 in that case. It was claimed in the closing remark of [Moura 2010] that similar arguments implied that equation (5-10) therein also gave the outer multiplicities in the case that $\boldsymbol{\omega}$ is incoherent. Part (d) of Proposition 2.4.6 implies this is false. The one step that was overlooked in the closing remark of [Moura 2010] was the proof of the existence of the second surjective map in the statement of the corresponding incoherent analogue of [Moura 2010, Proposition 5.14]. That map actually does not exist in the case where $\boldsymbol{\omega}$ is incoherent, while it is easily seen to exist if $\boldsymbol{\omega}$ is coherent using [Moura 2010, Corollary 4.4].   ◇

4.3. *Incoherent tensor products of boundary KR-modules.* Continuing our preparation to prove the last two parts of Proposition 2.4.6, we will conduct a partial study of the simple factors of

$$(4\text{-}3\text{-}1) \qquad W = \bigotimes_{i \in \partial I} V_q(Y_{i,r_i,\lambda_i}),$$

for certain choices of the parameters r_i, λ_i (recall (4-1-7)). Namely, setting

$$\boldsymbol{\omega} = \prod_{i \in \partial I} Y_{i,r_i,\lambda_i},$$

we study W in the cases that $\text{mo}(\boldsymbol{\omega}) = 2$ and $\lambda_i \neq 0$ for all $i \in \partial I$. We treat the case where $\boldsymbol{\omega}$ is incoherent here, leaving the coherent case to Section 4.4. As usual, we let $k \in \partial I$ be the node such that $\boldsymbol{\omega}$ is not k-minimal.

The incoherence of $\boldsymbol{\omega}$ implies there exists a unique choice of $l, m \in \partial I_k$ such that[2]

$$(4\text{-}3\text{-}2) \qquad r_l = r_k + 2\lambda_k + d(k, l) \quad \text{and} \quad r_k = r_m + 2\lambda_m + d(k, m).$$

In particular,

$$r_m < r_k < r_l.$$

Recall (4-1-10) and (4-1-11) and set

$$(4\text{-}3\text{-}3) \quad \boldsymbol{\omega}_l = Y_{m,r_m,\lambda_m}(I_l)Y_{k,r_k,\lambda_k}Y_{l,r_l,\lambda_l} \quad \text{and} \quad \boldsymbol{\omega}_m = Y_{m,r_m,\lambda_m}Y_{k,r_k,\lambda_k}(I_m)Y_{l,r_l,\lambda_l}.$$

Recalling that $I_l = [m, k]$ and $I_m = [l, k]$, one easily checks using (4-3-2) that

$$(4\text{-}3\text{-}4) \qquad \begin{aligned} \boldsymbol{\omega}_l &= Y_{m,r_m,\lambda_m-1}Y_{k,r_k+2,\lambda_k-1}Y_{l,r_l,\lambda_l}Y_{l_*,r_m+2(\lambda_m-1)+d(l_*,m)}, \\ \boldsymbol{\omega}_m &= Y_{m,r_m,\lambda_m}Y_{k,r_k,\lambda_k-1}Y_{l,r_l+2,\lambda_l-1}Y_{m_*,r_k+2(\lambda_k-1)+d(m_*,k)}. \end{aligned}$$

Here l_* is the element of $(i_*, l]$ closest to i_* and similarly for m_*. In particular, $\boldsymbol{\omega}_m, \boldsymbol{\omega}_l \in \mathcal{P}_q^+$. Let $\lambda = \text{wt}(\boldsymbol{\omega})$ and $\nu = \lambda - \vartheta_I$.

Lemma 4.3.1. *Let $\boldsymbol{\varpi} \in \text{wt}_\ell(W)$.*

(a) *If $\text{wt}(\boldsymbol{\varpi}) \geq \nu$, $\dim(W_{\boldsymbol{\varpi}}) = 1$. In particular, $\dim(W_\nu) = \#\text{wt}_\ell(W_\nu)$.*

(b) *If $\boldsymbol{\varpi} \in \mathcal{P}_q^+ \setminus \{\boldsymbol{\omega}, \boldsymbol{\omega}_m, \boldsymbol{\omega}_l\}$ and $V_q(\boldsymbol{\varpi})$ is an irreducible factor of W, $\text{wt}(\boldsymbol{\varpi}) \not\geq \nu$.*

Proof. Set

$$D = \{\boldsymbol{\mu} \in \text{wt}_\ell(W) : \text{wt}(\boldsymbol{\mu}) \geq \nu\}$$

and, as in the proof of (3-1-6), let $\mathcal{J}$ be the set of triples $(J_i)_{i \in \partial I}$ of disjoint connected subdiagrams of I satisfying

$$i \notin J_i \quad \Leftrightarrow \quad J_i = \varnothing.$$

---

[2]All results of this section remain valid if $\lambda_k = 0$ as long as one defines r_k as in the second equality of (4-3-2).

Given $J \in \mathscr{J}$, define

(4-3-5) $$\boldsymbol{\omega}(J) = \prod_{i \in \partial I} Y_{i,r_i,\lambda_i}(J_i) \quad \text{and} \quad \operatorname{supp}(J) = \{i \in \partial I : J_i \neq \varnothing\}.$$

It easily follows from (3-4-1) and Lemma 4.1.1 that

(4-3-6) $$D = \{\boldsymbol{\omega}(J) : J \in \mathscr{J}\}.$$

Thus, part (a) is equivalent to showing that the map $\mathscr{J} \to \mathcal{P}_q$, $J \mapsto \boldsymbol{\omega}(J)$, is injective. In preparation for proving that as well as part (b), we first collect some information about the elements $\boldsymbol{\omega}(J)$.

If $\operatorname{supp}(J) = \varnothing$, there is nothing to do. Otherwise, there exists $s \in I$ such that $s \in \partial_a J_a$ for some $a \in \partial I$. Evidently, given such s, a is uniquely determined. Set

(4-3-7) $\pi(J, s) =$

$$Y_{s,r_a+2\lambda_a+d(s,a)}^{-1} \left(\prod_{b \in \partial I_a : d(s, J_b)=1} Y_{s,r_b+2(\lambda_b-1)+d(s,b)} \right) \left(\prod_{b \in \partial I} (Y_{s,r_s,\lambda_s-\delta_{s,a}})^{\delta_{s,b}} \right),$$

where $d(s, J_b) = \min\{d(s, i) : i \in J_b\}$ if $J_b \neq \varnothing$ and $d(s, J_b) = \infty$ otherwise. By definition, the s-th entry of $\boldsymbol{\omega}(J)$ coincides with that of $\pi(J, s)$. We begin by proving that

(4-3-8) $Y_{s,r_a+2\lambda_a+d(s,a)}^{-1}$ appears in $\boldsymbol{\omega}(J)$ unless $s \in \partial I_a$ and $(s, a) \in \{(l, k), (k, m)\}$.

Indeed,

$$s \notin \partial I \quad \Rightarrow \quad \pi(J, s) = Y_{s,r_a+2\lambda_a+d(s,a)}^{-1} \left(\prod_{b \in \partial I_a : d(s, J_b)=1} Y_{s,r_b+2(\lambda_b-1)+d(s,b)} \right)$$

and, hence, $Y_{s,r_a+2\lambda_a+d(s,a)}^{-1}$ does not appear if and only if there exists $b \in \partial I_a$ such that $d(s, J_b) = 1$ and

(4-3-9) $$r_b + 2(\lambda_b - 1) + d(s, b) = r_a + 2\lambda_a + d(s, a).$$

Note that (4-3-9) implies

(4-3-10) $$r_b < r_a + 2\lambda_a + d(a, b) \quad \text{and} \quad r_a < r_b + 2\lambda_b + d(a, b).$$

Indeed,

$$r_b < r_b + 2(\lambda_b - 1) + d(s, b) \overset{(4\text{-}3\text{-}9)}{=} r_a + 2\lambda_a + d(s, a) < r_a + 2\lambda_a + d(a, b)$$

and

$$r_a < r_a + 2\lambda_a + d(s, a) \overset{(4\text{-}3\text{-}9)}{=} r_b + 2(\lambda_b - 1) + d(s, b) < r_b + 2\lambda_b + d(a, b).$$

However, (4-3-10) contradicts (4-3-2), thus proving (4-3-8) when $s \notin \partial I$. Indeed, the contradiction is clear if $(a, b) \in \{(l, k), (k, l), (k, m), (m, k)\}$, while

$$r_l \overset{(4\text{-}3\text{-}2)}{=} r_m + 2\lambda_m + 2\lambda_k + d(k, m) + d(k, l)$$
$$= r_m + 2\lambda_m + 2\lambda_k + d(m, l) + 2d(k, i_*)$$
$$> r_m + 2\lambda_m + d(m, l),$$

revealing the contradiction with (4-3-10).

On the other hand,

$$s \in \partial I \quad \Rightarrow \quad \pi(J, s) = Y_{s, r_a + 2\lambda_a + d(s, a)}^{-1} Y_{s, r_s, \lambda_s - \delta_{s, a}},$$

and, hence, there will be a cancellation if, and only if,

there exists $0 \le p < \lambda_s - \delta_{s, a}$ such that $r_s + 2p = r_a + 2\lambda_a + d(s, a)$.

But then,

$$r_s = r_a + 2\lambda_a + d(s, a) - 2p \le r_a + 2\lambda_a + d(s, a),$$

and (4-3-2) implies that $p = 0$ and $(s, a) \in \{(l, k), (k, m)\}$ completing the proof of (4-3-8). Moreover,

(4-3-11) $(s, a) \in \{(l, k), (k, m)\} \quad \Rightarrow \quad \pi(J, s) = Y_{s, r_s, \lambda_s - 1}.$

Let $J, J' \in \mathscr{J}$ be such that

$$\omega(J) = \omega(J').$$

To see that $J = J'$, thus proving part (a), we will show that

(4-3-12) $\partial J_a = \partial J'_a \quad$ for all $a \in \partial I.$

Let $s \in \partial_a J_a$ for some $a \in \partial I$. If $Y_{s, r_a + 2\lambda_a + d(s, a)}^{-1}$ appears in $\omega(J)$, then $s \in \partial_b J'_b$ for some $b \in \partial I$ and

$$r_a + 2\lambda_a + d(s, a) = r_b + 2\lambda_b + d(s, b).$$

If it were $a \ne b$, this would imply (4-3-10), which is a contradiction as seen before. Hence, we must have $b = a$. If $Y_{s, r_a + 2\lambda_a + d(s, a)}^{-1}$ does not appear in $\omega(J)$, then $(s, a) \in \{(l, k), (k, m)\}$ and $\pi(J', s) = Y_{s, r_s, \lambda_s - 1}$. The latter implies that $s \in \partial_b J'_b$ for some $b \in \partial I$ and $(s, b) \in \{(l, k), (k, m)\}$. Hence, $b = a$, completing the proof of (4-3-12).

We now show that

(4-3-13) $D \cap \mathcal{P}_q^+ = \{\omega, \omega_m, \omega_l\},$

which proves part (b). Let $J \in \mathscr{J}$ be such that $\omega(J) \in \mathcal{P}_q^+$ and assume $\mathrm{supp}(J) \ne \varnothing$.

It follows from (4-3-8) that

(4-3-14)   either $[k, m] = J_m$ or $[l, k] = J_k$.

For the former, it follows that $J_k = \varnothing$ and

$$Y_{k, r_k, \lambda_k}(J_k) Y_{m, r_m, \lambda_m}(J_m) \in \mathcal{P}_q^+.$$

Since $Y_{l, r_l, \lambda_l}(J_l)$ is right negative if $J_l \neq \varnothing$, it follows that $\boldsymbol{\omega}(J) = \boldsymbol{\omega}_l$. Similarly, if $[l, k] = J_k$, it follows that $\boldsymbol{\omega}(J) = \boldsymbol{\omega}_m$. $\square$

We prove next that

(4-3-15) $\dim(V_q(\boldsymbol{\omega}_l)_v) = d(l_*, l) + 2$ and $\dim(V_q(\boldsymbol{\omega}_m)_v) = d(m_*, m) + 2,$

Plugging l_* in in place of i and l in place of j in Lemma 4.1.2, the first equality follows provided

(4-3-16) $r_l - (r_m + 2(\lambda_m - 1) + d(l_*, m)) \neq 2 + d(l_*, l).$

But (4-3-2) implies the left-hand side is strictly larger than the right-hand side. For the second statement in (4-3-15), we plug m in in place of i and m_* in place of j in Lemma 4.1.2 and check, using (4-3-2), that

$$r_k + 2(\lambda_k - 1) + d(m_*, k) - r_m \neq 2\lambda_m + d(m_*, m).$$

Remark 4.3.2. For $\mathfrak{g}$ of type D and assuming that both elements of ∂I_k are spin nodes, it was proved in [Pereira 2014, Lemma 4.6.3] that $V_q(\boldsymbol{\omega})$ is ℓ-minuscule. This implies that the qcharacter of $V_q(\boldsymbol{\omega})$ can be computed by means of the FM algorithm. A sketchy use of the algorithm was then used to prove (4-3-15) by performing a counting of ℓ-weights. The argument presented here replaces this counting by a combination of Lemma 3.3.1, which is part of the background of the FM algorithm, with the results of Section 3.1. It is interesting to note that the proof of [Pereira 2014, Lemma 4.6.3] also relies on special cases of the results from Section 3.1 whose proof in [Pereira 2014, Lemma 6.1] was sketched by making use of Nakajima's monomial realization of Kashiwara's crystals $B(\lambda)$, $\lambda \in P^+$. The proofs we gave in Section 3.1 are completely classical. Although the strategy developed here for proving Proposition 2.4.6(d) does not rely on whether $V_q(\boldsymbol{\omega})$ is ℓ-minuscule or not, it would be interesting to check if this is true in the generality we are working in here. To keep the length of the present paper within reasonable limits, we shall leave this topic to a future work. $\diamond$

Proof of Proposition 2.4.6(d). Since $m_v(V) = m_{v_{I^\lambda}}(V_q(\boldsymbol{\omega}_{I^\lambda}))$, we can assume $I^\lambda = I$ and, hence, $\text{supp}(\lambda) = \partial I$. By (4-2-7), we have

$$\dim(V_v) = \dim(V(\lambda)_v) + \dim(V(v_k)_v) + \xi \quad \text{with } 0 \leq \xi \leq 1,$$

and, after (2-4-8), we need to show that $\xi = 0$. By (3-1-9),

$$\dim(V(\nu_k)_\nu) = d(k, i_*),$$

while (3-1-6) implies

$$\dim(W_\nu) = \dim(V(\lambda)_\nu) + n + 1.$$

On the other hand, (3-1-10) is equivalent to

$$d(k, i_*) = n - 3 - d(l_*, l) - d(m_*, m)$$

and, hence,

$$\dim(W_\nu) - \dim(V_\nu) \quad = \quad d(l_*, l) + d(m_*, m) + 4 - \xi$$

$$\overset{(4\text{-}3\text{-}15)}{=} \dim(V_q(\omega_m)_\nu) + \dim(V_q(\omega_l)_\nu) - \xi > \dim(V_q(\omega_i)_\nu)$$

for $i = l, m$. Lemma 4.3.1 implies that V, $V_q(\omega_m)$, and $V_q(\omega_l)$ are the only possible irreducible factors of W having ν as a weight. Thus, the above computation shows that both $V_q(\omega_m)$ and $V_q(\omega_l)$ are indeed irreducible factors of W and, so, $\xi = 0$. $\square$

Remark 4.3.3. At the end of the above proof, we have shown that both $V_q(\omega_m)$ and $V_q(\omega_l)$ are irreducible factors of W. It is interesting to observe that, in the case that $d(l, i_*), d(m, i*) > 1$ (in particular $\mathfrak{g}$ is of type E), (4-1-5) would suffice in the above proof without the need of the sharper (4-1-6) and, hence, independently of the results of Section 4.2. The same comments apply to the coherent case treated in the next subsection. $\diamond$

4.4. Coherent tensor products of boundary KR-modules. We recall the following well-known proposition which is easily proved by considering pull-backs by the automorphisms given by [Chari 1995, Propositions 1.5 and 1.6] together with dualization (cf. [Moura and Pereira 2017, Section 4.1]).

Proposition 4.4.1. Let $\lambda \in P^+$, $\boldsymbol{\omega} = \prod_{i \in I} \boldsymbol{\omega}_{i, a_i, \lambda(h_i)}$, and $\boldsymbol{\varpi} = \prod_{i \in I} \boldsymbol{\omega}_{i, b_i, \lambda(h_i)}$, with $a_i, b_i \in \mathbb{C}^\times$. If there exists $\varepsilon = \pm 1$ such that

$$\frac{a_i}{a_j} = \left(\frac{b_j}{b_i}\right)^\varepsilon \quad \text{for all } i, j \in I,$$

then $V_q(\boldsymbol{\omega}) \cong_{U_q(\mathfrak{g})} V_q(\boldsymbol{\varpi})$. $\square$

Let W be as in Section 4.3 but this time assume $\boldsymbol{\omega}$ is coherent. Thus, letting $k \in \partial I$ be the node such that $\boldsymbol{\omega}$ is not k-minimal, by Proposition 4.4.1, we may assume

$$(4\text{-}4\text{-}1) \qquad r_l = r_k + 2\lambda_k + d(k, l) \quad \text{for all } l \in \partial I_k.$$

Using (4-1-10), set

$$\boldsymbol{\omega}_l = Y_{k, r_k, \lambda_k}(I_l) \prod_{i \in \partial I_k} Y_{i, r_i, \lambda_i} \quad \text{and} \quad \boldsymbol{\omega}' = Y_{k, r_k, \lambda_k}(I) \prod_{i \in \partial I_k} Y_{i, r_i, \lambda_i}.$$

Since
$$d(k, l) - d(k, l_*) = d(l, l_*),$$

(4-4-1) implies

$$(4\text{-}4\text{-}2) \qquad \omega_l = Y_{k,r_k,\lambda_k-1} Y_{m,r_m+2,\lambda_m-1} Y_{l,r_l,\lambda_l} Y_{l_*,r_l-2-d(l,l_*)},$$

where $m \in \partial I_k$, $m \neq l$, and

$$(4\text{-}4\text{-}3) \qquad \omega' = Y_{k,r_k,\lambda_k-1} Y_{i_*,r_k+2\lambda_k+d(i_*,k)} \prod_{i \in \partial I_k} Y_{i,r_i+2,\lambda_i-1}.$$

If there exists $l \in \partial I_k$ such that

$$(4\text{-}4\text{-}4) \qquad r_l + 2\lambda_l + d(l, m) = r_m + 2p \quad \text{for some } 0 \leq p < \lambda_m,$$

where $m \in \partial I_k$, $m \neq l$, set also

$$\omega'' = Y_{k,r_k,\lambda_k} Y_{l,r_l,\lambda_l} (I_k) Y_{m,r_m,\lambda_m}$$
$$= Y_{k,r_k,\lambda_k} Y_{k_*,r_l+2(\lambda_l-1)+d(k_*,l)} Y_{l,r_l,\lambda_l-1} Y_{m,r_m,p} Y_{m,r_m+2(p+1),\lambda_m-p-1}.$$

Note that if such l exists, it is unique.

Lemma 4.4.2. *Let $\varpi \in \mathrm{wt}_\ell(W)$.*

(a) *If $\varpi \in \{\omega, \omega', \omega'', \omega_i : i \in \partial I_k\}$, $\dim(W_\varpi) = 1$.*

(b) *If $\mathrm{wt}(\varpi) \geq v$ and $\varpi \notin \{\omega, \omega', \omega'', \omega_i : i \in \partial I_k\}$, then $\varpi \notin \mathcal{P}_q^+$. In particular, if $V_q(\varpi)$ is an irreducible factor of W, $\mathrm{wt}(\varpi) \not\geq v$.*

Proof. Defining D and $\omega(J)$ as in the proof of Lemma 4.3.1, (4-3-6) remains valid. As before, we start by collecting some information about the elements $\omega(J)$ such that $\mathrm{supp}(J) \neq \varnothing$. For $\pi(J, s)$ defined as in (4-3-7), we will prove that there exists at least one choice of (a, s) with $a \in \mathrm{supp}(J)$ and $s \in \partial_a J_a$ such that

$$(4\text{-}4\text{-}5) \qquad Y_{s,r_a+2\lambda_a+d(s,a)}^{-1} \quad \text{appears in } \omega(J),$$

unless $\# \mathrm{supp}(J) = 1$ and one of the following options holds:

(i) $J_k = I$.

(ii) $J_k = I_l$ for some $l \in \partial I_k$.

(iii) If $l \in \mathrm{supp}(J)$, $l \neq k$, then $J_l = I_k = [l, m]$ and the pair (l, m) satisfies (4-4-4).

To prove this, suppose (4-4-5) does not hold for all choices of (a, s). Assume first that

$$(4\text{-}4\text{-}6) \qquad \text{there exists } a \in \mathrm{supp}(J) \text{ such that } \partial_a J_a \cap \partial I_a = \varnothing.$$

Since none of the options (i)–(iii) satisfy this hypothesis, we need to show this yields a contradiction. Fix such a and let $s \in \partial_a J_a$, which implies $s \notin \partial I_a$. As seen

in the proof of Lemma 4.3.1, there must exist $b \in \partial I_a$ such that $d(s, J_b) = 1$ for which (4-3-9) holds. As before, this implies (4-3-10) which, this time, contradicts (4-4-1) if $(a, b) \in \{(l, k), (k, l) : l \in \partial I_k\}$. In particular, $a, b \neq k$ and $k \notin \mathrm{supp}(J)$. Note that, if $\{l, m\} = \{a, b\}$ and $t \in \partial_l J_l \cap (i_*, k)$, then $d(t, J_m) > 1$, which implies (4-4-5) holds with (l, t) in place of (a, s). Thus, either $J_a \cup J_b = I_k$ or $k \in J_b$. If $J_a \cup J_b = I_k$, letting $t \in \partial_b J_b$, it follows that $d(s, t) = 1$ and

$$\pi(J, t) = Y_{t, r_b + 2\lambda_b + d(t, b)}^{-1} Y_{t, r_a + 2(\lambda_a - 1) + d(t, a)}.$$

We claim (4-4-5) holds with (b, t) in place of (a, s), yielding the desired contradiction. Indeed, this is not the case if and only if

$$r_b + 2\lambda_b + d(t, b) = r_a + 2(\lambda_a - 1) + d(t, a),$$

which contradicts (4-3-9) since $d(s, b) = d(t, b) + 1$ and $d(t, a) = d(s, a) + 1$. If $k \in J_b$ and there exists $t \in \partial_b J_b \setminus \{k\}$, the same argument yields a contradiction. It remains to deal with the case $J_b = I_a$ and $J_a = I \setminus I_a = (i_*, a]$. In this case, we check that (4-4-5) holds with (b, k) in place of (a, s). Indeed,

$$(4\text{-}4\text{-}7) \qquad\qquad \pi(J, k) = Y_{k, r_b + 2\lambda_b + d(k, b)}^{-1} Y_{k, r_k, \lambda_k}.$$

Thus, $Y_{k, r_b + 2\lambda_b + d(k, b)}^{-1}$ is canceled if and only if

$$(4\text{-}4\text{-}8) \qquad r_b + 2\lambda_b + d(k, b) = r_k + 2p \quad \text{for some } 0 \leq p < \lambda_k.$$

One easily checks that (4-4-1) implies that there is no such p.

Assume now

$$(4\text{-}4\text{-}9) \qquad\qquad \partial_a J_a \cap \partial I_a \neq \varnothing \quad \text{for all } a \in \mathrm{supp}(J)$$

which implies $\# \mathrm{supp}(J) = 1$ and J_a is either I or I_m for some $m \in \partial I_a$. In particular, if $k \in \mathrm{supp}(J)$, then either option (i) or (ii) holds and we are done. Otherwise, we need to show we are in case (iii). Indeed, letting l be the element of $\mathrm{supp}(J)$, we must have $J_l = I$ or $J_l = I_m$ for some $m \in \partial I_l$. If $k \in \partial_l J_l$, then (4-4-7) holds with $b = l$ and we have a contradiction as before. Hence, we must have $J_l = I_k = [l, m]$ which implies

$$\pi(J, m) = Y_{m, r_l + 2\lambda_l + d(m, l)}^{-1} Y_{m, r_m, \lambda_m},$$

and we see that $Y_{m, r_l + 2\lambda_l + d(m, l)}^{-1}$ is canceled if and only (4-4-4) holds, thus proving we are in case (iii). This completes the proof of (4-4-5).

Note that $\boldsymbol{\omega}(J) = \boldsymbol{\omega}'$ if J is as in (i), $\boldsymbol{\omega}(J) = \boldsymbol{\omega}_l$ if J is as in (ii), and $\boldsymbol{\omega}(J) = \boldsymbol{\omega}''$ if J is as in (iii). Since (4-4-5) implies $\boldsymbol{\omega}(J) \notin \mathcal{P}_q^+$ if J does not satisfy one of these three conditions, all claims of the lemma follow. $\square$

Remark 4.4.3. Notice part (a) of Lemma 4.4.2 is weaker than that of Lemma 4.3.1. Indeed, that stronger statement is false in the context of this subsection. Similar

arguments to those employed in the proof of Lemma 4.3.1 can be used to show that, if $\mathrm{wt}(\varpi) \geq \nu$, then $\dim(W_\varpi) \leq 2$ and equality holds if and only if there exists $s \in I$ such that

$$(4\text{-}4\text{-}10) \quad 2\lambda_l + d(l, k) + d(s, l) = 2\lambda_m + d(m, k) + d(s, m) \quad \text{with } \{l, m\} = \partial I_k$$

and one of the following conditions holds:

(i) There exists such s in $[l, m]$ and $\varpi = \omega(J)$ for some $J \in \mathcal{J}$ such that $s \in \partial_l J_l$ and $[m, s) \subseteq J_m$;

(ii) Such s exists only in $(i_*, k]$ and $\varpi = \omega(J)$ for some $J \in \mathcal{J}$ such that $J_l = [l, s]$ and $J_m = [m, i_*)$.

In case (i), we have $\varpi = \omega(J')$ with $J'_l = J_l \setminus \{s\}$, $J'_m = J_m \cup \{s\}$, and $J'_k = J_k$. In case (ii), $\varpi = \omega(J')$ with $J'_l = [l, i_*)$, $J'_m = [m, s]$, and $J'_k = J_k$. Since these facts will not play a role in this paper, we omit the details. It might be interesting to observe that, for generic λ, there is no $s \in I$ satisfying (4-4-10) and, hence, part (a) of Lemma 4.3.1 is valid in the present context for "most" values of λ. ◇

Lemma 4.4.4. *If* (4-4-4) *holds,* $\omega'' \in \mathrm{wt}_\ell(V_q(\omega))$.

Proof. Observe that $\mathrm{wt}(\omega'') = \nu_k$. By (3-1-6) and Proposition 3.1.1,

$$\dim(W_{\nu_k}) = \dim(V(\lambda)_{\nu_k}) + 1 = d(l, m) + 2.$$

On the other hand, Lemma 4.1.2 implies $\dim(V_q(\omega)_{\nu_k}) = d(l, m) + 2$, completing the proof of the lemma. To see that Lemma 4.1.2 implies what we claimed, it suffices to check that $p > 0$ in (4-4-4). Indeed, we have

$$2p \overset{(4\text{-}4\text{-}4)}{=} r_l + 2\lambda_l + d(l, m) - r_m \overset{(4\text{-}4\text{-}1)}{=} d(k, l) - d(k, m) + d(l, m) + 2\lambda_l.$$

Since

$$d(l, m) = d(k, l) + d(k, m) - 2d(k, i_*),$$

it follows that

$$2p = 2(d(k, l) - d(k, i_*) + \lambda_l) > 0. \qquad \square$$

Proof of Proposition 2.4.6(c). Proceeding similarly to the proof of (4-3-15), this time it follows from Lemma 4.1.2 that

$$(4\text{-}4\text{-}11) \qquad \dim(V_q(\omega_l)_\nu) = d(l_*, l) + 1 \quad \text{for } l \in \partial I_k.$$

Evidently,

$$(4\text{-}4\text{-}12) \qquad \qquad \dim(V_q(\omega')_\nu) = 1.$$

Write $\partial I_k = \{l, m\}$. Proceeding as in the proof of part (d) of the proposition given

in the previous subsection, we get

$$\dim(W_\nu) - \dim(V_\nu) = d(l_*, l) + d(m_*, m) + 4 - \xi$$
$$= \dim(V_q(\omega')_\nu) + \dim(V_q(\omega_m)_\nu) + \dim(V_q(\omega_l)_\nu) + 1 - \xi.$$

Lemmas 4.4.2 and 4.4.4 imply that V, $V_q(\omega_l)$, $V_q(\omega_m)$, and $V_q(\omega')$ are the only possible irreducible factors of W having ν as a weight and all of them occur with multiplicity at most 1. Thus, the above computation shows that all of them are indeed irreducible factors of W as well as $\xi = 1$, thus proving (2-4-8).   □

4.5. *Proof of Theorem 2.4.4.* Let ω be as in Conjecture 2.4.2, choose $m \in \partial I \setminus \{k\}$, and let $\varpi \in \mathcal{P}_q^+$ be such that wt$(\varpi) = \lambda$,

$$\omega_{I_m} = \varpi_{I_m}, \quad V_q(\varpi_{I_l}) \text{ is minimal for } l \neq k, \quad \text{and} \quad \varpi \text{ is incoherent.}$$

One easily sees that such ϖ is unique for each choice of m. To simplify the writing, we will assume further that

$$(4\text{-}5\text{-}1) \qquad\qquad d(m, i_*) \leq d(l, i_*) \quad \text{for } l \in \partial I_k.$$

Under the hypothesis of Theorem 2.4.4, we will show that

$$(4\text{-}5\text{-}2) \qquad\qquad V_q(\omega) > V_q(\varpi),$$

which proves the theorem. We remark that (4-5-2) holds even if we did not choose m satisfying (4-5-1) (in fact, the two choices give rise to equivalent affinizations by Proposition 4.4.1). The reason behind this choice is that, under the hypotheses of Theorem 2.4.4, (4-5-1) implies

$$(4\text{-}5\text{-}3) \qquad\qquad \text{supp}(\lambda) \cap [i_*, m] = \{m\},$$

which simplifies part of the argument. Note that (4-5-1) also implies $d(i_*, m) \leq 2$, independently of the hypotheses of Theorem 2.4.4. Moreover, under the hypotheses of Theorem 2.4.4, $d(i_*, m) = 2$ only for $\mathfrak{g}$ of type E_6 with $d(k, i_*) = 1$, a case that can happen only under hypothesis (iii) of the theorem.

We have to show that, for all $\mu \in P^+$ such that $\mu < \lambda$,

$$(4\text{-}5\text{-}4) \quad \text{either } m_\mu(V_q(\omega)) \geq m_\mu(V_q(\varpi)) \text{ or}$$
$$\text{there exists } \mu' > \mu \text{ such that } m_{\mu'}(V_q(\varpi)) < m_{\mu'}(V_q(\omega)).$$

It obviously suffices to consider the case that $m_\mu(V_q(\varpi)) > 0$. For $i \in \partial I$, let $j_i \in [i_*, i]$ be the element satisfying

$$J_\mu \cap (j_i, i] = \varnothing \quad \text{and} \quad [i_*, j_i] \subseteq J_\mu$$

given by Lemma 3.2.4(b). Lemma 2.4.7 implies

$$l_\lambda \in (i_*, j_l] \quad \text{for } l \in \partial I_k.$$

If $k_\lambda \in [i_*, j_k]$, then $\mu \leq \nu$ and, since $m_\mu(V_q(\varpi)) > 0$, it follows from part (d) of Proposition 2.4.6 that $\mu < \nu$. Parts (c) and (d) of Proposition 2.4.6 imply that the second option in (4-5-4) is satisfied with $\mu' = \nu$. We claim that

$$k_\lambda \notin [i_*, j_k] \quad \Rightarrow \quad m_\mu(V_q(\omega)) = m_\mu(V_q(\varpi)),$$

which completes the proof of (4-5-4) and, hence, of Theorem 2.4.4. The claim clearly follows if we prove that, for $\pi \in \{\omega, \varpi\}$, we have

$$(4\text{-}5\text{-}5) \qquad k_\lambda \notin [i_*, j_k] \quad \Rightarrow \quad V_q(\pi_{J_\mu}) \cong V_q(\pi_{J_\mu}^{\{m\}}) \otimes V_q(\pi_{J_\mu}^{[i_*,l]}),$$

where l is the unique element of $\partial I \setminus \{k, m\}$. Note that we have used (4-5-3) here.

Suppose first that

$$j_k = i_*$$

which implies that J_μ is of type A. In order to prove (4-5-5), we shall use Theorem 3.4.1. For $i \in I$, set $\lambda_i = \lambda(h_i)$. Since ω is coherent, there exist $a \in \mathbb{F}^\times$ and $\epsilon \in \{-1, 1\}$ such that

$$\omega = \omega_{k_\lambda, a, \lambda_{k_\lambda}} \; \omega_{m, a_m, \lambda_m} \prod_{i \in (i_*, l]} \omega_{i, a_i, \lambda_i}$$

with

$$(4\text{-}5\text{-}6) \quad a_m = a q^{\epsilon(\lambda_{k_\lambda} + \lambda_m + d(m, i_*) + 1)} \quad \text{and} \quad a_i = a q^{\epsilon\left(\lambda_{k_\lambda} + \lambda_i + d(i, i_*) + 1 + 2 \sum_{j \in (i_*, i)} \lambda_j\right)}$$

for all $i \in (i_*, l]$. Then, by definition of ϖ, we have

$$\varpi = \omega_{k_\lambda, a, \lambda_{k_\lambda}} \; \omega_{m, b_m, \lambda_m} \prod_{i \in (i_*, l]} \omega_{i, a_i, \lambda_i}$$

with

$$(4\text{-}5\text{-}7) \qquad\qquad b_m = a q^{-\epsilon(\lambda_{k_\lambda} + \lambda_m + d(m, i_*) + 1)}.$$

Since the irreducibility of $V_q(\pi_{J_\mu})$ is independent of the value of ϵ, in order to avoid stating the version of Theorem 3.4.1 for decreasing minimal affinizations (see Remark 3.4.2), we shall choose ϵ so that we place ourselves in the context of Theorem 3.4.1 as stated here. Thus, we identify J_μ with a diagram of type A_n by letting l be identified with 1 and m with n and choose $\epsilon = -1$. With these choices, (4-5-5) follows if the pair $(\pi_{J_\mu}^{[i_*,l]}, \pi_{J_\mu}^{\{m\}})$ does not satisfy any of the conditions of Theorem 3.4.1 for $\pi \in \{\omega, \varpi\}$. Note also that l_λ corresponds to j in the statement of Theorem 3.4.1 and, hence, a_{l_λ} corresponds to a there and the ratio $q^s = b/a$ corresponds to a_m/a_{l_λ}, in the case $\pi = \omega$, and b_m/a_{l_λ} for $\pi = \varpi$. This gives

$$(4\text{-}5\text{-}8) \qquad \begin{aligned} \pi = \omega \quad &\Rightarrow \quad s = \lambda_{l_\lambda} - \lambda_m + d(l_\lambda, i_*) - d(m, i_*), \\ \pi = \varpi \quad &\Rightarrow \quad s = 2\lambda_{k_\lambda} + \lambda_{l_\lambda} + \lambda_m + d(m, i_*) + d(l_\lambda, i_*) + 2. \end{aligned}$$

Since λ_m corresponds to η in the statement of Theorem 3.4.1, the negation of its condition (i) is

$$(4\text{-}5\text{-}9)\quad s + \lambda_m + d(i,m) + 2 + \lambda_{l_\lambda} + d(i,l_\lambda) + 2\sum_{p\in[i,l_\lambda)}\lambda_p \neq 2t$$

$$\text{for all } i \in \operatorname{supp}(\lambda)\cap[l,i_*],\ 1\leq t\leq\min\{\lambda_i,\lambda_m\},$$

while that of condition (ii) is

$$(4\text{-}5\text{-}10)\quad s - \lambda_m - \lambda_{l_\lambda} - d(l_\lambda,m) - 2 \neq -2t\quad\text{for all } 1\leq t\leq\min\{{}_l|\lambda|_{i_*},\lambda_m\}.$$

Thus, to prove (4-5-5) in the case $d(k_\lambda,i_*)=1$, it suffices to check that (4-5-9) and (4-5-10) hold for s as in (4-5-8). For $\boldsymbol{\pi}=\boldsymbol{\omega}$, (4-5-9) becomes

$$2(\lambda_i - t) + 2\lambda_{l_\lambda} + d(l_\lambda,i_*) + d(i,m) + d(i,l_\lambda) + (2 - d(i_*,m)) + 2\sum_{p\in(i,l_\lambda)}\lambda_p \neq 0,$$

for all $1\leq t\leq\min\{\lambda_i,\lambda_m\}$, which is true since, for such t, all the summands above are nonnegative and several of them are not zero (e.g., $\lambda_{l_\lambda}\neq 0$). Equation (4-5-10) for $\boldsymbol{\pi}=\boldsymbol{\omega}$ becomes

$$2(\lambda_m - t) + (d(l_\lambda,m) - d(l_\lambda,i_*)) + d(m,i_*) + 2 \neq 0$$

for all $1\leq t\leq\min\{{}_l|\lambda|_{i_*},\lambda_m\}$. As before, we see that, for such t, all summands are nonnegative and several are positive, completing the proof of (4-5-5) in the case $d(k_\lambda,i_*)=1$ and $\boldsymbol{\pi}=\boldsymbol{\omega}$. For $\boldsymbol{\pi}=\boldsymbol{\varpi}$, (4-5-9) becomes

$$2(\lambda_{k_\lambda}+\lambda_{l_\lambda}+\lambda_i+\lambda_m-t)+d(l_\lambda,i_*)+d(i,m)+d(i,l_\lambda)+d(m,i_*)+4+2\sum_{p\in(i,l_\lambda)}\lambda_p\neq 0$$

for all $1\leq t\leq\min\{\lambda_i,\lambda_m\}$, which is easily seen to be true as before. On the other hand, (4-5-10) becomes

$$2\lambda_{k_\lambda} + 2t + (d(l_\lambda,i_*) - d(l_\lambda,m)) + d(m,i_*) \neq 0,$$

which clearly holds for all $t\geq 1$.

Finally, suppose

$$j_k \neq i_*$$

which, together with the hypothesis in (4-5-5), implies we must be under hypothesis (i) or (iii) of Theorem 2.4.4. In particular, J_μ is of type D_n with $n\geq 5$, $d(m,i_*)=1$, and $l_\lambda = l$. Hence, we need to check that $(\boldsymbol{\pi}_{J_\mu}^{\{m\}},\boldsymbol{\pi}_{J_\mu}^{\{l\}})$ does not satisfy any of the conditions of Corollary 3.4.6(a), in case hypothesis (i) is satisfied, and of Corollary 3.4.6(b), in case hypothesis (iii) is satisfied, with l corresponding to the nonspin node. This time there exist $a\in\mathbb{F}^\times$ and $\epsilon\in\{-1,1\}$ such that

$$\boldsymbol{\omega}=\boldsymbol{\omega}_{k_\lambda,a,\lambda_{k_\lambda}}\,\boldsymbol{\omega}_{m,a_m,\lambda_m}\,\boldsymbol{\omega}_{l,a_l,\lambda_l}\quad\text{and}\quad\boldsymbol{\varpi}=\boldsymbol{\omega}_{k_\lambda,a,\lambda_{k_\lambda}}\,\boldsymbol{\omega}_{m,b_m,\lambda_m}\,\boldsymbol{\omega}_{l,a_l,\lambda_l}$$

with

(4-5-11) $\dfrac{a_m}{a} = q^{\epsilon(\lambda_{k_\lambda} + \lambda_m + d(m,k_\lambda))} = \dfrac{a}{b_m}$ and $\dfrac{a_l}{a} = q^{\epsilon(\lambda_{k_\lambda} + \lambda_l + d(m,k_\lambda) + t)}$,

where $t = 0$ for hypothesis (i) (both l and m are spin nodes) and $t = 1$ for hypothesis (iii) (l and k are the extremal nodes of the subdiagram of type A_5). For checking (4-5-5) with $\pi = \omega$, in case (i), we need to check that

$$\lambda_m - \lambda_l \neq \pm(\lambda_m + \lambda_l + 2(2s - p)) \quad \text{for all } 1 \le p \le \min\{\lambda_m, \lambda_l\},\ 1 \le s \le \lfloor(\#J_\mu - 1)/2\rfloor.$$

But equality holds if and only if there exist $i \in \{m, l\}$ and s, p in the above ranges such that

$$\lambda_i - p + 2s = 0,$$

which is impossible since $\lambda_i - p \ge 0$ and $s > 0$. Similarly, in case (iii), one easily checks that

$$\lambda_m - \lambda_l - 1 \neq \pm(\lambda_m + \lambda_l + \#J_\mu - 2p) \quad \text{for all } 1 \le p \le \min\{\lambda_m, \lambda_l\}.$$

For checking (4-5-5) with $\pi = \varpi$, in case (i), we need to check that

$$2\lambda_{k_\lambda} + \lambda_m + \lambda_l + 2d(k_\lambda, m) \neq \pm(\lambda_m + \lambda_l + 2(2s - p))$$
$$\text{for all } 1 \le p \le \min\{\lambda_m, \lambda_l\},\ 1 \le s \le \lfloor(\#J_\mu - 1)/2\rfloor.$$

But equality is impossible because

$$\lambda_{k_\lambda} + d(k_\lambda, m) \ge d(j_k, m) + 2 = \#J_\mu \quad \text{while} \quad 2s - p \le \#J_\mu - 1$$

and

$$\lambda_{k_\lambda} + \lambda_m + \lambda_l + d(k_\lambda, m) > \#J_\mu + 2 + 2\min\{\lambda_m, \lambda_l\} \quad \text{while} \quad p - 2s \le \min\{\lambda_m, \lambda_l\}.$$

In case (iii), we have $\#J_\mu = 5$, $d(k_\lambda, m) = d(k, m) = 3$ and one can check that

$$2\lambda_{k_\lambda} + \lambda_m + \lambda_l + 2d(k_\lambda, m) + 1 \neq \pm(\lambda_m + \lambda_l + \#J_\mu - 2p) \quad \text{for all } 1 \le p \le \min\{\lambda_m, \lambda_l\}.$$

This completes the proof of Theorem 2.4.4.

Acknowledgements

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INTERIOR GRADIENT ESTIMATES FOR WEAK SOLUTIONS OF QUASILINEAR p-LAPLACIAN TYPE EQUATIONS

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We study the interior weighted Sobolev regularity for weak solutions of the quasilinear equations of the form div $A(x, u, \nabla u) = \operatorname{div} F$. The vector field A is allowed to be discontinuous in x, Hölder continuous in u and its growth in the gradient variable is like the p-Laplace operator with $1 < p < \infty$. We establish interior weighted $W^{1,q}$-regularity estimates for weak solutions to the equations for every $q > p$ assuming that the weak solutions are in the local John–Nirenberg BMO space. This paper therefore improves available results because it replaces the boundedness or continuity assumption on weak solutions by the borderline BMO one. Our regularity estimates also recover known results in which A is independent of the variable u. Our regularity theory complements the classical $C^{1,\alpha}$-regularity theory developed by many mathematicians including DiBenedetto and Tolksdorf for this general class of quasilinear elliptic equations.

1. Introduction

This paper establishes interior regularity estimates in weighted Sobolev spaces for weak solutions to the following general quasilinear p-Laplacian type equations:

$$(1\text{-}1) \qquad \operatorname{div}[A(x, u, \nabla u)] = \operatorname{div}[|F|^{p-2}F] \quad \text{in } B_{2R},$$

where B_{2R} is the ball in $\mathbb{R}^n$ centered at the origin and with radius $2R$ for some $R > 0$, F is a given measurable vector field function, u is an unknown solution, and

$$A = A(x, z, \xi) : B_{2R} \times \mathbb{K} \times \mathbb{R}^n \to \mathbb{R}^n$$

is a given vector field. We assume that $A(\,\cdot\,, z, \xi)$ is measurable in B_{2R} for every $(z, \xi) \in \mathbb{K} \times (\mathbb{R}^n \setminus \{0\})$, $A(x, \cdot\,, \xi)$ Hölder continuous in $\mathbb{K}$ for a.e. $x \in B_{2R}$ and for all $\xi \in \mathbb{R}^n \setminus \{0\}$, and $A(x, z, \cdot)$ differentiable in $\mathbb{R}^n \setminus \{0\}$ for each $z \in \mathbb{K}$ and for a.e. $x \in B_{2R}$. Here, $\mathbb{K}$ is an open interval in $\mathbb{R}$, which could be the same as $\mathbb{R}$. We

assume in addition that there exist constants $\Lambda > 0$, $\alpha \in (0, 1]$, and $1 < p < \infty$ such that A satisfies the natural growth conditions

(1-2)
$$\langle \partial_\xi A(x, z, \xi)\eta, \eta \rangle \geq \Lambda^{-1}|\xi|^{p-2}|\eta|^2$$
$$\text{for a.e. } x \in B_{2R}, \ \forall z \in \mathbb{K}, \ \forall \xi, \eta \in \mathbb{R}^n \setminus \{0\},$$

(1-3) $\quad |A(x, z, \xi)| + |\xi| \, |\partial_\xi A(x, z, \xi)| \leq \Lambda|\xi|^{p-1}$
$$\text{for a.e. } x \in B_{2R}, \ \forall z \in \mathbb{K}, \ \forall \xi \in \mathbb{R}^n \setminus \{0\},$$

(1-4) $\quad |A(x, z_1, \xi) - A(x, z_2, \xi)| \leq \Lambda|\xi|^{p-1}|z_1 - z_2|^\alpha$
$$\text{for a.e. } x \in B_{2R}, \ \forall z_1, z_2 \in \mathbb{K}, \ \forall \xi \in \mathbb{R}^n \setminus \{0\}.$$

Observe that under the conditions (1-2)–(1-4), the class of equations of the form (1-1) contains the well-known p-Laplace equations.

The focus of this paper is to investigate the regularity in weighted Sobolev spaces for weak solutions u of (1-1) when the nonlinearity of A depends on u as its variable. In this perspective, we would like to point out that, on the one hand, the $C^{1,\alpha}$-regularity theory for *bounded, weak solutions* of this class of equations has been investigated extensively, assuming some regularity of A in both x and z variables; see [DiBenedetto 1983; Evans 1982; Lewis 1983; Lieberman 1988; Gilbarg and Trudinger 1983; Ladyzhenskaya and Ural'tseva 1968; Malý and Ziemer 1997; Tolksdorf 1984; Ural'tseva 1968; Uhlenbeck 1977]. On the other hand, when A is discontinuous in x or F is not sufficiently regular, one does not expect those mentioned Schauder's type estimates for weak solutions of (1-1) to hold, and it is natural to search for L^q-estimates for the gradients instead; see [Gilbarg and Trudinger 1983; Ladyzhenskaya and Ural'tseva 1968; Maugeri et al. 2000; Krylov 2007; Malý and Ziemer 1997], for example. In this line of research, we note that in case $A = A_0$ for some A_0 which is independent of the variable $z \in \mathbb{K}$, the equation (1-1) is reduced to

(1-5)
$$\text{div}\,[A_0(x, \nabla u)] = \text{div}\,[|F|^{p-2}F] \quad \text{in } B_{2R},$$

and the $W^{1,q}$-regularity estimates of Calderón–Zygmund type for weak solutions to the class of equations (1-5) has been studied by many authors; for example, see [Iwaniec 1983; DiBenedetto and Manfredi 1993; Byun and Wang 2012; Byun et al. 2007; Caffarelli and Peral 1998; Di Fazio 1996; Duzaar and Mingione 2010; 2011; Kinnunen and Zhou 1999; Maugeri et al. 2000; Mengesha and Phuc 2012; Dong and Kim 2010; Krylov 2007; 2008]. However, if A depends on the z-variable as in (1-1) and even with $F = 0$, the $W^{1,q}$-regularity estimates become much more challenging, and not very well understood. This is due to the fact that the Calderón–Zygmund theory relies heavily on the scaling and dilation invariances of the considered class of equations; see [Wang 2003] for the geometric intuition of this fact. Since the

class of equations (1-5) is invariant under the scalings

$$(1\text{-}6) \qquad u \mapsto u/\lambda \quad \text{and} \quad u(x) \mapsto \frac{u(rx)}{r} \qquad \text{for all positive numbers } r, \lambda,$$

the $W^{1,q}$-regularity of Calderón–Zygmund for weak solutions of (1-5) is therefore naturally expected. Meanwhile, the invariant homogeneity with respect to (1-6) is no longer available for (1-1). This fact presents a serious obstacle in obtaining $W^{1,q}$-estimates for the weak solutions of (1-1) as they do not generate enough estimates to carry out the proof by using existing methods.

In the recent work [Hoang et al. 2015; Nguyen and Phan 2016], the $W^{1,q}$-regularity estimates for weak solutions of (1-1) are addressed, and the $W^{1,q}$-regularity estimates are established assuming that the weak solutions are bounded. To overcome the loss of the homogeneity that we mentioned, we introduced in [Hoang et al. 2015; Nguyen and Phan 2016] some "double-scaling parameter" technique. Essentially, we study an enlarged class of "double parameter" equations of the type (1-1). Then, by a compactness argument, we successfully applied the perturbation method in [Caffarelli and Peral 1998] to tackle the problem. Careful analysis is required to ensure that all intermediate steps in the perturbation process are uniform with respect to the scaling parameters. See also [Byun et al. 2017; Phan 2017] for further implementation of this idea, and [Dong and Kim 2011] for some other related results in this line of research. In the papers [Hoang et al. 2015; Nguyen and Phan 2016; Byun et al. 2017], the a priori boundedness assumption on the weak solutions is essential to start the investigation of $W^{1,q}$-theory. This is because the approach uses the maximum principle for the unperturbed equations to implement the perturbation technique of [Caffarelli and Peral 1998]. We also would like to reference [Bögelein 2014], where the same $W^{1,p}$-theory for parabolic equations of type (1-1) is also achieved for continuous weak solutions.

A natural question arises from the mentioned work: *Is it necessary to assume that solutions are bounded, both for Sobolev regularity theory and Schauder's regularity*? In this paper, we give an answer to this question in the Sobolev regularity setting. In particular, we establish the $W^{1,q}$-regularity estimates for weak solutions of (1-1) by assuming that the solutions are in the BMO John–Nirenberg space, i.e., the borderline case. This is achieved in Theorem 1.1 below. Our paper therefore generalizes all results in [Bögelein 2014; Byun et al. 2017; Hoang et al. 2015; Nguyen and Phan 2016]. Moreover, this paper also simplifies many technical issues in [Hoang et al. 2015; Nguyen and Phan 2016], and gives a generic approach to unify and treat both classes of equations (1-1) and (1-5) at the same time. Unlike [Byun et al. 2017; Hoang et al. 2015; Nguyen and Phan 2016], we only use "one parameter" in the class of our equations. Precisely, we investigate the equation

$$(1\text{-}7) \qquad \operatorname{div}\left[\boldsymbol{A}(x, \lambda u, \nabla u)\right] = \operatorname{div}\left[|\boldsymbol{F}|^{p-2}|\boldsymbol{F}\right] \quad \text{in } B_{2R},$$

with the parameter $\lambda \geq 0$. The class of equations (1-7) is indeed the smallest one that is invariant with respect to the scalings and dilation (1-6) and that includes (1-1). When $\lambda = 0$, the equation (1-7) clearly becomes the equation (1-5). This paper therefore recovers known results such as [Iwaniec 1983; DiBenedetto and Manfredi 1993; Byun and Wang 2012; Byun et al. 2007; Caffarelli and Peral 1998; Di Fazio 1996; Duzaar and Mingione 2010; 2011; Kinnunen and Zhou 1999; Maugeri et al. 2000; Mengesha and Phuc 2012] regarding the interior regularity of weak solutions of (1-5).

From now on, the notation A_q with $q \geq 1$ stands for the class of Muckenhoupt weights, whose definition is recalled in Definition 2.3. Also, $B_R(y)$ is the ball in $\mathbb{R}^n$ with radius $R > 0$ and centered at $y \in \mathbb{R}^n$. For simplicity, we also write $B_R = B_R(0)$. Moreover, for some locally integrable function $f : U \to \mathbb{R}$ with some measurable set $U \subset \mathbb{R}^n$ and with $\rho_0 > 0$, the BMO seminorm of bounded mean oscillation of f is defined by

$$\llbracket f \rrbracket_{\mathrm{BMO}(U, \rho_0)} = \sup_{y \in U, \, 0 < \rho < \rho_0} \frac{1}{|B_\rho(y)|} \int_{B_\rho(y) \cap U} |f(x) - \bar{f}_{B_\rho(y) \cap U}| \, dx,$$

$$\text{where } \bar{f}_{B_\rho(y) \cap U} = \frac{1}{|B_\rho(y)|} \int_{B_\rho(y) \cap U} f(x) \, dx.$$

The main result of this paper is the following interior regularity estimates for weak solutions of (1-7) in weighted Lebesgue spaces.

Theorem 1.1. *Let $\Lambda > 0$, $M > 0$, $p, q > 1$, $\gamma \geq 1$, and $\alpha \in (0, 1]$. Then there exists a sufficiently small constant $\delta = \delta(p, q, n, \Lambda, M, \gamma, \alpha) > 0$ such that the following statement holds true. Assume that $A : B_{2R} \times \mathbb{K} \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory map satisfying (1-2)–(1-4) and*

(1-8) $\llbracket A \rrbracket_{\mathrm{BMO}(B_R, R)}$
$$:= \sup_{0 < \rho \leq R} \sup_{y \in B_R} \frac{1}{|B_\rho(y)|} \int_{B_\rho(y)} \left[\sup_{z \in \mathbb{K}, \, \xi \in \mathbb{R}^n \setminus \{0\}} \frac{|A(x, z, \xi) - \overline{A}_{B_\rho(y)}(z, \xi)|}{|\xi|^{p-1}} \right] dx$$
$$\leq \delta$$

for some $R > 0$ and for some open interval $\mathbb{K} \subset \mathbb{R}$. Then for every $F \in L^p(B_{2R}, \mathbb{R}^n)$, if u is a weak solution of

$$\mathrm{div}\,[A(x, \lambda u, \nabla u)] = \mathrm{div}\,[|F|^{p-2}F] \quad \text{in } B_{2R}$$

with $\llbracket \lambda u \rrbracket_{\mathrm{BMO}(B_R, R)} \leq M$ for some $\lambda \geq 0$, the weighted regularity estimate

$$\int_{B_R} |\nabla u|^{pq} \omega(x) \, dx \leq C \left[\int_{B_{2R}} |F|^{pq} \omega(x) \, dx + \omega(B_{2R}) \left(\frac{1}{|B_{2R}|} \int_{B_{2R}} |\nabla u|^p \, dx \right)^q \right]$$

holds, as long as its right-hand side is finite, where $\omega \in A_q$ *with*

$$[\omega]_{A_q} \leq \gamma, \qquad \overline{A}_{B_\rho(y)}(z, \xi) := \fint_{B_\rho(y)} A(x, z, \xi)\, dx,$$

and C is a constant depending only on q, p, n, Λ, α, M, $\mathbb{K}$, R, *and* γ.

We emphasize that the significant contribution in Theorem 1.1 is that it relaxes and do not requires the considered weak solutions to be bounded as in [Bögelein 2014; Byun et al. 2017; Hoang et al. 2015; Nguyen and Phan 2016]. This is completely new even for the case $\omega = 1$, in comparison to the known work that we already mentioned for both the Schauder and the Sobolev regularity theories regarding weak solutions of (1-1). Certainly, removing the boundedness assumption on solutions and replacing it by the condition that weak solutions are in BMO is valuable in the critical cases in which the L^∞-bound for solutions are not available; see [DiBenedetto and Manfredi 1993], for example. When $p = n$, our weak solutions are in $W^{1,n}$, and hence they are in BMO by the Sobolev embedding theorem. Therefore, in this case, our theorem is applicable directly, while results [Bögelein 2014; Byun et al. 2017; Hoang et al. 2015; Nguyen and Phan 2016] may not be. Note that M is not required to be small: our $[\![\lambda u]\!]_{\text{BMO}(B_R, R)}$ is not necessarily small. When $\lambda = 0$, the condition $[\![\lambda u]\!]_{\text{BMO}(B_R, R)} \leq M$ is certainly held for every function u. Therefore, Theorem 1.1 recovers results in [Iwaniec 1983; Byun and Wang 2012; Byun et al. 2007; Caffarelli and Peral 1998; Di Fazio 1996; Duzaar and Mingione 2010; 2011; Kinnunen and Zhou 1999; Mengesha and Phuc 2012], in which the case that A is independent of $z \in \mathbb{K}$ is studied. This paper therefore unifies $W^{1,q}$-regularity estimates for both (1-1) and (1-5). We also would like to note that the fact that A is defined in $z \in \mathbb{K}$ only is important in many applications. A simple example is $\mathbb{K} = (0, \infty)$, meaning that (1-2)–(1-4) only hold for positive solutions u. In the study of cross-diffusion equations in [Hoang et al. 2015], $K = (0, M_0)$ for some $M_0 > 0$.

We remark that the smallness condition (1-8) on the mean oscillation of A with respect to the x-variable is necessary as there is a counterexample provided in [Meyers 1963] for linear equations. In this regard, we also would like to point out that in [Dong and Kim 2010], regularity estimates for weak solutions of equations with measurable coefficients that are small in partial BMO-seminorm are established.

This paper follows the perturbation approach of [Caffarelli and Peral 1998] and makes use of the Hardy–Littlewood maximal function; see also [Byun and Wang 2012; Byun et al. 2017; Nguyen and Phan 2016; Hoang et al. 2015; Phan 2017; Wang 2003]. One can also find in [Krylov 2007; 2008; Dong and Kim 2010; 2011] for a similar perturbation approach which uses the Fefferman–Stein sharp function. To overcome the loss of boundedness of solutions due to our assumption, instead of applying the maximum principle during the perturbation process as in prior work,

we directly derive and delicately use Hölder's regularity estimates for solutions of the corresponding homogeneous equations; see the estimates (3-4) and (3-15), for example. The well-known John–Nirenberg's theorem and reverse Hölder's inequality also play a very important role in our approach.

We now conclude this section by outlining the organization of this paper. Section 2 reviews some definitions and some known results needed in the paper. Intermediate steps in the approximation estimates required in the proof of Theorem 1.1 are established and proved in Section 3. Finally, Section 4 gives the proof of Theorem 1.1.

2. Definitions and preliminaries

Scaling invariances, and definitions of weak solutions. Let $\lambda' \geq 0$, and let us consider a function $u \in W^{1,p}_{\text{loc}}(U)$ satisfying

$$\text{div}\,[A(x, \lambda'u, \nabla u)] = \text{div}\,[|F|^{p-2}F] \quad \text{in } U,$$

in the sense of distribution, for some open bounded set $U \subset \mathbb{R}^n$. Then for some fixed $\lambda > 0$, the rescaled function

$$(2\text{-}1) \qquad\qquad v(x) = \frac{u(x)}{\lambda} \quad \text{for } x \in U$$

solves the equation

$$\text{div}\,[\hat{A}(x, \hat{\lambda}v, \nabla v)] = \text{div}\,[|\hat{F}|^{p-2}\hat{F}] \quad \text{in } U$$

in the distributional sense, where $\hat{\lambda} = \lambda\lambda' \geq 0$ and $\hat{A} : U \times \mathbb{K} \times \mathbb{R}^n \to \mathbb{R}^n$ is defined by

$$(2\text{-}2) \qquad \hat{A}(x, z, \xi) = \frac{A(x, z, \lambda\xi)}{\lambda^{p-1}} \quad \text{and} \quad \hat{F}(x) = \frac{F(x)}{\lambda^{p-1}}.$$

Remark 2.1. If $A : U \times \mathbb{K} \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies the conditions (1-2)–(1-4) on $U \times \mathbb{K} \times \mathbb{R}^n$, then the rescaled vector field $\hat{A} : U \times \mathbb{K} \times \mathbb{R}^n \to \mathbb{R}^n$ defined in (2-2) also satisfies the structural conditions (1-2)–(1-4) with the same constants Λ, p, and α. Moreover, $[\![A]\!]_{\text{BMO}(U, \rho_0)} = [\![\hat{A}]\!]_{\text{BMO}(U, \rho_0)}$ for any $\rho_0 > 0$.

In this paper, $C_0^\infty(U)$ is the set of all smooth compactly supported functions in U, $L^p(U, \mathbb{R}^n)$ with $1 \leq p < \infty$ is the Lebesgue space consisting of all measurable functions $f : U \to \mathbb{R}^n$ such that $|f|^p$ is integrable on U, and $W^{1,p}(U)$ is the standard Sobolev space on U. Moreover, $\langle\,\cdot\,,\cdot\,\rangle$ is the Euclidean inner product in $\mathbb{R}^n$. Let us now recall the definitions of weak solutions that we use throughout the paper.

Definition 2.2. Let $\mathbb{K} \subset \mathbb{R}$ be an interval, and let $\Lambda > 0$, $p > 1$, $\alpha \in (0, 1]$. Also, let $U \subset \mathbb{R}^n$ be an open bounded set in $\mathbb{R}^n$ with sufficiently smooth boundary ∂U, and let $A : U \times \mathbb{K} \times \mathbb{R}^n \to \mathbb{R}^n$ satisfy conditions (1-2)–(1-4) on $U \times \mathbb{K} \times \mathbb{R}^n$.

(i) For every $\boldsymbol{F} \in L^p(U; \mathbb{R}^n)$ and $\lambda \geq 0$, a function $u \in W^{1,p}_{\mathrm{loc}}(U)$ is called a *weak solution* of

$$\mathrm{div}\,[\boldsymbol{A}(x, \lambda u, \nabla u)] = \mathrm{div}\,[|\boldsymbol{F}|^{p-2}\boldsymbol{F}] \quad \text{in } U$$

if $\lambda u(x) \in \mathbb{K}$ for a.e. $x \in U$, and

$$\int_U \langle \boldsymbol{A}(x, \lambda u, \nabla u), \nabla \varphi \rangle\, dx = \int_U \langle |\boldsymbol{F}|^{p-2}\boldsymbol{F}, \nabla \varphi \rangle\, dx \quad \forall \varphi \in C_0^\infty(U).$$

(ii) For every $\boldsymbol{F} \in L^p(U; \mathbb{R}^n)$, $g \in W^{1,p}(U)$, and $\lambda \geq 0$, a function $u \in W^{1,p}(U)$ is a *weak solution* of

$$\begin{cases} \mathrm{div}\,[\boldsymbol{A}(x, \lambda u, \nabla u)] = \mathrm{div}\,[|\boldsymbol{F}|^{p-2}\boldsymbol{F}] & \text{in } U, \\ \qquad\qquad\qquad\qquad u = g & \text{on } \partial U, \end{cases}$$

if $\lambda u(x) \in \mathbb{K}$ for a.e. $x \in U$, $u - g \in W^{1,p}_0(U)$, and

$$\int_U \langle \boldsymbol{A}(x, \lambda u, \nabla u), \nabla \varphi \rangle\, dx = \int_U \langle \boldsymbol{F}, \nabla \varphi \rangle\, dx \quad \forall \varphi \in C_0^\infty(U).$$

Muckenhoupt weights, weighted inequalities, and the crawling ink-spots lemma.
This section recalls several analysis results and definitions that are needed in the paper. Firstly, we recall the definition of the A_p-Muckenhoupt class of weights introduced in [Muckenhoupt 1972].

Definition 2.3. Let $1 \leq p < \infty$. A nonnegative and locally integrable function $\omega : \mathbb{R}^n \to [0, \infty)$ is said to be in *the class A_p of Muckenhoupt weights* if

$$[\omega]_{A_p} := \sup_{\text{balls } B \subset \mathbb{R}^n} \left(\fint_B \omega(x)\, dx \right) \left(\fint_B \omega(x)^{1/(1-p)}\, dx \right)^{p-1} < \infty \quad \text{if } p > 1,$$

$$[\omega]_{A_1} := \sup_{\text{balls } B \subset \mathbb{R}^n} \left(\fint_B \omega(x)\, dx \right) \|\omega^{-1}\|_{L^\infty(B)} < \infty \qquad\qquad \text{if } p = 1.$$

It turns out that the class of A_p-Muckenhoupt weights satisfies the reverse Hölder's inequality and the doubling properties. In particular, a measure of any A_p-weight is comparable with the Lebesgue measure in some sense. This is in fact a well-known result due to R. Coifman and C. Fefferman, and it is an important ingredient in the paper.

Lemma 2.4 [Coifman and Fefferman 1974]. *For $1 < p < \infty$, the following statements hold true*:

(i) *If $\mu \in A_p$, then for every ball $B \subset \mathbb{R}^n$ and every measurable set $E \subset B$,*

$$\mu(B) \leq [\mu]_{A_p} \left(\frac{|B|}{|E|} \right)^p \mu(E).$$

(ii) *If $\mu \in A_p$ with $[\mu]_{A_p} \le \gamma$ for some given $\gamma \ge 1$, then there are $C = C(\gamma, n)$ and $\beta = \beta(\gamma, n) > 0$ such that*

$$\mu(E) \le C \left(\frac{|E|}{|B|} \right)^{\beta} \mu(B)$$

for every ball $B \subset \mathbb{R}^n$ and every measurable set $E \subset B$.

Observe that in the above statement and in this paper, the notation

$$|U| = \int_U dx, \qquad \mu(U) = \int_U \mu(x) \, dx,$$

for every measurable set $U \subset \mathbb{R}^n$ is used.

Secondly, we state a standard result in measure theory.

Lemma 2.5. *Assume that $g \ge 0$ is a measurable function in a bounded subset $U \subset \mathbb{R}^n$. Let $\theta > 0$ and $N > 1$ be given constants. If μ is a weight function in $\mathbb{R}^n$, then for any $1 \le p < \infty$,*

$$g \in L^p(U, \mu) \Leftrightarrow S := \sum_{j \ge 1} N^{pj} \mu(\{x \in U : g(x) > \theta N^j\}) < \infty.$$

Moreover, there exists a constant $C > 0$ depending only on θ, N, and p such that

$$C^{-1} S \le \|g\|_{L^p(U, \mu)}^p \le C(\mu(U) + S),$$

where $L^p(U, \mu)$ is the weighted Lebesgue space with norm

$$\|g\|_{L^p(U, \mu)} = \left(\int_U |g(x)|^p \mu(x) \, dx \right)^{1/p}.$$

Thirdly, we discuss the Hardy–Littlewood maximal operator and its boundedness in weighted spaces. For a given locally integrable function $f : \mathbb{R}^n \to \mathbb{R}$, the Hardy–Littlewood maximal function is defined as

$$(2\text{-}3) \qquad \mathcal{M} f(x) = \sup_{\rho > 0} \fint_{B_\rho(x)} |f(y)| \, dy \quad \text{for } x \in \mathbb{R}^n.$$

For a function f that is defined on a bounded domain U, we write

$$\mathcal{M}_U f(x) = \mathcal{M}(f \chi_U)(x),$$

where χ_U is the characteristic function of the set U. The following boundedness of the Hardy–Littlewood maximal operator $\mathcal{M} : L^q(\mathbb{R}^n, \omega) \to L^q(\mathbb{R}^n, \omega)$ is classical.

Lemma 2.6. *Let $\gamma \ge 1$ and $\omega \in A_q$ with $[\omega]_{A_q} \le \gamma$.*

(i) *Strong (q, q): Let $1 < q < \infty$. Then there exists a constant $C = C(\gamma, q, n)$ such that*

$$\|\mathcal{M}\|_{L^q(\mathbb{R}^n, \omega) \to L^q(\mathbb{R}^n, \omega)} \le C.$$

(ii) *Weak* (1, 1): *There exists a constant* $C = C(n)$ *such that for any* $\lambda > 0$, *we have*

$$\left| \{x \in \mathbb{R}^n : \mathcal{M}(f) > \lambda\} \right| \leq \frac{C}{\lambda} \int_{\mathbb{R}^n} |f| \, dx.$$

Finally, we recall the following important lemma. This lemma is usually referred to as the "crawling ink-spots" lemma, and is originally due to N. V. Krylov and M. V. Safonov [Krylov and Safonov 1979; Safonov 1980].

Lemma 2.7 (crawling ink-spots). *Suppose that* $\omega \in A_q$ *with* $[\omega]_{A_q} \leq \gamma$ *for some* $1 < q < \infty$ *and some* $\gamma \geq 1$. *Suppose also that* $R > 0$ *and that* C, D *are measurable sets satisfying* $C \subset D \subset B_R$. *Assume that there are* $\rho_0 \in (0, R/2)$ *and* $0 < \epsilon < 1$ *such that*

(i) $\omega(C) < \epsilon \omega(B_{\rho_0}(y))$ *for almost every* $y \in B_R$, *and*

(ii) *for all* $x \in B_R$ *and* $\rho \in (0, \rho_0)$, *if* $\omega(C \cap B_\rho(x)) \geq \epsilon \omega(B_\rho(x))$, *then*

$$B_\rho(x) \cap B_R \subset D.$$

Then

$$\omega(C) \leq \epsilon_1 \omega(D) \quad \text{for } \epsilon_1 = \epsilon 20^{nq} \gamma^2.$$

Hölder regularity and self-improving regularity. We recall some classical regularity results. The first is about the interior Hölder regularity for weak solutions of homogeneous p-Laplacian type equations (1-5). This result is indeed a consequence of the well-known De Giorgi–Nash–Moser theory; see [Giusti 2003, Theorem 7.6; Ladyzhenskaya and Ural'tseva 1968, Theorem 1.1, p. 251].

Lemma 2.8. *Let* $\Lambda > 0$, $p > 1$, *and let* $\mathbb{A}_0 : B_r \times \mathbb{R}^n \to \mathbb{R}^n$ *be a Carathéodory map and satisfy* (1-2)–(1-3) *on* $B_r \times \mathbb{R}^n$ *with some* $r > 0$. *If* $v \in W^{1,p}(B_r)$ *is a weak solution of the equation*

$$\operatorname{div}[\mathbb{A}_0(x, \nabla v)] = 0 \quad \text{in } B_r,$$

then there is $C_0 > 0$ *depending only on* Λ, n, p *such that*

$$\|v\|_{L^\infty(B_{5r/6})} \leq C_0 \left(\fint_{B_r} |v|^p \, dx \right)^{1/p}.$$

Moreover, there is a constant $\beta \in (0, 1)$ *depending only on* Λ, n, p, *and* $\|v\|_{L^\infty(B_{5r/6})}$ *such that*

$$|v(x) - v(y)| \leq C_0 \|v\|_{L^\infty(B_{5r/6})} \left(\frac{|x - y|}{r} \right)^\beta \quad \forall x, y \in \bar{B}_{2r/3}.$$

We now recall a classical result on self-improving regularity estimates for weak solutions of p-Laplacian type equations. The following result is due to N. Meyers and A. Elcrat [1975, Theorem 1]; see also [DiBenedetto and Manfredi 1993] and, for the parabolic version, [Kinnunen and Lewis 2000].

Lemma 2.9. *Let* $\Lambda > 0$, $p > 1$. *Then there exists* $p_0 = p_0(\Lambda, n, p) > p$ *such that the following statement holds true. Suppose that* $\mathbb{A}_0 : B_{2r} \times \mathbb{R}^n \to \mathbb{R}^n$ *is a Carathéodory map satisfying* (1-2)–(1-3) *on* $B_{2r} \times \mathbb{R}^n$ *with some* $r > 0$. *If* $v \in W^{1,p}(B_{2r})$ *is a weak solution of the equation*

$$\text{div}\,[\mathbb{A}_0(x, \nabla v)] = 0 \quad \text{in } B_{2r},$$

then for every $p_1 \in [p, p_0]$, *there exists a constant* $C = C(\Lambda, p_1, p, n) > 0$ *such that*

$$\left(\frac{1}{|B_r|} \int_{B_r} |\nabla v|^{p_1}\, dx\right)^{1/p_1} \le C \left(\frac{1}{|B_{2r}|} \int_{B_{2r}} |\nabla v|^p\, dx\right)^{1/p}.$$

Some simple energy estimates. In this section we derive some elementary estimates which will be used frequently in the paper.

Lemma 2.10. *Let* $\Lambda > 0$, $p > 1$, *and let* $U \subset \mathbb{R}^n$ *be a bounded open set and* $\mathbb{K}$ *an interval in* $\mathbb{R}$. *Assume that* $A : U \times \mathbb{K} \times \mathbb{R}^n \to \mathbb{R}^n$ *satisfies* (1-2)–(1-3) *on* $U \times \mathbb{K} \times \mathbb{R}^n$. *Then for any functions* $u, v \in W^{1,p}(U)$ *and any nonnegative function* $\phi \in C(\overline{U})$, *the following hold*:

(i) *If* $1 < p < 2$, *then for every* $\tau > 0$,

$$\int_U |\nabla u - \nabla v|^p \phi\, dx \le \tau \int_U |\nabla u|^p \phi\, dx$$
$$+ C(\Lambda, p)\tau^{(p-2)/p} \int_U \langle A(x, u, \nabla u) - A(x, u, \nabla v), \nabla u - \nabla v\rangle \phi\, dx.$$

(ii) *If* $p \ge 2$, *then*

$$\int_U |\nabla u - \nabla v|^p \phi\, dx \le C(\Lambda, p) \int_U \langle A(x, u, \nabla u) - A(x, u, \nabla v), \nabla u - \nabla v\rangle \phi\, dx.$$

Proof. This lemma is well-known; see [Tolksdorf 1984, Lemma 1; Nguyen and Phan 2016, Lemma 3.1]. However, because it is important and also for completeness, we provide the proof. We first claim that from (1-2), the monotonicity property

$$(2\text{-}4) \quad \langle A(x, z, \xi) - A(x, z, \eta), \xi - \eta\rangle$$
$$\ge \begin{cases} \gamma_0 |\xi - \eta|^p & \text{if } p \ge 2, \\ \gamma_0 (|\xi| + |\xi - \eta|)^{p-2}|\xi - \eta|^2 & \text{if } 1 < p < 2 \end{cases}$$

of A holds true for all $(x, z) \in U \times \mathbb{K}$ and all $\xi, \eta \in \mathbb{R}^n \setminus \{0\}$, where $\gamma_0 = \gamma_0(\Lambda, p) > 0$ is a constant. To prove the claim, observe that for each $(x, z) \in U \times \mathbb{K}$ and each $\xi, \eta \in \mathbb{R}^n \setminus \{0\}$, we can write

$$(2\text{-}5) \quad \langle A(x, z, \xi) - A(x, z, \eta), \xi - \eta\rangle$$
$$= \int_0^1 \langle A_\xi(x, z, \xi + t(\eta - \xi))(\xi - \eta), \xi - \eta\rangle\, dt,$$

where $A_\xi(x, z, \cdot)$ is the matrix of partial derivatives of A with respect to the third component variable in $\mathbb{R}^n \setminus \{0\}$ of A. It follows from (1-2) that

$$(2\text{-}6) \quad \langle A_\xi(x, z, \xi + t(\eta - \xi))(\xi - \eta), \xi - \eta \rangle \geq \Lambda^{-1} |\xi + t(\eta - \xi)|^{p-2} |\xi - \eta|^2.$$

Then, if $p \in (1, 2)$, we see that $|\xi + t(\eta - \xi)| \leq |\xi| + |\xi - \eta|$, and therefore,

$$\langle A(x, z, \xi) - A(x, z, \eta), \xi - \eta \rangle \geq \Lambda^{-1} (|\xi| + |\xi - \eta|)^{p-2} |\xi - \eta|^2.$$

Hence, the second estimate in (2-4) is proved. On the other hand, when $p \geq 2$, by (2-5)–(2-6), we see that

$$\langle A(x, z, \xi) - A(x, z, \eta), \xi - \eta \rangle \geq \Lambda^{-1} |\xi - \eta|^2 \int_0^{1/4} |\xi + t(\eta - \xi)|^{p-2} \, dt.$$

We may now assume without loss of generality that $|\xi - \eta| \neq 0$ and $|\eta| \leq |\xi|$. Let us define $t_0 = |\xi|/|\xi - \eta|$. Note that if $|\xi - \eta| \leq 2|\xi|$, then $t_0 \geq \frac{1}{2}$ and

$$|\xi + t(\eta - \xi)| \geq \left| |\xi| - t|\xi - \eta| \right| = |t - t_0| |\xi - \eta| \geq \tfrac{1}{4} |\xi - \eta| \quad \forall t \in \left(0, \tfrac{1}{4}\right).$$

Otherwise, we have $|\eta| \leq |\xi| \leq \frac{1}{2}|\xi - \eta|$, and then

$$\begin{aligned} |\xi + t(\eta - \xi)| &= |(1 - t)(\xi - \eta) + \eta| \\ &\geq (1 - t)|\xi - \eta| - |\eta| \\ &\geq \tfrac{3}{4}|\xi - \eta| - \tfrac{1}{2}|\xi - \eta| = \tfrac{1}{4}|\xi - \eta| \quad \forall t \in \left(0, \tfrac{1}{4}\right). \end{aligned}$$

Hence, in conclusion, we have $|\xi + t(\eta - \xi)| \geq \frac{1}{4}|\xi - \eta|$ for all $t \in \left(0, \frac{1}{4}\right)$, and therefore,

$$\langle A(x, z, \xi) - A(x, z, \eta), \xi - \eta \rangle \geq \frac{1}{4^{p-1}\Lambda} |\xi - \eta|^p.$$

This proves the first estimate in (2-4) when $p \geq 2$, completing the proof of (2-4).

Finally, observe that from (2-4), (ii) becomes trivial. Therefore, it remains to prove (i) with $1 < p < 2$. In this case, for each $\xi, \eta \in \mathbb{R}^n \setminus \{0\}$ and each $\tau \in (0, 1)$, we can use Young's inequality to obtain

$$\begin{aligned} |\xi - \eta|^p &= (|\xi| + |\xi - \eta|)^{p(2-p)/2} (|\xi| + |\xi - \eta|)^{p(p-2)/2} |\xi - \eta|^p \\ &\leq \frac{\tau}{3-p} (|\xi| + |\xi - \eta|)^p + C_p \tau^{(p-2)/p} (|\xi| + |\xi - \eta|)^{p-2} |\xi - \eta|^2. \end{aligned}$$

From this and (2-4), we infer that

$$\begin{aligned} |\xi - \eta|^p &\leq \tau |\xi|^p + C_p \tau^{(p-2)/p} (|\xi| + |\xi - \eta|)^{p-2} |\xi - \eta|^2 \\ &\leq \tau |\xi|^p + C(\Lambda, p) \tau^{(p-2)/p} \langle A(x, z, \xi) - A(x, z, \eta), \xi - \eta \rangle. \end{aligned}$$

Then (i) follows and the proof of Lemma 2.10 is complete. □

Lemma 2.11 (Caccioppoli type estimates). *Let $\Lambda > 0$, $p > 1$ be fixed. Then for every $r > 0$ and every $A_0 : B_r \times \mathbb{R}^n$ satisfying (1-2)–(1-3) on $B_r \times \mathbb{R}^n$, if $v \in W^{1,p}(B_r)$ is a weak solution of*

$$\operatorname{div}[A_0(x, \nabla v)] = 0 \quad in \ B_r$$

then it holds that

$$\int_{B_r} |\nabla v|^p \phi(x)^p \, dx \le C(\Lambda, p) \int_{B_r} |v - k|^p |\nabla \phi(x)|^p \, dx$$

for all $\phi \in C_0^1(B_r)$, $\phi \ge 0$, and for all $k \in \mathbb{R}$.

Proof. Since $(v - k)\phi \in W_0^{1,p}(B_r)$, we can use it as a test function. From this, together with Hölder's inequality and Young's inequality, we can infer that

$$\int_{B_r(x_0)} \langle A_0(x, \nabla v) - A_0(x, 0), \nabla v \rangle \phi^p \, dx$$
$$= -p \int_{B_r(x_0)} \langle A_0(x, \nabla v), \nabla \phi \rangle (v - k) \phi^{p-1} \, dx$$
$$\le C(\Lambda, p) \int_{B_r(x_0)} |\nabla v|^{p-1} \phi^{p-1} |\nabla \phi| |v - k| \, dx$$
$$\le \frac{1}{4} \int_{B_r(x_0)} |\nabla v|^p \phi^p(x) \, dx + C(\Lambda, p) \int_{B_r(x_0)} |v - k|^p |\nabla \phi|^p \, dx.$$

Now, by Lemma 2.10, it follows that

$$\int_{B_r(x_0)} |\nabla v|^p \phi^p \, dx \le \frac{1}{4} \int_{B_r(x_0)} |\nabla v|^p \phi^p \, dx$$
$$+ C(\Lambda, p) \int_{B_r(x_0)} \langle A_0(x, \nabla v) - A_0(x, 0), \nabla v \phi^p \rangle \, dx$$
$$\le \frac{1}{2} \int_{B_r(x_0)} |\nabla v|^p \phi^p \, dx + C(\Lambda, p) \int_{B_r(x_0)} |v - k|^p |\nabla \phi|^p \, dx.$$

Therefore,

$$\int_{B_r(x_0)} |\nabla v|^p \phi(x)^p \, dx \le C(\Lambda, p) \int_{B_r(x_0)} |v - k|^p |\nabla \phi(x)|^p \, dx,$$

as desired. □

A known approximation estimate. We recall a known approximation estimate established in [Byun and Wang 2012; Byun et al. 2007] and many other papers for the solutions of equations of the type (1-5) in which the vector field A_0 is independent of the variable $z \in \mathbb{K}$. This approximation estimate will be used in an intermediate step for the proof of Theorem 1.1.

Lemma 2.12. *Let* $\Lambda > 0$, $p > 1$ *be fixed. Then for every* $\epsilon \in (0, 1)$, *there exists a sufficiently small number* $\delta_0 = \delta_0(\epsilon, \Lambda, n, p) \in (0, \epsilon)$ *such that the following holds. Assume that* $\mathbf{A}_0 : B_{2R} \times \mathbb{R}^n \to \mathbb{R}^n$ *is such that* (1-2)–(1-3) *hold, and*

$$\sup_{\substack{\xi \in \mathbb{R}^n \\ \xi \neq 0}} \sup_{\substack{x \in B_{2R} \\ 0 < \rho < R}} \frac{1}{|B_\rho(x)|} \int_{B_\rho(x)} \frac{|\mathbf{A}_0(y, \xi) - \overline{\mathbf{A}}_{0, B_\rho(x)}|}{|\xi|^{p-1}} \, dy \leq \delta_0.$$

Then for every $x_0 \in B_R$ *and* $r \in (0, R/2)$, *and for* $\mathbf{G} \in L^p(B_{2R}, \mathbb{R}^n)$, *if* w *is a weak solution in* $W^{1,p}(B_{2r}(x_0))$ *of*

$$\operatorname{div}[\mathbf{A}_0(x, \nabla w)] = \operatorname{div}[|\mathbf{G}|^{p-2}\mathbf{G}] \quad \text{in } B_{2r}(x_0)$$

satisfying

$$\frac{1}{|B_{2r}(x_0)|} \int_{B_{2r}(x_0)} |\nabla w|^p \, dx \leq 1,$$

and if

$$\frac{1}{|B_{2r}(x_0)|} \int_{B_{2r}(x_0)} |\mathbf{G}|^p \, dx \leq \delta_0^p,$$

then there is $h \in W^{1,p}(B_{7r/4}(x_0))$ *such that*

$$\frac{1}{|B_{7r/4}(x_0)|} \int_{B_{7r/4}(x_0)} |\nabla w - \nabla h|^p \, dx \leq \epsilon^p, \quad \|\nabla h\|_{L^\infty(B_{3r/2}(x_0))} \leq C(\Lambda, n, p).$$

3. Interior approximation estimates

In this section, let $\mathbf{A} : B_{2R} \times \mathbb{K} \times \mathbb{R}^n \to \mathbb{R}^n$ satisfy (1-2)–(1-4) on $B_{2R} \times \mathbb{K} \times \mathbb{R}^n$ for some $R > 0$ and some open interval $\mathbb{K} \subset \mathbb{R}$. We study the weak solutions $u \in W^{1,p}(B_{2R})$ of the scaling parameter equation

$$(3\text{-}1) \qquad \operatorname{div}[\mathbf{A}(x, \lambda u, \nabla u)] = \operatorname{div}[|\mathbf{F}|^{p-2}\mathbf{F}] \quad \text{in } B_{2R},$$

with the parameter $\lambda \geq 0$. Our goal in this section is to provide necessary estimates for proving Theorem 1.1. Our approach is based on the perturbation technique introduced in [Caffarelli and Peral 1998] together with the "scaling parameter" technique introduced in [Hoang et al. 2015; Nguyen and Phan 2016]. The approach is also influenced by recent developments [Bögelein 2014; Byun and Wang 2012; Byun et al. 2007; 2017; Phan 2017]. In our first step, we fix u in $\mathbf{A}$ and then approximate the solution u of (3-1) by a solution of the corresponding homogeneous equations with the fixed u coefficient, as in [Bögelein 2014; Byun et al. 2017].

Lemma 3.1. *Let* $\Lambda, M > 0$, $p > 1$ *be fixed and* $\kappa \in (0, 1]$. *Then, for every small* $\epsilon \in (0, 1)$, *there exists a sufficiently small number* $\delta_1 = \delta_1(\epsilon, \Lambda, n, p, \kappa) \in (0, \epsilon)$ *such that the following holds. Assume that* $\mathbf{A} : B_{2R} \times \mathbb{K} \times \mathbb{R}^n \to \mathbb{R}^n$ *satisfies* (1-2)–(1-4) *with some* $\mathbb{K} \subset \mathbb{R}$ *and some* $R > 0$, *and that* $\mathbf{F} \in L^p(B_{2R}, \mathbb{R}^n)$ *satisfies*

$$\fint_{B_r(x_0)} |\boldsymbol{F}|^p \, dx \leq \delta_1^p$$

for some $x_0 \in B_R$ and $r \in (0, R)$. Suppose also that $u \in W^{1,p}(B_{2R})$ is a weak solution of (3-1) *satisfying*

$$\fint_{B_r(x_0)} |\nabla u|^p \, dx \leq 1 \quad \text{and} \quad \lambda \left(\fint_{B_r(x_0)} |u - \bar{u}_{B_r(x_0)}|^p \right)^{1/p} \leq M$$

for some $\lambda \geq 0$. Then

$$(3\text{-}2) \qquad \fint_{B_r(x_0)} |\nabla u - \nabla v|^p \, dx \leq \epsilon^p \kappa^n,$$

where $v \in W^{1,p}(B_r)$ is the weak solution of

$$(3\text{-}3) \qquad \begin{cases} \text{div}\, [\boldsymbol{A}(x, \lambda u, \nabla v)] = 0 & \text{in } B_r(x_0), \\ v = u - \bar{u}_{B_r(x_0)} & \text{on } \partial B_r(x_0). \end{cases}$$

Moreover, it also holds that

$$(3\text{-}4) \qquad \lambda \left(\fint_{B_r(x_0)} |v|^p \, dx \right)^{1/p} \leq C(n, p)[M + \lambda r \epsilon \kappa^{n/p}].$$

Proof. Note that for $\widetilde{\boldsymbol{A}}_0(x, \xi) := \boldsymbol{A}(x, \lambda u(x), \xi)$, we see that $\widetilde{\boldsymbol{A}}_0$ is independent of the variable $z \in \mathbb{K}$, and it satisfies the assumptions (1-2)–(1-3). The equation (3-3) is written as

$$(3\text{-}5) \qquad \begin{cases} \text{div}\, [\widetilde{\boldsymbol{A}}_0(x, \nabla v)] = 0 & \text{in } B_r(x_0), \\ v = u - \bar{u}_{B_r(x_0)} & \text{on } \partial B_r(x_0), \end{cases}$$

and we note that the existence of the weak solution v of (3-5) follows from the standard theory in calculus of variation. Therefore, it remains to prove the estimates (3-2) and (3-4). Since $v - [u - \bar{u}_{B_r(x_0)}] \in W_0^{1,p}(B_r(x_0))$, we can take it as a test function for (3-3); we obtain

$$\int_{B_r(x_0)} \langle \boldsymbol{A}(x, \lambda u, \nabla v), \nabla u - \nabla v \rangle \, dx = 0.$$

Similarly, we can use $v - [u - \bar{u}_{B_r(x_0)}]$ as a test function for the equation for (3-1) to see that

$$\int_{B_r(x_0)} \langle \boldsymbol{A}(x, \lambda u, \nabla u), \nabla u - \nabla v \rangle \, dx = \int_{B_r(x_0)} \langle |\boldsymbol{F}|^{p-2} \boldsymbol{F}, \nabla u - \nabla v \rangle \, dx.$$

It then follows from these two identities that

$$(3\text{-}6) \quad \int_{B_r(x_0)} \langle \boldsymbol{A}(x, \lambda u, \nabla u) - \boldsymbol{A}(x, \lambda u, \nabla v), \nabla v - \nabla u \rangle \, dx$$

$$= \int_{B_r(x_0)} \langle |\boldsymbol{F}|^{p-2} \boldsymbol{F}, \nabla u - \nabla v \rangle \, dx.$$

We only consider the case $1 < p < 2$, because the case $p \geq 2$ is similar, and simpler. It follows from Lemma 2.10(i), Remark 2.1, and (3-6) that for each $\tau \in (0, 1)$,

$$
\int_{B_r(x_0)} |\nabla u - \nabla v|^p \, dx
$$
$$
\leq \tau \int_{B_r(x_0)} |\nabla u|^p \, dx
$$
$$
+ C(\Lambda, \tau, p) \int_{B_r(x_0)} \langle A(x, \lambda u, \nabla u) - A(x, \lambda u, \nabla v), \nabla v - \nabla u \rangle \, dx
$$
$$
\leq \tau \int_{B_r(x_0)} |\nabla u|^p \, dx + C(\Lambda, \tau, p) \int_{B_r(x_0)} \left| \langle |F|^{p-2} F, \nabla u - \nabla v \rangle \right| \, dx
$$
$$
\leq \tau \int_{B_r(x_0)} |\nabla u|^p \, dx + \frac{1}{2} \int_{B_r(x_0)} |\nabla u - \nabla v|^p \, dx + C(\Lambda, \tau, p) \int_{B_r(x_0)} |F|^p \, dx,
$$

where in the last step, we have used Hölder's inequality and Young's inequality. Hence, by canceling similar terms, we obtain

$$
\fint_{B_r(x_0)} |\nabla u - \nabla v|^p \, dx \leq 2\tau \fint_{B_r(x_0)} |\nabla u|^p \, dx + C(\Lambda, \tau, p) \fint_{B_r(x_0)} |F|^p \, dx.
$$

Now, choosing $\tau = \epsilon^p \kappa^n / 4$ and $\delta_1 = \delta_1(\epsilon, \Lambda, n, p, \kappa) \in (0, \epsilon)$ sufficiently small so that $C(\Lambda, \tau, p)\delta^p < \epsilon^p \kappa^n / 2$, the estimate (3-2) follows. It remains to prove (3-4). By Poincaré's inequality, we see that

$$
\left(\fint_{B_r(x_0)} |v|^p \, dx \right)^{1/p}
$$
$$
\leq C(p) \left[\left(\fint_{B_r(x_0)} \left| v - [u - \bar{u}_{B_r(x_0)}] \right|^p \, dx \right)^{1/p} + \left(\fint_{B_r(x_0)} |u - \bar{u}_{B_r(x_0)}|^p \, dx \right)^{1/p} \right]
$$
$$
\leq C(n, p) \left[r \left(\fint_{B_r(x_0)} |\nabla v - \nabla u|^p \, dx \right)^{1/p} + \left(\fint_{B_r(x_0)} |u - \bar{u}_{B_r(x_0)}|^p \, dx \right)^{1/p} \right].
$$

From this and since $\kappa \in (0, 1)$, it follows that

$$
\lambda \left(\fint_{B_r(x_0)} |v|^p \, dx \right)^{1/p} \leq C(n, p)[M + r \lambda \epsilon \kappa^{n/p}],
$$

as desired. □

Next, we approximate the solution u by the solution w of the following equation, whose principal part is a vector field that is independent of w and has small oscillation with respect to the x-variable:

$$
\text{(3-7)} \qquad \begin{cases} \operatorname{div}\left[A(x, \lambda \bar{u}_{B_{\kappa r}(x_0)}, \nabla w) \right] = 0 & \text{in } B_{\kappa r}(x_0), \\ w = v & \text{on } \partial B_{\kappa r}(x_0), \end{cases}
$$

where v is the weak solution of (3-3) and $\kappa \in \left(0, \frac{1}{3}\right)$ is sufficiently small to be determined. Our next result is in the same fashion as Lemma 3.1.

Lemma 3.2. *Let Λ, $M > 0$, $p > 1$, and $\alpha \in (0, 1]$ be fixed, and let $\epsilon \in (0, 1)$. There exist positive, sufficiently small numbers $\kappa = \kappa(\epsilon, \Lambda, M, p, n, \alpha) \in \left(0, \frac{1}{3}\right)$ and $\delta_2 = \delta_2(\epsilon, \Lambda, M, n, \alpha, p) \in (0, \epsilon)$ such that the following holds. Assume that $A : B_{2R} \times \mathbb{K} \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies* (1-2)–(1-4) *with some $R > 0$ and some open interval $\mathbb{K} \subset \mathbb{R}$, and assume that $F \in L^p(B_{2R}, \mathbb{R}^n)$ and*

$$\fint_{B_r(x_0)} |F|^p \, dx \leq \delta_2^p$$

for some $x_0 \in B_R$ and $r \in (0, R/2)$. Then, for every $\lambda > 0$, if $u \in W^{1,p}(B_{2R})$ is a weak solution of (3-1) *satisfying*

$$\fint_{B_{2\kappa r}(x_0)} |\nabla u|^p \, dx \leq 1, \quad \fint_{B_r(x_0)} |\nabla u|^p \, dx \leq 1, \quad and \quad [\![\lambda u]\!]_{\mathrm{BMO}(B_R, R)} \leq M,$$

then it holds that

$$(3\text{-}8) \quad \left(\fint_{B_{\kappa r}(x_0)} |\nabla u - \nabla w|^p \, dx\right)^{1/p} \leq \epsilon \quad and \quad \left(\fint_{B_{\kappa r}} |\nabla w|^p \, dx\right)^{1/p} \leq C_0(n, p),$$

where w is the weak solution of (3-7).

Proof. For a given sufficiently small $\epsilon > 0$, let $\epsilon' \in (0, \epsilon/2)$ and $\kappa \in \left(0, \frac{1}{3}\right)$, both sufficiently small and depending on $\epsilon, \Lambda, M, n, \alpha, p$, which will be determined. Then, let $\delta_2 = \delta_1(\epsilon', \Lambda, n, p, \kappa) > 0$, where δ_1 is defined as in Lemma 3.1. Let v be the solution of (3-3). By using Lemma 3.1, we see that

$$(3\text{-}9) \quad \begin{aligned} \fint_{B_r(x_0)} |\nabla u - \nabla v|^p \, dx &\leq (\epsilon')^p \kappa^n, \\ \lambda \left(\fint_{B_r(x_0)} |v|^p \, dx\right)^{1/p} &\leq C(n, p)[r\epsilon'\lambda\kappa^{n/p} + M]. \end{aligned}$$

Observe also that the first inequality in (3-9), the assumption in the lemma, and the fact that both ϵ and κ are small imply that

$$(3\text{-}10) \quad \begin{aligned} \left(\fint_{B_{2\kappa r}(x_0)} |\nabla v|^p \, dx\right)^{1/p} \\ &\leq \left(\fint_{B_{2\kappa r}(x_0)} |\nabla u - \nabla v|^p \, dx\right)^{1/p} + \left(\fint_{B_{2\kappa r}(x_0)} |\nabla u|^p \, dx\right)^{1/p} \\ &\leq \left(\frac{1}{2^n \kappa^n} \fint_{B_r(x_0)} |\nabla u - \nabla v|^p \, dx\right)^{1/p} + \left(\fint_{B_{2\kappa r}(x_0)} |\nabla u|^p \, dx\right)^{1/p} \\ &\leq \frac{\epsilon'}{2^{n/p}} + 1 \leq 2. \end{aligned}$$

On the other hand, from the Caccioppoli type estimate in Lemma 2.11, (3-9), and $\kappa \in \left(0, \frac{1}{3}\right)$, we also see that

$$(3\text{-}11) \quad \left(\frac{1}{|B_{2\kappa r}(x_0)|}\int_{B_{2\kappa r}(x_0)}|\nabla v|^p\,dx\right)^{1/p} \leq \frac{C(\Lambda,n,p)}{(1-2\kappa)r\kappa^{n/p}}\left(\frac{1}{|B_r(x_0)|}\int_{B_r(x_0)}|v|^p\,dx\right)^{1/p}$$
$$\leq C(\Lambda,n,p)\left[\epsilon' + M(\lambda\kappa^{n/p}r)^{-1}\right].$$

Now, let w be the weak solution of (3-7). As in the proof of Lemma 3.1, the existence of w is assured. Therefore, it remains to prove the estimate (3-8). Taking $w - v \in W_0^{1,p}(B_{\kappa r}(x_0))$ as a test function for (3-7) and (3-3), we obtain

$$(3\text{-}12) \quad \int_{B_{\kappa r}(x_0)} \langle A(x,\lambda u,\nabla v),\nabla w - \nabla v\rangle\,dx$$
$$= \int_{B_{\kappa r}(x_0)} \langle A(x,\lambda\bar{u}_{B_{\kappa r}(x_0)},\nabla w),\nabla w - \nabla v\rangle\,dx = 0.$$

Again, we only need to consider the case $1 < p < 2$, as $p \geq 2$ can be done similarly using (ii) of Lemma 2.10. From now on, for simplicity, we write $\hat{u} = u - \bar{u}_{B_{\kappa r}(x_0)}$. We can use Lemma 2.10(i), the condition (1-4), and (3-12) to obtain, with some $\tau > 0$ sufficiently small to be determined,

$$\int_{B_{\kappa r}(x_0)}|\nabla v - \nabla w|^p\,dx$$
$$\leq \tau\int_{B_{\kappa r}(x_0)}|\nabla v|^p\,dx + \left(C(\Lambda,p)\tau^{(p-2)/p}\right.$$
$$\left. \times \int_{B_{\kappa r}(x_0)}\langle A(x,\lambda\bar{u}_{B_{\kappa r}(x_0)},\nabla v) - A(x,\lambda\bar{u}_{B_{\kappa r}(x_0)},\nabla w),\nabla v - \nabla w\rangle\,dx\right)$$
$$\leq \tau\int_{B_{\kappa r}(x_0)}|\nabla v|^p\,dx + \left(C(\Lambda,p)\tau^{(p-2)/p}\right.$$
$$\left. \times \int_{B_{\kappa r}(x_0)}\langle A(x,\lambda\bar{u}_{B_{\kappa r}(x_0)},\nabla v) - A(x,\lambda u,\nabla v),\nabla v - \nabla w\rangle\,dx\right)$$
$$\leq \tau\int_{B_{\kappa r}(x_0)}|\nabla v|^p\,dx + C(\Lambda,p)\tau^{(p-2)/p}\int_{B_{\kappa r}(x_0)}|\lambda\hat{u}|^\alpha|\nabla v|^{p-1}|\nabla v - \nabla w|\,dx$$
$$\leq \frac{1}{2}\int_{B_{\kappa r}(x_0)}|\nabla v - \nabla w|^p\,dx + \tau\int_{B_{\kappa r}(x_0)}|\nabla v|^p\,dx$$
$$+ C(\Lambda,p)\tau^{(p-2)/(p-1)}\int_{B_{\kappa r}(x_0)}|\lambda\hat{u}|^{\alpha p/(p-1)}|\nabla v|^p\,dx,$$

where in the last step, we have used the Hölder's inequality and Young's inequality. Hence, by canceling similar terms, we obtain

$$\frac{1}{|B_{\kappa r}(x_0)|}\int_{B_{\kappa r}(x_0)}|\nabla v - \nabla w|^p\,dx$$
$$\leq \frac{2\tau}{|B_{\kappa r}(x_0)|}\int_{B_{\kappa r}(x_0)}|\nabla v|^p\,dx + \frac{C(\Lambda,p)\tau^{(p-2)/(p-1)}}{|B_{\kappa r}(x_0)|}\int_{B_{\kappa r}(x_0)}|\lambda\hat{u}|^{\alpha p/(p-1)}|\nabla v|^p\,dx.$$

For q_1 greater than but sufficiently close to p and depending only on Λ, p, we write

$$q_1 = \frac{\alpha p p_1}{(p-1)(p_1 - p)} > p.$$

Using Hölder's inequality, the self-improving regularity estimate (i.e., Lemma 2.9), and (3-10), we then obtain

$$\frac{1}{|B_{\kappa r}(x_0)|} \int_{B_{\kappa r}(x_0)} |\nabla v - \nabla w|^p \, dx$$

$$\leq \frac{2\tau}{|B_{\kappa r}(x_0)|} \int_{B_{\kappa r}(x_0)} |\nabla v|^p \, dx + \left[C(\Lambda, p)\tau^{(p-2)/(p-1)} \right.$$

$$\left. \times \left(\frac{1}{|B_{\kappa r}(x_0)|} \int_{B_{\kappa r}(x_0)} |\lambda \hat{u}|^{q_1} \right)^{(p_1 - p)/p_1} \left(\frac{1}{|B_{\kappa r}(x_0)|} \int_{B_{\kappa r}(x_0)} |\nabla v|^{p_1} \, dx \right)^{p/p_1} \right]$$

$$\leq C(\Lambda, n, p) \left[2\tau + \tau^{(p-2)/(p-1)} \left(\frac{1}{|B_{\kappa r}(x_0)|} \int_{B_{\kappa r}(x_0)} |\lambda \hat{u}|^{q_1} \, dx \right)^{(p_1 - p)/p_1} \right]$$

$$\times \left(\frac{1}{|B_{2\kappa r}(x_0)|} \int_{B_{2\kappa r}(x_0)} |\nabla v|^p \, dx \right).$$

Now, from the well-known John–Nirenberg theorem, we further write

$$\frac{1}{|B_{\kappa r}(x_0)|} \int_{B_{\kappa r}(x_0)} |\lambda \hat{u}|^{q_1} \, dx$$

$$= \frac{1}{|B_{\kappa r}(x_0)|} \int_{B_{\kappa r}(x_0)} |\lambda \hat{u}|^{p/2} |\lambda \hat{u}|^{q_1 - p/2} \, dx$$

$$\leq \left(\frac{1}{|B_{\kappa r}(x_0)|} \int_{B_{\kappa r}(x_0)} |\lambda \hat{u}|^p \, dx \right)^{1/2} \left(\frac{1}{|B_{\kappa r}(x_0)|} \int_{B_{\kappa r}(x_0)} |\lambda \hat{u}|^{2q_1 - p} \, dx \right)^{1/2}$$

$$\leq C(n, \alpha, p) [\![\lambda u]\!]_{\mathrm{BMO}(B_R, R)}^{q_1 - p/2} \left(\frac{1}{|B_{\kappa r}(x_0)|} \int_{B_{\kappa r}(x_0)} |\lambda \hat{u}|^p \, dx \right)^{1/2}$$

$$= C(n, M, \alpha, p) \left(\frac{1}{|B_{\kappa r}(x_0)|} \int_{B_{\kappa r}(x_0)} |\lambda \hat{u}|^p \, dx \right)^{1/2}.$$

Therefore,

$$(3\text{-}13) \quad \frac{1}{|B_{\kappa r}(x_0)|} \int_{B_{\kappa r}(x_0)} |\nabla v - \nabla w|^p \, dx$$

$$\leq C(\Lambda, n, \alpha, p) \left[2\tau + \tau^{(p-2)/(p-1)} \left(\frac{1}{|B_{\kappa r}(x_0)|} \int_{B_{\kappa r}(x_0)} |\lambda \hat{u}|^p \, dx \right)^{(p_1 - p)/2p_1} \right]$$

$$\times \left(\frac{1}{|B_{2\kappa r}(x_0)|} \int_{B_{2\kappa r}(x_0)} |\nabla v|^p \, dx \right).$$

From (3-11) and $[\![\lambda u]\!]_{\mathrm{BMO}(B_{2R},R)} \le M$, we can take $\tau = \frac{1}{2}$ in (3-13) to obtain in particular

$$\left(\frac{1}{|B_{\kappa r}(x_0)|} \int_{B_{\kappa r}(x_0)} |\nabla v - \nabla w|^p \, dx \right)^{1/p}$$
$$\le C(\Lambda, M, n, \alpha, p) \left(\frac{1}{|B_{2\kappa r}(x_0)|} \int_{B_{2\kappa r}(x_0)} |\nabla v|^p \, dx \right)^{1/p}$$
$$\le C_1(\Lambda, M, n, p) \left[\epsilon' + M(r\kappa^{n/p}\lambda)^{-1} \right].$$

Hence, if $\dfrac{\epsilon \kappa^{n/p} \lambda r}{4MC_1(\Lambda, M, n, p)} \ge 1$, we choose ϵ' sufficiently small so that

$$C_1(\Lambda, n, p)\epsilon' < \frac{\epsilon}{4}.$$

Then

$$\left(\frac{1}{|B_{\kappa r}(x_0)|} \int_{B_{\kappa r}(x_0)} |\nabla v - \nabla w|^p \, dx \right)^{1/p} \le \frac{\epsilon}{2}.$$

From this, the first estimate in (3-9), and the triangle inequality, the first estimate of (3-8) follows. Therefore, it remains to consider the case

$$(3\text{-}14) \qquad \qquad \lambda \kappa^{n/p} r \epsilon \le 4MC_1(\Lambda, M, n, p).$$

In this case, we first note that from our choice that $\epsilon' \le \epsilon$, we particularly have

$$\lambda \kappa^{n/p} \epsilon' r \le C(\Lambda, M, n, p).$$

Then, it follows from the second estimate in (3-9) that

$$\lambda \left(\frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} |v|^p \, dx \right)^{1/p} \le C(\Lambda, M, n, p).$$

On the other hand, from (3-3), and the scaling invariances discussed at the beginning of Section 2, we observe that $\tilde{v}(x) = \lambda v(x - x_0)$ is a weak solution of

$$\operatorname{div}[\hat{A}_0(x, \nabla \tilde{v})] = 0 \quad \text{in } B_r,$$

where $\hat{A}_0(x, \xi) = \lambda^{p-1} A(x - x_0, \lambda u(x - x_0), \lambda^{-1}\xi)$ for all $x \in B_r, \xi \in \mathbb{R}^n$. From this and Remark 2.1, we can apply Hölder's regularity theory in Lemma 2.8 for the solution $\tilde{v}$ to find that there is $\beta \in (0, 1)$ depending only on Λ, M, n, p such that

$$(3\text{-}15) \qquad \begin{aligned} \lambda \|v\|_{L^\infty(B_{5r/6}(x_0))} &\le C(\Lambda, M, n, p), \\ \lambda |v(x) - v(y)| &\le C(\Lambda, M, p, n)\kappa^\beta \quad \forall x, y \in \bar{B}_{\kappa r}(x_0). \end{aligned}$$

The estimate (3-15), (3-10), and (3-13) imply that

$$(3\text{-}16) \quad \frac{1}{|B_{\kappa r}(x_0)|} \int_{B_{\kappa r}(x_0)} |\nabla v - \nabla w|^p \, dx$$

$$\leq C(\Lambda, M, n, \alpha, p)$$

$$\times \left[2\tau + \tau^{(p-2)/(p-1)} \left(\frac{1}{|B_{\kappa r}(x_0)|} \int_{B_{\kappa r}(x_0)} |\lambda \hat{u}|^p \, dx \right)^{(p_1-p)/2p_1} \right].$$

On the other hand, for $v' = v + \bar{u}_{B_{\kappa r}}$, we can write

$$\frac{1}{|B_{\kappa r}(x_0)|} \int_{B_{\kappa r}(x_0)} |\lambda \hat{u}|^p \, dx$$

$$\leq C(p) \left[\frac{1}{|B_{\kappa r}(x_0)|} \int_{B_{\kappa r}(x_0)} |\lambda(u - v')|^p \, dx + \frac{1}{|B_{\kappa r}(x_0)|} \int_{B_{\kappa r}(x_0)} |\lambda(v' - \bar{v}'_{B_{\kappa r}(x_0)})|^p \, dx \right.$$

$$\left. + \frac{1}{|B_{\kappa r}(x_0)|} \int_{B_{\kappa r}(x_0)} |\lambda(\bar{u}_{B_{\kappa r}(x_0)} - \bar{v}'_{B_{\kappa r}(x_0)})|^p \, dx \right]$$

$$\leq C(n, p) \left[\frac{1}{\kappa^n |B_r(x_0)|} \int_{B_r(x_0)} |\lambda(u - v')|^p \, dx \right.$$

$$\left. + \frac{1}{|B_{\kappa r}(x_0)|} \int_{B_{\kappa r}(x_0)} |\lambda(v - \bar{v}_{B_{\kappa r}(x_0)})|^p \, dx \right].$$

Since $u - v' \in W_0^{1,2}(B_r(x_0))$, we can use Poincaré's inequality for the first term in the right-hand side of the last estimate to obtain

$$\left(\frac{1}{|B_{\kappa r}(x_0)|} \int_{B_{\kappa r}(x_0)} |\lambda \hat{u}|^p \, dx \right)^{1/p}$$

$$\leq C(\Lambda, n, p) \left[\frac{\lambda r}{\kappa^{n/p}} \left(\frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} |\nabla u - \nabla v|^p \, dx \right)^{1/p} \right.$$

$$\left. + \lambda \sup_{x, y \in \bar{B}_{\kappa r}(x_0)} |v(x) - v(y)| \right].$$

From this estimate, (3-9), and (3-15), we infer that

$$\left(\frac{1}{|B_{\kappa r}(x_0)|} \int_{B_{\kappa r}(x_0)} |\lambda \hat{u}|^p \, dx \right)^{1/p} \leq C(\Lambda, p, n) \left[\lambda r \epsilon' + \kappa^\beta \right].$$

From this, we can control the estimate in (3-16) as

$$\frac{1}{|B_{\kappa r}(x_0)|} \int_{B_{\kappa r}(x_0)} |\nabla v - \nabla w|^p \, dx$$

$$\leq C(\Lambda, M, n, \alpha, p) \left[2\tau + \tau^{(p-2)/(p-1)} (\lambda r \epsilon' + \kappa^\beta)^{p(p_1-p)/2p_1} \right].$$

Then, combining this last estimate with (3-14), we obtain

$$\frac{1}{|B_{\kappa r}(x_0)|} \int_{B_{\kappa r}(x_0)} |\nabla v - \nabla w|^p \, dx$$

$$\leq C_2(\Lambda, M, \alpha, p, n) \left[\tau + \tau^{(p-2)/(p-1)} \left(\frac{\epsilon'}{\epsilon \kappa^{n/p}} + \kappa^\beta \right)^{p(p_1-p)/2p_1} \right].$$

We firstly choose $\tau > 0$ so that

$$C_2(\Lambda, M, n, \alpha, p)\tau = \frac{1}{2} \left(\frac{\epsilon}{2} \right)^p.$$

Next, we choose κ sufficiently small depending only on Λ, n, α, p, and ϵ so that

$$\kappa^\beta \leq \frac{1}{2} \left[\frac{(\epsilon/2)^p}{4C_2(\Lambda, M, p, \alpha, n)\tau^{(p-2)/(p-1)}} \right]^{2p_1/p(p_1-p)},$$

and finally we choose $\epsilon' \in (0, \epsilon/2)$ sufficiently small so that

$$\epsilon' \leq \frac{\kappa^{n/p}\epsilon}{2} \left[\frac{(\epsilon/2)^p}{4C_2(\Lambda, M, p, \alpha, n)\tau^{(p-2)/(p-1)}} \right]^{2p_1/p(p_1-p)}.$$

From these choices, it follows that

$$\left(\frac{1}{|B_{\kappa r}(x_0)|} \int_{B_{\kappa r}(x_0)} |\nabla v - \nabla w|^p \, dx \right)^{1/p} \leq \frac{\epsilon}{2}.$$

The first estimate (3-8) then holds thanks to this estimate, the first estimate in (3-9), and the triangle inequality.

Finally, to complete the proof, it remains to verify the second estimate of (3-8). By using the triangle inequality, the assumption of the lemma, and the fact that $\epsilon \in (0, 1)$, we see that

$$\left(\fint_{B_{\kappa r}(x_0)} |\nabla w|^p \, dx \right)^{1/p} \leq \left(\fint_{B_{\kappa r}(x_0)} |\nabla w - \nabla u|^p \, dx \right)^{1/p} + \left(\fint_{B_{\kappa r}(x_0)} |\nabla u|^p \, dx \right)^{1/p}$$

$$\leq \epsilon + \left(2^n \fint_{B_{2\kappa r}(x_0)} |\nabla u|^p \, dx \right)^{1/p}$$

$$\leq \epsilon + 2^{n/p} \leq 1 + 2^{n/p} = C_0(n, p).$$

The proof is therefore complete. $\square$

Summarizing these efforts, we can state and prove the main result of the section.

Proposition 3.3. *Let* $\Lambda > 0$, $p > 1$, *and* $\alpha \in (0, 1]$ *be fixed. Then, for every* $\epsilon \in (0, 1)$, *there exist sufficiently small numbers* $\kappa = \kappa(\epsilon, \Lambda, M, p, n, \alpha) \in \left(0, \frac{1}{2}\right]$ *and* $\delta = \delta(\epsilon, \Lambda, M, \alpha, n, p) \in (0, \epsilon)$ *such that the following holds. Assume that* $A : B_{2R} \times \mathbb{K} \times \mathbb{R}^n \to \mathbb{R}^n$ *such that* (1-2)–(1-4) *and* (1-8) *hold for some* $R > 0$ *and*

some open interval $\mathbb{K} \subset \mathbb{R}$, *and assume that*

$$\fint_{B_{2r}(x_0)} |\boldsymbol{F}|^p \, dx \leq \delta^p$$

for some $x_0 \in \bar{B}_R$ *and some* $r \in (0, R/2)$. *Then for every* $\lambda \geq 0$, *if* $u \in W^{1,p}(B_{2R})$ *is a weak solution of* (3-1) *satisfying*

$$\fint_{B_{4\kappa r}(x_0)} |\nabla u|^p \, dx \leq 1, \qquad \fint_{B_{2r}(x_0)} |\nabla u|^p \, dx \leq 1, \quad and \quad [\![\lambda u]\!]_{\mathrm{BMO}(B_R, R)} \leq M,$$

then there is $h \in W^{1,p}(B_{7\kappa r/2}(x_0))$ *such that*

$$(3\text{-}17) \qquad \fint_{B_{7\kappa r/2}(x_0)} |\nabla u - \nabla h|^p \, dx \leq \epsilon^p, \qquad \|\nabla h\|_{L^{\infty}(B_{3\kappa r}(x_0))} \leq C(\Lambda, n, p).$$

Proof. For given ϵ, let

$$\delta = \min\{\delta_0(\epsilon/[2C_0(n, p)], \Lambda, n, p), \delta_2(\epsilon/2, \Lambda, M, \alpha, p)\},$$

where δ_0 is defined in Lemma 2.12, δ_2 is defined in Lemma 3.2, and $C_0(n, p) > 1$ is a constant defined in (3-8). We now prove our Lemma 3.2 with this choice of δ, κ. Note that since both numbers δ_0, δ_2 are independent of λ, so are δ, κ. If $\lambda = 0$, then our proposition follows directly from Lemma 2.12 with $\boldsymbol{G}$ replaced by $\boldsymbol{F}$ and for $\kappa = \frac{1}{2}$. Also, when $\lambda > 0$, let κ be a number defined as in Lemma 3.2. Then our proposition follows directly by applying Lemma 3.2 with r replaced by $2r$, Lemma 2.12 with $\boldsymbol{G} = 0$ and r replaced by $2\kappa r$, and the triangle inequality. $\qquad \square$

4. Level set estimates and proof of Theorem 1.1

Level set estimates. Recall that the Hardy–Littlewood maximal function $\mathcal{M}(f)$ is defined in (2-3), and $\mathcal{M}_U(f) = \mathcal{M}(f\chi_U)$ for an open set U and its characteristic function χ_U. Our first result of this subsection is the following important lemma on the density of the level sets of a solution u of (3-1).

Lemma 4.1. *Let* Λ, M *be positive numbers,* $p, \gamma > 1, \alpha \in (0, 1]$, *and let* $\epsilon > 0$ *be sufficiently small. Then there exist a sufficiently large number* $N = N(\Lambda, n, p) \geq 1$ *and two positive sufficiently small numbers* $\kappa = \kappa(\epsilon, \Lambda, M, p, n, \gamma, \alpha) \in \left(0, \frac{1}{2}\right]$ *and* $\delta = \delta(\epsilon, \Lambda, M, p, n, \gamma, \alpha) \in (0, \epsilon)$ *such that the following statement holds. Suppose that* $\boldsymbol{A} : B_{2R} \times \mathbb{K} \times \mathbb{R}^n \to \mathbb{R}^n$ *such that* (1-2)–(1-4) *and* (1-8) *hold for some* $R > 0$ *and some open interval* $\mathbb{K} \subset \mathbb{R}$. *Suppose also that* $u \in W^{1,p}(B_{2R})$ *is a weak solution of* (3-1) *satisfying* $[\![\lambda u]\!]_{\mathrm{BMO}(B_R, R)} \leq M$ *with some* $\lambda \geq 0$. *If* $y \in B_R$ *and* $\rho \in (0, \kappa_0)$ *such that*

$$B_\rho(y) \cap \left\{ B_R : \mathcal{M}_{B_{2R}}(|\nabla u|^p) \leq 1 \right\} \cap \left\{ B_R : \mathcal{M}_{B_{2R}}(|\boldsymbol{F}|^p) \leq \delta^p \right\} \neq \varnothing$$

for $\kappa_0 = \min\{1, R\}\kappa/6$, then

$$(4\text{-}1) \qquad \omega\big(\{x \in B_R : \mathcal{M}_{B_{2R}}(|\nabla u|^p) > N\} \cap B_\rho(y)\big) \leq \epsilon\omega(B_\rho(y))$$

for $\omega \in A_q$ with $[\omega]_{A_q} \leq \gamma$ and $q > 1$.

Proof. The proof is standard using Proposition 3.3. However, as Proposition 3.3 is stated differently compared to the other similar approximation estimates in the literature, details of the proof of this lemma are required. For a given $\epsilon > 0$, let $\epsilon' > 0$ be a positive number to be determined depending only on ϵ, Λ, n, p, and γ. Then let $\kappa = \kappa(\epsilon', \Lambda, M, p, n, \alpha)$ and $\delta = \delta(\epsilon', \Lambda, M, p, n, \alpha)$ be the numbers defined in Proposition 3.3. We prove the lemma with this choice of δ, κ. By the assumption, we can find

$$(4\text{-}2) \qquad x_0 \in B_\rho(y) \cap \big\{B_R : \mathcal{M}_{B_{2R}}(|\nabla u|^p) \leq 1\big\} \cap \big\{B_R : \mathcal{M}_{B_{2R}}(|F|^p) \leq \delta^p\big\}.$$

Let $r = \kappa^{-1}\rho \in (0, R/6)$. Since $\rho \in (0, \kappa_0)$ and κ is sufficiently small, we have $B_{4r}(y) \subset B_{5r}(x_0) \subset B_{2R}$. From this and (4-2), it follows that

$$\fint_{B_{4r}(y)} |\nabla u|^p \, dx \leq \frac{|B_{5r}(x_0)|}{|B_{4r}(y)|} \fint_{B_{5r}(x_0)} |\nabla u|^p \, dx \leq \left(\frac{5}{4}\right)^n,$$

$$\fint_{B_{4r}(y)} |F|^p \, dx \leq \frac{|B_{5r}(x_0)|}{|B_{4r}(y)|} \fint_{B_{5r}(x_0)} |F|^p \, dx \leq \left(\frac{5}{4}\right)^n \delta^p.$$

Moreover, we also have $B_{8\rho}(y) \subset B_{9\rho}(x_0) \subset B_{2R}$ and therefore

$$\fint_{B_{8\kappa r}(y)} |\nabla u|^p \, dx = \fint_{B_{8\rho}(y)} |\nabla u|^p \, dx \leq \frac{|B_{9\rho}(x_0)|}{|B_{8\rho}(y)|} \fint_{B_{9\rho}(x_0)} |\nabla u|^p \, dx \leq \left(\frac{9}{8}\right)^n.$$

Hence, all conditions in Proposition 3.3 are satisfied with some suitable scaling. From this and our choice of κ, δ, we can apply Proposition 3.3 to find a function $h \in W^{1,p}(B_{7\rho/2}(y))$ satisfying

$$\fint_{B_{7\rho/2}(y)} |\nabla u - \nabla h|^p \, dx \leq (\epsilon')^p \left(\frac{3}{2}\right)^n, \qquad \|\nabla h\|_{L^\infty(B_{3\rho}(y))} \leq C_*(\Lambda, n, p),$$

where in the above estimates, we have used the fact that $\kappa r = \rho$. Let us now denote

$$N = \max\{2^p C_*^p, 2^n\}.$$

We prove (4-1) with this choice of N. To this end, we firstly prove that

$$(4\text{-}3) \quad \big\{x \in B_\rho(y) : \mathcal{M}_{B_{7\rho/2}(y)}(|\nabla u - \nabla h|^p)(x) \leq C_*^p\big\}$$
$$\subset \big\{x \in B_\rho(y) : \mathcal{M}_{B_{2R}}(|\nabla u|^p)(x) \leq N\big\}.$$

To prove this statement, let x be a point in the set on the left side of (4-3). We verify that

$$(4\text{-}4) \qquad\qquad\qquad \mathcal{M}_{B_{2R}}(|\nabla u|^p)(x) \le N.$$

Let $\rho' > 0$ be any number. If $\rho' < 2\rho$, then $B_{\rho'}(x) \subset B_{3\rho}(y) \subset B_{2R}$, and it follows that

$$\left(\fint_{B_{\rho'}(x)} |\nabla u(z)|^p\, dz \right)^{1/p}$$
$$\le \left(\fint_{B_{\rho'}(x)} |\nabla u(z) - \nabla h(z)|^p\, dz \right)^{1/p} + \left(\fint_{B_{\rho'}(x)} |\nabla h(z)|^p\, dz \right)^{1/p}$$
$$\le \left(\mathcal{M}_{B_{7\rho/2}(y)}(|\nabla u - \nabla h|^p)(x) \right)^{1/p} + \|\nabla h\|_{L^\infty(B_{3\rho}(y))} \le 2C_* \le N^{1/p}.$$

On the other hand, if $\rho' \ge 2\rho$, we note that $B_{\rho'}(x) \subset B_{2\rho'}(x_0)$, and it follows from this and (4-2) that

$$\frac{1}{|B_{\rho'}(x)|} \int_{B_{\rho'}(x) \cap B_{2R}} |\nabla u(z)|^p\, dz$$
$$\le \frac{|B_{2\rho'}(x_0)|}{|B_{\rho'}(x)|} \frac{1}{|B_{2\rho'}(x_0)|} \int_{B_{2\rho'}(x_0) \cap B_{2R}} |\nabla u(z)|^p\, dz \le 2^n \le N.$$

Hence, (4-4) is verified and therefore (4-3) is proved. Observe that (4-3) is in fact equivalent to

$$(4\text{-}5) \quad \left\{ x \in B_\rho(y) : \mathcal{M}_{B_{2R}}(|\nabla u|^p)(x) > N \right\}$$
$$\subset E := \left\{ x \in B_\rho(y) : \mathcal{M}_{B_{7\rho/2}(y)}(|\nabla u - \nabla h|^p)(x) > C_*^p \right\}.$$

On the other hand, from the weak type $(1, 1)$ estimate of the Hardy–Littlewood maximal function (see Lemma 2.6), it is true that

$$\frac{|E|}{|B_\rho(y)|} \le \frac{C(n)}{C_*^p} \fint_{B_{7\rho/2}(y)} |\nabla u - \nabla h|^p\, dz \le C_1(\Lambda, n, p)(\epsilon')^p.$$

From this and the doubling property of A_q-weights as in (ii) of Lemma 2.4, it follows that

$$\frac{\omega(E)}{\omega(B_\rho(y))} \le C(n, \gamma) \left(\frac{|E|}{|B_\rho(y)|} \right)^\beta \le C'(\Lambda, n, p, \gamma)(\epsilon')^{p\beta}$$

for some $\beta = \beta(\gamma, n) > 0$. Therefore, by choosing ϵ' depending on $\epsilon, \Lambda, n, p, \gamma$ such that

$$C'(\Lambda, n, p, \gamma)(\epsilon')^{p\beta} = \epsilon,$$

we obtain

$$\omega(E) \le \epsilon\,\omega(B_\rho(y)).$$

From this estimate and the definition of E in (4-5), the estimate (4-1) follows and the proof is complete. □

The following level set estimate is a direct corollary of Lemma 4.1 and Lemma 2.7, which is also the main result of this subsection.

Lemma 4.2. *Let* Λ, M *be positive numbers,* $p, \gamma > 1, \alpha \in (0, 1]$, *and let* $\epsilon > 0$ *be sufficiently small. Then there exist a sufficiently large number* $N = N(\Lambda, n, p) \geq 1$ *and a sufficiently small number* $\delta = \delta(\epsilon, \Lambda, M, p, n, \alpha) \in (0, \epsilon)$ *such that the following statement holds. Assume that* $A : B_{2R} \times \mathbb{K} \times \mathbb{R}^n \to \mathbb{R}^n$ *is such that* (1-2)–(1-4) *and* (1-8) *hold for some* $R > 0$ *and some open interval* $\mathbb{K} \subset \mathbb{R}$. *Suppose also that for any* $\lambda \geq 0$, *if* $u \in W^{1,p}(B_{2R})$ *is a weak solution of* (3-1) *satisfying*

$$(4\text{-}6) \qquad [\![\lambda u]\!]_{\mathrm{BMO}(B_R, R)} \leq M,$$
$$\omega(\{B_R : \mathcal{M}_{B_{2R}}(|\nabla u|^p) > N\}) \leq \epsilon \omega(B_{\kappa_0}(y)) \quad \forall y \in \bar{B}_R,$$

for some $\omega \in A_q$ *with* $q > 1$ *and* $[\omega]_{A_q} \leq \gamma$, *then*

$$(4\text{-}7) \quad \omega(\{B_R : \mathcal{M}_{B_{2R}}(|\nabla u|^p) > N\})$$
$$\leq \epsilon_1 [\omega(\{B_R : \mathcal{M}_{B_{2R}}(|\nabla u|^p) > 1\}) + \omega(\{B_R : \mathcal{M}_{B_{2R}}(|F|^p) > \delta^p\})],$$

with ϵ_1 *as defined in Lemma 2.7 and* κ_0 *as defined in Lemma 4.1.*

Proof. Let N, κ_0, δ be defined as in Lemma 4.1. We apply Lemma 2.7 with

$$C = \{x \in B_R : \mathcal{M}_{B_{2R}}(|\nabla u|^p)(x) > N\}$$

and

$$D = \{x \in B_R : \mathcal{M}_{B_{2R}}(|\nabla u|^p)(x) > 1\} \cup \{x \in B_R : \mathcal{M}_{B_{2R}}(|F|^p)(x) > \delta^p\}.$$

Observe that by the second condition in (4-6), (i) of Lemma 2.7 is satisfied. On the other hand, by Lemma 4.1, (ii) of Lemma 2.7 also holds true. Therefore, both conditions of Lemma 2.7 are valid, and (4-7) follows directly from Lemma 2.7. □

Proof of the interior $W^{1,q}$-regularity estimates. From Lemma 4.2 and an iterating procedure, we obtain the following lemma:

Lemma 4.3. *Let* $\Lambda, M, p, \alpha, \epsilon, N, \delta, \kappa, \kappa_0$ *and* A, R *be as in Lemma 4.2. Then, for any* $\lambda \geq 0$, *if* $u \in W^{1,p}(B_{2R})$ *is a weak solution of* (3-1) *satisfying*

$$[\![\lambda u]\!]_{\mathrm{BMO}(B_R, R)} \leq M \quad \text{and} \quad \omega(\{B_R : \mathcal{M}_{B_{2R}}(|\nabla u|^p) > N\}) \leq \epsilon \omega(B_{\kappa_0}(y)) \quad \forall y \in \bar{B}_R$$

for some $\omega \in A_q$ *with* $q > 1$ *and* $[\omega]_{A_q} \leq \gamma$, *then with* ϵ_1 *defined as in Lemma 2.7, and for any* $k \in \mathbb{N}$, *the following estimate holds:*

$$(4\text{-}8) \quad \omega(\{B_R : \mathcal{M}_{B_{2R}}(|\nabla u|^p) > N^k\})$$
$$\leq \epsilon_1^k \omega(\{B_R : \mathcal{M}_{B_{2R}}(|\nabla u|^p) > 1\}) + \sum_{i=1}^{k} \epsilon_1^i \omega(\{B_R : \mathcal{M}_{B_{2R}}(|F|^p) > \delta^p N^{k-i}\}).$$

Proof. The proof is based on induction on $k \in \mathbb{N}$, and an iteration of Lemma 4.2. See, for example, [Phan 2017, Lemma 4.10]. □

We now can complete the proof of Theorem 1.1.

Proof of Theorem 1.1. The proof now is quite standard. However, we include it here for completeness, and for the transparency regarding the role of the scaling parameter λ. Let $N = N(\Lambda, p, n)$ be defined as in Lemma 4.3. For $q > 1$, we choose $\epsilon > 0$ sufficiently small and depending only on Λ, n, p, q, and γ such that

$$\epsilon_1 N^q = \frac{1}{2},$$

where ϵ_1 is defined as in Lemma 4.3. With this ϵ, we can now choose

$$\delta = \delta(\epsilon, \Lambda, M, p, q, n, \alpha), \quad \kappa = \kappa(\epsilon, \Lambda, M, p, q, n, \gamma, \alpha), \quad \kappa_0 = \min\{1, R\}\kappa/6$$

as determined by Lemma 4.3. Assume that the assumptions of Theorem 1.1 hold with this choice of δ. For $\lambda \geq 0$, let us assume that u is a weak solution of (3-1) satisfying $[\![\lambda u]\!]_{\mathrm{BMO}(B_R)} \leq M$, and let

$$(4\text{-}9) \qquad E = E(\lambda, N) = \big\{ B_R : \mathcal{M}_{B_{2R}}(|\nabla u|^p) > N \big\}.$$

We now prove the estimate in Theorem 1.1 with the additional assumption that

$$(4\text{-}10) \qquad \omega(E) \leq \epsilon \omega(B_{\kappa_0}(y)) \quad \forall y \in \overline{B}_R.$$

Let us now consider the sum

$$S = \sum_{k=1}^{\infty} N^{qk} \omega\big(\big\{ B_R : \mathcal{M}_{B_{2R}}(|\nabla u|^p) > N^k \big\}\big).$$

From (4-10), we can apply Lemma 4.3 to obtain

$$S \leq \sum_{k=1}^{\infty} N^{kq} \sum_{i=1}^{k} \epsilon_1^i \omega\big(\big\{ B_R : \mathcal{M}_{B_{2R}}(|F|^p) > \delta^p N^{k-i} \big\}\big)$$
$$+ \sum_{k=1}^{\infty} (N^q \epsilon_1)^k \omega\big(\big\{ B_R : \mathcal{M}_{B_{2R}}(|\nabla u|^p) > 1 \big\}\big).$$

By Fubini's theorem, the above estimate can be rewritten as

$$(4\text{-}11) \quad S \leq \sum_{j=1}^{\infty} (N^q \epsilon_1)^j \sum_{k=j}^{\infty} N^{q(k-j)} \omega\big(\big\{ B_R : \mathcal{M}_{B_{2R}}(|F|^p) > \delta^p N^{k-j} \big\}\big)$$
$$+ \sum_{k=1}^{\infty} (N^q \epsilon_1)^k \omega\big(\big\{ B_R : \mathcal{M}_{B_{2R}}(|\nabla u|^p) > 1 \big\}\big).$$

Observe that

$$\omega\big(\big\{ B_R : \mathcal{M}_{B_{2R}}(|\nabla u|^p) > 1 \big\}\big) \leq \omega(B_R).$$

From this, the choice of ϵ, Lemma 2.5, and (4-11) it follows that

$$S \leq C\big[\|\mathcal{M}_{B_{2R}}(|\boldsymbol{F}|^p)\|_{L^q(B_R,\omega)}^q + \omega(B_R)\big].$$

Applying Lemma 2.5 again, we infer that

$$\|\mathcal{M}_{B_{2R}}(|\nabla u|^p)\|_{L^q(B_R,\omega)}^q \leq C\big[\|\mathcal{M}_{B_{2R}}(|\boldsymbol{F}|^p)\|_{L^q(B_{2R},\omega)}^q + \omega(B_R)\big].$$

Also, by Lebesgue's differentiation theorem, it is true that

$$|\nabla u(x)|^p \leq \mathcal{M}_{B_{2R}}(|\nabla u|^p)(x) \quad \text{a.e. } x \in B_R.$$

Hence,

$$\|\nabla u\|_{L^{pq}(B_R,\omega)}^{pq} \leq C\big[\|\mathcal{M}_{B_{2R}}(|\boldsymbol{F}|^p)\|_{L^q(B_R,\omega)}^q + \omega(B_R)\big].$$

From this and Lemma 2.6, it follows that

$$(4\text{-}12) \qquad \|\nabla u\|_{L^{pq}(B_R,\omega)} \leq C\big[\|\boldsymbol{F}\|_{L^{pq}(B_{2R},\omega)} + \omega(B_R)^{1/q}\big].$$

Summarizing the efforts, we conclude that (4-12) holds true as long as u is a weak solution of (3-1) for $\lambda \geq 0$ and (4-10) holds.

It now remains to remove the additional assumption (4-10). To this end, assume all assumptions in Theorem 1.1 hold, and let u be a weak solution of (3-1) with some $\lambda \geq 0$. Let $\mu > 0$ sufficiently large to be determined, and let $\lambda' = \lambda\mu \geq 0$, $u_\mu = u/\mu$, and $\boldsymbol{F}_\mu = \boldsymbol{F}/\mu$. We note that u_μ is a weak solution of

$$(4\text{-}13) \qquad \text{div}\,[\hat{A}(x, \lambda' u_\mu, \nabla u_\mu)] = \text{div}\,[|\boldsymbol{F}_\mu|^{p-2}\boldsymbol{F}_\mu] \quad \text{in } B_{2R},$$

where

$$\hat{A}(x, z, \xi) = \frac{A(x, z, \mu\xi)}{\mu^{p-1}}.$$

Note that by Remark 2.1, $\hat{A}$ satisfies (1-2)–(1-4) with the same constants Λ, p, α. Moreover, $\hat{A}$ also satisfies (1-8). We then denote

$$E_\mu = \big\{B_R : \mathcal{M}_{B_{2R}}(|\nabla u_\mu|^p) > N\big\}$$

and assume that

$$(4\text{-}14) \qquad K_0 = \left(\frac{1}{|B_{2R}|}\int_{B_{2R}}|\nabla u|^p\,dx\right)^{1/p} > 0.$$

We claim that we can choose $\mu = CK_0$ with some sufficiently large constant C depending only on Λ, M, p, q, n, and R/κ_0 such that

$$(4\text{-}15) \qquad \omega(E_M) \leq \epsilon\omega(B_{\kappa_0}(y)) \quad \forall y \in \bar{B}_R.$$

If this holds, we can apply (4-12) for u_μ, which is a weak solution of (4-13), to obtain

$$\|\nabla u_\mu\|_{L^{pq}(B_R,\omega)} \leq C\big[\|\boldsymbol{F}_\mu\|_{L^{pq}(B_{2R},\omega)} + \omega(B_R)^{1/q}\big].$$

Then, by multiplying this equality with μ, we obtain

$$\|\nabla u\|_{L^{pq}(B_R,\omega)} \le C\big[\|F\|_{L^{pq}(B_{2R},\omega)} + \omega(B_R)^{1/q} K_0\big].$$

The proof of Theorem 1.1 is therefore complete if we can prove (4-15). To this end, using the doubling property of $\omega \in A_q$ as in (i) of Lemma 2.4, we have

$$\frac{\omega(E_\mu)}{\omega(B_{\kappa_0}(y))} = \frac{\omega(E_\mu)}{\omega(B_{2R})} \frac{\omega(B_{2R})}{\omega(B_{\kappa_0}(y))} \le \gamma \frac{\omega(E_\mu)}{\omega(B_{2R})} \left(\frac{2R}{\kappa_0}\right)^{nq}.$$

From this, and using (ii) of Lemma 2.4 again, we can find $\beta = \beta(\gamma, n) > 0$ such that

$$(4\text{-}16) \qquad \frac{\omega(E_\mu)}{\omega(B_{\kappa_0}(y))} \le C(\gamma, n) \left(\frac{2R}{\kappa_0}\right)^{nq} \left(\frac{|E_\mu|}{|B_{2R}|}\right)^{\beta/p}.$$

Now, by the definition of E_μ and the weak type $(1, 1)$ estimate for the maximal function, we see that

$$\frac{|E_\mu|}{|B_{2R}|} = \big|\big\{B_R : \mathcal{M}_{B_{2R}}(|\nabla u|^p) > N\mu^p\big\}\big| \,\big/\, |B_{2R}|$$

$$= \frac{C(n, p)}{N\mu^p} \frac{1}{|B_{2R}|} \int_{B_{2R}} |\nabla u|^p \, dx \le \frac{C(p, n)K_0^p}{N\mu^p},$$

where K_0 is defined in (4-14). From this estimate and (4-16), it follows that

$$\frac{\omega(E_\mu)}{\omega(B_{\kappa_0}(y))} \le C^*(\Lambda, \gamma, p, n) \left(\frac{2R}{\kappa_0}\right)^{nq} \left(\frac{K_0}{\mu}\right)^\beta.$$

Now we choose μ such that

$$\mu = K_0 \bigg[\epsilon^{-1} C^*(\Lambda, \gamma, p, n) \left(\frac{2R}{\kappa_0}\right)^{nq}\bigg]^{1/\beta}.$$

Then it follows that

$$\omega(E_\mu) \le \epsilon \omega(B_{\kappa_0}(y)) \quad \forall y \in \bar{B}_R.$$

This proves (4-15) and completes the proof of Theorem 1.1.   □

Acknowledgements

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LOCAL UNITARY PERIODS AND RELATIVE DISCRETE SERIES

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Let F be a p-adic field ($p \neq 2$), let E be a quadratic Galois extension of F, and let $n \geq 2$. We construct representations in the discrete spectrum of the p-adic symmetric space $H \backslash G$, where $G = \mathbf{GL}_{2n}(E)$ and $H = \mathbf{U}_{E/F}(F)$ is a quasisplit unitary group over F.

1. Introduction

Let F be a p-adic field ($p \neq 2$) and let E be a quadratic Galois extension of F. Let $G = \mathbf{GL}_n(E)$ and let H be the group of F-points of a quasisplit unitary group $\mathbf{U}_{E/F}$. In this paper, we are concerned with constructing irreducible representations of G that occur in the discrete spectrum[1] $L^2_{\mathrm{disc}}(H \backslash G)$ of the p-adic symmetric space $H \backslash G$. Such representations are referred to as relative discrete series (RDS) representations for $H \backslash G$. Of course, the issue of characterizing the discrete spectrum is of interest more generally. One would eventually strive for a recipe to construct RDS representations for any connected reductive group $G = \mathbf{G}(F)$ and symmetric

[1]To make sense of $L^2_{\mathrm{disc}}(H \backslash G)$, one also needs to take the quotient by the center Z_G of G and consider square integrable functions on the quotient $Z_G H \backslash G$. Moreover, we must consider representations that admit a (unitary) central character.

subgroup $H = \mathbf{G}^\theta(F)$, where θ is an F-involution of $\mathbf{G}$. For the symmetric spaces

$$(\mathbf{GL}_n(F) \times \mathbf{GL}_n(F)) \backslash \mathbf{GL}_{2n}(F) \quad \text{and} \quad \mathbf{GL}_n(F) \backslash \mathbf{GL}_n(E),$$

the author has carried out in [Smith 2018, Theorem 6.3] a construction of RDS analogous to the one proved in this paper (which we state as Theorem 5.11 immediately below). Sakellaridis and Venkatesh [2017] have considered (and answered) much more general questions in the harmonic analysis on p-adic spherical varieties; however, they do not give an explicit description of the discrete spectrum. Understanding the discrete series for p-adic symmetric spaces is a natural first step towards the general picture for spherical varieties. Moreover, it is known from [Kato and Takano 2010] that H-distinguished[2] discrete series representations of G are automatically RDS. We are thus interested in constructing RDS that do not occur in the discrete spectrum of G.

In the case of number fields, distinction by unitary groups, that is, nonvanishing of the period integral attached to $\mathbf{U}_{E/F}$, has deep connections with quadratic base change [Arthur and Clozel 1989]. The study of global period integrals has largely been pioneered by H. Jacquet and his collaborators; see, for instance, [Jacquet and Lai 1985; Jacquet 2001; 2005; 2010; 1999]. Feigon, Lapid and Offen provide in [Feigon et al. 2012] a detailed discussion of both local and global aspects of the theory, and a very nice treatment of the history of the subject and the contributions of Jacquet et al. We will recall some of the results of that reference in Section 5A1, restricting ourselves to those required to prove our main theorem. We encourage the reader to consult [Feigon et al. 2012] for more details, including a discussion of the failure of local multiplicity-one in §13 of that paper.

We now give a statement of our main result. Let $n \geq 2$ be an integer. Let $Q = P_{(n,n)}$ be the upper triangular parabolic subgroup of $\mathbf{GL}_{2n}(E)$, with standard Levi factorization $L = M_{(n,n)}$, $U = N_{(n,n)}$.

Theorem 5.11. *Let $\pi = \iota_Q^G \tau$ be a parabolically induced representation, where $\tau = \tau' \otimes {}^\sigma\tau'$, and τ' is a discrete series representation of $\mathbf{GL}_n(E)$ such that τ' is not Galois invariant, i.e., $\tau' \ncong {}^\sigma\tau'$. The representation π is a relative discrete series representation for $\mathbf{U}_{E/F}(F) \backslash \mathbf{GL}_{2n}(E)$ that does not occur in the discrete series of $\mathbf{GL}_{2n}(E)$.*

Theorem 5.11 is the direct analogue of [Smith 2018, Theorem 6.3] and the overall method of proof is the same. The idea of the proof is to reduce the verification of the relative Casselman's criterion (Theorem 3.4) of Kato and Takano to the usual Casselman's criterion for the inducing discrete series. In contrast to the work in [Smith 2018], here we can construct representations only by inducing from the maximal parabolic subgroup $P_{(n,n)}$ due to the nature of the symmetric

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[2]See Definition 2.4.

space $H\backslash G$ and the (lack of) existence of θ-elliptic Levi subgroups (Definition 4.4, Lemma 4.20). Similarly, the lack of θ-elliptic Levi subgroups prevents us from considering $\mathbf{GL}_n(E)$, when n is odd.

Remark 1.1. In light of recent work of Raphaël Beuzart-Plessis, announced in his 2017 cours Peccot, our construction of RDS via Theorem 5.11 exhausts all relative discrete series for the symmetric pair $\mathbf{U}_{E/F}(F)\backslash\mathbf{GL}_{2n}(E)$ that lie outside of $L^2_{\mathrm{disc}}(G)$. The remaining relative discrete series are the $\mathbf{U}_{E/F}(F)$-distinguished discrete series representations of $\mathbf{GL}_{2n}(E)$. We discuss the exhaustion of the discrete spectrum in Section 5D.

We also prove the following corollary to Theorem 5.11.

Corollary 5.13. *Let $n \geq 2$ be an integer. There exist infinitely many equivalence classes of RDS representations of the form constructed in Theorem 5.11 and such that the discrete series τ is not supercuspidal.*

We now give an outline of the contents of the paper. In Section 2, we set notation and recall basic facts regarding parabolic induction and distinguished representations. We review the relative Casselman's criterion of [Kato and Takano 2010] in Section 3. We also discuss the invariant forms on Jacquet modules constructed in [Kato and Takano 2008] and [Lagier 2008]. In Section 4, we recall the required structural results on p-adic symmetric spaces. In particular, we discuss (θ, F)-split tori, θ-split parabolic subgroups, θ-elliptic Levi subgroups, and the relative root system. The main results of the paper (Theorem 5.11 and Corollary 5.13) are stated and proved in Section 5. In Section 6 we prove the remaining technical results needed to establish the main theorem.

2. Notation and preliminaries

Let F be a nonarchimedean local field of characteristic zero and odd residual characteristic. Let $\mathcal{O}_F$ be the ring of integers of F. Let E be a quadratic Galois extension of F. Let $\sigma \in \mathrm{Gal}(E/F)$ be the nontrivial element of the Galois group of E over F.

For now, let $\mathbf{G}$ be an arbitrary connected reductive group defined over F and let $G = \mathbf{G}(F)$ denote the group of F-points. We will restrict to the case that $G = \mathbf{GL}_n(E)$ from Section 5 onwards. We let Z_G denote the center of G, and let A_G denote the F-split component of Z_G. Let θ be an F-involution of $\mathbf{G}$. Define $\mathbf{H} = \mathbf{G}^\theta$ to be the (closed) subgroup of θ-fixed points of $\mathbf{G}$. The quotient $H\backslash G$ is a p-adic symmetric space.

We will routinely abuse notation and identify an algebraic group defined over F with its group of F-points. When the distinction is to be made, we will use boldface

to denote the algebraic group and regular typeface to denote the group of F-points. For any F-torus $\mathbf{A}$, we let A^1 denote the group of $\mathcal{O}_F$-points of $\mathbf{A}$.

Let $\mathbf{GL}_n$ denote the general linear group of n by n invertible matrices. We write $\mathbf{P}_{(\underline{m})}$ for the block upper triangular parabolic subgroup of $\mathbf{GL}_n$, corresponding to a partition $(\underline{m}) = (m_1, \ldots, m_k)$ of n. The group $\mathbf{P}_{(\underline{m})}$ has block-diagonal Levi subgroup $\mathbf{M}_{(\underline{m})} \cong \prod_{i=1}^{k} \mathbf{GL}_{m_i}$ and unipotent radical $\mathbf{N}_{(\underline{m})}$. We use $\mathrm{diag}(a_1, \ldots, a_n)$ to denote an $n \times n$ diagonal matrix with entries $a_1, \ldots, a_n$.

For any $g, x \in G$, we write ${}^g x = g x g^{-1}$. For any subset X of G, we write ${}^g X = \{{}^g x : x \in X\}$. Let $C_G(X)$ denote the centralizer of X in G and let $N_G(X)$ be the normalizer of X in G. Given a real number r we let $\lfloor r \rfloor$ denote the greatest integer that is less than or equal to r. We use $\widehat{(\cdot)}$ to denote that a symbol is omitted. For instance, $\mathrm{diag}(\widehat{a_1}, a_2, \ldots, a_n)$ may be used to denote the diagonal matrix $\mathrm{diag}(a_2, \ldots, a_n)$.

2A. *Induced representations of p-adic groups.*

We now briefly review some necessary background of the representation theory of G and discuss the representations that are relevant in the harmonic analysis on $H \backslash G$. We will only consider representations on complex vector spaces. A representation (π, V) of G is smooth if for every $v \in V$ the stabilizer of v in G is an open subgroup. A smooth representation (π, V) of G is admissible if, for every compact open subgroup K of G, the subspace V^K of K-invariant vectors is finite-dimensional. All of the representations that we consider are smooth and admissible. A quasicharacter of G is a one-dimensional representation. Let (π, V) be a smooth representation of G. If ω is a quasicharacter of Z_G, then (π, V) is an ω-representation if π has central character ω.

Let P be a parabolic subgroup of G with Levi subgroup M and unipotent radical N. Given a smooth representation (ρ, V_ρ) of M we may inflate ρ to a representation of P, also denoted ρ, by declaring that N acts trivially. We define the representation $\iota_P^G \rho$ of G to be the (normalized) parabolically induced representation $\mathrm{Ind}_P^G(\delta_P^{1/2} \otimes \rho)$. We normalize by the square root of the modular character δ_P of P. The character δ_P is given by $\delta_P(p) = |\det \mathrm{Ad}_{\mathfrak{n}}(p)|$, for all $p \in P$, where $\mathrm{Ad}_{\mathfrak{n}}$ denotes the adjoint action of P on the Lie algebra $\mathfrak{n}$ of N [Casselman 1995]. Let (π, V) be a smooth representation of G. Let (π_N, V_N) denote the normalized Jacquet module of π along P. Precisely, V_N is the quotient of V by the P-stable subspace

$$V(N) = \mathrm{span}\{\pi(n)v - v : n \in N, v \in V\},$$

and the action of P on V_N is normalized by $\delta_P^{-1/2}$. The unipotent radical of N acts trivially on (π_N, V_N) and we will regard (π_N, V_N) as a representation of the Levi factor $M \cong P/N$ of P.

The geometric lemma (Lemma 2.1) is a fundamental tool in the study of induced representations. Let $P = MN$ and $Q = LU$ be two parabolic subgroups of G with

Levi factors M and L, and unipotent radicals N and U respectively. Following [Roche 2009], let

$$S(M, L) = \{y \in G : M \cap {}^yL \text{ contains a maximal } F\text{-split torus of } G\}.$$

There is a canonical bijection between the double-coset space $P\backslash G/Q$ and the set $M\backslash S(M, L)/L$. Let $y \in S(M, L)$. The subgroup $M \cap {}^yQ$ is a parabolic subgroup of M and $P \cap {}^yL$ is a parabolic subgroup of ${}^yL$. The unipotent radical of $M \cap {}^yQ$ is $M \cap {}^yU$ and the unipotent radical of $P \cap {}^yL$ is $N \cap {}^yL$; moreover, $M \cap {}^yL$ is a Levi subgroup of both $M \cap {}^yQ$ and $P \cap {}^yL$. Given a representation ρ of L, we obtain a representation ${}^y\rho = \rho \circ \operatorname{Int} y^{-1}$ of ${}^yL$.

Lemma 2.1 (the geometric lemma [Bernstein and Zelevinsky 1977, Lemma 2.12]). *Let ρ be a smooth representation of L. There is a filtration of the space of the representation $(\iota_Q^G\rho)_N$ such that the associated graded object is isomorphic to the direct sum*

$$(2\text{-}1) \qquad \bigoplus_{y \in M\backslash S(M,L)/L} \iota_{M \cap {}^yQ}^M(({}^y\rho)_{N \cap {}^yL}).$$

Remark 2.2. We write $\mathcal{F}_N^y(\rho)$ for the smooth representation $\iota_{M \cap {}^yQ}^M(({}^y\rho)_{N \cap {}^yL})$ of M.

Let Φ be the root system of G (relative to a choice of maximal F-split torus A_0). Fix a base Δ_0 for Φ and let W_0 be the Weyl group of G (with respect to A_0). The choice of Δ_0 determines a system Φ^+ of positive roots. Given a subset Θ of Δ_0, we may associate a standard parabolic subgroup $P_\Theta = M_\Theta N_\Theta$, with Levi factor M_Θ and unipotent radical N_Θ, in the usual way. The following lemma gives a good choice of Weyl group representatives to use when applying Lemma 2.1 for standard parabolic subgroups.

Lemma 2.3 [Casselman 1995, Proposition 1.3.1]. *Let Θ and Ω be subsets of Δ_0. The set*

$$[W_\Theta\backslash W_0/W_\Omega] = \{w \in W_0 : w\Omega, w^{-1}\Theta \subset \Phi^+\}$$

gives a choice of Weyl group representatives for the double-coset space $P_\Theta\backslash G/P_\Omega$.

We always use the choice of "nice" representatives $[W_\Theta\backslash W_0/W_\Omega] \subset S(M_\Theta, M_\Omega)$ for the double-coset space $P_\Theta\backslash G/P_\Omega \simeq M_\Theta\backslash S(M_\Theta, M_\Omega)/M_\Omega$.

2B. *Distinguished (induced) representations.* Let π be a smooth representation of G. We also let π denote its restriction to H. Let χ be a quasicharacter of H.

Definition 2.4. The representation π is said to be (H, χ)-distinguished if the space $\operatorname{Hom}_H(\pi, \chi)$ is nonzero.

If π is $(H, 1)$-distinguished, where 1 is the trivial character of H, then we will simply call π H-distinguished. The elements of $\mathrm{Hom}_H(\pi, 1)$ are H-invariant linear forms on the space of π.

2B1. *Relative matrix coefficients.* Let (π, V) be a smooth H-distinguished representation of G. Let $\lambda \in \mathrm{Hom}_H(\pi, 1)$ be a nonzero H-invariant linear form on V and let v be a nonzero vector in V. In analogy with the usual matrix coefficients, define a complex-valued function $\varphi_{\lambda,v}$ on G by $\varphi_{\lambda,v}(g) = \langle \lambda, \pi(g)v \rangle$. We refer to the functions $\varphi_{\lambda,v}$ as λ-relative matrix coefficients. When λ is understood, we will drop it from the terminology. The representation π is smooth; therefore, the relative matrix coefficients $\varphi_{\lambda,v}$ lie in $C^\infty(G)$, for every $v \in V$. In addition, since λ is H-invariant, the functions $\varphi_{\lambda,v}$ descend to well-defined functions on the quotient $H \backslash G$.

Let ω be a unitary character of Z_G and suppose that π is an ω-representation. Since the central character ω is unitary, the function $Z_G H \cdot g \mapsto |\varphi_{\lambda,v}(g)|$ is well defined on $Z_G H \backslash G$. The center Z_G of G is unimodular since it is abelian. The fixed point subgroup H is also reductive (see [Digne and Michel 1994, Theorem 1.8]) and thus unimodular. It follows that there exists a G-invariant measure on the quotient $Z_G H \backslash G$ by [Robert 1983, Proposition 12.8].

Definition 2.5. The representation (π, V) is said to be

(1) (H, λ)-relatively square integrable if and only if all of the λ-relative matrix coefficients are square integrable modulo $Z_G H$,

(2) H-relatively square integrable if and only if π is (H, λ)-relatively square integrable for every $\lambda \in \mathrm{Hom}_H(\pi, 1)$.

When H is understood, we drop it from the terminology and speak of relatively square integrable representations. If (π, V) is H-distinguished and (H, λ)-relatively square integrable, then the morphism that sends $v \in V$ to the relative matrix coefficient $\varphi_{\lambda,v}$ is an intertwining operator from π to the right regular representation of G on $L^2(Z_G H \backslash G, \omega)$, where $L^2(Z_G H \backslash G, \omega)$ is the space of functions on $H \backslash G$, square integrable modulo Z_G, that are Z_G-eigenfunctions with eigencharacter ω.

Definition 2.6. If (π, V) is an irreducible subrepresentation of $L^2(Z_G H \backslash G)$, then we say that (π, V) occurs in the discrete spectrum of $H \backslash G$. In this case, we say that (π, V) is a relative discrete series (RDS) representation.

The main goal of the present paper is to construct RDS representations for $\mathbf{U}_{E/F}(F) \backslash \mathbf{GL}_{2n}(E)$, when $\mathbf{U}_{E/F}$ is a quasisplit unitary group over F.

2B2. *Invariant forms on induced representations.* Lemma 2.7 is well known and follows from an explicit version of Frobenius reciprocity due to Bernstein and Zelevinsky [1976, Proposition 2.29]. Let $Q = LU$ be a θ-stable parabolic subgroup

with θ-stable Levi factor L and unipotent radical U. Note that the identity component of $Q^\theta = L^\theta U^\theta$ is a parabolic subgroup of the identity component H° of H, with the expected Levi decomposition; see for example [Helminck and Wang 1993] or [Gurevich and Offen 2016, Lemma 3.1]. Let μ be a positive quasi-invariant measure on the quotient $Q^\theta \backslash H$ [Bernstein and Zelevinsky 1976, Theorem 1.21].

Lemma 2.7. *Let ρ be a smooth representation of L and let $\pi = \iota_Q^G \rho$. The map $\lambda \mapsto \lambda^G$ is an injection of $\mathrm{Hom}_{L^\theta}(\delta_Q^{1/2} \rho, \delta_{Q^\theta})$ into $\mathrm{Hom}_H(\pi, 1)$, where for any function ϕ in the space of π, λ^G is given explicitly by*

$$\langle \lambda^G, \phi \rangle = \int_{Q^\theta \backslash H} \langle \lambda, \phi(h) \rangle \, d\mu(h).$$

Corollary 2.8. *If $\delta_Q^{1/2}$ restricted to L^θ is equal to δ_{Q^θ}, then the map $\lambda \mapsto \lambda^G$ is an injection of $\mathrm{Hom}_{L^\theta}(\rho, 1)$ into $\mathrm{Hom}_H(\pi, 1)$. In particular, if ρ is L^θ-distinguished, then π is H-distinguished.*

Proof. Observe that $\mathrm{Hom}_{L^\theta}(\delta_Q^{1/2} \rho, \delta_{Q^\theta}) = \mathrm{Hom}_{L^\theta}(\rho, \delta_Q^{-1/2}|_{L^\theta} \delta_{Q^\theta})$. $\square$

In fact, the H-invariant linear form on $\pi = \iota_Q^G \rho$ arises from the closed orbit in $Q \backslash G / H$ via the Mackey theory.

3. Background on RDS: the relative Casselman's criterion

3A. *Exponents (of induced representations).* Let (π, V) be a finitely generated admissible representation of G. Let χ be a quasicharacter of the F-split component A_G of the center of G. For $n \in \mathbb{N}$, $n \geq 1$, define the subspace

$$V_{\chi,n} = \{v \in V : (\pi(z) - \chi(z))^n v = 0, \text{ for all } z \in A_G\},$$

and set

$$V_\chi = \bigcup_{n=1}^\infty V_{\chi,n}.$$

Each $V_{\chi,n}$ is a G-stable subspace of V and V_χ is the generalized χ-eigenspace in V for the A_G-action on V. By [Casselman 1995, Proposition 2.1.9],

(1) V is a direct sum $V = \bigoplus_\chi V_\chi$, where χ ranges over quasicharacters of A_G, and

(2) since V is finitely generated, there are only finitely many χ such that $V_\chi \neq 0$. Moreover, there exists $n \in \mathbb{N}$ such that $V_\chi = V_{\chi,n}$, for each χ.

Let $\mathcal{E}xp_{A_G}(\pi)$ be the (finite) set of quasicharacters of A_G such that $V_\chi \neq 0$. The quasicharacters that appear in $\mathcal{E}xp_{A_G}(\pi)$ are called the exponents of π. The second item above implies that V has a finite filtration such that the quotients are χ-representations, for $\chi \in \mathcal{E}xp_{A_G}(\pi)$.

Lemma 3.1. *The characters χ of A_G that appear in $\mathcal{E}xp_{A_G}(\pi)$ are the central quasicharacters of the irreducible subquotients of π.*

Let (π, V) be a finitely generated admissible representation of G. Let $P = MN$ be a parabolic subgroup of G with Levi factor M and unipotent radical N. It is a theorem of Jacquet that (π_N, V_N) is also finitely generated and admissible (see [Casselman 1995, Theorem 3.3.1]). Applying (1) and (2) to (π_N, V_N), we obtain a direct sum decomposition,

$$V_N = \bigoplus_{\chi \in \mathcal{E}xp_{A_M}(\pi_N)} (V_N)_\chi,$$

where the set $\mathcal{E}xp_{A_M}(\pi_N)$ of quasicharacters of A_M, such that $(V_N)_\chi \neq 0$, is finite. The quasicharacters of A_M appearing in $\mathcal{E}xp_{A_M}(\pi_N)$ are called the exponents of π along P.

We are ultimately interested in the exponents of parabolically induced representations. For a proof of the following lemma, see [Smith 2018, Lemma 4.16].

Lemma 3.2. *Let $P = MN$ be a parabolic subgroup of G, let (ρ, V_ρ) be a finitely generated admissible representation of M and let $\pi = \iota_P^G \rho$. The quasicharacters $\chi \in \mathcal{E}xp_{A_G}(\pi)$ are the restriction to A_G of characters η of A_M appearing in $\mathcal{E}xp_{A_M}(\rho)$.*

3B. *Invariant linear forms on Jacquet modules.* Let (π, V) be an admissible H-distinguished representation of G. Let λ be a nonzero H-invariant linear form on V. Let P be a θ-split parabolic subgroup of G. Recall that a parabolic subgroup P of G is θ-split if $\theta(P)$ is opposite to P (see Section 4B). Let N be the unipotent radical of P, and let $M = P \cap \theta(P)$ be a θ-stable Levi factor of P. Kato and Takano, and independently Lagier, defined an M^θ-invariant linear form λ_N on the Jacquet module (π_N, V_N). The construction of λ_N relies on Casselman's canonical lifting [1995, Proposition 4.1.4].

We next recall Proposition 5.6 of [Kato and Takano 2008]; see the cited works for details of the construction of λ_N.

Proposition 3.3 [Kato and Takano 2008; Lagier 2008]. *Let $\lambda \in \mathrm{Hom}_H(\pi, 1)$ be nonzero and let P be a θ-split parabolic subgroup of G with unipotent radical N and θ-stable Levi component $M = P \cap \theta(P)$. Let (π, V) be an admissible H-distinguished representation of G.*

(1) *The linear functional $\lambda_N : V_N \to \mathbb{C}$ is M^θ-invariant.*

(2) *The mapping $\mathrm{Hom}_H(\pi, 1) \to \mathrm{Hom}_{M^\theta}(\pi_N, 1)$, sending λ to λ_N, is linear.*

3C. *The relative Casselman's criterion.* Let (π, V) be a finitely generated admissible H-distinguished representation of G. Fix a nonzero H-invariant form λ on V.

For any closed subgroup Z of the center of G, Kato and Takano [2010] define

$$(3\text{-}1) \qquad \mathcal{E}xp_Z(\pi, \lambda) = \{\chi \in \mathcal{E}xp_Z(\pi) : \lambda|_{V_\chi} \neq 0\},$$

and refer to the set $\mathcal{E}xp_Z(\pi, \lambda)$ as exponents of π relative to λ.

The following appears as [Kato and Takano 2010, Theorem 4.7], see Section 4A for the definition of the set $S_M^- \setminus S_G S_M^1$.

Theorem 3.4 (relative Casselman's criterion). *Let ω be a unitary character of Z_G. Let (π, V) be a finitely generated admissible H-distinguished ω-representation of G. Fix a nonzero H-invariant linear form λ on V. The representation (π, V) is (H, λ)-relatively square integrable if and only if the condition*

$$(3\text{-}2) \qquad |\chi(s)| < 1 \text{ for all } \chi \in \mathcal{E}xp_{S_M}(\pi_N, \lambda_N) \text{ and all } s \in S_M^- \setminus S_G S_M^1$$

is satisfied for every proper θ-split parabolic subgroup $P = MN$ of G.

It is an immediate corollary of Theorem 3.4 that if (π, V) is an H-distinguished discrete series representation of G, then π is H-relatively square integrable. For a proof of the following, see [Smith 2018, Proposition 4.23].

Proposition 3.5. *Let (π, V) be a finitely generated admissible representation of G. Let $\chi \in \mathcal{E}xp_{Z_G}(\pi)$ and assume that none of the irreducible subquotients of (π, V) with central character χ are H-distinguished. Then for any $\lambda \in \mathrm{Hom}_H(\pi, 1)$, the restriction of λ to V_χ is equal to zero, i.e., $\lambda|_{V_\chi} \equiv 0$.*

4. *p*-adic symmetric spaces and parabolic subgroups

In this section, we discuss the tori, root systems, and parabolic subgroups relevant to our study of $H \backslash G$. We briefly review some general notions before turning our attention to the case of $\mathbf{U}_{E/F}(F) \backslash \mathbf{GL}_{2n}(E)$ in Section 4C.

4A. (θ, *F*)-split tori and the relative root system.

We say that an element $g \in G$ is θ-split if $\theta(g) = g^{-1}$. Recall that an F-torus S contained in G is (θ, F)-split if S is F-split and every element of S is θ-split. Let S_0 be a maximal (θ, F)-split torus of G. Fix a θ-stable maximal F-split torus A_0 of G that contains S_0 [Helminck and Wang 1993, Lemma 4.5(iii)]. Let $\Phi_0 = \Phi(G, A_0)$ be the root system of G with respect to A_0, and let W_0 be the associated Weyl group.

Let M be any Levi subgroup of G. Let A_M be the F-split component of the center of M. The (θ, F)-split component of M is the largest (θ, F)-split torus S_M contained in Z_M. The torus S_M is the connected component of the subgroup of θ-split elements in A_M. Explicitly,

$$S_M = (\{x \in A_M : \theta(x) = x^{-1}\})^\circ,$$

where $(\cdot)^\circ$ indicates the Zariski-connected component of the identity.

There is an action of θ on the F-rational characters $X^*(A_0)$ of A_0. Indeed, since A_0 is θ-stable, for $\chi \in X^*(A_0)$, the character

$$(\theta\chi)(a) = \chi(\theta(a))$$

is well defined for all $a \in A_0$. In addition, $\Phi_0 \subset X^*(A_0)$ is stable under the action of θ. Let Φ_0^θ be the set of θ-fixed roots. Recall that a choice Δ_0 of base for Φ_0 determines a system Φ_0^+ of positive roots.

Definition 4.1. A base Δ_0 of Φ_0 is called a θ-base if for every $\alpha \in \Phi_0^+$, such that $\alpha \neq \theta(\alpha)$, we have $\theta(\alpha) \in \Phi_0^-$.

Let Δ_0 be a θ-base of Φ_0 (existence of a θ-base is proved in [Helminck 1988]). Let $p : X^*(A_0) \to X^*(S_0)$ be the morphism defined by restricting the F-rational characters of A_0 to the subtorus S_0. The map p is surjective and the kernel of p is the submodule $X^*(A_0)^\theta$ consisting of θ-fixed F-rational characters. The restricted root system of $H\backslash G$ (relative to our choice of (A_0, S_0, Δ_0)) is defined to be

$$\overline{\Phi}_0 = p(\Phi_0) \setminus \{0\} = p(\Phi_0 \setminus \Phi_0^\theta).$$

The set $\overline{\Phi}_0$ coincides with the set $\Phi(G, S_0)$ of roots in G with respect to S_0. The set $\overline{\Phi}_0$ is a root system by [Helminck and Wang 1993, Proposition 5.9]; however, $\overline{\Phi}_0$ is not necessarily reduced. The set

$$\overline{\Delta}_0 = p(\Delta_0) \setminus \{0\} = p(\Delta_0 \setminus \Delta_0^\theta)$$

forms a base for $\overline{\Phi}_0$. The linear independence of $\overline{\Delta}_0$ follows from the fact that Δ_0 is a θ-base and that $\ker p = X^*(A_0)^\theta$ consists of θ-fixed characters.

Given a subset $\overline{\Theta} \subset \overline{\Delta}_0$, define the subset

$$[\overline{\Theta}] = p^{-1}(\overline{\Theta}) \cup \Delta_0^\theta$$

of Δ_0. Subsets of Δ_0 of the form $[\overline{\Theta}]$, where $\overline{\Theta} \subset \overline{\Delta}_0$, are called θ-split. The maximal θ-split subsets of Δ_0 are of the form $[\overline{\Delta}_0 \setminus \{\bar{\alpha}\}]$, where $\bar{\alpha} \in \overline{\Delta}_0$.

4B. θ-split parabolic subgroups and θ-elliptic Levi factors. As above, let Δ_0 be a θ-base of Φ_0. To any subset Θ of Δ_0, we may associate a Δ_0-standard parabolic subgroup P_Θ of G with unipotent radical N_Θ and standard Levi factor $M_\Theta = C_G(A_\Theta)$, where A_Θ is the F-split torus

$$A_\Theta = \left(\bigcap_{\alpha \in \Theta} \ker \alpha \right)^\circ.$$

Let Φ_Θ be the subsystem of Φ_0 generated by the simple roots Θ. Let Φ_Θ^+ be the system of Θ-positive roots. The unipotent radical N_Θ of P_Θ is generated by the

root groups N_α, where $\alpha \in \Phi_0^+ \setminus \Phi_\Theta^+$. The torus A_Θ is the F-split component of the center of M_Θ and Φ_Θ is the root system of A_0 in M_Θ.

Definition 4.2. A parabolic subgroup P of G is θ-split if $\theta(P)$ is opposite to P.

If P is a θ-split parabolic subgroup, then $M = P \cap \theta(P)$ is a θ-stable Levi subgroup of both P and the opposite parabolic $P^{\mathrm{op}} = \theta(P)$. If $\Theta \subset \Delta_0$ is θ-split, then the Δ_0-standard parabolic subgroup $P_\Theta = M_\Theta N_\Theta$ is θ-split. Any Δ_0-standard θ-split parabolic subgroup arises this way [Kato and Takano 2008, Lemma 2.5(1)]. Following [Kato and Takano 2010, §1.5], the (θ, F)-split component of M_Θ is equal to

$$S_\Theta = (\{s \in A_\Theta : \theta(s) = s^{-1}\})^\circ = \left(\bigcap_{\bar{\alpha} \in p(\Theta)} \ker(\bar{\alpha} : S_0 \to F^\times) \right)^\circ.$$

For any $0 < \epsilon \leq 1$, define

(4-1) $S_\Theta^-(\epsilon) = \{s \in S_\Theta : |\alpha(s)|_F \leq \epsilon, \text{ for all } \alpha \in \Delta_0 \setminus \Theta\}.$

Let S_Θ^- denote $S_\Theta^-(1)$. The set S_Θ^- is referred to as the dominant part of S_Θ.

By [Helminck and Helminck 1998, Theorem 2.9], the subset Δ_0^θ of θ-fixed roots in Δ_0 determines the (Δ_0-standard) minimal θ-split parabolic subgroup $P_0 = P_{\Delta_0^\theta}$. By [Helminck and Wang 1993, Proposition 4.7(iv)], the minimal θ-split parabolic subgroup P_0 has standard θ-stable Levi $M_0 = C_G(S_0)$. Let N_0 be the unipotent radical of P_0. We have $P_0 = M_0 N_0$.

Lemma 4.3 [Kato and Takano 2008, Lemma 2.5]. *Let $S_0 \subset A_0$, Δ_0, and $P_0 = M_0 N_0$ be as above.*

(1) *Any θ-split parabolic subgroup P of G is conjugate to a Δ_0-standard θ-split parabolic subgroup by an element $g \in (\mathbf{HM_0})(F)$.*

(2) *If the group of F-points of the product $(\mathbf{HM_0})(F)$ is equal to HM_0, then any θ-split parabolic subgroup of G is H-conjugate to a Δ_0-standard θ-split parabolic subgroup.*

Let $P = MN$ be a θ-split parabolic subgroup. Pick $g \in (\mathbf{HM_0})(F)$ such that $P = gP_\Theta g^{-1}$ for some θ-split subset $\Theta \subset \Delta_0$. Since $g \in (\mathbf{HM_0})(F)$ we have $g^{-1}\theta(g) \in \mathbf{M_0}(F)$, and we have $S_M = gS_\Theta g^{-1}$. For a given $\epsilon > 0$, one may extend the definition of S_Θ^- in (4-1) to the torus S_M. Set

$$S_M^-(\epsilon) = gS_\Theta^-(\epsilon)g^{-1}$$

and define $S_M^- = S_M^-(1)$. Recall that we write S_M^1 to denote the $\mathcal{O}_F$-points $S_M(\mathcal{O}_F)$.

The next definition is made in analogy with the notion of an elliptic Levi subgroup. The following terminology is from [Murnaghan 2017].

Definition 4.4. A θ-stable Levi subgroup L of G is θ-elliptic if and only if L is not contained in any proper θ-split parabolic subgroup of G.

The next lemma follows immediately from Definition 4.4.

Lemma 4.5. *If a θ-stable Levi subgroup L of G contains a θ-elliptic Levi subgroup, then L is θ-elliptic.*

The next proposition appears in [Murnaghan 2017, Proof of Proposition 8.4].

Proposition 4.6. *Let Q be a parabolic subgroup of G. If Q admits a θ-elliptic Levi factor L, then Q is θ-stable.*

Proof. By definition, L is θ-stable. One can show that for any root α of A_L in G we have $\theta(\alpha) = \alpha$. It follows that the unipotent radical of Q is also θ-stable. $\square$

4C. *Structure of $\mathbf{U}_{E/F}(F)\backslash\mathbf{GL}_{2n}(E)$.* Let $\mathbf{G} = R_{E/F}\mathbf{GL}_n$ be the restriction of scalars from E to F of $\mathbf{GL}_n$. We will restrict to the case that n is even from Section 5 onward. We identify the group $G = \mathbf{G}(F)$ with the set $\mathbf{GL}_n(E)$ of E-points of $\mathbf{GL}_n$. The nontrivial element σ of the Galois group $\mathrm{Gal}(E/F)$ gives rise to an F-involution of $\mathbf{G}$ given by entrywise Galois conjugation on $\mathbf{GL}_n(E)$. We denote the Galois involution of $\mathbf{G}$ by σ. Explicitly,

$$\sigma(g) = (\sigma(g_{ij})), \quad \text{where } g = (g_{ij}) \in G.$$

Following [Feigon et al. 2012], let $\mathbf{X}$ denote the F-variety of Hermitian matrices in $\mathbf{G}$,

$$(4\text{-}2) \qquad\qquad \mathbf{X} = \{x \in \mathbf{G} : {}^t\sigma(x) = x\}.$$

Here ${}^tg$ denotes the transpose of $g \in \mathbf{G}$. There is a right action of $\mathbf{G}$ on $\mathbf{X}$ given by $x \cdot g = {}^t\sigma(g)xg$, where $x \in \mathbf{X}$ and $g \in \mathbf{G}$. Write $X = \mathbf{X}(F)$ for the F points of $\mathbf{X}$. There is a finite set X/G of G-orbits in X indexed by $F^\times/N_{E/F}(E^\times)$ [Feigon et al. 2012]. By local class field theory, $F^\times/N_{E/F}(E^\times)$ is isomorphic to $\mathrm{Gal}(E/F)$, and thus consists of two elements.

Given $x \in X$, define an F-involution θ_x of G by

$$(4\text{-}3) \qquad\qquad \theta_x(g) = x^{-1}{}^t\sigma(g)^{-1}x,$$

for all $g \in G$. Let $\mathbf{H}^x = \mathbf{G}^{\theta_x}$ be the subgroup of θ_x-fixed elements. The group of F-points $H^x = \mathbf{H}^x(F)$ is a unitary group associated to E/F and x.

Remark 4.7. In the literature, $\mathbf{U}_{E/F,x}$ is often used to denote the unitary group $\mathbf{H}^x$ associated to E/F and x. We will use the $\mathbf{U}_{E/F,x}$ notation for unitary groups that appear as subgroups of Levi factors of G.

Definition 4.8. An involution θ_1 of G is G-equivalent to another involution θ_2 if there exists $g \in G$ such that $\theta_1 = \text{Int } g^{-1} \circ \theta_2 \circ \text{Int } g$, where $\text{Int } g$ denotes the inner F-automorphism of $\mathbf{G}$ given by $\text{Int } g(x) = gxg^{-1}$, for all $x \in \mathbf{G}$. We write $g \cdot \theta$ to denote the involution $\text{Int } g^{-1} \circ \theta \circ \text{Int } g$.

Two involutions θ_{x_1} and θ_{x_2} are G-equivalent if and only if x_1 and x_2 lie in the same G-orbit in X/G. Indeed, if there exists $g \in G$ such that $y = x \cdot g = {}^t\sigma(g)xg$, then one can check that θ_y is equal to the involution $g \cdot \theta_x = \text{Int } g^{-1} \circ \theta_x \circ \text{Int } g$. Note that the G-action $\theta \mapsto g \cdot \theta$ on involutions is also a right-action. Since X/G has order two, there are two G-equivalence classes of involutions of the form θ_x. It is well known that when n is odd, $\mathbf{H}^x$ is always quasisplit over F. When n is even there are two isomorphism classes of unitary group associated to E/F, one of which is quasisplit.

We fix $\theta = \theta_{w_\ell}$, where w_ℓ is the permutation matrix in G with unit antidiagonal, and write $\mathbf{H} = \mathbf{G}^\theta$. The group $\mathbf{H} = \mathbf{U}_{E/F,w_\ell}$ is quasisplit over F. Write $H = \mathbf{H}(F)$ for the group of F-points of $\mathbf{H}$.

Let J_r be the $r \times r$ permutation matrix with unit antidiagonal

$$
J_r = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}
$$

and note that $w_\ell = J_n$. For any positive integer r, there exists $\gamma_r \in \mathbf{GL}_r(E)$ such that ${}^t\sigma(\gamma_r)J_r\gamma_r$ lies in the diagonal F-split torus of $\mathbf{GL}_r(E)$. For instance if r is even, we set

$$
\gamma_r = \begin{pmatrix} 1 & & & & & & 1 \\ & \ddots & & & & \cdot^{\cdot^{\cdot}} & \\ & & 1 & 1 & & & \\ & & 1 & -1 & & & \\ & \cdot^{\cdot^{\cdot}} & & & \ddots & & \\ 1 & & & & & & -1 \end{pmatrix}
$$

and if r is odd, we take

$$
\gamma_r = \begin{pmatrix} 1 & & & & & & 1 \\ & \ddots & & & & \cdot^{\cdot^{\cdot}} & \\ & & 1 & 0 & 1 & & \\ & & 0 & 1 & 0 & & \\ & & 1 & 0 & -1 & & \\ & \cdot^{\cdot^{\cdot}} & & & & \ddots & \\ 1 & & & & & & -1 \end{pmatrix}.
$$

Define $\gamma = \gamma_n$ and notice that

$$(4\text{-}4) \qquad {}^t\sigma(\gamma)w_\ell\gamma = \mathrm{diag}(\underbrace{2,\dots,2}_{\lfloor n/2 \rfloor}, \widehat{1}, \underbrace{-2,\dots,-2}_{\lfloor n/2 \rfloor})$$

lies in the diagonal F-split torus A_T of G.

Let $\mathbf{T}$ be the maximal (nonsplit) diagonal F-torus of $\mathbf{G}$. The torus $\mathbf{T}$ is obtained by restriction of scalars of the diagonal torus of $\mathbf{GL}_n$. Let $T = \mathbf{T}(F)$, and identify T with the diagonal matrices in $\mathbf{GL}_n(E)$. Let A_T be the F-split component of T. Define $T_0 = {}^\gamma T$, then the F-split component of T_0 is $A_0 = {}^\gamma A_T$. The tori T, A_T, T_0 and A_0 are all θ-stable. Observe that A_0 is a maximal F-split torus of G that is θ-split. In particular, A_0 is a maximal (θ, F)-split torus of G. Indeed, ${}^t\sigma(\gamma)w_\ell\gamma$ lies in the abelian subgroup A_T; therefore, for any $\gamma t \gamma^{-1} \in A_0$, we have

$$\begin{aligned}
\theta(\gamma t \gamma^{-1}) &= w_\ell^{-1}{}^t\sigma(\gamma)^{-1}({}^t\sigma(t)^{-1}){}^t\sigma(\gamma)w_\ell \\
&= \gamma({}^t\sigma(\gamma)w_\ell\gamma)^{-1}t^{-1}({}^t\sigma(\gamma)w_\ell\gamma)\gamma^{-1} \\
&= (\gamma t \gamma^{-1})^{-1},
\end{aligned}$$

where we've used that ${}^t\sigma(t)^{-1} = t^{-1}$, for any $t \in A_T$.

Lemma 4.9. *For any $x \in X$, the (θ_x, F)-split component of G, which we denote by $S_{G,x}$, is equal to the F-split component A_G of the center of G.*

Proof. Let $z \in A_G$. Since z is a diagonal matrix with entries in $F^\times$, we have ${}^t\sigma(z) = z$; moreover, since z is central in G,

$$\theta_x(z) = x^{-1}{}^t\sigma(z)^{-1}x = x^{-1}z^{-1}x = z^{-1}.$$

It follows that $S_{G,x} = (A_G)^\circ = A_G$ (see Section 4A). $\qquad\square$

Let $\Phi = \Phi(G, A_T)$ be the root system of G with respect to A_T, with standard base $\Delta = \{\epsilon_i - \epsilon_{i+1} : 1 \le i \le n-1\}$. Let $\Phi_0 = \Phi(G, A_0)$ be the root system of G with respect to $A_0 = {}^\gamma A_T$. Observe that $\Phi_0 = {}^\gamma\Phi$. Set $\Delta_0 = {}^\gamma\Delta$. The set of positive roots of Φ_0 with respect to Δ_0 is denoted Φ_0^+. We have $\Phi_0^+ = {}^\gamma\Phi^+$, where Φ^+ is the set of positive roots in Φ determined by Δ. Our current aim is to use Φ_0 to determine the (standard) θ-split parabolic subgroups of G. First, we note the following.

Lemma 4.10. *For any $\alpha \in \Phi_0$, we have $\theta(\alpha) = -\alpha$.*

Proof. Let $\alpha \in \Phi_0$. For any $a \in A_0$, we have $\theta(a) = a^{-1}$; therefore,

$$(\theta\alpha)(a) = \alpha(\theta(a)) = \alpha(a^{-1}) = \alpha(a)^{-1} = (-\alpha)(a).$$

Since $a \in A_0$ was arbitrary, we have $\theta(\alpha) = -\alpha$. $\qquad\square$

Two corollaries of Lemma 4.10 follow immediately (cf. Definition 4.1):

Corollary 4.11. *The set Φ_0^θ of θ-fixed roots in Φ_0 is empty.*

Corollary 4.12. *Any set of simple roots in Φ_0 is a θ-base for Φ_0. In particular, Δ_0 is a θ-base.*

Explicitly, $\Delta_0 = \{^\gamma(\epsilon_i - \epsilon_{i+1}) : 1 \le i \le n - 1\}$ and, by Corollary 4.12, the set of simple roots Δ_0 is a θ-base for Φ_0. Since the maximal F-split torus A_0 is a maximal (θ, F)-split torus, the restricted root system of $H\backslash G$ is just the root system Φ_0 of G. The next proposition now follows immediately.

Proposition 4.13. *Every parabolic subgroup of G standard with respect to Δ_0 is a θ-split parabolic subgroup. Any such parabolic subgroup is the γ-conjugate of the usual block upper triangular parabolic subgroups of G.*

By Lemma 4.3(1), any θ-split parabolic subgroup of G is $(\mathbf{HT}_0)(F)$-conjugate to a Δ_0-standard θ-split parabolic subgroup.

We now consider θ-stable parabolic subgroups; in particular, we are concerned with determining which proper θ-stable parabolic subgroups admit a θ-elliptic Levi factor.

Definition 4.14. Let $(\underline{n}) = (n_1, \dots, n_r)$ be a partition of n; we say that $(\underline{n})$ is balanced if $n_i = n_{r+1-i}$, for $1 \le i \le r$.

Let $(\underline{n}) = (n_1, \dots, n_r)$ be a partition of n. The opposite partition to $(\underline{n})$ is $(\underline{n})^{\mathrm{op}} = (n_r, \dots, n_1)$. This terminology reflects that the standard upper triangular parabolic subgroup that is $\mathbf{GL}_n$-conjugate to the (lower triangular) opposite parabolic of $P_{(\underline{n})}$ is precisely $P_{(\underline{n})^{\mathrm{op}}}$. Observe that $(\underline{n})$ is balanced if and only if $(\underline{n})^{\mathrm{op}} = (\underline{n})$.

Lemma 4.15. *The θ-stable block upper triangular parabolic subgroups of G correspond to balanced partitions of n. The only such parabolic that has a θ-stable θ-elliptic Levi subgroup is $P_{(n/2,n/2)}$, in the case that n is even.*

Proof. Let $(\underline{n}) = (n_1, \dots, n_r)$ be a partition of n. Let $A = A_{(\underline{n})}$ be the diagonal F-split torus corresponding to $(\underline{n})$. The parabolic subgroup $P = P_{(\underline{n})}$ is θ-stable if and only if its standard Levi subgroup $M = C_G(A)$ and unipotent radical N are θ-stable; moreover, M is θ-stable if and only if A is θ-stable. Observe that A is θ-stable if and only if $w_\ell \in N_G(A)$. Indeed, $\theta(a) = w_\ell^{-1}{}^t\sigma(a)^{-1}w_\ell$ and A is stable under the involution $a \mapsto {}^t\sigma(a)^{-1} = a^{-1}$. It is immediate that $\theta(A) = A_{(\underline{n})^{\mathrm{op}}}$; moreover, A is θ-stable if and only if $(\underline{n})^{\mathrm{op}} = (\underline{n})$ if and only if $(\underline{n})$ is balanced. Therefore, it suffices to show that N is θ-stable if and only if $(\underline{n})$ is balanced. First note that N is stable under the map $n \mapsto \sigma(n)^{-1}$; on the other hand, the transpose map sends N to the opposite unipotent radical N^{op}. In particular, N is θ-stable if and only if $N = w_\ell^{-1}N^{\mathrm{op}}w_\ell$. A simple matrix computation shows that this occurs if and only if $(\underline{n})^{\mathrm{op}} = (\underline{n})$, i.e., $(\underline{n})$ is balanced.

Let $(\underline{n}) = (n_1, \dots, n_{\lfloor r/2 \rfloor}, \widehat{n_\bullet}, n_{\lfloor r/2 \rfloor}, \dots, n_1)$ be a balanced partition of n. Now, we show that M is θ-elliptic if and only if $(\underline{n}) = (n/2, n/2)$ by applying [Smith 2018, Lemma 3.8], which states that a θ-stable Levi subgroup M is θ-elliptic if and

only if $S_M = S_G$. An element $a = \operatorname{diag}(a_1, \ldots, a_n)$ of A_T is θ-split if and only if a centralizes w_ℓ. Indeed, since A_T is pointwise fixed by taking the transpose-Galois conjugates, we have

$$\theta(a) = w_\ell^{-1} \operatorname{diag}(a_1^{-1}, \ldots, a_n^{-1}) w_\ell = \operatorname{diag}(a_n^{-1}, \ldots, a_1^{-1}),$$

which is equal to a^{-1} if and only if $a_i = a_{n+1-i}$, for all $1 \leq i \leq n$. It follows that $a^{-1} = \theta(a)$ if and only if $a \in C_{A_T}(w_\ell)$, where

$$C_{A_T}(w_\ell) = \{\operatorname{diag}(a_1, \ldots, a_{\lfloor n/2 \rfloor}, \widehat{a}_\bullet, a_{\lfloor n/2 \rfloor}, \ldots, a_1) : a_i \in F^\times, 1 \leq i \leq \lfloor n/2 \rfloor\}.$$

The (θ, F)-split component S_M of M is thus equal to the identity component of $A \cap C_{A_T}(w_\ell)$; this intersection in turn is equal to the F-split torus

(4-5)
$$\left\{\operatorname{diag}\left(\underbrace{a_1, \ldots, a_1}_{n_1}, \ldots, \underbrace{a_{\lfloor r/2 \rfloor}, \ldots, a_{\lfloor r/2 \rfloor}}_{n_{\lfloor r/2 \rfloor}}, \underbrace{\widehat{a_\bullet, \ldots, a_\bullet}}_{n_\bullet}, \underbrace{a_{\lfloor r/2 \rfloor}, \ldots, a_{\lfloor r/2 \rfloor}}_{n_{\lfloor r/2 \rfloor}}, \ldots, \underbrace{a_1, \ldots, a_1}_{n_1}\right)\right\}.$$

In particular, $S_M = A \cap C_{A_T}(w_\ell)$. By Lemma 4.9, we have $S_G = A_G$ and observe that S_M is equal to A_G if and only if $r = 2$, that is, n is even and $(\underline{n}) = (n/2, n/2)$, as claimed. $\square$

Remark 4.16. When n is even, we set $L = M_{(n/2, n/2)}$ and reiterate that L is the only proper block-diagonal θ-elliptic Levi subgroup of G.

Corollary 4.17. *The minimal parabolic (Borel) subgroup Q_0 of G consisting of the upper triangular matrices is a θ-stable minimal parabolic of G. In particular, $\mathbf{Q}_0 = R_{E/F}\mathbf{B}$, where $\mathbf{B}$ is the upper triangular Borel subgroup of $\mathbf{GL}_n$.*

Proof. The partition $(1, \ldots, 1)$ is balanced; apply Lemma 4.15. $\square$

Corollary 4.18. *The F-subgroup $Q_0 \cap H$ of H, consisting of the upper triangular elements of H, is a Borel subgroup of H.*

Proof. See, for instance, [Gurevich and Offen 2016, Lemma 3.1]. $\square$

Lemma 4.19. *There are no proper θ-elliptic Levi subgroups of G that contain A_0.*

Proof. The maximal F-split torus A_0 of G is (θ, F)-split. The lemma follows from [Smith 2018, Lemma 3.8]. $\square$

Recall that a parabolic subgroup P of G is called A_T-semistandard if P contains A_T. If P is an A_T-semistandard parabolic subgroup, then there is a unique Levi factor M of P that contains A_T. We refer to M as the A_T-semistandard Levi factor of P.

Lemma 4.20. *Let $n \geq 2$ be an integer.*

(1) *If n is odd, then there are no proper θ-elliptic A_T-semistandard Levi subgroups of $G = \mathbf{GL}_n(E)$.*

(2) *If n is even, then $L = M_{(n/2,n/2)}$ is the only maximal proper θ-elliptic A_T-semistandard Levi subgroup of $G = \mathbf{GL}_n(E)$, up to conjugacy by Weyl group elements $w \in W = W(G, A_T)$, such that $w^{-1}w_\ell w \in N_G(L) \setminus L$.*

Proof. We give a sketch of the proof. We identify the Weyl group $W = W(G, A_T)$ of A_T in G with the subgroup of permutation matrices in G. By Lemma 4.15, if n even, then $L = M_{(n/2,n/2)}$ is θ-elliptic.

First, let $P = MN$ be a θ-stable maximal proper A_T-semistandard parabolic subgroup of G with θ-stable A_T-semistandard Levi subgroup M. It is well known that P is W-conjugate to a unique standard (block upper triangular) maximal parabolic subgroup of G. In particular, $M = wM_{(n_1,n_2)}w^{-1}$ for some partition (n_1, n_2) of n and some $w \in W$. Moreover, M is θ-stable if and only if its F-split component $A_M = wA_{(n_1,n_2)}w^{-1}$ is contained in a torus $A_{(\underline{n})}$, for some balanced partition $(\underline{n})$ of n. Assume that M is θ-stable and let $(\underline{n})$ be the coarsest partition such that $wA_{(n_1,n_2)}w^{-1}$ is contained in $A_{(\underline{n})}$. One may regard the F-split component $A_M = wA_{(n_1,n_2)}w^{-1}$ of M as being obtained by a two-coloring of $(\underline{n})$. That is, regard

$$\mathrm{diag}(\underbrace{a, \ldots, a}_{n_1}, \underbrace{b, \ldots, b}_{n_2}) \to A_{(\underline{n})}$$

as a two-coloring of the partition $(\underline{n})$. One may readily verify that:

(1) If n is odd, we see by considering (4-5) that $wA_{(n_1,n_2)}w^{-1}$ contains at least a rank-one F-split torus, noncentral in G, consisting of θ-split elements. (Indeed, since n is odd, the central segment in (4-5) must appear.) In particular, M cannot be θ-elliptic by [Smith 2018, Lemma 3.4]. Moreover, by Lemma 4.5, M cannot contain a θ-elliptic Levi subgroup of G. It follows that there are no A_T-semistandard θ-elliptic Levi subgroups when n is odd.

(2) Suppose that n is even. In light of (4-5), we see that M is θ-elliptic if and only if $(\underline{n})$ is a refinement of the partition $(n/2, n/2)$ of n. It readily follows that $n_1 = n_2 = n/2$ and M is conjugate to L.

Observe that $\theta(w) = w_\ell^{-1}ww_\ell$, for any $w \in W$. It is straightforward to check that $M = wLw^{-1}$ is θ-stable if and only if $w^{-1}w_\ell w \in N_G(L)$. It can be verified that a θ-stable conjugate $M = wLw^{-1}$ of L is θ-elliptic if and only if $w^{-1}w_\ell w \notin C_G(A_L) = L$. Thus L is the only maximal A_T-semistandard θ-elliptic Levi subgroup of G (up to conjugacy). $\square$

We observe that $L = M_{(n/2,n/2)}$ does not contain any proper θ-elliptic Levi subgroups. We argue by contradiction. Suppose that $L' \subsetneq L$ is a θ-elliptic Levi subgroup of G. Notice that L' is also θ-elliptic in L. Since L' is proper in L, it follows from Lemma 4.5 that L' is contained in a θ-stable maximal proper Levi subgroup L'' of L. Without loss of generality, $L'' \cong M_{(k_1,k_2)} \times \mathbf{GL}_{n/2}(F)$. However,

considering the action of θ on L described in (4-8), we observe that no such Levi subgroup L'' can be θ-stable.

Lemma 4.20 does not give a complete characterization of the θ-elliptic Levi subgroups of G. The following lemma takes us closer to the desired result; in particular, up to H-conjugacy (and choice of standard parabolic) we have all the relevant θ-stable parabolic subgroups.

Lemma 4.21. *Let* $\mathbf{P}$ *be any* θ*-stable parabolic subgroup of* $\mathbf{G}$, *then* $P = \mathbf{P}(F)$ *is* H*-conjugate to a* θ*-stable* A_T*-semistandard parabolic subgroup.*

Proof. First, note that by Corollary 4.17 and [Helminck and Wang 1993, Lemma 3.5], the torus $(\mathbf{A}_T \cap \mathbf{H})^\circ$ is a maximal F-split torus of H. Let $\mathbf{P} = \mathbf{M}\mathbf{N}$ be a θ-stable parabolic subgroup with the indicated Levi factorization, where $\mathbf{M}$ and $\mathbf{N}$ are both θ-stable. Let $\mathbf{P}_\bullet$ be a minimal θ-stable parabolic subgroup of $\mathbf{G}$ contained in $\mathbf{P}$. Let $\mathbf{A}_\bullet$ be a θ-stable maximal F-split torus contained in $\mathbf{P}_\bullet$ [Helminck and Wang 1993, Lemma 2.5]. By [Helminck and Wang 1993, Corollary 5.8], there exists $g = nh \in (N_{\mathbf{G}}(\mathbf{A}_\bullet) \cap N_{\mathbf{G}}((\mathbf{A}_\bullet \cap \mathbf{H})^\circ))(F)\mathbf{H}(F)$ such that $g^{-1}\mathbf{P}_\bullet g = \mathbf{Q}_0$. Note that n normalizes $(\mathbf{A}_\bullet \cap \mathbf{H})^\circ$ and all of $\mathbf{A}_\bullet$, while h is θ-fixed. Observe that

$$(4\text{-}6) \qquad g^{-1}A_\bullet g = h^{-1}n^{-1}A_\bullet nh = h^{-1}A_\bullet h \subset Q_0;$$

in particular, $g^{-1}A_\bullet g$ is a θ-stable maximal F-split torus. Let $\mathbf{U}_0$ be the unipotent radical of $\mathbf{Q}_0$. By [Helminck and Wang 1993, Lemma 2.4], $g^{-1}A_\bullet g$ is $(\mathbf{H} \cap \mathbf{U}_0)(F)$-conjugate to A_T. It follows that there exists $h' \in (\mathbf{H} \cap \mathbf{U}_0)(F)$ such that

$$(4\text{-}7) \qquad A_T = h'^{-1}g^{-1}A_\bullet gh' = h'^{-1}h^{-1}A_\bullet hh' = (hh')^{-1}A_\bullet(hh');$$

moreover, $A_T = (hh')^{-1}A_\bullet(hh')$ is contained in $(hh')^{-1}\mathbf{P}(hh')$ and $\mathbf{P}$ is H-conjugate to a θ-stable A_T-semistandard parabolic subgroup. $\qquad\square$

Let $n \geq 2$ be an even integer. Let $l = \mathrm{diag}(x, y) \in L = M_{(n/2, n/2)}$. We compute that

$$(4\text{-}8) \qquad \theta(l) = w_\ell^{-1}({}^t\sigma(l)^{-1})w_\ell = \begin{pmatrix} J_{n/2}^{-1}{}^t\sigma(y)^{-1}J_{n/2} & 0 \\ 0 & J_{n/2}^{-1}{}^t\sigma(x)^{-1}J_{n/2} \end{pmatrix}.$$

It follows immediately from (4-8) that l is θ-fixed if and only if

$$y = \theta_{J_{n/2}}(x) = J_{n/2}^{-1}({}^t\sigma(x)^{-1})J_{n/2},$$

and observe that

$$(4\text{-}9) \qquad L^\theta = \left\{ \begin{pmatrix} x & 0 \\ 0 & \theta_{J_{n/2}}(x) \end{pmatrix} : x \in \mathbf{GL}_{n/2}(E) \right\} \cong \mathbf{GL}_{n/2}(E).$$

From (4-9), we immediately obtain a characterization of the θ-fixed points of the associate Levi subgroup $M = {}^\gamma L$ of the Δ_0-standard parabolic subgroup $P = {}^\gamma Q$.

Lemma 4.22. *The Levi subgroup $M = {}^\gamma L$ is the θ-stable Levi subgroup of a standard θ-split parabolic $P = MN = {}^\gamma P_{(n/2,n/2)}$. The θ-fixed points of M are isomorphic to a product of two copies of the unitary group*

$$\mathbf{U}_{E/F,1_{n/2}} = \{x \in \mathbf{GL}_{n/2}(E) : x^{-1} = {}^t\sigma(x)\},$$

where $1_{n/2}$ is the $n/2 \times n/2$ identity matrix.

Proof. Let ${}^\gamma m \in M$, where $m \in L$. We have

$$\begin{aligned}
\theta({}^\gamma m) &= w_\ell^{-1}\, {}^t\sigma(\gamma)^{-1}\, {}^t\sigma(m)^{-1}\, {}^t\sigma(\gamma)\, w_\ell \\
&= \gamma({}^t\sigma(\gamma)w_\ell\gamma)^{-1}\, {}^t\sigma(m)^{-1}\, ({}^t\sigma(\gamma)w_\ell\gamma)\gamma^{-1} \\
&= \gamma\, {}^t\sigma(m)^{-1}\gamma^{-1},
\end{aligned}$$

where the last equality holds since ${}^t\sigma(\gamma)w_\ell\gamma \in A_L$ centralizes $m \in L$. It follows that ${}^\gamma m = \theta({}^\gamma m)$ if and only if $m = {}^t\sigma(m)^{-1}$. Writing m as a block-diagonal matrix $m = \mathrm{diag}(x, y)$, we have $m = {}^t\sigma(m)^{-1}$ if and only if $x = {}^t\sigma(x)^{-1}$ and $y = {}^t\sigma(y)^{-1}$. It follows that

$$(4\text{-}10) \quad M^\theta = \left\{ \gamma \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \gamma^{-1} : x, y \in \mathbf{GL}_{n/2}(E),\, x = {}^t\sigma(x)^{-1},\, y = {}^t\sigma(y)^{-1} \right\},$$

and $M^\theta \cong \mathbf{U}_{E/F,1_{n/2}} \times \mathbf{U}_{E/F,1_{n/2}}$, as claimed. $\qquad\square$

In Lemma 5.8, we will determine the θ-fixed points of the Levi subgroup of an arbitrary maximal θ-split parabolic subgroup that has θ-stable Levi factor associate to L.

5. Relative discrete series for $\mathbf{U}_{E/F}(F)\backslash\mathbf{GL}_{2n}(E)$

From now on, let $G = \mathbf{GL}_{2n}(E)$, where $n \geq 2$, and let $H = \mathbf{U}_{E/F,w_\ell}(F)$ be the θ-fixed points of G. Recall that H is (the F-points of) a quasisplit unitary group. Let $Q = P_{(n,n)}$ be the type (n, n) block upper triangular parabolic subgroup of G, with block-diagonal Levi factor $L = M_{(n,n)}$ and unipotent radical $U_{(n,n)}$.

In this section we prove Theorem 5.11, the main result of the paper. We construct representations in the discrete series of $H\backslash G$ via parabolic induction from L^θ-distinguished discrete series representations of L.

The parabolic subgroup Q is conjugate to the Δ_0-standard maximal θ-split parabolic P_Ω, where $\Omega = \Delta_0\backslash\{{}^\gamma(\epsilon_n - \epsilon_{n+1})\}$. In particular, $Q = \gamma^{-1}P_\Omega\gamma$, $L = \gamma^{-1}M_\Omega\gamma$ and $U = \gamma^{-1}N_\Omega\gamma$. For any $\Theta \subset \Delta_0$, we use the representatives $[W_\Theta\backslash W_0/W_\Omega]$ for the double-coset space $P_\Theta\backslash G/P_\Omega$ (see Lemma 2.3). There is an isomorphism $P_\Theta\backslash G/P_\Omega \cong P_\Theta\backslash G/Q$ given by $w \mapsto w\gamma$.

5A. *A few more ingredients.* Here, we assemble the remaining representation theoretic results needed to state and prove Theorem 5.11.

5A1. *The inducing discrete series representations.* The irreducible representations distinguished by arbitrary unitary groups are characterized in the paper [Feigon et al. 2012], continuing the work of Jacquet et al., for instance in [Jacquet and Lai 1985; Jacquet 2001; 2010]. Feigon, Lapid and Offen study both local and global distinction, largely using global methods. In particular, they show that an irreducible square integrable representation π of G is H^x-distinguished if and only if π is Galois invariant. Although Feigon, Lapid and Offen prove much stronger results, we'll recall only what we need for our application. The following appears as [Feigon et al. 2012, Corollary 13.5]. Recall from (4-2) that X is the variety of Hermitian matrices in $\mathbf{GL}_n(E)$.

Theorem 5.1 (Feigon, Lapid and Offen). *Let π be an irreducible admissible essentially square integrable representation of $\mathbf{GL}_n(E)$. For any $x \in X$, the following conditions are equivalent*:

(1) *The representation π is Galois invariant, that is $\pi \cong {}^\sigma\pi$.*

(2) *The representation π is H^x-distinguished.*

In addition, $\dim \mathrm{Hom}_{H^x}(\pi, 1) \leq 1$.

The multiplicity-one statement appears as [Feigon et al. 2012, Proposition 13.3]. It is known that local multiplicity-one for unitary groups does not hold in general; see [Feigon et al. 2012, Corollary 13.16] for instance, which gives a lower bound for the dimension of $\mathrm{Hom}_{H^x}(\pi, 1)$ for Galois invariant generic representations π. On the other hand, Feigon, Lapid and Offen are able to extend Theorem 5.1 to all ladder representations (see [Feigon et al. 2012, Theorem 13.11]).

5A2. *Distinction of inducing representations.*

Proposition 5.2. *An irreducible admissible representation $\pi_1 \otimes \pi_2$ of L is L^θ-distinguished if and only if π_2 is equivalent to the Galois-twist of π_1, that is, if and only if $\pi_2 \cong {}^\sigma\pi_1$.*

We actually prove a slightly more general result from which Proposition 5.2 is a trivial corollary, by taking into account the description of L^θ given in (4-9).

Lemma 5.3. *Let $m \geq 1$ be an integer. Let $G' = \mathbf{GL}_m(E) \times \mathbf{GL}_m(E)$, $x \in \mathbf{GL}_m(E)$ be a Hermitian matrix, and define*

$$H' = \left\{ \begin{pmatrix} A & 0 \\ 0 & \theta_x(A) \end{pmatrix} : A \in \mathbf{GL}_m(E) \right\}.$$

An irreducible admissible representation $\pi_1 \otimes \pi_2$ of G' is H'-distinguished if and only if π_2 is equivalent to the Galois-twist of π_1, i.e., $\pi_2 \cong {}^\sigma\pi_1$.

Proof. First, note that a representation $\pi_1 \otimes \pi_2$ is H'-distinguished if and only if π_2 is equivalent to $^{\theta_x}\widetilde{\pi}_1$, the θ_x-twist of the contragredient of π_1. It suffices to show that for any Hermitian matrix x in $\mathbf{GL}_m(E)$, and any irreducible admissible representation π of $\mathbf{GL}_m(E)$, the Galois-twisted representation $^\sigma\pi$ is equivalent to $^{\theta_x}\widetilde{\pi}$. By Theorem 2 of [Gel'fand and Kajdan 1975], $\widetilde{\pi}$ is equivalent to the representation $\widehat{\pi}$ defined by $\widehat{\pi}(g) = \pi(^tg^{-1})$, acting on the space V of π. Since π is admissible, we have $\widetilde{\widetilde{\pi}} \cong \pi$; thus, $\widehat{(\widetilde{\pi})} \cong \pi$. On the other hand, the representation $^{\theta_x}\widetilde{\pi}$ on $\widetilde{V}$ is given by $^{\theta_x}\widetilde{\pi}(g) = \widetilde{\pi}(\theta_x(g))$. Using that x is Hermitian, it is readily verified that

$$^{\theta_x}\widetilde{\pi}(g) = {}^\sigma\widehat{(\widetilde{\pi})}(xgx^{-1}) = {}^{x^{-1}}({}^\sigma\widehat{(\widetilde{\pi})})(g).$$

We observe that $^{\theta_x}\widetilde{\pi}$ is equivalent to $^\sigma\widehat{(\widetilde{\pi})}$ since $\operatorname{Int} x^{-1}$ is an inner automorphism of $\mathbf{GL}_m(E)$. It is also clear that taking Galois twists commutes with the map sending π to $\widehat{\pi}$ (and twisting by $\operatorname{Int} x^{-1}$, since x is Hermitian). Finally, we have shown

$$^{\theta_x}\widetilde{\pi} \cong {}^{x^{-1}}({}^\sigma\widehat{(\widetilde{\pi})}) \cong {}^\sigma\widehat{(\widetilde{\pi})} \cong {}^\sigma\pi,$$

as claimed. $\qquad\square$

5A3. *H-distinction of an induced representation.* Let τ' be an irreducible admissible representation of $\mathbf{GL}_n(E)$ and define $\tau = \tau' \otimes {}^\sigma\tau'$. By Proposition 5.2, the irreducible admissible representation τ of L is L^θ-distinguished. Let λ be a nonzero element of $\operatorname{Hom}_{L^\theta}(\tau, 1)$. The invariant form λ is defined using the pairing of τ' with its contragredient. By [Lapid and Rogawski 2003, Proposition 4.3.2], the restriction of $\delta_Q^{1/2}$ to L^θ is $\delta_{Q \cap H} = \delta_{Q^\theta}$. By Corollary 2.8, we have the following result.

Proposition 5.4. *Let τ' be an irreducible admissible representation of $\mathbf{GL}_n(E)$. If $\tau = \tau' \otimes {}^\sigma\tau'$, then the induced representation $\pi = \iota_Q^G \tau$ is H-distinguished.*

5B. *Computing exponents and distinction of Jacquet modules.* Let $P = MN$ be a proper θ-split parabolic subgroup of G, with θ-stable Levi factor M and unipotent radical N. By the geometric lemma (Lemma 2.1) and Lemma 3.1, the exponents of $\pi = \iota_Q^G \tau$ along P are given by

$$\mathcal{E}xp_{A_M}(\pi_N) = \bigcup_{y \in M \backslash S(M,L)/L} \mathcal{E}xp_{A_M}(\mathcal{F}_N^y(\tau)),$$

and the exponents $\mathcal{E}xp_{A_M}(\mathcal{F}_N^y(\tau))$ are the central characters of the irreducible subquotients of the representations $\mathcal{F}_N^y(\tau)$ (see Remark 2.2). Note that restriction of characters from A_M to the (θ, F)-split component S_M provides a surjection from $\mathcal{E}xp_{A_M}(\pi_N)$ to $\mathcal{E}xp_{S_M}(\pi_N)$; for instance see [Smith 2018, Lemma 4.15].

Let $y \in M \backslash S(M, L)/L$. There are two situations that we need to consider:

Case (1): $P \cap {}^yL = {}^yL$.

Case (2): $P \cap {}^y L \subsetneq {}^y L$ is a proper parabolic subgroup of ${}^y L$.

By Lemma 4.3, there exist a θ-split subset $\Theta \subset \Delta_0$ and an element $g \in (\mathbf{T}_0 \mathbf{H})(F)$ such that $P = g P_\Theta g^{-1}$. We choose the representative y to have the form $y = gw\gamma$, where $w \in [W_\Theta \backslash W_0 / W_\Omega]$. Recall that $Q = \gamma^{-1} P_\Omega \gamma$, $U = \gamma^{-1} N_\Omega \gamma$, and $L = \gamma^{-1} M_\Omega \gamma$, where $\Omega = \Delta_0 \backslash \{^\gamma (\epsilon_n - \epsilon_{n+1})\}$.

5B1. *Case (1).* Suppose that $P \cap {}^y L = {}^y L$. Then $M \cap {}^y L = {}^y L \cong M_{(n,n)}$. Moreover, the Levi subgroup M must be a maximal proper Levi subgroup of G that is associate to L. It follows that, in this case, $\Theta = \Omega$.

There are exactly two elements $w \in [W_\Omega \backslash W_0 / W_\Omega]$ such that $y = gw\gamma$ satisfies $M \cap {}^y L = {}^y L$: the identity and $^\gamma w_L$, where

$$w_L = \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix} \in N_G(L),$$

and 1_n is the $n \times n$ identity matrix. It follows that, in Case (1), there are two elements y that we need to consider: $y = ge\gamma = g\gamma$ and $y = g^\gamma w_L \gamma = g\gamma w_L$.

Note. For a representation τ' of $\mathbf{GL}_n(E)$, we have $^{w_L}(\tau' \otimes {}^\sigma \tau') \cong {}^\sigma \tau' \otimes \tau'$.[3]

Lemma 5.5. *Let* $\tau = \tau' \otimes {}^\sigma \tau'$ *be an irreducible admissible representation of L. Let* $P = {}^g P_\Omega$, $g \in (\mathbf{T}_0 \mathbf{H})(F)$, *be a maximal θ-split parabolic subgroup with Levi associate to L. Let* $\pi = \iota_Q^G \tau$.

(1) *If* $y = g\gamma$, *then* $\mathcal{F}_N^y(\tau) = {}^{g\gamma} \tau = {}^{g\gamma}(\tau' \otimes {}^\sigma \tau')$.

(2) *If* $y = g\gamma w_L$, *then* $\mathcal{F}_N^y(\tau) = {}^{g\gamma w_L} \tau = {}^{g\gamma}({}^\sigma \tau' \otimes \tau')$.

Proof. For $y = g\gamma x$, where x normalizes L, we have

$$M \cap {}^y Q = M \cap g\gamma Q \gamma^{-1} g^{-1} = M \cap P = M$$

and

$$P \cap {}^y L = P \cap g\gamma L \gamma^{-1} g^{-1} = P \cap M = M = {}^y L,$$

so

$$\mathcal{F}_N^y(\tau) = \iota_M^M(({}^y \tau)_{\{e\}}) = {}^y \tau. \qquad \square$$

If τ is an irreducible unitary (e.g., a discrete series) representation of L, then the two subquotients $\mathcal{F}_N^{g\gamma}(\tau) = {}^{g\gamma}(\tau' \otimes {}^\sigma \tau')$ and $\mathcal{F}_N^{g\gamma w_L}(\tau) = {}^{g\gamma}({}^\sigma \tau' \otimes \tau')$ of π_N are irreducible and unitary.

5B2. *Case (2).* Suppose that $P \cap {}^y L \subsetneq {}^y L$ is a proper parabolic subgroup of ${}^y L$. In particular, the Levi subgroup $M \cap {}^y L$ of $P \cap {}^y L$ is properly contained in ${}^y L$. The following is the direct analogue of [Smith 2018, Proposition 8.5]. The idea of the proof is to realize the exponents of π along P as restrictions of the exponents of τ along $P \cap {}^y L$ (see Lemma 3.2) and to apply Casselman's criterion to the discrete

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[3]The Weyl group element w_L has the property that $^{w_L} Q = Q^{\mathrm{op}}$.

series τ.

Proposition 5.6. *Let P be a maximal θ-split parabolic subgroup of G and let $y \in P \backslash G / Q$ be such that $P \cap {}^y L$ is a proper parabolic subgroup of ${}^y L$. Let τ be a discrete series representation of L. The exponents of $\pi = \iota_Q^G \tau$ along P contributed by the subquotient $\mathcal{F}_N^y(\tau) = \iota_{M \cap {}^y Q}^M ({}^y \tau)_{N \cap {}^y L}$ of π_N satisfy the condition in* (3-2).

Proof. If τ is a discrete series representation of L, then τ satisfies Casselman's criterion (see [1995, Theorem 6.5.1]). In light of Lemma 6.2, the argument that the exponents of $\pi = \iota_Q^G \tau$ along P satisfy the relative Casselman's criterion (Theorem 3.4) follows exactly as in the proof of [Smith 2018, Proposition 8.5]. $\square$

5B3. *Distinction of $\mathcal{F}_N^y(\tau)$.* A consequence of Proposition 5.6 is that we need only consider distinction of (subquotients of) the Jacquet module of $\pi = \iota_Q^G \tau$ in Case (1) (see Theorem 3.4 and Proposition 3.5). We now determine the θ-fixed points of M and study M^θ-distinction of the irreducible subquotients of π_N in the case where $P \cap {}^y L = {}^y L$.

Lemma 5.7. *The intersection $N_G(L) \cap Q^{\mathrm{op}}$ of the normalizer of L in G with the opposite parabolic Q^{op} of Q is equal to L.*

Proof. Suppose that $q = \begin{pmatrix} A & 0 \\ B & C \end{pmatrix} \in N_G(L) \cap Q^{\mathrm{op}}$, and let $l = \mathrm{diag}(l_1, l_2)$ be an arbitrary element of L. We have that

$$q l q^{-1} = \begin{pmatrix} A l_1 A^{-1} & 0 \\ B l_1 A^{-1} - C l_2 C^{-1} B A^{-1} & C l_2 C^{-1} \end{pmatrix} \in L,$$

and we see that

$$B l_1 A^{-1} - C l_2 C^{-1} B A^{-1} = 0,$$

for all $l_1, l_2 \in \mathbf{GL}_n(E)$. This occurs if and only if $B = C l_2 C^{-1} B l_1^{-1}$ for all $l_1, l_2 \in \mathbf{GL}_n(E)$. If we take $l_2 = 1_n$ to be the identity, then $B l_1 = B$ for any $l_1 \in \mathbf{GL}_n(E)$, which occurs if and only if $B(l_1 - 1_n) = 0$ for any $l_1 \in \mathbf{GL}_n(E)$. In particular, since there exists $l_1 \in \mathbf{GL}_n(E)$ such that $l_1 - 1_n$ is invertible, we must have $B = 0$. The lemma follows. $\square$

Lemma 5.8. *Let $M = g \gamma M_{(n,n)} \gamma^{-1} g^{-1}$, where $g \in (\mathbf{HT}_0)(F)$.*

(1) *The subgroup M^θ of θ-fixed points in M is the $g\gamma$-conjugate of $L^{g\gamma \cdot \theta}$.*

(2) *We have that $L^{g\gamma \cdot \theta} = L^{\theta_{x_L}}$ is equal to the product $\mathbf{U}_{E/F,x_1} \times \mathbf{U}_{E/F,x_2}$ of unitary groups.*

(3) *Explicitly, $M^\theta = g\gamma (\mathbf{U}_{E/F,x_1} \times \mathbf{U}_{E/F,x_2}) \gamma^{-1} g^{-1}$ is isomorphic to a product of unitary groups.*

(4) *Let τ be an irreducible admissible representation of L. Then ${}^{g\gamma} \tau$ is M^θ-distinguished if and only if τ is $\mathbf{U}_{E/F,x_1} \times \mathbf{U}_{E/F,x_2}$-distinguished.*

Proof. By Lemma 5.7, we have that $x_L = \gamma^{-1}g^{-1}\theta(g)\gamma = \text{diag}(x_1, x_2) \in L$; moreover, x_L is Hermitian. Indeed, since ${}^t\gamma = \gamma = \sigma(\gamma)$ and $w_\ell = {}^tw_\ell = \sigma(w_\ell)$, we have

$$\begin{aligned}
{}^t\sigma(x_L) &= {}^t\sigma(\gamma^{-1}g^{-1}\theta(g)\gamma) \\
&= {}^t\sigma(\gamma^{-1}g^{-1}w_\ell^{-1}({}^t\sigma(g)^{-1})w_\ell\gamma) \\
&= \sigma({}^t\gamma)\sigma({}^tw_\ell)g^{-1}\sigma({}^tw_\ell)^{-1}\sigma({}^tg)^{-1}\sigma({}^t\gamma)^{-1} \\
&= \gamma w_\ell g^{-1}w_\ell^{-1}{}^t\sigma(g)^{-1}\gamma^{-1} \\
&= \gamma w_\ell g^{-1}w_\ell^{-1}{}^t\sigma(g)^{-1}w_\ell w_\ell^{-1}\gamma^{-1} \\
&= \gamma w_\ell g^{-1}\theta(g)w_\ell^{-1}\gamma^{-1} \\
&= z\gamma^{-1}g^{-1}\theta(g)\gamma z^{-1} \\
&= zx_L z^{-1} = x_L,
\end{aligned}$$

where $z = {}^t\sigma(\gamma)w_\ell\gamma = \gamma w_\ell\gamma \in A_L$. In fact, we have that x_1 and x_2 are Hermitian elements of $\mathbf{GL}_n(E)$.

Upon restriction to L, $g\gamma \cdot \theta = x_L \cdot \theta_e = \text{Int}\, x_L^{-1} \circ {}^t\sigma(\ )^{-1}$. Note also that $x_L \cdot \theta_e = \theta_{e \cdot x_L} = \theta_{x_L}$. In particular, $l \in L$ is $g\gamma \cdot \theta$-fixed if and only if l is θ_{x_L}-fixed. Explicitly, $l \in L$ is θ_{x_L}-fixed if and only if $l = x^{-1}{}^t\sigma(l)^{-1}x_L$. Since x_L is Hermitian, we have that $L^{g\gamma \cdot \theta} = L^{\theta_{x_L}}$ is equal to the product $\mathbf{U}_{E/F,x_1} \times \mathbf{U}_{E/F,x_2}$ of unitary groups. Now, observe that $M^\theta = g\gamma L^{g\gamma \cdot \theta}(g\gamma)^{-1}$. Indeed, if $m = g\gamma l\gamma^{-1}g^{-1} \in M$, where $l \in L$, then m is θ-fixed if and only if $l = (g\gamma \cdot \theta)(l)$ is $(g\gamma \cdot \theta)$-fixed. $\square$

It is interesting to note that, even though we're interested in distinction by the quasisplit unitary group $\mathbf{H} = \mathbf{U}_{E/F,w_\ell}$, we will need to consider distinction by (possibly nonquasisplit) unitary groups for Jacquet modules.

Define a representation ρ of a Levi subgroup M of G to be regular if for every nontrivial element $w \in N_G(M)/M$ we have that the twist ${}^w\rho = \rho(w^{-1}(\cdot)w)$ is not equivalent to ρ. A representation $\pi_1 \otimes \pi_2$ of L is regular if and only if $\pi_1 \not\cong \pi_2$. The next lemma is [Smith 2018, Lemma 8.2].

Lemma 5.9. *Assume that τ is a regular unitary irreducible admissible representation of L. Let $P = MN$ be a θ-split parabolic subgroup with Levi associate to L. If $y \in M\backslash S(M, L)/L$ is such that $P \cap {}^yL = {}^yL$, then $\mathcal{F}_N^y(\tau)$ is irreducible and the central character $\chi_{N,y}$ of $\mathcal{F}_N^y(\tau)$ is unitary.*

From Lemma 5.5, it follows that M^θ-distinction of ${}^{g\gamma}\tau$ (respectively, ${}^{g\gamma w_L}\tau$) is equivalent to $\mathbf{U}_{E/F,x_1}$-distinction of τ' and $\mathbf{U}_{E/F,x_2}$-distinction of ${}^\sigma\tau'$ (respectively, $\mathbf{U}_{E/F,x_1}$-distinction of ${}^\sigma\tau'$ and $\mathbf{U}_{E/F,x_2}$-distinction of τ'). If $\tau = \tau'\otimes{}^\sigma\tau'$ is a regular discrete series representation, then $\tau' \not\cong {}^\sigma\tau'$. It follows from Theorem 5.1 that neither τ' nor ${}^\sigma\tau'$ can be distinguished by any unitary group. If $P = MN$ is any θ-split parabolic such that M is associate to L, then by Lemma 5.8, neither of the

irreducible subquotients of π_N, described in Lemma 5.5, can be M^θ-distinguished. We records this as the following.

Corollary 5.10. *Let* $\pi = \iota_Q^G \tau$, *where* $\tau = \tau' \otimes^\sigma \tau'$ *is a discrete series representation such that* $\tau' \not\cong {}^\sigma\tau'$. *Let* $P = {}^g P_\Omega$, *where* $g \in (\mathbf{HT}_0)(F)$, *be any maximal* θ-split *parabolic subgroup with* θ-stable Levi $M = P \cap \theta(P)$ associate to L. Neither *of the two irreducible unitary subquotients of* π_N, *twists of* $\tau' \otimes^\sigma \tau'$ *and* ${}^\sigma\tau' \otimes \tau'$ *(see Lemma 5.5), can be* M^θ-distinguished.

5C. *Constructing relative discrete series.* The following theorem is our main result. The argument is the same as the proof of [Smith 2017, Theorem 6.3].

Theorem 5.11. *Let* $Q = P_{(n,n)}$ *be the upper triangular parabolic subgroup of* G *with standard Levi factor* $L = M_{(n,n)}$ *and unipotent radical* $U = N_{(n,n)}$. *Let* $\pi = \iota_Q^G \tau$, *where* $\tau = \tau' \otimes^\sigma \tau'$, *and* τ' *is a discrete series representation of* $\mathbf{GL}_n(E)$ *such that* τ' *is not Galois invariant, i.e.,* $\tau' \not\cong {}^\sigma\tau'$. *The representation* π *is a relative discrete series representation for* $H \backslash G$ *that does not occur in the discrete series of* G.

Proof. Since τ is unitary and regular, π is irreducible by a result from [Bruhat 1961] (see [Casselman 1995, Theorem 6.6.1]). In addition, π is H-distinguished by Proposition 5.4. The representation π does not occur in the discrete series of G by Zelevinsky's classification [1980]. Let λ denote a fixed H-invariant linear form on π. It suffices to show that π satisfies the relative Casselman's criterion Theorem 3.4. Let $P = MN$ be a proper θ-split parabolic subgroup of G. The exponents of π along P are the central characters of the irreducible subquotients of the representations $\mathcal{F}_N^y(\tau)$ given by the geometric lemma (Lemma 2.1) (see Section 5B). By [Kato and Takano 2010, Lemma 4.6] and Proposition 5.6, the condition (3-2) is satisfied when $P \cap {}^y L$ is a proper parabolic subgroup of ${}^y L$. As in Lemma 5.9, the only unitary exponents of π along P occur when $P \cap {}^y L = {}^y L$. By Corollary 5.10, the only irreducible unitary subquotients of π_N cannot be M^θ-distinguished when $P \cap {}^y L = {}^y L$. In the latter case, by Proposition 3.5, the unitary exponents of π along P do not contribute to $\mathcal{E}xp_{S_M}(\pi_N, \lambda_N)$. Therefore, (3-2) is satisfied for every proper θ-split parabolic subgroup of G. Finally, by Theorem 3.4, the representation π appears in the discrete spectrum of $H \backslash G$. In particular, π is (H, λ)-relatively square integrable for all nonzero $\lambda \in \mathrm{Hom}_H(\pi, 1)$. $\qquad\square$

In addition, we note the following existence results. First, we recall the structure of the representations in the discrete spectrum of $\mathbf{GL}_n(E)$. Let ρ be an irreducible unitary supercuspidal representation of $\mathbf{GL}_r(E)$, $r \geq 1$. For an integer $k \geq 2$, write $\mathrm{St}(k, \rho)$ for the unique irreducible (unitary) quotient of the parabolically induced representation

$$\iota_{P_{(r,\ldots,r)}}^{\mathbf{GL}_{kr}(E)}(\nu^{\frac{1-k}{2}} \rho \otimes \nu^{\frac{3-k}{2}} \rho \otimes \ldots \otimes \nu^{\frac{k-1}{2}} \rho)$$

of $\mathbf{GL}_{kr}(E)$ (see [Zelevinsky 1980, Proposition 2.10, §9.1]), where $\nu(g) = |\det(g)|_E$,

for any $g \in \mathbf{GL}_r(E)$. The representations $\mathrm{St}(k, \rho)$ are the generalized Steinberg representations and they are precisely the nonsupercuspidal discrete series representations of $\mathbf{GL}_{kr}(E)$ [Zelevinsky 1980, Theorem 9.3]. The usual Steinberg representation St_n of $\mathbf{GL}_n(E)$ is obtained as $\mathrm{St}(n, 1)$.

Proposition 5.12. *Let $n \geq 2$ be an integer. There exist infinitely many equivalence classes of nonsupercuspidal discrete series representations τ of $\mathbf{GL}_n(E)$ that are not Galois invariant.*

Before giving a proof of Proposition 5.12, we note the following results.

Corollary 5.13. *Let $n \geq 2$ be an integer. There exist infinitely many equivalence classes of RDS representations of the form constructed in Theorem 5.11 and such that the discrete series τ is not supercuspidal.*

Proof. Apply Proposition 5.12 and [Zelevinsky 1980, Theorem 9.7(b)]. $\qquad\square$

Proposition 5.14. *Let ρ be an irreducible supercuspidal representation of $\mathbf{GL}_r(E)$, $r \geq 1$. For $k \geq 2$, the generalized Steinberg representation $\mathrm{St}(k, \rho)$ of $\mathbf{GL}_{kr}(E)$ is Galois invariant if and only if ρ is Galois invariant.*

Proof. First, observe that the twisted representation $^\sigma\mathrm{St}(k, \rho)$ is equivalent to the generalized Steinberg representation $\mathrm{St}(k, {}^\sigma\rho)$. It follows that $\mathrm{St}(k, \rho)$ is Galois invariant if and only if $\mathrm{St}(k, \rho) \cong {}^\sigma\mathrm{St}(k, \rho) \cong \mathrm{St}(k, {}^\sigma\rho)$. The result now follows from [Zelevinsky 1980, Theorem 9.7(b)], which gives us that $\mathrm{St}(k, \rho) \cong \mathrm{St}(k, {}^\sigma\rho)$ if and only if $\rho \cong {}^\sigma\rho$. $\qquad\square$

Proposition 5.15. *For $n \geq 2$, there exist infinitely many unitary twists of the Steinberg representation St_n of $\mathbf{GL}_n(E)$ that are not Galois invariant.*

Proof. Let $\chi : E^\times \to \mathbb{C}^\times$ be a (unitary) character of $E^\times$. By [Zelevinsky 1980, Theorem 9.7(b)], $\chi \, \mathrm{St}_n \cong {}^\sigma(\chi \, \mathrm{St}_n)$ if and only if $\chi = {}^\sigma\chi$. We have that $^\sigma\chi = \chi$ if and only if χ is trivial on the kernel of the norm map $N_{E/F} : E^\times \to F^\times$. Note that $\ker N_{E/F}$ is a nontrivial closed subgroup of $E^\times$. We can extend any nontrivial unitary character of $\ker N_{E/F}$ to $E^\times$ to obtain a unitary character χ of $E^\times$ such that $^\sigma\chi \neq \chi$. Given a unitary character of $\ker N_{E/F}$ there are infinitely many distinct extensions to $E^\times$. $\square$

The following is the main ingredient needed to prove Proposition 5.12.

Theorem 5.16 (Hakim and Murnaghan). *For $n \geq 1$, there exist infinitely many distinct equivalence classes of Galois invariant (respectively, non-Galois invariant) irreducible supercuspidal representations of $\mathbf{GL}_n(E)$.*

Proof. If $n = 1$, argue as in the proof of Proposition 5.15. For $n \geq 2$, use [Murnaghan 2011, Proposition 10.1] to obtain the existence of infinitely many pairwise inequivalent Galois invariant irreducible supercuspidal representations of $\mathbf{GL}_n(E)$. To complete the proof, apply [Hakim and Murnaghan 2002, Theorem 1.1]. $\qquad\square$

Proof of Proposition 5.12. If n is prime, then by Proposition 5.15 there exist infinitely many twists of the Steinberg representation St_n of $\mathbf{GL}_n(E)$ that are not Galois invariant. If n is composite, then Proposition 5.15 still applies; however, by Proposition 5.14 and Theorem 5.16 there are infinitely many classes of non-Galois invariant generalized Steinberg representations of $\mathbf{GL}_n(E)$.   □

Remark 5.17. A representation (π, V) is H-relatively supercuspidal if and only if the λ-relative matrix coefficients of π are compactly supported modulo $Z_G H$, for every nonzero $\lambda \in \mathrm{Hom}_H(\pi, 1)$. If we further assume that τ' is supercuspidal in Theorem 5.11, then $\pi = \iota_Q^G \tau$ is a nonsupercuspidal H-relatively supercuspidal representation of G; see [Smith 2018, Corollary 6.7]; this can be shown through a slight modification of the proof of Theorem 5.11. Indeed, when $P \cap {}^y L$ is proper in ${}^y L$, the subquotients $\mathcal{F}_N^y(\tau)$ of the Jacquet module vanish since τ is supercuspidal. Otherwise, $\mathcal{F}_N^y(\tau)$ cannot be M^θ-distinguished. By Proposition 3.5, $\lambda_N = 0$, for every proper θ-split parabolic subgroup P of G and any $\lambda \in \mathrm{Hom}_H(\pi, 1)$. Finally, by Theorem 6.2 of [Kato and Takano 2008], π is relatively supercuspidal; and since π is parabolically induced, π is not supercuspidal.

This modification of Theorem 5.11 can be obtained by more direct methods; see [Murnaghan 2017], for instance.

5D. *Exhaustion of the discrete spectrum.* Beuzart-Plessis [2017], in his 2017 Cours Peccot, announced Plancherel formulas for the two p-adic symmetric spaces $\mathbf{GL}_n(F)\backslash\mathbf{GL}_n(E)$ and $\mathbf{U}_{n,E/F}(F)\backslash\mathbf{GL}_n(E)$, where $\mathbf{U}_{n,E/F}(F)$ is a quasisplit unitary group. The two Plancherel formulas are realized in terms of the appropriate base change maps. Both results are obtained by a comparison of local relative trace formulas. Presently, we are concerned with the second case and only when n is even.

As above, let $G = \mathbf{GL}_{2n}(E)$ and $H = \mathbf{U}_{E/F, w_\ell}(F)$, where $n \geq 2$. Building on [Jacquet 2001; Feigon et al. 2012], Beuzart-Plessis has shown that the Plancherel formula for $H\backslash G$ is the push-forward of the Whittaker–Plancherel formula for $\mathbf{GL}_{2n}(F)$ via quadratic base change. As a consequence, the discrete spectrum of $H\backslash G$ consists of the quadratic base changes of the discrete series of $\mathbf{GL}_{2n}(F)$.

Let $\mathrm{Irr}(\mathbf{GL}_n(E))$ denote the set of equivalence classes of irreducible admissible representations of $\mathbf{GL}_n(E)$ (likewise for $\mathbf{GL}_n(F)$), and let $\mathrm{Irr}^\sigma(\mathbf{GL}_n(E))$ denote the subset $\{\pi \in \mathrm{Irr}(\mathbf{GL}_n(E)) : \pi \cong {}^\sigma\pi\}$ of Galois invariant representations. The quadratic base change map $\mathrm{bc} : \mathrm{Irr}(\mathbf{GL}_n(F)) \to \mathrm{Irr}^\sigma(\mathbf{GL}_n(E))$ sends (classes of) irreducible representations of $\mathbf{GL}_n(F)$ to (classes of) irreducible Galois invariant representations of $\mathbf{GL}_n(E)$. Moreover, the map $\mathrm{bc} : \mathrm{Irr}(\mathbf{GL}_n(F)) \to \mathrm{Irr}^\sigma(\mathbf{GL}_n(E))$ is surjective. Let $\eta_{E/F} : F^\times \to \mathbb{C}^\times$ be the quadratic character associated to the extension E/F by local class field theory. For any $\pi' \in \mathrm{Irr}(\mathbf{GL}_n(F))$, we have that $\mathrm{bc}(\pi') = \mathrm{bc}(\pi' \otimes \eta_{E/F})$. We refer the reader to [Arthur and Clozel 1989, Chapter 1, Section 6] for more information about quadratic base change and its properties; in

particular, Theorem 6.2 therein summarizes the basic properties of base change for tempered representations.

In the language of [Feigon et al. 2012], the RDS representations $\pi = \iota_Q^G(\tau' \otimes {}^\sigma\tau')$, with $\tau' \not\cong {}^\sigma\tau'$, constructed in Theorem 5.11 are *totally σ-isotropic*, that is, the cuspidal support of π is a tensor product of non-Galois invariant supercuspidal representations (see Proposition 5.14). Moreover, this means that π is the base change of a unique discrete series representation $\pi' \cong \pi' \otimes \eta_{E/F}$ of $\mathbf{GL}_{2n}(F)$, and π is not distinguished by the nonquasisplit unitary group [Feigon et al. 2012, Theorem 0.2, Lemma 3.4].

The following theorem frames the work of Beuzart-Plessis in terms of Theorem 5.11 and gives a complete description of $L^2_{\mathrm{disc}}(H\backslash G)$.

Theorem 5.18. *Let π be a relative discrete series representation for the quotient $\mathbf{U}_{E/F}(F)\backslash\mathbf{GL}_{2n}(E)$. Then π is either a $\mathbf{U}_{E/F}(F)$-distinguished discrete series representation of $\mathbf{GL}_{2n}(E)$, or π is equivalent to a representation of the form constructed in Theorem 5.11.*

Proof. Beuzart-Plessis has shown that the relative discrete series representations for $H\backslash G$ are precisely the images of the discrete series of $\mathbf{GL}_{2n}(F)$ under quadratic base change [Beuzart-Plessis 2017]. Let $\pi' \in \mathrm{Irr}(\mathbf{GL}_{2n}(F))$ be a discrete series representation of $\mathbf{GL}_{2n}(F)$. By [Arthur and Clozel 1989, Proposition 6.6], $\pi = \mathrm{bc}(\pi') \in \mathrm{Irr}^\sigma(G)$ is a discrete series representation of G if and only if $\pi' \not\cong \pi' \otimes \eta_{E/F}$. In this case, $\pi = \mathrm{bc}(\pi')$ is an H-distinguished discrete series representation of G [Feigon et al. 2012, Corollary 13.5] and π is known to be relatively discrete [Kato and Takano 2010, Proposition 4.10]. Otherwise, it must be the case that $\pi' \cong \pi' \otimes \eta_{E/F}$. If $\pi' \in \mathrm{Irr}(\mathbf{GL}_{2n}(F))$ is a discrete series representation such that $\pi' \cong \pi' \otimes \eta_{E/F}$, then there exists a (nonunique) discrete series representation $\tau' \in \mathrm{Irr}(\mathbf{GL}_n(E))$ such that $\tau' \not\cong {}^\sigma\tau'$ and $\mathrm{bc}(\pi') = \iota_{P_{(n,n)}}^G(\tau' \otimes {}^\sigma\tau')$ is equivalent to a relative discrete series representation constructed in Theorem 5.11; see Section 3.2 of [Feigon et al. 2012], particularly Lemma 3.4. (In this case, the representation π' is the *automorphic induction* of the discrete series τ'.) □

6. A technical lemma

Finally, we give two technical results required to prove Proposition 5.6, which allows us to reduce the relative Casselman's criterion for $\pi = \iota_Q^G\tau$ to the usual Casselman's criterion for τ. The set $A_{M\cap{}^yL}^{-{}^yL} \setminus A_{M\cap{}^yL}^1 A_{{}^yL}$ that appears in Lemma 6.2 is the dominant part of $A_{M\cap{}^yL}$ in $M\cap{}^yL$, and is precisely the cone on which we must consider the exponents of τ in order to apply Casselman's criterion. Lemma 6.1 is the analogue of [Smith 2018, Lemma 8.4] and the proof is essentially the same (see [Smith 2017, Lemma 5.2.13]). In the present setting, we must also consider non-Δ_0-standard θ-split parabolic subgroups in our analysis of the exponents of π.

In Lemma 6.2, we will explain how to adapt Lemma 6.1 to handle the nonstandard case.

In order to discuss Casselman's criterion for the inducing data of $\pi = \iota_Q^G \tau$ we use the following notation. If $\Theta_1 \subset \Theta_2 \subset \Delta_0$, then we define

$$A_{\Theta_1}^- = \{a \in A_{\Theta_1} : |\alpha(a)| \leq 1, \text{ for all } \alpha \in \Delta_0 \setminus \Theta_1\}$$

and

$$A_{\Theta_1}^{-\Theta_2} = \{a \in A_{\Theta_1} : |\beta(a)| \leq 1, \text{ for all } \beta \in \Theta_2 \setminus \Theta_1\}.$$

The set $A_{\Theta_1}^-$ is the dominant part of A_{Θ_1} in G, while $A_{\Theta_1}^{-\Theta_2}$ is the dominant part of A_{Θ_1} in M_{Θ_2}.

Lemma 6.1. *Let P_Θ, given by $\Theta \subset \Delta_0$, be any maximal θ-split Δ_0-standard parabolic subgroup. Let $w \in [W_\Theta \backslash W_0 / W_\Omega]$ be such that $M_\Theta \cap {}^w M_\Omega = M_{\Theta \cap w\Omega}$ is a proper Levi subgroup of ${}^w M_\Omega = M_{w\Omega}$. We have the containment*

$$(6\text{-}1) \qquad S_\Theta^- \setminus S_\Theta^1 S_{\Delta_0} \subset A_{\Theta \cap w\Omega}^{-w\Omega} \setminus A_{\Theta \cap w\Omega}^1 A_{w\Omega}.$$

Recall that $S_\Theta^1 = S_\Theta(\mathcal{O}_F)$ and $A_{\Theta \cap w\Omega}^1 = A_{\Theta \cap w\Omega}(\mathcal{O}_F)$. Let $P = MN$ be a maximal θ-split parabolic subgroup of G. By Lemma 4.3, there exists $g \in (\mathbf{H T}_0)(F)$ such that $P = g P_\Theta g^{-1}$, where P_Θ is a Δ_0-standard maximal θ-split parabolic subgroup. We may take $S_M = g S_\Theta g^{-1}$ and then we have $S_M^- = g S_\Theta^- g^{-1}$. Let $y \in P \backslash G / Q$, given by $y = gw\gamma$, where $w \in [W_\Theta \backslash W_0 / W_\Omega]$. We observe that ${}^y L = g(M_{w\Omega}) g^{-1}$. In particular, $M \cap {}^y L = g(M_{\Theta \cap w\Omega}) g^{-1}$ and $A_{M \cap {}^y L} = g(A_{\Theta \cap w\Omega}) g^{-1}$. The dominant part of the torus $A_{M \cap {}^y L}$ will be denoted by $A_{M \cap {}^y L}^{-{}^y L}$ and is determined by the simple roots $gw\Omega$ of the maximal (θ, F)-split torus ${}^g A_0$ in ${}^y L$.

Lemma 6.2. *Let $P = MN$ be any maximal θ-split parabolic subgroup of G with θ-stable Levi M and unipotent radical N. Choose a maximal subset Θ of Δ_0 and an element $g \in (\mathbf{H T}_0)(F)$ such that $P = g P_\Theta g^{-1}$. Let $y = gw\gamma \in P \backslash G / Q$, where $w \in [W_\Theta \backslash W_0 / W_\Omega]$, such that $M \cap {}^y L$ is a proper Levi subgroup of ${}^y L$. Then we have the containment*

$$(6\text{-}2) \qquad S_M^- \setminus S_M^1 S_G \subset A_{M \cap {}^y L}^{-{}^y L} \setminus A_{M \cap {}^y L}^1 A_{{}^y L}.$$

Proof. The (θ, F)-split component S_Θ of M_Θ is equal to its F-split component A_Θ; moreover, we have that $S_M = A_M$. We also have $S_M^- = g S_\Theta^- g^{-1}$ and $S_M^1 = g S_\Theta^1 g^{-1}$; moreover, since $S_G = S_{\Delta_0}$ is central in G we obtain

$$(6\text{-}3) \qquad S_M^- \setminus S_G S_M^1 = g(S_\Theta^-) g^{-1} \setminus S_G g(S_\Theta^1) g^{-1}.$$

By Lemma 6.1, we have

$$S_\Theta^- \setminus S_\Theta^1 S_{\Delta_0} \subset A_{\Theta \cap w\Omega}^{-w\Omega} \setminus A_{\Theta \cap w\Omega}^1 A_{w\Omega}.$$

By the equality in (6-3), it suffices to show that

$$(6\text{-}4) \qquad A_{M\cap {}^yL}^{-{}^yL} \setminus A_{M\cap {}^yL}^1 A_{{}^yL} = g(A_{\Theta\cap w\Omega}^{-w\Omega})g^{-1} \setminus g(A_{\Theta\cap w\Omega}^1)g^{-1}\, g(A_{w\Omega})g^{-1}.$$

Indeed, if (6-4) holds, then we have

$$
\begin{aligned}
S_M^- \setminus S_G S_M^1 &= g\, S_\Theta^- g^{-1} \setminus S_G g\, S_\Theta^1 g^{-1} \\
&\subset g(A_{\Theta\cap w\Omega}^{-w\Omega})g^{-1} \setminus g(A_{\Theta\cap w\Omega}^1)g^{-1}\, g(A_{w\Omega})g^{-1} \qquad \text{(by Lemma 6.1)} \\
&= A_{M\cap {}^yL}^{-{}^yL} \setminus A_{M\cap {}^yL}^1 A_{{}^yL},
\end{aligned}
$$

as claimed. The truth of (6-4) immediately follows from how we determine the dominant part of $A_{M\cap {}^yL}$. As above, we have that $M \cap {}^yL = g(M_{\Theta\cap w\Omega})g^{-1}$ and $A_{M\cap {}^yL} = g(A_{\Theta\cap w\Omega})g^{-1}$. Moreover, $A_{M\cap {}^yL}^1 = g(A_{\Theta\cap w\Omega}^1)g^{-1}$. Given a root $\alpha \in \Phi_0$ we obtain a root $g\alpha$ of ${}^gA_0$ in G by setting $g\alpha = \alpha \circ \operatorname{Int} g^{-1}$, as usual. Explicitly, we have

$$(6\text{-}5) \qquad A_{M\cap {}^yL}^{-{}^yL} = \{a \in A_{M\cap {}^yL} : |g\beta(a)| \leq 1, \ \beta \in w\Omega \setminus \Theta \cap w\Omega\}.$$

In fact, we have that $M \cap {}^yL$ is determined (as a Levi subgroup of ${}^yL$) by the simple roots $g(\Theta \cap w\Omega) \subset gw\Omega$ of ${}^gA_0$ in ${}^yL = gwM_\Omega$. It is immediate that

$$(6\text{-}6) \qquad A_{M\cap {}^yL}^{-{}^yL} = g(A_{\Theta\cap w\Omega}^{-w\Omega})g^{-1},$$

from which (6-4) follows, completing the proof of the lemma. $\qquad\square$

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