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YAMABE FLOW WITH PRESCRIBED SCALAR CURVATURE

INAS AMACHA AND RACHID REGBAOUI

We study the Yamabe flow corresponding to the prescribed scalar curvature problem on compact Riemannian manifolds with negative scalar curvature. The long time existence and convergence of the flow are proved under appropriate conditions on the prescribed scalar curvature function.

1. Introduction

The prescribed scalar curvature problem on a compact Riemannian manifold (M, g_0) of dimension $n \geq 3$, consists of finding a conformal metric g to g_0 whose scalar curvature R_g is equal to a given function $f \in C^\infty(M)$. If we set $g = u^{4/(n-2)} g_0$, where $0 < u \in C^\infty(M)$, then we have

$$R_g = u^{-\frac{n+2}{n-2}}(-c_n \Delta u + R_0 u),$$

where Δ is the Laplace operator associated with g_0, R_0 is the scalar curvature of g_0 and $c_n = 4\frac{n-1}{n-2}$.

Then the prescribed scalar curvature problem,

$$R_g = f,$$

is equivalent to solving the nonlinear PDE

$$-c_n \Delta u + R_0 u = f u^{\frac{n+2}{n-2}} \tag{1-1}$$

on the space of smooth positive functions on M. The solvability of this equation depends on R_0 and the prescribed function f. When f is constant, (1-1) becomes the famous Yamabe equation whose resolution has been a challenging problem in geometric analysis for a long time. See [Aubin 1976; Hebey and Vaugon 1993; Lee and Parker 1987; Schoen 1984; 1991] for more details on the Yamabe problem, and [Ambrosetti and Malchiodi 1999; Bismuth 2000; Escobar and Schoen 1986; Kazdan and Warner 1975; Rauzy 1995; Vázquez and Véron 1991], concerning the prescribed scalar curvature problem.

By changing g_0 conformally if necessary, we may always assume that R_0 satisfies one of the conditions, $R_0 > 0$, $R_0 = 0$ or $R_0 < 0$ everywhere on M. Equation (1-1)

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has a variational structure since there are different functionals whose Euler–Lagrange equations are equivalent to (1-1). When $R_0 < 0$, the following functional seems more appropriate to handle the prescribed scalar curvature problem:

$$\mathcal{E}(g) = \int_M R_g \, dV_g - \frac{n-2}{n} \int_M f \, dV_g, \tag{1-2}$$

where $g = u^{4/(n-2)} g_0$ belongs to the conformal class $[g_0]$ of g_0, R_g is the scalar curvature of g and $dV_g = u^{2n/(n-2)} dV_{g_0}$ is the volume element of g.

Simple computations ([Besse 1987]) show the L^2-gradient of $\mathcal{E}$ is $\frac{n-2}{2n}(R_g - f)g$, and then, after changing time by a constant scale, the associated negative gradient flow equation is

$$\begin{cases} \partial_t g = -(R_g - f)g, \\ g(0) = g^0, \end{cases} \tag{1-3}$$

where $g^0 = u_0^{4/(n-2)} g_0$ is a given metric in the conformal class of g_0.

Since (1-3) preserves the conformal structure of M, then any smooth solution of (1-3) is of the form $g(t) = u(t)^{4/(n-2)} g_0$, where $0 < u(t) \in C^\infty(M)$. For simplicity we have used the notation $u(t) := u(t, \cdot)$, $t \in I$ for any function u defined on $I \times M$, where I is a subset of $\mathbb{R}$. In terms of $u(t)$, the flow (1-3) may be written in the equivalent form:

$$\begin{cases} \partial_t u^N = \frac{n+2}{4}(c_n \Delta u - R_0 u + f u^N), \\ u(0) = u_0 \in C^\infty(M), \ u_0 > 0, \end{cases} \tag{1-4}$$

where $N = \frac{n+2}{n-2}$.

Our aim in this paper is to investigate this gradient flow by proving its longtime existence and analysing its asymptotic behaviour when $t \to +\infty$.

Our first result is the following existence theorem:

Theorem 1.1. *Suppose that $R_0 < 0$ and let $f \in C^\infty(M)$. Then for any $g^0 = u_0^{4/(n-2)} g_0$ with $0 < u_0 \in C^\infty(M)$, there exists a unique solution $g(t) = u(t)^{4/(n-2)} g_0$ of* (1-3) *defined on* $[0, +\infty)$, *where* $0 < u \in C^\infty([0, +\infty) \times M)$. *Moreover, the functional $\mathcal{E}$ is decreasing along the solution $g(t)$, that is,*

$$\frac{d}{dt}\mathcal{E}(g(t)) \le 0 \quad \textit{for all } t \in [0, +\infty).$$

We note here that apart from the smoothness of f, no further assumptions on the function f are needed in Theorem 1.1. However, for the longtime behaviour, it is necessary to assume additional conditions in order to get the convergence of the flow. Indeed, if $f \ge 0$, by applying the maximum principle to (1-4), we can easily check that

$$u(t) \ge \big(\min_M u_0^{4/(n-2)} + \min_M |R_0| t\big)^{\frac{n-2}{4}} \to +\infty \ \text{ as } t \to +\infty.$$

So if we want to get the convergence of the flow, it is necessary to assume at least that f is negative somewhere on M. We note that this last condition is also necessary for the resolution of (1-1) since it is well known that if the negative gradient flow associated with a functional $\mathcal{F}$ converges (in some sense), then its limit is a critical point of $\mathcal{F}$.

Before giving conditions on f ensuring the convergence of the flow, let us fix some notation: if $\Omega \subset M$ is an open set, we denote by λ_Ω the first eigenvalue of the conformal Laplacian $L = -c_n\Delta + R_0$ on Ω with zero Dirichlet boundary conditions, that is,

$$\lambda_\Omega = \inf_{0\neq u\in H_0^1(\Omega)} \frac{\int_M (c_n|\nabla u|^2 + R_0u^2)\, dV_{g_0}}{\int_M u^2\, dV_{g_0}}.$$

We then assume the following conditions on f:

There exists an open set $\Omega \subset M$ such that

(H1) $$\lambda_\Omega > 0 \text{ and } f < 0 \text{ on } M \setminus \Omega$$

and

(H2) $$\sup_{x\in\Omega} f(x) \leq C_\Omega \inf_{x\in M\setminus\Omega} |f(x)|,$$

where C_Ω is a positive constant depending only on Ω.

We then have the following result:

Theorem 1.2. *Suppose that $R_0 < 0$ and that $f \in C^\infty(M)$ satisfies conditions* (**H1**) *and* (**H2**). *Then there exists a function $0 < \bar{u} \in C^\infty(M)$ such that for any smooth metric $g^0 = u_0^{4/(n-2)} g_0$ with $0 < u_0 \leq \bar{u}$, the flow $g(t) = u(t)^{4/(n-2)} g_0$ given by Theorem 1.1 converges in the C^∞-topology to a conformal metric $g_\infty = u_\infty^{4/(n-2)} g_0$ whose scalar curvature is f, that is, $R_{g_\infty} = f$.*

A particular interesting case is when the function f satisfies $f(x) < 0$ for almost all $x \in M$. In this case conditions (**H1**) and (**H2**) are automatically satisfied and then we have the following corollary:

Corollary 1.3. *Suppose that $R_0 < 0$ and $f \in C^\infty(M)$ such that $f < 0$ almost everywhere on M. Then there exists a function $0 < \bar{u} \in C^\infty(M)$ such that for any smooth metric $g^0 = u_0^{4/(n-2)} g_0$ with $0 < u_0 \leq \bar{u}$, the flow $g(t) = u(t)^{4/(n-2)} g_0$ given by Theorem 1.1 converges in the C^∞-topology to a conformal metric $g_\infty = u_\infty^{4/(n-2)} g_0$ whose scalar curvature is f, that is, $R_{g_\infty} = f$.*

It is natural to ask if conditions (**H1**) and (**H2**) in Theorem 1.2 are necessary. The following theorem gives a partial answer to this question:

Theorem 1.4. *Suppose that $R_0 < 0$ and let $f \in C^\infty(M)$ such that condition* (**H1**) *is not satisfied, that is, for any open set $\Omega \subset M$ such that $f > 0$ on $M \setminus \Omega$, we* *suppose* $\lambda_\Omega \le 0$. *Then for any* $0 < u_0 \in C^\infty(M)$, *the solution* $u(t)$ *of* (1-4) *satisfies, for some constant* $C > 0$ *depending only on* u_0, g_0, f,

$$\max_{x \in M} u(t,x) \ge C t^{\frac{n-2}{n+2}} \to +\infty \quad \text{as } t \to +\infty.$$

We note here that condition (**H1**) is conformally invariant. Similar conditions to (**H1**) and (**H2**) were found by many authors to solve (1-1) by the direct method of elliptic PDEs; see [Bismuth 2000; Rauzy 1995; Vázquez and Véron 1991] for more details. To our knowledge, the only known results on Yamabe type flow on dimension $n \ge 3$ concern the case where f is constant or $M = \mathbb{S}^n$. The Yamabe flow was first introduced by Hamilton [1988] and has been the subject of several studies; see [Brendle 2005; 2007; Chow 1992; Schwetlick and Struwe 2003; Ye 1994]. For the case when f is nonconstant, we mention the work of Struwe [2005] about the Nirenberg's problem on the sphere $\mathbb{S}^2$, and the results of Chen and Xu [2012] concerning $\mathbb{S}^n$, $n \ge 3$. A general evolution problem related to the prescribed Gauss curvature on surfaces was studied by Baird, Fardoun and Regbaoui [Baird et al. 2004].

The paper is organized as follows. In Section 2 we prove the global existence of the flow by establishing local C^k-estimates on the solution u of (1-4). In Section 3, we study the asymptotic behaviour of the flow when $t \to +\infty$. In particular we prove uniform C^k-estimates on u which are necessary to get the convergence of the flow.

2. Global existence of the flow

In this section we shall establish some estimates on the solution u of (1-4) which will be an important tool in proving that the flow $g(t)$ is globally defined on $[0, +\infty)$. In this section we suppose that $R_0 < 0$ and $f \in C^\infty(M)$.

As already mentioned in the previous section, (1-3) is equivalent to (1-4), so it suffices to prove the existence of a solution $u(t)$ of (1-4) defined on $[0, +\infty)$ to obtain a metric $g(t)$ solution of (1-3) defined on $[0, +\infty)$. Since (1-4) is a parabolic equation (on the set of smooth positive functions on $[0, T) \times M$, for any $T > 0$), there exists a smooth solution $u(t)$ of (1-4) defined on a maximal interval $[0, T^*)$ satisfying $u(t) > 0$ on $[0, T^*)$. Thus we have a solution $g(t) = u^{4/(n-2)} g_0$ of (1-3) defined on a maximal interval $[0, T^*)$. For simplicity, we shall write u instead of $u(t)$ and g instead of $g(t)$.

Now, we derive some properties of g which will be important later. One can check by using (1-4) that the scalar curvature R_g satisfies the equation

$$\partial_t R_g = (n-1)\Delta_g (R_g - f) + R_g (R_g - f), \tag{2-1}$$

where Δ_g is the Laplacian associated with $g(t)$.

A simple computation using (2-1) gives

$$\frac{d}{dt}\mathcal{E}(g) = -\frac{n-2}{2} \int_M (R_g - f)^2 \, dV_g, \tag{2-2}$$

so the functional $\mathcal{E}$ is decreasing along the flow $g(t)$. If we set

$$E(u) := \mathcal{E}(g) = \mathcal{E}\big(u^{\frac{4}{n-2}} g_0\big) = \int_M \Big(c_n|\nabla u|^2 + R_0 u^2 - \frac{n-2}{n} f u^{\frac{2n}{n-2}}\Big)\, dV_{g_0},$$

then (2-2) can be written in terms of u:

$$\frac{d}{dt}E(u) = -\frac{8}{n-2}\int_M |\partial_t u|^2 u^{\frac{4}{n-2}}\, dV_{g_0} \le 0. \tag{2-3}$$

The following lemma will be very useful to prove integral estimates on the solution g.

Lemma 2.1. *We have for any* $p > 1$,

$$\frac{d}{dt}\int_M |R_g - f|^p\, dV_g = -\frac{4(n-1)(p-1)}{p}\int_M |\nabla_g |R_g - f|^{\frac{p}{2}}|_g^2\, dV_g + \Big(p - \frac{n}{2}\Big)\int_M (R_g - f)|R_g - f|^p\, dV_g + p\int_M f|R_g - f|^p\, dV_g,$$

where ∇_g *is the gradient with respect to the metric* g *and* $|\cdot|_g$ *is the Riemannian norm with respect to* g.

Proof. We have for any $p \ge 1$

$$\frac{d}{dt}\int_M |R_g - f|^p\, dV_g = p\int_M |R_g - f|^{p-2}(R_g - f)\partial_t R_g\, dV_g + \tfrac{1}{2}\int_M R_g\, tr_g(\partial_t g)\, dV_g.$$

Using (1-3) and (2-1), it follows that

$$\begin{aligned}
\frac{d}{dt}&\int_M |R_g - f|^p\, dV_g \\
&= (n-1)p\int_M |R_g - f|^{p-2}(R_g - f)\Delta_g(R_g - f) \\
&\qquad + p\int_M R_g |R_g - f|^p - \frac{n}{2}\int_M |R_g - f|^p (R_g - f)\, dV_g \\
&= -\frac{4(n-1)(p-1)}{p}\int_M |\nabla_g |R_g - f|^{\frac{p}{2}}|_g^2\, dV_g \\
&\qquad + \Big(p - \frac{n}{2}\Big)\int_M (R_g - f)|R_g - f|^p\, dV_g + p\int_M f|R_g - f|^p\, dV_g. \qquad \square
\end{aligned}$$

In order to prove that the solution $g(t) = u(t)^{4/(n-2)} g_0$ is globally defined on $[0, +\infty)$, we need upper and lower bounds on $u(t)$.

Proposition 2.2. *Let* $g(t) = u(t)^{4/(n-2)} g_0$ *be the solution of* (1-3) *defined on a maximal interval* $[0, T^*)$. *Then we have for any* $t \in [0, T^*)$,

$$\min\big(C_0, \min_M u_0\big) \le u(t) \le \max\big(1, \max_M u_0\big) e^{C_1 t}, \tag{2-4}$$

where

$$C_0 = \big(\min_M |R_0| / \max_M |f|\big)^{\frac{n-2}{4}}, \qquad C_1 = \frac{n-2}{4}\big(\max_M |R_0| + \max_M |f|\big).$$

Proof. The proof uses an elementary maximum principal argument. Indeed, fix $t \in [0, T)$ and let $(t_0, x_0) \in [0, t] \times M$ such that $u(t_0, x_0) = \min_{[0,t]\times M} u$. If $t_0 = 0$,

$$\min_{[0,t]\times M} u = \min_M u_0,$$

so the first inequality in (2-4) is proved in this case. Now suppose that $t_0 > 0$. We have then $\partial_t u(t_0, x_0) \le 0$ and $\Delta u(t_0, x_0) \ge 0$. Thus after substituting in (1-4) we obtain that

$$0 \ge -R_0(x_0)u(t_0, x_0) + f(x_0)u^N(t_0, x_0)$$

which implies

$$u(t_0, x_0) \ge \big(\min_M |R_0| / \max_M |f|\big)^{\frac{1}{N-1}},$$

where $N = \frac{n+2}{n-2}$. This proves the first inequality in (2-4). In order to prove the second inequality we set $v = e^{-C_1 t}u$ instead of u, where $C_1 = \frac{4}{n-2}(\max_M |R_0| + \max_M |f|)$. As above, fix $t \in [0, T)$ and let $(t_0, x_0) \in [0, t]\times M$ such that $v(t_0, x_0) = \max_{[0,t]\times M} v$. If $t_0 = 0$, then $\max_{[0,t]\times M} v = \max_M u_0$, which implies

$$\max_{[0,t]\times M} u \le \max_M u_0 \, e^{C_1 t},$$

so the second inequality in (2-4) is proved in this case. Now suppose that $t_0 > 0$. We have then $\partial_t v(t_0, x_0) \ge 0$ and $\Delta v(t_0, x_0) \le 0$, that is, $\partial_t u(t_0, x_0) \ge C_1 u(t_0, x_0)$ and $\Delta u(t_0, x_0) \le 0$. We obtain after substituting in (1-4) that

$$NC_1 u^N(t_0, x_0) \le \frac{n+2}{4}(-R_0(x_0)u(t_0, x_0) + f(x_0)u^N(t_0, x_0))$$

which implies that

$$u(t_0, x_0) \le 1 \tag{2-5}$$

since $NC_1 = \frac{n+2}{4}(\max_M |R_0| + \max_M |f|)$. It is clear that (2-5) implies that

$$\max_{[0,t]\times M} u \le e^{C_1 t}.$$

The proof of Proposition 2.2 is then complete. □

Now we prove integral estimates on R_g which will imply estimates on $\partial_t u$:

Proposition 2.3. *Let $g(t)$ be the solution of* (1.3) *defined on a maximal interval* $[0, T^*)$. *Then we have for any $t \in [0, T^*)$,*

$$\int_M |R_{g(t)} - f|^p \, dV_{g(t)} \le Ce^{Ct} \tag{2-6}$$

where $p = \frac{n^2}{2(n-2)}$ and C is a positive constant depending only on f, g_0, u_0.

Proof. In what follows C denotes a positive constant depending on f, g_0, u_0, whose value may change from line to line.

We have by Lemma 2.1 for any $t \in [0, T^*)$

$$\frac{d}{dt}\int_M |R_g-f|^p\,dV_g = -\frac{4(n-1)(p-1)}{p}\int_M \bigl|\nabla_g|R_g-f|^{\frac{p}{2}}\bigr|_g^2\,dV_g + \Bigl(p-\frac{n}{2}\Bigr)\int_M (R_g-f)|R_g-f|^p\,dV_g + p\int_M f|R_g-f|^p\,dV_g, \tag{2-7}$$

where ∇_g is the gradient with respect to the metric g and $|\cdot|_g$ is the Riemannian norm with respect to g. It follows from (2-7) that

$$\frac{d}{dt}\int_M |R_g-f|^p\,dV_g + \frac{4(n-1)(p-1)}{p}\int_M \bigl|\nabla_g|R_g-f|^{\frac{p}{2}}\bigr|_g^2\,dV_g \le \Bigl|p-\frac{n}{2}\Bigr|\int_M |R_g-f|^{p+1}\,dV_g + C\int_M |R_g-f|^p\,dV_g. \tag{2-8}$$

By (2-4) we have

$$\int_M \bigl|\nabla_g|R_g-f|^{\frac{p}{2}}\bigr|_g^2\,dV_g = \int_M \bigl|\nabla|R_g-f|^{\frac{p}{2}}\bigr|^2 u^2\,dV_{g_0} \ge C\int_M \bigl|\nabla|R_g-f|^{\frac{p}{2}}\bigr|^2\,dV_{g_0} \tag{2-9}$$

and

$$\int_M |R_g-f|^p\,dV_g = \int_M |R_g-f|^p u^{\frac{2n}{n-2}}\,dV_{g_0} \ge C\int_M |R_g-f|^p\,dV_{g_0}. \tag{2-10}$$

By Sobolev's inequality we have

$$\Bigl(\int_M |R_g-f|^{\frac{pn}{n-2}}\,dV_{g_0}\Bigr)^{\frac{n-2}{n}} \le C\Bigl(\int_M \bigl|\nabla|R_g-f|^{\frac{p}{2}}\bigr|^2\,dV_{g_0} + \int_M |R_g-f|^p\,dV_{g_0}\Bigr)$$

which, using (2-4), gives

$$\Bigl(\int_M |R_g-f|^{\frac{pn}{n-2}}\,dV_g\Bigr)^{\frac{n-2}{n}} \le Ce^{Ct}\Bigl(\int_M \bigl|\nabla|R_g-f|^{\frac{p}{2}}\bigr|^2\,dV_{g_0} + \int_M |R_g-f|^p\,dV_{g_0}\Bigr). \tag{2-11}$$

It follows from (2-8), (2-9) , (2-10) and (2-11) that

$$\frac{d}{dt}\int_M |R_g-f|^p\,dV_g + C^{-1}e^{-Ct}\Bigl(\int_M |R_g-f|^{\frac{pn}{n-2}}\,dV_g\Bigr)^{\frac{n-2}{n}} \le \Bigl(p-\frac{n}{2}\Bigr)\int_M |R_g-f|^{p+1}\,dV_g + C\int_M |R_g-f|^p\,dV_g. \tag{2-12}$$

By taking $p = \frac{n}{2}$ in (2-12) we get

$$\frac{d}{dt}\int_M |R_g - f|^p \, dV_g \le C \int_M |R_g - f|^p \, dV_g,$$

which implies that

$$\int_M |R_g - f|^{\frac{n}{2}} \, dV_g \le Ce^{Ct}. \tag{2-13}$$

Now taking again $p = \frac{n}{2}$ in (2-12) and integrating on $[0,t]$, $t \in [0,T^*)$, by using (2-13) we obtain

$$\int_0^t \left(\int_M |R_{g(s)} - f|^{\frac{n^2}{2(n-2)}} \, dV_{g(s)} \right)^{\frac{n-2}{n}} ds \le Ce^{Ct}. \tag{2-14}$$

We have by Hölder's inequality and Young's inequality, for any $\varepsilon > 0$ and $p > \frac{n}{2}$,

$$\begin{aligned} &\int_M |R_g - f|^{p+1} \, dV_g \\ &\qquad \le \varepsilon \left(\int_M |R_g - f|^{\frac{pn}{n-2}} \, dV_g \right)^{\frac{n-2}{n}} + \varepsilon^{-\frac{n}{2p-n}} \left(\int_M |R_g - f|^p \, dV_g \right)^{\frac{2p-n+2}{2p-n}}. \end{aligned} \tag{2-15}$$

If we combine (2-15) with (2-12) and taking $\varepsilon = (p - \frac{n}{2})^{-1} C^{-1} e^{-Ct}$, we get

$$\frac{d}{dt}\int_M |R_g - f|^p \, dV_g \le Ce^{Ct} \left(\int_M |R_g - f|^p \, dV_g \right)^{\frac{2p-n+2}{2p-n}} + C \int_M |R_g - f|^p \, dV_g,$$

that is,

$$\frac{d}{dt} \log\left(\int_M |R_g - f|^p \, dV_g \right) \le C\left(e^{Ct} \left(\int_M |R_g - f|^p \, dV_g \right)^{\frac{2}{2p-n}} + 1 \right)$$

In particular by choosing $p = \frac{n^2}{2(n-2)}$ and integrating on $[0,t]$, $t \in [0,T^*)$, we obtain

$$\begin{aligned} \log\left(\int_M |R_{g(t)} - f|^{\frac{n^2}{2(n-2)}} \, dV_{g(t)} \right) &\le \log\left(\int_M |R_{g(0)} - f|^{\frac{n^2}{2(n-2)}} \, dV_{g(0)} \right) \\ &\quad + Ce^{Ct} \int_0^t \left(\int_M |R_{g(s)} - f|^{\frac{n^2}{2(n-2)}} \, dV_{g(s)} \right)^{\frac{n-2}{n}} ds + Ct \end{aligned}$$

which by using (2-14) gives

$$\log\left(\int_M |R_{g(t)} - f|^{\frac{n^2}{2(n-2)}} \, dV_{g(t)} \right) \le Ce^t.$$

This proves Proposition 2.3. □

With the estimates of Proposition 2.2 we would like to apply the classical Schauder estimates for parabolic equations. To this end we need C^{α}-estimates:

Proposition 2.4. *Let* $g(t) = u(t)^{4/(n-2)} g_0$ *be the solution of* (1-3) *defined on a maximal interval* $[0, T^*)$. *Then we have for some* $\alpha \in (0, 1)$ *and any* $T \in [0, T^*)$

$$\|u\|_{C^{\alpha}([0,T]\times M)} \leq Ce^{CT},$$

where C *is a positive constant depending only on* u_0, g_0 *and* f.

Proof. By using Propositions 2.2 and 2.3, the proof is identical to that of Proposition 2.6 in Brendle [2005]. □

Proof of Theorem 1.1. Let $g(t) = u(t)^{4/(n-2)} g_0$ be the solution of (1-3) defined on a maximal interval $[0, T^*)$. Assume by contradiction that $T^* < +\infty$. Then by using Propositions 2.2 and 2.4 we have

$$\|u\|_{C^{\alpha}([0,T^*)\times M)} \leq Ce^{CT^*} \quad \text{and} \quad \min_{[0,T^*)\times M} u \geq \min\bigl(C_0, \min_M u_0\bigr)$$

for some $\alpha \in (0, 1)$, where C is a positive constant depending u_0, f, g_0. The classical theory of linear parabolic equations applied to (1-4) implies that u is bounded in $C^k([0, T^*) \times M)$ for any $k \in \mathbb{N}$, that is,

$$\|u\|_{C^k([0,T^*)\times M)} \leq C_k, \tag{2-16}$$

where C_k is a positive constant depending only on u_0, g_0, f and k. It is clear that (2-16) allows us to extend the solution beyond T^* contradicting thus the maximality of T^*. We see from (2-2) that the functional $\mathcal{E}$ is decreasing along the flow. The proof of Theorem 1.1 is then complete. □

3. Long Time behaviour of the flow

In this section we study the asymptotic behaviour of the flow $g(t)$ when $t \to +\infty$. First we prove the following proposition which gives a super solution of (1-1) when conditions (**H1**) and (**H2**) are satisfied.

Proposition 3.1. *Suppose that there exists an open set* $\Omega \subset M$ *such that conditions* (**H1**) *and* (**H2**) *are satisfied. Then there exists a conformal metric* $\bar{g} = \bar{u}^{4/(n-2)} g_0$, $0 < \bar{u} \in C^{\infty}(M)$, *satisfying*

$$R_{\bar{g}} - f \geq 0 \tag{3-1}$$

or equivalently

$$-c_n \Delta \bar{u} + R_0 \bar{u} - f \bar{u}^N \geq 0, \;\; N = \frac{n+2}{n-2}. \tag{3-2}$$

Proof. By hypothesis, there is an open set $\Omega \subset M$ satisfying (**H1**) and (**H2**), that is,

$$\text{(H1)} \qquad \lambda_\Omega > 0 \ \text{ and } \ f < 0 \text{ on } M \setminus \Omega$$

and

$$\text{(H2)} \qquad \sup_{x \in \Omega} f(x) \leq C_\Omega \inf_{x \in M \setminus \Omega} |f(x)|,$$

where C_Ω is a positive constant depending only on Ω.

Let $\varepsilon > 0$ and set

$$\Omega_\varepsilon = \{x \in M : d(x, \Omega) < \varepsilon\}.$$

For $\varepsilon > 0$ sufficiently small we have from (**H1**) that $\lambda_{\Omega_\varepsilon} > 0$, where $\lambda_{\Omega_\varepsilon}$ is the first eigenvalue of the operator $-c_n\Delta + R_0$ on Ω_ε with zero Dirichlet boundary conditions. Let $D \subset M$ be an open set of smooth boundary such that $\bar{\Omega} \subset D \subset \Omega_\varepsilon$. Then we have $\lambda_D \geq \lambda_{\Omega_\varepsilon} > 0$. Let φ_0 an eigenfunction associated with λ_D, that is,

$$-c_n\Delta\varphi_0 + R_0\varphi_0 = \lambda_D\varphi_0.$$

Then we have that $\varphi_0 \in C^\infty(\bar{D})$ and using the maximum principle of elliptic equations we have $\varphi_0 > 0$ on D. By normalising if necessary, we may suppose that

$$0 < \varphi_0 \leq 1 \ \text{ on } D. \tag{3-3}$$

Let $\chi \in C_0^\infty(D)$ such that $0 \leq \chi \leq 1$ and $\chi = 1$ on $\bar{\Omega}$. We define the function $\bar{u} \in C^\infty(M)$ by setting

$$\bar{u} = \delta(\chi\varphi_0 + 1 - \chi),$$

where $\delta > 0$ will be chosen later. By (3-3) and the definition of χ it is easy to check

$$m_0 := \inf_M(\chi\varphi_0 + 1 - \chi) > 0,$$

so

$$\bar{u} \geq \delta m_0. \tag{3-4}$$

Now let us prove that $\bar{u}$ satisfies (3-2). If we set

$$\mathcal{L}(\bar{u}) = -c_n\Delta\bar{u} + R_0\bar{u} - f\bar{u}^{\frac{n+2}{n-2}},$$

then (3-2) is equivalent to $\mathcal{L}(\bar{u}) \geq 0$.

A simple computation shows that we have on Ω (using the fact that $\chi = 1$ on $\bar{\Omega}$):

$$\mathcal{L}(\bar{u}) = \lambda_D\delta\varphi_0 - f\delta^N\varphi_0^N = \delta\varphi_0(\lambda_D - \delta^{N-1}f\varphi_0^{N-1})$$

and by using (3-3) it follows that

$$\mathcal{L}(\bar{u}) \geq \delta\varphi_0\big(\lambda_D - \delta^{N-1}\sup_{x \in \Omega} f(x)\big). \tag{3-5}$$

It follows from (3-5) that if we want $\mathcal{L}(\bar{u}) \geq 0$ on Ω, we have to choose $\delta > 0$ satisfying

$$\delta^{N-1} \sup_{x\in\Omega} f(x) \leq \lambda_D. \tag{3-6}$$

Now we examine the sign of $\mathcal{L}(\bar{u})$ on $M\setminus\Omega$. We have from the definition of $\bar{u}$ that

$$\mathcal{L}(\bar{u}) = \delta(-c_n\Delta + R_0)(\chi\varphi_0 + 1 - \chi) - f\bar{u}^N. \tag{3-7}$$

By using (3-4) and the fact that $f < 0$ on $M \setminus \Omega$, it follows from (3-7) that

$$\mathcal{L}(\bar{u}) \geq -\delta m_1 + \delta^N m_0^N \inf_{x\in M\setminus\Omega} |f(x)|,$$

where

$$m_1 = \sup_M |(-c_n\Delta + R_0)(\chi\varphi_0 + 1 - \chi)|.$$

Thus, if we want $\mathcal{L}(\bar{u}) \geq 0$ on $M \setminus \Omega$, we have to assume

$$-m_1 + \delta^{N-1} m_0^N \inf_{x\in M\setminus\Omega} |f(x)| \geq 0,$$

that is,

$$\delta^{N-1} \inf_{x\in M\setminus\Omega} |f(x)| \geq m_1 m_0^{-N}. \tag{3-8}$$

It is clear that the existence of $\delta > 0$ satisfying both (3-6) and (3-8) is equivalent to condition (**H2**) with $C_\Omega = \lambda_D m_0^N / m_1$. □

Proposition 3.1 allows us to prove uniform L^∞-estimates on the flow.

Proposition 3.2. *Let $0 < u_0 \in C^\infty(M)$ such that $u_0 \leq \bar{u}$ where $\bar{u}$ is given by Proposition 3.1. Then the solution u of* (1-4) *satisfies, for any $(t, x) \in [0, +\infty) \times M$,*

$$\min\bigl(C_0, \min_M u_0\bigr) \leq u(t, x) \leq \max_M \bar{u}, \tag{3-9}$$

where

$$C_0 = \bigl(\min_M |R_0| / \max_M |f|\bigr)^{\frac{n-2}{4}}.$$

Proof. First observe that the first inequality in (3-9) has already been proved in Proposition 2.2. It remains then to prove the second inequality, that is,

$$u(t, x) \leq \max_M \bar{u}.$$

Let $v = \bar{u} - u$. Since u satisfies (1-4) and $\bar{u}$ satisfies (3-2), we have

$$\partial_t(\bar{u}^N - u^N) \geq \frac{n+2}{4}(c_n\Delta v - R_0 v + f(\bar{u}^N - u^N)). \tag{3-10}$$

We have $\bar{u}^N - u^N = av$, where

$$a(t, x) = N \int_0^1 \bigl(s\bar{u}(t, x) + (1 - s)u(t, x)\bigr)^{N-1} ds,$$

so it follows from (3-10) that

$$\partial_t(av) \geq \frac{n+2}{4}(c_n\Delta v - R_0 v + afv). \tag{3-11}$$

Since $v(0,x) = \bar{u}(x) - u_0(x) \geq 0$, by applying the maximum principle to (3-11) we get $v(t,x) \geq 0$ for any $t \geq 0$, that is,

$$u(t,x) \leq \bar{u}(x). \qquad \square$$

Now we prove that the integral estimate (2-6) in Proposition 2.3 can be improved when $t \to +\infty$. More precisely, we have:

Proposition 3.3. *Let $0 < u_0 \in C^\infty(M)$ such that $u_0 \leq \bar{u}$ where $\bar{u}$ is given by Proposition 3.1. Let $g(t)$ be the solution of* (1-3) *given by Theorem 1.1 such that $g(0) = u_0^{4/(n-2)} g_0$. Then we have, for any $p \geq 1$,*

$$\lim_{t\to+\infty} \int_M |R_{g(t)} - f|^p \, dV_{g(t)} = 0. \tag{3-12}$$

Proof. In what follows C denotes a positive constant depending only on u_0, g_0, f, p, and its value may change from line to line.

We have by (2-2) for any $t \geq 0$,

$$\frac{n-2}{2}\int_0^t \int_M |R_g - f|^2 \, dV_g = \mathcal{E}(g(0)) - \mathcal{E}(g(t)). \tag{3-13}$$

On the other hand, we have

$$\mathcal{E}(g(t)) = \int_M \left(c_n |\nabla u|^2 + R_0 u^2 - \frac{n-2}{n} f u^{\frac{2n}{n-2}} \right) dV_{g_0},$$

and since u is uniformly bounded by Proposition 3.2, we have $\mathcal{E}(g(t)) \geq -C$. So it follows from (3-13) that

$$\int_0^{+\infty}\int_M |R_{g(t)} - f|^2 \, dV_{g(t)} \leq C. \tag{3-14}$$

Since by Proposition 3.2 the volume of $g(t)$ is uniformly bounded, it suffices to prove (3-12) for a sequence $p_k \to +\infty$. We shall prove (3-12) by induction when $p = p_k$, where

$$p_k := \frac{n}{2}\left(\frac{n}{n-2}\right)^k, \quad k \in \mathbb{N}.$$

First we prove (3-12) for $p_0 = \frac{n}{2}$. As in the proof of Proposition 2.3, if we use Lemma 2.1 and the fact that u is uniformly bounded by Proposition 3.2, then we have for any $p > 1$:

$$\text{(3-15)}\quad \frac{d}{dt}\int_M |R_g - f|^p\, dV_g + C^{-1}\left(\int_M |R_g - f|^{\frac{pn}{n-2}}\, dV_g\right)^{\frac{n-2}{n}} \le C\int_M |R_g - f|^p\, dV_g + \left(p - \frac{n}{2}\right)\int_M |R_g - f|^{p+1}\, dV_g.$$

Set

$$\phi_p(t) = \int_M |R_g - f|^p\, dV_g.$$

If $p_0 < 2$, then by using Hölder's inequality and the fact that u is uniformly bounded, we have

$$\text{(3-16)}\qquad \phi_{p_0} \le C\phi_2^{p_0/2}.$$

So it follows from (3-15) by taking $p = p_0 = \frac{n}{2}$ that

$$\text{(3-17)}\qquad \frac{d}{dt}\phi_{p_0}^{2/p_0} \le C\phi_2.$$

By (3-14) there is a sequence $t_\nu \to +\infty$ such that $\int_{t_\nu}^{+\infty} \phi_2(s)\, ds \to 0$ and $\phi_2(t_\nu) \to 0$. So by integrating (3-17) on $[t_\nu, t]$ and using (3-16) we get

$$\phi_{p_0}^{2/p_0}(t) \le \phi_{p_0}^{2/p_0}(t_\nu) + C\int_{t_\nu}^t \phi_2(s)\, ds \le C\phi_2(t_\nu) + C\int_{t_\nu}^t \phi_2(s)\, ds$$

Letting $t \to +\infty$ and $\nu \to +\infty$ we obtain $\phi_{p_0}(t) \to 0$ as $t \to +\infty$.

If $p_0 \ge 2$, by using Hölder's inequality and Young's inequality we have, for any $\varepsilon > 0$,

$$\int_M |R_g - f|^{p_0}\, dV_g \le \varepsilon\left(\int_M |R_g - f|^{p_0 n/n-2}\, dV_g\right)^{\frac{n-2}{n}} + \varepsilon^{-n(p_0-2)/4}\left(\int_M |R_g - f|^2\, dV_g\right)^{p_0/2}.$$

By taking $\varepsilon = \frac{1}{2}C^{-1}$, where C^{-1} is the constant appearing in (3-15), we obtain from (3-15)(where we take $p = p_0 = \frac{n}{2}$),

$$\text{(3-18)}\qquad \frac{d}{dt}\phi_{p_0} + C^{-1}\phi_{p_0 n/(n-2)}^{(n-2)/n} \le C\phi_2^{p_0/2}.$$

But by Hölder's inequality, since the volume of g is uniformly bounded, we have

$$\phi_2 \le C\phi_{p_0}^{2/p_0} \quad\text{and}\quad \phi_{p_0} \le C\phi_{p_0 n/(n-2)}^{(n-2)/n}.$$

Thus it follows from (3-18) that

$$\text{(3-19)}\qquad \frac{d}{dt}\phi_{p_0}^{2/p_0} + C^{-1}\phi_{p_0}^{2/p_0} \le C\phi_2.$$

If we integrate (3-19) on $[0, t]$ and use (3-14) we get

$$\int_0^t \phi_{p_0}^{2/p_0}(s)\,ds \le C,$$

which implies, since $t \ge 0$ is arbitrary,

$$\int_0^{+\infty} \phi_{p_0}^{2/p_0}(s)\,ds \le C.$$

Thus there exists a sequence $t_\nu \to +\infty$ such that $\phi_{p_0}^{2/p_0}(t_\nu) \to 0$ as $\nu \to +\infty$. If we integrate again (3-19) on $[t_\nu, t]$, we obtain

$$\phi_{p_0}^{2/p_0}(t) \le \phi_{p_0}^{2/p_0}(t_\nu) + C\int_{t_\nu}^t \phi_2(s)\,ds.$$

By using (3-14), it follow that $\phi_{p_0}^{2/p_0}(t) \to 0$ as $t \to +\infty$.

Now suppose by induction that

$$\lim_{t\to+\infty} \phi_{p_k}(t) = 0. \tag{3-20}$$

First let us prove that

$$\lim_{t\to+\infty} \int_t^{t+1} \phi_{p_{k+1}}^{(n-2)/n}(s)\,ds = 0. \tag{3-21}$$

We may suppose $k \ge 1$. Indeed, if $k = 0$ (that is, $p_k = p_0 = \frac{n}{2}$), then (3-21) follows directly from (3-15) (with $p = \frac{n}{2}$) by integrating on $[t, t+1]$ and using (3-20). Thus let us prove (3-21) when $k \ge 1$.

By using Hölder's inequality and Young's inequality we have for any $p > \frac{n}{2}$ and $\varepsilon > 0$,

$$\begin{aligned}\int_M |R_g - f|^{p+1}\,dV_g \\ \le \varepsilon\biggl(\int_M |R_g - f|^{\frac{pn}{n-2}}\,dV_g\biggr)^{\frac{n-2}{n}} + \varepsilon^{-\frac{n}{2p-n}}\biggl(\int_M |R_g - f|^p\,dV_g\biggr)^{1+\frac{2}{2p-n}}\end{aligned} \tag{3-22}$$

By taking $p = p_k$, $\varepsilon = \frac{1}{2}C^{-1}$, where C is the constant appearing in (3-15), from (3-15) we obtain

$$\frac{d}{dt}\phi_{p_k} + \tfrac{1}{2}C^{-1}\phi_{p_{k+1}}^{(n-2)/n} \le C\phi_{p_k}^{1+2/(2p_k-n)} + C\phi_{p_k}.$$

Then (3-21) follows by integrating on $[t, t+1]$ and using (3-20).

Now if we apply (3-22) by taking $p = p_{k+1}$ and $\varepsilon = C^{-1}/(p_{k+1} - n/2)$, where C is the constant appearing in (3-15), we obtain from (3-15) (where we take $p = p_{k+1}$),

$$\frac{d}{dt}\phi_{p_{k+1}} \le C\phi_{p_{k+1}}^{1+\alpha_k} + C\phi_{p_{k+1}},$$

where $\alpha_k = \frac{2}{2p_{k+1}} - n$. The last inequality is equivalent to

$$\frac{d}{dt}\log\phi_{p_{k+1}} \le C\big(\phi_{p_{k+1}}^{\alpha_k} + 1\big). \tag{3-23}$$

By (3-21) there is a sequence $t_\nu \to +\infty$ such that $\nu \le t_\nu \le \nu + 1$ satisfying $\phi_{p_{k+1}}(t_\nu) \to 0$ as $\nu \to +\infty$. If we integrate (3-23) on $[t_\nu, t]$ where $t \in [\nu, \nu+1]$, we obtain

$$\log\frac{\phi_{p_{k+1}}(t)}{\phi_{p_{k+1}}(t_\nu)} \le C\left(\int_\nu^{\nu+1} \phi_{p_{k+1}}^{\alpha_k}(s)\,ds + 1\right). \tag{3-24}$$

We note here that $\alpha_k \le \frac{n-2}{n}$, so by Hölder's inequality we have

$$\int_\nu^{\nu+1} \phi_{p_{k+1}}^{\alpha_k}(s)\,ds \le \left(\int_\nu^{\nu+1} \phi_{p_{k+1}}^{(n-2)/n}(s)\,ds\right)^{\frac{n\alpha_k}{n-2}} \to 0 \ \text{ as } \ \nu \to +\infty$$

by (3-21). Thus it follows from (3-24) that

$$\log\frac{\phi_{p_{k+1}}(t)}{\phi_{p_{k+1}}(t_\nu)} \le C,$$

which implies that $\phi_{p_{k+1}}(t) \to 0$ as $t \to +\infty$. The proof of Proposition 3.3 is then complete. □

Now we can prove uniform C^α-estimates on the solution.

Proposition 3.4. *Let $0 < u_0 \in C^\infty(M)$ such that $u_0 \le \bar{u}$ where $\bar{u}$ is given by Proposition 3.1. Then the solution u of* (1-4) *satisfies, for some $\alpha \in (0, 1)$,*

$$\|u\|_{C^\alpha([0,+\infty)\times M)} \le C,$$

where C is a positive constant depending only on u_0, g_0 and f.

Proof. By using Propositions 3.2 and 3.3, the proof is identical to that of Proposition 2.6 in Brendle [2005]. □

Now we are in position to prove Theorem 1.2.

Proof of Theorem 1.2. Let $g = u^{4/(n-2)}g_0$ be the solution of (1-3) given by Theorem 1.1. By Proposition 3.2 we have that u is bounded from below and above uniformly on $[0, +\infty)$. As in the proof of Theorem 1.1, this implies that (1-4) is uniformly parabolic and by Proposition 3.4 we have a uniform C^α-bound on the solution u on $[0, +\infty) \times M$. We then apply the classical regularity theory of linear parabolic equations to obtain uniform C^k-bound for any $k \in \mathbb{N}$, that is,

$$\|u(t)\|_{C^k(M)} \le C_k, \tag{3-25}$$

for some constant C_k independent of t. It follows from (3-25) that there is a sequence $t_\nu \to +\infty$ such that $u(t_\nu)$ converges in $C^k(M)$ for any $k \in \mathbb{N}$, to some function

$u_\infty \in C^\infty(M)$. Since $u(t)$ is uniformly bounded from below by Proposition 3.2, then we have $u_\infty > 0$. By using Proposition 3.3 and passing to the limit when $\nu \to \infty$, we see that $R_{g_\infty} = f$, where $g_\infty = u_\infty^{4/(n-2)} g_0$, that is, f is the scalar curvature of g_∞. By the general result of Simon [1983] on evolution equations, u_∞ is the unique limit of $u(t)$ when $t \to +\infty$. □

Proof of Corollary 1.3. Since $f < 0$ almost everywhere on M, then for $\varepsilon > 0$ small enough, the open set

$$\Omega_\varepsilon = \{x \in M : f(x) > -\varepsilon\}$$

has arbitrary small volume. This implies that the first eigenvalue μ_{Ω_ε} of $-c_n\Delta$ on Ω_ε with zero Dirichlet conditions is arbitrarily large if ε is small enough. But since

$$\lambda_{\Omega_\varepsilon} \geq \mu_{\Omega_\varepsilon} + \min_M R_0,$$

we have $\lambda_{\Omega_\varepsilon} > 0$ if ε is small enough. Thus the condition (**H1**) is satisfied with $\Omega = \Omega_\varepsilon$. Condition (**H2**) is also satisfied since by continuity of f we have $f \leq 0$ everywhere on M. □

Proof of Theorem 1.4. Suppose that condition (**H1**) is not satisfied, that is, for any open set $\Omega \subset M$ such that $f < 0$ on $M \setminus \Omega$, we suppose $\lambda_\Omega \leq 0$. For $\varepsilon > 0$, consider the following family of open sets:

$$\Omega_\varepsilon = \{x \in M : f(x) > -\varepsilon\}.$$

For simplicity of notation we set $\lambda_\varepsilon = \lambda_{\Omega_\varepsilon}$. According to our hypothesis we have

$$\lambda_\varepsilon \leq 0 \quad \text{for all } \varepsilon > 0. \tag{3-26}$$

By using Sard's theorem, there exists a sequence $\varepsilon_n \to 0$ such that ε_n is a regular value of f and then Ω_{ε_n} has a smooth boundary

$$\partial\Omega_{\varepsilon_n} = \{x \in M : f(x) = -\varepsilon_n\}.$$

Let φ_n an eigenfunction of $-c_n\Delta + R_0$ associated with λ_{ε_n}. As already mentioned in the proof of Proposition 3.2, we have by the maximum principle that

$$\varphi_n > 0 \text{ on } \Omega_{\varepsilon_n} \quad \text{and} \quad \frac{\partial \varphi_n}{\partial \nu} \leq 0 \text{ on } \partial\Omega_{\varepsilon_n}, \tag{3-27}$$

where ν is the outer normal vector to $\partial\Omega_{\varepsilon_n}$. By normalising if necessary, we may assume that

$$\int_{\Omega_{\varepsilon_n}} \varphi_n \, dV_{g_0} = 1. \tag{3-28}$$

If we multiply (1-4) by φ_n and integrate on Ω_{ε_n}, we have

$$\text{(3-29)}\quad \frac{d}{dt}\int_{\Omega_{\varepsilon_n}} u^N\varphi_n\,dV_{g_0} = \frac{n+2}{4}\int_{\Omega_{\varepsilon_n}} (c_n\Delta u - R_0 u)\varphi_n\,dV_{g_0} + \frac{n+2}{4}\int_{\Omega_{\varepsilon_n}} f u^N\varphi_n\,dV_{g_0}.$$

An integration by parts gives

$$\int_{\Omega_{\varepsilon_n}} (c_n\Delta u - R_0 u)\varphi_n\,dV_{g_0} = -\lambda_{\varepsilon_n}\int_{\Omega_{\varepsilon_n}} u\varphi_n\,dV_{g_0} - c_n\int_{\partial\Omega_{\varepsilon_n}} \frac{\partial\varphi_n}{\partial\nu} u\,dV_{g_0}.$$

Since $\lambda_{\varepsilon_n} \le 0$, by using (3-27) and (3-28) we then obtain

$$\text{(3-30)}\quad \int_{\Omega_{\varepsilon_n}} (c_n\Delta u - R_0 u)\varphi_n\,dV_{g_0} \ge -\lambda_{\varepsilon_n}\inf_M u - c_n\inf_M u\int_{\partial\Omega_{\varepsilon_n}} \frac{\partial\varphi_n}{\partial\nu}\,dV_{g_0}.$$

On the other hand we have

$$c_n\int_{\partial\Omega_{\varepsilon_n}} \frac{\partial\varphi_n}{\partial\nu}\,dV_{g_0} = c_n\int_{\Omega_{\varepsilon_n}} \Delta\varphi_n\,dV_{g_0} = \int_{\Omega_{\varepsilon_n}} (-\lambda_{\varepsilon_n} + R_0)\varphi_n\,dV_{g_0}$$

and by using (3-28) we get, since $R_0 < 0$,

$$\text{(3-31)}\quad -c_n\int_{\partial\Omega_{\varepsilon_n}} \frac{\partial\varphi_n}{\partial\nu}\,dV_{g_0} \ge \lambda_{\varepsilon_n} + \inf_M |R_0|.$$

Combining (3-30) and (3-31) we obtain

$$\int_{\Omega_{\varepsilon_n}} (c_n\Delta u - R_0 u)\varphi_n\,dV_{g_0} \ge \inf_M |R_0|\inf_M u.$$

If we substitute in (3-29) we get

$$\text{(3-32)}\quad \frac{d}{dt}\int_{\Omega_{\varepsilon_n}} u^N\varphi_n\,dV_{g_0} \ge \frac{n+2}{4}\inf_M |R_0|\inf_M u + \frac{n+2}{4}\int_{\Omega_{\varepsilon_n}} f u^N\varphi_n\,dV_{g_0}.$$

By Proposition 3.2 we have $u \ge C_0$, where C_0 is a positive constant depending only on u_0, g_0 and f. Using the fact that $f > -\varepsilon_n$ on Ω_{ε_n}, it follows from (3-32) that

$$\frac{d}{dt}\int_{\Omega_{\varepsilon_n}} u^N\varphi_n\,dV_{g_0} \ge C - \frac{n+2}{4}\,\varepsilon_n\int_{\Omega_{\varepsilon_n}} u^N\varphi_n\,dV_{g_0},$$

where C is a positive constant depending only on u_0, g_0 and f. By integrating this differential inequality on $[0, t]$, we get

$$\int_{\Omega_{\varepsilon_n}} u^N(t)\varphi_n\,dV_{g_0} \ge \int_{\Omega_{\varepsilon_n}} u_0^N\varphi_n\,dV_{g_0} + Ct - \frac{n+2}{4}\,\varepsilon_n\int_0^t\int_{\Omega_{\varepsilon_n}} u^N(s)\varphi_n\,dV_{g_0}\,ds,$$

which, using (3-28), implies

$$\max_{x\in M} u^N(t,x) \geq Ct - \frac{n+2}{4}\,\varepsilon_n \int_0^t \max_{x\in M} u^N(s,x)\,ds.$$

Letting $n \to +\infty$, we obtain

$$\max_{x\in M} u^N(t,x) \geq Ct.$$

The proof of Theorem 1.4 is complete. □

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RIGIDITY OF PROPER HOLOMORPHIC MAPPINGS BETWEEN GENERALIZED FOCK–BARGMANN–HARTOGS DOMAINS

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A generalized Fock–Bargmann–Hartogs domain $D_n^{m,p}(\mu)$ is defined as a domain fibered over $\mathbb{C}^n$ with the fiber over $z \in \mathbb{C}^n$ being a generalized complex ellipsoid $\Sigma_z(m, p)$. In general, a generalized Fock–Bargmann–Hartogs domain is an unbounded nonhyperbolic domain without smooth boundary. The main contribution of this paper is as follows. By using the explicit formula of Bergman kernels of the generalized Fock–Bargmann–Hartogs domains, we obtain the rigidity results of proper holomorphic mappings between two equidimensional generalized Fock–Bargmann–Hartogs domains. We therefore exhibit an example of unbounded weakly pseudoconvex domains on which the rigidity results of proper holomorphic mappings can be built.

1. Introduction

A holomorphic map $F : \Omega_1 \to \Omega_2$ between two domains Ω_1, Ω_2 in $\mathbb{C}^n$ is said to be proper if $F^{-1}(K)$ is compact in Ω_1 for every compact subset $K \subset \Omega_2$. In particular, an automorphism $F : \Omega \to \Omega$ of a domain Ω in $\mathbb{C}^n$ is a proper holomorphic mapping of Ω into Ω. There are many works about proper holomorphic mappings between various bounded domains with some requirements of the boundary, e.g., [Bedford and Bell 1982; Diederich and Fornæss 1982; Dini and Selvaggi Primicerio 1997; Tu and Wang 2015]. However, very little seems to be known about proper holomorphic mapping between the unbounded weakly pseudoconvex domains. There are also some works about automorphism groups of hyperbolic domains, e.g., [Isaev 2007; Isaev and Krantz 2001; Kim and Verdiani 2004]. In this paper, we mainly focus our attention on some unbounded nonhyperbolic weakly pseudoconvex domains.

The Fock–Bargmann–Hartogs domain $D_{n,m}(\mu)$ is defined by

$$D_{n,m}(\mu) = \left\{(z, w) \in \mathbb{C}^n \times \mathbb{C}^m : \|w\|^2 < e^{-\mu\|z\|^2}\right\} \quad \text{for } \mu > 0,$$

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where $\|\cdot\|$ is the standard Hermitian norm. The Fock–Bargmann–Hartogs domains $D_{n,m}(\mu)$ are strongly pseudoconvex domains in $\mathbb{C}^{n+m}$ with smooth real-analytic boundary. We note that each $D_{n,m}(\mu)$ contains $\{(z, 0) \in \mathbb{C}^n \times \mathbb{C}^m\} \cong \mathbb{C}^n$. Thus each $D_{n,m}(\mu)$ is not hyperbolic in the sense of Kobayashi and $D_{n,m}(\mu)$ can not be biholomorphic to any bounded domain in $\mathbb{C}^{n+m}$. Therefore, each Fock–Bargmann–Hartogs domain $D_{n,m}(\mu)$ is an unbounded nonhyperbolic domain in $\mathbb{C}^{n+m}$.

Yamamori [2013] gave an explicit formula for the Bergman kernels of the Fock–Bargmann–Hartogs domains in terms of the polylogarithm functions. By checking that the Bergman kernel ensures the revised Cartan's theorem, Kim, Ninh and Yamamori [Kim et al. 2014] determined the automorphism group of the Fock–Bargmann–Hartogs domains as follows.

Theorem 1.1 [Kim et al. 2014]. *The automorphism group* $\mathrm{Aut}(D_{n,m}(\mu))$ *is exactly the group generated by all automorphisms of* $D_{n,m}(\mu)$ *as follows*:

$$\begin{aligned} \varphi_U &: (z, w) \mapsto (Uz, w), && U \in \mathcal{U}(n), \\ \varphi_{U'} &: (z, w) \mapsto (z, U'w), && U' \in \mathcal{U}(m), \\ \varphi_v &: (z, w) \mapsto (z+v, e^{-\mu\langle z,v\rangle-(\mu/2)\|v\|^2}w), && v \in \mathbb{C}^n, \end{aligned}$$

where $\mathcal{U}(k)$ *is the unitary group of degree* k *and* $\langle\cdot,\cdot\rangle$ *is the standard Hermitian inner product on* $\mathbb{C}^n$.

Recently, [Tu and Wang 2014] has established the rigidity of the proper holomorphic mappings between two equidimensional Fock–Bargmann–Hartogs domains.

Theorem 1.2 [Tu and Wang 2014]. *If* $D_{n,m}(\mu)$ *and* $D_{n',m'}(\mu')$ *are two equidimensional Fock–Bargmann–Hartogs domains with* $m \geq 2$ *and* f *is a proper holomorphic mapping of* $D_{n,m}(\mu)$ *into* $D_{n',m'}(\mu')$, *then* f *is a biholomorphism between* $D_{n,m}(\mu)$ *and* $D_{n',m'}(\mu')$.

A generalized complex ellipsoid (also called generalized pseudoellipsoid) is a domain of the form

$$\Sigma(\boldsymbol{n};\boldsymbol{p}) = \left\{(\zeta_1,\ldots,\zeta_r) \in \mathbb{C}^{n_1}\times\cdots\times\mathbb{C}^{n_r} : \sum_{k=1}^{r}\|\zeta_k\|^{2p_k} < 1\right\},$$

where $\boldsymbol{n} = (n_1,\ldots,n_r) \in \mathbb{N}^r$ and $\boldsymbol{p} = (p_1,\ldots,p_r) \in (\mathbb{R}_+)^r$. In the special case where all the $p_k = 1$, the generalized complex ellipsoid $\Sigma(\boldsymbol{n};\boldsymbol{p})$ reduces to the unit ball in $\mathbb{C}^{n_1+\cdots+n_r}$. Also, it is known that a generalized complex ellipsoid $\Sigma(\boldsymbol{n};\boldsymbol{p})$ is homogeneous if and only if $p_k = 1$ for all $1 \leq k \leq r$ [Kodama 2014]. In general, a generalized complex ellipsoid is not strongly pseudoconvex and its boundary is not smooth. The automorphism group $\mathrm{Aut}(\Sigma(\boldsymbol{n};\boldsymbol{p}))$ of $\Sigma(\boldsymbol{n};\boldsymbol{p})$ has been studied by Dini and Selvaggi Primicerio [1997], Kodama [2014] and Kodama, Krantz and Ma [Kodama et al. 1992].

Kodama [2014] obtained the result as follows.

Theorem 1.3 [Kodama 2014]. (i) *If* 1 *does not appear in* $p_1, \dots, p_r$, *then any automorphism* $\varphi \in \mathrm{Aut}(\Sigma(\boldsymbol{n}; \boldsymbol{p}))$ *is of the form*

$$\varphi(\zeta_1, \dots, \zeta_r) = (\gamma_1(\zeta_{\sigma(1)}), \dots, \gamma_r(\zeta_{\sigma(r)})), \tag{1-1}$$

where $\sigma \in S_r$ *is a permutation of the r numbers* $\{1, \dots, r\}$ *such that* $n_{\sigma(i)} = n_i$, $p_{\sigma(i)} = p_i$ *for* $1 \le i \le r$ *and* $\gamma_1, \dots, \gamma_r$ *are unitary transformations of* $\mathbb{C}^{n_1}, \dots, \mathbb{C}^{n_r}$, *respectively.*

(ii) *If* 1 *appears in* $p_1, \dots, p_r$, *we can assume*, *without loss of generality*, *that* $p_1 = 1$, $p_2 \ne 1, \dots, p_r \ne 1$. *Then* $\mathrm{Aut}(\Sigma(\boldsymbol{n}; \boldsymbol{p}))$ *is generated by elements of the form* (1-1) *and automorphisms of the form*

$$\varphi_a(\zeta_1, \zeta_2, \dots, \zeta_r) = \left(T_a(\zeta_1), \zeta_2(\psi_a(\zeta_1))^{1/2p_2}, \dots, \zeta_r(\psi_a(\zeta_1))^{1/2p_r}\right), \tag{1-2}$$

where T_a *is an automorphism of the ball* $\mathbb{B}^{n_1}$ *in* $\mathbb{C}^{n_1}$ *which sends a point* $a \in \mathbb{B}^{n_1}$ *to the origin and*

$$\psi_a(\zeta_1) = \frac{1 - \|a\|^2}{(1 - \langle \zeta_1, a\rangle)^2}.$$

In this paper, we define the generalized Fock–Bargmann–Hartogs domains $D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)$ as

$$D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu) = \left\{(z, w_{(1)}, \dots, w_{(\ell)}) \in \mathbb{C}^{n_0} \times \mathbb{C}^{n_1} \times \cdots \times \mathbb{C}^{n_\ell} : \sum_{j=1}^{\ell} \|w_{(j)}\|^{2p_j} < e^{-\mu\|z\|^2}\right\} \quad (\mu > 0),$$

where $\boldsymbol{p} = (p_1, \dots, p_\ell) \in (\mathbb{R}_+)^\ell$, $\boldsymbol{n} = (n_1, \dots, n_\ell)$ and $w_{(j)} = (w_{j1}, \dots, w_{jn_j}) \in \mathbb{C}^{n_j}$, in which n_j is a positive integer for $1 \le j \le \ell$. Here and henceforth, with no loss of generality, we always assume that $p_i \ne 1$ $(2 \le i \le \ell)$ for $D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)$.

Obviously, each generalized Fock–Bargmann–Hartogs domain $D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)$ is an unbounded nonhyperbolic domain. In general, a generalized Fock–Bargmann–Hartogs domain is not a strongly pseudoconvex domain and its boundary is not smooth.

In this paper, we prove the following results.

Theorem 1.4. *Suppose* $D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)$ *and* $D_{m_0}^{\boldsymbol{m},\boldsymbol{q}}(\nu)$ *are two equidimensional generalized Fock–Bargmann–Hartogs domains. Let*

$$f : D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu) \to D_{m_0}^{\boldsymbol{m},\boldsymbol{q}}(\nu)$$

be a biholomorphic mapping. Then there exists $\phi \in \mathrm{Aut}(D_{m_0}^{\boldsymbol{m},\boldsymbol{q}}(\nu))$ *such that*

$$\phi \circ f(z,w) = (z, w_{(\sigma(1))}, \dots, w_{(\sigma(\ell))}) \begin{pmatrix} A & & & & \\ & \Gamma_1 & & & \\ & & \Gamma_2 & & \\ & & & \ddots & \\ & & & & \Gamma_\ell \end{pmatrix}, \tag{1-3}$$

where $\sigma \in S_\ell$ *is a permutation such that* $n_{\sigma(j)} = m_j$, $p_{\sigma(j)} = q_j$ $(1 \le j \le \ell)$, $\sqrt{\nu/\mu}\,A \in \mathcal{U}(n)$ $(n := n_0 = m_0)$, *and* $\Gamma_i \in \mathcal{U}(m_i)$ $(1 \le i \le \ell)$.

Corollary 1.5. *Let* $f : D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu) \to D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)$ *be a biholomorphic mapping with* $f(0) = 0$. *Then we have*

$$f(z,w) = (z, w_{(\sigma(1))}, \dots, w_{(\sigma(\ell))}) \begin{pmatrix} A & & & & \\ & \Gamma_1 & & & \\ & & \Gamma_2 & & \\ & & & \ddots & \\ & & & & \Gamma_\ell \end{pmatrix},$$

where $\sigma \in S_\ell$ *is a permutation such that* $n_{\sigma(j)} = n_j$, $p_{\sigma(j)} = p_j$ $(1 \le j \le \ell)$, $A \in \mathcal{U}(n_0)$ *and* $\Gamma_i \in \mathcal{U}(n_i)$ $(1 \le i \le \ell)$.

As a consequence, it is easy for us to prove the following results.

Theorem 1.6. *The automorphism group* $\mathrm{Aut}(D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu))$ *is generated by the following mappings*:

$$\varphi_A : (z, w_{(1)}, \dots, w_{(\ell)}) \mapsto (zA, w_{(1)}, \dots, w_{(\ell)});$$
$$\varphi_D : (z, w_{(1)}, \dots, w_{(\ell)}) \mapsto (z, (w_{(\sigma(1))}, \dots, w_{(\sigma(\ell))})D),$$
$$\varphi_a : (z,w) \mapsto \big(z + a, w_{(1)}(e^{-2\mu<z,a>-\mu\|a\|^2})^{1/2p_1}, \dots, w_{(\ell)}(e^{-2\mu\langle z,a\rangle-\mu\|a\|^2})^{1/2p_\ell}\big),$$

where $a \in \mathbb{C}^{n_0}$, $A \in \mathcal{U}(n_0)$, $\sigma \in S_\ell$ *is a permutation such that* $n_{\sigma(j)} = n_j$, $p_{\sigma(j)} = p_j$ $(1 \le j \le \ell)$, *and*

$$D = \begin{pmatrix} \Gamma_1 & & & \\ & \Gamma_2 & & \\ & & \ddots & \\ & & & \Gamma_\ell \end{pmatrix},$$

in which $\Gamma_i \in \mathcal{U}(n_i)$ $(1 \le i \le \ell)$.

Now, for $\boldsymbol{p}$ and $\boldsymbol{q}$, we introduce the notation

$$\epsilon = \begin{cases} 1, & p_1 = 1, \\ 0, & p_1 \ne 1, \end{cases} \qquad \delta = \begin{cases} 1, & q_1 = 1 \\ 0, & q_1 \ne 1. \end{cases}$$

Theorem 1.7. *Suppose* $D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)$ *and* $D_{m_0}^{\boldsymbol{m},\boldsymbol{q}}(\nu)$ *are two equidimensional generalized Fock–Bargmann–Hartogs domains with* $\min\{n_{1+\epsilon}, n_2, \dots, n_\ell, n_1 + \dots + n_\ell\} \ge 2$

and $\min\{m_{1+\delta}, m_2, \cdots, m_\ell, m_1 + \cdots + m_\ell\} \geq 2$. *Then any proper holomorphic mapping between* $D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)$ *and* $D_{m_0}^{\boldsymbol{m},\boldsymbol{q}}(\nu)$ *must be a biholomorphism.*

Remark 1.1. The conditions $\min\{n_{1+\epsilon}, n_2, \ldots, n_\ell\} \geq 2$ cannot be removed. For example, $n_1 = 1$ (i.e, $w_{(1)} \in \mathbb{C}$), $p_1 \neq 1$, and

$$F(z, w) : (z, w_{(1)}, \ldots, w_{(\ell)}) \to (z, w_{(1)}^2, w_{(2)}, \ldots, w_{(\ell)}).$$

Then F is a proper holomorphic mapping between $D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)$ and $D_{n_0}^{\boldsymbol{n},\boldsymbol{q}}(\mu)$, where $\boldsymbol{q} = (p_1/2, p_2, \ldots, p_\ell)$. F is not a biholomorphism.

Corollary 1.8. *Suppose* $D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)$ *is a generalized Fock–Bargmann–Hartogs domain with*

$$\min\{n_{1+\epsilon}, n_2, \ldots, n_\ell, n_1 + \cdots + n_\ell\} \geq 2.$$

Then any proper holomorphic self-mapping of $D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)$ *must be an automorphism.*

Remark 1.2. The conditions $n_1 + \cdots + n_\ell \geq 2$ cannot be removed. For instance, with no loss of generality, we can assume $n_1 = 1$ and $n_i = 0$ $(2 \leq i \leq \ell)$. Then

$$F : (z, w_{(1)}) \to \big(\sqrt{2}z, w_{(1)}^2\big)$$

is a proper holomorphic self-mapping of $D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)$ which is not an automorphism.

The paper is organized as follows. In Section 2, using the explicit formula for the Bergman kernels of the generalized Fock–Bargmann–Hartogs domains, we prove that a proper holomorphic mapping between two equidimensional generalized Fock–Bargmann–Hartogs domains extends holomorphically to their closures, and check that Cartan's theorem holds also for the generalized Fock–Bargmann–Hartogs domains. In Section 3, we exploit the boundary structure of generalized Fock–Bargmann–Hartogs domains to prove our results in this paper.

2. Preliminaries

The Bergman kernel of the domain $\boldsymbol{D_{n_0}^{n,p}(\mu)}$**.** For a domain Ω in $\mathbb{C}^n$, let $A^2(\Omega)$ be the Hilbert space of square integrable holomorphic functions on Ω with the inner product

$$\langle f, g\rangle = \int_\Omega f(z)\overline{g(z)}\, dV(z) \quad (f, g \in \mathcal{O}(\Omega)),$$

where dV is the Euclidean volume form. The Bergman kernel $K(z, w)$ of $A^2(\Omega)$ is defined as the reproducing kernel of the Hilbert space $A^2(\Omega)$, that is, for all $f \in A^2(\Omega)$ we have

$$f(z) = \int_\Omega f(w)K(z, w)\, dV(w) \quad (z \in \Omega).$$

For a positive continuous function p on Ω, let $A^2(\Omega, p)$ be the weighted Hilbert space of square integrable holomorphic functions with respect to the weight function p with the inner product

$$\langle f, g\rangle = \int_\Omega f(z)\overline{g(z)}p(z)\,dV(z) \quad (f, g \in \mathcal{O}(\Omega)).$$

Similarly, the weighted Bergman kernel $K_{A^2(\Omega,p)}$ of $A^2(\Omega, p)$ is defined as the reproducing kernel of the Hilbert space $A^2(\Omega, p)$. For a positive integer m, define the Hartogs domain $\Omega_{m,p}$ over Ω by

$$\Omega_{m,p} = \{(z, w) \in \Omega \times \mathbb{C}^m : \|w\|^2 < p(z)\}.$$

Ligocka [1985; 1989] showed that the Bergman kernel of $\Omega_{m,p}$ can be expressed as infinite sum in terms of the weighted Bergman kernel of $A^2(\Omega, p^k)$ $(k=1, 2, \dots)$ as follows.

Theorem 2.1 [Ligocka 1989]. *Let K_m be the Bergman kernel of $\Omega_{m,p}$ and let $K_{A^2(\Omega,p^k)}$ be the weighted Bergman kernel of $A^2(\Omega, p^k)$ $(k = 1, 2, \dots)$. Then*

$$K_m((z, w), (t, s)) = \frac{m!}{\pi^m}\sum_{k=0}^{\infty}\frac{(m+1)_k}{k!}K_{A^2(\Omega,p^{k+m})}(z, t)\langle w, s\rangle^k,$$

where $(a)_k$ denotes the Pochhammer symbol $(a)_k = a(a+1)\cdots(a+k-1)$.

The Fock–Bargmann space is the weighted Hilbert space $A^2(\mathbb{C}^n, e^{-\mu\|z\|^2})$ on $\mathbb{C}^n$ with the Gaussian weight function $e^{-\mu\|z\|^2}$ $(\mu > 0)$. The reproducing kernel of $A^2(\mathbb{C}^n, e^{-\mu\|z\|^2})$, called the Fock–Bargmann kernel, is $\mu^n e^{\mu\langle z,t\rangle}/\pi^n$; see [Bargmann 1967]. Thus, the Fock–Bargmann–Hartogs domain

$$D_{n,m} = \left\{(z, w) \in \mathbb{C}^n \times \mathbb{C}^m : \|w\|^2 < e^{-\mu\|z\|^2}\right\} \quad (\mu > 0)$$

and the Fock–Bargmann space $A^2(\mathbb{C}^n, e^{-\mu\|z\|^2})$ are closely related. Using the above Theorem 2.1 and the expression of the Fock–Bargmann kernel, Yamamori [2013] gave the Bergman kernel of the Fock–Bargmann–Hartogs domain $D_{n,m}$ as follows.

Theorem 2.2 [Yamamori 2013]. *The Bergman kernel of the Fock–Bargmann–Hartogs domain $D_{n,m}$ is given by*

$$K_{D_{n,m}}((z, w), (t, s)) = \frac{m!\mu^n}{\pi^{m+n}}\sum_{k=0}^{\infty}\frac{(m+1)_k(k+m)^n}{k!}e^{\mu(k+m)\langle z,t\rangle}\langle w, s\rangle^k,$$

where $(a)_k$ denotes the Pochhammer symbol $(a)_k = a(a+1)\cdots(a+k-1)$.

Following the idea of Theorem 2.1, we compute the Bergman kernel for the generalized Fock–Bargmann–Hartogs domain $D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)$. In order to compute the Bergman kernel, we first introduce some notation.

Let

$$\alpha = (\alpha_{(1)}, \dots, \alpha_{(\ell)}) \in (\mathbb{R}_+)^{n_1} \times \cdots \times (\mathbb{R}_+)^{n_\ell},$$

where $\alpha_{(i)} = (\alpha_{i1}, \dots, \alpha_{in_i}) \in (\mathbb{R}_+)^{n_i}$ for $1 \le i \le \ell$. For $\alpha \in (\mathbb{R}_+)^n$, we define

$$\beta(\alpha) = \frac{\prod_{i=1}^{n} \Gamma(\alpha_i)}{\Gamma(|\alpha|)};$$

see [D'Angelo 1994]. Here Γ is the usual Euler Gamma function.

Lemma 2.3 [D'Angelo 1994, Lemma 1]. *Suppose $\alpha \in (\mathbb{R}_+)^n$. Then we have*

$$\int_{B_+^n} r^{2\alpha-1}\, dV(r) = \frac{\beta(\alpha)}{2^n|\alpha|},$$

$$\int_{S_+^{n-1}} w^{2\alpha-1}\, d\sigma(w) = \frac{\beta(\alpha)}{2^{n-1}},$$

where dV is the Euclidean n-dimensional volume form, $d\sigma$ is the Euclidean $(n-1)$-dimensional volume form, and the subscript "+" denotes that all the variables are positive, that is, $B_+^n = B^n \cap (\mathbb{R}_+)^n$ and $S_+^{n-1} = S^{n-1} \cap (\mathbb{R}_+)^n$, in which B^n is the unit ball in $\mathbb{R}^n$ and S^{n-1} is the unit sphere in $\mathbb{R}^n$.

Theorem 2.4. *Suppose $\alpha = (\alpha_{(1)}, \dots, \alpha_{(\ell)}) \in (\mathbb{R}_+)^{n_1} \times \cdots \times (\mathbb{R}_+)^{n_\ell}$, with each $\alpha_{(i)} = (\alpha_{i1}, \dots, \alpha_{in_i}) \in (\mathbb{R}_+)^{n_i}$ $(1 \le i \le \ell)$. Then we have the formula*

$$\text{(2-1)} \quad \int_{\sum_{j=1}^{\ell} \|w_{(j)}\|^{2p_j} < t} w^\alpha \bar{w}^\alpha \, dV(w) = (\pi)^{n_1+\cdots+n_\ell} \frac{\prod_{i=1}^{\ell}\prod_{j=1}^{n_i} \Gamma(\alpha_{ij}+1) \prod_{i=1}^{\ell} \Gamma\big((|\alpha_{(i)}|+n_i)/p_i\big)}{\prod_{i=1}^{\ell} p_i \prod_{i=1}^{\ell} \Gamma(|\alpha_{(i)}|+n_i)\Gamma\Big(\sum_{i=1}^{\ell}((|\alpha_{(i)}|+n_i)/p_i)+1\Big)} \times t^{\sum_{i=1}^{\ell}(|\alpha_{(i)}|+n_i)/p_i}.$$

Proof. For the integral

$$\text{(2-2)} \qquad \int_{\sum_{j=1}^{\ell} \|w_{(j)}\|^{2p_j} < t} w^\alpha \bar{w}^\alpha \, dV(w),$$

by applying the polar coordinates $w = se^{i\theta}$ (namely, $w_{ij} = s_{ij}e^{i\theta_{ij}}$, $1 \le j \le n_i$, $1 \le i \le \ell$, $s = (s_{(1)}, \dots, s_{(\ell)})$), we have

$$\text{(2-2)} = (2\pi)^{n_1+\cdots+n_\ell} \int_{\substack{\sum_{j=1}^{\ell} \|s_{(j)}\|^{2p_j} < t \\ s_{ji}>0,\ 1\le i\le n_j,\ 1\le j\le \ell}} s^{2\alpha+1}\, dV(s).$$

Using the spherical coordinates in the variables $s_{(1)}, s_{(2)}, \dots, s_{(\ell)}$, we get

$$\begin{aligned}\int_{\substack{\sum_{j=1}^{\ell}\|s_{(j)}\|^{2p_j}<t\\ s_{ji}>0,\,1\le i\le n_j,\,1\le j\le \ell}} s^{2\alpha+1}\,dV(s) &= \int_{\substack{\sum_{i=1}^{\ell}\rho_i^{2p_i}<t\\ \rho_i>0,\,1\le i\le\ell}} \rho_1^{2|\alpha_{(1)}|+2n_1-1}\cdots\rho_\ell^{2|\alpha_{(\ell)}|+2n_\ell-1}\,d\rho_1\,d\rho_2\cdots d\rho_\ell\\ &\quad\times\int_{S_+^{n_1-1}}\cdots\int_{S_+^{n_\ell-1}} w_{(1)}^{2\alpha_{(1)}+1}\cdots w_{(\ell)}^{2\alpha_{(\ell)}+1}\,d\sigma(w_{(1)})\cdots d\sigma(w_{(\ell)}).\end{aligned}$$

Let ${\rho_i}^{p_i}=r_i$, $1\le i\le\ell$. Then we have $d\rho_i=\rho_i^{1-p_i}/p_i\,dr_i=r_i^{(1/p_i)-1}/p_i\,dr_i$. Therefore, Lemma 2.3 and the above formulas yield

$$\begin{aligned}\text{(2-2)} &= (2\pi)^{n_1+\cdots+n_\ell}\frac{1}{\prod_{i=1}^{\ell}p_i}\frac{\beta(\alpha_{(1)}+1)}{2^{n_1-1}}\cdots\frac{\beta(\alpha_{(\ell)}+1)}{2^{n_\ell-1}}\\ &\quad\times\int_{\substack{\sum_{i=1}^{\ell}|r_i|^2<t\\ r_i>0,1\le i\le\ell}} r_1^{(2|\alpha_{(1)}|+2n_1)/p_1-1}\cdots r_\ell^{(2|\alpha_{(\ell)}|+2n_\ell)/p_\ell-1}\,dr_1\cdots dr_\ell.\end{aligned}$$

Let $r=(r_1,r_2,\dots,r_\ell)\in(\mathbb{R}_+)^\ell$ and $k:=t^{-1/2}r$. Then $dr=t^{\ell/2}\,dk$. After a straightforward computation, we obtain that

$$\begin{aligned}\text{(2-2)} &= (2\pi)^{n_1+\cdots+n_\ell}\frac{1}{\prod_{i=1}^{\ell}p_i}\frac{\beta(\alpha_{(1)}+1)}{2^{n_1-1}}\cdots\frac{\beta(\alpha_{(\ell)}+1)}{2^{n_\ell-1}}\cdot t^{\sum_{i=1}^{\ell}(|\alpha_{(i)}|+n_i)/p_i}\\ &\quad\times\int_{B_+^{\ell}} k_1^{(2|\alpha_{(1)}|+2n_1)/p_1-1}\cdots k_\ell^{(2|\alpha_{(\ell)}|+2n_\ell)/p_\ell-1}\,dk_1\cdots dk_\ell.\end{aligned}$$

Applying Lemma 2.3 to the above formula, we get

$$\begin{aligned}\text{(2-3)}\quad \text{(2-2)} &= (\pi)^{n_1+\cdots+n_\ell}\beta(\alpha_{(1)}+1)\cdots\beta(\alpha_{(\ell)}+1)\frac{\beta(\alpha')}{|\alpha'|\prod_{i=1}^{\ell}p_i}\cdot t^{\sum_{i=1}^{\ell}(|\alpha_{(i)}|+n_i)/p_i}\\ &= (\pi)^{n_1+\cdots+n_\ell}\frac{1}{\prod_{i=1}^{\ell}p_i}\frac{\prod_{i=1}^{\ell}\prod_{j=1}^{n_i}\Gamma(\alpha_{ij}+1)\prod_{i=1}^{\ell}\Gamma((|\alpha_{(i)}|+n_i)/p_i)}{\prod_{i=1}^{\ell}\Gamma(|\alpha_{(i)}|+n_i)\Gamma\Big(\sum_{i=1}^{\ell}(|\alpha_{(i)}|+n_i)/p_i+1\Big)}\\ &\qquad\times t^{\sum_{i=1}^{\ell}(|\alpha_{(i)}|+n_i)/p_i},\end{aligned}$$

where $\alpha'=((|\alpha_{(1)}|+n_1)/p_1,\dots,(|\alpha_{(\ell)}|+n_\ell)/p_\ell)\in(\mathbb{R}_+)^\ell$. □

Now we consider the Hilbert space $A^2(D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu))$ of square-integrable holomorphic functions on $D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)$.

Lemma 2.5. *Let* $f \in A^2(D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu))$. *Then*

$$f(z, w) = \sum_{\alpha} f_\alpha(z) w^\alpha,$$

where the series is uniformly convergent on compact subsets of the domain $D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)$, $f_\alpha(z) \in A^2(\mathbb{C}^{n_0}, e^{-\mu\lambda_\alpha\|z\|^2})$ *for any* $\alpha = (\alpha_{(1)}, \dots, \alpha_{(\ell)}) \in \mathbb{N}^{n_1} \times \cdots \times \mathbb{N}^{n_\ell}$ *with* $\alpha_{(i)} = (\alpha_{i1}, \dots, \alpha_{in_i}) \in \mathbb{N}^{n_i}$ $(1 \le i \le \ell)$ *and* $\lambda_\alpha = \sum_{i=1}^{\ell} (|\alpha_{(i)}| + n_i)/p_i$, *in which* $A^2(\mathbb{C}^n, e^{-\mu\lambda_\alpha\|z\|^2})$ *denotes the space of square-integrable holomorphic functions on* $\mathbb{C}^n$ *with respect to the measure* $e^{-\mu\lambda_\alpha\|z\|^2}\, dV_{2n}$.

Proof. Since $D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)$ is a complete Reinhardt domain, each holomorphic function on $D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)$ is the sum of a locally uniformly convergent power series. Thus, for $f \in A^2(D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu))$, we have

$$f(z, w) = \sum_{\alpha} f_\alpha(z) w^\alpha,$$

where the series is uniformly convergent on compact subsets of $D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)$. We choose a sequence of compact subsets D_k $(1 \le k < \infty)$

$$D_k := \left\{ (z, w_{(1)}, \dots, w_{(\ell)}) \in \mathbb{C}^{n_0} \times \mathbb{C}^{n_1} \times \cdots \times \mathbb{C}^{n_\ell} : \sum_{j=1}^{\ell} \|w_{(j)}\|^{2p_j} \le e^{-\mu\|z\|^2} - \frac{1}{k} \right\} \cap \overline{B(0, k)},$$

where $B(0, k)$ is the ball in $\mathbb{C}^{n_0+n_1+\cdots+n_\ell}$ of the radius k. Obviously, $D_k \Subset D_{k+1}$ and $\bigcup_{k=1}^{\infty} D_k = D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)$. Since D_k is a circular domain,

$$f_\alpha(z) w^\alpha \perp f_\beta(z) w^\beta \quad (\alpha \ne \beta)$$

in the Hilbert space $A^2(D_k)$. Hence we have

$$\|f\|^2_{L^2(D_k)} = \sum_{|\alpha|=0}^{\infty} \|f_\alpha(z) w^\alpha\|^2_{L^2(D_k)}.$$

Since $f(z, w) \in A^2(D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu))$, we have

$$\|f_\alpha(z) w^\alpha\|^2_{L^2(D_k)} \le \|f\|^2_{L^2(D_k)} \le \|f\|^2_{L^2(D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu))}.$$

Then $f_\alpha(z) w^\alpha \in A^2(D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu))$. Therefore,

$$\int_{D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)} |f_\alpha(z)|^2 w^\alpha \bar{w}^\alpha \, dV < \infty \\ \Longrightarrow \int_{\mathbb{C}^{n_0}} |f_\alpha(z)|^2 \, dV(z) \int_{\sum_{j=1}^{\ell} \|w_{(j)}\|^{2p_j} < e^{-\mu\|z\|^2}} w^\alpha \bar{w}^\alpha \, dV(w) < \infty.$$

By (2-1), it follows that

$$\int_{\mathbb{C}^{n_0}} |f_\alpha(z)|^2 e^{-\mu\lambda_\alpha \|z\|^2}\, dV(z) < \infty.$$

Consequently, $f_\alpha(z) \in A^2(\mathbb{C}^{n_0}, e^{-\mu\lambda_\alpha\|z\|^2})$, where $\lambda_\alpha = \sum_{i=1}^{\ell}(|\alpha_{(i)}| + n_i)/p_i$. □

Lemma 2.5 implies that $\{f(z)w^\alpha : f(z) \in A^2(\mathbb{C}^{n_0}, e^{-\mu\lambda_\alpha\|z\|^2})\}$ forms a linearly dense subset of $A^2(D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu))$. Now we can express the Bergman kernel of $D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)$.

Theorem 2.6. *The Bergman kernel of $D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)$ can be expressed by the form*

$$K_{D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)}[(z,w),(s,t)] = \sum_{|\alpha|=0}^{\infty} c_\alpha \frac{\lambda_\alpha^{n_0}\mu^{n_0}}{\pi^{n_0}} e^{\lambda_\alpha \mu\langle z,s\rangle} w^\alpha \bar{t}^\alpha, \tag{2-4}$$

where $\alpha = (\alpha_{(1)}, \dots, \alpha_{(\ell)}) \in \mathbb{N}^{n_1} \times \cdots \times \mathbb{N}^{n_\ell}$, $\alpha_{(i)} = (\alpha_{i1}, \dots, \alpha_{in_i}) \in \mathbb{N}^{n_i}$, $1 \le i \le \ell$, and

$$c_\alpha = \frac{\prod_{i=1}^{\ell} p_i \prod_{i=1}^{\ell} \Gamma(|\alpha_{(i)}| + n_i) \Gamma\Big(\sum_{i=1}^{\ell}(|\alpha_{(i)}| + n_i)/p_i + 1\Big)}{(\pi)^{n_1+\cdots+n_\ell} \prod_{i=1}^{\ell}\prod_{j=1}^{n_i} \Gamma(\alpha_{ij}+1) \prod_{i=1}^{\ell} \Gamma((|\alpha_{(i)}| + n_i)/p_i)}, \quad \lambda_\alpha = \sum_{i=1}^{\ell} \frac{|\alpha_{(i)}| + n_i}{p_i}.$$

Proof. Since $D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)$ is a complete Reinhardt domain, it follows that

$$K_{D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)}[(z,w),(s,t)] = \sum_{|\beta|=0}^{\infty} c_\beta g_\beta(z,s) w^\beta \bar{t}^\beta,$$

where the sum is locally uniformly convergent, by the invariance of the Bergman kernel $K_{D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)}$ on $D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)$ under the unitary subgroup action

$$(z_1, \dots, z_{n_0+|\boldsymbol{n}|}) \to \big(e^{\sqrt{-1}\theta_1} z_1, \dots, e^{\sqrt{-1}\theta_{n_0+|\boldsymbol{n}|}} z_{n_0+|\boldsymbol{n}|}\big) \quad (\theta_1, \dots, \theta_{n_0+|\boldsymbol{n}|} \in \mathbb{R}).$$

For any $\alpha = (\alpha_{(1)}, \dots, \alpha_{(\ell)}) \in \mathbb{N}^{n_1} \times \cdots \times \mathbb{N}^{n_\ell}$ with $\alpha_{(i)} = (\alpha_{i1}, \dots, \alpha_{in_i}) \in \mathbb{N}^{n_i}$ $(1 \le i \le \ell)$, and any $f(z) \in A^2(\mathbb{C}^{n_0}, e^{-\mu\lambda_\alpha\|z\|^2})$ for $\lambda_\alpha = \sum_{i=1}^{\ell}(|\alpha_{(i)}| + n_i)/p_i$, we have $f(z)w^\alpha \in A^2(D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu))$. Thus

$$\begin{aligned} f(z)w^\alpha &= \int_{D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)} f(s)t^\alpha K_{D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)}[(z,w),(s,t)]\, dV \\ &= \int_{\mathbb{C}^{n_0}} f(s) \sum_{\beta=0}^{\infty} c_\beta g_\beta(z,s) w^\beta\, dV(s) \int_{\sum_{j=1}^{\ell}\|t_{(j)}\|^{2p_j} < e^{-\mu\|s\|^2}} t^\alpha \bar{t}^\beta\, dV(t) \\ &= w^\alpha \int_{\mathbb{C}^{n_0}} f(s) g_\alpha(z,s) \big[e^{-\mu\|s\|^2}\big]^{\sum_{i=1}^{\ell}(|\alpha_{(i)}|+n_i)/p_i}\, dV(s) \quad \text{(by (2-1))}. \end{aligned}$$

By [Bargmann 1967], we get that the Bergman kernel of $A^2(\mathbb{C}^{n_0}, e^{-\mu\lambda_\alpha\|z\|^2})$ can be described by the form

$$K_\alpha(z, w) = \frac{\lambda_\alpha^{n_0} \mu^{n_0}}{\pi^{n_0}} e^{\lambda_\alpha \mu \langle z, w \rangle}. \tag{2-5}$$

Thus we obtain

$$g_\alpha(z, s) = \frac{\lambda_\alpha^{n_0} \mu^{n_0}}{\pi^{n_0}} e^{\lambda_\alpha \mu \langle z, s \rangle}.$$

This completes the proof. □

The transformation rule for Bergman kernels under proper holomorphic mapping (e.g., Theorem 1 in [Bell 1982]) is also valid for unbounded domains (e.g., see Corollary 1 in [Trybuła 2013]). Note that the coordinate functions play a key role in the approach of [Bell 1982] to extend proper holomorphic mapping, but, in general, are no longer square integrable on unbounded domains. In order to overcome this difficulty, by combining the transformation rule for Bergman kernels under proper holomorphic mapping in [Bell 1982] and our explicit form (2-4) of the Bergman kernel function for $D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)$, we prove that a proper holomorphic mapping between two equidimensional generalized Fock–Bargmann–Hartogs domains extends holomorphically to their closures as follows.

Lemma 2.7. *Suppose that $f : D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu) \to D_{m_0}^{\boldsymbol{m},\boldsymbol{q}}(\nu)$ is a proper holomorphic mapping between two equidimensional generalized Fock–Bargmann–Hartogs domains. Then f extends holomorphically to a neighborhood of the closure of $D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)$.*

In fact, using the explicit form (2-4) of the Bergman kernel function for $D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)$, we immediately have Lemma 2.7 by a slight modification of the proof of Theorem 2.5 in [Tu and Wang 2014].

Cartan's theorem on $D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\boldsymbol{\mu})$. Suppose D is a domain in $\mathbb{C}^N$ and let $K_D(z, w)$ be its Bergman kernel. From [Ishi and Kai 2010], we know that if the conditions

(a) $K_D(0, 0) > 0$,

(b) $T_D(0, 0)$ is positive definite,

are satisfied, where T_D is an $N \times N$ matrix

$$T_D(z, w) := \begin{pmatrix} \partial^2 \log K_D(z, w)/\partial z_1 \partial \overline{w_1} & \cdots & \partial^2 \log K_D(z, w)/\partial z_1 \partial \overline{w_N} \\ \vdots & \ddots & \vdots \\ \partial^2 \log K_D(z, w)/\partial z_N \partial \overline{w_1} & \cdots & \partial^2 \log K_D(z, w)/\partial z_N \partial \overline{w_N} \end{pmatrix}.$$

Then Cartan's theorem can also be applied to the case of unbounded circular domains. The above conditions are obviously satisfied by the bounded domain.

Kim, Ninh and Yamamori [Kim et al. 2014] proved the following result.

Lemma 2.8 [Kim et al. 2014, Theorem 4]. *Suppose that D is a circular domain and its Bergman kernel satisfies the above conditions* (a) *and* (b). *If $\varphi \in \operatorname{Aut}(D)$ preserves the origin*, *then φ is a linear mapping.*

Ishi and Kai [2010] proved the following generalization of Lemma 2.8.

Lemma 2.9 [Ishi and Kai 2010, Proposition 2.1]. *Let D_k be a circular domain (not necessarily bounded) in $\mathbb{C}^N$ with $0 \in D_k$ $(k = 1, 2)$, and let $\varphi : D_1 \to D_2$ be a biholomorphism with $\varphi(0) = 0$. If $K_{D_k}(0, 0) > 0$ and $T_{D_k}(0, 0)$ is positive definite $(k = 1, 2)$, then φ is linear.*

Therefore, by using the expressions of Bergman kernels of generalized Fock–Bargmann–Hartogs domains, we have the following result.

Theorem 2.10. *Suppose that $\varphi : D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu) \to D_{m_0}^{\boldsymbol{m},\boldsymbol{q}}(\nu)$ be a biholomorphic mapping between two equidimensional generalized Fock–Bargmann–Hartogs domains with $\varphi(0) = 0$. Then φ is linear.*

Proof. By using the expressions (2-4) of Bergman kernels of generalized Fock–Bargmann–Hartogs domains and a straightforward computation, we show that the Bergman kernel of every generalized Fock–Bargmann–Hartogs domain satisfies the above conditions (a) and (b). So we get Theorem 2.10 by Lemma 2.9. □

3. Proof of the main theorem

To begin, we exploit the boundary structure of $D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)$, which is comprised of

$$bD_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu) = b_0 D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu) \cup b_1 D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu) \cup b_2 D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu),$$

where

$$b_0 D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu) := \Big\{(z, w_{(1)}, \dots, w_{(\ell)}) \in \mathbb{C}^{n_0} \times \cdots \times \mathbb{C}^{n_\ell} : \sum_{j=1}^{\ell} \|w_{(j)}\|^{2p_j} = e^{-\mu\|z\|^2},\ \|w_{(j)}\|^2 \neq 0,\ 1+\epsilon \le j \le \ell\Big\},$$

$$b_1 D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu) := \bigcup_{j=1+\epsilon}^{\ell} \Big\{(z, w_{(1)}, \dots, w_{(\ell)}) \in \mathbb{C}^{n_0} \times \cdots \times \mathbb{C}^{n_\ell} : \sum_{j=1}^{\ell} \|w_{(j)}\|^{2p_j} = e^{-\mu\|z\|^2},\ \|w_{(j)}\|^2 = 0,\ p_j > 1\Big\},$$

$$b_2 D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu) := \bigcup_{j=1+\epsilon}^{\ell} \Big\{(z, w_{(1)}, \dots, w_{(\ell)}) \in \mathbb{C}^{n_0} \times \cdots \times \mathbb{C}^{n_\ell} : \sum_{j=1}^{\ell} \|w_{(j)}\|^{2p_j} = e^{-\mu\|z\|^2},\ \|w_{(j)}\|^2 = 0,\ p_j < 1\Big\}.$$

Proposition 3.1. (1) *The boundary $b_0 D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)$ is a real analytic hypersurface in $\mathbb{C}^{n_0+n_1+\cdots+n_\ell}$ and $D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)$ is strongly pseudoconvex at all points of $b_0 D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)$.*

(2) *$D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)$ is weakly pseudoconvex but not strongly pseudoconvex at any point of $b_1 D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)$ and is not smooth at any point of $b_2 D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)$.*

Proof. Let

$$\rho(z, w_{(1)}, \dots, w_{(\ell)}) := \sum_{j=1}^{\ell} \|w_{(j)}\|^{2p_j} - e^{-\mu\|z\|^2}.$$

Then ρ is a real analytic definition function of $b_0 D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)$. Fix a point

$$(z_0, w_{(1)0}, \dots, w_{(\ell)0}) \in b_0 D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)$$

and let $T = (\zeta, \eta_{(1)}, \dots, \eta_{(\ell)}) \in T^{1,0}_{(z_0, w_{(1)0}, \dots, w_{(\ell)0})}(b_0 D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu))$. Then by definition, we know that

$$w_{(j)0} \neq 0 \quad (j = 1+\epsilon, \dots, \ell), \tag{3-1}$$

$$\sum_{k=1}^{\ell} p_k \|w_{(k)0}\|^{2(p_k-1)} \overline{w_{(k)0}} \cdot \eta_{(k)} + \mu e^{-\mu\|z_0\|^2} \overline{z_0} \cdot \zeta = 0, \tag{3-2}$$

$$\sum_{j=1}^{\ell} \|w_{(j)0}\|^{2p_j} - e^{-\mu\|z_0\|^2} = 0. \tag{3-3}$$

Thanks to (3-1), (3-2) and (3-3), the Levi form of ρ at the point $(z_0, w_{(1)0}, \dots, w_{(\ell)0})$ can be computed as follows:

$$\begin{aligned}
&L_\rho(T, T)\\
&\quad := \sum_{i,j=1}^{n_0+n_1+\dots+n_\ell} \frac{\partial^2 \rho}{\partial T_i \partial \overline{T_j}}(z_0, w_{(1)0}, \dots, w_{(\ell)0}) T_i \overline{T_j}\\
&\quad = \sum_{k=1}^{\ell} p_k(p_k-1)\|w_{(k)0}\|^{2(p_k-2)} |\overline{w_{(k)0}} \cdot \eta_{(k)}|^2 + \sum_{k=1}^{\ell} p_k \|w_{(k)0}\|^{2(p_k-1)} \|\eta_{(k)}\|^2\\
&\qquad\qquad + \mu e^{-\mu\|z_0\|^2}\|\zeta\|^2 - \mu^2 e^{-\mu\|z_0\|^2} |\overline{z_0} \cdot \zeta|^2\\
&\quad = \sum_{k=1}^{\ell} {p_k}^2 \|w_{(k)0}\|^{2(p_k-2)} |\overline{w_{(k)0}} \cdot \eta_{(k)}|^2 + \mu e^{-\mu\|z_0\|^2}\|\zeta\|^2 - \mu^2 e^{-\mu\|z_0\|^2} |\overline{z_0} \cdot \zeta|^2\\
&\qquad\qquad + \sum_{k=1}^{\ell} p_k \|w_{(k)0}\|^{2(p_k-2)} \big(\|w_{(k)0}\|^2 \|\eta_{(k)}\|^2 - |\overline{w_{(k)0}} \cdot \eta_{(k)}|^2\big)\\
&\quad = \Big(\sum_{k=1}^{\ell} \|w_{(k)0}\|^{2p_k}\Big)^{-1} \Big(\sum_{k=1}^{\ell} {p_k}^2 \|w_{(k)0}\|^{2(p_k-2)} |\overline{w_{(k)0}} \cdot \eta_{(k)}|^2\Big) \Big(\sum_{k=1}^{\ell} \|w_{(k)0}\|^{2p_k}\Big)\\
&\qquad\qquad - \Big(\sum_{k=1}^{\ell} \|w_{(k)0}\|^{2p_k}\Big)^{-1} \Big|\sum_{k=1}^{\ell} p_k \|w_{(k)0}\|^{2(p_k-1)} \overline{w_{(k)0}} \cdot \eta_{(k)}\Big|^2\\
&\qquad + \sum_{k=1}^{\ell} p_k \|w_{(k)0}\|^{2(p_k-2)} \big(\|w_{(k)0}\|^2 \|\eta_{(k)}\|^2 - |\overline{w_{(k)0}} \cdot \eta_{(k)}|^2\big) + \mu e^{-\mu\|z_0\|^2}\|\zeta\|^2
\end{aligned}$$

$$
\begin{aligned}
&= \Big(\sum_{k=1}^{\ell}\|w_{(k)0}\|^{2p_k}\Big)^{-1}\bigg[\Big(\sum_{k=1}^{\ell} {p_k}^2\|w_{(k)0}\|^{2(p_k-2)}|\overline{w_{(k)0}}\cdot\eta_{(k)}|^2\Big)\Big(\sum_{k=1}^{\ell}\|w_{(k)0}\|^{2p_k}\Big)\\
&\qquad - \Big|\sum_{k=1}^{\ell} p_k\|w_{(k)0}\|^{2(p_k-1)}\overline{w_{(k)0}}\cdot\eta_{(k)}\Big|^2\bigg] + \mu e^{-\mu\|z_0\|^2}\|\zeta\|^2\\
&\qquad\qquad + \sum_{k=1}^{\ell} p_k\|w_{(k)0}\|^{2(p_k-2)}\big(\|w_{(k)0}\|^2\|\eta_{(k)}\|^2 - |\overline{w_{(k)0}}\cdot\eta_{(k)}|^2\big)\\
&\geq \mu e^{-\mu\|z_0\|^2}\|\zeta\|^2 \geq 0.
\end{aligned}
$$

by the Cauchy–Schwarz inequality, for all

$$T = (\zeta, \eta_{(1)}, \ldots, \eta_{(\ell)}) \in T^{1,0}_{(z_0, w_{(1)0}, \ldots, w_{(\ell)0})}(b_0 D^{\boldsymbol{n},\boldsymbol{p}}_{n_0}(\mu)).$$

Obviously, if $\zeta \neq 0$, then $L_\rho(T, T) > 0$.

On the other hand, combining with (3-1), (3-2) and (3-3), we know that the equality holds if and only if

$$\zeta = 0, \tag{3-4}$$

$$\|w_{(k)0}\|^2\|\eta_{(k)}\|^2 - |\overline{w_{(k)0}}\cdot\eta_{(k)}|^2 = 0, \tag{3-5}$$

$$
\begin{aligned}
\bigg[\Big(\sum_{k=1}^{\ell} {p_k}^2\|w_{(k)0}\|^{2(p_k-2)}|\overline{w_{(k)0}}\cdot\eta_{(k)}|^2\Big)\Big(\sum_{k=1}^{\ell}\|w_{(k)0}\|^{2p_k}\Big)&\\
- \Big|\sum_{k=1}^{\ell} p_k\|w_{(k)0}\|^{2(p_k-1)}\overline{w_{(k)0}}\cdot\eta_{(k)}\Big|^2\bigg] &= 0.
\end{aligned}
\tag{3-6}
$$

Suppose $\zeta = 0$. Then $T = (\zeta, \eta_{(1)}, \ldots, \eta_{(\ell)}) \neq 0$ implies that there exists $\eta_{(i_0)} \neq 0$. If $L_\rho(T, T) = 0$ for all

$$T \neq 0 \in T^{1,0}_{(z_0, w_{(1)0}, \ldots, w_{(\ell)0})}(b_0 D^{\boldsymbol{n},\boldsymbol{p}}_{n_0}(\mu)),$$

then by (3-1), (3-2), (3-3) and (3-6), we have $\eta_{(k)} = 0$ $(1 \leq k \leq \ell)$. This is a contradiction.

When there exists $j_0 \geq 1 + \epsilon$ such that $\|w_{(j_0)0}\|^2 = 0$ and $p_{j_0} > 1$, then $(z_0, w_{(1)0}, \ldots, w_{(\ell)0}) \in b_1 D^{\boldsymbol{n},\boldsymbol{p}}_{n_0}(\mu)$. Let $T_0 = (0, \ldots, \eta_{(j_0)}, 0, \ldots, 0)$, $\|\eta_{(j_0)}\| \neq 0$. Then $L_\rho(T_0, T_0) = 0$. Hence $D^{\boldsymbol{n},\boldsymbol{p}}_{n_0}(\mu)$ is weakly pseudoconvex but not strongly pseudoconvex on any point of $b_1 D^{\boldsymbol{n},\boldsymbol{p}}_{n_0}(\mu)$.

It is obvious that $D^{\boldsymbol{n},\boldsymbol{p}}_{n_0}(\mu)$ is not smooth at any point of $b_2 D^{\boldsymbol{n},\boldsymbol{p}}_{n_0}(\mu)$. The proof is completed. □

Lemma 3.1 [Tu and Wang 2015]. *Let $\Sigma(\boldsymbol{n}; \boldsymbol{p})$ and $\Sigma(\boldsymbol{m}; \boldsymbol{q})$ be two equidimensional generalized pseudoellipsoids, $\boldsymbol{n}, \boldsymbol{m} \in \mathbb{N}^\ell$, $\boldsymbol{p}, \boldsymbol{q} \in (\mathbb{R}_+)^\ell$ (where $p_k, q_k \neq 1$ for $2 \leq k \leq \ell$). Let $h : \Sigma(\boldsymbol{n}; \boldsymbol{p}) \to \Sigma(\boldsymbol{m}; \boldsymbol{q})$ be a biholomorphic linear isomorphism between $\Sigma(\boldsymbol{n}; \boldsymbol{p})$ and $\Sigma(\boldsymbol{m}; \boldsymbol{q})$. Then there exists a permutation $\sigma \in S_r$ such that*

$n_{\sigma(i)} = m_i$, $p_{\sigma(i)} = q_i$ *and*

$$h(\zeta_1, \dots, \zeta_r) = (\zeta_{\sigma(1)}, \dots, \zeta_{\sigma(r)}) \begin{pmatrix} U_1 & & & \\ & U_2 & & \\ & & \ddots & \\ & & & U_r \end{pmatrix},$$

where U_i *is a unitary transformation of* $\mathbb{C}^{m_i}$ $(m_i = n_{\sigma(i)})$ *for* $1 \le i \le r$.

Define

$$V_1 := \{(z, w_{(1)}, \dots, w_{(\ell)}) \in \mathbb{C}^{n_0} \times \mathbb{C}^{n_1} \times \cdots \times \mathbb{C}^{n_\ell} : w_{(1)} = \cdots = w_{(\ell)} = 0\},$$
$$V_2 := \{(z, w_{(1)}, \dots, w_{(\ell)}) \in \mathbb{C}^{m_0} \times \mathbb{C}^{m_1} \times \cdots \times \mathbb{C}^{m_\ell} : w_{(1)} = \cdots = w_{(\ell)} = 0\}$$

(so $V_1 \cong \mathbb{C}^{n_0}$ and $V_2 \cong \mathbb{C}^{m_0}$). Then we have the following lemma.

Lemma 3.2. *Suppose* $D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)$ *and* $D_{m_0}^{\boldsymbol{m},\boldsymbol{q}}(\nu)$ *are two equidimensional generalized Fock–Bargmann–Hartogs domains, and* $f : D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu) \to D_{m_0}^{\boldsymbol{m},\boldsymbol{q}}(\nu)$ *is a biholomorphic mapping. Then we have* $f(V_1) \subseteq V_2$ *and* $f|_{V_1} : V_1 \to V_2$ *is biholomorphic, and consequently,* $n_0 = m_0$.

Proof. Let $f(z, 0) = (f_1(z), f_2(z))$. Then we get

$$\sum_{i=1}^{\ell} \|f_{2i}\|^{2q_i} < e^{-\nu\|f_1(z)\|^2} \le 1.$$

Then we obtain that the bounded entire mapping $f_{2i}(z)$ on $\mathbb{C}^{n_0}$ is constant $(1 \le i \le \ell)$ by Liouville's theorem. Since $f(z)$ is biholomorphic, $f_1(z)$ is an unbounded function. Hence there exist $\{z_k\}$ such that $f_1(z_k) \to \infty$ as $k \to \infty$. It implies $f_2(z) \equiv 0$. This proves $f(V_1) \subseteq V_2$. Similarly, by making the same argument for f^{-1}, we have $f^{-1}(V_2) \subseteq V_1$. Namely, $f|_{V_1} : V_1 \to V_2$ is biholomorphic. Hence $n_0 = m_0$. □

Now we give the proof of Theorem 1.4.

Proof of Theorem 1.4. Let $f(0, 0) = (a, b)$ (thus $b = 0$ by Lemma 3.2) and define

$$\phi(z, w_{(1)}, \dots, w_{(\ell)}) := \bigl(z - a, w_{(1)}\bigl(e^{2\nu\langle z,a\rangle - \nu\|a\|^2}\bigr)^{1/2q_1}, \dots, w_{(\ell)}\bigl(e^{2\nu\langle z,a\rangle - \nu\|a\|^2}\bigr)^{1/2q_\ell}\bigr).$$

Obviously, $\phi \in \mathrm{Aut}(D_{m_0}^{\boldsymbol{m},\boldsymbol{q}}(\nu))$ and $\phi \circ f(0, 0) = (0, 0)$. Then $\phi \circ f$ is linear by Theorem 2.10. We describe $\phi \circ f$ as follows:

$$\phi \circ f(z, w) = (z, w)\begin{pmatrix} A & B \\ C & D \end{pmatrix} = (zA + wC, zB + wD).$$

According to Lemma 3.2, we have $f(z, 0) = (f_1(z), 0)$. Thus $B = 0$. Since $g := \phi \circ f$ is biholomorphic, A and D are invertible matrices. We write $g(z, w)$ as

$$g(z,w)=(z,w)\begin{pmatrix}A & 0\\ C & D\end{pmatrix}=(z,w_{(1)},\dots,w_{(\ell)})\begin{pmatrix}A & 0 & \cdots & 0\\ C_{11} & D_{11} & \cdots & D_{1\ell}\\ \vdots & \vdots & \ddots & \cdots\\ C_{\ell 1} & D_{\ell 1} & \cdots & D_{\ell\ell}\end{pmatrix},$$

which implies that

$$\begin{aligned}g^{-1}(z,w)&=(z,w)\begin{pmatrix}A^{-1} & 0\\ -D^{-1}CA^{-1} & D^{-1}\end{pmatrix}\\ &=(z,w_{(1)},\dots,w_{(\ell)})\begin{pmatrix}A^{-1} & 0 & \cdots & 0\\ E_{11} & G_{11} & \cdots & G_{1\ell}\\ \vdots & \vdots & \ddots & \vdots\\ E_{\ell 1} & G_{\ell 1} & \cdots & G_{\ell\ell}\end{pmatrix}.\end{aligned}$$

Set

$$\Sigma(\boldsymbol{n};\boldsymbol{p})=\Big\{(w_{(1)},\dots,w_{(\ell)})\in\mathbb{C}^{n_1}\times\cdots\times\mathbb{C}^{n_\ell}:\sum_{j=1}^{\ell}\|w_{(j)}\|^{2p_j}<1\Big\}.$$

Then, if $\sum_{j=1}^{\ell}\|w_{(j)}\|^{2p_j}<e^{-\mu\|0\|^2}=1$, we obtain

$$\sum_{j=1}^{\ell}\|w_{(1)}D_{1j}+\cdots+w_{(\ell)}D_{\ell j}\|^{2q_j}<e^{-\nu\|wC\|^2}<1,$$

and if $\sum_{j=1}^{\ell}\|w_{(j)}\|^{2q_j}<e^{-\nu\|0\|^2}=1$, we have

$$\sum_{j=1}^{\ell}\|w_{(1)}G_{1j}+\cdots+w_{(\ell)}G_{\ell j}\|^{2p_j}<e^{-\mu\|w(-D^{-1}CA^{-1})\|^2}<1.$$

Therefore, we conclude that the mapping $g_2(w):\Sigma(\boldsymbol{n};\boldsymbol{p})\to\Sigma(\boldsymbol{m};\boldsymbol{q})$ given by

$$g_2(w_{(1)},\dots,w_{(\ell)})=wD=(w_{(1)},\dots,w_{(\ell)})\begin{pmatrix}D_{11} & \cdots & D_{1\ell}\\ \vdots & \ddots & \vdots\\ D_{\ell 1} & \cdots & D_{\ell\ell}\end{pmatrix}$$

is a biholomorphic linear mapping. By Lemma 3.1, g_2 can be expressed in the form

$$g_2(w_{(1)},\dots,w_{(\ell)})=(w_{(\sigma(1))},\dots,w_{(\sigma(\ell))})\begin{pmatrix}\Gamma_1 & & & \\ & \Gamma_2 & & \\ & & \ddots & \\ & & & \Gamma_\ell\end{pmatrix},$$

where $\sigma\in S_\ell$ is a permutation with $n_{\sigma(j)}=m_j,\ p_{\sigma(j)}=q_j\ (j=1,\dots,\ell)$ and $\Gamma_i\in\mathcal{U}(m_i)\ (1\le i\le\ell)$. Hence g can be rewritten as follows:

$$g(z,w)=(z,w)\begin{pmatrix}A&0\\C&D\end{pmatrix}=(z,w_{(\sigma(1))},\dots,w_{(\sigma(\ell))})\begin{pmatrix}A&&&&\\C_{\sigma(1)1}&\Gamma_1&&&\\C_{\sigma(2)1}&&\Gamma_2&&\\\vdots&&&\ddots&\\C_{\sigma(\ell)1}&&&&\Gamma_\ell\end{pmatrix}.$$

Next we prove that $C=0$. The linearity of g yields $g(bD_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu))=bD_{m_0}^{\boldsymbol{m},\boldsymbol{q}}(\nu)$. Let $(0,w)=(0,0,\dots,w_{(j)},0,\dots,0)\in bD_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)$, namely, $\|w_{(j)}\|^2=(e^{-\mu\|0\|^2})^{1/p_j}=1$. As Γ_j $(1\le j\le\ell)$ are unitary matrices, moreover, assuming $\sigma(i_0)=j$, we have

$$\|w_{(j)}\|^{2p_j}=\|w_{(\sigma(i_0))}\Gamma_{i_0}\|^{2q_{i_0}}=e^{-\nu\|w_{(\sigma(i_0))}C_{\sigma(i_0)1}\|^2}=1.$$

This implies $w_{(j)}C_{j1}=0$ for all $\|w_{(j)}\|^2=1$. So $C_{j1}=0$ $(1\le j\le\ell)$. Thus we have

$$g(z,w_{(1)},\dots,w_{(\ell)})=(z,w_{(\sigma(1))},\dots,w_{(\sigma(\ell))})\begin{pmatrix}A&&&&\\&\Gamma_1&&&\\&&\Gamma_2&&\\&&&\ddots&\\&&&&\Gamma_\ell\end{pmatrix}.$$

Lastly, we show $\sqrt{\nu/\mu}\,A\in\mathcal{U}(n)$ $(n:=n_0=m_0)$. For $z\in\mathbb{C}^{n_0}$, take $(w_{(1)},\dots,w_{(\ell)})$ such that $e^{-\mu\|z\|^2}=\sum_{j=1}^{\ell}\|w_{(j)}\|^{2p_j}$. By $g(bD_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu))=bD_{m_0}^{\boldsymbol{m},\boldsymbol{q}}(\nu)$, we have

$$\sum_{j=1}^{\ell}\|w_{(\sigma(j))}\Gamma_j\|^{2q_j}=e^{-\nu\|zA\|^2}.$$

Since $\Gamma_j (j=1,\dots,\ell)$ are unitary matrices, we get

$$e^{-\mu\|z\|^2}=\sum_{j=1}^{\ell}\|w_{(\sigma(j))}\|^{2p_{\sigma(j)}}=\sum_{j=1}^{\ell}\|w_{(\sigma(j))}\Gamma_j\|^{2q_j}=e^{-\nu\|zA\|^2}.$$

Therefore, $\nu\|zA\|^2=\mu\|z\|^2$ $(z\in\mathbb{C}^n)$. Then we get $\sqrt{\nu/\mu}\,A\in\mathcal{U}(n)$, and the proof is completed. □

Proof of Corollary 1.5. In fact, the significance of the above ϕ is just to ensure that $\phi\circ f(0)=0$. Then the proof of Theorem 1.4 implies that Corollary 1.5 is obvious. □

Proof of Theorem 1.6. Obviously, φ_A, φ_D and φ_a are biholomorphic self-mappings of $D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)$. On the other hand, for $\varphi\in\mathrm{Aut}(D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu))$, we assume $\varphi(0,0)=(a,b)$ (then $b=0$ by Lemma 3.2). Hence $\varphi_{-a}\circ\varphi$ preserves the origin. Then by Corollary 1.5, we obtain $\varphi_{-a}\circ\varphi=\varphi_D\circ\varphi_A$ for some φ_A, φ_D. Hence $\varphi=\varphi_a\circ\varphi_D\circ\varphi_A$, and the proof is complete. □

Proof of Theorem 1.7. Let f be a proper holomorphic mapping between two equidimensional generalized Fock–Bargmann–Hartogs domains $D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)$ and $D_{m_0}^{\boldsymbol{m},\boldsymbol{q}}(\nu)$. Then by Lemma 2.7, f extends holomorphically to a neighborhood Ω of the closure $\overline{D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)}$ with

$$f(bD_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)) \subset bD_{m_0}^{\boldsymbol{m},\boldsymbol{q}}(\nu).$$

Then by Proposition 3.1 and Lemma 1.3 in [Pinchuk 1975], we have

$$f(M \cap b_0 D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)) \subset b_1 D_{m_0}^{\boldsymbol{m},\boldsymbol{q}}(\nu) \cup b_2 D_{m_0}^{\boldsymbol{m},\boldsymbol{q}}(\nu), \tag{3-7}$$

where $M := \{z \in \Omega : \det(\partial f_i/\partial z_j) = 0\}$ is the zero locus of the complex Jacobian of the holomorphic mapping f on Ω.

If $M \cap bD_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu) \neq \varnothing$, then from

$$\min\{n_{1+\epsilon}, n_2, \dots, n_\ell\} \geq 2,$$

we have $M \cap b_0 D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu) \neq \varnothing$. Take an irreducible component M' of M with $M' \cap b_0 D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu) \neq \varnothing$. Then the intersection $E_{M'}$ of M' with $b_0 D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)$ is a real analytic submanifold of dimensional $2(n_0 + n_1 + \cdots + n_\ell) - 3$ on a dense, open subset of $E_{M'}$. By (3-7), we have $f(E_{M'}) \subset b_1 D_{m_0}^{\boldsymbol{m},\boldsymbol{q}}(\nu) \cup b_2 D_{m_0}^{\boldsymbol{m},\boldsymbol{q}}(\nu)$. Hence

$$f(M' \cap D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)) \subset \bigcup_{j=1+\delta}^{\ell} \mathrm{Pr}_i(D_{m_0}^{\boldsymbol{m},\boldsymbol{q}}(\nu)), \tag{3-8}$$

where $\mathrm{Pr}_i(D_{m_0}^{\boldsymbol{m},\boldsymbol{q}}(\nu)) := \{(z, w_{(1)}, \dots, w_{(\ell)}) \in D_{m_0}^{\boldsymbol{m},\boldsymbol{q}}(\nu) : \|w_{(i)}\| = 0\}$ $(1+\delta \leq i \leq \ell)$ by the uniqueness theorem. Since $\operatorname{codim} M' = 1$,

$$\operatorname{codim}\left[\bigcup_{j=1+\delta}^{\ell} \mathrm{Pr}_i(D_{m_0}^{\boldsymbol{m},\boldsymbol{q}}(\nu))\right] \geq \min\{m_{1+\delta}, \dots, m_\ell, m_1 + \cdots + m_\ell\} \geq 2$$

and $f : D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu) \to D_{m_0}^{\boldsymbol{m},\boldsymbol{q}}(\nu)$ is proper, this contradicts (3-8). Thus we have $M \cap bD_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu) = \varnothing$.

Let $S := M \cap D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)$. Then we have

$$S \subset D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu), \qquad \bar{S} \cap bD_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu) = \varnothing.$$

If $S \neq \varnothing$, then S is a complex analytic set in $\mathbb{C}^{n_0+n_1+\cdots+n_\ell}$ also. For any $(z, w) \in S$, we have $|w_{\ell n_\ell}|^{2p_\ell} \leq \sum_{j=1}^{\ell} \|w_{(j)}\|^{2p_j} \leq e^{-\mu\|z\|^2} \leq 1$. Thus

$$|w_{\ell n_\ell}|^2 \leq 1 \leq 1 + \|(z, w')\|, \tag{3-9}$$

where $w = (w', w_{\ell n_\ell})$. Then S is an algebraic set of $\mathbb{C}^{n_0+n_1+\cdots+n_\ell}$ by §7.4, Theorem 3 of [Chirka 1989].

Suppose S_1 is an irreducible component of S. Let $\overline{S_1}$ be the closure of S_1 in $\mathbb{P}^{n_0+n_1+\cdots+n_\ell}$. Then by §7.2, Proposition 2 of [Chirka 1989], $\overline{S_1}$ is a projective algebraic set and $\dim \overline{S_1} = n_0+n_1+\cdots+n_\ell-1$. Let $[\xi, z, w]$ be the homogeneous coordinate in $\mathbb{P}^{n_0+n_1+\cdots+n_\ell}$. We embed $\mathbb{C}^{n_0+n_1+\cdots+n_\ell}$ into $\mathbb{P}^{n_0+n_1+\cdots+n_\ell}$ as the affine piece $U_0 = \{[\xi, z, w] \in \mathbb{P}^{n_0+n_1+\cdots+n_\ell} : \xi \neq 0\}$ by $(z, w) \hookrightarrow [1, z, w]$. Then we have

$$D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu) \cap U_0 = \left\{ [\xi, z, w] : \xi \neq 0,\ \sum_{j=1}^{\ell} \frac{\|w_{(j)}\|^{2p_j}}{|\xi|^{2p_j}} < e^{-\mu\|z\|^2/|\xi|^2} \right\}.$$

Let $H = \{\xi = 0\} \subset \mathbb{P}^{n_0+n_1+\cdots+n_\ell}$. Consider another affine piece

$$U_1 = \{[\xi, z, w] \in \mathbb{P}^{n_0+n_1+\cdots+n_\ell} : z_1 \neq 0\}$$

with affine coordinate $(\zeta, t, s) = (\zeta, t_2, \dots, t_{n_0}, s_{(1)}, \dots, s_{(\ell)})$. Let $t' = (1, t_2, \dots, t_{n_0})$. Since

$$\frac{\|w_{(j)}\|^{2p_j}}{|\xi|^{2p_j}} = \frac{\|w_{(j)}\|^{2p_j}}{|z_1|^{2p_j}} \frac{|z_1|^{2p_j}}{|\xi|^{2p_j}} = \frac{\|s_{(j)}\|^{2p_j}}{|\zeta|^{2p_j}},$$

$$e^{-\mu\|z\|^2/|\xi|^2} = e^{-\mu(\|z\|^2/|z_1|^2)(|z_1|^2/|\xi|^2)} = e^{-\mu(1+|t_2|^2+\cdots+|t_{n_0}|^2)/|\zeta|^2},$$

we obtain

$$D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu) \cap U_0 \cap U_1 = \left\{ (\zeta, t_2, \dots, t_{n_0}, s_{(1)}, \dots, s_{(\ell)}) \in \mathbb{C}^{n_0+n_1+\cdots+n_\ell} : \sum_{j=1}^{\ell} \frac{\|s_{(j)}\|^{2p_j}}{|\zeta|^{2p_j}} < e^{-\mu\|t'\|^2/|\zeta|^2} \right\}. \tag{3-10}$$

Let $S' = \overline{S_1} \cap U_1$ and $H_1 = H \cap U_1 = \{\zeta = 0\}$ (note $\xi = \zeta/z_1$). For every $u \in S' \cap H_1$, there exists a sequence of points $\{u_k\} \subset \overline{S_1} \cap ((U_0 \cap U_1)\backslash H_1)$ such that $u_k \to u$ $(k \to \infty)$. The formula (3-10) implies

$$\|s_{(j)}(u_k)\|^{2p_j} \le |\zeta(u_k)|^{2p_j} e^{-\mu\|t'\|^2/|\zeta(u_k)|^2} \quad (1 \le j \le \ell). \tag{3-11}$$

Since $u \in H_1$, that means $\zeta(u) = 0$ and $\zeta(u_k) \to 0$ $(k \to \infty)$. Therefore we have $\|s_{(j)}(u)\|^{2p_j} \le 0$ $(1 \le j \le \ell)$ as $k \to \infty$. Hence

$$S' \cap H_1 \subset \{\zeta = 0 : s_{(1)} = \cdots = s_{(\ell)} = 0\}.$$

Then $\dim(S' \cap H_1) \le n_0 - 1$. Theorem 6 in §6.2 of [Shafarevich 1974] implies

$$n_0 - 1 \ge \dim(S' \cap H_1) \ge \dim S' + \dim H_1 - n_0 - n_1 - \cdots - n_\ell \ge \dim S' - 1.$$

This means $\dim S' \le n_0$, and thus $n_0 + n_1 + \cdots + n_\ell - 1 = \dim S' \le n_0$. Therefore, we get $n_1 + \cdots + n_\ell \le 1$, contradicting the assumption that

$$\min\{n_{1+\epsilon}, n_2, \dots, n_\ell, n_1 + \cdots + n_\ell\} \ge 2.$$

Therefore, $S = \varnothing$ and thus f is unbranched. Since the generalized Fock–Bargmann–Hartogs domain is simply connected, $f : D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu) \to D_{m_0}^{\boldsymbol{m},\boldsymbol{q}}(\nu)$ is a biholomorphism. The proof is completed. □

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GALOISIAN METHODS FOR TESTING IRREDUCIBILITY OF ORDER TWO NONLINEAR DIFFERENTIAL EQUATIONS

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We provide a criterion to compute the Malgrange pseudogroup, the nonlinear analog of the differential Galois group, for classes of second order differential equations. Let G_k be the differential Galois groups of their k-th variational equations along an algebraic solution Γ. We show that if the dimension of one of the G_k is large enough, then the Malgrange pseudogroup is known. This in turn proves the irreducibility of the original nonlinear differential equation. To make the criterion applicable, we give a method to compute the dimensions of the variational Galois groups G_k via constructive reduced form theory. As an application, we reprove the irreducibility of the second and third Painlevé equations for special values of their parameter. In the appendices, we recast the various notions of variational equations found in the literature and prove their equivalences.

Introduction

The Malgrange pseudogroup of a vector field may be seen as a nonlinear analog of the Galois group of linear differential equations. Our aim in this work is to provide a criterion to compute Malgrange pseudogroups using an approach initiated by Casale: we study variational equations along a given algebraic solution curve Γ and use the fact that their Galois groups lie, in a certain sense, in $\mathrm{Mal}(X)$. Our main

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theorem below shows that if the dimension of these Galois groups is large enough, then $\mathrm{Mal}(X)$ is large and known.

In previous works, Casale applied this to integrability. We apply it to a stronger notion; the irreducibility of nonlinear differential equations.

Irreducibility of differential equations. The first formalized definition of reducibility appears in the Stockholm lessons of Paul Painlevé [1897]. A complete algebraization of this definition was given by K. Nishioka [1988] and H. Umemura [1988]. Note that Nishioka's concept of decomposable extension may be more general than reducibility. The first application was the proof of the irreducibility of the first Painlevé equation [Painlevé 1900; Nishioka 1988; Umemura 1988; 1990]. Umemura gave a simple criterion to prove irreducibility and the Japanese school applied it to all Painlevé equations [Noumi and Okamoto 1997; Umemura and Watanabe 1997; 1998; Watanabe 1995; 1998]. These papers deal with reducibility of *solutions*; in this paper, we will emphasize the (stronger) notion of *reducibility of an equation* (see next section for proper definitions).

Painlevé [1902] suggested that irreducibility of a differential equation can be proved by the computation of its (hypothetical) "rationality group", as (incorrectly) defined by J. Drach [1898]. Such a group-like object was finally defined in [Umemura 1996] (where it is a group functor) and in [Malgrange 2001] (where it is an algebraic pseudogroup); see also [Pommaret 1983].

The Malgrange pseudogroup. Let X denote a vector field on a manifold M. The smallest algebraic pseudogroup containing the flows of X is the Malgrange pseudogroup, denoted by $\mathrm{Mal}(X)$ (see Appendix C2, and references therein for a more precise definition).

The computation of the Malgrange pseudogroup of a differential equation is a difficult (and currently wide open) problem. In this paper, we use differential Galois groups of the variational equations along an algebraic solution of equations of the form $y'' = f(x, y)$ to determine their Malgrange pseudogroup.

The study of an equation through its linearization is ancient. Applications to integrability of differential equations were revived by S. L. Ziglin [1982], followed by many authors, notably J. J. Morales-Ruiz and J.-P. Ramis [2001a; 2001b] and then jointly with C. Simó [Morales-Ruiz et al. 2007] using the differential Galois group of the variational equations along a solution.

Casale [2009] proved that these Galois groups provide a lower bound for the Malgrange pseudogroup in the following way. This pseudogroup acts on the phase space and the algebraic solution (along which we linearize) parametrizes a curve $\mathscr{C}$ in this space. Then the group of k-jets of elements fixing a point in $\mathscr{C}$ contains the Galois group of the k-th order variational equation along $\mathscr{C}$.

Using techniques developed in [Morales-Ruiz and Ramis 2001a; 2001b; 2007] and the Malgrange pseudogroup following [Casale 2009], we will prove the following theorem, which is the main result of this work.

Theorem 1. *Let M be a smooth irreducible algebraic 3-fold over $\mathbb{C}$ and X be a rational vector field on M such that there exist a closed rational 1-form α with $\alpha(X) = 1$ and a closed rational 2-form γ with $\iota_X \gamma = 0$.*

Assume $\mathscr{C}$ is an algebraic X-invariant curve with $X_{\mathscr{C}} \not\equiv 0$. Assume that the following two conditions are satisfied:

(i) *The differential Galois group of the first variational equation of X along $\mathscr{C}$ is not virtually solvable*;

(ii) *There exists an integer k such that the dimension of the differential Galois group of the k-th variational equation is at least 6.*

Then, the Malgrange pseudogroup is

$$\mathrm{Mal}(X) = \{\varphi \mid \varphi^*\alpha = \alpha,\ \varphi^*\gamma = \gamma\}.$$

Moreover, if there exist rational coordinates x, y, z on M such that

$$X = \frac{\partial}{\partial x} + z\frac{\partial}{\partial y} + f(x, y, z)\frac{\partial}{\partial z}$$

then the equation $y'' = f(x, y, y')$ is irreducible.

The proof will be given in Appendix C, essentially because it requires a number of definitions and clarifications which we give there.

Another way to express the conclusion of the theorem is that the singular holomorphic foliation $\mathscr{F}_X$ of M defined by trajectories of X has no transversal rational geometric structure except the transversal rational volume form given by γ.

This theorem can be applied to compute the Malgrange pseudogroup of equations of the form $y'' = f(x, y)$. Solutions $x \mapsto (x, y(x), y'(x))$ of such an equation are trajectories of the vector field $\frac{\partial}{\partial x} + z\frac{\partial}{\partial y} + f(x, y)\frac{\partial}{\partial z}$ on the phase space. The forms $\alpha = dx$ and $\gamma = \iota_X(dx \wedge dy \wedge dz)$ are closed and $\alpha(X) = 1,\ \iota_X\gamma = 0$. To apply the theorem, a particular solution is needed.

Irreducibility and Malgrange pseudogroup. After Umemura, the Japanese school proved irreducibility of solutions of Painlevé equations using the so-called J condition.

Theorem [Umemura 1990]. *If $E \in \mathbb{C}[x, y, y', y'']$ is a second order differential equation and φ is a nonalgebraic reducible solution then there exists a differential field extension $\mathbb{C}(x) \subset L$ and a first order differential equation $F \in L[y, y']$ such that $F(\varphi, \varphi') = 0$.*

Casale improved this condition under some assumptions:

Theorem [Casale 2009]. *If $E \in \mathbb{C}[x, y, y', y'']$ is a second order differential equation and φ is a nonalgebraic reducible solution and the Malgrange pseudogroup of E is big enough then there exists a first order differential equation $F \in \mathbb{C}[x, y, y']$ such that $F(\varphi, \varphi') = 0$.*

This theorem can be rephrased as follows:

> *If the Malgrange pseudogroup of an equation is big enough then a reducible solution cannot be a general solution.*

This leads us to define reducibility of an equation as the existence of a reducible general solution. A link with the Malgrange pseudogroup is given by the following.

Theorem 2 [Casale 2009]. *Let M be a smooth irreducible algebraic 3-fold over $\mathbb{C}$ and X be a rational vector field on M such that there exist a closed rational 1-form α with $\alpha(X) = 1$ and a closed rational 2-form γ with $\iota_X \gamma = 0$.*

If the Malgrange pseudogroup is

$$\mathrm{Mal}(X) = \{\varphi \mid \varphi^*\alpha = \alpha,\ \varphi^*\gamma = \gamma\}$$

and there exist rational coordinates x, y, z on M such that

$$X = \frac{\partial}{\partial x} + z\frac{\partial}{\partial y} + f(x, y, z)\frac{\partial}{\partial z},$$

then the equation $y'' = f(x, y, y')$ is irreducible.

Casale [2008] applied this philosophy to reprove the irreducibility of the first Painlevé equation. S. Cantat and F. Loray [2009] used this to reprove the irreducibility of the sixth Painlevé equation.

The strongest notion of solvability used in differential equations is Liouville integrability of Hamiltonian systems (and derived notions such as Bogoyavlenskij integrability [1998]). In this case, $\mathrm{Mal}(X)$ is commutative. In the case when the system is integrable by quadratures, $\mathrm{Mal}(X)$ is solvable. In the reducible case, the consequence on $\mathrm{Mal}(X)$ is more technical (see Appendix C and the proof of the main theorem in [Casale 2009]).

Applications. The second Painlevé equation with parameter a is

$$(P_{\mathrm{II}}(a)) \qquad y'' = xy + 2y^3 + a.$$

There is a Bäcklund transformation ([Noumi and Okamoto 1997]) linking $(P_{\mathrm{II}}(a))$ and $(P_{\mathrm{II}}(a + 1))$. Hence, determining the Malgrange pseudogroup for $(P_{\mathrm{II}}(a))$ determines it for all $(P_{\mathrm{II}}(a + n))$, $n \in \mathbb{Z}$.

M. Noumi and K. Okamoto [1997] proved that, apart from the rational solutions when $\alpha \in \mathbb{Z}$ and hypergeometric solutions when $\alpha \in 1/2 + \mathbb{Z}$, the solutions of this equation are irreducible in the sense of Nishioka and Umemura.

Painlevé equations can be presented as Hamiltonian systems with two degrees of freedom. Morales-Ruiz applied Morales–Ramis theory to the Hamiltonian form of $(P_{\mathrm{II}}(n))$ in order to prove its non-Liouville-integrability. His work has been continued in [Stoyanova and Christov 2007; Horozov and Stoyanova 2007; Żołądek and Filipuk 2015]. These computations can be reinterpreted following [Casale 2009]: the nonsolvability of the Galois group of the first variational equation implies the nonsolvability of the Malgrange pseudogroup, and hence the nonintegrability by quadratures of $(P_{\mathrm{II}}(n))$.

For an ordinary differential equation, reducibility is much more general than integrability by quadratures and the corresponding property of its Malgrange pseudogroup is less easy to formulate precisely. The seminal work of Morales-Ruiz has to be continued further and Galois groups of higher order variational equations must be computed.

The approach presented here uses the Malgrange pseudogroup of the rational vector field X on $M = \mathbb{C}^3$ given by

$$X_2 = \frac{\partial}{\partial x} + z\frac{\partial}{\partial y} + (xy + 2y^3)\frac{\partial}{\partial z},$$

whose trajectories are parametrized by solutions of $(P_{\mathrm{II}}(n))$. Using the notation of the theorem, $\alpha = dx$, $\gamma = \iota_X(dx \wedge dy \wedge dz)$ and $\mathscr{C} = \{y = z = 0\}$, we prove that

$$\mathrm{Mal}(X) = \{\varphi \mid \varphi^*\alpha = \alpha,\ \varphi^*\gamma = \gamma\}.$$

This equality implies the irreducibility of $(P_{\mathrm{II}}(n))$. Note that this property of the Malgrange pseudogroup is stronger than irreducibility in the sense of Nishioka and Umemura. However, it is not a purely algebraic property: it is formulated for the differential field $\mathbb{C}(x)$ and seems to be specific to differential fields which are finitely generated over the constants, whereas the definition of irreducibility can be stated over any differential field.

The application of our theorem to prove the irreducibility of the second Painlevé equation requires two steps.

First, one needs to check whether the Galois group of the first variational equation is solvable after an algebraic extension (or virtually solvable). This differential equation reduces to the Airy equation $y'' = xy$ and it is easy, for example by using the Kovacic algorithm [1986], to show that its differential Galois group is $\mathrm{SL}(2, \mathbb{C})$.

Then, to check the dimension condition seems more hazardous at first sight. We would need to compute Galois groups of higher order variational equations until we found a Galois group of dimension at least 6. Until now, no bound is known on the order of the required variational equation that one would have to study to prove this. Moreover, the size of the (linearized) variational equations grows quickly and, even though there are theoretical methods to compute differential Galois groups in [Hrushovski 2002], the computation of differential Galois groups of such big systems is still unfeasible in general.

In our case, the situation is better because the methods of P. H. Berman and M. F. Singer [Berman and Singer 1999; Berman 2002] could allow us to determine the differential Galois group. We choose another approach, following the works of A. Aparicio, E. Compoint, T. Dreyfus and J.-A. Weil on reduced forms of linear differential systems (see [Aparicio and Weil 2012; Aparicio et al. 2013]), notably [Aparicio and Weil 2011; Aparicio et al. 2016] where new effective techniques allow the computation of the Lie algebra of the differential Galois group of a variational equation of order k when the variational equation of order $k-1$ has an abelian differential Galois group. We show how to extend their method to our situation.

These computations can be reused to compute the Malgrange pseudogroup and prove irreducibility of a larger class of differential equations: $y'' = xy + y^n P(x, y)$. We will then show how this technique can be used to compute the Malgrange pseudogroup of a family of Painlevé III equations and prove their irreducibility.

Organization of the paper. Section 1 contains the definitions of reducibility, variational equations and their differential Galois groups in order to state the main theorem. The proof of the main theorem is postponed until Appendix C. In Section 2, we elaborate a simple irreducibility criterion for equations of the form $y'' = xy + y^n P(x, y)$ and give two irreducibility proofs for a Painlevé II equation. In Section 3, we apply a similar scheme to prove the irreducibility of a Painlevé III equation from statistical physics.

In the appendices, we detail the constructions needed to prove the main theorem. In Appendix A, we recast the Galois groups in the context of G-principal connections. In Appendix B, we describe and compare various notions of variational equations (arc space and frame bundle viewpoints), as the literature is occasionally hazy on this point. In Appendix C, we give the definition of the Malgrange pseudogroup of a vector field and give some of its properties regarding the reducibility and the variational equations. Together with the Cartan classification of pseudogroups in dimension 2 (in a neighborhood of a generic point), this allows us to finally prove Theorem 1.

1. Definitions

1.1. *Irreducibility.* In the 21st of his Stockholm lessons, Painlevé [1897] defined different classes of transcendental functions and gave the definition of second order differential equations reducible to first order differential equations. Then he proved that the so-called Picard–Painlevé equation, a special case of Painlevé's sixth equation discovered by E. Picard, is irreducible. This proof relies on the fact that this equation has no moving singularities and that its flow gives bimeromorphic transformations of the plane $\mathbb{C}^2$. In this situation, reducible equations have a flow sending a foliation by algebraic curves onto another algebraic one. This is not the case for the Picard–Painlevé equation.

Later, Painlevé claimed without proof that the computation of Drach's rationality group [Drach 1898] would prove the irreducibility of an equation. He tried to compute it for the first Painlevé equation in [Painlevé 1902].

Definition 3 [Painlevé 1897; Nishioka 1988; Umemura 1988]. Let (K, δ) be an ordinary differential field, y be a differential indeterminate and

$$(E) : \delta^2 y = F(y, \delta y) \in K(y, \delta y)$$

be a second order differential equation defined on K. A solution of the equation (E) is called a *reducible solution* if it lies in a differential extension L of K built in the following way:

$$K = K_0 \subset K_1 \subset \cdots \subset K_m = L$$

with one of the following elementary extensions for any i. Either

- $K_i \subset K_{i+1}$ is an algebraic extension, or
- $K_i \subset K_{i+1}$ is a linear extension, i.e., $K_{i+1} = K_i(f_j^p; 1 \leq p, j \leq n)$ with $\delta f_j^p = \sum_k A_j^k f_k^p$, $A_j^k \in K_i$, or
- $K_i \subset K_{i+1}$ is an abelian extension, i.e., $K_{i+1} = K_i(\varphi_j(a_1, \dots, a_n); 1 \leq j \leq n)$ with φ's a basis of periodic functions on $\mathbb{C}^n$ given by the field of rational functions on an abelian variety over $\mathbb{C}$ and a's in K_i, or
- $K_i \subset K_{i+1}$ has transcendence degree 1, i.e., $K_{i+1} = K_i(z, \delta z)$ with $P(z, \delta z) = 0$, $P \in K_i[X, Y] - \{0\}$.

Note that Nishioka's definition of decomposable extension seems more general than reducibility. We do not know any example of a decomposable irreducible extension nor any proof of the equivalence of the two notions. In the articles of Umemura, the notion of reducibility appears together with the notion of classical functions. The latter is similar except that the last kind of elementary extension is not allowed.

This definition may not be the most relevant to understand the geometry of the differential equation: a second order differential equation may have two functionally independent first integrals in a Picard–Vessiot extension of $\mathbb{C}(x, y, z)$ without being reducible. This is the case for the Picard–Painlevé equation as it is explained in the 21st lesson of Painlevé [1897]; see also [Casale 2007] and [Watanabe 1998].

The above definition is a property of individual solutions; however, the equation may have an exceptional solution which is reducible whereas the others are not. For example, any equation $\delta^2 y = yF(y, \delta y) + \delta y G(y, \delta y) \in K[y, \delta y]$ admits $y = 0$ as a solution. Therefore we will introduce a notion of *reducibility of the equation* which translates, in algebraic terms, the idea that the general solution of the equation is reducible.

Definition 4. Let (K, δ) be an ordinary differential field and

$$(E) : \delta^2 y = F(y, \delta y) \in K(y, \delta y)$$

be a second order differential equation defined on K. The equation (E) is called a *reducible differential equation over* K if there exists a reducible solution f such that the transcendence degree of $K(f, \delta f)/K$ equals 2 (i.e., the general solution of the equation is reducible).

Example 1. Consider the equation $\delta^2 y = 0$. We want to show that it is reducible over $\big(\mathbb{C}(x), \delta = \partial/\partial x\big)$. Its general solution is $f = ax + b$ for arbitrary (i.e., transcendental) constants a and b. Here, $K = \mathbb{C}(x)$ and $K(f, \delta f) = \mathbb{C}(a, b)(x)$ (with a and b transcendental over $\mathbb{C}$) so that the transcendence degree of $K(f, \delta f)/K$ does indeed equal 2. This is why, in the second condition for reducibility of solutions (in Definition 3 above), we allow *linear* extensions with possibly new constants (and not only Picard–Vessiot extensions).

Remark 5. Solutions of a reducible second order differential equation are reducible. Reducibility of the equation means that one can choose a geometric model (M, X) for the differential field K_m and a dominant rational map π from M to $\mathbb{A}^2_K$ such that the rational vector field X is π-projectable on $\partial + y' \frac{\partial}{\partial y} + F(y, y') \frac{\partial}{\partial y'}$. A solution is an integral curve of this vector field. Now the image of a rational map is constructible so that either the solution is algebraic or it is in the image of π. In each case, the solution is reducible.

Using the Malgrange pseudogroup of a vector field and É. Cartan's classification of pseudogroups, Casale proved the following theorem.

Theorem 6 [Casale 2008, Annexe A]. *Let X be a rational vector field on M, a smooth irreducible algebraic 3-fold. Assume there exist a rational closed 1-form α such that $\alpha(X) = 1$ and a rational closed 2-form γ such that $\iota_X \gamma = 0$. Then one of the following holds.*

- *There exists a 1-form ω with coefficients in the algebraic closure $\overline{\mathbb{C}(M)}^{\text{alg}}$ such that $\omega(X) = 0$ and for any local determination of algebraic functions $\omega \wedge d\omega = 0$.*
- *There exist θ_1 and θ_2, two rational 1-forms vanishing on X, and a traceless 2×2 matrix (θ_i^j) of rational 1-forms such that $\theta_i(X) = 0$, $d\theta_i = \sum_k \theta_i^k \wedge \theta_k$ and $d\theta_i^j = \sum_k \theta_k^j \wedge \theta_i^k$, for all $(i, j) \in \{1, 2\}^2$.*
- *The Malgrange pseudogroup is* $\mathrm{Mal}(X) = \{\varphi \mid \varphi^*\alpha = \alpha,\ \varphi^*(\gamma) = \gamma\}$.

The systems of PDE given in the first two items of the statement are the analog of the resolvent equations in classical Galois theory. The existence of a rational solution to the resolvent equations would imply that the Malgrange pseudogroup is small. Then in [Casale 2009], the claim of Painlevé is proved.

Theorem 7 [Casale 2009]. *Let E be a rational equation of order two,*

$$y'' = F(x, y) \in \mathbb{C}(x, y),$$

and $X = \partial/\partial x + z\,\partial/\partial y + F(x, y)\,\partial/\partial z$ be the rational vector field on $\mathbb{C}^3$ associated to E. If $\mathrm{Mal}(X) = \{\varphi \mid \varphi^* dx = dx,\ \varphi^*(\iota_X dx \wedge dy \wedge dz) = \iota_X dx \wedge dy \wedge dz\}$ then E is irreducible.

1.2. ***Variational equations.*** Let X be a vector field on an algebraic manifold M and $\mathscr{C} \subset M$ an algebraic X-invariant curve such that $X_{\mathscr{C}} \not\equiv 0$. Variational equations can be written easily in local coordinates. Intrinsic versions will be given in the appendices. In local coordinates $(x_1, \dots, x_n)$ on M, the flow equations of $X = \sum a_i(x)\partial/\partial x_i$ are

$$\frac{d}{dt} x_i = a_i(x), \quad i = 1, \dots, n.$$

This flow can be used to move germs of analytic curves on M pointwise. Let $\epsilon \mapsto x(\epsilon)$ be such a germ defined on $(\mathbb{C}, 0)$. For any ϵ small enough, one has

$$\frac{d}{dt} x_i(\epsilon) = a_i(x(\epsilon)) \quad i = 1, \dots, n.$$

Analyticity allow us to expand this equality. Let

$$x(\epsilon) = \left(\sum_k x_1^{(k)} \frac{\epsilon^k}{k!}, \dots, \sum_k x_n^{(k)} \frac{\epsilon^k}{k!} \right),$$

then

$$(\mathrm{VE}_k) \qquad \begin{cases} \dfrac{d}{dt} x_i^0 &= a_i(x^0), \\ \dfrac{d}{dt} x_i^{(1)} &= \sum_j \dfrac{\partial a_i}{\partial x_j}(x^0) x_j^{(1)}, \\ \dfrac{d}{dt} x_i^{(2)} &= \sum_j \dfrac{\partial a_i}{\partial x_j}(x^0) x_j^{(2)} + \sum_{j,\ell} \dfrac{\partial^2 a_i}{\partial x_j \partial x_\ell}(x^0) x_j^{(1)} x_\ell^{(1)}, \\ \dfrac{d}{dt} x_i^{(3)} &= \sum_j \dfrac{\partial a_i}{\partial x_j}(x^0) x_j^{(3)} + \sum_{j,\ell} 3 \dfrac{\partial^2 a_i}{\partial x_j \partial x_\ell}(x^0) x_j^{(2)} x_\ell^{(1)}, \\ & \qquad + \sum_{j,\ell,m} \dfrac{\partial^3 a_i}{\partial x_j \partial x_\ell \partial x_m}(x^0) x_j^{(1)} x_\ell^{(1)} x_m^{(1)}, \\ \vdots & \\ \dfrac{d}{dt} x_i^{(k)} &= F_k(\partial^\beta a_i(x_0), x_i^{(\ell)} \mid i = 1, \dots, n, |\beta| \le k,\ \ell \le k), \end{cases}$$

where the F's are given by Faa di Bruno formulas (see formula (14) on page 860 in [Morales-Ruiz et al. 2007]). The k-th order variational equation is the differential system on k-th order jets of parametrized curves on M obtained in this way. Because $\mathscr{C}$ is an algebraic X-invariant curve, the space of parametrized curves with $x^0 \in \mathscr{C}$ is an algebraic subvariety invariant by the variational equation. The variational equation gives a nonlinear connection on the bundle over $\mathscr{C}$ of parametrized curves pointed on $\mathscr{C}$. This restriction is the variational equation along $\mathscr{C}$.

The system (VE_k) is a rank nk nonlinear system but it can be linearized. For instance the third order variational equation is linearized using new unknowns $z_{\ell,k,j} = x_\ell^{(1)} x_k^{(1)} x_j^{(1)}$, $z_{k,j} = x_k^{(2)} x_j^{(1)}$ and $z_k = x_k^{(3)}$, which amounts to performing some tensor constructions on lower order linearized variational equations (such as symmetric powers of the first variational equation); see [Simon 2014; Aparicio and Weil 2011; Morales-Ruiz et al. 2007]. The linear system obtained is

$$(\mathrm{LVE}_3)\quad \begin{cases} \dfrac{d}{dt} x_i^0 &= a_i(x^0), \\ \dfrac{d}{dt} z_{\ell,k,j} &= \sum_{b,c,d} \Big(\dfrac{\partial a_\ell}{\partial x_b} + \dfrac{\partial a_k}{\partial x_c} + \dfrac{\partial a_j}{\partial x_d} \Big)(x^0) z_{b,c,d}, \\ \dfrac{d}{dt} z_{k,j} &= \sum_{b,c} \Big(\dfrac{\partial a_\ell}{\partial x_b} + \dfrac{\partial a_k}{\partial x_c} \Big)(x^0) z_{b,c} + \sum_{c,d} \dfrac{\partial^2 a_k}{\partial x_c \partial x_d}(x^0) z_{c,d,j}, \\ \dfrac{d}{dt} z_i &= \sum_j \dfrac{\partial a_i}{\partial x_j}(x^0) z_j + \sum_{j,\ell} 3 \dfrac{\partial^2 a_i}{\partial x_j \partial x_\ell}(x^0) z_{j,\ell} \\ & \qquad + \sum_{j,\ell,m} \dfrac{\partial^3 a_i}{\partial x_j \partial x_\ell \partial x_m}(x^0) z_{j,\ell,m}. \end{cases}$$

When X preserves a transversal fibration $\pi : M \to B$, the parametrized curves $\epsilon \to x(\epsilon)$ included in fibers of π give a subset of curves invariant by X. The restriction of the variational equation to this subset is called the π-normal variational equation. The main case of interest is the normal variational equation of an ODE. Such a differential equation gives a vector field $\partial/\partial x_1 + \cdots$ where x_1 is the independent coordinate. The normal variational equation (with respect to the projection on the curve of the independent coordinate) is obtained from the variational equation by setting $x_1^{(k)} = 0$ when $k \geq 1$.

The k-th order linearized normal variational equation is obtained from the k-th order linearized variational equation by setting $z_\alpha = 0$ when a coordinate of $\alpha \in \mathbb{N}^k$ is equal to 1. The induced system will be denoted by NLVE_k.

1.3. ***The Galois group and the main theorem.*** Following E. Picard and E. Vessiot, the differential Galois group of a linear differential system $\frac{d}{dt} Y = AY$ with $A \in \mathrm{GL}(n, \mathbb{C}(t))$ can be defined in the following way.

Select a regular point t_0 of the differential system and a fundamental matrix $F(t) \in \mathrm{GL}(\mathbb{C}\{t - t_0\})$ of holomorphic solutions at this point. Then the splitting field, called the Picard–Vessiot extension, is $L = \mathbb{C}(t, F_i^j(t) \mid 1 \le i, j \le n)$ and the differential Galois group G is the group of $\mathbb{C}(t)$-automorphisms of L commuting with d/dt.

Picard proved that this group G is a linear algebraic subgroup of $\mathrm{GL}(n, \mathbb{C})$ and Vessiot proved the Galois correspondence. In our context, the linearized normal variational equation is a subsystem of the linearized variational equation so the Galois correspondence implies that its Galois group is a quotient of the Galois group of the variational equation.

Introductions to this theory may be found in [Magid 1994] or the reference book [van der Put and Singer 2003]. Other variations on that theme can be found, for example, in [Katz 1990; Kolchin 1973; Bertrand 1996]. We propose an overview of the theory from the "principal bundle" point of view in the appendices.

The statement of our main theorem involves the Malgrange pseudogroup of a vector field. We recall its definition in Appendix C2.

Theorem 1. *Let M be a smooth irreducible algebraic 3-fold over $\mathbb{C}$ and X be a rational vector field on M such that there exist a closed rational 1-form α with $\alpha(X) = 1$ and a closed rational 2-form γ with $\iota_X \gamma = 0$.*

Assume $\mathscr{C}$ is an algebraic X-invariant curve with $X_{\mathscr{C}} \not\equiv 0$. Assume that the following two conditions are satisfied:

(i) *The differential Galois group of the first variational equation of X along $\mathscr{C}$ is not virtually solvable;*

(ii) *There exists an integer k such that the dimension of the differential Galois group of the k-th variational equation is at least 6.*

Then, the Malgrange pseudogroup is

$$\mathrm{Mal}(X) = \{\varphi \mid \varphi^*\alpha = \alpha,\ \varphi^*\gamma = \gamma\}.$$

Moreover, if there exist rational coordinates x, y, z on M such that

$$X = \frac{\partial}{\partial x} + z\frac{\partial}{\partial y} + f(x, y, z)\frac{\partial}{\partial z}$$

then the equation $y'' = f(x, y, y')$ is irreducible.

In the application, we will compute the Galois group of the normal variational equation. As this group is a quotient of the group used in the theorem, one can replace (VE_1) by (NVE_1) and (LVE_k) by (NLVE_k) without changing the conclusion of the theorem. We postpone the proof of the theorem to the appendices because it requires additional technology which is recalled there. In the next two sections, we show applications of this theorem to the irreducibility of second order equations such as the Painlevé equations (P_{II}) and (P_{III}).

2. Irreducibility of $d^2y/dx^2 = f(x, y)$ and the Painlevé II equation

We will compute the differential Galois group of some normal variational equation along the solution $y = 0$ of differential equations of the form

$$\frac{d^2y}{dx^2} = xy + y^n P(x, y) \text{ with } P \in \mathbb{C}(x, y) \text{ without poles along } y = 0 \text{ and } n \geq 2.$$

The vector field of our equation is

$$X = \frac{\partial}{\partial x} + z\frac{\partial}{\partial y} + (xy + y^n P(x, y))\frac{\partial}{\partial z}.$$

This equation has a solution $y = z = 0$. The first normal variational equation along this curve is

$$\frac{\partial}{\partial x} + z^{(1)}\frac{\partial}{\partial y^{(1)}} + xy^{(1)}\frac{\partial}{\partial z^{(1)}}$$

Using a parametrization $x = t$ of this curve, we get a linear system,

$$\frac{d}{dt}Y = AY, \qquad \text{with } A = \begin{pmatrix} 0 & 1 \\ t & 0 \end{pmatrix}.$$

It is easily seen, from the form of the equation, that the variational equations of order less than n bring no new information, because of the term in y^n. Letting

$$y = \sum_{i=1}^{n} y^{(i)}\epsilon^i/(i!) \quad \text{and} \quad p(x) = n!P(x, 0),$$

we have

$$xy \;+\; y^n P(x, y) = \sum_{i=1}^{n-1} xy^{(i)}\epsilon^i + (xy^{(n)} + (y^{(1)})^n p(x))\epsilon^n + o(\epsilon^n)$$

and the n-th order normal variational equation along the solution $y = z = 0$ is

$$\frac{\partial}{\partial x} + \left(\sum_{k=1}^{n-1} z^{(k)}\frac{\partial}{\partial y^{(k)}} + xy^{(k)}\frac{\partial}{\partial z^{(k)}}\right) + z^{(n)}\frac{\partial}{\partial y^{(n)}} + (xy^{(n)} + p(x)(y^{(1)})^n)\frac{\partial}{\partial z^{(n)}}.$$

The linearized normal variational system can be reduced to

$$(\mathrm{NVE}_n) \quad \frac{d}{dt}\left(\begin{array}{c} \vdots \\ \vdots \\ \binom{n}{k}(y^{(1)})^{n-k}(z^{(1)})^k \\ \vdots \\ \hline y^{(n)} \\ z^{(n)} \end{array}\right) = \left(\begin{array}{ccc|cc} & & & \vdots & \\ & \mathtt{sym}^n\begin{pmatrix} 0 & 1 \\ t & 0 \end{pmatrix} & & 0 & \\ & & & \vdots & \\ \hline 0 & \cdots & 0 & 0 & 1 \\ p(t) & \cdots & 0 & t & 0 \end{array}\right)\left(\begin{array}{c} \vdots \\ \vdots \\ \binom{n}{k}(y^{(1)})^{n-k}(z^{(1)})^k \\ \vdots \\ \hline y^{(n)} \\ z^{(n)} \end{array}\right).$$

Example 2. For example, in the case of the second Painlevé equation with $a = 0$ or (P_{II}), we have $n = 3$ and the linearized variational system is

$$(\mathrm{PNVE}_3) \qquad \frac{d}{dt}Y = \mathcal{A}\cdot Y, \quad \text{with } \mathcal{A} = \left(\begin{array}{cccc|cc} 0 & 1 & 0 & 0 & 0 & 0 \\ 3t & 0 & 2 & 0 & 0 & 0 \\ 0 & 2t & 0 & 3 & 0 & 0 \\ 0 & 0 & t & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \\ 12 & 0 & 0 & 0 & t & 0 \end{array}\right).$$

2.1. ***Reduced forms and a first irreducibility proof of*** (P_{II})**.** We introduce material from [Aparicio et al. 2013; Aparicio and Weil 2012] concerning the Kolchin–Kovacic reduced forms of linear differential systems.

Consider a differential system $[A] : Y' = AY$ with $A \in \mathrm{Mat}(n, k)$. Let G denote its differential Galois group and $\mathfrak{g}$ its Lie algebra. Given a matrix $P \in \mathrm{GL}(n, \bar{k})$, the change of variable $Y = P.Z$ transforms $[A]$ into a system $Z' = B.Z$, where

$$B = PAP^{-1} - P'P^{-1}.$$

The standard notation is $B = P[A]$. The systems $[A]$ and $[P[A]]$ are called *equivalent over* $\bar{k}$. The Galois group may change but its Lie algebra $\mathfrak{g}$ is preserved under this transformation.

We say that $[A]$ is *in reduced form* if $A \in \mathfrak{g}(k)$. When this is not the case, we say that a matrix $B \in \mathrm{Mat}(n, \bar{k})$ is a *reduced form* of $[A]$ if there exists $P \in \mathrm{GL}(n, \bar{k})$ such that $B = P[A]$ and $B \in \mathfrak{g}(\bar{k})$. Our technique to find $\mathfrak{g}$, for the variational equations, will be to transform them into reduced form.

Example 3. The first variational equation of Painlevé II has matrix

$$A_1 = \begin{pmatrix} 0 & 1 \\ t & 0 \end{pmatrix}.$$

This corresponds to the Airy equation and its Galois group is known to be $\mathrm{SL}(2, \mathbb{C})$. Obviously, $A_1 \in \mathfrak{sl}(2, \mathbb{C}(t))$ so the first variational equation is in reduced form.

Let $a_1(x), \ldots, a_r(x) \in k$ be a basis of the $\mathbb{C}$-vector space generated by the coefficients of A. We may decompose A as $A = \sum_{i=1}^{r} a_i(x)M_i$, where the M_i are constant matrices. The *Lie algebra associated to* A, denoted $\mathrm{Lie}(A)$, is the *algebraic Lie algebra generated by the* M_i: it is the smallest Lie algebra which contains the M_i and is also the Lie algebra of some (connected) linear algebraic group $\mathcal{H}$; see [Aparicio et al. 2013].

Example 4. We compute the Lie algebra $\mathrm{Lie}(\mathcal{A})$ associated to $\mathcal{A}$ in the system (PNVE_3). Let

$$X := \left(\begin{array}{cccc|cc} 0&1&0&0&0&0\\ 0&0&2&0&0&0\\ 0&0&0&3&0&0\\ 0&0&0&0&0&0\\ \hline 0&0&0&0&0&1\\ 0&0&0&0&0&0 \end{array}\right) = \left(\begin{array}{c|c} \mathtt{sym}^n\begin{pmatrix}0&1\\0&0\end{pmatrix} & 0\\ \hline 0 & \begin{matrix}0&1\\0&0\end{matrix} \end{array}\right),$$

$$Y := \left(\begin{array}{cccc|cc} 0&0&0&0&0&0\\ 3&0&0&0&0&0\\ 0&2&0&0&0&0\\ 0&0&1&0&0&0\\ \hline 0&0&0&0&0&0\\ 0&0&0&0&1&0 \end{array}\right) = \left(\begin{array}{c|c} \mathtt{sym}^n\begin{pmatrix}0&0\\1&0\end{pmatrix} & 0\\ \hline 0 & \begin{matrix}0&0\\1&0\end{matrix} \end{array}\right),$$

and

$$H := [X, Y] = \left(\begin{array}{cccc|cc} 3&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&-1&0&0&0\\ 0&0&0&-3&0&0\\ \hline 0&0&0&0&1&0\\ 0&0&0&0&0&-1 \end{array}\right)$$

this is the standard $\mathfrak{sl}_2$ triplet with $[H, X] = 2X,\ [H, Y] = -2Y$), and we introduce the off-diagonal matrices

$$E_i = \left(\begin{array}{c|c} \begin{matrix}\ddots&&\\&0&\\&&\ddots\end{matrix} & \begin{matrix}\vdots\\0\\\vdots\end{matrix}\\ \hline B_i & 0 \end{array}\right),$$

where

$$B_0 = \begin{pmatrix}0&0&0&0\\1&0&0&0\end{pmatrix},\quad B_1 = \begin{pmatrix}+1&0&0&0\\0&-1&0&0\end{pmatrix},\quad B_2 = \begin{pmatrix}0&-1&0&0\\0&0&1&0\end{pmatrix},$$

$$B_3 = \begin{pmatrix}0&0&+1&0\\0&0&0&-1\end{pmatrix},\quad B_4 = \begin{pmatrix}0&0&0&-1\\0&0&0&0\end{pmatrix}.$$

We have $[X, E_i]{=}(i+1)E_{i+1}$, $[Y, E_i]{=}(5-i)E_{i-1}$, $[H, E_i]{=}(-4+2i)E_i$ and $[E_i, E_j]{=}0$ (with $E_{-1} = E_5 = 0$). We now show that $\mathrm{Lie}(\mathcal{A})$ is generated (as a Lie algebra) by X, Y and E_1. Indeed, $\mathrm{Lie}(\mathcal{A})$ is generated (as a Lie algebra) by

$M_1 := X + E_1$ and $M_2 := Y$; thus $[M_1, M_2] = H$ and $[M_1, H] = -2X - 4E_1$ so $[M_1, H] + 2M_1 = -2E_1$ and so $E_1 \in \mathrm{Lie}(\mathcal{A})$.

The above calculations then show that $\mathrm{Lie}(\mathcal{A})$ has dimension 8 and that it has $\{X, Y, H, E_0, \dots, E_4\}$ as a basis. We admit, as is proved later, that this Lie algebra is actually an algebraic Lie algebra: there exists an algebraic group $\mathcal{H}$ whose Lie algebra $\mathfrak{h}$ is equal to $\mathrm{Lie}(\mathcal{A})$.

A theorem of Kolchin ([van der Put and Singer 2003, Proposition 1.31]), shows that $\mathfrak{g} \subseteq \mathrm{Lie}(\mathcal{A})$ (and that $G \subseteq H$). A reduced form is obtained when we achieve equality in that inclusion. Moreover, when G is connected (which will be the case in this paper), the reduction theorem of Kolchin and Kovacic ([van der Put and Singer 2003, Corollaire 1.32]), shows that a reduced form exists and that the reduction matrix P may be chosen in $\mathcal{H}(k)$.

Let us now continue the above examples with the third variational equation of Painlevé II. Denote by $\mathfrak{h}_{\mathrm{diag}}$ the Lie algebra generated by the block-diagonal elements X, Y, H. Similarly, let $\mathfrak{h}_{\mathrm{sub}}$ be the Lie algebra generated by the off-diagonal matrices E_i (closed under conjugation by $\mathfrak{h}_{\mathrm{diag}}$). Of course, $\mathfrak{h}_{\mathrm{diag}}$ is $\mathfrak{sl}_2$ in its representation on a direct sum $\mathrm{Sym}^n(\mathbb{C}^2) \oplus \mathbb{C}^2$.

We see that $\mathfrak{h} = \mathfrak{h}_{\mathrm{diag}} \oplus \mathfrak{h}_{\mathrm{sub}}$. It is easily seen that $\mathfrak{h}_{\mathrm{diag}}$ is a Lie subalgebra of $\mathfrak{h}$ and that $\mathfrak{h}_{\mathrm{sub}}$ is an ideal in $\mathfrak{h}$.

We have seen that $\mathfrak{g} \subset \mathrm{Lie}(\mathcal{A})$. Furthermore, $\mathfrak{h}_{\mathrm{diag}} \subset \mathfrak{g}$ (because VE_1 has Galois group $\mathrm{SL}_2(\mathbb{C})$). It follows that $\mathfrak{g} = \mathfrak{h}_{\mathrm{diag}} \oplus \tilde{\mathfrak{g}}$, where $\tilde{\mathfrak{g}} \subset \mathfrak{h}_{\mathrm{sub}}$ is an ideal in $\mathfrak{g}$; in particular, it is closed under the bracket with elements of $\mathfrak{h}_{\mathrm{diag}}$ (adjoint action of $\mathfrak{h}_{\mathrm{diag}}$ on $\mathfrak{h}_{\mathrm{sub}}$).

Now the only invariant subsets of $\mathfrak{h}_{\mathrm{sub}}$ under this adjoint action are seen to be $\{0\}$ and $\mathfrak{h}_{\mathrm{sub}}$ (this is reproved and generalized in Proposition 9 below and its lemmas). So the Lie algebra $\mathfrak{g}$ is either $\mathfrak{sl}_2$ (of dimension 3) or $\mathfrak{h}$ (of dimension 8).

As the Galois group of the block-diagonal part is connected, the differential Galois group G of $[\mathcal{A}]$ is connected. Hence we know (by the reduction theorem of Kolchin and Kovacic cited above) that there exists a reduction matrix $P \in H(k)$. Furthermore, as the block-diagonal part of $\mathcal{A}$ is already in reduced form, the block-diagonal part of the reduction matrix P may be chosen to be the identity. So there exists a reduction matrix of the form

$$P = \mathrm{Id} + \sum_{i=1}^{5} f_i(t) E_i, \quad \text{with} \quad f_i(t) \in \mathbb{C}(t).$$

A simple calculation shows that $P\mathcal{A}P^{-1} = \mathcal{A} + \sum_{i=1}^{5} f_i(t)[E_i, X + tY]$ and so

$$P[\mathcal{A}] = X + tY + E_1 + \sum_{i=1}^{5} f_i(t)[E_i, X + tY] - \sum_{i=1}^{5} f_i'(t) E_i.$$

We see that the case $\mathfrak{g} = \mathfrak{sl}_2$ happens if and only if we can find $f_i \in \mathbb{C}(t)$ such that $\sum_{i=1}^{5} f_i'(t)E_i = \sum_{i=1}^{5} f_i(t)[X + tY, E_i] + E_1$. Let Ψ denote the matrix of the adjoint action $[X + tY, \bullet]$ of $X + tY$ on $\mathfrak{h}_{\mathrm{sub}}$. We see that $\mathfrak{g} = \mathfrak{sl}_2$ if and only if we can find an $F \in \mathbb{C}(t)^5$ solution of the differential system

$$F' = \Psi \cdot F + \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

We now gather the properties of (P_{II}) elaborated in this sequence of examples.

Proposition 8. *The Painlevé II equation is irreducible when the parameter $a = 0$.*

Proof. Using the Barkatou algorithm and its Maple implementation [Barkatou 1999; Barkatou et al. 2012], one easily sees that the above differential system does not have a rational solution. If follows that, using the notations of the above examples, we have $\mathfrak{g} = \mathfrak{h}$ of dimension 8. So, for (P_{II}), we have: the Galois group of the first variational equation is $\mathrm{SL}(2, \mathbb{C})$ which is not virtually solvable; the Galois group of the third variational equation has dimension $8 > 5$. Theorem 1 thus shows that the Painlevé II equation is irreducible. □

2.2. ***The Galois group of the n-th variational equation.*** We will now generalize this process to all equations of the form $\frac{d^2y}{dx^2} = xy + y^n P(x, y)$. We will elaborate a much easier irreducibility criterion, which will allow to reprove the above proposition without having to trust a computer.

The aim of this subsection is to prove the following:

Proposition 9. *The Galois group of the n-th variational equation* (LNVE_n) *is either* $\mathrm{SL}_2(\mathbb{C})$ *or its dimension is $n + 5$ and then the differential equation $y'' = xy + y^n P(x, y)$ is irreducible.*

2.2.1. *Adjoint action.*

Lemma 10. *Let A be a 2×2 matrix of rational function of the variable t such that the Galois group G_1 of the differential system $\frac{dY}{dt} = AY$ has Lie algebra $\mathfrak{sl}_2$. Consider a system*

$$\frac{d}{dt}\begin{pmatrix} Z \\ Y \end{pmatrix} = \left(\begin{array}{c|c} \mathtt{sym}^n A & 0 \\ B & A \end{array}\right)\begin{pmatrix} Z \\ Y \end{pmatrix}$$

with differential Galois group G. Then G has dimension 3 or $n + 3$ or $n + 5$ or $2n + 5$.

Proof. Let $\mathfrak{g}$ be the Lie algebra of G. It has a block lower triangular form shaped by the form of the system, i.e., $\mathfrak{g} \subset \mathfrak{h} \subset \mathfrak{gl}_{n+3}$ where

$$\mathfrak{h} = \left\{ \left(\begin{array}{c|c} \mathtt{sym}^n a & 0 \\ \hline b & a \end{array} \right),\ a \in \mathfrak{sl}_2, b \in (\mathbb{C}^2)^\vee \otimes \mathrm{Sym}^n(\mathbb{C}^2) \right\}.$$

The south-east block A defines a subsystem, thus G contains a subgroup isomorphic to SL_2. The north-west block defines a quotient system so there is a surjective group morphism from G onto SL_2. The kernel of this map is a commutative ideal (the off-diagonal matrices E_i, in our examples) and inherits the structure of an $\mathfrak{sl}_2$-module for the inclusion of $\mathfrak{sl}_2$ in $\mathfrak{h}$ via $\mathfrak{g}$. As a representation, $\mathfrak{g} \cap ((\mathbb{C}^2)^\vee \otimes \mathrm{Sym}^n(\mathbb{C}^2))$ is a subspace of $(\mathbb{C}^2)^\vee \otimes \mathrm{Sym}^n(\mathbb{C}^2)$. This representation is nothing but the adjoint representation. The decomposition in irreducible representations is

$$(\mathbb{C}^2)^\vee \otimes \mathrm{Sym}^n(\mathbb{C}^2) = \mathrm{Sym}^{n-1}(\mathbb{C}^2) \oplus \mathrm{Sym}^{n+1}(\mathbb{C}^2)$$

(see [Fulton and Harris 1991, Example 11.11]). So the Lie algebra $\mathfrak{g}$ is either $\mathfrak{sl}_2$, or $\mathfrak{sl}_2 \rtimes \mathrm{Sym}^{n-1}(\mathbb{C}^2)$, or $\mathfrak{sl}_2 \rtimes \mathrm{Sym}^{n+1}(\mathbb{C}^2)$ or $\mathfrak{sl}_2 \rtimes (\mathrm{Sym}^{n-1}(\mathbb{C}^2) \oplus \mathrm{Sym}^{n-1}(\mathbb{C}^2))$. Its dimension is then 3 or $n+3$ or $n+5$ or $2n+5$. □

2.2.2. *Vector field interpretation.* To simplify our computations, we will use the following identification. The Lie algebra $\mathfrak{sl}_2$ may be viewed as a Lie algebra of linear vector fields on $\mathbb{C}^2$, namely,

$$\mathbb{C}X + \mathbb{C}H + \mathbb{C}Y$$

with $X = x\partial/\partial y$, $H = x\partial/\partial x - y\partial/\partial y$ and $Y = y\partial/\partial x$. These are the same standard X, Y and H as the matrices of Example 4.

The dual representation $\mathbb{C}^2 \otimes \mathrm{Sym}^n((\mathbb{C}^2)^\vee)$ is the space of vector fields on $\mathbb{C}^2$ whose coefficients are homogeneous polynomials of degree n. The decomposition in irreducible representation is the decomposition of any vector field in $\mathbb{C}^2 \otimes \mathrm{Sym}^n((\mathbb{C}^2)^\vee)$ as

$$A\frac{\partial}{\partial x} + B\frac{\partial}{\partial y} = G(x, y)\Big(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\Big) + \frac{\partial K}{\partial y}\frac{\partial}{\partial x} - \frac{\partial K}{\partial x}\frac{\partial}{\partial y}$$

with $G \in \mathrm{Sym}^{n-1}((\mathbb{C}^2)^\vee)$ and $K \in \mathrm{Sym}^{n+1}((\mathbb{C}^2)^\vee)$.[1]

The *symplectic gradient* of a polynomial K will be denoted by

$$J\nabla K := \frac{\partial K}{\partial y}\frac{\partial}{\partial x} - \frac{\partial K}{\partial x}\frac{\partial}{\partial y}.$$

If we define

$$K_i := \left(\binom{n+1}{i}\right) x^{n+1-i} y^i \quad\text{and}\quad E_i := \frac{1}{n+1} J\nabla(K_i),$$

[1]Using $G = \frac{1}{n+1}\big(\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y}\big)$ and $K = \frac{1}{n+1}(yA - xB)$.

then calculation shows that, as in the previous section,

$$[X, E_i]=(i+1)E_{i+1}, \quad [Y, E_i]=(n+2-i)E_{i-1} \quad \text{and} \quad [H, E_i]=(2i-n-1)E_i.$$

Lemma 11. *Let* $\mathfrak{h} := \mathrm{Lie}(\mathcal{A})$ *be the Lie algebra associated to the matrix* $\mathcal{A}$ *of* (LNVE$_n$). *Let* G *denote the differential Galois group of* (LNVE$_n$) *and* $\mathfrak{g}$ *be its Lie algebra. With the standard notation of Example 4, we have*:

(1) $\mathfrak{h}$ *is generated, as a Lie algebra, by* X, Y *and* E_0 *and* $\mathfrak{h} = \mathfrak{sl}_2 \rtimes \mathrm{Sym}^{n+1}(\mathbb{C}^2)$.

(2) *Either* $\mathfrak{g} = \mathfrak{sl}(2)$ (*of dimension* 3) *or* $\mathfrak{g} = \mathfrak{h}$ (*of dimension* $n+5$).

Proof. With the matrices of Example 4, we have $\mathcal{A}=X+tY+p(t)E_0$. As $[X, E_i]=(n+1-i)E_{i+1}$, the Lie algebra generated by X, Y and E_0 has dimension $n+5$ and may be identified with $\mathfrak{sl}_2 \rtimes \mathrm{Sym}^{n+1}(\mathbb{C}^2)$. Moreover, a Lie algebra containing X, Y and any of the E_i contains $\mathfrak{sl}_2 \rtimes \mathrm{Sym}^{n+1}(\mathbb{C}^2)$ (because $[Y, E_i] = (n+2-i)E_{i-1}$).

If 1, t and $p(t)$ are linearly independent over $\mathbb{C}$ then $\mathrm{Lie}(\mathcal{A})$ is the algebraic envelope of the Lie algebra generated by X, Y and E_0; because the latter is algebraic (it is $\mathfrak{sl}_2 \rtimes \mathrm{Sym}^{n+1}(\mathbb{C}^2)$), we have $\mathrm{Lie}(\mathcal{A}) = \mathfrak{sl}_2 \rtimes \mathrm{Sym}^{n+1}(\mathbb{C}^2)$. Now, we have seen that $\mathfrak{g}$ is either $\mathfrak{sl}_2$, or $\mathfrak{sl}_2 \rtimes \mathrm{Sym}^{n-1}(\mathbb{C}^2)$, or $\mathfrak{sl}_2 \rtimes \mathrm{Sym}^{n+1}(\mathbb{C}^2)$ or $\mathfrak{sl}_2 \rtimes (\mathrm{Sym}^{n-1}(\mathbb{C}^2) \oplus \mathrm{Sym}^{n-1}(\mathbb{C}^2))$. Among these, only $\mathfrak{sl}_2$ and $\mathfrak{sl}_2 \rtimes \mathrm{Sym}^{n+1}(\mathbb{C}^2)$ are in $\mathrm{Lie}(\mathcal{A})$, which proves the lemma in that case.

We are left with the case $p(t) = a + bt$ with $(a, b) \in \mathbb{C}^2$. Then $\mathrm{Lie}(\mathcal{A})$ is the algebraic envelope of the Lie algebra generated by $M_1 := X + aE_0$ and $M_2 := Y + bE_0$. If $b = 0$, then $[M_1, M_2] = H$ and $[M_1, H] = 2X + a(n+1)E_0$ so $[M_1, H] - 2M_1 = a(n-1)E_0$. So $\mathrm{Lie}(\mathcal{A})$ contains E_0 and we are done. If $b \neq 0$ then let $M_3 := [M_1, M_2] = H + (n+1)^2 bE_1$; then $[M_3, Y] = -2Y - (n+1)bE_0$ so $2M_2 - [M_3, Y]$ is a multiple of E_0 and the result is again true. □

Proof of Proposition 9. This follows from the above two lemmas and Theorem 1. □

2.3. ***Irreducibility criteria.*** Thanks to Proposition 9, to show the irreducibility of $y'' = xy + y^n P(x, y)/(Q(x, y))$, it is enough to show that (LNVE)$_n$ has a Lie algebra not isomorphic to $\mathfrak{sl}(2)$. Using the Kolchin–Kovacic reduction theory, we achieve this by proving (as in our first proof of irreducibility of Painlevé II) that there is no reduction matrix that transforms our system to one with Lie algebra $\mathfrak{sl}(2)$. This gives us the following simple irreducibility criterion.

Theorem 12. *We consider the equation* $(E): \ y'' = xy + y^n P(x, y)$. *Let* $p(t) := n!P(t, 0)$. *Let* $L_{n+1} := \mathtt{sym}^{n+1}(\partial_t^2 - t)$ *denote the* $(n+1)$*-th symmetric power of the Airy equation. Assume the equation* $L_{n+1}(f) = p(t)$ *does not admit a rational solution. Then, if* X *is the vector fields* $\partial/\partial x + z\partial/\partial y + (xy + y^n P(x, y))\partial/\partial z$, $\alpha = dx$ *and* $\gamma \iota_X dx \wedge dy \wedge dz$, *we have*

$$\mathrm{Mal}(X) = \{\varphi \mid \varphi^*\alpha = \alpha,\ \varphi^*\gamma = \gamma\}.$$

Corollary 13. *Under the assumption of Theorem 12, the equation (E) is irreducible.*

Lemma 14. *Let $A_1 = \left(\begin{smallmatrix} 0 & 1 \\ t & 0 \end{smallmatrix}\right)$ denote the companion matrix of the Airy equation. Let $A_1^\vee := -A_1^T$ denote the matrix of the dual system. In a convenient basis, the matrix Ψ of the adjoint action $[\mathcal{A}_{\mathrm{diag}}, \bullet]$ of $\mathcal{A}_{\mathrm{diag}}$ on $\mathfrak{h}_{\mathrm{sub}}$ is $\Psi = \mathtt{sym}^{n+1}(A_1)^\vee$.*

Proof. We have

$$\mathtt{sym}^{n+1}(A)^\vee = \begin{pmatrix} 0 & -(n+1)t & \ddots & & & \\ -1 & \ddots & -nt & 0 & & \\ \ddots & -2 & \ddots & \ddots & \ddots & \\ & 0 & \ddots & \ddots & -t & \\ & & \ddots & -n & 0 & \end{pmatrix}.$$

We choose the following basis of $\mathfrak{h}_{\mathrm{sub}}$, using again the vector field representation. Start from the same matrix $F_0 := E_0$ and set $F_{i+1} := -\frac{1}{i+1}[X, F_i]$. Then, one can check that $[Y, F_i] = -(n+2-i)F_{i-1}$. So the matrix Ψ of the map $[X + tY, \bullet]$ on the basis (F_i) is naturally $\mathtt{sym}^{n+1}(A)^\vee$. □

Proof of Theorem 12. Let us go backwards: assume that the equation $y'' = xy + y^n P(x, y)$ is reducible. Then we must have $\mathfrak{g}_3 = \mathfrak{sl}_2$ (otherwise, the dimension of the Lie Algebra $\mathfrak{g}_3$ would be $n+5$, thus exceeding the bound of Theorem 1). Reducing to $\mathfrak{sl}_2$ implies that we can find a rational solution to $Y' = \Psi Y + \vec{b}$, where $\vec{b} = (p(t), 0, \ldots, 0)^T$. Transforming the latter to an operator, via the cyclic vector $(0, \ldots, 0, 1)$ reduces the system to the equation $\mathtt{sym}^{n+1}(\partial_t^2 - t)^\vee = p(t)$. But $\partial^2 - t$ is selfadjoint, hence the result. □

The proof of the corollary is a direct result of Theorem 2.

Corollary 15. *The equation $(P_{\mathrm{II}}) : y'' = xy + 2y^3$ is irreducible.*

Proof. In this case, $n = 3$ and $L_4 = \partial^5 - 20t\partial^3 - 30\partial^2 + 64t^2\partial + 64t$. A solution of $L_4(y) = 12$ would be a polynomial (because L_4 has no finite singularity); now the image of a polynomial of degree N by L_4 is a polynomial of degree $N+1$ so 12 cannot be in the image of L_4. As equation $L_4(y) = 12$ has no rational solution, Theorem 12 shows that (P_{II}) is irreducible. □

Corollary 16. *Assume that $p(t)$ has a pole of order k, $1 \le k \le n+2$. Then the equation $y'' = xy + y^n P(x, y)$ is irreducible.*

Proof. As Airy has no finite singularity, neither does L_{n+1}. Thus, if a function $f \in \mathbb{C}(t)$ has a pole of order $d > 0$, then $L(f)$ has this pole of order $d + n + 2$. So if $p(t)$ is in the image of f by L then all its poles have order at least $n+3$. □

3. Irreducibility of Painlevé III equations

The third Painlevé equation is

$$(P_{\mathrm{III}}) \qquad \frac{d^2y}{dx^2} = \frac{1}{y}\left(\frac{dy}{dx}\right)^2 - \frac{1}{x}\frac{dy}{dx} + \frac{1}{x}(\alpha y^2 + \beta) + \gamma y^3 + \delta \frac{1}{y}$$

with $(\alpha, \beta, \gamma, \delta) \in \mathbb{C}^4$. For special values $(\alpha, \beta, \gamma, \delta) = (2\mu - 1, -2\mu + 1, 1, -1)$, $\mu \in \mathbb{C}$, this equation has a solution: $y = 1$. For $\mu = \frac{1}{2}$, this equation is related to the 2D Ising model in statistical physics; see [McCoy et al. 1977; Tracy and Widom 2011]. We will show that the latter equation is irreducible (in fact, we prove its irreducibility for $\mu \notin \mathbb{Z}$).

This equation has a time-dependent Hamiltonian form (see, e.g., [Clarkson 2006; 2010]). Letting

$$xH(x, y, z) = 2y^2z^2 - (xy^2 - 2\mu y - x)z - \mu xy,$$

we may consider the time-dependent Hamiltonian system

$$\left\{\frac{dy}{dx} = \frac{\partial H}{\partial z}, \quad \frac{dz}{dx} = -\frac{\partial H}{\partial y}\right\}.$$

Eliminating z between these equations shows that y satisfies (P_{III}). It also means that solutions of P_{III} parametrize curves

$$x \mapsto \left(x,\ y(x),\ \frac{xy'(x) + xy(x)^2 - 2\mu y(x) - x}{4y(x)^2}\right)$$

which are integral curves of the vector field $X = \partial/\partial x + \partial H/\partial z\, \partial/\partial y - \partial H/\partial y\, \partial/\partial z$.

Proposition 17. *Let* $(\alpha, \beta, \gamma, \delta) := (2\mu - 1, -2\mu + 1, 1, -1)$, *where* $\mu \notin \mathbb{Z}$. *The third Painlevé equation* (P_{III}) *with parameters* $(\alpha, \beta, \gamma, \delta)$ *is irreducible.*

Before we prove the theorem, we remark that it includes the case $\mu = 1/2$: the third Painlevé equation (P_{III}), as it appears in the study of the 2D Ising model in statistical physics in [McCoy et al. 1977; Tracy and Widom 2011], is irreducible.

Proof. This vector field X satisfies the hypothesis of our theorem with the forms $\alpha = dx$, $\gamma = dy \wedge dz + dH \wedge dx$ and the algebraic invariant curve (Γ) given by $y = 1, z = -\frac{\mu}{2}$.

The first variational equation along Γ has matrix

$$A_1 = \begin{pmatrix} -2 - 2\,\dfrac{\mu}{x} & \dfrac{4}{x} \\ -\mu - \dfrac{\mu^2}{x} & 2 + 2\,\dfrac{\mu}{x} \end{pmatrix}.$$

Conjugation by

$$Q_1 := \begin{pmatrix} -2\mu & 1 \\ -\mu^2 & 0 \end{pmatrix}$$

puts it in Jordan normal formal at 0, giving us

$$\tilde{A}_1 := Q_1^{-1}.A_1.Q_1 = \begin{pmatrix} 0 & \frac{1}{\mu}+\frac{1}{x} \\ 4\mu & 0 \end{pmatrix} = \frac{1}{x}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \frac{1}{\mu} \\ 4\mu & 0 \end{pmatrix}.$$

We have $\mathrm{Trace}(\tilde{A}_1) = 0$ so $\mathrm{Gal}(VE_1) \subset \mathrm{SL}(2, \mathbb{C})$. This first variational equation is equivalent to the differential operator

$$L_2 := \left(\frac{d}{dx}\right)^2 - 4 - 4\,\frac{\mu}{x}.$$

This L_2 is reducible for integer μ (it then has an exponential solution $e^{\pm 2x} P_\mu(x)$, where P_μ is a polynomial of degree $|\mu|$) and it is irreducible otherwise. Moreover, it admits a log in its local solution at 0, as shown by the Jordan form structure of the local matrix at 0. So, for $\mu \notin \mathbb{Z}$, the criterion of Boucher and Weil [2003] shows $\mathrm{Gal}(VE_1) = \mathrm{SL}(2, \mathbb{C})$ and that the first variational equation is in reduced form.

Let A_2 be the matrix of the second variational equation. As A_1 is in reduced form, we let

$$Q_2 := \left(\begin{array}{c|c} \mathrm{Sym}^2(Q_1) & 0_{3\times 2} \\ \hline 0_{2\times 3} & Q_1 \end{array}\right)$$

and $\tilde{A}_2 := Q_2^{-1} \cdot A_2 \cdot Q_2$. We obtain $\tilde{A}_2 = C_\infty + \frac{1}{x} C_0$, where C_i are constant matrices. Indeed, setting $M_1 := C_0$ and $M_2 := C_\infty - \frac{1}{\mu} C_0$, we have

$$M_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -4\mu^2 & 2\mu & 0 & 0 & 1 \\ 0 & 4\mu^2 & -2\mu & 0 & 0 \end{pmatrix} \text{ and } M_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 8\mu & 0 & 0 & 0 & 0 \\ 0 & 4\mu & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ -12\mu^2 & 0 & 1 & 4\mu & 0 \end{pmatrix}.$$

Now, letting $M_3 := \frac{1}{8\mu}[M_1, M_2]$, a simple calculation shows $[M_1, M3] = -M_1$ and $[M_2, M_3] = M_2$. It follows that the Lie algebra $\mathrm{Lie}(\tilde{A}_2)$ is equal to $\mathrm{SL}(2, \mathbb{C})$ (in a 5-dimensional representation). It follows that $\mathrm{Gal}(VE_2) \subseteq \mathrm{SL}(2, \mathbb{C})$. However, we know that $\mathrm{SL}(2, \mathbb{C}) \subseteq \mathrm{Gal}(VE_2)$ (because $\mathrm{Gal}(VE_1) = \mathrm{SL}(2, \mathbb{C})$) so $\mathrm{Gal}(VE_2) = \mathrm{SL}(2, \mathbb{C})$ and $\tilde{A}_2$ is in reduced form.

We thus need to go to the third variational equation. Its matrix has the form

$$A_3 = \left(\begin{array}{c|c|c} \mathtt{sym}^3(A_1) & & \\ \hline B_2^{(3)} & \mathtt{sym}^2(A_1) & \\ \hline B_3^{(3)} & B_2^{(2)} & A_1 \end{array}\right), \quad \text{where } A_2 = \left(\begin{array}{c|c} \mathtt{sym}^2(A_1) & \\ \hline B_2^{(2)} & A_1 \end{array}\right)$$

and $B_2^{(3)}$ comes from $B_2^{(2)}$ so the really new part is the south-west block $B_3^{(3)}$.

The situation is strikingly similar to the (P_{II}) case from the previous section. Let

$$N_1 = \left(\begin{array}{c|c} 0_{7\times 7} & 0_{7\times 2} \\ \hline \begin{matrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{matrix} & 0_{2\times 2} \end{array}\right), \; N_2 = \left(\begin{array}{c|c} 0_{7\times 7} & 0_{7\times 2} \\ \hline \begin{matrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{matrix} & 0_{2\times 2} \end{array}\right),$$

$$N_3 = \left(\begin{array}{c|c} 0_{7\times 7} & 0_{7\times 2} \\ \hline \begin{matrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{matrix} & 0_{2\times 2} \end{array}\right), N_4 = \left(\begin{array}{c|c} 0_{7\times 7} & 0_{7\times 2} \\ \hline \begin{matrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{matrix} & 0_{2\times 2} \end{array}\right),$$

$$N_5 = \left(\begin{array}{c|c} 0_{7\times 7} & 0_{7\times 2} \\ \hline \begin{matrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{matrix} & 0_{2\times 2} \end{array}\right).$$

As in the study of the variational equation, we form a block-diagonal partial reduction matrix Q_3 with blocks $\mathtt{sym}^3(Q_1)$, $\mathtt{sym}^2(Q_1)$, Q_1 and let $\tilde{A}_3 = Q_3^{(-1)} \cdot A_3 \cdot Q_3$. Again, we obtain $\tilde{A}_3 = C_\infty + \frac{1}{x} C_0$, where C_i are constant matrices. We set $M_1 := C_0$ and $M_2 := \frac{1}{4\mu} C_\infty - \frac{1}{\mu} C_0$ and $M_3 := [M_1, M_2]$. Then, direct inspection shows that $\mathrm{Lie}(\tilde{A}_3)$ is equal to $\mathrm{vect}_{\mathbb{C}}(M_1, M_2, M_3, N_1, \ldots, N_5)$ and has dimension 8. Using the results of the previous section, it follows that we have either $\mathfrak{g}_3 = \mathfrak{sl}(2)$ of dimension 3 or $\mathfrak{g}_3 = \mathrm{Lie}(\tilde{A}_3)$ of dimension 8.

The adjoint maps $\mathrm{Ad}_{M_i} := [M_i, \bullet]$ acting on $\mathrm{vect}_{\mathbb{C}}(N_1, \ldots, N_5)$ have respective matrices

$$\Psi_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 \end{pmatrix} \text{ and } \Psi_2 = \begin{pmatrix} 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and $\Psi_3 = [\Psi_1, \Psi_2]$ (this follows from the Jacobi identities on Lie brackets). The matrix of the adjoint action of $\tilde{A}_3$ on $\mathrm{vect}_k(N_1, \ldots, N_5)$ is $\Psi := \left(\frac{1}{\mu} + \frac{1}{x}\right)\Psi_1 + 4\mu\Psi_2$.

In order to have $\mathfrak{g}_3 = \mathfrak{sl}(2)$, we would need to find a find a gauge transformation matrix $P = \mathrm{Id}_{9\times 9} + \sum_{i=1}^{5} f_i N_i$ (with $f_i \in \bar{k}$) such that $\mathrm{Lie}(P[\tilde{A}_3]) = \mathrm{vect}_C(M_1, M_2, M_3)$. Let $\vec{b} = (b_1, \ldots, b_5)^T$ be defined by $\tilde{B}_3^{(3)} = \sum_{i=1}^{5} b_i N_i$, namely

$$\vec{b} = \left(-32\,\frac{\mu^4}{x},\ -8\,\frac{\mu^3}{x},\ 4/3\,\frac{\mu^2}{x},\ 0,\ 0\right)^T.$$

Then, letting $\vec{F} = (f_1, \ldots, f_5)^T$, the method developed for (P_{II}) in the previous section shows $\mathrm{Lie}(P[\tilde{A}_3]) = \mathrm{vect}_C(M_1, M_2, M_3)$ if and only if the 5×5 system $\vec{F}' = \Psi \cdot \vec{F} + \vec{b}$ has an algebraic solution.

It is easily seen that the latter is impossible. For example, the above system converts to $L(f_1) = g$ where

$$g = 8192\,\frac{\mu^4}{x} + 5120\,\frac{(4\,\mu+1)\mu^4}{x^2} + 512\,\frac{(24\,\mu^2+16\,\mu-7)\mu^4}{x^3} - 256\,\frac{(31\,\mu+3)\mu^4}{x^4} + 768\,\frac{\mu^4}{x^5}$$

and $L = \mathtt{sym}^4(L_2)$ where $L_2 = \left(\frac{d}{dx}\right)^2 - 4 - 4\,\frac{\mu}{x}$. When $\mu \notin \mathbb{Z}$ (as assumed here), the differential Galois group of L_2 (and hence of L) is $\mathrm{SL}(2, \mathbb{C})$). So the equation $L(f_1) = g$ has an algebraic solution if and only if it has a rational solution. Let us prove that the latter is impossible.

The exponents of L at zero are positive integers; it follows that, if f_1 had a pole of order $n \geq 1$ at zero, $L(f_1)$ would have a pole of order $n+5 \geq 6$ at zero. As g only has a pole of order 5 at zero, f_1 must be a polynomial. But then $L(f_1)$ would have a pole of order at most 4 at zero, contradicting the relation $L(f_1) = g$.

Reasoning as in Section 2, it follows that the Lie algebra of the Galois group of the third variational equation is $\mathfrak{g}_3 = \mathrm{Lie}(\tilde{A}_3)\,\mathrm{vect}_{\mathbb{C}}(M_1, M_2, M_3, N_1, \ldots, N_5)$ and has dimension 8. Our Theorem 1 thus implies that (P_{III}) is irreducible for these values of its parameters. □

Appendix A: Review on principal connections

The G-principal connections are the version of linear differential systems in fundamental form for an algebraic group G that may not be a linear group or not be canonically embedded in a GL_n. They are a geometric version of Kolchin's strongly normal extensions [1973]. The differential systems in vector form appear as a quotient of this fundamental (or principal) form.

A1. ***G-principal partial connection.*** Consider an algebraic group G and a smooth algebraic manifold M. A *principal G-bundle* is a bundle $P \xrightarrow{\pi} M$ over M such that G acts on P and the map $P \times G \to P \times_M P$ given by $(p, g) \mapsto (p, pg)$ is an isomorphism.

Let $\mathscr{F}$ be an algebraic singular foliation on M. A *connection along $\mathscr{F}$* (or a *partial connection*) on a bundle $P \xrightarrow{\pi} M$ is a lift of vector field tangent to $\mathscr{F}$ on P. If $0 \to T(P/M) \to TP \xrightarrow{\pi_*} TM \times_M P \to 0$ is the tangential exact sequence then a connection along $\mathscr{F}$ is a splitting above $\mathscr{F}$ given by $\nabla : T\mathscr{F} \times_M P \to TP$. Such a partial connection is called a *rational partial connection* when the splitting is rational.

We are interested in the case where $\mathscr{F}$ is defined by a rational vector field X on M. In this situation, it is enough to lift X to P by a rational vector field ∇_X such that $\pi_* \nabla_X = X$. Then ∇ is defined on a vector collinear to X by linearity.

A *G-principal connection along $\mathscr{F}$* is a G-equivariant splitting $\nabla : T\mathscr{F} \times_M P \to TP$ such that $\nabla(X)(pg) = g_* \nabla(X)(p)$ where $g_* : TP \to TP$ is the map induced by the action of g on P.

If $G \subset H$ is an inclusion of algebraic groups and P is a G-principal bundle then one defines an H-principal bundle $HP = (H \times P)/G$ where $(h, p)g = (hg, pg)$. A partial G-connection $\nabla : T\mathscr{F} \times_M P \to TP$ can be composed with the inclusion $H \times TP \subset T(H \times P)$ and we obtain a map $T\mathscr{F} \times_M (H \times P) \to T(H \times P)$. This map is G-equivariant. By taking quotients, we get the induced H-principal connection along $\mathscr{F}$, given by $H\nabla : T\mathscr{F} \times_M (HP) \to T(HP)$. It is the extension of ∇ to H. In particular, if G is a linear group then the extension of a partial G-principal connection to $\mathrm{GL}(n, \mathbb{C})$ is a usual linear connection in fundamental form with respect to variables tangent to $\mathscr{F}$.

A2. ***G-connections and their Galois groups.*** In this paper, a G-bundle $E \to M$ is given by: the typical fiber E_* (an affine variety with an action of G), a G-principal bundle $P \to M$ and a quotient $E = (P \times E_*)/G$ for the diagonal action of G. A principal connection along $\mathscr{F}$ on P will induce a connection along $\mathscr{F}$ on E. Such a connection is called a *partial G-connection on E*.

A connection ∇ on a bundle $E \to M$ may be viewed as a G-connection for many different groups (and maybe for no group). If we know that such a group G exists, we denote by GE the principal bundle and $G\nabla$ the G-principal connection. The Galois group of the G-connection will be a good candidate for such a group.

If $\mathbb{C}(M)^{\mathscr{F}} = \mathbb{C}$, i.e., when the foliation has no rational first integrals, then there exists a smallest algebraic group $\mathrm{Gal}\,\nabla \subset G$ such that ∇ is birational to a $\mathrm{Gal}\,\nabla$-connection. This group is well defined up to conjugation in G and is called the *Galois group of* ∇. Its existence is proved following the classical Picard–Vessiot theory in the following way. A $\mathrm{Gal}\,\nabla$-principal bundle is obtained as a minimal $G\nabla$-invariant algebraic subvariety $Q \subset GE$ dominating M and $\mathrm{Gal}\,\nabla$ is the stabilizer of Q in G. It is easy to prove that this group is a well-defined subgroup of G up to conjugacy.

When a connection ∇ is given, it is not easy to find a group G which would endow ∇ with a structure of G-connection. If such groups exist, we have to prove that our result does not depend on the choice of one of these groups. In the case of linear connections, there is a canonical choice (up to the choice of a point on M).

Given a vector bundle E, we say that ∇ is a *linear connection* when, for any $X \in \mathcal{F}$, $\nabla(X)$ preserves the module $\mathcal{E}$ of functions on E which are linear on the fibers. Then there is a canonical way to obtain a principal connection. Following Picard and Vessiot, if E_m is the fiber of E at $m \in M$ then the tensor product $E \otimes E_m^*$ of our vector bundle with the dual of the trivial vector bundle $M \times E_m$ is endowed with

- a connection given by the connection on the first factor,

- an action of $\mathrm{GL}(E_*)$ on the second factor, thus preserving the connection,
- a canonical point $id \in E_m \otimes E_m^*$ in the fiber at m.

Then the space $\max(E \otimes E_m^*)$ of tensors of maximal rank is a $\mathrm{GL}(E_m)$-principal bundle endowed with a principal partial connection. From the action of $\mathrm{GL}(E_m)$ on E_m, we see that linear connections are GL-connections.

The Galois group obtained from a linear connection using this principal bundle and the minimal invariant subvariety Q containing id is called $\mathrm{Gal}_m \nabla$.

A3. ***Principal bundle and groupoids.*** Given $P \xrightarrow{\pi} M$, a G-principal bundle over M, one obtains a groupoid $\mathcal{G}$ by taking the quotient $\mathcal{G} := (P \times P)/G$ of the cartesian product by the diagonal action of G (see [Mackenzie 1987] for more details; the main example is described in Appendices B2 and C2). The identity is the subvariety quotient of the diagonal in $P \times P$. From a G-principal connection ∇ on P, one derives a connection $\nabla \oplus \nabla$ on the product $P \times P$ defined in an obvious way from the decomposition $T(M \times M) \times_{M \times M} (P \times P) = TM \times_M P \oplus_P TM \times_M P$. This connection is the so-called *flows matrix equation*.

Let $G \subset H$ be an inclusion of algebraic groups and $HP \to M$ and $H\nabla$ be an extension of the principal connection to H. One gets a groupoid inclusion $\mathcal{G} \to \mathcal{H}$ such that $(H\nabla \oplus H\nabla)|_{\mathcal{G}} = \nabla \oplus \nabla$.

Remark 18. The following claims are not used in this paper. They may help the reader to understand the links between the various definitions of differential Galois groups appearing in the literature [Bertrand 1996; Katz 1990; Pillay 2004; Cartier 2009].

- The smallest algebraic subvariety of $\mathcal{G}$ which contains the identity and is $\nabla \oplus \nabla$-invariant is the Galois groupoid of ∇.
- Its restriction above $\{x\} \times M \subset M \times M$ is the Picard–Vessiot extension pointed at $x \in M$.
- Its restriction over the diagonal $M \subset M \times M$ is a $\mathcal{D}_M$-group bundle called the intrinsic Galois group of ∇ in the sense of Pillay [2004].

Appendix B: Variational equations

Various types of variational equations appear in the literature. Morales-Ruiz, Ramis and Simó discuss three of them in [Morales-Ruiz et al. 2007]. More precisely, there are various ways to obtain a linear system from the variational equation seen as an equation on germs of curves. In this paper, for the theoretical result, we consider the *frame variational equation* (see below) as the principal connection coming from the variational equation. However, for practical calculations, one generally linearizes the variational equation.

In this appendix, we give the definitions and the proofs needed to compare these different approaches. Some of these results can be found in [Morales-Ruiz et al. 2007, Propositions 8 to 12].

B1. *Arc bundles and the variational equation.* This variational equation does not appear in [Morales-Ruiz et al. 2007]. It has been used by several authors as a perturbative variational equation; see [Boucher and Weil 2003] for references.

The set of all parametrized curves on M is denoted by $CM = \{c : \widehat{(\mathbb{C}, 0)} \to M\}$. It has a natural structure of proalgebraic variety. Let $\mathbb{C}[M]$ be the coordinate ring of M and $\mathbb{C}[\delta]$ be the $\mathbb{C}$-vector space of linear ordinary differential operators with constant coefficients. The coordinate ring of CM is $\mathrm{Sym}(\mathbb{C}[M] \otimes \mathbb{C}[\delta])/L$, where

- the tensor product is a tensor product of $\mathbb{C}$-vector spaces,
- $\mathrm{Sym}(V)$ is the $\mathbb{C}$-algebra generated by the vector space V,
- $\mathrm{Sym}(\mathbb{C}[M] \otimes \mathbb{C}[\delta])$ has a structure of a δ-differential algebra via the right composition of differential operators,
- the Leibniz ideal L is the δ-ideal generated by $fg \otimes 1 - (f \otimes 1)(g \otimes 1)$ for all $(f, g) \in \mathbb{C}[M]^2$.

Local coordinates $(x_1, \dots, x_n)$ on M induce local coordinates on CM via the Taylor expansion of curves c at 0:

$$c(\epsilon) = \left(\sum c_1^{(k)} \frac{\epsilon^k}{k!}, \dots, \sum c_n^{(k)} \frac{\epsilon^k}{k!}\right).$$

Let $x_i^{(k)} : CM \to \mathbb{C}$ be defined by $x_i^{(k)}(c) = c_i^{(k)}$. This function is the element $x_i \otimes \delta^k$ in $\mathbb{C}[CM]$ and we have the following facts.

(1) $\mathbb{C}[CM]$ is the δ-algebra generated by $\mathbb{C}[M]$. The action of $\delta : \mathbb{C}[CM] \to \mathbb{C}[CM]$ can be written in local coordinates and gives the total derivative operator $\sum_{i,k} x_i^{(k+1)} \partial/\partial x_i^{(k)}$.

(2) Morphisms and derivations of $\mathbb{C}[M]$ act on $\mathbb{C}[CM]$ as morphisms and derivations, respectively, via the first factor (it can be easily checked that the Leibniz ideal is preserved).

(3) The vector space $\mathbb{C}[\delta]$ is filtered by the spaces $\mathbb{C}[\delta]^{\leq k}$ of operators of order less than k. This gives a filtration of $\mathbb{C}[CM]$ by $\mathbb{C}$-algebras of finite type.

(4) These algebras are coordinate rings of the space of k-jets of parametrized curves $C_k M = \{j_k c \mid c \in CM\}$.

(5) The action of δ has degree $+1$ with respect to the filtration.

(6) Prolongations of morphisms and derivations of $\mathbb{C}[M]$ on $\mathbb{C}[CM]$ have degree 0.

Set theoretically, the prolongations are obtained in the following way. Any holomorphic map $\varphi : U \to V$ between open subsets of M can be prolonged on open sets $C_k U$ of $C_k M$ of curves through points in U by $C_k\varphi : C_k U \to C_k V$; $j_k c \mapsto j_k(\varphi \circ c)$. One easily checks that $C_k(\varphi_1 \circ \varphi_2) = C_k\varphi_1 \circ C_k\varphi_2$. This equality can be used to prolong holomorphic vector fields defined on open subsets $U \subset M$ by the infinitesimal generator of the local 1-parameter group obtained by prolongation of the flow of X, i.e., $C_k(\exp(tX)) = \exp(tC_k X)$.

When X is a rational vector field on M, its prolongation $C_k X$ on $C_k M$ is also rational. Let $X = \sum a_i(x)\partial/\partial x_i$ be a vector field on M in local coordinates. One gets $C_k X = \sum_{i,\ell \le k} \delta^\ell(a_i)\partial/\partial x_i^\ell$.

If $\mathscr{F}$ is the foliation by integral curves of X on M then $C_k X$ defines a rational connection along $\mathscr{F}$ on $C_k M$: for a vector V tangent to $\mathscr{F}$ at $x \in M$ with $X(x) \neq 0$ or ∞, one defines $\nabla_V(j_k c) = V/(X(x)) R_k X(j_k c)$. It is the *k-th order variational connection/equation* of X.

Usually, the variational equation is studied along a given integral curve of X: if $\mathscr{C}$ is an invariant curve and if $C_k M_{\mathscr{C}}$ is the subspace of $C_k M$ of curves through points in $\mathscr{C}$, the vector field $C_k X$ preserves $C_k M_{\mathscr{C}}$. Its restriction to $C_k M_{\mathscr{C}}$ is called *the k-th order variational connection/equation along $\mathscr{C}$.*

B2. ***Frame bundles and the frame variational equation.*** This variational equation is the one used in the theoretical part of [Morales-Ruiz et al. 2007, Section 3.4] as well as in [Casale 2009].

The set of all formal frames on M is denoted by

$$RM = \{r : \widehat{(\mathbb{C}^n, 0)} \to M \mid \det(\mathrm{Jac}(r)) \neq 0\}.$$

Like the arc spaces, this set has a natural structure of proalgebraic variety. Let $\mathbb{C}[\partial_1, \dots, \partial_n]$ be the $\mathbb{C}$-vector space of linear partial differential operators with constant coefficients. The coordinate ring of RM is

$$(\mathrm{Sym}(\mathbb{C}[M] \otimes \mathbb{C}[\partial_1, \dots, \partial_n])/L)[1/\mathrm{Jac}],$$

where

- the tensor product is a tensor product of $\mathbb{C}$-vector spaces;
- $\mathrm{Sym}(V)$ is the $\mathbb{C}$-algebra generated by the vector space V;
- $\mathrm{Sym}(\mathbb{C}[M] \otimes \mathbb{C}[\partial_1, \dots, \partial_n])$ has a structure of a $\mathbb{C}[\partial_1, \dots, \partial_n]$-differential algebra via the right composition of differential operators;
- the Leibniz ideal L is the $\mathbb{C}[\partial_1, \dots, \partial_n]$-ideal generated by

  $$fg \otimes 1 - (f \otimes 1)(g \otimes 1)$$

  for all $(f, g) \in \mathbb{C}[M]^2$;

- the quotient is then localized by Jac, the sheaf of ideals (not differential!), generated by $\det([x_i \otimes \partial_j]_{i,j})$ for a transcendental basis $(x_1, \ldots, x_n)$ of $\mathbb{C}(M)$ on a Zariski open subset of M where such a basis is defined.

Local coordinates $(x_1, \ldots, x_n)$ on M induce local coordinates on RM via the Taylor expansion of maps r at 0:

$$r(\epsilon_1 \ldots, \epsilon_n) = \left(\sum r_1^\alpha \frac{\epsilon^\alpha}{\alpha!}, \ldots, \sum r_n^\alpha \frac{\epsilon^\alpha}{\alpha!} \right).$$

Let $x_i^\alpha : RM \to \mathbb{C}$ be defined by $x_i^\alpha(r) = r_i^\alpha$. This function is the element $x_i \otimes \partial^\alpha$ in $\mathbb{C}[RM]$.

(1) The action of $\partial_j : \mathbb{C}[RM] \to \mathbb{C}[RM]$ can be written in local coordinates and gives the total derivative operator $\sum_{i,\alpha} x_i^{\alpha+1_j} \partial/(\partial x_i^\alpha)$.

(2) We leave to the reader the translation of the properties from Appendix B1 in this multivariate situation.

All the remarks we have made about arc spaces extend *mutatis mutandis* to the frame bundle. There is one important difference: RM is a principal bundle over M. Let us describe this structure here.

The proalgebraic group

$$\Gamma = \{\gamma : \widehat{(\mathbb{C}^n, 0)} \xrightarrow{\sim} \widehat{(\mathbb{C}^n, 0)}, \text{ where } \gamma \text{ is formal invertible}\}$$

is the projective limit of groups

$$\Gamma_k = \{ j_k\gamma \mid \gamma : (\mathbb{C}^n, 0) \xrightarrow{\sim} (\mathbb{C}^n, 0), \text{ where } \gamma \text{ is holomorphic invertible}\}.$$

It acts on RM and the map $RM \times \Gamma \to RM \times_M RM$ sending (r, γ) to $(r, r \circ \gamma)$ is an isomorphism. The action of $\gamma \in \Gamma$ on RM is denoted by $S\gamma : RM \to RM$ as it acts as a change of source coordinates of frames. At the coordinate ring level, this action is given by the action of formal change of coordinates on $\mathbb{C}[\partial_1, \ldots, \partial_n]$ followed by the evaluation at 0 in order to get an operator with constant coefficients. This action has degree 0 with respect to the filtration induced by the order of differential operators. For any k, this means that the bundle of order k frames R_kM is a principal bundle over M for the group Γ_k.

When X is a rational vector field on M, its prolongation R_kX on R_kM is also rational. Let $X = \sum a_i(x)\partial/\partial x_i$ be a vector field on M in local coordinates. One gets $R_kX = \sum_{i,|\alpha|\le k} \partial^\alpha(a_i)\partial/\partial x_i^\alpha$.

If $\mathscr{F}$ is the foliation by integral curves of X on M then R_kX defines a rational connection along $\mathscr{F}$ on R_kM. Moreover the prolongation is defined by an action of the first factor on a tensor product whereas Γ_k acts on the other factor. These two actions commute, meaning that R_kX is a Γ_k-principal connection along $\mathscr{F}$. It is the *k-th order frame variational connection/equation* of X.

As for variational equations, one can restrict this connection above an integral curve $\mathscr{C}$ of X: one gets *the k-th order frame variational connection/equation along $\mathscr{C}$*. This connection is a principal connection of a bundle on $\mathscr{C}$. After choosing a point $m \in \mathscr{C}$ where X is defined, we obtain a Galois group $\mathrm{Gal}_m(R_k X|_{\mathscr{C}}) \subset \Gamma_k$.

From a frame $r : \widehat{(\mathbb{C}^n, 0)} \to M$, one derives many parametrized curves such that CM is a Γ-bundle. More precisely: if V_k denotes the vector space of k-jets of maps $\widehat{(\mathbb{C}, 0)} \to \widehat{(\mathbb{C}^n, 0)}$ then $C_k M = (R_k M \times V_k)/\Gamma$. The k-th order variational connection is a Γ_k-connection.

B3. ***The linearized variational equations.*** The variational equations are usually given in the linearized form described in Section 1.2. In [Morales-Ruiz et al. 2007], another linear variational equation is introduced, using a faithful linear representation of Γ. Let us recall these constructions and their relations with the frame variational equations above.

B3.1 *The Morales–Ramis–Simó linearization.* The theoretically easier linearization of variational equation is done through linearization of frame variational equations. This is the approach followed by Morales-Ruiz, Ramis and Simó. It is based on the fact that Γ_k is a linear group. Let V_k be the set of k-th order jets of map form $(\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ without invertibility condition. Using coordinates on $(\mathbb{C}^n, 0)$ one can check that V_k is a vector space (the addition depends on the choice of coordinates) and, using Faa di Bruno formulas, one can check that $(j_k s, j_k \gamma) \mapsto j_k(s \circ \gamma)$ defines a faithful representation of Γ_k on V_k. Then, from this inclusion $\Gamma_k \subset \mathrm{GL}(V_k)$, one gets an extension of the principal variational equation to the *jet-linearized principal variational connection*.

B3.2 *The geometric explanation of the linearization.* The second linearization (see Section 1) is done in the following way. The coordinate ring of the arc space has a natural degree from the filtration. It is defined on generators of the algebra by $d^\circ(f \otimes \delta^i) = i$. The Leibniz rule implies that the Leibniz ideal L (see Appendix B2) is generated by homogeneous elements and then the degree is well defined on $\mathbb{C}[CM]$.

This degree gives a decomposition $\mathbb{C}[CM] = \oplus_k \mathcal{E}_k$ in subspaces of homogeneous functions of degree k called jet differentials of degree k [Green and Griffiths 1980]. It is a straightforward to verify the following properties:

- $\mathcal{E}_k$ is a locally free $\mathbb{C}[M]$-module of finite rank.
- If $\varphi : U \to V$ is a biholomorphism on open sets of M then $C\varphi$, sending $\mathcal{O}_V \otimes_{\mathbb{C}[M]} \mathbb{C}[CM]$ to $\mathcal{O}_U \otimes_{\mathbb{C}[M]} \mathbb{C}[CM]$ preserves $\mathcal{O}_M \otimes \mathcal{E}_k$.
- If X is an holomorphic vector field on a open set U of M then $\mathcal{E}_k$ is CX-invariant.

Now X is a rational vector field on M and E_k is the dual vector bundle of $\mathcal{E}_k$. From these properties, we find that E_k, endowed with the action of X through CX, is a linear $\mathscr{F}$-connection. It is the *degree-linearized k-th order variational equation*.

The choice of an invariant curve $\mathscr{C}$ and a point $c \in \mathscr{C}$, such that X_c is defined and not zero, will give the Galois group of the degree-linearized variational equation along $\mathscr{C}$ at c denoted by $\mathrm{Gal}_c(LV_k\mathscr{C})$ (even though it depends on X).

The right composition with ∂ gives an inclusion $\mathcal{E}_k \to \mathcal{E}_{k+1}$ and thus a projection $\mathrm{Gal}_c(LV_{k+1}\mathscr{C})$ onto $\mathrm{Gal}_c(LV_k\mathscr{C})$. The inductive limit of differential systems is denoted by $LV_{\mathscr{C}}$, it is the *degree-linearized variational equation*. The projective limit of groups is denoted by $\mathrm{Gal}_c(LV\mathscr{C})$.

Proposition 19. *The Galois group of the degree-linearized variational equation is isomorphic to the Galois group of the frame variational equation.*

The proof will be given in Appendix C.

B4. ***The covariational equations.*** This variational equation is the one used in [Morales-Ruiz et al. 2007, p. 861] to linearize the variational equation.

The set of all formal functions on M is denoted by

$$FM = \{f : \widehat{(M, m)} \to \widehat{(\mathbb{C}, 0)}\}.$$

Its structural ring is $\mathbb{C}[FM] = \mathrm{Sym}(\mathcal{D}_M^{\geq 1})$, the $\mathbb{C}[M]$-algebra generated by the module of differential operators on M generated as an operator algebra by derivations. From this definition, FM is the vector bundle over M dual of $\mathcal{D}_M^{\geq 1}$. It is a projective limit of F_kM, the bundle of k-jets of functions (the dual of operators of order less than k).

A vector field X on M acts on $\mathcal{D}_M^{\geq 1}$ by the commutator $P \mapsto [X, P]$ and this action preserves the order. This gives a linear $\mathscr{F}$-connection on each F_kM. This is the linearized variational equation of [Morales-Ruiz et al. 2007]. In this paper, it is called the *covariational equation*. The following comparison result is proved in Appendix C1 below:

Proposition 20. *The covariational equation of order k and the variational equation of order k have the same Galois group.*

B5. ***Normal variational and normal covariational equations.*** When the vector field X preserves a foliation $\mathscr{G}$ on M then its prolongation C_kX on the space C_kM of k-jets of parametrized curves on M preserves the subspace $C_k\mathscr{G}$ of curves contained in leaves of $\mathscr{G}$. The subspace $C_k\mathscr{G}$ is an algebraic subvariety of C_kM and the restriction of C_kX on $C_k\mathscr{G}$ is the k-th order variational equation tangent to $\mathscr{G}$. When $\mathscr{G}$ is generically transversal to the trajectories of X, this equation is called the *k-th order normal variational equation*. We don't know how this equation depends on the choice of such a foliation $\mathscr{G}$. However, in our situation of a vector field given by a differential equation, there is a canonical transversal foliation given by the levels of the independent variable.

Let B be the curve with local coordinate x and $\pi : M \dashrightarrow B$ be the phase space of a differential equation with independent variable x. The foliation $\mathscr{G}$ is given by the level subsets of π. Using local coordinates $x_1, \dots, x_n$ on M such that $x_1 = x$, the subvariety $C_k\mathscr{G} \subset C_kM$ is described by the equations $x_1^\ell = 0$, $1 \le \ell \le k$. The variational equation in local coordinates is the system (VE_k) page 307. By setting $x_1^\ell = 0$, $1 \le \ell \le k$ into this system, one gets the differential system for the normal variational equation.

The linearization of the normal variational equation is done by the linearization of the variational equation. Let $I \subset \mathbb{C}[C_kM]$ be the ideal defining the subvariety $C_k\mathscr{G}$. Then $\mathscr{E}_k \cap I \subset \mathscr{E}_k \subset \mathbb{C}[C_kM]$ are finite rank linear spaces invariant under the action of C_kX. The induced action on the quotient $\mathscr{E}_k/(\mathscr{E}_k \cap I)$ is the *linearized k-th order normal variational equation.*

The normal covariational equation is more intrinsic. Let $F_k^XM \subset F_kM$ be the space of k-jets of first integrals f of X,

$$f : \widehat{(M, x)} \to \widehat{(\mathbb{C}, 0)},$$

such that $X.f = 0$. It is a linear subspace defined by its annihilator

$$\mathcal{D}_M^{\ge 1} \cdot X \subset \mathcal{D}_M^{\ge 1}.$$

The commutator $P \to [X, P]$ preserves $\mathcal{D}_M^{\ge 1} \cdot X$. So, it defines a linear connection on F_k^XM: this is the *normal covariational equation.*

Appendix C: The proofs

We recall the definitions and results of Casale [2009] using the frame bundle RM of M. It has a central place in the theory. In this appendix, it is used to present the Malgrange pseudogroup and in the previous one it was used to have the variational equation in fundamental form.

Because it is a principal bundle, it has an associated groupoid: $\mathrm{Aut}(M) = (RM \times RM)/\Gamma$. The Γ-orbit of a couple of frames (r, s) is the set of all $(r\circ\gamma, s\circ\gamma)$ for $\gamma \in \Gamma$. It is characterized by the formal map $r \circ s^{-1} : \widehat{(M, s(0))} \to \widehat{(M, r(0))}$. The quotient $\mathrm{Aut}(M)$ is the space of formal selfmaps on M with its natural structure of groupoid. For an $m \in M$, we define $\mathrm{Aut}(M)_{m,M}$ to be the space of maps with source at m and target anywhere on M. The choice of a frame $r : \widehat{(\mathbb{C}^n, 0)} \to \widehat{(M, m)}$ gives an isomorphism between $\mathrm{Aut}(M)_{m,M}$ and RM.

C1. *Proofs of the comparison propositions.*

Proof of Proposition 19. We will first compare these variational equations for a fixed order, then study their projective limits.

In order to compare all the variational equations, we will need to look more carefully at the frame bundle. The proof is then just another way to write the properties of $\mathscr{E}_k$. Its second property says that we have a group inclusion $\mathrm{Aut}_k(M)_{c,c} \to \mathrm{GL}(E_k(c))$ and a compatible inclusion of principal bundles,

$$\mathrm{Aut}_k(M)_{c,\mathscr{C}} \to E_k(\mathscr{C}) \otimes (E_k(c))^*.$$

This inclusion is compatible with the action of the vector field X. This means that the fundamental form of the k-th order degree-linearized variational equation (i.e., $C_k X$ action on $\max(E_k(\mathscr{C}) \otimes (E_k(c))^*)$) is an extension of the frame variational equation. Thus, their Galois groups are the same.

The comparison of limit groups is not direct because the family $(\mathrm{GL}(E_k(c))_k$ is not a projective system. The module $\mathcal{E}_k$ is filtered by

$$\mathcal{E}_0 \circ \delta^k \subset \mathcal{E}_1 \circ \delta^{k-1} \subset \cdots \subset \mathcal{E}_k.$$

Let $T_k \subset \mathrm{GL}(E_k(c))$ denote the subgroup preserving this filtration. Now,

- there is a natural projection $T_k \to T_{k-1}$,
- the Galois group of the k-th order degree-linearized variational equation is a subgroup of T_k, and
- the inclusion $\mathrm{Aut}_k(M)_{c,c} \to T_k$ is compatible with the projections.

This proves the proposition. □

Proof of Proposition 20. There is a direct way to see that the variational equation and the covariational equation will have the same Galois group. Instead of using the Picard–Vessiot principal bundle for the covariational equation, one can build a better principal bundle. Consider the bundle of coframes

$$R^{-1}M = \{f : \widehat{(M,m)} \to \widehat{(\mathbb{C}^n,0)}, \text{ where } f \text{ is formal and invertible}\}$$

whose coordinate ring is $\mathrm{Sym}(\mathcal{D}_M^{\geq 1} \otimes \mathbb{C}^n)[1/\mathrm{Jac}]$. This is a Γ-principal bundle. The action of X by the commutator defines a Γ-principal connection. This connection is called the *coframe variational equation*. The map $R \to R^{-1}$ sending a frame r to its inverse r^{-1} is an isomorphism of principal bundles (up to changing the side of the group action) conjugating the frame and the coframe variational equations.

Now let F_k be the vector space of k-jets of formal maps $\widehat{(\mathbb{C}^n,0)} \to \widehat{(\mathbb{C},0)}$ and C_k be the vector space of k-jets of formal maps $\widehat{(\mathbb{C},0)} \to \widehat{(\mathbb{C}^n,0)}$; then one has

$$F_k M = (R_k^{-1} M \times F_k)/\,\Gamma_k$$

and $C_k M = (R_k M \times C_k)/\,\Gamma_k$. Moreover these two isomorphisms are compatible with the various variational equations. So, the Galois group of the covariational equation equals the one of the variational equation. □

C2. *The Malgrange pseudogroup.* The *Malgrange pseudogroup of a vector field X on M* is a subgroupoid of $\mathrm{Aut}(M)$. We choose here to call it a pseudogroup as its elements are formal diffeomorphisms between the formal neighborhoods of points of M satisfying the definition of a pseudogroup; see [Casale 2009].

It is defined by means of differential invariants of X, i.e., rational functions $H \in \mathbb{C}(RM)$ such that $RX \cdot H = 0$. Let $\mathrm{Inv}(X) \subset \mathbb{C}(RM)$ be the subfield of differential invariants of X. Let W be a model for Inv and $\pi : RM \dashrightarrow W$ be the dominant map from the inclusion $\mathrm{Inv} \subset \mathbb{C}(RM)$. Let $\mathrm{Mal}(X)$ be $(RM \times_W RM)/\Gamma \subset \mathrm{Aut}\, M$. To define properly this fiber product, one needs to restrict $\pi : (RM)^o \to W$ on its domain of definition. Then, $RM \times_W RM$ is defined to be the Zariski closure of $(RM)^o \times_W (RM)^o$ in $RM \times RM$. By construction, any Taylor expansion of the flow of X belongs to $\mathrm{Mal}(X)$.

Malgrange [2001] showed that there exists a Zariski open subset M^o of M such that the restriction on $\mathrm{Mal}(X)$ to $\mathrm{Aut}(M^o)$ is a subgroupoid. This result was extended by Casale [2009] and allows us to view the Malgrange pseudogroup as a set-theoretical subgroupoid of $\mathrm{Aut}(M)$.

From the Cartan classification of pseudogroups [1908], one gets the following theorem for rank two differential system; see the appendix of [Casale 2008] for a proof.

Theorem 21 [Cartan 1908; Casale 2008]. *Let M be a smooth irreducible algebraic 3-fold over $\mathbb{C}$ and X be a rational vector field on M such that there exist a closed rational 1-form α with $\alpha(X) = 1$ and a closed rational 2-form γ with $\iota_X \gamma = 0$. One of the following statements holds.*

- *On a covering $\widetilde{M} \xrightarrow{\pi} M$ of a Zariski open subset of M, there exists a rational 1-form ω such that $\omega \wedge d\omega = 0$ and $\omega(\pi^* X) = 0$. Then*

  $$\mathrm{Mal}(X) \subset \{\varphi \mid \varphi^*\alpha = \alpha,\ (\widetilde{\varphi}^*\omega) \wedge \omega = 0\},$$

  *where $\widetilde{\varphi}$ stands for any lift of φ to $\widetilde{M}$. The vector field is said to be **transversally imprimitive**.*

- *There exists a vector of rational 1-forms $\Theta = \binom{\theta_1}{\theta_2}$ such that $\theta_1 \wedge \theta_2 = \gamma$ and a trace free matrix of 1-forms Ω such that*

  $$d\Theta = \Omega \wedge \Theta \quad \textit{and} \quad d\Omega = -\Omega \wedge \Omega.$$

  *One has $\mathrm{Mal}(X) \subset \{\varphi \mid \varphi^*\alpha = \alpha,\ \varphi^*\Theta = D\Theta,\ \textit{and}\ dD = [D, \Omega]\}$. The vector field is said to be **transversally affine**.*

- $\mathrm{Mal}(X) = \{\varphi \mid \varphi^*\alpha = \alpha,\ \varphi^*\gamma = \gamma\}$.

In order to compute dimensions of Malgrange pseudogroups and of Galois groups of variational equations, it will be easier to work with the Lie algebra of the Malgrange pseudogroup. Roughly speaking, $\mathfrak{mal}(X)$ is the sheaf of Lie algebra of vector fields whose flows belongs to Mal(X). The reader is invited to read [Malgrange 2001] for a formal definition.

C3. ***Proof of our main theorem.***

Proof. The assumptions made on X, α and γ ensure that the first variational equation is reducible to a block-diagonal matrix with a first block in $\mathrm{SL}_2(\mathbb{C})$ and a second equal to the identity matrix.

Using theorems above, the proof of this theorem is reduced to Lemmas 22 and 26.

Lemma 22. *If there exists a finite map $\pi : \widetilde{M} \to M$ and an integrable 1-form on $\widetilde{M}$ vanishing on $\pi^* X$ then the Lie algebra of the Galois group of the first variational equation along any solution is solvable.*

Proof. This lemma is proved in the spirit of Ziglin and Morales and Ramis (see, for example, [Audin 2001]).

First we have to descend from the covering to M. Remark that ω is a 1-form on M whose coefficients are algebraic functions. Letting $\omega_i,\ i = 1, \dots, \ell$, be the conjugates of ω, the product $\bar{\omega} = \prod \omega_i \in \mathrm{Sym}^\ell \Omega^1(M)$ is a well-defined rational symmetric form on M. For any holomorphic function f on open subsets of M, $f\bar{\omega}$ satisfies all hypotheses so that one can assume that the rational symmetric form $\bar{\omega}$ is holomorphic at the generic point of $\mathscr{C}$.

At generic $p \in \mathscr{C}$, the section $\bar{\omega} : M \dashrightarrow \mathrm{Sym}^\ell T^* M$ vanishes at order k, thus one can write $\bar{\omega} = \bar{\omega}_k(p) + \cdots$, where $\bar{\omega}_k(p)$ is lowest order homogeneous part of $\bar{\omega}$. It is a well-defined symmetric form ℓ on the tangent space $T_p M$, that is, $\bar{\omega}_k(p) : T_p M \to \mathrm{Sym}^\ell T^*(T_p M)$; note that $\bar{\omega}_k(p)(\lambda v)(\mu w) = \lambda^k \mu^\ell \bar{\omega}_k(p)(v)$ for any $v \in TpM$ and any $w \in T_v(T_p M)$. We may say that $\bar{\omega}$ is a symmetric ℓ-form on the space $T_p M$, homogeneous of degree k.

Sublemma 23. *The vanishing order of $\bar{\omega}$ is constant on a Zariski open subset of $\mathscr{C}$.*

Proof. Let $p \in \mathscr{C}$ be such that $X(p) \neq 0$. One can choose rectifying coordinates $x_1, \dots, x_n$ such that $x(p) = 0,\ X = \partial/\partial x_1$ and

$$\bar{\omega} = \sum_{\substack{\alpha \in \times \mathbb{N}^{n-1} \\ |\alpha| = \ell}} w_\alpha(x) \prod_{i=2}^{n} dx_i^{\alpha_i}.$$

Since $\mathscr{L}_X \bar{\omega} \wedge \bar{\omega} = 0$, one has that, for any $c \in \mathbb{C}$ small enough, $w_\alpha(x_1 + c, \dots, x_n) = f_c(x) w_\alpha(x_1, \dots, x_n)$ where f_c is a holomorphic function depending on c not on α. Now $f_0 = 1$ thus $f_c(0) \neq 0$ for c small enough and the vanishing order of ω at 0 equals the one at $(c, 0, \dots, 0)$. □

This sublemma enables us to define a rational section

$$\bar{\omega}_k : T_{\mathscr{C}} M \dashrightarrow \mathrm{Sym}^\ell V^*(T_{\mathscr{C}} M),$$

where $V^*(T_{\mathscr{C}} M) = T^*(T_{\mathscr{C}} M)/T^* \mathscr{C}$.

Remember that from X, we get a vector field C_1X on $T_{\mathscr{C}}M$ called the first order variational equation along $\mathscr{C}$.

Sublemma 24. $$\mathscr{L}_{C_1X}\bar{\omega}_k \wedge \bar{\omega}_k = 0.$$

Proof. Here again, we will prove the sublemma in local analytic coordinates. Let $p \in \mathscr{C}$ be such that $X(p) \neq 0$. One can choose rectifying coordinates $x_1, \dots, x_n$ such that $x(p) = 0$, $X = \partial/\partial x_1$ and

$$\bar{\omega} = \sum_{\substack{\alpha \in \times \mathbb{N}^{n-1} \\ |\alpha| = \ell}} w_\alpha(x) \prod_{i=2}^{n} dx_i^{\alpha_i}.$$

For any $c \in \mathbb{C}$ small enough, $w_\alpha(x_1 + c, \dots, x_n) = f_c(x) w_\alpha(x_1, \dots, x_n)$ so the zero set of $\bar{\omega}$ in $T_{\mathscr{C}}M$ is a subvariety invariant under translations collinear to X. One can get local equations for this zero set in the form

$$\eta = \sum_{\substack{\alpha \in \times \mathbb{N}^{n-1} \\ |\alpha| = \ell}} n_\alpha(x_2, \dots, x_n) \prod_{i=2}^{n} dx_i^{\alpha_i}$$

and there exists a holomorphic h such that $\bar{\omega} = h\eta$. Now, by taking the lowest order homogeneous parts, one gets $\bar{\omega}_k = h_{k_1}\eta_{k_2}$. Since η is x_1-independent so is η_{k_2}. In local coordinates induced on $T_{\mathscr{C}}M$, $C_1X = \partial/\partial x_1$ so $\mathscr{L}_{C_1X}\eta_{k_2} = 0$ and a direct computation proves that $\mathscr{L}_{C_1X}\bar{\omega}_k \wedge \bar{\omega}_k = 0$. □

Sublemma 25. $\mathrm{Gal}(C_1X)$ *is virtually solvable.*

Proof. The rational form $\bar{\omega}_k$ defines in each fiber of $T_{\mathscr{C}}M$ a homogeneous ℓ-web. This fiberwise rational web is C_1X-invariant. This implies that the action of the Galois group on a fiber T_pM must preserve this web. In other words, the Galois group at p preserves the set of symmetric forms on T_pM which are rational multiples of $\bar{\omega}_k(p)$.

The form η given in the previous sublemma shows that the web is a pull-back of a web defined on the normal bundle of $\mathscr{C}$ in M. The group $\mathrm{Gal}(C_1X)$ is included in a block diagonal group with a block (1) and a 2×2 block given by a subgroup of $\mathrm{SL}(2, \mathbb{C})$. As $\mathrm{SL}(2, \mathbb{C})$ does not preserve a web on $\mathbb{C}^2$, the 2×2 block is a proper subgroup of $\mathrm{SL}(2, \mathbb{C})$. This proves the sublemma, and thus concludes the proof of Lemma 22. □

Lemma 26. *If X is transversally affine then the Galois group of the formal variational equation along any solution has dimension smaller than* 5.

Proof. We will see that this lemma is a consequence of theorems and lemmas from [Casale 2009]. From Theorem 2.4 there, the Galois group of the formal variational equation along $\mathscr{C}$ is a subgroup of

$$\mathrm{Mal}(X)_p = \{\varphi : (M, p) \to (M, p) \mid \varphi \in \mathrm{Mal}(X)\}$$

for a generic $p \in \mathscr{C}$. Its Lie algebra is included in

$$\mathfrak{mal}(X)_p^0 = \{Y \text{ a vector field on } (M, p) \mid Y(p) = 0, Y \in \mathfrak{mal}(X)\}.$$

From Lemma 3.8 of [Casale 2009], the dimension of

$$\mathfrak{mal}(X)_p = \{Y \text{a vector field on } (M, p) \mid Y \in \mathfrak{mal}(X)\}$$

for $p \in \mathscr{C}$ is smaller than the dimension of the same Lie algebra for generic $p \in M$.

Assume X is transversally affine and choose a point $p \in M$ such that the 1-form α and the forms Θ^0 and Θ^1 from the definition are holomorphic and $\alpha \wedge \theta_1^0 \wedge \theta_2^0 \neq 0$. Then we choose local analytic coordinates such that $\alpha = dx_1$ and $\left[\begin{smallmatrix} dx_2 \\ dx_3 \end{smallmatrix}\right] = F\Theta^0$ with $dF + F\Theta^1 = 0$. In these coordinates, $\varphi \in \mathrm{Mal}$ satisfies $\varphi^*\alpha = \alpha$, $\varphi^*\Theta^0 = D\Theta^0$ and $dD = [D, \Theta^1]$ if and only if

$$\varphi(x_1, x_2, x_3) = (x_1 + c_0, c_1x_2 + c_2x_3 + c_3, c_4x_1 + c_5x_2 + c_6)$$

with $c \in \mathbb{C}^7$ such that $\det\left[\begin{smallmatrix} c_1 & c_2 \\ c_4 & c_5 \end{smallmatrix}\right] = 1$. The infinitesimal version of these calculations shows that the dimension of $\mathfrak{mal}(X)_p$ is smaller than 6. The Lie algebra $\mathfrak{mal}(X)_p^0$ is strictly smaller than $\mathfrak{mal}(X)_p$ as it does not contain $X = \partial/\partial x_1$ so the dimension of the Galois group of the formal variational equation is smaller than or equal to 5. □

Combining these two lemmas, we see Theorem 1 follows from Theorem 21. □

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COMBINATORIAL CLASSIFICATION OF QUANTUM LENS SPACES

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We answer the question of how large the dimension of a quantum lens space must be, compared to the primary parameter r, for the isomorphism class to depend on the secondary parameters. Since classification results in C^*-algebra theory reduce this question to one concerning a certain kind of SL-equivalence of integer matrices of a special form, our approach is entirely combinatorial and based on the counting of certain paths in the graphs shown by Hong and Szymański to describe the quantum lens spaces.

1. Introduction

In a seminal paper by Hong and Szymański [2003], an important class of *quantum lens spaces* $C(L_q(r; (m_1, \dots, m_n)))$ was given a description as C^*-algebras arising from certain graphs — or their adjacency matrices — in the vein of Cuntz and Krieger [1980]. These graphs can be read off directly from the data $(r; (m_1, \dots, m_n))$ determining the quantum lens space, where $r > 2$ are integers and m_i are units of $\mathbb{Z}/r\mathbb{Z}$. Using this characterisation, it is easy to see that $C(L_q(r; (m_1, \dots, m_n)))$ can only be isomorphic to $C(L_q(r'; (m'_1, \dots, m'_{n'})))$ when $r = r'$ and $n = n'$, and this raises the important question of to what extent the choice of the units can influence the C^*-algebras.

To answer such questions, one appeals naturally to the classification theory for C^*-algebras by K-theory, as indeed a large class of Cuntz–Krieger algebras were classified by Restorff [2006]. Unfortunately, the quantum lens spaces fall outside this class, and indeed, outside any class considered at the time [Hong and Szymański 2003] was written. Thus, apart from noting that the m_i can obviously not influence the C^*-algebras when $n \le 3$, Hong and Szymański left the question open.

Quantum lens spaces are still a subject of interest, however, see for instance Arici, Brain, and Landi [Arici et al. 2015] and Brzeziński and Szymański [2018], and using recent classification results obtained for Cuntz–Krieger algebras with

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uncountably many ideals, Eilers, Restorff, Ruiz, and Sørensen in [Eilers et al. 2018] managed to reduce this question to elementary matrix algebra and to prove that when $n = 4$ there are precisely two different $C(L_q(r; (m_1, \dots, m_n)))$ when r is a multiple of 3, and only one when r is not.

Søren Eilers conducted computer experiments for other r and n which suggested that the quantum lens spaces are unique when $n < s$ for s the smallest even number strictly larger than the smallest divisor of r which is not 2, and that at least two choices of m_i give different C^*-algebras when $n \geq s$. It is the aim of the paper at hand to provide the combinatorial insight needed to prove that this in fact is the case, and to study the number of different C^*-algebras that can be obtained by varying the m_i.

We will not work directly on questions of isomorphism of the C^*-algebras, and hence, *no prior knowledge on C^*-algebras or their classification theory is required.* Instead we study the equivalent notion of SL equivalence of the graphs associated to the given data. Indeed, a result of [Eilers et al. 2018] states that the following are equivalent:

- $C(L_q(r; (m_1, \dots, m_n))) \otimes \mathbb{K} \simeq C(L_q(r; (m'_1, \dots, m'_n))) \otimes \mathbb{K}$.
- There exist integer matrices U, V both of the form

$$\begin{pmatrix} 1 & * & * & \cdots & * \\ & 1 & * & & \\ & & \ddots & \ddots & \vdots \\ & & & 1 & * \\ & & & & 1 \end{pmatrix}$$

  so that $U(\mathsf{A}_{(r;(m_1,\dots,m_n))} - I) = (\mathsf{A}_{(r;(m'_1,\dots,m'_n))} - I)V$.

The exact notation and definitions will be given in Section 2 together with the rudimentary results needed for our classification. Section 3 handles the most general case, basically establishing the influence of the odd prime divisors of the parameter r on the number of C^*-algebras emerging by varying the m_i. A lower bound on the number of such C^*-algebras is found and for $4 \nmid r$ the exact s such that the C^*-algebra is unique for $n < s$ is determined. The special case of finding s when $4 \mid r$ is then dealt with in Section 4.

The main result of the paper is Theorem 5.1 which combines the results of Sections 3 and 4 to find for every $r > 2$ the s such that the C^*-algebra is unique for every $n < s$. The other major achievement is Theorem 3.9 which bounds the number of different quantum lens spaces arising for some $r > 2$ and $n \in \mathbb{N}$. Based on computer experiments, we conjecture that this bound is in fact an equality when $4 \nmid r$ (Conjecture 5.3).

2. Preliminaries

We dedicate this section to setting the stage. We establish notation, definitions, and find initial results that will assist in showing the later sections' classification results.

Number theoretical notation.

Definition 2.1. We let Z_n denote the multiplicative group of integers modulo n. That is $Z_n = (\mathbb{Z}/n\mathbb{Z})^*$.

Notation 2.2. We write $p^k \| n$ if $p^k \mid n$ and $p^{k+1} \nmid n$, i.e., k is the greatest power of p dividing n.

Notation 2.3. To ease notation, we write the reduction of an integer a calculated modulo r as $[a]_r$, i.e., we always have $0 \le [a]_r \le r-1$.

The graph. This section will introduce a definition of the graph $\mathsf{M}_{(r;(m_1,\dots,m_n))}$, arising from the quantum lens space $C(L_q(r;(m_1,\dots,m_n)))$ as defined in [Hong and Szymański 2003]. Further, we introduce another graph $\mathsf{N}_{(r;(m_1,\dots,m_n))}$, which is easier to work with in the combinatorial setting, but has similar properties in a sense that will be made clear.

Definition 2.4. Let $r > 2$ and $\overline{m} = (m_1, \dots, m_n) \in (Z_r)^n$ for some $n \in \mathbb{N}$. Then we define a directed graph $\mathsf{M}_{(r;\overline{m})}$ in the following way:

- For every pair s, t with $1 \le s \le n$ and $0 \le t < r$ there is a vertex $g_{s,t}$.
- There is a directed edge from g_{s_1,t_1} to g_{s_2,t_2} if and only if $s_1 \le s_2$ and $t_2 = [t_1 + m_{s_1}]_r$.

For every $s \in \mathbb{N}$ we will call the subgraph consisting of the vertices $\{g_{s,x} \mid 0 \le x < r\}$ the s-th subgraph of $\mathsf{M}_{(r;\overline{m})}$, and we will call a vertex of the form $g_{s,c}$ a c-vertex.

An example of the graph $\mathsf{M}_{(5;(1,2,1))}$ is sketched in Figure 1.

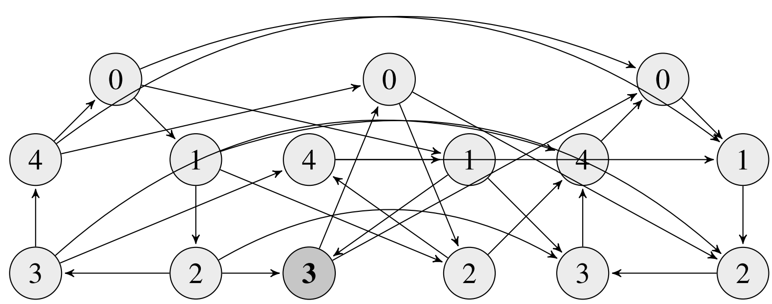

Figure 1. Example of $\mathsf{M}_{(r;\overline{m})}$ with $n = 3$, $r = 5$ and $m = (1, 2, 1)$. The bold 3 in a darker circle denotes the vertex $g_{2,3}$.

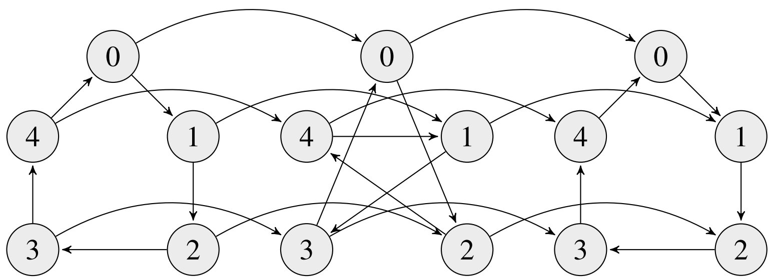

Figure 2. Example of $\mathsf{N}_{(r;\overline{m})}$ where $n = 3,\ r = 5$, and $m = (1, 2, 1)$.

Definition 2.5. Let $r > 2$ and $\overline{m} = (m_1, \ldots, m_n) \in (Z_r)^n$ for some $n \in \mathbb{N}$. Then we define a directed graph $\mathsf{N}_{(r;\overline{m})}$ in the following way:

- For every pair s, t with $1 \le s \le n$ and $0 \le t < r$ there is a vertex $c_{s,t}$.
- There is a directed edge from c_{s_1,t_1} to c_{s_2,t_2} in the following two cases:
  - ◊ $s_1 + 1 = s_2$ and $t_2 = t_1$.
  - ◊ $s_1 = s_2$ and $t_2 = [t_1 + m_{s_1}]_r$.

For every s we will call the subgraph consisting of the vertices $\{c_{s,x} \mid 0 \le x < r\}$ the s-th subgraph of $\mathsf{N}_{(r;\overline{m})}$, and we will call a vertex of the form $c_{s,t}$ a t-vertex.

Here is the graph we would rather look at. Instead of having edges from a subgraph to all the subgraphs after it, it only has edges to the one just after it. This edge will always go from $c_{s,t}$ to $c_{s+1,t}$. We show an example of the graph on Figure 2.

Definition 2.6. Let $r > 2$ and $\overline{m} = (m_1, \ldots, m_n) \in (Z_r)^n$ for some $n \in \mathbb{N}$. Then we let $\mathsf{A}_{(r;\overline{m})}$ be the matrix such that $\mathsf{A}_{(r;\overline{m})}\langle i, j\rangle$ is the number of directed paths in $\mathsf{M}_{(r;\overline{m})}$ from the 0-vertex of the i-th subgraph to the 0-vertex of the j-th subgraph that does not pass through the 0-vertex of any other subgraph. We call a path that satisfies these criteria *legal*.

Definition 2.7. Let $r > 2$ and $\overline{m} = (m_1, \ldots, m_n) \in (Z_r)^n$ for some $n \in \mathbb{N}$. Then we let $\mathsf{B}_{(r;\overline{m})}$ be the matrix such that $\mathsf{B}_{(r;\overline{m})}\langle i, j\rangle$ is the number of directed paths on $\mathsf{N}_{(r;\overline{m})}$ from the 0-vertex of the i-th subgraph to the 0-vertex of the j-th subgraph which do not exclusively visit 0-vertices and which do not visit the 0-vertex of any other subgraph except if all the following vertices of the path are 0-vertices. We will call a path that satisfies these criteria *legal*.

We introduce this new graph definition $\mathsf{N}_{(r;\overline{m})}$ because it is easier to work with than $\mathsf{M}_{(r;\overline{m})}$. Note we always calculate indices in subgraphs modulo r.

Lemma 2.8. *Let $r > 2$ and $\overline{m} \in (Z_r)^n$ be given. Then $\mathsf{A}_{(r;\overline{m})} = \mathsf{B}_{(r;\overline{m})}$.*

Proof. There is a bijection between the edges of $\mathsf{M}_{(r;\overline{m})}$ and paths of $\mathsf{N}_{(r;\overline{m})}$ as follows. The edge $g_{s_1,t_1} \to g_{s_2,t_1+m_{s_1}}$ of $\mathsf{M}_{(r;\overline{m})}$ corresponds to the path

$$c_{s_1,t_1} \to c_{s_1,t_1+m_{s_1}} \to c_{s_1+1,t_1+m_{s_1}} \to \cdots \to c_{s_2,t_1+m_{s_1}}$$

on $\mathsf{N}_{(r;\overline{m})}$.That this is a bijection follows immediately from the fact that the edge and path are both uniquely determined by s_1, s_2, and t_1.

Now, we need to establish a bijection between the legal paths on $\mathsf{M}_{(r;\overline{m})}$ and the legal paths on $\mathsf{N}_{(r;\overline{m})}$. This happens naturally by translating any edge in a legal path on $\mathsf{M}_{(r;\overline{m})}$ into a subpath of the form above of a legal path on $\mathsf{N}_{(m;\overline{r})}$. That this map has an inverse follows easily since any legal path in $\mathsf{N}_{(r;\overline{m})}$ consists of subpaths of the above form where a new subpath starts whenever we stay in the same subgraph. Further, we have that the constraint of Definition 2.6 translates into the constraint of Definition 2.7: an edge from the t-th subgraph to the 0-vertex of the n-th subgraph in Definition 2.6 corresponds to going to the 0-vertex in the t-th subgraph and then visiting 0-vertices exclusively until reaching the 0-vertex of the n-th subgraph in Definition 2.7. □

Equivalence classes. The overall aim of the article is to classify the quantum lens spaces, which is a problem that Theorem 7.1 of Section 7.2 of [Eilers et al. 2018] reduces to a question of SL equivalence, hence elementary matrix algebra.

Theorem 2.9 (Eilers, Restorff, Ruiz, and Sørensen). *Let $r > 2$ and $\overline{m}, \overline{m}' \in (Z_r)^n$ be given. The following are equivalent*:

- $C(L_q(r; (m_1, \dots, m_n))) \otimes \mathbb{K} \simeq C(L_q(r; (m'_1, \dots, m'_n))) \otimes \mathbb{K}$.
- *There exist matrices U, V both of the form*

$$\begin{pmatrix} 1 & * & * & \cdots & * \\ & 1 & * & & \\ & & \ddots & \ddots & \vdots \\ & & & 1 & * \\ & & & & 1 \end{pmatrix}$$

so that $U(\mathsf{A}_{(r;(m_1,\dots,m_n))} - I) = (\mathsf{A}_{(r;(m'_1,\dots,m'_n))} - I)V$.

Thus, determining whether two quantum lens spaces, $C(L_q(r; (m'_1, \dots, m'_n)))$ and $C(L_q(r; (m_1, \dots, m_n)))$, are isomorphic comes down to whether or not the matrices $\mathsf{A}_{(r;\overline{m})}$ and $\mathsf{A}_{(r;\overline{m}')}$ (or $\mathsf{B}_{(r;\overline{m})}$ and $\mathsf{B}_{(r;\overline{m}')}$ by Lemma 2.8) are equivalent with respect to the equivalence relation, $\sim$, defined below.

Definition 2.10. We will say that two matrices C and D are upper triangular equivalent, written $C \cong D$, if there exist upper triangular matrices, X, Y, with 1 in every entry of the diagonal such that $XC = DY$.

Equivalently, the matrices C and D are upper triangular equivalent, if there is a series of pivots transforming C into D with the restrictions that

(1) a multiple of row k can only be added to row l if $k > l$,

(2) a multiple of column k can only be added to column l if $k < l$.

Note that this is clearly an equivalence relation since such upper triangular matrices are invertible.

Definition 2.11. We say that two matrices, A and C, are equivalent, if

$$A - I \cong C - I.$$

In that case we write $A \sim C$.

In particular, we are interested in efficiently deciding the number of equivalence classes given n and $r > 2$ and deciding whether or not two graphs belong to the same equivalence class.

Definition 2.12. Let $r > 2$ and $n \in \mathbb{N}$ be given. Then we define

$$S_{r,n} = \{\mathsf{A}_{(r;\overline{m})} \mid \overline{m} \in (Z_r)^n\} = \{\mathsf{B}_{(r;\overline{m})} \mid \overline{m} \in (Z_r)^n\}$$

as the set of all matrices produced by vectors of length n with parameter r.

Definition 2.13. Let $r > 2$ and $n \in \mathbb{N}$ be given. Then $\varphi_r(n)$ denotes the number of elements of $S_{r,n}/\sim$ and $\widetilde{\varphi}(r)$ denotes the least n such that $\varphi_r(n) > 1$.

Thus, our goal in this paper is to find a bound for φ_r given r and to express $\widetilde{\varphi}$ in closed form.

Invariants. In this section we establish some invariants and properties in relation to changes to the vector $\overline{m}$ in $\mathsf{N}_{(r;\overline{m})}$.

Lemma 2.14. *The matrix* $\mathsf{B}_{(r;\overline{m})}$ *does not depend on the choice of* m_1 *and* m_n.

Proof. If $n = 1$ this is obvious, so assume $n > 1$. Consider legal paths in $\mathsf{N}_{(r;(m_1,\dots,m_n))}$ from the 0-vertex of the first subgraph to the 0-vertex of the j-th subgraph for $j > 1$. No matter what m_1 is there is exactly one way to reach any of the vertices of the second subgraph from the 0-vertex of the first subgraph. Thus, the number of such directed paths is independent of m_1 and the first part follows.

Now, consider the last subgraph. Once it is reached, there is exactly one way to reach the 0-vertex, once it is reached, so this does not depend on m_n. □

Lemma 2.15. *Let* $r > 2$, $\overline{m} \in (Z_r)^n$, *and* $b \in Z_r$. *Then* $\mathsf{B}_{(r;\overline{m})} = \mathsf{B}_{(r;b\cdot\overline{m})}$.

Proof. We will show that there is a bijection between the legal paths of $\mathsf{B}_{(r;\overline{m})}$ and $\mathsf{B}_{(r;b\cdot\overline{m})}$ as follows. Let γ be a legal path

$$c_{s_1,0} = c_{s_1,t_1} \to c_{s_2,t_2} \to \cdots \to c_{s_q,t_q} = c_{s_q,0}$$

on $\mathsf{N}_{(r;\overline{m})}$. Our bijection sends the legal γ to the path ω on $\mathsf{N}_{(r;(b\cdot\overline{m}))}$ given by

$$c_{s_1,0} = c_{s_1,[b\cdot t_1]_r} \to c_{s_2,[b\cdot t_2]_r} \to \cdots \to c_{s_q,[b\cdot t_q]_r} = c_{s_q,0}.$$

That the map is injective follows since multiplication by $b \in Z_r$ is an injection $Z_r \to Z_r$. Further, it is easy to see that all legal paths on $\mathsf{N}_{(r;\overline{m})}$ will be mapped to legal paths on $\mathsf{N}_{(r;(b\cdot\overline{m}))}$ since multiplication by b does not change the positions of the 0-vertices in a path. Thus, there is an injection from the legal paths on $\mathsf{N}_{(r;\overline{m})}$ to the legal paths on $\mathsf{N}_{(r;(b\cdot\overline{m}))}$ and by the same argument there must be an injection from the legal paths on $\mathsf{N}_{(r;(b\cdot\overline{m}))}$ to the legal paths on $\mathsf{N}_{(r;\overline{m})}$. It follows that said map is a bijection and we are done. □

Corollary 2.16. *Let* $\overline{m} = (m_1, m_2, \ldots, m_{n-1}, m_n) \in (Z_r)^n$. *Then there exists an* $\overline{m}' \in (Z_r)^n$ *with* 1 *in the first, last and k-th index, i.e.,*

$$\overline{m}' = (1, m'_2, \ldots, m'_{k-1}, 1, m'_{k+1}, \ldots, m'_{n-1}, 1),$$

such that $\mathsf{B}_{(r;\overline{m})} = \mathsf{B}_{(r;\overline{m}')}$.

Proof. Take b to be the inverse in Z_r of m_k in Lemma 2.15. Then

$$\mathsf{B}_{(r;\overline{m})} = \mathsf{B}_{(r;m_k^{-1}\cdot\overline{m})} = \mathsf{B}_{(r;\overline{m}')},$$

where the last equality follows from Lemma 2.14. □

Entry specific properties and formulae. To proceed with any further results we need some combinatorial formulae and properties to be in place.

Theorem 2.17. *Let* $\overline{1} = (1, \ldots, 1)$. *Then* $\mathsf{B}_{(r;\overline{1})}\langle i, j\rangle = \binom{r-1+(j-i)}{j-i}$.

Proof. Since every directed edge in $\mathsf{N}_{(r;\overline{1})}$ either goes from $c_{s,t}$ to $c_{s+1,t}$ or from $c_{s,t}$ to $c_{s,t+1}$, we can characterise any directed path from the 0-vertex of the i-th subgraph to the 0-vertex of the j-th subgraph satisfying Definition 2.7 by a $j-i+1$-tuple $(a_0, \ldots, a_{j-i})$, such that a_s is the number of edges of the form $c_{s,t} \to c_{s,t+1}$ that occur in the path. A necessary and sufficient condition for such a tuple to characterise a directed path of the desired form is that $\sum_{l=0}^{j-i} a_l = r,\ a_s \geq 0$ for all $s > 0$, and $a_0 > 0$.

Thus, $\mathsf{B}_{(r;\overline{1})}\langle i, j\rangle$ is equal to the number of ways that r can be written as the sum of $j-i+1$ nonnegative integers, where the first one has to be at least 1. This is equivalent to the number of ways to write $r-1$ as the sum of $j-i+1$ nonnegative integers. The latter being a known combinatorial problem, we get

$$\mathsf{B}_{(r;\overline{1})}\langle i, j\rangle = \binom{r-1+(j-i)}{j-i}.$$ □

Corollary 2.18. *Let* $r>2$ *and* $\overline{m} \in (Z_r)^n$. *Then* $\mathsf{B}_{(r;\overline{m})}\langle i, i\rangle = 1,\ \mathsf{B}_{(r;\overline{m})} = \langle i, i+1\rangle = r$, *and* $\mathsf{B}_{(r;\overline{m})}\langle i, i+2\rangle = \frac{r(r+1)}{2}$ *for all* i.

Proof. When we consider only $\mathsf{B}_{(r;\overline{m})}\langle i, i\rangle$, $\mathsf{B}_{(r;\overline{m})}\langle i, i+2\rangle$, and $\mathsf{B}_{(r;\overline{m})}\langle i, i+2\rangle$, their values depend solely on the vector (m_i, m_{i+1}, m_{i+2}), so we can assume by Corollary 2.16 that

$$m_i = m_{i+1} = m_{i+2} = 1.$$

The conclusion then follows trivially from Theorem 2.17 □

From the corollary we immediately obtain the following result, which also appeared in [Eilers et al. 2018] and we will note for future use:

Corollary 2.19. *Let $r > 2$. Then $\widetilde{\varphi}(r) \geq 4$.*

Proof. By Corollary 2.18 we have that for $n \leq 3$ the matrices $\mathsf{B}_{(m;\overline{r})}$ do not depend on $\overline{m}$. Thus, they are all equal when $\overline{m}$ varies and we can only have one equivalence class. □

In fact, Eilers et al. [2018] established that $\widetilde{\varphi}(r) = 4$ if and only if $3 \mid r$. As stated earlier, we shall see a general closed expression for $\widetilde{\varphi}$ in a later section.

Equivalence of matrices. To show equivalence of matrices we need to do some manipulations with matrices that might be a bit technical. So the following lemma simply establishes the equivalence of two matrices where every entry except for the diagonal is divisible by either r or $\frac{r}{2}$ when r is even.

Lemma 2.20. *Let $r > 2$ be given such that $r = 2^t s$ for some $t \in \{0, 1\}$ and odd $s \in \mathbb{N}$. Suppose that the two $n \times n$ upper triangular integer matrices A, B have 1's in their diagonal, r on the diagonal from $\langle 1, 2\rangle$ to $\langle n-1, n\rangle$, and $\frac{r(r+1)}{2}$ on the diagonal from $\langle 1, 3\rangle$ to $\langle n-2, n\rangle$. Further, suppose s divides every entry of $A - I$ and $B - I$. Then $A \sim B$.*

Proof. We need to show that we can transform the matrix $A - I$ into $B - I$ by integer row and column operations. If r is odd, every entry of $A - I$ and $B - I$ is divisible by r by assumption, and the matrices are of the form

$$r\begin{pmatrix} 0 & 1 & & & \\ 0 & 0 & 1 & & \\ \vdots & \vdots & & \ddots & \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

with integer entries in the upper right corner. All such matrices can easily be transformed into an upper triangular matrix with zeros everywhere except for the diagonal from $\langle 1, 2\rangle$ to $\langle n-1, n\rangle$ by row and column operations, so since $\sim$ is an equivalence relation, we get $A \sim B$.

Now, assume that $2 \mid r$, but $4 \nmid r$. Then the matrices $A-I$ and $B-I$ are of the form

$$A - I = \frac{r}{2}\begin{pmatrix} 0 & 2 & r{+}1 & & & \\ 0 & 0 & 2 & r{+}1 & & \\ \vdots & \vdots & & \ddots & \ddots & \\ 0 & 0 & \cdots & 0 & 2 & r{+}1 \\ 0 & 0 & \cdots & 0 & 0 & 2 \\ 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix}$$

with integer entries in the upper-right corner. We show that such a matrix can be transformed by row and column operations into the matrix

$$C - I := \frac{r}{2}\begin{pmatrix} 0 & 2 & r{+}1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 2 & r{+}1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 2 & r{+}1 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 2 & r{+}1 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 2 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{pmatrix},$$

which by transitivity shows $A \sim B$.

We proceed by induction on n. For $n = 1, 2, 3$ all matrices on the described form will be identical and thus equivalent to C.

Now, assume that for $k < n$ every matrix on the described form is equivalent to C, and consider the $n \times n$ matrix A on that form. By the induction hypothesis and considering A as an $n-1 \times n-1$ matrix with an added row and column, we can reduce A by row and column operations to a matrix with diagonals like $C - I$ and zeroes everywhere else except for in the rightmost column.

Using column operations we can now make every entry of that rightmost column (except for the $r-1$-entry) even without changing the rest of the matrix. If an entry of the column is odd, we can just subtract $r-1$ from it by subtracting the appropriate column.

Having made all those entries even they can all be eliminated by subtracting the 2 in the $(n-1)$-st row an appropriate amount of times. Then the matrix $C - I$ is achieved, which concludes the proof. □

Another useful result on when matrices are not equivalent is the following:

Lemma 2.21. *Let A and B be $n \times n$ upper-rectangular matrices with 1 in their diagonal. If every entry of $A - I$ and $B - I$ except for the entry $\langle 1, n\rangle$ is divisible by $k \in \mathbb{N}$ and $(A - I)\langle 1, n\rangle \not\equiv (B - I)\langle 1, n\rangle \pmod{k}$, then $A \not\sim B$.*

Proof. Since every entry of $A - I$ and $B - I$ except the upper right is divisible by k, the upper right entry is invariant modulo k under row and column operations. □

3. The general case

In general it is very difficult to find an explicit formula for $\mathsf{B}_{(r;\overline{m})}\langle i, j\rangle$ given arbitrary r and $\overline{m}$. However, for the purpose of bounding $\varphi_r(n)$ from below and deciding $\widetilde{\varphi}(r)$ in the case where $4 \nmid r$, it turns out to be sufficient to be able to compute $\mathsf{B}_{(r;\overline{m})}\langle i, j\rangle$ modulo r.

Thus, this section aims to develop techniques for assessing $\mathsf{B}_{(r;\overline{m})}\langle i, j\rangle$ modulo r. The main technical result is Theorem 3.2 from which the exact value of $\widetilde{\varphi}(r)$ follows for $4 \nmid r$ and a lower bound on $\varphi_r(n)$, which appears to be an equality when $4 \nmid r$ (see Conjecture 5.3). Throughout the section, we will define $0^0 = 1$ and $0! = 1$ for the sake of simplicity.

We start with the following lemma, which formally captures the technique which will be used multiple times in the proof of Theorem 3.2:

Lemma 3.1. *Suppose D is a finite set, p is a prime, $s, b\colon D \to \mathbb{Z}$ are functions, $k, j \in \mathbb{N}$, and $a : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{Z}$ satisfies $\gcd(a(m, m), p) = 1$ for all $m \le j$. Define the function*

$$w(l) = \sum_{d\in D} s(d) \sum_{t=0}^{l} a(t, l) b(d)^t$$

and assume that $p^k \mid w(l)$ for $0 \le l < j$. Then

$$w(j) \equiv \sum_{d\in D} a(j, j) s(d) b(d)^j \pmod{p^k}.$$

Proof. First, we show by strong induction over t that $p^k \mid \sum_{d\in D} s(d)b(d)^t$ for all $t < j$. For $t = 0$ we have $p^k \mid \sum_{d\in D} s(d)a(0, 0)$ and since $\gcd(p^k, a(0, 0)) = 1$ we get $p^k \mid \sum_{d\in D} s(d)$. Now, assume that $p^k \mid \sum_{d\in D} s(d)b(d)^t$ for all t satisfying $0 \le t < m$ for some $m < j$. Then,

$$0 \equiv w(m) = \sum_{d\in D} s(d) \sum_{t=0}^{m} a(t, m) b(d)^t \equiv \sum_{d\in D} a(m, m) s(d) b(d)^m \pmod{p^k},$$

so $p^k \mid \sum_{d\in D} s(d)b(d)^m$ as $\gcd(a(m, m), p^k) = 1$.

Second, the fact that $p^k \mid \sum_{d\in D} s(d)b(d)^t$ for all $t < j$ yields

$$w(j) = \sum_{d\in D} s(d) \sum_{t=0}^{j} a(t, j) b(d)^t \equiv \sum_{d\in D} a(j, j) s(d) b(d)^j \pmod{p^k}. \qquad \square$$

Having proved the lemma we now turn to the main technical theorem of the section from which the remaining results follow naturally.

Theorem 3.2. *Let p be an odd prime, $r > 2$ and $n \le p + 1$ be given, and $\overline{m} = (m_1, \dots, m_n)$. Suppose that $p^k \mid r$ and $p^k \mid \mathsf{B}_{(r;\overline{s})}\langle 1, a\rangle$ for every $\bar{s} \in (Z_r)^{p+1}$ and* *every $a < n$. Then*

$$\mathsf{B}_{(r;\overline{m})}\langle 1, n\rangle \equiv \binom{r+n-2}{n-1}\prod_{k=2}^{n-1} m_k^{-1} \pmod{p^k}.$$

Proof. For every vector $\overline{m}$ we can reduce the problem to considering a vector $\overline{m}'$ which satisfies $m'_1 = m'_2 = m'_n = 1$ as follows. First, recall that no matter what vector we consider, we can always assume without loss of generality that its first and last entry is 1 since it does not affect any of the sides of the above expression by Lemma 2.14. Second, as in the proof of Corollary 2.16 we can multiply $\overline{m}$ by a m_2^{-1} to get $\overline{m}' = m_2^{-1}\cdot\overline{m}$, which means that $m'_2 = 1$. Then the left hand side will not change since $\mathsf{B}_{(r;\overline{m})}\langle 1, n\rangle = \mathsf{B}_{(r;\overline{m'})}\langle 1, n\rangle$ by Lemma 2.15 and the right hand side will satisfy

$$\binom{r+n-2}{n-1}\prod_{k=2}^{n-1} m_k^{-1} \equiv \binom{r+n-2}{n-1}\prod_{k=2}^{n-1} bm_k^{-1} \pmod{p^k}$$

since for $n < p+1$, $p^k \mid \binom{r+n-2}{n-1}$, and for $n = p+1$, $b^{n-2} \equiv 1 \pmod{p}$ and $p^{k-1} \mid \binom{r+n-2}{n-1}$. Now, assuming $m'_1 = m'_n = 1$ yields the above. Thus, for the remaining proof we will assume that $m_1 = m_2 = m_n = 1$.

Now, let $\overline{n}_j$ denote the vector $(m_1, \ldots, m_{n-j}, \mathbf{1}_j)$, where $\mathbf{1}_j$ is the vector of 1's of length j, and note that with the above assumption, $\overline{n}_1 = \overline{m}$ and $\overline{n}_{n-2} = \overline{1}$. Our approach will be to show that for all $1 \le j < n-1$ we have

$$\mathsf{B}_{(r;\overline{n_j})}\langle 1, n\rangle \equiv m_{n-j}^{-1} g_j(m_1, \ldots, m_{n-j-1}) \pmod{p^k} \tag{1}$$

for some integer function $g_j : (Z_r)^{n-j-1} \to \mathbb{Z}$ which is independent of m_{n-j} and where m_{n-j}^{-1} is the inverse of m_{n-j} modulo p^k. Noting that (1) yields $\mathsf{B}_{(r;\overline{n_{j+1}})}\langle 1, n\rangle \equiv g(j, m_1, \ldots, m_{n-j-1}) \pmod{p^k}$, we get

$$\mathsf{B}_{(r;\overline{n_j})}\langle 1, n\rangle \equiv \mathsf{B}_{(r;\overline{n_{j+1}})}\langle 1, n\rangle m_{n-j}^{-1} \pmod{p^k},$$

and applying this together with Theorem 2.17 and $m_2 = 1$ gives us

$$\begin{aligned}
\mathsf{B}_{(r;\overline{m})}\langle 1, n\rangle &= \mathsf{B}_{(r;\overline{n_1})}\langle 1, n\rangle \equiv \mathsf{B}_{(r;\overline{n_2})}\langle 1, n\rangle m_{n-1}^{-1}\\
&\;\vdots\\
&\equiv \mathsf{B}_{(r;\overline{n_{n-2}})}\langle 1, n\rangle \prod_{k=3}^{n-1} m_k^{-1} = \mathsf{B}_{(r;\overline{1})}\langle 1, n\rangle \prod_{k=3}^{n-1} m_k^{-1}\\
&= \binom{r+n-2}{n-1}\prod_{k=2}^{n-1} m_k^{-1} \pmod{p^k}.
\end{aligned}$$

Thus, all we need to do is prove that we can indeed write an expression for $\mathsf{B}_{(r;\overline{n_j})}\langle 1, n\rangle$ of the form (1).

To do so, fix a j with $1 \leq j < n-1$ and consider the graph $\mathsf{N}_{(r;\overline{n_j})}$. We may write

$$\mathsf{B}_{(r;\overline{n_j})}\langle 1, n\rangle = \sum_{q=0}^{r-1} L_j(q) S_j(q), \tag{2}$$

where $S_j(q)$ denotes the number of paths on $\mathsf{N}_{(r;\overline{n_j})}$ from $c_{1,0}$ to $c_{n-j,q}$ that are subpaths of a legal path from $c_{1,0}$ to $c_{n,0}$ and that do not traverse any edges in the $(n-j)$-th subgraph, and similarly $L_j(q)$ is the number of paths on $\mathsf{N}_{(r;\overline{n_j})}$ from $c_{n-j,q}$ to $c_{n,0}$ that are subpaths of a legal path from $c_{1,0}$ to $c_{n,0}$.

We start our analysis by finding a formula for $L_j(q)$. First, we consider the $(n-j+1)$-th subgraph of $\mathsf{N}_{(r;\overline{n_j})}$ and count the number of paths from $c_{n-j+1,i}$ to $c_{n,0}$ on $\mathsf{N}_{(r;\overline{n_j})}$ that are subpaths of a legal path from $c_{1,0}$ to $c_{n,0}$ for each $0 \leq i < r$. As in the proof of Theorem 2.17 one can see choosing such a path as choosing a partition of $[r-i]_r$ into a sum of $j-1$ nonnegative integers since $m_{n-j+1} = m_{n-j+2} = \cdots = m_n = 1$. Thus, the number of such paths equals $\binom{[r-i]_r+j-1}{j-1}$.

Second, there are three cases to consider. When $i, q > 0$ there is exactly one path from $c_{n-j,q}$ to $c_{n-j+1,i}$ not traversing any edges in the $(n-j+1)$-th subgraph that is a subpath of a legal path from $c_{1,0}$ to $c_{n,0}$ if and only if $[m_{n-j}^{-1}q]_r \leq [m_{n-j}^{-1}i]_r$. Otherwise there are none. This is clear since such a path would be of the form

$$c_{n-j,q} \to c_{n-j,[q+m_{n-j}]_r} \to \cdots \to c_{n-j,i} \to c_{n-j+1,i}$$

and zero is not a member of $\{q, q+m_{n-j}, \ldots, i\}$ if and only if $[m_{n-j}^{-1}q]_r \leq [m_{n-j}^{-1}i]_r$. For $i = 0$ there is exactly one such subpath for every q and for $q = 0$ there is exactly one such subpath if and only if $i = 0$. Thus, for $q > 0$,

$$\begin{aligned} L_j(q) &= \sum_{i=0}^{r-1} (1_{\{[m_{n-j}^{-1}q]_r \leq [m_{n-j}^{-1}i]_r\}} + 1_{\{i=0\}}) \binom{[r-i]_r+j-1}{j-1} \\ &= \sum_{i=[m_{n-j}^{-1}q]_r}^{r} \binom{[r-im_{n-j}]_r+j-1}{j-1}, \end{aligned}$$

where 1_{boolean} is an indicator function assuming the value 1 if true and 0 otherwise, and where we changed $i = 0$ terms into $i = r$ terms. Introducing the new variable $\sigma = r - i$ we rewrite the sum as

$$L_j(q) = \sum_{\sigma=0}^{[-m_{n-j}^{-1}q]_r} \binom{[\sigma m_{n-j}]_r+j-1}{j-1}. \tag{3}$$

Since evidently $L_j(0) = 1$, this formula holds even for $q = 0$ and thus for all $0 \leq q < r$.

For $j = 1$, (3) yields $L_j(q) = [-m_{n-1}^{-1}q]_r$ and inserting this into (2) yields

$$\mathsf{B}_{(r;\overline{n_1})} = \sum_{q=0}^{r-1} [-m_{n-1}^{-1}q]_r S_j(q) \equiv -m_{n-1}^{-1} \sum_{q=0}^{r-1} S_j(q)q \pmod{p^k}.$$

Since $S_j(q)q$ only depends on $m_1, m_2, \ldots, m_{n-2}$, it follows that we can write $\mathsf{B}_{(r;\overline{n_1})}$ in the form (1).

So let us consider the case when $j > 1$. Inserting the expression (3) into (2) and substituting $d = r - q$ and noting that the $d = 0$ is equal to the $d = r$ term yields

$$(j-1)!\,\mathsf{B}_{(r;\overline{m})}\langle 1, n\rangle = \sum_{d=0}^{r-1} \sum_{\sigma=0}^{[m_{n-j}^{-1}d]_r} S_j([r-d]_r) \prod_{i=1}^{j-1}([\sigma m_{n-j}]_r + i).$$

Expanding the product and introducing $s(d) = S_j([r-d]_r)$ we get the sum

$$(j-1)!\,\mathsf{B}_{(r;\overline{n_j})}\langle 1, n\rangle = \sum_{d=0}^{r-1} \sum_{\sigma=0}^{[m_{n-j}^{-1}d]_r} \sum_{t=0}^{j-1} a(t, j-1)([\sigma m_{n-j}]_r)^t s(d)$$

for an integer function $a\colon \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{Z}$, where $a(j-1, j-1) = 1$. Now, note that by the same reasoning we must also have for every $0 \le l < j-1$ that

$$\begin{aligned} w(l) &:= l!\,\mathsf{B}_{(r;\overline{n_j})}\langle 1, n-j+l+1\rangle \\ &= \sum_{d=0}^{r-1} \sum_{\sigma=0}^{[m_{n-j}^{-1}d]_r} s(d) \sum_{t=0}^{l} a(t, l)([\sigma m_{n-j}]_r)^t, \end{aligned}$$

where $a(l, l) = 1$. By assumption, p^k divides $\mathsf{B}_{(r;\overline{n_j})}\langle 1, n-j+l+1\rangle$ for every $l < j-1$. Hence, $p^k \mid w(l)$ for $0 \le l < j-1$. Applying Lemma 3.1 with the functions $s, a, b(\sigma) := [\sigma m_{n-j}]_r$, prime p, and exponent t, we get

$$\begin{aligned} (4) \qquad (j-1)!\,\mathsf{B}_{(r;\overline{n_j})}\langle 1, n\rangle &= w(j-1) \\ &\equiv \sum_{d=0}^{r-1} \sum_{\sigma=0}^{[m_{n-j}^{-1}d]_r} a(j-1, j-1)s(d)([\sigma m_{n-j}]_r)^{j-1} \\ &\equiv m_{n-j}^{j-1} \sum_{d=0}^{r-1} \sum_{\sigma=0}^{[m_{n-j}^{-1}d]_r} s(d)\sigma^{j-1} \pmod{p^k}. \end{aligned}$$

By Faulhaber's formula ([Graham et al. 1989, Chapter 6.5]) in the convention $B_1 = \frac{1}{2}$, we can write

$$\sum_{\sigma=0}^{[m_{n-j}^{-1}d]_r} \sigma^{j-1} = \frac{1}{j} \sum_{t=0}^{j-1} \binom{j}{j-t-1} B_{j-t-1}([m_{n-j}^{-1}d]_r)^{t+1},$$

where B_n is the n-th Bernoulli number. Inserting this into (4), noting that $p^k \mid r$ and multiplying both sides by j, we find that

$$j!\,\mathsf{B}_{(r;\overline{n_j})}\langle 1, n\rangle \equiv m_{n-j}^{j-2} \sum_{d=0}^{r-1} s(d)d \sum_{t=0}^{j-1} \binom{j}{j-t-1} B_{j-t-1}(m_{n-j}^{-1}d)^t \pmod{p^k}.$$

As it is a well-known fact that $j!B_l$, $l < j$, is an integer, we multiply by $j!(m_{n-j}^{j-2})^{-1}$ to ensure that each factor of each term is an integer

$$(m_{n-j}^{j-2})^{-1} j!^2\,\mathsf{B}_{(r;\overline{n_j})}\langle 1, n\rangle \equiv \sum_{d=0}^{r-1} s(d)d \sum_{t=0}^{j-1} \binom{j}{j-t-1} j!B_{j-t-1}(m_{n-j}^{-1}d)^t \pmod{p^k}.$$

To apply Lemma 3.1 again we write

$$\begin{aligned}\tilde{s}(d) &:= s(d)d,\\ \tilde{a}(t,l) &:= \binom{l+1}{l-t}(l+1)!B_{l-t},\\ \tilde{b}(d) &:= [m_{n-j}^{-1}d]_r,\end{aligned}$$

and considering the vectors $\overline{v_{l+1}} = (m_1, m_2, \ldots, m_{n-j}, \mathbf{1}_{l+1})$ one finds:

$$\begin{aligned}\tilde{w}(l) &:= \sum_{d=0}^{r-1} \tilde{s}(d) \sum_{t=0}^{l} \tilde{a}(t,l)\tilde{b}(d)^t\\ &\equiv (m_{n-j}^{l-1})^{-1}(l+1)!^2\,\mathsf{B}_{(r;\overline{v_{l+1}})}\langle 1, n-j+l+1\rangle \pmod{p^k}.\\ &\equiv (m_{n-j}^{l-1})^{-1}(l+1)!^2\,\mathsf{B}_{(r;\overline{n_j})}\langle 1, n-j+l+1\rangle \pmod{p^k}.\end{aligned}$$

Now,

$$\tilde{a}(l,l) = \binom{l+1}{0}(l+1)!B_0 = (l+1)!$$

so for $0 \le l < j-1 < n-1 \le p$ we have $\gcd(\tilde{a}(l,l), p) = 1$. Further, by assumption, $p^k \mid \mathsf{B}_{(r;\overline{n_j})}\langle 1, n-j+l+1\rangle$ for $0 \le l < j-1$, so $p^k \mid \tilde{w}(l)$ for $0 \le l < j-1$. Thus, Lemma 3.1 yields

$$\begin{aligned}(m_{n-j}^{j-2})^{-1} j!^2\,\mathsf{B}_{(r;\overline{n_j})}\langle 1, n\rangle &= \tilde{w}(j-1)\\ &\equiv \sum_{q=0}^{r-1} s(d)d\tilde{a}(j-1, j-1)(m_{n-j}^{-1}d)^{j-1}\\ &\equiv m_{n-j}^{1-j} \sum_{q=0}^{r-1} j!s(d)d^j \pmod{p^k}.\end{aligned}$$

This means that

$$\mathsf{B}_{(r;\overline{n_j})}\langle 1, n\rangle \equiv m_{n-j}^{-1} \sum_{q=0}^{r-1} j!^{-1} s(d) d^j \pmod{p^k},$$

where we note that $j!^{-1}$ is well defined because $j < n-1 \le p$ so $\gcd(j!, p) = 1$. Since $s(d)$ only depends on $m_1, \dots, m_{n-j-1}$, it is clear that we can find g_j satisfying (1) and we are done. □

Having proved the above theorem we can apply it to find $\widetilde{\varphi}(r)$ whenever $4 \nmid r$. We first use the theorem to prove the following lemma, which will give the first half of the proof.

Lemma 3.3. *Let $r > 2$, p be an odd prime, and $p^k \| r$ for some $k \in \mathbb{N}$. For every vector $\overline{m}$ with entries in Z_r and every pair a, b satisfying $0 < b - a < p$, we have*

$$p^k \mid \mathsf{B}_{(r;\overline{m})}\langle a, b\rangle.$$

Proof. We proceed by induction on the difference $n = b - a$. When $n = 1$ we have $p^k \mid \mathsf{B}_{(r;\overline{m})}\langle a, b\rangle = r$. Now, suppose that $p^k \mid \mathsf{B}_{(r;\overline{m})}\langle a', b'\rangle$ for every a', b' satisfying $0 < b' - a' < n$ for some n with $1 < n < p$ and let $b - a = n$. Then we can apply Theorem 3.2 with the indices $\langle 1, n\rangle$ shifted to $\langle a, b\rangle$ to get

$$\begin{aligned}\mathsf{B}_{(r;\overline{m})}\langle a, b\rangle &\equiv \binom{r-1+(b-a)}{b-a} \prod_{k=a+1}^{b-1} m_k^{-1}\\ &\equiv \frac{r\cdots(r-1+(b-a))}{(b-a)!} \prod_{k=a+1}^{b-1} m_k^{-1}\\ &\equiv 0 \pmod{p^k},\end{aligned}$$

where the last equivalence follows since p^k divides

$$\frac{r\cdots(r-1+(b-a))}{(b-a)!}$$

because $b - a < p$ and r divides the numerator. □

Now, using the previous lemma and Theorem 3.2 we obtain an upper bound on $\widetilde{\varphi}$ simply by pointing to two graphs that are not equivalent. In Theorem 3.9 below we will establish a lower bound for the number of equivalence classes from which the result will follow. But for clarity we now give a short independent proof.

Theorem 3.4. *Let $r > 2$ be given and let p be the smallest odd prime dividing r. Then $\widetilde{\varphi}(r) \le p + 1$.*

Proof. Let k be such that $p^k \| r$, set

$$\bar{a} = (\mathbf{1}_{p+1}) \text{ and } \bar{b} = (1, -1, \mathbf{1}_{p-1}),$$

and consider the matrices $A = \mathsf{B}_{(r;\overline{a})}$, $B = \mathsf{B}_{(r;\overline{b})}$. Then by Lemma 3.3 we have $p^k \mid A\langle a, b\rangle$ and $p^k \mid B\langle a, b\rangle$ for $a < b$ and $\langle a, b\rangle \neq \langle 1, p+1\rangle$. Using Theorem 3.2 twice and noting that $(r+1)\cdots(r+p-1) \equiv (p-1)! \pmod{p^k}$, we get

$$\begin{aligned} A\langle 1, p+1\rangle &= \binom{r+p-1}{p} \prod_{k=2}^{p} a_k^{-1} \\ &= \frac{r}{p} \pmod{p^k}, \end{aligned}$$

and

$$\begin{aligned} B\langle 1, p+1\rangle &\equiv \binom{r+p-1}{p} \prod_{k=2}^{p} b_k^{-1} \\ &\equiv -\frac{r}{p} \pmod{p^k}, \end{aligned}$$

since $b_2 = -1$. It follows that p^k divides every entry of $A - I$ and $B - I$ except for the entry $\langle 1, p+1\rangle$. Applying Lemma 2.21 we get $\mathsf{B}_{(r;\overline{a})} \not\sim \mathsf{B}_{(r;\overline{b})}$ implying $\varphi_r(p+1) > 1$ and the conclusion follows. □

Now, using Theorem 3.4, we determine $\widetilde{\varphi}(r)$ whenever $4 \nmid r$.

Theorem 3.5. *Let $r > 2$ be given such that $4 \nmid r$ and let p be the smallest odd prime dividing r. Then $\widetilde{\varphi}(r) = p + 1$.*

Proof. It follows from Lemmas 2.20 and 3.3 that for every $n \le p$ and every $\overline{m} \in (Z_r)^n$ we have $\mathsf{B}_{(r;\overline{m})} \sim \mathsf{B}_{(r;\overline{1})}$, so $\varphi_r(n) = 1$ for $n \le p$. Thus, $\varphi(r) > p$. The conclusion now follows from Theorem 3.4. □

The remainder of this section deals with the number of equivalence classes, $\varphi_r(n)$.

Notation 3.6. Let $A = (a_{ij})$ be a matrix. Then we denote by $A[c, d]$ the partial square matrix

$$\begin{pmatrix} a_{cc} & \cdots & a_{cd} \\ \vdots & & \vdots \\ a_{dc} & \cdots & a_{dd} \end{pmatrix}.$$

Lemma 3.7. *Let A, B be upper triangular matrices with $A \sim B$. Then $A[b, b+c] \sim B[b, b+c]$ for $b, c \in \mathbb{N}$ whenever the partial matrices are well defined.*

Proof. By definition, we have $A \sim B$ if and only if $A - I$ can be transformed into $B - I$ by pivots where a row can only be added to a row above it and a column can only be added to a column on its right. Noting that any such series of pivots on A will act on the submatrix $(A - I)[b, b+c]$ as though they were simply pivots carried out on $(A - I)[b, b+c]$ as an independent matrix, it follows that $(A - I)[b, b+c] = A[b, b+c] - I$ can be transformed into $B[b, b+c] - I$ with pivots as described in our definition and the result follows. □

We introduce a necessary condition for two vectors $\overline{m}$ and $\overline{n}$ to have graphs with equivalent matrices.

Theorem 3.8. *Let $r > 2$ have prime factorisation $r = 2^j p_1^{\alpha_1} \cdots p_k^{\alpha_k}$, $j \in \mathbb{N}_0$, for distinct odd primes p_i. Further, let $\overline{m}, \overline{m}' \in (Z_r)^n$ be given such that $\mathsf{B}_{(r;\overline{m})} \sim \mathsf{B}_{(r;\overline{m}')}$. Then for every i with $1 \leq i \leq k$ and every t with $1 \leq t \leq n - p_i$ we have*

$$\prod_{l=t+1}^{t+p_i-1} m_l \equiv \prod_{l=t+1}^{t+p_i-1} m'_l \pmod{p_i}.$$

Proof. Assume for contradiction that for some i, t we have

$$\prod_{l=t+1}^{t+p_i-1} m_l \not\equiv \prod_{l=t+1}^{t+p_i-1} m'_l \pmod{p_i}$$

and consider the matrices

$$A = \mathsf{B}_{(r;\overline{m})}[t, t+p_i][t, t+p_i] \quad \text{and} \quad B = \mathsf{B}_{(r;\overline{m}')}[t, t+p_i][t, t+p_i].$$

By Lemma 3.7 we must have $A \sim B$ and by Lemma 3.3, $p_i^{\alpha_i}$ divides every entry of $A - I$ and $B - I$ except the entry $\langle 1, p_i\rangle$. For the entry $\langle 1, p_i\rangle$ note that $p^{\alpha_i - 1} \| \binom{r+p_i-1}{p_i}$ and that given integers a, b, c such that $a \not\equiv b \pmod{p}$ and $p^{\alpha-1} \| c$ for a prime p, then $ac \not\equiv bc \pmod{p^\alpha}$. Combining these two observations yields

$$\begin{aligned} A\langle 1, p_i\rangle &\equiv \binom{r+p_i-1}{p_i} \prod_{l=t+1}^{t+p_i-1} m_l^{-1} \\ &\not\equiv \binom{r+p_i-1}{p_i} \prod_{l=t+1}^{t+p_i-1} m_l'^{-1} \\ &\equiv B\langle 1, p_i\rangle \pmod{p_i^{\alpha_i}}. \end{aligned}$$

Thus, by Lemma 2.21 we have $A \not\sim B$, which is a contradiction. □

This necessary condition on equivalence translates directly into a lower bound on the number of equivalence classes, $\varphi_r(n)$.

Theorem 3.9. *Let $r > 2$ have prime factorisation $r = 2^j p_1^{\alpha_1} \cdots p_k^{\alpha_k}$, $j \in \mathbb{N}_0$, for odd distinct primes p_i. Then*

$$\varphi_r(n) \geq \prod_{i=1}^{k} \lceil (p_i - 1)^{n-p_i} \rceil.$$

Proof. For every $1 \le i \le k$ we define a function $T_i\colon (Z_{p_i})^n \to (Z_{p_i})^{n-p_i}$ given by

$$T_i(\overline{m}) = \left(\left[\prod_{l=t+1}^{t+p_i-1} m_l\right]_{p_i}\right)_{t=1}^{n-p_i}.$$

In the case $n \le p_i$, T_i is simply the function $T_i\colon (Z_{p_i})^n \to \{1\}$. To show that each T_i is surjective, let $\overline{m}' \in (Z_{p_i})^{n-p_i}$ be a vector and define $\overline{m} \in (Z_{p_i})^n$ as follows:

$$m_l = \begin{cases} 1, & l < p_i \\ m'_{l-p_i+1}\left[\displaystyle\prod_{q=l-p_i+2}^{l-1} m_q^{-1}\right]_{p_i}, & l \ge p_i. \end{cases}$$

Since the t-th entry of $T_i(\overline{m})$ is given by

$$\left[\prod_{l=t+1}^{t+p_i-1} m_l\right]_{p_i} = \left[m_{t+p_i-1}\prod_{l=t+1}^{t+p_i-2} m_l\right]_{p_i} = m'_t,$$

it follows that $\overline{m}' \in T_i((Z_{p_i})^n)$ and thus, T_i is surjective.

Now, define the map $T\colon (Z_r)^n \to (Z_{p_1})^{n-p_1} \times \cdots \times (Z_{p_k})^{n-p_k}$ by $T(\overline{m}) = (T_1(\overline{m}), T_2(\overline{m}), \dots, T_k(\overline{m}))$ in the natural way. Since each T_i is surjective on $(Z_{p_i})^n \to (Z_{p_i})^{n-p_i}$ it follows by the Chinese remainder theorem that T is also surjective. Now, for any two vectors $\overline{m}, \overline{n} \in (Z_r)^n$ such that $\mathsf{B}_{(r;\overline{m})} \sim \mathsf{B}_{(r;\overline{n})}$ we must have $T(\overline{m}) = T(\overline{n})$ by Theorem 3.8. Thus, T is an invariant of the equivalence relation $\sim$, it is surjective, and its codomain has $\prod_{i=1}^{k} \lceil (p_i - 1)^{n-p_i} \rceil$ elements and it follows that, indeed,

$$\varphi_r(n) \ge \prod_{i=1}^{k} \lceil (p_i - 1)^{n-p_i} \rceil. \qquad \square$$

By Theorem 3.9, we now have a lower bound on the number of equivalence classes, but we conjecture that the condition in Theorem 3.8 is actually sufficient whenever $4 \nmid r$. This would then result in equality in Theorem 3.9; see Conjectures 5.3 and 5.2. Note further that using the inequality we can obtain Theorems 3.4 and 3.5 since when $n = p + 1$, where p is the least odd prime dividing r, we will get at least $(p - 1)$ classes.

4. The case of multiples of 4

Until now we have not determined $\widetilde{\varphi}(r)$ in the special case where 4 divides r. This section will show that for $4 \mid r$ we have $\widetilde{\varphi}(r) \le 6$ with equality if and only if $3 \nmid r$. To this end, we start with a few lemmas regarding specific entries of $\mathsf{B}_{(r;\overline{m})}$. Throughout the section we will change our notation slightly to make our calculations more natural, identifying the r-th vertex of any subgraph of $\mathsf{N}_{(r;\overline{m})}$ with the 0-th.

Lemma 4.1. *Let $r > 2$ be given with $2^t \mid r,\ t > 1$, and let $\overline{m} \in (Z_r)^4$. Then*

$$2^t \mid \mathsf{B}_{(r;\overline{m})}\langle 1, 4\rangle.$$

Proof. By Corollary 2.16 we can assume without loss of generality that $\overline{m} = (1, m_2, 1, 1)$ for some $m_2 \in Z_r$. We calculate $\mathsf{B}_{(r;\overline{m})}\langle 1, 4\rangle$ by counting the number of legal paths from $c_{1,0}$ to $c_{4,0}$. We will sum over the last vertex $q,\ 1 \le q \le r$, of the second subgraph that each path visits. Denote by $S_2(q)$ the number of paths from $c_{1,0}$ to $c_{2,q}$ that are subpaths of a legal path from $c_{1,0}$ to $c_{4,0}$ and similarly, let $L_2(q)$ denote the number of paths from $c_{2,q}$ to $c_{4,0}$ that do not traverse any edges of the second subgraphs and that are subpaths of a legal path from $c_{1,0}$ to $c_{4,0}$.Then

$$\mathsf{B}_{(r;\overline{m})}\langle 1, 4\rangle = \sum_{q=1}^{r} S_2(q)L_2(q).$$

First, it is not hard to see that $L_2(q) = r - q + 1$ as $m_3 = 1$. Second, if we write $q = [tm_2]_r,\ 1 \le t \le r$ we can see that for every subpath ϕ counted by $S_2(q)$ there must be a first vertex $c_{2,v}$ of the second subgraph that it visits. We must have $v \in \{[wm_2]_r \mid 1 \le w \le t\}$ for else ϕ could never legally visit $c_{2,q}$. Further, there is exactly one subpath ϕ going through $c_{2,v}$ as specified, the path

$$c_{1,0} \to c_{1,1} \to \cdots \to c_{1,v} \to c_{2,v} \to c_{2,[v+m_2]_r} \to c_{2,t}.$$

It follows that

$$S_2(q) = \left|\{[wm_2]_r \mid 1 \le w \le t\}\right| = t \equiv qm_2^{-1} \pmod r,$$

so we can calculate

$$\mathsf{B}_{(r;\overline{m})}\langle 1, 4\rangle \equiv \sum_{q=1}^{r} qm_2^{-1}(r-q+1) \equiv m_2^{-1}\sum_{q=1}^{r} q(r-q+1) \pmod r.$$

By noting that $\mathsf{B}_{(r;\overline{1})}\langle 1, 4\rangle \equiv \sum_{q=1}^{r} q(r-q+1) \pmod r$, it follows that

$$\mathsf{B}_{(r;\overline{m})}\langle 1, 4\rangle \equiv m_2^{-1}\,\mathsf{B}_{(r;\overline{1})}\langle 1, 4\rangle \equiv m_2^{-1}\binom{r+2}{3} \equiv 0 \pmod{2^t}$$

by use of Theorem 2.17. □

Lemma 4.2. *Let $r > 2$ be given and assume that $2^t \| r$ for a $t > 1$ and let $\overline{m} \in (Z_r)^5$. Then*

$$2^{t-2} \| \mathsf{B}_{(r;\overline{m})}\langle 1, 5\rangle.$$

Proof. By Corollary 2.16 we can assume without loss of generality that $\overline{m} = (1, m_2, 1, m_4, 1)$. We calculate $\mathsf{B}_{(r;\overline{m})}\langle 1, 4\rangle$ by counting the number of legal paths from $c_{1,0}$ to $c_{5,0}$. We will sum over the last vertex $q,\ 1 \le q \le r$, of the second subgraph that each path visits. Denote by $S_2(q)$ be the number of paths from $c_{1,0}$

to $c_{2,q}$ that are subpaths of a legal path from $c_{1,0}$ to $c_{5,0}$ and similarly, let $L_2(q)$ denote the number of paths from $c_{2,q}$ to $c_{5,0}$ that do not traverse any edges of the second subgraph and are subpaths of a legal path from $c_{1,0}$ to $c_{5,0}$. Then,

$$\mathsf{B}_{(r;\overline{m})}\langle 1,5\rangle = \sum_{q=1}^{r} S_2(q)L_2(q).$$

As in the proof of Lemma 4.1, $S_2(q) \equiv qm_2^{-1} \pmod r$. It follows that

$$\mathsf{B}_{(r;\overline{m})}\langle 1,5\rangle \equiv \sum_{q=1}^{r} qm_2^{-1}L_2(q) \equiv m_2^{-1}\,\mathsf{B}_{(r;(1,1,1,m_4,1))}\langle 1,5\rangle \pmod r. \tag{5}$$

We proceed to calculate $\mathsf{B}_{(r;(1,1,1,m_4,1))}\langle 1,5\rangle$ by almost the same approach as before. Write

$$\mathsf{B}_{(r;(1,1,1,m_4,1))}\langle 1,5\rangle = \sum_{q=1}^{r} S_3(q)L_4(q),$$

where $S_3(q)$ is the number of paths on $\mathsf{N}_{(r;((1,1,1,m_4,1)))}$ from $c_{1,0}$ to $c_{3,q}$ that are subpaths of a legal path from $c_{1,0}$ to $c_{5,0}$. Further, $L_4(q)$ is the number of paths from $c_{3,q}$ to $c_{5,0}$ that do not traverse any edge of the third subgraph and are subpaths of a legal path from $c_{1,0}$ to $c_{5,0}$. Let ϕ be a path counted by $L_3(q)$ and let $c_{4,v}$ be the last vertex of the fourth subgraph that ϕ visits. By $L_3(q,v)$ we count the number of such ϕ. Then,

$$\begin{aligned}\mathsf{B}_{(r;(1,1,1,m_4,1))}\langle 1,5\rangle &= \sum_{q=1}^{r} S_3(q)\sum_{v=1}^{r} L_3(q,v) = \sum_{v=1}^{r}\sum_{q=1}^{r} S_3(q)L_3(q,v)\\ &= \sum_{q=1}^{r} S_3(q)L_3(q,r) + \sum_{v=1}^{r-1}\sum_{q=1}^{r} S_3(q)L_3(q,v).\end{aligned}$$

Since $\sum_{q=1}^{r} S_3(q)L_3(q,r)$ simply counts the number of legal paths from $c_{1,0}$ to $c_{4,r} = c_{4,0}$ that are subpaths of a legal path from $c_{1,0}$ to $c_{5,0}$, we have

$$\sum_{q=1}^{r} S_3(q)L_3(q,r) = \mathsf{B}_{(r;(1,1,1,m_4,1))}\langle 1,4\rangle = \binom{r+2}{3} \equiv 0 \pmod{2^t}$$

by Theorem 2.17 and Lemma 2.14. Considering the case $1 \le v < r$ yields that $L_3(q,v) = 0$ if $[qm_4^{-1}]_r > [vm_4^{-1}]_r$ since there is no legal path from $c_{4,q}$ to $c_{4,v}$ because such a path would visit $c_{4,0}$ and $v \neq 0$. Further, if $[qm_4^{-1}]_r \le [vm_4^{-1}]_r$ we have $L_3(q,v) = 1$ since only the path

$$c_{3,q} \to c_{4,q} \to c_{4,q+m_4} \to \cdots \to c_{4,v} \to c_{5,v} \to \cdots \to c_{5,0}$$

satisfies the criteria. It follows that

$$\begin{aligned}
\mathsf{B}_{(r;(1,1,1,m_4,1))}\langle 1,5\rangle &\equiv \sum_{v=1}^{r-1}\sum_{q=1}^{r} S_3(q)L_3(q,v)\\
&\equiv \sum_{q=1}^{r}\sum_{\substack{1\le v<r\\ [qm_4^{-1}]_r\le[vm_4^{-1}]_r}} S_3(q)\\
&= \sum_{q=1}^{r}[r-qm_4^{-1}]_r S_3(q) \pmod{2^t},
\end{aligned}$$

where the last equality follows since multiplying by m_4^{-1} modulo r induces a bijection on the set $\{1,\dots,r-1\}$, yielding

$$\left|\{v \mid 1\le v<r \wedge [qm_4^{-1}]_r \le [vm_4^{-1}]_r\}\right| = [r-qm_4^{-1}]_r.$$

So we get

$$\begin{aligned}
\mathsf{B}_{(r;(1,1,1,m_4,1))}\langle 1,5\rangle &\equiv \sum_{q=1}^{r}[r-qm_4^{-1}]_r S_3(q)\\
&\equiv m_4^{-1}\sum_{q=1}^{r} -qS_3(q)\\
&\equiv m_4^{-1}\,\mathsf{B}_{(r;\overline{1})}\langle 1,5\rangle \pmod{2^t}.
\end{aligned}$$

Inserting this into (5) then finally yields

$$\begin{aligned}
\mathsf{B}_{(r;\overline{m})}\langle 1,5\rangle &\equiv m_2^{-1}\,\mathsf{B}_{(r;(1,1,1,m_4,1))}\langle 1,5\rangle\\
&\equiv m_2^{-1}m_4^{-1}\,\mathsf{B}_{(r;\overline{1})}\langle 1,5\rangle\\
&\equiv m_2^{-1}m_4^{-1}\binom{r+3}{4}\\
&\equiv s2^{t-2} \pmod{2^t}
\end{aligned}$$

for some odd integer s since $2^{t-2}\|\binom{r+3}{4}$ as $4 \mid r$. □

Lemma 4.3. *Let $r>2$ be given. Then*

$$\mathsf{B}_{(r;(1,1,-1,1,1,1))}\langle 1,6\rangle = \tfrac{11}{20}r+\tfrac{3}{8}r^2-\tfrac{1}{8}r^3+\tfrac{1}{8}r^4+\tfrac{3}{40}r^5$$

Proof. In the graph $\mathsf{N}_{(r;(1,1,-1,1,1,1))}$, again let $S_3(q)$ be the number of paths from vertex $c_{1,0}$ to $c_{3,q}$ that are subpaths of a legal path from $c_{1,0}$ to $c_{6,0}$ such that $c_{3,q}$ is the last vertex visited in the third subgraph and let $L_3(q)$ be the number of paths from $c_{3,q}$ to $c_{6,0}$ that does not traverse any edges of the third subgraph and are subpaths of a legal path from $c_{1,0}$ to $c_{6,0}$. We will find a closed form for each function.

First, let $0 < q < r$. Counting the paths of $S_3(q)$, we notice that there is exactly one path from $c_{1,0}$ to $c_{3,q}$ for every path from $c_{1,0}$ to $c_{2,i}$ for $p > i \geq q$. This is the path

$$c_{1,0} \to \cdots c_{2,i} \to c_{3,i} \to c_{3,i-1} \to \cdots \to c_{3,q}.$$

Since $m_1 = m_2 = 1$ in this case, the number of paths from $c_{1,0}$ to $c_{2,i}$ that are part of a legal path from $c_{1,0}$ to $c_{6,0}$ is i. Thus,

$$S_3(q) = \sum_{i=q}^{r-1} i = \frac{(r-q)(r+q-1)}{2}, \quad 0 < q < r.$$

The function $L_3(q)$ is only counting paths that are traversing subgraphs with parameter $m_i = 1$. We see by Corollary 2.18 that

$$L_3(q) = \mathsf{B}_{(r-q+1;\bar{1})}\langle 1, 3\rangle = \frac{(r-q+1)(r-q+2)}{2}.$$

Second, for $q = 0$ we have $S_3(0) = \frac{r(r+1)}{2}$ by Corollary 2.18 since this is simply $\mathsf{B}_{(r;(1,1,-1,1,1))}\langle 1, 3\rangle$. Further, there is only one legal subpath from $c_{4,0}$ to $c_{6,0}$ of a legal path from $c_{1,0}$ to $c_{6,0}$ so $L_3(0) = 1$.

Thus, we have

$$\begin{aligned}\mathsf{B}_{(r;(1,1,-1,1,1))}\langle 1, 6\rangle &= \sum_{q=0}^{r-1} S_3(q)L_3(q) \\ &= \frac{r(r+1)}{2} + \sum_{q=1}^{r-1} \frac{(r-q)(r-q+1)(r-q+2)(r+q-1)}{4} \\ &= \tfrac{11}{20}r + \tfrac{3}{8}r^2 - \tfrac{1}{8}r^3 + \tfrac{1}{8}r^4 + \tfrac{3}{40}r^5,\end{aligned}$$

where the last equality follows by writing out the expression and applying Faulhaber's formula ([Graham et al. 1989, Chapter 6.5]). □

Theorem 4.4. *Let $r > 2$ be given such that $4 \mid r$. Then $\widetilde{\varphi}(r) \leq 6$ with equality if and only if $3 \nmid r$.*

Proof. If $3 \mid r$, we have $\widetilde{\varphi}(r) \leq 4$ by Theorem 3.4, so we will now only consider the case when $3 \nmid r$.

First, we show that $\widetilde{\varphi}(r) > 5$. Let $\overline{m}, \overline{m}' \in (Z_r)^5$ be given and let $X = \mathsf{A}_{(r;\overline{m})}$ and $Y = \mathsf{A}_{(r;\overline{m}')}$. We will demonstrate that $X \sim Y$, proving that $\varphi_r(5) = 1$. Since $3 \nmid r$ it follows from Lemma 3.3 that if $r = s2^t$, $2 \nmid s$, then s will divide every entry of $\mathsf{B}_{(r;\overline{m})}$ and $\mathsf{B}_{(r;\overline{m}')}$ except for the diagonal. Thus, by Lemmas 4.1 and 4.2 the matrices are of the form

$$X-I=\begin{pmatrix} 0 & r & \frac{1}{2}r(r+1) & x_1 r & \frac{1}{4}x_2 r \\ 0 & 0 & r & \frac{1}{2}r(r+1) & x_3 r \\ 0 & 0 & 0 & r & \frac{1}{2}r(r+1) \\ 0 & 0 & 0 & 0 & r \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$Y-I=\begin{pmatrix} 0 & r & \frac{1}{2}r(r+1) & y_1 r & \frac{1}{4}y_2 r \\ 0 & 0 & r & \frac{1}{2}r(r+1) & y_3 r \\ 0 & 0 & 0 & r & \frac{1}{2}r(r+1) \\ 0 & 0 & 0 & 0 & r \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

for integers $x_1, x_2, x_3, y_1, y_2, y_3$, where $2 \nmid x_2, y_2$. Now, reducing according to Definition 2.11 in a number of steps, we get

$$X-I \overset{1}{\cong} \begin{pmatrix} 0 & r & \frac{1}{2}r & 0 & \frac{1}{4}x_2 r \\ 0 & 0 & r & \frac{1}{2}r(r+1) & 0 \\ 0 & 0 & 0 & r & \frac{1}{2}r \\ 0 & 0 & 0 & 0 & r \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\overset{2}{\cong} \begin{pmatrix} 0 & r & \frac{1}{2}r & 0 & \frac{1}{4}y_2 r \\ 0 & 0 & r & \frac{1}{2}r(r+1) & 0 \\ 0 & 0 & 0 & r & \frac{1}{2}r \\ 0 & 0 & 0 & 0 & r \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \overset{3}{\cong} Y-I.$$

Step 1 reduces the entries of the first row and last column of $X-I$ modulo r by subtracting the fourth row and second column from the others. Step 2 adds the third column to the last column $\frac{1}{2}(y_2-x_2)$ times and then subtracts the fourth row from the second $\frac{1}{2}(y_2-x_2)$ times. Step 3 is simply the reverse of Step 1 except with $Y-I$ instead of $X-I$. It follows that $X \sim Y$.

Second, we show that $\widetilde{\varphi}(r) \leq 6$, which completes the proof. Suppose that $5 \mid r$. Then it follows by Theorem 3.4 that $\widetilde{\varphi}(r) \leq 6$. So assume that $3, 5 \nmid r$. Now, since $4 \mid r$, Theorem 2.17 yields

$$r \mid \mathsf{B}_{(r;\bar{1})}\langle 1, 6\rangle = \binom{r+4}{5}.$$

Using Lemmas 4.1, 4.2, and 4.3 and noting that since $4 \mid r$ we have

$$\tfrac{11}{20}r + \tfrac{3}{8}r^2 - \tfrac{1}{8}r^3 + \tfrac{1}{8}r^4 + \tfrac{3}{40}r^5 \equiv \pm\tfrac{1}{4}r \pmod r$$

we get by Lemma 4.3 that

$$\mathsf{B}_{(r;(1,1,-1,1,1,1))}-I \overset{1}{\cong} \begin{pmatrix} 0 & r & \frac{1}{2}r & 0 & \frac{1}{4}x_1r & \pm\frac{1}{4}r \\ 0 & 0 & r & \frac{1}{2}r(r+1) & x_2r & \frac{1}{4}x_3r \\ 0 & 0 & 0 & r & \frac{1}{2}r(r+1) & 0 \\ 0 & 0 & 0 & 0 & r & \frac{1}{2}r \\ 0 & 0 & 0 & 0 & 0 & r \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\overset{2}{\cong} \begin{pmatrix} 0 & r & \frac{1}{2}r & 0 & \frac{1}{4}x_1r & \frac{1}{4}r \\ 0 & 0 & r & \frac{1}{2}r & x_2r & \frac{1}{4}x_3r \\ 0 & 0 & 0 & r & \frac{1}{2}r & 0 \\ 0 & 0 & 0 & 0 & r & \frac{1}{2}r \\ 0 & 0 & 0 & 0 & 0 & r \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \overset{3}{\cong} \begin{pmatrix} 0 & r & \frac{1}{2}r & 0 & \frac{1}{4}r & \frac{1}{4}r \\ 0 & 0 & r & \frac{1}{2}r & 0 & \frac{1}{4}r \\ 0 & 0 & 0 & r & \frac{1}{2}r & 0 \\ 0 & 0 & 0 & 0 & r & \frac{1}{2}r \\ 0 & 0 & 0 & 0 & 0 & r \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

and

$$\mathsf{B}_{(r;(1,1,1,1,1,1))}-I \overset{1}{\cong} \begin{pmatrix} 0 & r & \frac{1}{2}r & 0 & \frac{1}{4}y_1r & 0 \\ 0 & 0 & r & \frac{1}{2}r(r+1) & y_2r & \frac{1}{4}y_3r \\ 0 & 0 & 0 & r & \frac{1}{2}r(r+1) & 0 \\ 0 & 0 & 0 & 0 & r & \frac{1}{2}r \\ 0 & 0 & 0 & 0 & 0 & r \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\overset{2}{\cong} \begin{pmatrix} 0 & r & \frac{1}{2}r & 0 & \frac{1}{4}y_1r & 0 \\ 0 & 0 & r & \frac{1}{2}r & y_2r & \frac{1}{4}y_3r \\ 0 & 0 & 0 & r & \frac{1}{2}r & 0 \\ 0 & 0 & 0 & 0 & r & \frac{1}{2}r \\ 0 & 0 & 0 & 0 & 0 & r \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \overset{3}{\cong} \begin{pmatrix} 0 & r & \frac{1}{2}r & 0 & \frac{1}{4}r & 0 \\ 0 & 0 & r & \frac{1}{2}r & 0 & \frac{1}{4}r \\ 0 & 0 & 0 & r & \frac{1}{2}r & 0 \\ 0 & 0 & 0 & 0 & r & \frac{1}{2}r \\ 0 & 0 & 0 & 0 & 0 & r \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

for odd $x_1, x_2, x_3, y_1, y_2, y_3$ by the following steps. Step 1 reduces the first row and last column modulo r by subtracting the second column and fifth row repeatedly from the other columns and rows. Step 2 subtracts the third column (fourth row) $\frac{r}{2}$ times from the fourth column (third row) and adds the second column (fifth row) $\frac{r}{4}$ times to the fourth column (third row). Step 3 reduces the entries $\langle 1, 5\rangle$, $\langle 2, 5\rangle$, and $\langle 2, 6\rangle$ modulo $\frac{r}{2}$ by subtracting the fourth column and third row repeatedly from the fifth and sixth column and first and second row repeatedly. Note that the changes to entries $\langle 4, 1\rangle$, $\langle 4, 2\rangle$, $\langle 5, 3\rangle$, and $\langle 6, 3\rangle$ can be inverted by adding the second and third column to the fourth column and by adding the fourth and fifth row to the third row.

Now, dividing every entry by $\frac{r}{4}$, it follows that we have $\mathsf{B}_{(r;\overline{1})} \sim \mathsf{B}_{(r;(1,1,-1,1,1,1)}$ if and only if

$$\begin{pmatrix} 0 & 4 & 2 & 0 & 1 & \pm 1 \\ 0 & 0 & 4 & 2 & 0 & 1 \\ 0 & 0 & 0 & 4 & 2 & 0 \\ 0 & 0 & 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \cong \begin{pmatrix} 0 & 4 & 2 & 0 & 1 & 0 \\ 0 & 0 & 4 & 2 & 0 & 1 \\ 0 & 0 & 0 & 4 & 2 & 0 \\ 0 & 0 & 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

However, this can be checked to not be the case simply by solving the system of linear equations induced by Definition 2.11 and finding that there are no solutions. Our conclusion follows. □

5. Concluding remarks

Combining the results of the previous sections, we arrive at our main result, which answers the question of for which parameters n and r there only is a single, unique quantum lens space.

Theorem 5.1. *Let $r > 2$ and let p be the smallest odd prime dividing r. Then*

$$\widetilde{\varphi}(r) = \begin{cases} p+1, & 4 \nmid r \\ \min\{6, p+1\}, & 4 \mid r. \end{cases}$$

Proof. For $4 \nmid r$ this follows directly from Theorem 3.5. Thus, let $4 \mid r$. By Corollary 2.19, $\widetilde{\varphi}(r) \geq 4$, and it follows that if $p = 3$ we have $\widetilde{\varphi}(r) = 4$ by Theorem 3.4 and if $p \neq 3$ we have $\widetilde{\varphi}(r) = 6$ by Theorem 4.4. □

We recall that $\widetilde{\varphi}(r)$ is the minimum n for which there exists an m such that $C(L_q(r, \overline{1})) \otimes \mathbb{K} \not\cong C(L_q(r, \overline{m})) \otimes \mathbb{K}$ so that our result explains exactly how to find the smallest dimension where the m-vector influences the stable isomorphism class of the quantum lens space for any fixed r. In fact, using Proposition 14.5 in [Eilers et al. 2016] we get that $\widetilde{\varphi}(r)$ is the minimum n for which there is an m such that $C(L_q(r, \overline{1})) \not\cong C(L_q(r, \overline{m}))$.

Further, for the case when the quantum lens space is not uniquely given, we studied the number of equivalence classes arising by varying the parameter $\overline{m} \in (Z_r)^n$. In Theorem 3.8, a lower bound on the number of such equivalence classes was found by giving a necessary condition for two quantum lens spaces to be isomorphic. However, computer experiments suggest that this necessary condition is in fact even sufficient when $4 \nmid r$. We thus conjecture the following which we have confirmed by computer experiments for $r \in \{3, 5, 6, 9\}$ and $n \leq 8$, and for $r \in \{10, 15, 21\}$ and $n \leq 7$.

Conjecture 5.2. *Let $r = 2^t \cdot p_1^{\alpha_1} \cdots p_k^{\alpha_k}$, $t \in \{0, 1\}$. Further, let $\overline{m}, \overline{m}' \in (Z_r)^n$ be given. Then $\mathsf{B}_{(r;\overline{m})} \sim \mathsf{B}_{(r;\overline{m}')}$ if and only if for every $1 \le i \le k$ and $1 \le t \le n - p_i$,*

$$\prod_{l=t+1}^{t+p_i-1} m_l \equiv \prod_{l=t+1}^{t+p_i-1} m_l' \pmod{p_i}.$$

This conjecture is true if and only if we have equality in Theorem 3.9 when $4 \nmid r$, so an equivalent conjecture is the following.

Conjecture 5.3. *Let $r > 2$ have the prime factorisation $r = 2^t \cdot p_1^{\alpha_1} \cdots p_k^{\alpha_k}$, $t \in \{0, 1\}$. Then*

$$\varphi_r(n) = \prod_{i=1}^{k} \lceil (p_i - 1)^{n-p_i} \rceil.$$

Proving these conjectures seems hard to do using the methods of this paper, however, since determining equivalence of matrices once they become sufficiently large is a complex task unless one can find better invariants to rely on. Worth noting is that proving Conjectures 5.2 and 5.3 would yield the following satisfactory result, which resounds well with the overall findings of this paper.

Conjecture 5.4. *The equivalence classes of $S_{r,n}/\sim$ all have the same number of members.*

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ON GENERIC QUADRATIC FORMS

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Based on Totaro's computation of the Chow ring of classifying spaces for orthogonal groups, we compute the Chow rings of all orthogonal Grassmannians associated with a generic quadratic form of any dimension. This closes the gap between the known particular cases of the quadric and the highest orthogonal Grassmannian. We also relate two different notions of generic quadratic forms.

1. Introduction

Let k be a field of characteristic different from 2 and let $F_g = k(t_1, \dots, t_n)$ be the field of rational functions over k in variables $t_1, \dots, t_n$ for some $n \geq 2$. We call *generic* the diagonal quadratic form $q_g := \langle t_1, \dots, t_n \rangle$ over F_g. Thus q_g is the n-dimensional quadratic form $F_g^n \to F_g$ on the vector space F_g^n given by the formula

$$q_g : (x_1, \dots, x_n) \mapsto \sum_{1 \leq i \leq n} t_i x_i^2.$$

The Chow ring of the projective quadric defined by q_g has been computed in [Karpenko 1991, Corollary 2.2]. The Chow ring of the highest orthogonal Grassmannian of a generic quadratic form has been computed in [Petrov 2016] (see also [Smirnov and Vishik 2014]), but this was done for a different notion of generic, which we call here *standard generic*. As shown in Section 3, the n-dimensional standard generic quadratic form q lives over the field of rational functions $F = k(t_{ij})_{1 \leq i \leq j \leq n}$ in $n(n+1)/2$ variables t_{ij} and can be defined (in arbitrary characteristic including characteristic 2) by the formula

$$F^n \to F, \quad (x_1, \dots, x_n) \mapsto \sum_{1 \leq i \leq j \leq n} t_{ij} x_i x_j.$$

In the present paper we determine the Chow ring $\mathrm{CH}\, X$ of all orthogonal Grassmannians X associated with the generic and the standard generic quadratic forms.

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(The characteristic $\neq 2$ assumption is removed in the latter case; the characteristic 2 analog for the first case is provided in Section 9.) Namely, our main theorem (Theorem 6.1; see also Corollary 8.2 and Proposition 9.2) affirms that the ring $\operatorname{CH} X$ is generated by the Chern classes of the tautological vector bundle of X. A complete list of relations satisfied by these Chern classes (in general, not only in the generic situation) is provided in Theorem 2.1. All the (well-known) relations that hold over an algebraic closure of the base field actually already hold over the base field itself. This way we obtain a description of the ring $\operatorname{CH} X$ in terms of generators and relations. It also follows that the additive group of $\operatorname{CH} X$ is torsion-free (see Corollary 6.2).

To prove the main theorem, we use the computation of the Chow ring of classifying spaces for orthogonal groups $O(n)$ performed in [Pandharipande 1998], as well as in [Totaro 1999] over the field of complex numbers, and later in [Molina Rojas and Vistoli 2006] over an arbitrary field of characteristic $\neq 2$. We actually need only a piece of this computation which is made in [Totaro 1999] over arbitrary fields (of arbitrary characteristic); see Section 5.

Note that the algebraic group $O(n)$ over a field k is not connected if n is even or $\operatorname{char} k \neq 2$. In the remaining case (when n is odd and $\operatorname{char} k = 2$), the algebraic group $O(n)$ is not smooth. In contrast, the special orthogonal group $O^+(n)$ is always smooth and connected. But since $O(n)$-torsors correspond to all nondegenerate n-dimensional quadratic forms while $O^+(n)$-torsors correspond to quadratic forms of trivial discriminant, it is more appropriate to work with $O(n)$ for the question raised in this paper. On the other hand, since orthogonal Grassmannians depend only on the similarity class of the quadratic form in question and any odd-dimensional quadratic form is similar to that of trivial discriminant, $O(n)$ can be replaced by $O^+(n)$ for odd n.

2. Tautological Chern subring

In this section we consider an arbitrary nondegenerate quadratic form $q : V \to F$ of an arbitrary dimension $n \geq 2$ over an arbitrary field F. (Characteristic 2 is not excluded; nondegenerate quadratic forms are defined as in [Elman et al. 2008, §7.A]. We require $n \geq 2$ everywhere in the paper because the varieties we are interested in, introduced below, do not occur for $n = 1$.) In particular, V is an n-dimensional F-vector space. We fix an integer $1 \leq m \leq n/2$ and write X for the orthogonal Grassmannian of isotropic m-planes (i.e., totally isotropic m-dimensional subspaces) in V. Note that the variety X is smooth projective; it is geometrically connected if and only if $m \neq n/2$.

Let $\mathcal{T} = \mathcal{T}_X$ be the tautological (rank-m) vector bundle on X: the fiber of $\mathcal{T}$ over a point of X, given by an isotropic m-plane, is this very m-plane itself. We define the *tautological Chern subring* $\operatorname{CT} X$ in the Chow ring $\operatorname{CH} X$ as the subring generated by the Chern classes $c_1(\mathcal{T}), \dots, c_m(\mathcal{T})$. The goal of this section is to determine the ring $\mathrm{CT}\, X$ by providing a list of defining relations on its generators.

The variety X is a closed subvariety of the usual Grassmannian Γ of all m-planes in V. The Chow ring $\mathrm{CH}\,\Gamma$ is known to be generated by the Chern classes of the tautological (rank-m) vector bundle on Γ. Therefore the pull-back $\mathrm{CH}\,\Gamma \to \mathrm{CH}\, X$ with respect to the closed imbedding $X \hookrightarrow \Gamma$ provides an epimorphism $\mathrm{CH}\,\Gamma \twoheadrightarrow \mathrm{CT}\, X$. Since a description of the ring $\mathrm{CH}\,\Gamma$ by generators and relations is available (see [Buch et al. 2009, Lemma 1.2] or [Fulton 1984, Example 14.6.6]), we fulfill our goal if we describe the kernel of the epimorphism $\mathrm{CH}\,\Gamma \twoheadrightarrow \mathrm{CT}\, X$ in terms of generators of $\mathrm{CH}\,\Gamma$. For this, it is more convenient to use the generators $c_1, \dots, c_{n-m} \in \mathrm{CH}\,\Gamma$ given by the Chern classes of $-[\mathcal{T}]$ rather than of $\mathcal{T} = \mathcal{T}_\Gamma$ itself. By $[\mathcal{T}]$ here we mean the class of $\mathcal{T}$ in the Grothendieck ring $K(\Gamma)$. The Chern classes of $-[\mathcal{T}]$ are the Segre classes of $\mathcal{T}$, i.e., the components of the multiplicative inverse to the total Chern class $c(\mathcal{T})$. The tautological vector bundle $\mathcal{T}$ is a subbundle of the trivial (rank-n) vector bundle V and $c_1, \dots, c_{n-m}$ are the Chern classes of the quotient $V/\mathcal{T}$. We define $c_i \in \mathrm{CH}^i(\Gamma)$ for every integer i by setting $c_i := c_i(-[\mathcal{T}]) = c_i(V/\mathcal{T})$. Therefore $c_0 = 1$ and $c_i = 0$ for $i < 0$ as well as for $i > n - m$.

Theorem 2.1. *The kernel of the epimorphism* $\mathrm{CH}\,\Gamma \twoheadrightarrow \mathrm{CT}\, X$ *is generated by the elements*

$$(2.2)\qquad c_i^2 - 2c_{i-1}c_{i+1} + 2c_{i-2}c_{i+2} - \cdots + (-1)^i 2c_0c_{2i}, \qquad \text{with } i > n/2 - m,$$

and c_{n-m}. *The abelian group* $\mathrm{CT}\, X$ *is free with a basis consisting of the images of the products* $c_1^{\alpha_1} \cdots c_{n-m-1}^{\alpha_{n-m-1}}$ *with* $\alpha_1 + \cdots + \alpha_{n-m-1} \le m$ *and* $\alpha_i \le 1$ *for* $i > n/2 - m$.

Proof. Let us first check that the elements (2.2) lie in the kernel. The i-th element is mapped to the Chern class $c_{2i}(-[\mathcal{T}] - [\mathcal{T}^\vee]) \in \mathrm{CT}\, X$, where $\mathcal{T} = \mathcal{T}_X$ and $\mathcal{T}^\vee$ is the dual vector bundle. The isomorphism $V/\mathcal{T}^\perp = \mathcal{T}^\vee$, where $\mathcal{T}^\perp$ is the vector bundle given by the orthogonal complement, shows that $-[\mathcal{T}] - [\mathcal{T}^\vee] = -[\mathcal{T}] + [\mathcal{T}^\perp] = [\mathcal{T}^\perp/\mathcal{T}]$. Since the rank of the quotient $\mathcal{T}^\perp/\mathcal{T}$ is $n - 2m$ (see [Elman et al. 2008, Proposition 1.5]), its Chern classes vanish in degrees $> n - 2m$.

In order to show that c_{n-m} is in the kernel, we proceed similarly to [Vishik 2009, proof of Proposition 2.1]. Notice that the projective bundle $\mathbb{P}(\mathcal{T})$ over X can be identified with the variety of flags of totally isotropic subspaces in V of dimensions 1 and m. In particular, besides the projection $\pi : \mathbb{P}(\mathcal{T}) \to X$, we have a projection $\pi_1 : \mathbb{P}(\mathcal{T}) \to X_1$ to the projective quadric X_1 (the orthogonal Grassmannian of 1-planes). Moreover, the tautological line bundle on the projective bundle $\mathbb{P}(\mathcal{T})$ is the pull-back $\pi_1^*(\mathcal{T}_1)$ of the tautological line bundle $\mathcal{T}_1$ on X_1. It follows by [Elman et al. 2008, §58] or [Fulton 1984, Chapter 3] that $c_i(-[\mathcal{T}]) = \pi_*(\pi_1)^* c_{i+m-1}(-[\mathcal{T}_1])$ for any i. Since $\dim X_1 = n - 2$, the Chern class $c_{n-1}(-[\mathcal{T}_1])$ vanishes, implying the vanishing of $c_{n-m}(-[\mathcal{T}])$.

In order to show that the kernel is generated by the elements (2.2) and c_{n-m}, we construct additive generators of the quotient C of the ring $\mathrm{CH}\,\Gamma$ by the ideal generated by the elements (2.2) and c_{n-m}. We recall that the group $\mathrm{CH}\,\Gamma$ is free; a basis is given by the products $c_1^{\alpha_1}\cdots c_{n-m}^{\alpha_{n-m}}$ with $\alpha_1+\cdots+\alpha_{n-m}\le m$. Using the additional relations in C, we can eliminate squares of c_i for $i>n/2-m$. Indeed, in the quotient of C by the subgroup generated by the products satisfying the additional condition, any element is divisible by an arbitrary 2-power and therefore is 0 since C is finitely generated.

It follows that the group C is generated by the products $c_1^{\alpha_1}\cdots c_{n-m-1}^{\alpha_{n-m-1}}$ satisfying the additional condition $\alpha_i\le 1$ for $i>n/2-m$. It turns out that these are free generators. Moreover, they remain free when we map them to $\mathrm{CT}\,X$ and this finishes the proof of the theorem.

Our products are free in $\mathrm{CT}\,X$ because their images in the $\mathbb{Q}$-vector space $\mathbb{Q}\otimes\mathrm{CH}\,\overline{X}$ are free, where $\overline{X}$ is X over an algebraic closure of F. For odd n this follows from [Buch et al. 2009, Theorem 2.2(b) and formula (15)] (see Remark 2.3). For even n it follows from [Buch et al. 2009, Theorem 3.2(b) and formula (40)]. □

Remark 2.3. The paper [Buch et al. 2009], applied in the above proof, actually deals with the singular cohomology ring instead of the Chow ring. The link is explained by the following two well-known facts: the variety $\overline{X}$ is cellular and the ring $\mathrm{CH}\,\overline{X}$ does not depend on the base field. If the base field is $\mathbb{C}$, then the cycle map from $\mathrm{CH}\,\overline{X}$ to the corresponding singular cohomology ring is an isomorphism, [Fulton 1984, Example 19.1.11(b)].

Remark 2.4. In the case of the highest orthogonal Grassmannian, the ring $\mathrm{CH}\,\overline{X}$ has been described in [Vishik 2005] (see also [Elman et al. 2008, Proposition 86.16 and Theorem 86.12]).

Remark 2.5. Theorem 2.1 shows that the ring $\mathrm{CT}\,X$ only depends on the integers n and m.

Remark 2.6. For odd n, the ring $\mathrm{CT}\,X$ can be identified with the full Chow ring $\mathrm{CH}\,Y$ of the variety of isotropic m-planes in an $n-1$-dimensional vector space endowed with a nondegenerate alternating bilinear form: there is an isomorphism $\mathrm{CH}\,Y\to\mathrm{CT}\,X$ mapping the Segre classes of the tautological vector bundle on Y to the Segre classes of $\mathcal{T}_X$. (See [Buch et al. 2009, Theorem 1.2] for a description of the ring $\mathrm{CH}\,Y$ by generators and relations.) This funny observation in the case of the highest orthogonal Grassmannian turned out to be very useful in [Totaro 2005]. We do not use it here.

Our next and ultimate goal is to show that $\mathrm{CT}\,X=\mathrm{CH}\,X$ in the case of generic q. First we need to clarify what is meant by *generic*. We start with the notion of the standard generic quadratic form.

3. The standard generic quadratic form

For a field k (of any characteristic) and an integer $n \geq 2$, the standard generic n-dimensional quadratic form is defined as follows.

We consider the orthogonal group $O(n)$ over k and its tautological imbedding into the general linear group $\mathrm{GL}(n)$. The generic fiber of the quotient map

$$\mathrm{GL}(n) \to \mathrm{GL}(n)/O(n)$$

is an $O(n)$-torsor over the function field $F := k(\mathrm{GL}(n)/O(n))$. It determines an n-dimensional quadratic form over F (via the identification of [Demazure and Gabriel 1970, Chapitre III, §5, 2.1]; for the case of smooth $O(n)$ see also [Knus et al. 1998, (29.28)]) which we call the *standard generic* one.

In order to describe it explicitly, we first recall the interpretation of the quotient variety $\mathrm{GL}(n)/O(n)$ as the variety Q of nondegenerate quadratic forms on the vector space $V := k^n$.[1] The variety of all quadratic forms on V is an affine space (of dimension $n(n+1)/2$) and Q is its open subvariety. The group $\mathrm{GL}(n)$ acts on Q in the evident way. The action is such that for any algebraically closed field $K \supset k$, the abstract group $\mathrm{GL}(n)(K)$ of K-points of $\mathrm{GL}(n)$ acts transitively on the set $Q(K)$ of K-points of Q. Finally, the algebraic group $O(n)$, by its very definition, is the stabilizer of the split quadratic form $q_0 \in Q(k)$, defined by the formulas (7.1) and (7.2).[2] It follows by [Demazure and Gabriel 1970, Proposition 2.1 of Chapter III §3] that Q is the quotient variety $\mathrm{GL}(n)/O(n)$.

For any field extension L/k, an L-point of Q is a nondegenerate quadratic form q on the L-vector space V_L; the fiber of the quotient map $\mathrm{GL}(n) \to Q$ over this point is an $O(n)$-torsor E over L, and q is the quadratic form corresponding to E. In particular, the quadratic form given by the generic fiber of $\mathrm{GL}(n) \to Q$ is defined over the field of rational functions $F = k(t_{ij})_{1\leq i\leq j\leq n}$ (where t_{ij} are indeterminates, and F/k is purely transcendental of the transcendence degree $n(n+1)/2$) by the formula

$$(x_1, \dots, x_n) \mapsto \sum_{1\leq i\leq j\leq n} t_{ij} x_i x_j .$$

4. Chow rings of classifying spaces

Let F be a field (of arbitrary characteristic) and let G be an affine algebraic group over F, not necessarily smooth. The Chow ring CH_G of the classifying space of G, introduced in [Totaro 1999], is the G-equivariant Chow ring $\mathrm{CH}_G(\operatorname{Spec} F)$. This is a graded ring; the grading is given by codimension of cycles.

The ring CH_G if cofunctorial in G: a homomorphism $G' \to G$ of affine algebraic

[1] This interpretation is mentioned in [Totaro 1999, §15]; however, due to the context, $k = \mathbb{C}$ there.

[2] A quadratic form over k is called *split* if it is isomorphic to q_0.

groups produces a homomorphism of graded rings $\mathrm{CH}_G \to \mathrm{CH}_{G'}$ (see [Molina Rojas and Vistoli 2006, §2]).

By [Edidin and Graham 1997, Lemma 4] (see also [Karpenko 2012, §3]), if G is a split torus, the homomorphism of graded rings $S(\hat{G}) \to \mathrm{CH}_G$ is an isomorphism, where $\hat{G}$ is the character lattice of G, $S(\hat{G})$ is the symmetric $\mathbb{Z}$-algebra, and a character $\chi \in \hat{G} = S^1(\hat{G})$, viewed as a G-equivariant line bundle over Spec F, is mapped to its first equivariant Chern class in CH^1_G.

We return to the situation where G is an arbitrary affine algebraic group over F:

Proposition 4.1. *Let G' be a closed normal subgroup of G such that the quotient $T := G/G'$ is a split torus. Then the restriction homomorphism $\mathrm{CH}_G \to \mathrm{CH}_{G'}$ is surjective and its kernel is generated by some elements in CH^1_G. More precisely, the kernel is generated by the image of the* (*additive*) *homomorphism*

$$\hat{T} = S^1(\hat{T}) = \mathrm{CH}^1_T \to \mathrm{CH}^1_G$$

induced by the quotient homomorphism $G \to T$.

Proof. For any integer i, let us consider a generically free G-representation V possessing an open G-equivariant subset $U \subset V$ such that $\mathrm{codim}_V(V \setminus U) \geq i$ and there are a G-torsor $U \to U/G$ and a G'-torsor $U \to U/G'$. By definition of CH_G (and similarly for G' in place of G), we have a ring homomorphism $\mathrm{CH}_G \to \mathrm{CH}(U/G)$ which is bijective in codimensions $< i$. Moreover, the diagram

$$\begin{array}{ccc} \mathrm{CH}_G & \longrightarrow & \mathrm{CH}_{G'} \\ \downarrow & & \downarrow \\ \mathrm{CH}(U/G) & \longrightarrow & \mathrm{CH}(U/G') \end{array}$$

commutes, where the bottom map is the pull-back homomorphism with respect to the T-torsor $U/G' \to U/G$. Therefore, in order to prove surjectivity of $\mathrm{CH}_G \to \mathrm{CH}_{G'}$, it suffices to prove surjectivity of $\mathrm{CH}(U/G) \to \mathrm{CH}(U/G')$. Moreover, to get the description of the kernel for $\mathrm{CH}_G \to \mathrm{CH}_{G'}$ it suffices to prove the similar description for the kernel of $\mathrm{CH}(U/G) \to \mathrm{CH}(U/G')$, where the homomorphism $\hat{T} \to \mathrm{CH}^1(U/G)$ is the composition $\hat{T} \to \mathrm{CH}^1_G \to \mathrm{CH}^1(U/G)$.

Let us first consider the case of $T = \mathbb{G}_{\mathrm{m}}$. Let $\mathcal{L}$ be the line bundle

$$((U/G') \times \mathbb{A}^1)/T$$

over U/G. Then U/G' is an open subvariety in $\mathcal{L}$ and its complement is the zero section. By the homotopy invariance and the localization property of Chow groups ([Elman et al. 2008, Theorem 57.13 and Proposition 57.9]) we have an exact sequence,

$$\mathrm{CH}(U/G) \to \mathrm{CH}(U/G) \to \mathrm{CH}(U/G') \to 0,$$

where the first map is the multiplication by the first Chern class of $\mathcal{L}$. This finishes the proof for $T = \mathbb{G}_{\mathrm{m}}$.

In the general case, we induct on the rank of T. We decompose T as $\mathbb{G}_m \times T_1$ and define an intermediate subgroup G_1 with $G' \subset G_1 \subset G$ as the kernel of the composition $G \to T \to T_1$. The quotient G/G_1 is then T_1 and the quotient G_1/G' is $\mathbb{G}_m$. The homomorphism $\mathrm{CH}_G \to \mathrm{CH}_{G'}$ decomposes into the composition $\mathrm{CH}_G \to \mathrm{CH}_{G_1} \to \mathrm{CH}_{G'}$. The surjectivity statement follows because both maps in the composition are surjective by induction. It remains to determine the kernel.

Let $x \in \mathrm{CH}_G$ be an element vanishing in $\mathrm{CH}_{G'}$, then the image of x in CH_{G_1} is the product yx_1 for some $x_1 \in \mathrm{CH}_{G_1}$, where $y \in \mathrm{CH}^1_{G_1}$ is the image of a character of $\mathbb{G}_m$. Extending the character to T, we get an element $y' \in \mathrm{CH}_G$ lying in the image of $\hat{T} \to \mathrm{CH}^1_G$ and mapped to y. Using the surjectivity of $\mathrm{CH}_G \to \mathrm{CH}_{G_1}$, we find an element $x'_1 \in \mathrm{CH}_G$ mapped to x_1. The difference $x - y'x'_1$ is then in the kernel of $\mathrm{CH}_G \to \mathrm{CH}_{G_1}$ and therefore, by induction, lies in the ideal generated by the image of $\hat{T}$. It follows that x itself lies in the ideal. □

Corollary 4.2. *In the situation of Proposition 4.1, if the ring* $\mathrm{CH}_{G'}$ *is generated by Chern classes* (*in the sense of* [Karpenko 2018, §5]), *then the ring* CH_G *is also generated by Chern classes.*

Proof. For any $i \geq 0$ and any $x \in \mathrm{CH}^i_G$, since the image of x in $\mathrm{CH}^i_{G'}$ is a polynomial in Chern classes, there exists an element $x' \in \mathrm{CH}^i_G$, lying in the Chern subring, such that the difference $x - x'$ vanishes in $\mathrm{CH}_{G'}$. By Proposition 4.1, $x - x'$ belongs to the ideal in CH_G generated by CH^1_G so we can induct on i. □

Example 4.3. Taking G to be a split connected reductive algebraic group and $G' \subset G$ to be the semisimple group given by the commutator subgroup of G, we are in the situation of Proposition 4.1: G/G' is a split torus. Therefore Proposition 4.1 describes the relation between the Chow ring of the classifying space of a split reductive group G and that of its semisimple part G'. In particular, by Corollary 4.2, if $\mathrm{CH}_{G'}$ is generated by Chern classes, then CH_G is also generated by Chern classes. This has been proved (by a different method) in [Karpenko 2018, Proposition 5.5] in the case of special (split reductive) G, where *special* means that every G-torsor over any field extension of the base field is trivial.

5. Chow rings of classifying spaces for orthogonal groups

The following proposition is a (slightly modified) particular case of [Totaro 1999, Proposition 14.2]. We provide a proof because it is shorter than that of the original statement.

Proposition 5.1. *For any algebraic group G* (*over any field*) *and any imbedding of G into a special algebraic group H, the homomorphism* $\mathrm{CH}_H \to \mathrm{CH}_G$ *is surjective provided that the Chow groups of the quotient H/G over any field extension of the base field are trivial in positive codimensions.*

Proof. As usual, we replace the homomorphism in question by the pull-back homomorphism $\mathrm{CH}(U/H) \to \mathrm{CH}(U/G)$ with respect to the morphism $U/G \to U/H$, where U is an open subvariety in an H-representation, an H-torsor over U/H, and a G-torsor over U/G. Since H is special, every H-torsor is Zariski–locally trivial, [Chevalley et al. 1958]. It follows that the fiber of $U/G \to U/H$ over any point $x \in U/H$ is isomorphic to the quotient variety H/G with scalars extended to the residue field of x and therefore has trivial Chow groups in positive codimensions. The statement follows from the spectral sequence of [Rost 1996, Corollary 8.2] (see also [Karpenko and Merkurjev 1990, §3]) computing the K-cohomology groups of the total space of the fibration $U/G \to U/H$ in terms of the K-cohomology groups of the base and of the fibers. □

We get the following statement for arbitrary base fields of arbitrary characteristic:

Corollary 5.2. *For any $n \geq 2$, the homomorphism $\mathrm{CH}_{\mathrm{GL}(n)} \to \mathrm{CH}_{O(n)}$, given by the tautological imbedding $O(n) \hookrightarrow \mathrm{GL}(n)$, is surjective.*

Proof. As explained in Section 3, the quotient variety $\mathrm{GL}(n)/O(n)$ is identified with the variety Q of n-dimensional nondegenerate quadratic forms. Since Q is an open subvariety in the affine space of all n-dimensional quadratic forms, we have $\mathrm{CH}^{>0}(Q) = 0$ by the homotopy invariance and the localization property of Chow groups. □

6. Main theorem and its consequences

In this section, k is a field (of any characteristic), n is an integer ≥ 2, F is the function field $k(\mathrm{GL}(n)/O(n))$, E is the standard generic $O(n)$-torsor given by the generic fiber of $\mathrm{GL}(n) \to \mathrm{GL}(n)/O(n)$, and q is the corresponding standard generic quadratic form.

For m with $1 \leq m \leq n/2$, let X be the m-th orthogonal Grassmannian of q. We would like to determine the ring $\mathrm{CH}\,X$. The main result is expressed in terms of the tautological (rank-m) vector bundle on X. Its proof will be given in the next section.

Theorem 6.1. *The ring $\mathrm{CH}\,X$ is generated by the Chern classes of the tautological vector bundle.*

Theorem 6.1 claims that $\mathrm{CH}\,X = \mathrm{CT}\,X$, and the ring $\mathrm{CT}\,X$ has been computed in Section 2.

Before proving Theorem 6.1, let us list some consequences. Let Y be any (partial) flag variety of totally isotropic subspaces in q. Let us consider the standard graded epimorphism $\mathrm{CH}\,Y \to GK(Y)$ onto the graded ring associated with the topological filtration (i.e., the filtration by codimension of support) on the Grothendieck ring $K(Y)$.

Corollary 6.2. *The abelian group* CH Y *is free and, in particular, torsion-free. The ring epimorphism* CH $Y \to GK(Y)$ *is an isomorphism. The topological filtration on* $K(Y)$ *coincides with the gamma filtration.*

Proof. The variety Y is the variety of flags of totally isotropic subspaces in q of some dimensions $m_1 < \cdots < m_d$. Let X be the orthogonal Grassmannian of m-planes with $m = m_d$. The projection $Y \to X$ is a partial flag variety of subspaces in the tautological vector bundle on X. Therefore, it suffices to prove Corollary 6.2 for X instead of Y.

We have that CH X = CT X (Theorem 6.1) and that CT X is a free abelian group (Theorem 2.1).

The kernel of the epimorphism is contained in the torsion subgroup. Since CH X is torsion-free, the epimorphism is an isomorphism.

Since the Chow ring CH X is generated by Chern classes, the topological filtration on $K(X)$ coincides with the gamma filtration; see [Karpenko 1998, Remark 2.17]. □

7. Proof of the main theorem

We continue to work over a field k of arbitrary characteristic. We realize the orthogonal group $O(n)$ as the automorphism group of the following split quadratic form $q_0 : V \to k$ on the k-vector space $V := k^n$:

$$k^n \ni (x_1, \ldots, x_{n/2}, y_{n/2}, \ldots, y_1) \mapsto x_1 y_1 + x_2 y_2 + \cdots + x_{n/2} y_{n/2} \tag{7.1}$$

if n is even, and

$$k^n \ni (x_1, \ldots, x_{(n-1)/2}, z, y_{(n-1)/2}, \ldots, y_1) \mapsto x_1 y_1 + x_2 y_2 + \cdots + x_{(n-1)/2} y_{(n-1)/2} + z^2 \tag{7.2}$$

if n is odd.

Instead of the m-th orthogonal Grassmannian X_0 (for some m with $1 \le m \le n/2$), we consider the variety Y_0 of flags of totally isotropic subspaces in q_0 of dimensions $1, \ldots, m$. The group $O(n)$ acts on Y_0 and for any algebraically closed field $K \supset k$ the action of the group $O(n)(K)$ on the set $Y_0(K)$ is transitive. Therefore, by [Demazure and Gabriel 1970, Proposition 2.1 of Chapter III §3], the variety Y_0 is the quotient $O(n)/P$, where P is the stabilizer of the rational point of Y_0 given by the standard flag $V_1 \subset \cdots \subset V_m$ with V_i being the span of the first i vectors in the standard basis of V. (Note that for any m, the variety X_0 is also the quotient of $O(n)$ by the stabilizer of any rational point on X_0; this includes $m = n/2$ even though neither X_0 nor Y_0 are connected in this case: recall that the orthogonal group $O(n)$ is also nonconnected for even n.)

Any orthogonal transformation stabilizing this flag also stabilizes the orthogonal complements

$$V_m^\perp = V_{n-m} \subset \cdots \subset V_1^\perp = V_{n-1}.$$

Let $\mathcal{F}$ be the variety of flags of all subspaces in V of dimensions

$$1, \dots, m, n-m, \dots, n-1.$$

The group $\mathrm{GL}(n)$ acts on $\mathcal{F}$ and $\mathcal{F} = \mathrm{GL}(n)/S$, where S is the stabilizer of the standard flag $V_1 \subset \cdots \subset V_m \subset V_{n-m} \subset \cdots \subset V_{n-1}$.

Let E be the standard generic $O(n)$-torsor given by the generic fiber of $\mathrm{GL}(n) \to \mathrm{GL}(n)/O(n)$. Let $\mathcal{E}$ be the corresponding $\mathrm{GL}(n)$-torsor obtained via the imbedding $O(n) \hookrightarrow \mathrm{GL}(n)$. We have a commutative square,

$$\begin{array}{ccc} \mathrm{CH}_S & \longrightarrow & \mathrm{CH}(\mathcal{E}/S) \\ \big\downarrow & & \big\downarrow \\ \mathrm{CH}_P & \longrightarrow & \mathrm{CH}(E/P) \end{array}$$

with surjective horizontal mappings (cf. [Karpenko 2017a, Lemma 2.1]).

We claim that the homomorphism $\mathrm{CH}_S \to \mathrm{CH}_P$ is surjective. Admitting the claim for the moment, we conclude that the pull-back homomorphism

$$\mathrm{CH}(\mathcal{E}/S) \to \mathrm{CH}(E/P) = \mathrm{CH}\,Y$$

from the above commutative square is surjective too. Since the group $\mathrm{GL}(n)$ is special, the $\mathrm{GL}(n)$-torsor $\mathcal{E}$ is trivial, implying that $\mathcal{E}/S = \mathcal{F}$. We get a surjection $\mathrm{CH}\,\mathcal{F} \twoheadrightarrow \mathrm{CH}\,Y$ implying that the ring $\mathrm{CH}\,Y$ is generated by the Chern classes of the m tautological vector bundles on Y (given by the components of the flags). It follows (see [Karpenko 2017b, Lemma 4.3]) that $\mathrm{CH}\,X = \mathrm{CT}\,X$.

We finish by proving the claim. The subgroup $S' := \mathbb{G}_{\mathrm{m}}^m \times \mathrm{GL}(n-2m) \times \mathbb{G}_{\mathrm{m}}^m \subset S$ is a Levi subgroup of S, its intersection with $P \subset S$ is $P' := \mathbb{G}_{\mathrm{m}}^m \times O(n-2m)$. The imbedding $P' \hookrightarrow S'$ is the product of the map $\mathbb{G}_{\mathrm{m}}^m \hookrightarrow \mathbb{G}_{\mathrm{m}}^m \times \mathbb{G}_{\mathrm{m}}^m$, $x \mapsto (x, x^{-1})$ and the tautological imbedding $O(n-2m) \hookrightarrow \mathrm{GL}(n-2m)$.

In the commutative square

$$\begin{array}{ccc} \mathrm{CH}_S & \longrightarrow & \mathrm{CH}_{S'} \\ \big\downarrow & & \big\downarrow \\ \mathrm{CH}_P & \longrightarrow & \mathrm{CH}_{P'} \end{array}$$

the horizontal maps are isomorphisms (see [Karpenko 2018, proof of Proposition 6.1]). Therefore, in order to prove the claim, it suffices to prove that the homomorphism $\mathrm{CH}_{S'} \to \mathrm{CH}_{P'}$ is surjective.

In the commutative square

$$\begin{array}{ccc} \mathrm{CH}_{S'} & \longrightarrow & \mathrm{CH}_{\mathrm{GL}(n-2m)} \\ \downarrow & & \downarrow \\ \mathrm{CH}_{P'} & \longrightarrow & \mathrm{CH}_{O(n-2m)} \end{array}$$

the horizontal maps are epimorphisms by Proposition 4.1. The map on the right is an epimorphism by Corollary 5.2. We can now prove the surjectivity of the map on the left in every codimension $i \geq 0$ using induction on i.

For $i = 0$ there is nothing to prove. For $i = 1$, we have a commutative diagram

$$\begin{array}{ccccc} \widehat{\mathbb{G}_{\mathrm{m}}^m \times \mathbb{G}_{\mathrm{m}}^m} & \longrightarrow & \mathrm{CH}^1_{S'} & \twoheadrightarrow & \mathrm{CH}^1_{\mathrm{GL}(n-2m)} \\ \downarrow\!\!\downarrow & & \downarrow & & \downarrow\!\!\downarrow \\ \widehat{\mathbb{G}_{\mathrm{m}}^m} & \longrightarrow & \mathrm{CH}^1_{P'} & \twoheadrightarrow & \mathrm{CH}^1_{O(n-2m)} \end{array}$$

with a surjection on the left. Since the lower row is exact (by Proposition 4.1),[3] the statement for $i = 1$ follows.

For $i \geq 2$, it suffices to show that any element $x \in \mathrm{CH}^i_{P'}$, vanishing in the group $\mathrm{CH}_{O(n-2m)}$, is in the image of $\mathrm{CH}_{S'}$. Since $x = y_1 x_1 + \cdots + y_r x_r$ for some $r \geq 0$, some $y_1, \ldots, y_r \in \mathrm{CH}^1_{P'}$, and some $x_1, \ldots, x_r \in \mathrm{CH}^{i-1}_{P'}$ by Proposition 4.1, we are done.

8. The generic quadratic form in characteristic $\neq 2$

For a field k of characteristic $\neq 2$ and an integer $n \geq 2$, we defined in the introduction the generic n-dimensional quadratic form $q_g := \langle t_1, \ldots, t_n \rangle$ over the field of rational functions $F_g := k(t_1, \ldots, t_n)$, and the standard generic quadratic form q over the field of rational functions $F := k(t_{ij})_{1 \leq i \leq j \leq n}$. Now we are going to compare q_g with q.

Proposition 8.1. *The field F_g can be k-identified with a subfield in F the way that the field extension F/F_g is purely transcendental and the generic quadratic form q_g with the scalars extended to the field F becomes isomorphic to the standard generic form q.*

Corollary 8.2. *Theorem 6.1 as well as Corollary 6.2 hold for the generic quadratic form in place of the standard generic one.*

Proof. In case of a purely transcendental field extension, the change of field homomorphism for Chow rings is an isomorphism (see [Kahn and Sujatha 2000, Lemma 1.4a]). □

[3] The upper row is also exact but we don't care.

Proof of Proposition 8.1. Let us apply the standard orthogonalization procedure to the standard basis $e_1, \dots, e_n$ of F^n, where the orthogonality refers to the symmetric bilinear form associated with q. This means that we construct an orthogonal basis $e'_1, \dots, e'_n$ by taking for e'_i the sum of e_i and a linear combination of $e_1, \dots, e_{i-1}$, where the coefficients of the linear combination are determined by the condition that e'_i is orthogonal to $e_1, \dots, e_{i-1}$. The procedure works for q because its restriction to the span of $e_1, \dots, e_i$ is nondegenerate for every i.

Then $t_i := q(e'_i)$ equals t_{ii} plus a rational function in $t_{11}, \dots, t_{i-1i-1}$ and in t_{rs} with $1 \le r < s \le n$. It follows that the elements t_{rs} $(1 \le r < s \le n)$ and $t_1, \dots, t_n$ all together generate the field F over k and therefore — since their number is the transcendence degree — are algebraically independent over k. In particular, $t_1, \dots, t_n$ are algebraically independent so that the field F_g is identified with the subfield $k(t_1, \dots, t_n) \subset F$. This identification has the required properties. □

9. The generic quadratic form in characteristic 2

In characteristic 2 (actually, in arbitrary characteristic), any nondegenerate quadratic form, depending on the parity of n, is isomorphic to the form

$$[a_1, a_2]\perp \cdots \perp[a_{n-1}, a_n] \quad \text{or} \quad [a_1, a_2]\perp \cdots \perp[a_{n-2}, a_{n-1}]\perp\langle a_n\rangle,$$

where $a_1, \dots, a_n$ are constants from the base field and $a_n \neq 0$ in the case of odd n. The notation $[a_1, a_2]$ stands for the 2-dimensional form

$$(x_1, x_2) \mapsto a_1x_1^2 + x_1x_2 + a_2x_2^2.$$

So, the generic n-dimensional quadratic form q_g will be defined as the form

$$[t_1, t_2]\perp \cdots \perp[t_{n-1}, t_n] \quad \text{or} \quad [t_1, t_2]\perp \cdots \perp[t_{n-2}, t_{n-1}]\perp\langle t_n\rangle \tag{9.1}$$

over the rational function field $F_g := k(t_1, \dots, t_n)$.

Proposition 9.2. *Proposition 8.1 and Corollary 8.2 hold in characteristic* 2 *as well.*

Proof. We only need to identify the field F_g with a subfield in $F = k(t_{ij})_{1\le i\le j\le n}$ (over k) the way that the field extension F/F_g is purely transcendental and the generic quadratic form q_g with the scalars extended to the field F becomes isomorphic to the standard generic q.

Starting with the standard basis $e_1, \dots, e_n$ of the vector space F^n, we construct a new basis $e'_1, \dots, e'_n$ as follows. For every odd i, the vector e'_i is e_i plus a linear combination of $e_1, \dots, e_{i-1}$ and if $i < n$ then the vector e'_{i+1} is e_{i+1} plus a linear combination of $e_1, \dots, e_{i-1}$, where the coefficients of the linear combinations are determined by the condition that the new vectors are orthogonal to each of $e_1, \dots, e_{i-1}$. Additionally, for every even i, we divide the vector e'_i by the nonzero scalar (e'_{i-1}, e'_i).

With respect to the new basis, the standard generic quadratic form q has the shape (9.1), where $t_i := q(e'_i)$. For odd i, t_i equals t_{ii} plus a rational function in $t_{11}, \dots, t_{i-1i-1}$ and in t_{rs} with $1 \le r < s \le n$. For even i, t_i equals t_{ii}/f_i plus a rational function in $t_{11}, \dots, t_{i-2i-2}$ and in t_{rs} with $1 \le r < s \le n$, where f_i is also a rational function in $t_{11}, \dots, t_{i-1i-1}$ and in t_{rs} with $1 \le r < s \le n$.

It follows that the elements t_{rs} $(1 \le r < s \le n)$ and $t_1, \dots, t_n$ all together generate the field F over k and therefore are algebraically independent over k. In particular, $t_1, \dots, t_n$ are algebraically independent so that the field F_g is identified with the subfield $k(t_1, \dots, t_n) \subset F$. This identification has the required properties. □

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RANKIN–COHEN BRACKETS AND IDENTITIES AMONG EIGENFORMS

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We investigate the cases for which the Rankin–Cohen brackets of two quasimodular eigenforms give rise to eigenforms. More precisely, we characterise all the cases in a subspace of the space of quasimodular forms for which Rankin–Cohen brackets of two quasimodular eigenforms are again eigenforms. In the process, we obtain some new polynomial identities among quasimodular eigenforms. To prove the results on quasimodular forms, we prove several results in the theory of nearly holomorphic modular forms. These new results in the theory of nearly holomorphic modular forms are of independent interest.

1. Introduction

For an even positive integer k, let M_k and S_k be the respective spaces of modular forms and cusp forms of weight k for the full modular group $\mathrm{SL}_2(\mathbb{Z})$. For an even positive integer k, the Eisenstein series of weight k is defined by

$$E_k(z) = 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) e^{2\pi i n z},$$

where B_k is the k-th Bernoulli number, $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$ and z is in the complex upper-half plane $\mathcal{H}$. We know that for $k \geq 4$, $E_k \in M_k$, but E_2 is not a modular form, rather it is a quasimodular form of weight 2 for $\mathrm{SL}_2(\mathbb{Z})$. There are numerous identities among modular forms. A direct implication of these identities are nice relations among Fourier coefficients of various modular forms. For example, it is well known that $E_4^2 = E_8$ and by comparing the Fourier coefficients of both sides of this identity, we obtain

$$\sigma_7(n) = \sigma_3(n) + 120 \sum_{m=1}^{n-1} \sigma_3(m)\sigma_3(n-m) \tag{1}$$

for $n \geq 1$. Since E_k is an eigenform for any even integer $k \geq 4$, the identity

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$E_4^2 = E_8$ can be interpreted as an identity where the product of two eigenforms results in an eigenform. So it is natural to look for other identities which can be obtained in this way, i.e., those identities in which the product of two eigenforms is an eigenform. The investigation for such identities in the space of modular forms for the full modular group $\mathrm{SL}_2(\mathbb{Z})$ has been done by Duke [1999] and Ghate [2000] independently. They explicitly provided all of the cases in which the product of two eigenforms for the full modular group $\mathrm{SL}_2(\mathbb{Z})$ gives an eigenform. The phenomenon of the product of two eigenforms giving rise to an eigenform can be generalized in two different ways. One generalization is, instead of taking two eigenforms, one may take products of an arbitrary number of eigenforms. Another generalization is by taking the Rankin–Cohen brackets of two eigenforms. Here we note that the product of two modular forms is a particular case of a Rankin–Cohen bracket of two modular forms. Both of these generalizations have been well studied, and we have satisfactory answers for them. Products of arbitrary numbers of eigenforms giving eigenforms have been classified by Emmons and Lanphier [2007], and Rankin–Cohen brackets of eigenforms have been studied by Lanphier and Takloo-Bighash [2004]. In this paper we study the Rankin–Cohen brackets of quasimodular eigenforms. We note that quasimodular forms are a generalization of modular forms. The motivation for studying Rankin–Cohen brackets of quasimodular eigenforms is the well-known identity $E_2\Delta = D\Delta$, where $D = \frac{1}{2\pi i}\frac{d}{dz}$ is the differential operator and

$$\Delta(z) = e^{2\pi i z}\prod_{n\geq 1}(1-e^{2\pi i n z})^{24} = \sum_{n=1}^{\infty}\tau(n)e^{2\pi i n z}$$

is the Ramanujan delta function. The identity $E_2\Delta = D\Delta$ is an identity between quasimodular forms for the group $\mathrm{SL}_2(\mathbb{Z})$ in which the product of two quasimodular eigenforms gives rise to an eigenform. The phenomenon of the product of two quasimodular eigenforms giving an eigenform has been studied in [Meher 2012] and [Das and Meher 2015]. The phenomenon of products of arbitrary numbers of quasimodular eigenforms giving eigenforms has been studied in [Kumar and Meher 2016]. To state our main result we first recall the notion of Rankin–Cohen brackets on quasimodular forms.

Rankin–Cohen brackets for quasimodular forms have been defined by Martin and Royer [2009]. Let f and g be two quasimodular forms of weights k and l and depths s and t respectively. Then for any integer $\nu \geq 0$, the ν-th Rankin–Cohen bracket of f and g is defined by

$$(2) \qquad [f, g]_\nu := \sum_{\alpha=0}^{\nu}(-1)^\alpha\binom{k-s+\nu-1}{\nu-\alpha}\binom{l-t+\nu-1}{\alpha}D^\alpha f D^{\nu-\alpha} g.$$

Let $\widetilde{M}_k^{\leq s}$ be the space of quasimodular forms of weight k and depth at most s for the full modular group $\mathrm{SL}_2(\mathbb{Z})$. Note that the differential operator D maps

$\widetilde{M}_k^{\le s}$ into $\widetilde{M}_{k+2}^{\le s+1}$. It is known from [Martin and Royer 2009] that if $f \in \widetilde{M}_k^{\le s}$ and $g \in \widetilde{M}_l^{\le t}$, then $[f, g]_\nu \in \widetilde{M}_{k+l+2\nu}^{\le s+t}$. We now state the main result of this paper:

Theorem 1.1. *Let f and g be two quasimodular eigenforms such that the depth of each of the forms f and g is strictly less than the half of the weight of the form. Then there are only finitely many triples (f, g, ν) with the property that f and g are quasimodular eigenforms and $[f, g]_\nu$ is again an eigenform. All the possible cases* (*up to some constant multiple*) *are the following*:

- $[E_4, E_4]_0 = E_8,\qquad [E_4, E_6]_0 = E_{10},$
  $[E_4, E_{10}]_0 = [E_6, E_8]_0 = E_{14},\qquad [E_4, DE_4]_0 = \frac{1}{2}DE_8.$
- *If $k, l \in \{4, 6, 8, 10, 14\}$ and $\nu \ge 1$ with $k+l+2\nu \in \{12, 16, 18, 20, 22, 26\}$, then*
  $$[E_k, E_l]_\nu = c_\nu(k, l)\Delta_{k+l+2\nu},$$
  where
  $$c_\nu(k, l) = -\frac{2l}{B_l}\binom{\nu+l-1}{\nu} + (-1)^{\nu+1}\frac{2k}{B_k}\binom{\nu+k-1}{\nu}.$$
- *If $k \in \{4, 6, 8, 10, 14\}$ and $\nu \ge 0$ with $l, k+l+2\nu \in \{12, 16, 18, 20, 22, 26\}$, then*
  $$[E_k, \Delta_l]_\nu = c_\nu(l)\Delta_{k+l+2\nu},$$
  where
  $$c_\nu(l) = \binom{\nu+l-1}{\nu}.$$
- $[E_4, DE_4]_1 = 960\Delta_{12},\qquad [E_4, DE_8]_1 = [E_8, DE_4]_1 = 1920\Delta_{16},$
  $[E_6, DE_6]_1 = -3024\Delta_{16},\qquad [E_4, DE_6]_2 = -5040\Delta_{16},$
  $[E_6, DE_4]_2 = 5040\Delta_{16},\qquad [E_4, DE_4]_3 = 4800\Delta_{16},$
  $[E_8, DE_8]_1 = 3840\Delta_{20},\qquad [E_6, DE_6]_3 = -28224\Delta_{20},$
  $[E_4, DE_4]_5 = 13440\Delta_{20}.$
- $[E_4, D^2E_4]_1 = 960D\Delta_{12},\qquad [E_4, DE_6]_1 = -2016D\Delta_{12},$
  $[E_6, DE_4]_1 = 1440D\Delta_{12},\qquad [E_4, DE_4]_2 = 2400D\Delta_{12},$
  $[E_6, D^2E_6]_1 = -3024D\Delta_{16},\qquad [E_6, DE_6]_2 = -10584D\Delta_{16},$
  $[E_4, D^2E_4]_3 = 4800D\Delta_{16},\qquad [E_4, DE_4]_4 = 8400D\Delta_{16},$
  $[E_8, D^2E_8]_1 = 3840D\Delta_{20}\qquad [E_8, DE_8]_2 = 17280D\Delta_{20},$
  $[E_6, D^2E_6]_3 = -28224D\Delta_{20},\qquad [E_6, DE_6]_4 = -63504D\Delta_{20},$
  $[E_4, D^2E_4]_5 = 13440D\Delta_{20},\qquad [E_4, DE_4]_6 = 20160D\Delta_{20}.$

We see from the list of identities in the above theorem that there are some new identities. These identities give new relations among the Fourier coefficients of modular forms. From the list we also see that in some cases the Rankin–Cohen brackets of two quasimodular forms give rise to modular forms. It would be interesting to further investigate the cases for which Rankin–Cohen brackets of two quasimodular forms give rise to modular forms.

The idea of the proof of the above theorem is to prove a similar result in the space of nearly holomorphic modular forms in certain cases and then use the isomorphism between the space of nearly holomorphic modular forms and the space of quasimodular forms to prove the result in the space of quasimodular forms. The advantage of using the space of nearly holomorphic modular forms is the existence of the Petersson inner product. To prove Theorem 1.1, we define the Rankin–Cohen brackets on nearly holomorphic modular forms and prove various results involving certain operators on nearly holomorphic modular forms. Rankin–Cohen brackets and properties of various operators on nearly holomorphic modular forms are of independent interest.

The article is organized as follows. In Section 2, we recall some basic results and prove some new results in the theory of nearly holomorphic modular forms. In Section 3, we state some basic results in the theory of quasimodular forms. In Section 4, we define the Rankin–Cohen brackets on nearly holomorphic modular forms and prove some basic results which will be useful for our purpose. In Section 5, we prove some results which are generalizations of a result of Shimura [1976] to the case of Rankin–Cohen brackets of nearly holomorphic modular forms. These results are the main ingredients in the proof of Theorem 1.1. In Section 6, we prove Theorem 1.1.

2. Nearly holomorphic modular forms

Notations and basic results.

Definition 2.1. A nearly holomorphic modular form f of weight k and depth $\le p$ for $\mathrm{SL}_2(\mathbb{Z})$ is a polynomial in $1/\mathrm{Im}(z)$ of degree $\le p$ whose coefficients are holomorphic functions on $\mathcal{H}$ with moderate growth such that

$$(cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right) = f(z),$$

for any $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and $z \in \mathcal{H}$, where $\mathrm{Im}(z)$ is the imaginary part of z.

We denote by $\widehat{M}_k^{\le p}$ the space of all nearly holomorphic modular forms of weight k and depth $\le p$ for $\mathrm{SL}_2(\mathbb{Z})$. We also denote by $\widehat{M}_k = \bigcup_p \widehat{M}_k^{\le p}$ the space of all nearly holomorphic modular forms of weight k.

Definition 2.2. The Maass–Shimura operator R_k on $f \in \widehat{M}_k$ is defined by

$$R_k f(z) = \frac{1}{2\pi i}\Big(\frac{k}{2i\,\mathrm{Im}(z)} + \frac{\partial}{\partial z}\Big) f(z).$$

The operator R_k takes $\widehat{M}_k$ into $\widehat{M}_{k+2}$. Thus it is also called the Maass-raising operator. We write $R_k^m := R_{k+2m-2} \circ \cdots \circ R_{k+2} \circ R_k$ with $R_k^0 = \mathrm{id}$ and $R_k^1 = R_k$, where id is the identity map. We state the following decomposition theorem of the space of nearly holomorphic modular forms [Shimura 2012, Theorem 5.2].

Theorem 2.3. *Let $k \geq 2$ be even. If $f \in \widehat{M}_k^{\leq p}$ and $p < k/2$ then*

$$\widehat{M}_k^{\leq p} = \bigoplus_{r=0}^{p} R_{k-2r}^r M_{k-2r},$$

and if $p \geq k/2$ then

$$\widehat{M}_k^{\leq p} = \bigoplus_{r=0}^{\frac{k}{2}-1} R_{k-2r}^r M_{k-2r} \oplus \mathbb{C} R_2^{\frac{k}{2}-1} E_2^*,$$

where $E_2^*(z) := E_2(z) - \frac{3}{\pi \mathrm{Im}(z)}$ is a nearly holomorphic modular form of weight 2 and depth 1 for the group $\mathrm{SL}_2(\mathbb{Z})$.

Following Shimura [2012, pp. 32], we define the slowly increasing and rapidly decreasing functions in $\widehat{M}_k$. Shimura has defined slowly increasing and rapidly decreasing functions in a broader space than $\widehat{M}_k$. Here we define those for $\widehat{M}_k$.

Definition 2.4. Let $f \in \widehat{M}_k$. Then f is called a

- **slowly increasing function** if for each $\alpha \in \mathrm{SL}_2(\mathbb{Q})$ there exist positive constants A, B and c depending on f and α such that

$$|\mathrm{Im}(\alpha z)^{k/2} f(\alpha z)| < A y^c \quad \text{if } y = \mathrm{Im}(z) > B;$$

- **rapidly decreasing function** if for each $\alpha \in \mathrm{SL}_2(\mathbb{Q})$ and a positive real number c, there exist positive constants A and B depending on f, α and c such that

$$|\mathrm{Im}(\alpha z)^{k/2} f(\alpha z)| < A y^{-c} \quad \text{if } y = \mathrm{Im}(z) > B.$$

Remark 2.5. If $f \in M_k$, then f is a slowly increasing function. In addition, if $f \in S_k$ then f is a rapidly decreasing function. From the above definitions we observe that the product of a rapidly decreasing function with any nearly holomorphic modular form gives a rapidly decreasing function.

We state the following result [Shimura 2012, Lemma 6.10].

Lemma 2.6. *If $f \in M_k$, then $R_k^m f$ is a slowly increasing function for any integer $m \geq 0$. Moreover, it is a rapidly decreasing function if $f \in S_k$.*

If $f, g \in \widehat{M}_k$ are such that the product fg is a rapidly decreasing function, then the Petersson inner product of f and g is defined by

$$\langle f, g\rangle := \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}} f(z)\overline{g(z)}y^k \frac{dxdy}{y^2}, \tag{3}$$

where $z = x + iy$. The above integral is convergent since fg is a rapidly decreasing function.

The Maass-lowering operator is the differential operator defined by

$$L := -y^2 \frac{\partial}{\partial\bar{z}}\cdot$$

The operator L maps $\widehat{M}_{k+2}$ to $\widehat{M}_k$. From the definition of L, it is clear that L annihilates any holomorphic function. We state the following result [Shimura 2012, Theorem 6.8] which shows that under certain conditions, the operators L and R_k are adjoint to each other with respect to the Petersson inner product.

Lemma 2.7. *Let $f \in \widehat{M}_k$ and $g \in \widehat{M}_{k-2}$ be such that fg, $f(R_{k-2}g)$ and $(Lf)g$ are rapidly decreasing functions. Then we have*

$$\langle f, R_{k-2}g\rangle = \langle Lf, g\rangle.$$

In a particular case of the above result, we obtain the following result which plays a crucial role in the proof of our main result.

Lemma 2.8. *Let $f \in S_k$. Then $\langle f, R_{k-2}g\rangle = 0$ for every $g \in \widehat{M}_{k-2}$ such that both g and $R_{k-2}g$ are slowly increasing functions.*

Eisenstein series. Let

$$\Gamma_\infty = \left\{\pm\begin{pmatrix}1 & m\\ 0 & 1\end{pmatrix} \,\middle|\, m \in \mathbb{Z}\right\}.$$

For any integer $k \geq 0$, $z \in \mathcal{H}$ and $s \in \mathbb{C}$, the Eisenstein series $E_k(z, s)$ is defined by

$$E_k(z, s) = \sum_{\gamma\in\Gamma_\infty\backslash \mathrm{SL}_2(\mathbb{Z})} j(\gamma, z)^{-k}|j(\gamma, z)|^{-2s},$$

where $j(\gamma, z) = (cz + d)$ for $\gamma = \left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)$. The series of $E_k(z, s)$ is absolutely convergent for $\mathrm{Re}(2s) > 2 - k$. It is well known that

$$R_k^r E_k(z) = (-4\pi y)^{-r} \frac{\Gamma(k + r)}{\Gamma(k)} E_{k+2r}(z, -r), \tag{4}$$

where $y = \mathrm{Im}(z)$.

Following [Diamantis and O'Sullivan 2010], we also recall the completed normalized nonholomorphic Eisenstein series, defined by

$$E_k^*(z,s)=\theta_k(s)\sum_{\gamma\in\Gamma_\infty\backslash\mathrm{SL}_2(\mathbb{Z})}(\mathrm{Im}(\gamma z))^s\left(\frac{j(\gamma,z)}{|j(\gamma,z)|}\right)^{-k}, \tag{5}$$

where

$$\theta_k(s)=\pi^{-s}\Gamma(s+k/2)\zeta(2s)\quad\text{and}\quad \gamma z:=\frac{az+b}{cz+d}\quad\text{for } \gamma=\begin{pmatrix}a&b\\c&d\end{pmatrix}.$$

We observe that

$$E_k(z,s)=\frac{y^{-s-k/2}}{\theta_k(s+k/2)}E_k^*(z,s+k/2). \tag{6}$$

Hecke operators. For $f\in\widehat{M}_k$ and any integer $n\geq 1$, the action of the n-th Hecke operator on f is defined by

$$(T_nf)(z)=n^{k-1}\sum_{d|n}d^{-k}\sum_{b=0}^{d-1}f\left(\frac{nz+bd}{d^2}\right). \tag{7}$$

For each integer $n\geq 1$, T_n maps $\widehat{M}_k$ to $\widehat{M}_k$. A nearly holomorphic modular form is called an eigenform if it is an eigenvector for each Hecke operator T_n ($n\geq 1$). We recall the following commuting relation between Maass–Shimura operators and Hecke operators [Beyerl et al. 2012, Propositions 2.4 and 2.5].

Proposition 2.9. *Let $f\in\widehat{M}_k$. Then*

$$(R_k^m(T_nf))(z)=\frac{1}{n^m}(T_n(R_k^mf))(z)$$

for any integer $m\geq 0$. Moreover, R_k^mf is an eigenvector for T_n if and only if f is. In this case, if λ_n is the eigenvalue of T_n corresponding to f then the eigenvalue of T_n corresponding to R_k^mf is $n^m\lambda_n$.

The following result characterizes all nearly holomorphic eigenforms for the full modular group $\mathrm{SL}_2(\mathbb{Z})$. The result has been proved in [Kumar and Meher 2016, Theorem 1.1].

Proposition 2.10. *Let f be a nearly holomorphic eigenform of weight k and depth p for the full modular group $\mathrm{SL}_2(\mathbb{Z})$. If $p<k/2$ then $f=R_{k-2p}^pf_p$, where $f_p\in M_{k-2p}$ is an eigenform, and if $p=k/2$ then $f\in\mathbb{C}R_2^{k/2-1}E_2^*$.*

Properties of raising and lowering operators. We first recall the following relation [Shimura 2012, 6.13c, pp. 34].

$$LR_k=R_{k-2}L+\frac{k}{4}\cdot \tag{8}$$

Using the above identity we prove the following lemma.

Lemma 2.11. *Let m and r be two positive integers. Then*:

- $L^r R_k = R_{k-2r} L^r + \frac{1}{4} r(k-r+1) L^{r-1}$.
- $L R_k^m = R_{k-2}^m L + \frac{1}{4} m(k+m-1) R_k^{m-1}$.
- *For $m \le r$ we have*

  $$L^r R_k^m = R_{k-2r}^m L^r + c_1 R_{k-2r+2}^{m-1} L^{r-1} + c_2 R_{k-2r+4}^{m-2} L^{r-2} + \cdots + c_m L^{r-m},$$

  where

  $$c_i = \frac{1}{4^i} r(r-1)\cdots(r-i+1)(k+i-r)(k+i-r+1)\cdots(k+2i-r-1).$$

- *For any positive integer r and any nonnegative integer m, we have*

  $$L^r R_k^{r+m} = R_{k+2m-2r}^r L^r R_k^m + c_1 R_{k+2m-2r+2}^{r-1} L^{r-1} R_k^m + \cdots + c_{r-1} R_{k+2m-2} L R_k^m + c_r R_k^m,$$

  where c_i is as defined in the previous identity.

Proof. For $r = 1$ the first identity is true by (8). Then the first identity can be proved by using induction on r. Similarly for $m = 1$, the second identity is true by (8), and then the second identity can be proved by using induction on m. The third identity can be proved by using the first identity and induction on m. The fourth identity is a direct application of the third identity. □

Using the above lemma, we now prove the following result which is of independent interest and is also useful for our purposes.

Theorem 2.12. *Let $f \in S_k$ and $g \in \widehat{M}_l$. Assume that r and s are positive integers such that $k + 2r = l + 2s$. Then*

$$\langle R_k^r f, R_l^s g \rangle = \begin{cases} c_r \langle f, g \rangle & \text{if } r = s, \\ 0 & \text{if } r \ne s, \end{cases}$$

where

$$c_r = \frac{r!}{4^r} k(k+1)\cdots(k+r-1).$$

Proof. If $r = s$ then $k = l$ and by using Lemma 2.7, we obtain

$$\langle R_k^r f, R_k^r g \rangle = \langle f, L_k^r R_k^r g \rangle.$$

Using the fourth identity of Lemma 2.11 in the above expression we obtain

$$\langle R_k^r f, R_k^r g \rangle = \langle f, R_{k+2m-2r}^r L^r g + c_1 R_{k+2m-2r+2}^{r-1} L^{r-1} g + \cdots + c_{r-1} R_{k+2m-2} L g + c_r g \rangle.$$

Now applying Lemma 2.8 to the right-hand side of the above expression, we obtain the required result in this case. If $r \neq s$, without loss of any generality we may assume that $r < s$. Let $r = s + m$ for some positive integer m. Then again by the fourth identity of Lemma 2.11 we get

$$\langle R_k^r f, R_k^s g\rangle = \big\langle f, R_{l+2m-2r}^r L^r R_l^m g + c_1 R_{l+2m-2r+2}^{r-1} L^{r-1} R_l^m g + \cdots + c_{r-1} R_{l+2m-2} L R_l^m g + c_r R_l^m g\big\rangle.$$

Applying Lemma 2.8 to the right-hand side of the above expression, we deduce that

$$\langle R_k^r f, R_k^s g\rangle = 0. \qquad \square$$

Let $\widehat{S}_k$ be the subspace of $\widehat{M}_k$ consisting of rapidly decreasing functions. As an application of the above theorem, we have the following result.

Corollary 2.13. *There exists an orthogonal basis of $\widehat{S}_k$ consisting of Hecke eigenforms with respect to the Petersson inner product.*

Proof. Using the property of rapidly decreasing functions and the decomposition theorem for the space of nearly holomorphic modular forms, given in Theorem 2.3, it follows that

$$\widehat{S}_k = \bigoplus_{r=0}^{k/2-1} R_{k-2r}^r S_{k-2r}.$$

Since S_k has an orthogonal basis consisting of Hecke eigenforms with respect to the Petersson inner product, the result follows from Proposition 2.10 and Theorem 2.12. $\square$

3. Quasimodular forms

Definition 3.1. A holomorphic function f on $\mathcal{H}$ is called a quasimodular form of weight k and depth p for $\mathrm{SL}_2(\mathbb{Z})$ if there exist holomorphic functions f_0, f_1, $f_2, \ldots, f_p$ on $\mathcal{H}$ with moderate growth such that

$$(cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right) = \sum_{j=0}^{p} f_j(z)\left(\frac{c}{cz+d}\right)^j$$

for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, and f_p is not identically vanishing.

We denote by $\widetilde{M}_k^{\leq p}$ the space of all quasimodular forms of weight k and depth $\leq p$ for the full modular group $\mathrm{SL}_2(\mathbb{Z})$. We also denote by $\widetilde{M}_k = \bigcup_p \widetilde{M}_k^{\leq p}$ the space of all quasimodular forms of weight k. Any quasimodular form f of weight k and depth p for $\mathrm{SL}_2(\mathbb{Z})$ can be written as

$$f(z) = g_0(z) + g_1(z) E_2(z) + \cdots + g_p(z) E_2^p(z), \tag{9}$$

where $g_i \in M_{k-2i}$ for $0 \le i \le p$ and $g_p \not\equiv 0$. For any integer $n \ge 1$, the action of the Hecke operator T_n on a quasimodular form is the same as the action on a nearly holomorphic modular form as given in (7). For each integer $n \ge 1$, T_n maps $\widetilde{M}_k$ to itself. A quasimodular form is called an eigenform if it is an eigenvector for each Hecke operator T_n $(n \ge 1)$. We state the following result [Das and Meher 2015, Proposition 3.1] which characterises all quasimodular eigenforms for the full modular group $\mathrm{SL}_2(\mathbb{Z})$.

Proposition 3.2. *Let f be a quasimodular eigenform of weight k and depth p for $\mathrm{SL}_2(\mathbb{Z})$. If $p < k/2$ then $f = D^p f_p$, where $f_p \in M_{k-2p}$ is an eigenform, and if $p = k/2$ then $f \in \mathbb{C}D^{k/2-1}E_2$.*

We also recall the following results on quasimodular eigenforms [Kumar and Meher 2016, Lemma 4.3, 4.4].

Lemma 3.3. *If $f = \sum_{n=0}^{\infty} a(n)e^{2\pi i n z} \in \widetilde{M}_k$ is a nonzero eigenform then $a(1) \neq 0$.*

Lemma 3.4. *A quasimodular eigenform $f \in \widetilde{M}_k$ with nonzero constant Fourier coefficient is an eigenform if and only if $f \in \mathbb{C}E_k$.*

Let $\widehat{M}_*^{\le p}$ be the space of all nearly holomorphic modular forms of depth at most p for the group $\mathrm{SL}_2(\mathbb{Z})$, and let $\widetilde{M}_*^{\le p}$ be the space of all quasimodular forms of depth at most p for the group $\mathrm{SL}_2(\mathbb{Z})$. Then there is an isomorphism between $\widehat{M}_*^{\le p}$ and $\widetilde{M}_*^{\le p}$ given in the next theorem [Ouled Azaiez 2008, Theorem 1].

Theorem 3.5. *The map*

$$f(z) = \sum_{j=0}^{p} \frac{f_j(z)}{\mathrm{Im}(z)^j} \mapsto f_0(z)$$

from $\widehat{M}_*^{\le p}$ to $\widetilde{M}_*^{\le p}$ is an isomorphism.

The map above induces a ring isomorphism between $\widehat{M}_*$ and $\widetilde{M}_*$. Also if $f \in M_k$, then the above isomorphism from $\widehat{M}_*$ and $\widetilde{M}_*$ maps $R_k^m f$ to $D^m f$ and $R_2^m E_2^*$ to $D^m E_2$ for any integer $m \ge 0$. Thus from Propositions 2.10 and 3.2 we have the following result.

Proposition 3.6. *A polynomial relation among eigenforms in $\widehat{M}_*$ gives rise to a corresponding polynomial relation in $\widetilde{M}_*$ and vice versa.*

4. Rankin–Cohen brackets and Rankin–Selberg L-functions

Rankin–Cohen brackets. Let F and G be two nearly holomorphic modular forms of weights k and l, and depths s and t, respectively, for the group $\mathrm{SL}_2(\mathbb{Z})$. Analogous to the Rankin–Cohen brackets defined for quasimodular forms in (2), we define the

Rankin–Cohen brackets for nearly holomorphic modular forms. For any integer $\nu \geq 0$, the ν-th Rankin–Cohen bracket of F and G is defined by

$$[F, G]_\nu := \sum_{\alpha=0}^{\nu} (-1)^\alpha \binom{k-s+\nu-1}{\nu-\alpha} \binom{l-t+\nu-1}{\alpha} (R_k^\alpha F)(R_l^{\nu-\alpha} G). \tag{10}$$

By abuse of notation, the ν-th Rankin–Cohen bracket of two nearly holomorphic modular forms is denoted by the same notation as the ν-th Rankin–Cohen bracket of two quasimodular forms.

Theorem 4.1. *Let F and G be as above. Then for any integer $\nu \geq 0$ we have $[F, G]_\nu \in \widehat{M}_{k+l+2\nu}^{\leq s+t}$.*

Proof. From the definition of Rankin–Cohen brackets in (10), it is easy to see that $[F, G]_\nu \in \widehat{M}_{k+l+2\nu}^{\leq s+t+\nu}$. Thus it remains to show that the depth of $[F, G]_\nu$ is in fact at most $s+t$. Let f and g be the respective constant coefficients of F and G when we write both F and G as polynomials in $1/\mathrm{Im}(z)$. Then we know that f and g are quasimodular forms of weights k and l and depths s and t, respectively. From (10) and (2) we see that if we write $[F, G]_\nu$ as a polynomial in $1/\mathrm{Im}(z)$, then the constant coefficient of $[F, G]_\nu$ is $[f, g]_\nu$. But we know that the depth of the quasimodular form $[f, g]_\nu$ is at most $s+t$. Hence by Theorem 3.5, the depth of $[F, G]_\nu$ is at most $s+t$. □

Rankin–Selberg L-functions. Let $f = \sum_{m=0}^{\infty} a(m) e^{2\pi i m z} \in M_k$. The L-function attached to f is defined by

$$L(f, s) = \sum_{m=1}^{\infty} \frac{a(m)}{m^s}.$$

If $f \in S_k$, then $L(f, s)$ is analytically continued to the whole complex plane and it satisfies the functional equation

$$L^*(f, s) := (2\pi)^{-s} \Gamma(s) L(f, s) = (-1)^{k/2} L^*(f, k-s).$$

If $f(z) = \sum_{m=0}^{\infty} a(m) e^{2\pi i m z}$ and $g(z) = \sum_{m=0}^{\infty} b(m) e^{2\pi i m z}$ are modular forms of weights k and l for $\mathrm{SL}_2(\mathbb{Z})$, respectively, then the Rankin–Selberg L-function associated with f and g is defined by

$$L(f \times g, s) := \sum_{m=1}^{\infty} \frac{a(m)\overline{b(m)}}{m^s}.$$

We recall a result of Zagier [1977, Proposition 6].

Theorem 4.2. *Let k_1, k_2, k and n be integers satisfying $k_2 \geq k_1 + 2 > 2$ and $k = k_1 + k_2 + 2n$. Let $f(z) = \sum_{m=1}^{\infty} a(m) e^{2\pi i m z} \in S_k$ and $g(z) = \sum_{m=0}^{\infty} b(m) e^{2\pi i m z} \in M_{k_1}$.*

Then

$$\langle f, [g, E_{k_2}]_n\rangle = (-1)^n \frac{\Gamma(k-1)\Gamma(k_2+n)}{(4\pi)^{k-1} n! \Gamma(k_2)} L(f \times g, k_1+k_2+n-1).$$

Remark 4.3. In the hypothesis of Theorem 4.2, the condition $k_2 \geq k_1 + 2 > 2$ can be removed if g is a cusp form.

If $g = E_{k_1}$, then we have the following result; see [Lanphier and Takloo-Bighash 2004, Theorem 2.2].

Theorem 4.4. *Let $k_1, k_2 \geq 4$ be even integers and let n be a nonnegative integer and $k = k_1 + k_2 + 2n$. Suppose that $f \in S_k$ is a normalized eigenform. Then*

$$\langle f, [E_{k_1}, E_{k_2}]_n\rangle = (-1)^{k_2/2+n} \frac{2k_1}{B_{k_1}} \frac{2k_2}{B_{k_2}} \frac{\Gamma(k-1)}{n! 2^{k-1} \Gamma(k-n-1)} L^*(f, k-n-1) L^*(f, k_2+n).$$

Remark 4.5. Note that we have an extra $(-1)^n$ appearing in the right-hand sides of both the expressions given in Theorems 4.2 and 4.4. This is because an extra $(-1)^n$ appears in the definition of Rankin–Cohen brackets given in [Zagier 1977].

We now recall an interesting nonvanishing result of the L-function $L(f, s)$ associated with the cusp form f [Lanphier and Takloo-Bighash 2004, Corollary 3.2] at the center critical point.

Lemma 4.6. *Suppose that $k > 20$ and $k \equiv 0 \pmod 4$. Then there are two eigenforms $f, g \in S_k$ with $L^*(f, k/2) \neq 0$ and $L^*(g, k/2) \neq 0$.*

Remark 4.7. We know that $[E_4, E_4]_2 = 4800\Delta_{12}$ for some nonzero constant $c \in \mathbb{R}$. Also we have

$$\langle \Delta_{12}, \Delta_{12}\rangle = 4800\langle \Delta_{12}, [E_4, E_4]_2\rangle \neq 0.$$

Thus by Theorem 4.4, $L^*(\Delta_{12}, 6) \neq 0$. Similarly one proves that $L^*(\Delta_{16}, 8) \neq 0$ and $L^*(\Delta_{20}, 10) \neq 0$. Therefore by Lemma 4.6 we deduce that for each integer $k \geq 12$ with $k \equiv 0 \pmod 4$, there exists a nonzero eigenform $f \in S_k$ such that $L^*(f, k/2) \neq 0$.

5. Preparatory results

We start the section with the following result of Shimura [1976, Theorem 2], who has proved the result for modular forms of higher level with characters. Here we state the result for the group $SL_2(\mathbb{Z})$ for our purpose.

Theorem 5.1. *Suppose $f \in S_k$, $g \in M_{k_1}$, and $k_1 + 2r_2 < k$ with a nonnegative integer r_2. Then*

$$\langle f, g \cdot R_{k_2}^{r_2} E_{k_2}\rangle = cL(f \times g, k-1-r_2),$$

where $k_2 = k - k_1 - 2r_2$, and $c = \Gamma(k-1-r_2)\Gamma(k-k_1-r_2)/\Gamma(k-k_1-2r_2)$.

The following result generalizes Theorem 5.1 and may be of independent interest. We follow the idea of Shimura to prove the result. We obtain Theorem 4.2 as a special case of the following result.

Theorem 5.2. *Let $k_1, k_2, k, r_1, r_2, \nu$ be nonnegative integers such that $k_2 \geq 4$, $k+2r = k_1+k_2+2r_1+2r_2+2\nu$. Suppose that $f = \sum_{n=1}^{\infty} a(n)e^{2\pi inz} \in S_k$ and $g = \sum_{n=0}^{\infty} b(n)e^{2\pi inz} \in M_{k_1}$. Assume that either g is a cusp form or $k_2 \geq k_1+2$. Then we have*

$$\langle R_k^r f, [R_{k_1}^{r_1} g, R_{k_2}^{r_2} E_{k_2}]_\nu \rangle = c(k, r; k_1, r_1, k_2, r_2) \cdot L\Big(f \times g, \frac{k}{2}+\frac{k_1}{2}+\frac{k_2}{2}-1\Big), \tag{11}$$

where

$$c(k,r;k_1,r_1,k_2,r_2) = \frac{(-1)^{r_2+\nu}}{(4\pi)^{k+2r-1}} \sum_{\alpha=0}^{\nu} A_\alpha \sum_{u=0}^{r} \sum_{v=0}^{r_1+\alpha} (-1)^{-u-v} P_{u,k}^{(r)} P_{v,k_1}^{(r_1+\alpha)} \Gamma(k+2r-r_2-\nu+\alpha-u-v-1),$$

with

$$A_\alpha = \binom{k_1+r_1+\nu-1}{\nu-\alpha}\binom{k_2+r_2+\nu-1}{\alpha}\frac{\Gamma(k_2+r_2+\nu-\alpha)}{\Gamma(k_2)}$$

and

$$P_{u,k}^{(r)} = \binom{r}{u}\frac{\Gamma(k+r)}{\Gamma(k+r-u)}.$$

Moreover, for $r=0$ we have

$$c(k,0;k_1,r_1,k_2,r_2) = \frac{(-1)^{r_2+\nu}}{(4\pi)^{k-1}} \frac{\Gamma(k_2+r_1+r_2+\nu)\Gamma(k_1+k_2+r_1+r_2+2\nu-1)}{\Gamma(k_2)\Gamma(\nu+1)} \neq 0. \tag{12}$$

Proof. Using the definitions of Rankin–Cohen brackets and the Petersson inner product we have

$$\langle R_k^r f, [R_{k_1}^{r_1} g, R_{k_2}^{r_2} E_{k_2}]_\nu \rangle = \sum_{\alpha=0}^{\nu} (-1)^\alpha \binom{k_1+r_1+\nu-1}{\nu-\alpha}\binom{k_2+r_2+\nu-1}{\alpha} \times \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}} R_k^r f\, \overline{R_{k_1}^{r_1+\alpha} g}\, \overline{R_{k_2}^{r_2+\nu-\alpha} E_{k_2}}\, y^{k+2r} \frac{dx\, dy}{y^2}.$$

Using the identity (4) for $R_{k_2}^{r_2+\nu-\alpha} E_{k_2}$ in the above expression we obtain

$$\langle R_k^r f, [R_{k_1}^{r_1} g, R_{k_2}^{r_2} E_{k_2}]_\nu \rangle = \sum_{\alpha=0}^{\nu} (-1)^{r_2+\nu} \frac{A_\alpha}{(4\pi)^{r_2+\nu-\alpha}} \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}} \sum_{\gamma\in\Gamma_\infty\backslash \mathrm{SL}_2(\mathbb{Z})} R_k^r f\, \overline{R_{k_1}^{r_1+\alpha} g} \times \overline{j(\gamma,z)^{-k_2-2r_2-2\nu+2\alpha}|j(\gamma,z)^{-2(-r_2-\nu+\alpha)}|}\, y^{k+2r-r_2-\nu+\alpha} \frac{dx\, dy}{y^2}. \tag{13}$$

To interchange the sum and integral in (13), we observe that

$$\int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}} \sum_{\gamma\in\Gamma_\infty\backslash\,\mathrm{SL}_2(\mathbb{Z})} \left| R_k^r f\ \overline{R_{k_1}^{r_1+\alpha} g}\ \overline{j(\gamma,z)^{-k_2-2r_2-2\nu+2\alpha}|j(\gamma,z)|^{2(r_2+\nu-\alpha)}}\right| \times y^{k+2r-r_2-\nu+\alpha}\,\frac{dx\,dy}{y^2}$$

$$\le \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}} \left|y^{\frac{1}{2}(k+2r+k_1+2r_1+2\alpha)} R_k^r f R_{k_1}^{r_1+\alpha} g\right| \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} |j(\gamma,z)|^{-k_2}\ y^{k_2/2}\,\frac{dx\,dy}{y^2}.$$

For $k_2 \ge 4$ note that

$$\sum_{\gamma\in\Gamma_\infty\backslash\,\mathrm{SL}_2(\mathbb{Z})} |j(\gamma,z)|^{-k_2} \le \zeta(k_2-1).$$

Also, since f is a cusp form, $R_k^r f$ is a rapidly decreasing function and so is $(R_k^r f)(R_{k_1}^{r_1+\alpha} g)$. Hence, the integral in the right-hand side of above inequality is a finite quantity. Interchanging the sum and integral in (13), we obtain

$$\langle R_k^r f, [R_{k_1}^{r_1} g, R_{k_2}^{r_2} E_{k_2}]_\nu\rangle = (-1)^{r_2+\nu}\sum_{\alpha=0}^{\nu}\frac{A_\alpha}{(4\pi)^{r_2+\nu-\alpha}} \sum_{\gamma\in\Gamma_\infty\backslash\mathrm{SL}_2(\mathbb{Z})}\int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}} R_k^r f(z)\ \overline{R_{k_1}^{r_1+\alpha} g(z)} \times \overline{j(\gamma,z)^{-k_2-2r_2-2\nu+2\alpha}|j(\gamma,z)|^{-2(-r_2-\nu+\alpha)}}\, y^{k+2r-r_2-\nu+\alpha}\,\frac{dx\,dy}{y^2}.$$

Changing the variable $z\mapsto\gamma^{-1}z$ in the above expression and unfolding we obtain

$$\langle R_k^r f, [R_{k_1}^{r_1} g, R_{k_2}^{r_2} E_{k_2}]_\nu\rangle = (-1)^{r_2+\nu}\sum_{\alpha=0}^{\nu}\frac{A_\alpha}{(4\pi)^{r_2+\nu-\alpha}}\int_0^\infty\int_0^1 R_k^r f(z)\overline{R_{k_1}^{r_1+\alpha} g(z)}\ y^{k+2r-r_2-\nu+\alpha-2}dx\,dy.$$

From the Fourier expansions of f and g we get

$$\langle R_k^r f, [R_{k_1}^{r_1} g, R_{k_2}^{r_2} E_{k_2}]_\nu\rangle = (-1)^{r_2+\nu}\sum_{\alpha=0}^{\nu}\frac{A_\alpha}{(4\pi)^{r_2+\nu-\alpha}}\sum_{u=0}^{r} P_{u,k}^{(r)}\sum_{v=0}^{r_1+\alpha} P_{v,k_1}^{(r_1+\alpha)}(-4\pi)^{-u-v} \times\int_0^\infty\int_0^1\sum_{m\ge1,\,n\ge0} a(m)\overline{b(n)}m^{r-u}n^{r_1+\alpha-v}e^{2\pi i x(m-n)}e^{-2\pi y(m+n)} \times y^{k+2r-r_2-\nu+\alpha-2-u-v}\,dx\,dy. \tag{14}$$

Since either g is a cusp form or $k_2 \ge k_1+2$, by using the bounds of Fourier coefficients we can interchange the sum and integration of the above expression, and using the fact that the integral over the variable x will be nonzero only when $m = n$, we obtain

$$\langle R_k^r f, [R_{k_1}^{r_1} g, R_{k_2}^{r_2} E_{k_2}]_\nu \rangle = (-1)^{r_2+\nu} \sum_{\alpha=0}^{\nu} \frac{A_\alpha}{(4\pi)^{r_2+\nu-\alpha}} \sum_{u=0}^{r} P_{u,k}^{(r)} \sum_{v=0}^{r_1+\alpha} P_{v,k_1}^{(r_1+\alpha)} (-4\pi)^{-u-v} \sum_{m\geq 1} a(m)\overline{b(m)} m^{r+r_1+\alpha-u-v} \int_0^\infty e^{-4\pi y m}\, y^{k+2r-r_2-\nu+\alpha-2-u-v}\, dy.$$

Using the definition of the Gamma function above gives (11). It remains to simplify the constant for $r = 0$. The proof is straightforward and purely combinatorial. We use the following two binomial identities, which hold for nonnegative integers x, j, n with $x, j \geq n$. The first identity (see [Quaintance and Gould 2016, pp. 74]) is

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} \binom{x-i}{j} = \binom{x-n}{j-n} \tag{15}$$

and second is the well-known Vandermonde's identity, given by

$$\sum_{k=0}^{n} \binom{x}{k} \binom{j}{n-k} = \binom{x+j}{n}. \tag{16}$$

In the expression of $c(k, 0; k_1, r_1, k_2, r_2)$, we first apply (15) to the sum over the variable v, simplify the obtained expression and then use (16) to obtain its required form as given in (12). □

Next we prove a result similar to Theorem 5.2 in the case when g is the Eisenstein series E_{k_1} and $k_1, k_2 \geq 4$ are any even integers. We first recall a result of Diamantis and O'Sullivan [2010, Proposition 2.1].

Lemma 5.3. *Let k_1, k_2 be even and nonnegative with $k = k_1 + k_2$. Then for any normalized eigenform $f \in S_k$ and for all $s, w \in \mathbb{C}$, we have*

$$\langle f, y^{-k/2} E_{k_1}^*(z, \bar{u}) E_{k_2}^*(z, \bar{v}) \rangle = (-1)^{k_2/2} 2\pi^{k/2} L^*(f, s) L^*(f, w),$$

where $2u = s + w - k + 1$ and $2v = -s + w + 1$.

We show the following result.

Theorem 5.4. *Let k_1, k_2 be even nonnegative integers and $r_1, r_2 \geq 0$ be integers. For any integer $\nu \geq 0$, let $k = k_1 + k_2 + 2r_1 + 2r_2 + 2\nu$. Then for any normalized eigenform $f \in S_k$, we have*

$$\langle f, [R_{k_1}^{r_1} E_{k_1}, R_{k_2}^{r_2} E_{k_2}]_\nu \rangle = c(k; k_1, r_1, k_2, r_2) \cdot L^*\Big(f, \frac{k}{2} + \frac{k_1}{2} - \frac{k_2}{2}\Big) L^*\Big(f, \frac{k}{2} + \frac{k_1}{2} + \frac{k_2}{2} - 1\Big),$$

where

$$c(k;k_1,r_1,k_2,r_2)=\frac{(-1)^{k_1/2-r_1}}{2^{k-1}}\frac{2k_1}{B_{k_1}}\frac{2k_2}{B_{k_2}}\binom{k_1+k_2+r_1+r_2+2\nu-2}{\nu}.$$

Proof. Using the definition of Rankin–Cohen brackets

$$\langle f,[R_{k_1}^{r_1}E_{k_1},R_{k_2}^{r_2}E_{k_2}]_\nu\rangle =\sum_{\alpha=0}^{\nu}(-1)^\alpha\binom{k_1+r_1+\nu-1}{\nu-\alpha}\binom{k_2+r_2+\nu-1}{\alpha}\langle f,(R_{k_1}^{r_1+\alpha}E_{k_1})(R_{k_2}^{r_2+\nu-\alpha}E_{k_2})\rangle. \tag{17}$$

For any $0\le\alpha\le\nu$, by (4) we write

$$\begin{aligned}\langle f,(R_{k_1}^{r_1+\alpha}E_{k_1})(R_{k_2}^{r_2+\nu-\alpha}E_{k_2})\rangle &=(-4\pi)^{-r_1-r_2-\nu}\frac{\Gamma(k_1+r_1+\alpha)\Gamma(k_2+r_2+\nu-\alpha)}{\Gamma(k_1)\Gamma(k_2)}\\ &\quad\times\langle f,y^{-r_1-r_2-\nu}E_{k_1+2r_1+2\alpha}(z,-r_1-\alpha)E_{k_2+2r_2+2\nu-2\alpha}(z,-r_2-\nu+\alpha)\rangle.\end{aligned}$$

Using the relation given in (6) between the Eisenstein series $E_k(z,s)$ and the completed Eisenstein series $E^*(z,s)$, the above identity can be rewritten as

$$\begin{aligned}\langle f,(R_{k_1}^{r_1+\alpha}E_{k_1})(R_{k_2}^{r_2+\nu-\alpha}E_{k_2})\rangle &=\frac{(-4\pi)^{-r_1-r_2-\nu}\Gamma(k_1+r_1+\alpha)\Gamma(k_2+r_2+\nu-\alpha)}{\Gamma(k_1)\Gamma(k_2)\theta_{k_1+2r_1+2\alpha}(k_1/2)\theta_{k_2+2r_2+2\nu-2\alpha}(k_2/2)}\\ &\quad\times\langle f,y^{-k/2}E^*_{k_1+2r_1+2\alpha}(z,k_1/2)E^*_{k_2+2r_2+2\nu-2\alpha}(z,k_2/2)\rangle.\end{aligned}$$

Now using Lemma 5.3 in the above identity and substituting it into (17), we obtain

$$\langle f,[R_{k_1}^{r_1}E_{k_1},R_{k_2}^{r_2}E_{k_2}]_\nu\rangle =c(k;k_1,r_1,k_2,r_2)\cdot L^*\Big(f,\frac{k}{2}+\frac{k_1}{2}-\frac{k_2}{2}\Big)L^*\Big(f,\frac{k}{2}+\frac{k_1}{2}+\frac{k_2}{2}-1\Big),$$

where

$$c(k;k_1,r_1,k_2,r_2) =\sum_{\alpha=0}^{\nu}(-1)^\alpha\binom{k_1+r_1+\nu-1}{\nu-\alpha}\binom{k_2+r_2+\nu-1}{\alpha}\frac{(-1)^{k_2/2+r_1+\alpha}2\pi^{k_1+k_2}}{4^{r+r_2+\nu}\zeta(k_1)\zeta(k_2)\Gamma(k_1)\Gamma(k_2)}.$$

Using (16) we further simplify the above expression and deduce that

$$c(k;k_1,r_1,k_2,r_2)=\frac{2(-1)^{k_2/2-r_1}\pi^{k_1+k_2}}{4^{r_1+r_2+\nu}\Gamma(k_1)\Gamma(k_2)\zeta(k_1)\zeta(k_2)}\binom{k_1+k_2+r_1+r_2+2\nu-2}{\nu}.$$

For any positive even integer m, we get the required result by using the well-known relation $\zeta(m)=-\frac{1}{2}(2\pi i)^m B_m/m!$, in the above expression. □

We also need the following result which we will use in the proof of Theorem 1.1.

Theorem 5.5. *Let k, k_1, k_2, r_1, r_2 and ν be as in the Theorem 5.4. Then for any normalized eigenform $f \in S_{k-2}$ we have*

$$\langle R_{k-2} f, [R_{k_1}^{r_1} E_{k_1}, R_{k_2}^{r_2} E_{k_2}]_\nu \rangle \\ = c^1(k; k_1, r_1, k_2, r_2) \cdot L^*\Big(f, \frac{k}{2} + \frac{k_1}{2} - \frac{k_2}{2} - 1\Big) L^*\Big(f, \frac{k}{2} + \frac{k_1}{2} + \frac{k_2}{2} - 1\Big),$$

where

$$c^1(k; k_1, r_1, k_2, r_2) = \frac{(-1)^{k_1/2}}{2^{k-1}} \frac{2k_1}{B_{k_1}} \frac{2k_2}{B_{k_2}} \sum_{\alpha=0}^{\nu} A_\alpha t_\alpha,$$

with $t_\alpha = (-1)^{r_1-1}(r_1+\alpha)(k_1+r_1+\alpha-1) + (-1)^{r_2+\nu-1}(r_2+\nu-\alpha)(k_2+r_2+\nu-\alpha-1)$ *and*

$$A_\alpha = \binom{k_1+r_1+\nu-1}{\nu-\alpha}\binom{k_2+r_2+\nu-1}{\alpha}.$$

Furthermore, we have

$$c^1(k; k_1, 0, k_2, r_2) \neq 0 \quad \textit{for } r_2 \neq 0. \tag{18}$$

Proof. Using the definition of Rankin–Cohen bracket and Lemma 2.7 we have

$$\langle R_{k-2} f, [R_{k_1}^{r_1} E_{k_1}, R_{k_2}^{r_2} E_{k_2}]_\nu \rangle \\ = \sum_{\alpha=0}^{\nu} (-1)^\alpha A_\alpha \big\{ \langle f, (L R_{k_1}^{r_1+\alpha} E_{k_1})(R_{k_2}^{r_2+\nu-\alpha} E_{k_2}) \rangle + \langle f, (R_{k_1}^{r_1+\alpha} E_{k_1})(L R_{k_2}^{r_2+\nu-\alpha} E_{k_2}) \rangle \big\}.$$

Now using the second identity of Lemma 2.11 and then Lemma 2.8 in the last two inner products, we obtain

$$\langle R_{k-2} f, [R_{k_1}^{r_1} E_{k_1}, R_{k_2}^{r_2} E_{k_2}]_\nu \rangle \\ = \sum_{\alpha=0}^{\nu} \frac{(-1)^\alpha A_\alpha}{4} \Big\{ (r_1+\alpha)(k_1+r_1+\alpha-1) \langle f, (R_{k_1}^{r_1+\alpha-1} E_{k_1})(R_{k_2}^{r_2+\nu-\alpha} E_{k_2}) \rangle \\ + (r_2+\nu-\alpha)(k_2+r_2+\nu-\alpha-1) \langle f, (R_{k_1}^{r_1+\alpha} E_{k_1})(R_{k_2}^{r_2+\nu-\alpha-1} E_{k_2}) \rangle \Big\}.$$

Applying the same method as used in the proof of Theorem 5.4 for both the terms on the right-hand side of the above identity, we get the main result. To complete the proof, we prove (18) by using simple combinatorial methods. Let $r_1 = 0$ and $r_2 \neq 0$. If $r_2 + \nu$ is even or $\nu = 0$, the result follows trivially and hence we assume that $r_2 + \nu$ is odd and $\nu \geq 1$. After simplifying the expression for t_α, we see that

$$\sum_{\alpha=0}^{\nu} A_\alpha t_\alpha = -(k-2) \sum_{\alpha=0}^{\nu} A_\alpha \alpha + (r_2+\nu)(k_2+r_2+\nu-1) \sum_{\alpha=0}^{\nu} A_\alpha.$$

Using Vandermonde's identity (16) for both the sums in the above expression, a simple calculation gives

$$\sum_{\alpha=0}^{\nu} A_\alpha t_\alpha = \frac{r_2(k_2+r_2+\nu-1)(k-\nu-r_2-2)}{\nu}\binom{k_1+k_2+r_2+2\nu-3}{\nu-1},$$

which is nonzero. This completes the proof. $\square$

6. Proof of Theorem 1.1

By Proposition 3.2, Lemma 3.3 and Lemma 3.4, to prove Theorem 1.1 we need to check only whether the following cases give eigenforms:

- $[E_{k_1}, D^{r_2}E_{k_2}]_\nu$ for $k_1 \neq k_2$.
- $[E_{k_1}, D^{r_1}E_{k_1}]_\nu$.
- $[D^{r_1}f, E_{k_2}]_\nu$, where $f \in S_{k_1}$ is an eigenform.

By Theorem 3.5 it is equivalent to check the following cases of Rankin–Cohen brackets of nearly holomorphic modular forms are eigenforms:

- $[E_{k_1}, R_{k_2}^{r_2}E_{k_2}]_\nu$ for $k_1 \neq k_2$.
- $[E_{k_1}, R_{k_1}^{r_1}E_{k_1}]_\nu$.
- $[R_{k_1}^{r_1}f, E_{k_2}]_\nu$, where $f \in S_{k_1}$ is an eigenform.

Consider the first case. Let $[E_{k_1}, R_{k_2}^{r_2}E_{k_2}]_\nu$ be an eigenform, where $k_1 \neq k_2$. Put $k = k_1+k_2+2r_2+2\nu$ and $a = k_1+k_2+r_2+\nu$. By Proposition 2.10 we have

$$[E_{k_1}, R_{k_2}^{r_2}E_{k_2}]_\nu = R_{k-2r'}^{r'}g, \tag{19}$$

where $g \in M_{k-2r'}$ is an eigenform and $r' \geq 0$ is an integer.

If $k = 24$ or $k \geq 28$, then the dimension of S_k is at least 2. Therefore if $r' = 0$ and $k = 24$ or $k \geq 28$ in (19), there exists a nonzero eigenform $h \in S_k$ such that

$$\langle h, g\rangle = \langle h, [E_{k_1}, R_{k_2}^{r_2}E_{k_2}]_\nu\rangle = 0.$$

Then by Theorem 5.4 we deduce that

$$L(h, a-1)L(h, a-k_1) = 0. \tag{20}$$

Since $L(h,s)$ has an Euler product in the region $\mathrm{Re}(s) > \frac{k+1}{2}$, $L(h,s)$ does not vanish for $\mathrm{Re}(s) > \frac{k+1}{2}$. We see that $a-1 > \frac{k+1}{2}$ and therefore $L(h, a-1) \neq 0$. We will also prove that $L(h, a-k_1) \neq 0$. First assume that $k_2 > k_1$. Since $k_2 > k_1$ and k_1 and k_2 are even positive numbers, we have $k_1 < k_2+1$. Thus $L(h, a-k_1) \neq 0$ as $a-k_1 > \frac{k+1}{2}$. If $k_1 > k_2$, then $k-a+k_1 > \frac{k+1}{2}$, and by the functional equation of L-functions, we deduce that $L(h, a-k_1) \neq 0$. Therefore, the above discussion gives a contradiction to (20). Thus if $r' = 0$ and $k = 24$ or $k \geq 28$, $[E_{k_1}, R_{k_2}^{r_2}E_{k_2}]_\nu$ is not an eigenform whenever $k_1 \neq k_2$.

For $r' = 0$, $k \neq 24$ and $k < 28$, we have finitely many cases to verify. We find that there are the following cases for which Rankin–Cohen brackets of nearly holomorphic eigenforms give rise to eigenforms:

- The holomorphic modular cases listed in Theorem 1.1 for which $k_1 \neq k_2$.
- The nonholomorphic cases given by $[E_4, R_8E_8]_1 = [E_8, R_4E_4]_1 = 1920\Delta_{16}$, $[E_4, R_6E_6]_2 = -5040\Delta_{16}$, $[E_6, R_4E_4]_2 = 5040\Delta_{16}$.

By Theorem 3.5, we get the corresponding cases for quasimodular eigenforms.

If $r' \geq 1$ and $k \neq 14$, by employing Lemma 2.8 as done in the case when $r' = 0$ and $k = 24$ or $k \geq 28$, we deduce that the Rankin–Cohen brackets of eigenforms do not result in eigenforms. If $r' \geq 1$ and $k = 14$, we get the following cases for which we get eigenforms:

$$[E_6, DE_4]_1 = 1440D\Delta_{12}, \quad [E_4, DE_6]_1 = -2016D\Delta_{12}.$$

Now consider the second case. Assume that

$$[E_{k_1}, R_{k_1}^{r_1}E_{k_1}]_\nu = R_{k-2r'}^{r'}f, \tag{21}$$

where $f \in M_{k-2r'}$ is an eigenform and $r' \geq 0$ is an integer. Put $k = 2k_1 + 2r_1 + 2\nu$.

If $\nu = 0$, the Rankin–Cohen bracket reduces to the product of two nearly holomorphic eigenforms. This has been done in [Kumar and Meher 2016]. By Theorem 3.5 we see that the only case for which the product of quasimodular eigenforms is an eigenform, is

$$E_4(DE_4) = \tfrac{1}{2}DE_8.$$

Assume that $\nu \geq 1$. If $r' = 0$, comparing the Fourier expansion of both sides of (21), we deduce that f has to be a cusp form. Since $\langle f, f\rangle \neq 0$, we have

$$\langle f, [E_{k_1}, R_{k_1}^{r_1}E_{k_1}]_\nu\rangle \neq 0.$$

Thus by Theorem 5.4 we have

$$L^*(f, k/2 + k_1 - 1)L^*(f, k/2) \neq 0.$$

Since $k/2 + k_1 - 1$ lies in the region in which $L(f, s)$ has an Euler product, we have $L(f, k/2 + k_1 - 1) \neq 0$. Thus $L^*(f, k/2) \neq 0$. From the functional equation of $L(f, s)$ we see that $L^*(f, k/2) = 0$ if $k \equiv 2 \pmod 4$. Therefore $k \equiv 0 \pmod 4$. If $k > 20$ and $k \equiv 0 \pmod 4$, by Lemma 4.6, there exist two eigenforms $g, h \in S_k$ such that

$$\langle g, f\rangle \neq 0 \quad \text{and} \quad \langle h, f\rangle \neq 0.$$

This contradicts the fact that f is an eigenform. If $k \leq 20$, there are only finitely many cases to verify, and we obtain the following cases for which the Rankin–Cohen brackets of two nearly holomorphic eigenforms give rise to eigenforms:

- The holomorphic modular cases listed in Theorem 1.1.

- The nonholomorphic modular cases

$$[E_4, R_4E_4]_1 = 960\Delta_{12}, \qquad [E_6, R_6E_6]_1 = -3024\Delta_{16},$$
$$[E_4, R_4E_4]_3 = 4800\Delta_{16}, \qquad [E_8, R_8E_8]_1 = 3840\Delta_{20},$$
$$[E_6, R_6E_6]_3 = -28224\Delta_{20}, \quad [E_4, R_4E_4]_5 = 13440\Delta_{20}.$$

Then by Theorem 3.5 we get the corresponding result for quasimodular eigenforms.

Let $r' \geq 1$. By (21) and Lemma 2.8, for any eigenform $g \in S_k$ we have

$$\langle g, [E_{k_1}, R_{k_1}^{r_1}E_{k_1}]_\nu\rangle = 0.$$

As done in the case when $r' = 0$, we deduce that $L^*(g, k/2) = 0$. By Remark 4.7 this implies that $k \equiv 2 \pmod 4$. If $r' = 1$ then $[E_{k_1}, R_{k_1}^{r_1}E_{k_1}]_\nu = R_{k-2}f$ and $k-2 \equiv 0 \pmod 4$. Also if $k-2 > 20$ and $k-2 \equiv 0 \pmod 4$, by Lemma 4.6 there exist two normalized eigenforms f_1 and f_2 in S_{k-2} such that

$$L^*\Big(f_1, \frac{k-2}{2}\Big) \neq 0 \quad \text{and} \quad L^*\Big(f_2, \frac{k-2}{2}\Big) \neq 0.$$

Then by Theorem 2.12 we have

$$\langle f_1, f\rangle = \frac{1}{c_1}\langle R_{k-2}f_1, R_{k-2}f\rangle = \frac{1}{c_1}\langle R_{k-2}f_1, [E_{k_1}, R_{k_1}^{r_1}E_{k_1}]_\nu\rangle,$$

and

$$\langle f_2, f\rangle = \frac{1}{c_1}\langle R_{k-2}f_2, R_{k-2}f\rangle = \frac{1}{c_1}\langle R_{k-2}f_1, [E_{k_1}, R_{k_1}^{r_1}E_{k_1}]_\nu\rangle.$$

Thus by applying Theorem 5.5 (in view of (18)) we deduce that there are two normalized eigenforms f_1 and f_2 in S_{k-2} such that

$$\langle f_1, f\rangle \neq 0 \quad \text{and} \quad \langle f_2, f\rangle \neq 0.$$

This gives a contradiction.

If $k-2 \leq 20$, we verify the finitely many remaining cases and deduce that if $r' = 1$ and $\nu \geq 1$, we get the following cases for which Rankin–Cohen brackets of two nearly holomorphic modular forms are again eigenforms:

- The holomorphic modular cases listed in Theorem 1.1.
- The nonholomorphic cases:

$$[E_4, R_4^2E_4]_1 = 960R_{12}\Delta_{12}, \qquad [E_4, R_4E_4]_2 = 2400R_{12}\Delta_{12},$$
$$[E_6, R_6^2E_6]_1 = -3024R_{16}\Delta_{16}, \quad [E_6, R_6E_6]_2 = -10584R_{16}\Delta_{16},$$
$$[E_4, R_4^2E_4]_3 = 4800R_{16}\Delta_{16}, \qquad [E_4, R_4E_4]_4 = 8400R_{16}\Delta_{16},$$

$$[E_8, R_8^2 E_8]_1 = 3840 R_{20}\Delta_{20}, \qquad [E_8, R_8 E_8]_2 = 17280 R_{20}\Delta_{20},$$
$$[E_6, R_6^2 E_6]_3 = -28224 R_{20}\Delta_{20}, \quad [E_6, R_6 E_6]_4 = -63504 R_{20}\Delta_{20},$$
$$[E_4, R_4^2 E_4]_5 = 13440 R_{20}\Delta_{20}, \qquad [E_4, R_4 E_4]_6 = 20160 R_{20}\Delta_{20}.$$

By Theorem 3.5 we have the corresponding cases for quasimodular forms.

Let $r' \geq 2$. If $g \in S_{k-2}$ is any eigenform, then Theorem 2.12 implies that

$$\langle R_{k-2}g, [E_{k_1}, R_{k_1}^{r_1} E_{k_1}]_\nu \rangle = \langle R_{k-2}g, R_{k-2r'}^{r'} f \rangle = 0.$$

Thus by Theorem 5.5 (in view of (18)), the above identity implies $L^*\big(g, \frac{k-2}{2}\big) = 0$. We have already proved that if $r' \geq 1$, then $k \equiv 2 \pmod 4$. Since $k-2 \equiv 0 \pmod 4$ and g is an arbitrary eigenform, if $k-2 > 20$, Lemma 4.6 gives a contradiction. If $k-2 \leq 20$, by checking the remaining finitely many cases, we deduce that if $r' \geq 2$, we do not get any case where Rankin–Cohen brackets of two nearly holomorphic eigenforms give rise to eigenforms. Thus by Theorem 3.5, we get the corresponding result in the case of quasimodular forms.

Now consider the third case. Let $[R_{k_1}^{r_1} f, E_{k_2}]_\nu$ be an eigenform, where $f \in S_k$ is an eigenform. Let $k = k_1 + k_2 + 2r_1 + 2\nu$. By Proposition 2.10 we have

$$[R_{k_1}^{r_1} f, E_{k_2}]_\nu = R_{k-2r'}^{r'} g,$$

where r' is a nonnegative integer and $g \in M_{k-2r'}$ is an eigenform. If either $k = 24$ or $k \geq 28$, the dimension of S_k is at least 2. Therefore if $r' = 0$ and either $k = 24$ or $k \geq 28$, there exists a nonzero eigenform $h \in S_k$ such that

$$\langle h, g \rangle = \langle h, [R_{k_1}^{r_1} f, E_{k_2}]_\nu \rangle = 0.$$

Applying Theorem 5.2 (in view of (12)), we get

$$L\Big(h \times f, \frac{k}{2} + \frac{k_1}{2} + \frac{k_2}{2} - 1\Big) = 0. \tag{22}$$

Since h and f are both eigenforms of weight k and k_1, respectively, $L(h \times f, s)$ has an Euler product in the region $\mathrm{Re}(s) > \frac{k}{2} + \frac{k_1}{2}$ and hence $L\big(h \times f, \frac{k}{2} + \frac{k_1}{2} + \frac{k_2}{2} - 1\big) \neq 0$. This contradicts (22). Therefore, when $r' = 0$ and either $k = 24$ or $k \geq 28$, the Rankin–Cohen brackets do not result in eigenforms. If $r' = 0$, $k \neq 24$ and $k < 28$, we verify these finitely many cases and deduce that we obtain only the modular cases listed in Theorem 1.1. If $r' \geq 1$, by employing Lemma 2.8 as done in the case when $r' = 0$ and either $k = 24$ or $k \geq 28$, we deduce that the Rankin–Cohen brackets of eigenforms do not result in eigenforms. This proves Theorem 1.1.

7. Further remarks

Although Theorem 1.1 is a result about quasimodular eigenforms, it is clear from the proof of Theorem 1.1 that one can state a similar result in the case of nearly holomorphic eigenforms. The result in the case of nearly holomorphic eigenforms is a generalization of the main result of [Beyerl et al. 2012] to the case of Rankin–Cohen brackets. Thus in this way we give a different proof of the main result of [Beyerl et al. 2012]. Our proof has the same flavor as the proofs of the main results given in [Duke 1999; Ghate 2000; Lanphier and Takloo-Bighash 2004].

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DUALITY FOR DIFFERENTIAL OPERATORS OF LIE–RINEHART ALGEBRAS

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Let (S, L) be a Lie–Rinehart algebra over a commutative ring R. This article proves that, if S is flat as an R-module and has Van den Bergh duality in dimension n, and if L is finitely generated and projective with constant rank d as an S-module, then the enveloping algebra of (S, L) has Van den Bergh duality in dimension $n + d$. When, moreover, S is Calabi–Yau and the d-th exterior power of L is free over S, the article proves that the enveloping algebra is skew Calabi–Yau, and it describes a Nakayama automorphism of it. These considerations are specialised to Poisson enveloping algebras. They are also illustrated on Poisson structures over two- and three-dimensional polynomial algebras and on Nambu–Poisson structures on certain two-dimensional hypersurfaces.

Introduction

Rinehart [1963] introduced the concept of Lie–Rinehart algebra (S, L) over a commutative ring R and defined its enveloping algebra U. This generalises both constructions of universal enveloping algebras of R-Lie algebras and algebras of differential operators of commutative R-algebras. Huebschmann [1999] investigated Poincaré duality on the (co)homology groups of (S, L). This duality is defined by

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the existence of a right U-module C, called the *dualising* module of (S, L) such that, for all left U-modules M and $k \in \mathbb{N}$,

$$\operatorname{Ext}^k_U(S, M) \cong \operatorname{Tor}^U_{d-k}(C, M). \tag{0-1}$$

Chemla [1999] proved that for Lie–Rinehart algebras arising from affine complex Lie algebroids, the algebra U has a rigid dualising complex, which she determined, and has Van den Bergh duality [1998]. Having Van den Bergh duality in dimension n for an R-algebra A means that

- A is homologically smooth, that is, A lies in the perfect derived category $\mathrm{per}(A^e)$ of the algebra $A^e := A \otimes_R A^{\mathrm{op}}$; and
- $\operatorname{Ext}^\bullet_{A^e}(A, A^e)$ is zero for $\bullet \neq 0$ and invertible as an A-bimodule if $\bullet = n$.

When this occurs, there is a functorial isomorphism, for all A-bimodules M and integers i (see [Van den Bergh 1998]),

$$\operatorname{Ext}^i_{A^e}(A, M) \cong \operatorname{Tor}^{A^e}_{n-i}(A, \operatorname{Ext}^n_{A^e}(A, A^e) \otimes_A M);$$

and $\operatorname{Ext}^n_{A^e}(A, A^e)$ is called the inverse dualising bimodule of A. Two classes of algebras with Van den Bergh duality are of particular interest, namely,

- *Calabi–Yau* algebras, for which $\operatorname{Ext}^n_{A^e}(A, A^e)$ is required to be isomorphic to A as an A-bimodule (see [Ginzburg 2006]); and
- *skew Calabi–Yau* algebras, for which there exists an automorphism

  $$\nu \in \operatorname{Aut}_{R-\mathrm{alg}}(A)$$

  such that $\operatorname{Ext}^n_{A^e}(A, A^e) \simeq A^\nu$ as A-bimodules (see [Reyes et al. 2014]); here A^ν denotes the A-bimodule obtained from A by twisting the action of A on the right by ν.

The relevance of these algebras comes from their role in the noncommutative geometry initiated in [Artin and Schelter 1987] and in the investigation of Calabi–Yau categories, and also from the specificities of their Hochschild cohomology when R is a field. For instance, it is proved in [Ginzburg 2006; Lambre 2010] that the Gerstenhaber bracket of the Hochschild cohomology of Calabi–Yau algebras have a BV generator.

This article investigates when the enveloping algebra U of a general Lie–Rinehart algebra (S, L) over a commutative ring R has Van den Bergh duality.

It considers Lie–Rinehart algebras (S, L) such that S has Van den Bergh duality and is flat as an R-module, and L is finitely generated and projective with constant rank d as an S-module. Under these conditions, it is proved that U has Van den Bergh duality. Note that, when R is a perfect field, the former condition amounts to saying that S is a smooth affine R-algebra [Krähmer 2007]. Note also that, under the latter condition, it is proved in [Huebschmann 1999, Theorem 2.10] that (S, L) has duality in the sense of (0-1). Under the additional assumption that S is Calabi–Yau and $\Lambda^d L$ is free as an S-module, it appears as a corollary that U is skew Calabi–Yau, and a Nakayama automorphism may be described explicitly. These considerations are specialised to the situation where the Lie–Rinehart algebra (S, L) arises from a Poisson structure on S. Also they are illustrated by detailed examples in the following cases:

- For Poisson brackets on polynomial algebras in two or three variables.
- For Nambu–Poisson structures on two-dimensional hypersurfaces of the shape $1 + T(x, y, z) = 0$, where T is a weight homogeneous polynomial.

Throughout the article, R denotes a commutative ring, (S, L) denotes a Lie–Rinehart algebra over R and U denotes its enveloping algebra. Given an R-Lie algebra $\mathfrak{g}$, its universal enveloping algebra is denoted by $\mathcal{U}_R(\mathfrak{g})$. For an R-algebra A, the category of left A-modules is denoted by $\mathrm{Mod}(A)$ and $\mathrm{Mod}(A^{\mathrm{op}})$ is identified with the category of right A-modules. For simplicity, the piece of notation $\otimes$ is used for $\otimes_R$. All complexes have differential of degree $+1$.

1. Main results

A *Lie–Rinehart* algebra over a commutative ring R is a pair (S, L) where S is a commutative R-algebra and L is a Lie R-algebra which is also a left S-module, endowed with a homomorphism of R-Lie algebras,

$$\begin{aligned} &L \to \mathrm{Der}_R(S), \\ &\alpha \mapsto \partial_\alpha := \alpha(-), \end{aligned} \tag{1-1}$$

such that, for all $\alpha, \beta \in L$ and $s \in S$,

$$[\alpha, s\beta] = s[\alpha, \beta] + \alpha(s)\beta.$$

Following [Huebschmann 1999], the *enveloping* algebra U of (S, L) is identified with the algebra

$$(S \rtimes L)/I,$$

where $S \rtimes L$ is the smash-product algebra of S by the action of L on S by derivations and I is the two-sided ideal of $S \rtimes L$ generated by

$$\{s \otimes \alpha - 1 \otimes s\alpha \mid s \in S,\ \alpha \in L\}$$

(see Lemma 3.0.1); it is proved in [Huebschmann 1999] that this set generates I as a right ideal.

As mentioned in the introduction, when L is a finitely generated S-module with constant rank d, the Lie–Rinehart algebra (S, L) has duality in the sense of (0-1) with $C = \Lambda_S^d L^\vee$. Here $-^\vee$ is the duality $\mathrm{Hom}_S(-, S)$ and $\Lambda_S^d L^\vee$ is considered as a right U-module using the Lie derivative λ_α, for $\alpha \in L$ (see [Huebschmann 1999, Section 2]),

$$\lambda_\alpha : \Lambda_S^\bullet L^\vee \to \Lambda_S^\bullet L^\vee ;$$

this is the derivation of $\Lambda_S^\bullet L^\vee$ such that, for all $s \in S$, $\varphi \in L^\vee$ and $\beta \in L$,

$$\lambda_\alpha(s) = \alpha(s) \quad \text{and} \quad \lambda_\alpha(\varphi)(\beta) = \alpha(\varphi(\beta)) - \varphi([\alpha, \beta]).$$

The right U-module structure of $\Lambda_S^d L^\vee$ is such that, for all $\varphi \in \Lambda_S^d L^\vee$ and $\alpha \in L$,

$$\varphi \cdot \alpha = -\lambda_\alpha(\varphi). \tag{1-2}$$

The first main result of the article gives sufficient conditions for U to have Van den Bergh duality. It also describes the inverse dualising bimodule. Here are some explanations on this description. On one hand, R-linear derivations $\partial \in \mathrm{Der}_R(S)$ act on $\mathrm{Ext}^n_{S^e}(S, S^e)$, $n \in \mathbb{N}$, by Lie derivatives (see Section 4),

$$\mathcal{L}_\partial : \mathrm{Ext}^n_{S^e}(S, S^e) \to \mathrm{Ext}^n_{S^e}(S, S^e).$$

Combining with the action of L on S yields an action $\alpha \otimes e \mapsto \alpha \cdot e$ of L on $\mathrm{Ext}^n_{S^e}(S, S^e)$ such that, for all $\alpha \in L$ and $e \in \mathrm{Ext}^n_{S^e}(S, S^e)$,

$$\alpha \cdot e = \mathcal{L}_{\partial_\alpha}(e).$$

Although this is not a U-module structure on $\mathrm{Ext}^n_{S^e}(S, S^e)$, it defines a left U-module structure on $\Lambda_S^d L^\vee \otimes_S \mathrm{Ext}^n_{S^e}(S, S^e)$, $d \in \mathbb{N}$, such that, for all $\alpha \in L$, $\varphi \in \Lambda_S^d L^\vee$ and $e \in \mathrm{Ext}^n_{S^e}(S, S^e)$,

$$\alpha \cdot (\varphi \otimes e) = -\varphi \cdot \alpha \otimes e + \varphi \otimes \alpha \cdot e.$$

On the other hand, consider the functor

$$F : \mathrm{Mod}(U) \to \mathrm{Mod}(U^e)$$

(see Section 3.3) such that, if $N \in \mathrm{Mod}(U)$, then $F(N)$ equals $U \otimes_S N$ in $\mathrm{Mod}(U)$ and has a right U-module structure defined by the following formula, for all $\alpha \in L$, $u \in U$ and $n \in N$:

$$(u \otimes n) \cdot \alpha = u\alpha \otimes n - u \otimes \alpha \cdot n.$$

This functor takes left U-modules which are invertible as S-modules to invertible U-bimodules (see Section 3.6). The main result of this article is the following.

Theorem 1. *Let R be a commutative ring. Let (S, L) be a Lie–Rinehart algebra over R. Denote by U the enveloping algebra of (S, L). Assume that*

- *S is flat as an R-module,*
- *S has Van den Bergh duality in dimension n,*
- *L is finitely generated and projective with constant rank d as an S-module.*

Then, U has Van den Bergh duality in dimension $n+d$ and there is an isomorphism of U-bimodules,

$$\mathrm{Ext}^{n+d}_{U^e}(U, U^e) \simeq F(\Lambda^d_S L^\vee \otimes_S \mathrm{Ext}^n_{S^e}(S, S^e)).$$

Note that when R is Noetherian and S is finitely generated as an R-algebra and projective as an R-module, then there is an isomorphism of S-(bi)modules,

$$\mathrm{Ext}^n_{S^e}(S, S^e) \simeq \Lambda^n_S \,\mathrm{Der}_R(S)\,;$$

this isomorphism is compatible with the actions by Lie derivatives (see Section 4.5). The above theorem was proved in [Chemla 1999, Theorem 4.4.1] when $R = \mathbb{C}$ and S is finitely generated as a $\mathbb{C}$-algebra.

The preceding theorem specialises to the situation where the involved invertible S-modules are free. On one hand, when $(\Lambda^d_S L)^\vee$ is free as an S-module with free generator φ_L, there is an associated *trace* mapping

$$\lambda_L : L \to S,$$

such that, for all $\alpha \in L$,

$$\varphi_L \cdot \alpha = \lambda_L(\alpha) \cdot \varphi_L,$$

where the action on the left-hand side is given by (1-2) and that on the right-hand side is just given by the S-module structure. On the other hand, when S is Calabi–Yau in dimension n, each generator of the free of rank one S-module $\mathrm{Ext}^n_{S^e}(S, S^e)$ determines a volume form $\omega_S \in \Lambda^n_S \Omega_{S/R}$, and the *divergence*

$$\mathrm{div} : \mathrm{Der}_R(S) \to S$$

associated with ω_S is defined by the following equality, for all $\partial \in \mathrm{Der}_R(S)$:

$$\mathcal{L}_\partial(\omega_S) = \mathrm{div}(\partial)\omega_S;$$

(see 4.5 for details). The second main result of the article then reads as follows.

Theorem 2. *Let R be a commutative ring. Let (S, L) be a Lie–Rinehart algebra over R. Denote by U the enveloping algebra of (S, L). Assume that*

- *S is flat as an R-module,*
- *S is Calabi–Yau in dimension n,*
- *L is finitely generated and projective with constant rank d and $\Lambda^d_S L$ is free as S-modules.*

Then, U is skew Calabi–Yau with a Nakayama automorphism $\nu \in \mathrm{Aut}_R(U)$ such that, for all $s \in S$, and $\alpha \in L$,

$$\begin{cases} \nu(s) = s, \\ \nu(\alpha) = \alpha + \lambda_L(\alpha) + \mathrm{div}(\partial_\alpha), \end{cases}$$

where λ_L is any trace mapping on $\Lambda^d_S L^\vee$ and div *is any divergence.*

Among all Lie–Rinehart algebras, those arising from Poisson structures on S play a special role because of the connection to Poisson (co)homology. Recall that any R-bilinear Poisson bracket $\{-,-\}$ on S defines a Lie–Rinehart algebra structure on $(S, L) = (S, \Omega_{S/R})$ such that, for all $s, t \in S$,

- $\partial_{ds} = \{s, -\}$;
- $[ds, dt] = d\{s, t\}$.

In this case, the formulations of Theorems 1 and 2 simplify because, when $\Omega_{S/R}$ is projective with constant rank n as an S-module, the right U-module structure of $\Lambda_S^n \Omega_{S/R}^\vee$ (see (1-2)) is given by classical Lie derivatives; that is, for all $s \in S$,

$$\lambda_{ds}(\varphi) = \mathcal{L}_{\{s,-\}}(\varphi). \tag{1-3}$$

More precisely, these theorems specialise as follows.

Corollary 1. *Let R be a Noetherian ring. Let $(S, \{-,-\})$ be a finitely generated Poisson algebra over R. Denote by U the enveloping algebra of the associated Lie Rinehart algebra $(S, \Omega_{S/R})$. Assume that*

- *S is projective in* $\mathrm{Mod}(R)$;
- $S \in \mathrm{per}(S^e)$;
- *$\Omega_{S/R}$, which is then projective in* $\mathrm{Mod}(S)$, *has constant rank n.*

Then, U has Van den Bergh duality in dimension $2n$ and there is an isomorphism of U-bimodules,

$$\mathrm{Ext}^{2n}_{U^e}(U, U^e) \simeq U \otimes_S \Lambda_S^n \mathrm{Der}_R(S) \otimes_S \Lambda_S^n \mathrm{Der}_R(S),$$

where the right-hand side term is a left U-module in a natural way and a right U-module such that, for all $u \in U$, $\varphi, \varphi' \in \Lambda_S^n \mathrm{Der}_R(S)$ and $s \in S$,

$$(u \otimes \varphi \otimes \varphi') \cdot ds = u\, ds \otimes \varphi \otimes \varphi' - u \otimes (\mathcal{L}_{\{s,-\}}(\varphi) \otimes \varphi' + \varphi \otimes \mathcal{L}_{\{s,-\}}(\varphi')).$$

In particular, if S has a volume form, then U is skew Calabi–Yau with a Nakayama automorphism $\nu : U \to U$ such that, for all $s \in S$,

$$\begin{cases} \nu(s) = s, \\ \nu(ds) = ds + 2\,\mathrm{div}(\{s,-\}), \end{cases}$$

where div *is the divergence of the chosen volume form.*

For the case where $R = \mathbb{C}$ and S is finitely generated as a $\mathbb{C}$-algebra, the above corollary is announced in [Lü et al. 2017, Theorem 0.7, Corollary 0.8] using the main results of [Chemla 1999].

This article is structured as follows. Section 2 presents useful information on the case where S has Van den Bergh duality. Section 3 is devoted to technical lemmas on

U-(bi)modules. In particular, it presents the above mentioned functor F and its right adjoint G, which play an essential role in the proof of the main results. Section 4 introduces the action of L on $\mathrm{Ext}^\bullet_{S^e}(S, S^e)$ by Lie derivatives. This structure is used in Section 5 in order to describe $\mathrm{Ext}^\bullet_{U^e}(U, U^e)$ and prove Theorem 1, Theorem 2 and Corollary 1. Finally, Section 6 applies this corollary to a class of examples of Nambu–Poisson surfaces.

2. Poincaré duality for S

As proved in [Van den Bergh 1998] when R is a field, if S has Van den Bergh duality in dimension n, then there is a functorial isomorphism, for all S-bimodules N,

$$\mathrm{Ext}^\bullet_{S^e}(S, N) \simeq \mathrm{Tor}^{S^e}_{n-\bullet}(S, \mathrm{Ext}^n_{S^e}(S, S^e) \otimes_S N).$$

It is direct to check that this is still the case without assuming that R is a field. In view of the proof of the main results of the article, Section 2.1 relates the above mentioned isomorphism to the fundamental class of S, following [Lambre 2010], and Section 2.2 relates Van den Bergh duality to the regularity of commutative algebras, following [Krähmer 2007].

2.1. ***Fundamental class and contraction.*** Consider a projective resolution $P^\bullet$ in $\mathrm{Mod}(S^e)$,

$$\cdots \to P^{-2} \to P^{-1} \to P^0 \xrightarrow{\epsilon} S,$$

and let $p^0 \in P^0$ be such that $\epsilon(p_0) = 1_S$. For all $M, N \in \mathrm{Mod}(S^e)$ and $n \in \mathbb{N}$, define the contraction

$$\begin{aligned}\mathrm{Tor}^{S^e}_n(S, M) \times \mathrm{Ext}^n_{S^e}(S, N) &\to \mathrm{Tor}^{S^e}_0(S, M \otimes_S N),\\ (\omega, e) &\mapsto \iota_e(\omega)\end{aligned}$$

as the mapping induced by the following one:

$$\begin{aligned}M \otimes_{S^e} P^{-n} &\to \mathrm{Hom}_R\big(\mathrm{Hom}_{S^e}(P^{-n}, N), (M \otimes_S N) \otimes_{S^e} P^0\big),\\ x \otimes p &\mapsto (\varphi \mapsto (x \otimes \varphi(p)) \otimes p^0).\end{aligned}$$

This makes sense because $P^\bullet$ is concentrated in nonpositive degrees. The construction depends neither on the choice of p^0 nor on that of $P^\bullet$.

Following the proof of [Lambre 2010, Proposition 3.3], when $S \in \mathrm{per}(S^e)$ and n is taken equal to $\mathrm{pd}_{S^e}(S)$, the contraction induces an isomorphism for all $N \in \mathrm{Mod}(S^e)$,

$$\begin{aligned}\mathrm{Tor}^{S^e}_n(S, \mathrm{Ext}^n_{S^e}(S, S^e)) &\to \mathrm{Hom}_{S^e}\big(\mathrm{Ext}^n_{S^e}(S, N), \mathrm{Tor}^{S^e}_0(S, \mathrm{Ext}^n_{S^e}(S, S^e) \otimes_S N)\big),\\ \omega &\mapsto \iota_?(\omega).\end{aligned}$$

In the particular case $N = S^e$, the *fundamental class* of S is the element $c_S \in \operatorname{Tor}_n^{S^e}(S, \operatorname{Ext}_{S^e}^n(S, S^e))$ such that

$$(\iota_?(c_S))_{|\operatorname{Ext}_{S^e}^n(S,S^e)} = \operatorname{Id}_{\operatorname{Ext}_{S^e}^n(S,S^e)}.$$

Following the arguments in the proof of [Lambre 2010, Théorème 4.2], when S has Van den Bergh duality in dimension n, which gives that $n = \operatorname{pd}_{S^e}(S)$, the contraction with c_S induces an isomorphism, for all $N \in \operatorname{Mod}(S^e)$,

$$\iota_?(c_S) : \operatorname{Ext}_{S^e}^n(S, N) \xrightarrow{\simeq} \operatorname{Tor}_0^{S^e}(S, \operatorname{Ext}_{S^e}^n(S, S^e) \otimes_S N). \tag{2-1}$$

When S is projective in $\operatorname{Mod}(R)$, the Hochschild complex $S^{\otimes \bullet+2}$ is a resolution of S in $\operatorname{Mod}(S^e)$ and the contraction

$$\operatorname{Tor}_n^{S^e}(S, M) \times \operatorname{Ext}_{S^e}^m(S, N) \to \operatorname{Tor}_{n-m}^{S^e}(S, M \otimes_S N),$$
$$(\omega, e) \mapsto \iota_e(\omega)$$

may be defined for all $M, N \in \operatorname{Mod}(S^e)$ and $m, n \in \mathbb{N}$, as the mapping induced at the level of Hochschild (co)chains by

$$M \otimes S^{\otimes n} \times \operatorname{Hom}_R(S^{\otimes m}, N) \to (M \otimes_S N) \otimes S^{\otimes (m-n)},$$
$$((x|s_1|\cdots|s_n), \psi) \mapsto (x \otimes \psi(s_1|\cdots|s_m)|s_{m+1}|\cdots|s_n).$$

When, in addition, S has Van den Bergh duality in dimension n, then [Lambre 2010, Théorème 4.2] asserts that the following mapping given by contraction with c_S is an isomorphism, for all $N \in \operatorname{Mod}(S^e)$ and $m \in \mathbb{N}$,

$$\iota_?(c_S) : \operatorname{Ext}_{S^e}^m(S, N) \to \operatorname{Tor}_{n-m}^{S^e}(S, \operatorname{Ext}_{S^e}^n(S, S^e) \otimes_S N).$$

2.2. *Relationship to regularity.* The main results of this article assume that S has Van den Bergh duality. For commutative algebras, this property is related to smoothness and regularity. The relationship is detailed in [Krähmer 2007] for the case where R is a perfect field, and is summarised below in the present setting.

Proposition 2.2.1 [Krähmer 2007]. *Let R be a Noetherian commutative ring. Let S be a finitely generated commutative R-algebra and projective as an R-module. Let $n \in \mathbb{N}$. The following properties are equivalent.*

(i) *S has Van den Bergh duality in dimension n.*

(ii) $\operatorname{gl.dim}(S^e) < \infty$ *and $\Omega_{S/R}$, which is then projective in* $\operatorname{Mod}(S)$, *has constant rank n.*

When these properties are true, $\operatorname{gl.dim}(S) < \infty$ *and* $\operatorname{Ext}_{S^e}^n(S, S^e) \simeq \Lambda_S^n \operatorname{Der}_R(S)$ *as S-modules.*

Proof. See [Krähmer 2007] for full details. Since S is projective over R, then $\mathrm{pd}_{(S^e)^e}(S^e) \leqslant 2\,\mathrm{pd}_{S^e}(S)$ [Cartan and Eilenberg 1956, Chap. IX, Proposition 7.4]; besides, using the Hochschild resolution of S in $\mathrm{Mod}(S^e)$ yields that

$$\mathrm{gl.\,dim}(S) \leqslant \mathrm{pd}_{S^e}(S) \leqslant \mathrm{gl.\,dim}(S^e);$$

thus

$$\begin{aligned} S \in \mathrm{per}(S^e) &\Leftrightarrow \mathrm{gl.\,dim}(S^e) < \infty \\ &\Rightarrow \mathrm{gl.\,dim}(S) < \infty. \end{aligned} \tag{2-2}$$

Note also that, following [Hochschild et al. 1962, Theorem 3.1],

$$\mathrm{gl.\,dim}(S^e) < \infty \quad \Rightarrow \quad \Omega_{S/R} \text{ is projective in } \mathrm{Mod}(S). \tag{2-3}$$

Denote by μ the multiplication mapping $S \otimes S \to S$. Assume $\mathrm{gl.\,dim}(S^e) < \infty$, let $\mathfrak{p} \in \mathrm{Spec}(S)$ ($\subseteq \mathrm{Spec}(S^e)$) and denote by d the rank of $(\Omega_{S/R})_\mathfrak{p}$. Since $\Omega_{S/R} \simeq \mathrm{Ker}(\mu)/\mathrm{Ker}(\mu)^2$ as modules over S ($\simeq S^e/\mathrm{Ker}(\mu)$) and $\mathrm{gl.\,dim}(S^e) < \infty$, the $(S^e)_\mathfrak{p}$-module $\mathrm{Ker}(\mu)_\mathfrak{p}$ is generated by a regular sequence having d elements. There results a Koszul resolution of $S_\mathfrak{p}$ in $\mathrm{Mod}((S^e)_\mathfrak{p})$. Using this resolution and the isomorphism $\mathrm{Ext}^\bullet_{S^e}(S, S^e)_\mathfrak{p} \simeq \mathrm{Ext}^\bullet_{(S^e)_\mathfrak{p}}(S_\mathfrak{p}, (S^e)_\mathfrak{p})$ in $\mathrm{Mod}((S^e)_\mathfrak{p})$ yields isomorphisms of $(S^e)_\mathfrak{p}$-modules,

$$\mathrm{Ext}^\bullet_{S^e}(S, S^e)_\mathfrak{p} \simeq \begin{cases} 0 & \text{if } \bullet \neq d, \\ S_\mathfrak{p} & \text{if } \bullet = d. \end{cases} \tag{2-4}$$

Now assume (i). Then, $\mathrm{gl.\,dim}(S^e) < \infty$ (see (2-2)), $\Omega_{S/R}$ is projective (see (2-3)) and has constant rank n (see (2-4)). Conversely, assume that $\mathrm{gl.\,dim}(S^e) < \infty$ and $\Omega_{S/R}$ has constant rank n. Then, $S \in \mathrm{per}(S^e)$ (see (2-2)) and the S-module (equivalently, the symmetric S-bimodule) $\mathrm{Ext}^\bullet_{S^e}(S, S^e)$ is zero if $\bullet \neq n$ and is invertible if $\bullet = n$ (see (2-4)). Thus,

$$\text{(i)} \Leftrightarrow \text{(ii)}.$$

Finally, assume that both (i) and (ii) are true. Then, $\mathrm{gl.\,dim}(S) < \infty$ (see (2-2)). Moreover, Van den Bergh duality [1998, Theorem 1] does apply here and provides an isomorphism of S-modules,

$$\mathrm{Ext}^0_{S^e}(S, \mathrm{Ext}^n_{S^e}(S; S^e)^{-1}) \simeq \mathrm{Tor}_n^{S^e}(S, S),$$

whereas [Hochschild et al. 1962, Theorem 3.1] yields an isomorphism of S-modules,

$$\mathrm{Tor}_n^{S^e}(S, S) \simeq \Lambda^n_S \Omega_{S/R}.$$

Thus, $\mathrm{Ext}^n_{S^e}(S, S^e) \simeq \Lambda^n_S \mathrm{Der}_R(S)$ in $\mathrm{Mod}(S)$. □

3. Material on U-(bi)modules

The purpose of this section is to introduce an adjoint pair of functors (F, G) between $\mathrm{Mod}(U)$ and $\mathrm{Mod}(U^e)$. In the proof of Theorem 1, the U-bimodule $\mathrm{Ext}^{\bullet}_{U^e}(U, U^e)$ is described as the image under F of a certain left U-module which is invertible as an S-module. This section develops the needed properties of F. Hence, Section 3.1 recalls the basic constructions of U-modules; Sections 3.2 and 3.3 introduce the functors G and F, respectively; Section 3.4 proves that (F, G) is adjoint; Section 3.5 introduces and collects basic properties of compatible left $S \rtimes L$-modules, these are applied in Section 4 to the action of L on $\mathrm{Ext}^{\bullet}_{S^e}(S, S^e)$ by Lie derivatives; and Section 3.6 proves that the functor F transforms left U-modules that are invertible as S-modules into invertible U-bimodules. These results are based on the description of U as a quotient of the smash-product $S \rtimes L$ given in the following lemma. This description is established in [Lambre et al. 2017, Proposition 2.10] in the case of Lie–Rinehart algebras arising from Poisson algebras.

Lemma 3.0.1. (1) *The identity mappings* $\mathrm{Id}_S : S \to S$ *and* $\mathrm{Id}_L : L \to L$ *induce an isomorphism of R-algebras*

$$(S \rtimes L)/I \to U, \tag{3-1}$$

where I is the two-sided ideal of the smash-product algebra $S \rtimes L$ generated by

$$\{s \otimes \alpha - 1 \otimes s\alpha \mid s \in S, \alpha \in L\}.$$

(2) *If L is projective as a left S-module, then U is projective both as a left and as a right S-module.*

Proof. (1) Recall (see [Rinehart 1963]) that U is defined as follows: Endow $S \oplus L$ with an R-Lie algebra structure such that, for all $s, t \in S$ and $\alpha, \beta \in L$,

$$[s + \alpha, t + \beta] = \alpha(t) - \beta(s) + [\alpha, \beta].$$

Then, U is the factor R-algebra of the subalgebra of the universal enveloping algebra $\mathcal{U}_R(S \oplus L)$ generated by the image of $S \oplus L$ by the two-sided ideal generated by the classes in $\mathcal{U}_R(S \oplus L)$ of the following elements, for $s, t \in S$ and $\alpha \in L$:

$$s \otimes t - st, \qquad s \otimes \alpha - s\alpha.$$

Recall also that $S \rtimes L$ is the R-algebra with underlying R-module

$$S \otimes \mathcal{U}_R(L),$$

such that the images of $S \otimes 1$ and $1 \otimes \mathcal{U}_R(L)$ are subalgebras, and the following hold, for all $s, t \in S$ and $\alpha, \beta \in S$:

$$\begin{cases} (s \otimes 1) \cdot (1 \otimes \alpha) = s \otimes \alpha, \\ (1 \otimes \alpha) \cdot (s \otimes 1) = \alpha(s) \otimes 1 + s \otimes \alpha. \end{cases}$$

Therefore, the natural mappings $S \to U$ and $L \to U$ induce an R-algebra homomorphism from $S \rtimes L$ to U. This homomorphism vanishes on I whence the R-algebra homomorphism (3-1).

Besides, the universal property of U stated in [Huebschmann 1999, Section 2, p. 110] yields an R-algebra homomorphism,

$$U \to (S \rtimes L)/I, \tag{3-2}$$

induced by the natural mappings $S \to (S \rtimes L/I)$ and $L \to (S \rtimes L)/I$. In view of the behaviour of (3-1) and (3-2) on the respective images of $S \cup L$, these algebra homomorphisms are inverse to each other.

(2) It is proved in [Rinehart 1963, Lemma 4.1] that U is projective as a left S-module. Consider the increasing filtration of U by the left S-submodules

$$0 \subseteq F_0U \subseteq F_1U \subseteq \cdots,$$

where F_pU is the image of $\oplus_{i=0}^{p} S \otimes L^{\otimes i}$ in U, for all $p \in \mathbb{N}$. In view of the equality

$$\alpha s = s\alpha + \alpha(s)$$

in U for all $s \in S$ and $\alpha \in L$, the left S-module F_pU is also a right S-submodule of U, and $F_pU/F_{p-1}U$ is a symmetric S-bimodule for all $p \in \mathbb{N}$. Therefore, the considerations used in the proof of [Rinehart 1963, Lemma 4.1] may be adapted in order to prove that U is projective as a right S-module. □

3.1. ***Basic constructions of U-modules.*** Left $S \rtimes L$-modules are identified with R-modules N endowed with a left S-module structure, and a left L-module structure such that, for all $n \in N$, $\alpha \in L$ and $s \in S$,

$$\alpha \cdot (s \cdot n) = \alpha(s) \cdot n + s \cdot (\alpha \cdot n).$$

Left U-modules are identified with left $S \rtimes L$-modules N such that, for all $n \in N$, $\alpha \in L$ and $s \in S$,

$$s \cdot (\alpha \cdot n) = (s\alpha) \cdot n.$$

Recall that the action of L endows S with a left U-module structure.

Right $S \rtimes L$-modules are identified with the R-modules M endowed with a right S-module structure and a right L-module structure such that, for all $m \in M$, $\alpha \in L$ and $s \in S$,

$$(m \cdot \alpha) \cdot s = m \cdot \alpha(s) + (m \cdot s) \cdot \alpha.$$

Right U-modules are identified with right $S \rtimes L$-modules M such that, for all $m \in M$, $s \in S$ and $\alpha \in L$,

$$(m \cdot s) \cdot \alpha = m \cdot (s\alpha).$$

The following constructions are classical. The corresponding U-module structures are introduced in [Huebschmann 1999, Section 2].

Let M, M' be right $S \rtimes L$-modules. Let N, N' be a left $S \rtimes L$-module. Then:

- N is a right $S \rtimes L$-module for the right L-module structure such that, for all $n \in N,\ s \in S$ and $\alpha \in L$,

$$n \cdot s = s \cdot n \quad \text{and} \quad n \cdot \alpha = -\alpha \cdot n. \tag{3-3}$$

- $\mathrm{Hom}_S(N, N')$ is a left $S \rtimes L$-module for the left L-module structure such that, for all $f \in \mathrm{Hom}_S(N, N'),\ n \in N$ and $\alpha \in L$,

$$(\alpha \cdot f)(n) = \alpha \cdot f(n) - f(\alpha \cdot n); \tag{3-4}$$

  moreover, this is a left U-module structure if N and N' are left U-modules.

- $\mathrm{Hom}_S(M, M')$ is a left $S \rtimes L$-module for the left L-module structure such that, for all $f \in \mathrm{Hom}_S(M, M'),\ m \in M$ and $\alpha \in L$,

$$(\alpha \cdot f)(m) = -f(m) \cdot \alpha + f(m \cdot \alpha). \tag{3-5}$$

- $\mathrm{Hom}_S(N, S)$ is a right $S \rtimes L$-module for the right L-module structure such that, for all $f \in \mathrm{Hom}_S(N, S),\ n \in N$ and $\alpha \in L$,

$$(f \cdot \alpha)(n) = -\alpha(f(n)) + f(\alpha \cdot n). \tag{3-6}$$

- $N \otimes_S N'$ is a left $S \rtimes L$-module for the left L-module structure such that, for all $n \in N,\ n' \in N$ and $\alpha \in L$,

$$\alpha \cdot (n \otimes n') = \alpha \cdot n \otimes n' + n \otimes \alpha \cdot n'; \tag{3-7}$$

  moreover, this is a left U-module structure if N and N' are left U-modules.

- $M \otimes_S N$ is a left $S \rtimes L$-module for the left L-module structure such that, for all $m \in M,\ n \in N$ and $\alpha \in L$,

$$\alpha \cdot (m \otimes n) = -m \cdot \alpha \otimes n + m \otimes \alpha \cdot n. \tag{3-8}$$

3.2. ***The functor*** $\boldsymbol{G = \mathrm{Hom}_{S^e}(S, -) : \mathrm{Mod}(U^e) \to \mathrm{Mod}(U)}$**.** Given $M \in \mathrm{Mod}(U^e)$, recall that

$$M^S = \{m \in M \mid (\text{for all } s \in S)\ (s \otimes 1 - 1 \otimes s) \cdot m = 0\}.$$

This is a symmetric S^e-submodule of M. Recall also the canonical isomorphisms that are inverse to each other:

$$\begin{aligned} M^S &\leftrightarrow \mathrm{Hom}_{S^e}(S, M) \\ m &\mapsto (s \mapsto (s \otimes 1) \cdot m) \\ \varphi(1) &\leftarrow\!\mid \varphi. \end{aligned} \tag{3-9}$$

Lemma 3.2.1. *Let* $M \in \mathrm{Mod}(U^e)$. *Then,*

(1) M^S *is a left U-module such that, for all* $m \in M^S$ *and* $\alpha \in L$,

$$\alpha \cdot m := (\alpha \otimes 1 - 1 \otimes \alpha) \cdot m; \tag{3-10}$$

(2) *the corresponding left U-module structure on* $\mathrm{Hom}_{S^e}(S, M)$ (*under the identification* (3-9)) *is such that, for all* $\varphi \in \mathrm{Hom}_{S^e}(S, M)$, $\alpha \in L$ *and* $s \in S$,

$$(\alpha \cdot \varphi)(s) = (\alpha \otimes 1 - 1 \otimes \alpha) \cdot \varphi(s) - \varphi(\alpha(s)).$$

Proof. (1) Given all $s \in S$ and $\alpha \in L$, denote

$$s \otimes 1 - 1 \otimes s \in U^e \quad \text{and} \quad \alpha \otimes 1 - 1 \otimes \alpha \in U^e$$

by ds and $d\alpha$, respectively; in particular

$$d\alpha \cdot ds = ds \cdot d\alpha + d(\alpha(s)),$$

and, for all $m \in M^S$,

$$ds \cdot (d\alpha \cdot m) = d\alpha \cdot (ds \cdot m) - d(\alpha(s)) \cdot m = 0,$$

which proves that $d\alpha \cdot m \in M^S$. Therefore, (3-10) defines a left L-module structure on M^S. Now, for all $m \in M^S$, $s \in S$ and $\alpha \in L$,

$$\begin{aligned}\alpha \cdot (s \otimes 1) \cdot m = d\alpha \cdot (s \otimes 1) \cdot m &= (\alpha(s) \otimes 1 + s\alpha \otimes 1 - s \otimes \alpha) \cdot m \\ &= (\alpha(s) \otimes 1) \cdot m + (s \otimes 1)(\alpha \otimes 1 - 1 \otimes \alpha) \cdot m \\ &= (\alpha(s) \otimes 1) \cdot m + (s \otimes 1) \cdot (\alpha \cdot m),\end{aligned}$$

$$\begin{aligned}(s \otimes 1) \cdot (\alpha \cdot m) &= (s \otimes 1) \cdot (\alpha \otimes 1 - 1 \otimes \alpha) \cdot m = (s\alpha \otimes 1) \cdot m - (s \otimes 1) \cdot (1 \otimes \alpha) \cdot m \\ &= (s\alpha \otimes 1) \cdot m - (1 \otimes \alpha) \cdot (s \otimes 1) \cdot m = (s\alpha \otimes 1) \cdot m - (1 \otimes \alpha) \cdot (1 \otimes s) \cdot m \\ &= (s\alpha \otimes 1 - 1 \otimes s\alpha) \cdot m = (s\alpha) \cdot m.\end{aligned}$$

Hence, this left L-module structure on M^S is a left U-module structure.

(2) By definition, $\mathrm{Hom}_{S^e}(S, M)$ is endowed with the left U-module structure such that (3-9) is an isomorphism in $\mathrm{Mod}(U)$. Let $\varphi \in \mathrm{Hom}_{S^e}(S, M)$, $\alpha \in L$ and $s \in S$. Then,

$$\begin{aligned}(\alpha \cdot \varphi)(s) &= (1 \otimes s) \cdot (\alpha \cdot \varphi(1)) = ((1 \otimes s)(\alpha \otimes 1 - 1 \otimes \alpha)) \cdot \varphi(1) \\ &= (\alpha \otimes s - 1 \otimes s\alpha - 1 \otimes \alpha(s)) \cdot \varphi(1) \\ &= ((\alpha \otimes 1 - 1 \otimes \alpha)(1 \otimes s) - 1 \otimes \alpha(s)) \cdot \varphi(1) \\ &= \alpha \cdot (1 \otimes s) \cdot \varphi(1) - (1 \otimes \alpha(s)) \cdot \varphi(1) = \alpha \cdot \varphi(s) - \varphi(\alpha(s)).\end{aligned}$$

□

Thus, the assignment $M \mapsto M^S$ defines a functor

$$\begin{aligned} G : \mathrm{Mod}(U^e) &\to \mathrm{Mod}(U), \\ M &\mapsto M^S. \end{aligned} \tag{3-11}$$

3.3. ***The functor $F = U \otimes_S - : \mathbf{Mod}(U) \to \mathbf{Mod}(U^e)$.*** Let $N \in \mathrm{Mod}(U)$. In view of [Huebschmann 1999, (2.4)], $U_U \otimes_S N$ is a right U-module such that, for all $u \in U,\ n \in N,\ s \in S$ and $\alpha \in L$,

$$(u \otimes n) \cdot s = u \otimes sn = us \otimes n \quad \text{and} \quad (u \otimes n) \cdot \alpha = u\alpha \otimes n - u \otimes \alpha \cdot n.$$

Besides, $U \otimes_S N$ is a left U-module such that, for all $u, u' \in U$ and $n \in N$,

$$u' \cdot (u \otimes n) = u'u \otimes n.$$

Therefore, $U \otimes_S N$ is a U-bimodule, and hence a left U^e-module. These considerations define a functor,

$$\begin{aligned} F : \mathrm{Mod}(U) &\to \mathrm{Mod}(U^e), \\ N &\mapsto U \otimes_S N. \end{aligned} \tag{3-12}$$

3.4. ***The adjunction between F and G.***

Proposition 3.4.1. *The functors $F = U \otimes_S -$ and $G = \mathrm{Hom}_{S^e}(S, -)$ introduced in Section 3.2 and Section 3.3 form an adjoint pair*,

$$\mathrm{Mod}\,U \underset{G}{\overset{F}{\rightleftarrows}} \mathrm{Mod}\,U^e$$

In particular, *there is a functorial isomorphism*, *for all $M \in \mathrm{Mod}(U^e)$ and $N \in \mathrm{Mod}(U)$*,

$$\mathrm{Hom}_U(N, G(M)) \xrightarrow{\sim} \mathrm{Hom}_{U^e}(F(N), M).$$

Proof. Given $f \in \mathrm{Hom}_U(N, G(M))$, denote by $\Phi(f)$ the well-defined mapping

$$\begin{aligned} U \otimes_S N &\to M, \\ u \otimes n &\mapsto (u \otimes 1) \cdot f(n). \end{aligned}$$

Consider $F(N)$ $(= U \otimes_S N)$ as a U-bimodule. Then, for all $u, u' \in U,\ n \in N,\ s \in S$ and $\alpha \in L$,

$$\begin{aligned} \Phi(f)(u' \cdot (u \otimes n)) = \Phi(f)(u'u \otimes n) &= (u'u \otimes 1) \cdot f(n) \\ &= (u' \otimes 1) \cdot \Phi(f)(u \otimes n), \end{aligned}$$

$$\begin{aligned}\Phi(f)((u\otimes n)\cdot s) &= \Phi(f)(u\otimes s\cdot n) = (u\otimes 1)\cdot f(s\cdot n)\\ &= (u\otimes 1)\cdot((1\otimes s)\cdot f(n)) = ((1\otimes s)\cdot(u\otimes 1))\cdot f(n)\\ &= (1\otimes s)\cdot \Phi(f)(u\otimes n) = (\Phi(f)(u\otimes n))\cdot s,\end{aligned}$$

$$\begin{aligned}\Phi(f)((u\otimes n)\cdot \alpha) &= \Phi(f)(u\alpha\otimes n - u\otimes \alpha\cdot n)\\ &= (u\alpha\otimes 1)\cdot f(n) - (u\otimes 1)\cdot f(\alpha\cdot n)\\ &= (u\alpha\otimes 1)\cdot f(n) - (u\otimes 1)\cdot(\alpha\otimes 1 - 1\otimes\alpha)\cdot f(n)\\ &= (u\otimes\alpha)\cdot f(n) = (1\otimes\alpha)\cdot\Phi(f)(u\otimes n)\\ &= (\Phi(f)(u\otimes n))\cdot\alpha.\end{aligned}$$

In other words,

$$\Phi(f)\in \mathrm{Hom}_{U^e}(F(N), M).$$

Given $g\in \mathrm{Hom}_{U^e}(F(N), M)$, then, for all $n\in N$ and $s\in S$,

$$(s\otimes 1 - 1\otimes s)\cdot g(1\otimes n) = g(s\otimes_S n - 1\otimes_S s\cdot n) = 0\,;$$

hence, denote by $\Psi(g)$ the well-defined mapping

$$\begin{aligned} N &\to M^S,\\ n &\mapsto g(1\otimes n).\end{aligned}$$

Therefore, for all $n\in N,\ s\in S$ and $\alpha\in L$,

$$\begin{aligned}\Psi(g)(s\cdot n) &= g(1\otimes s\cdot n) = g(s\otimes n) = g((s\otimes 1)\cdot(1\otimes n))\\ &= (s\otimes 1)\cdot g(1\otimes n) = (s\otimes 1)\cdot\Psi(g)(n),\end{aligned}$$

$$\begin{aligned}\Psi(g)(\alpha\cdot n) &= g(1\otimes\alpha\cdot n) = g(\alpha\otimes n - (1\otimes\alpha)\cdot(1\otimes n))\\ &= (\alpha\otimes 1)\cdot g(1\otimes n) - (1\otimes\alpha)\cdot g(1\otimes n) = \alpha\cdot\Psi(g)(n);\end{aligned}$$

in other words,

$$\Psi(g)\in \mathrm{Hom}_U(N, G(M)).$$

By construction, Ψ and Φ are inverse to each other. □

3.5. ***Compatible left $S \rtimes L$-modules.*** As explained in Section 1, the main results of this article are expressed in terms of the action of L on $\mathrm{Ext}^\bullet_{S^e}(S, S^e)$ by Lie derivatives and will be presented in Section 4. Although this action does not define a U-module structure on $\mathrm{Ext}^\bullet_{S^e}(S, S^e)$, it satisfies some compatibility with the S-module structure. The actions of L satisfying such a compatibility have specific properties that are used in the rest of the article and which are summarised below.

Define a *compatible* left $S \rtimes L$-module as a left $S \rtimes L$-module N such that, for all $n \in N$, $\alpha \in L$ and $s \in S$, the elements $s\alpha \in L$ and $\alpha(s) \in S$ satisfy

$$(s\alpha) \cdot n = s \cdot (\alpha \cdot n) - \alpha(s) \cdot n. \tag{3-13}$$

Note that a left $S \rtimes L$-module is both compatible and a left U-module if and only if L acts trivially, that is, by the zero action.

The two following lemmas present the properties of compatible left $S \rtimes L$-modules used in the rest of the article.

Lemma 3.5.1. *Let M be a right U-module. Let N be a compatible left $S \rtimes L$-module. Then*:

(1) *The right $S \rtimes L$-module $N^\vee = \mathrm{Hom}_S(N, S)$ is a right U-module.*

(2) *The left $S \rtimes L$-module $\mathrm{Hom}_S(N^\vee, M)$ is a left U-module.*

(3) *The left $S \rtimes L$-module $M \otimes_S N$ is a left U-module.*

(4) *The following canonical mapping is a morphism of left U-modules*:

$$\begin{aligned} \theta : M \otimes_S N &\to \mathrm{Hom}_S(N^\vee, M), \\ m \otimes n &\mapsto (\theta_{m\otimes n} : \varphi \mapsto m \cdot \varphi(n)). \end{aligned}$$

Proof. (1) Given $\varphi \in N^\vee$, $s \in S$ and $\alpha \in L$, then

$$\varphi \cdot (s\alpha) = (\varphi \cdot s) \cdot \alpha.$$

Indeed, for all $n \in N$,

$$\begin{aligned} (\varphi\cdot(s\alpha))(n) &= -(s\alpha)(\varphi(n)) + \varphi((s\alpha)\cdot n) \\ &= -s\,\alpha(\varphi(n)) + \varphi(s\cdot(\alpha\cdot n) - \alpha(s)\cdot n) \\ &= -s\,\alpha(\varphi(n)) + s\,\varphi(\alpha\cdot n) - \alpha(s)\varphi(n) \\ &= ((\varphi\cdot\alpha)\cdot s)(n) - (\varphi\cdot\alpha(s))(n) \\ &= ((\varphi\cdot s)\cdot\alpha)(n). \end{aligned}$$

(2) This is precisely [Huebschmann 1999, (2.3)].

(3) The $S \rtimes L$-module structure of $M \otimes_S N$ is described in (3-8). Given $m \in M$, $n \in N$, $s \in S$ and $\alpha \in L$, then

$$\begin{aligned} (s\alpha) \cdot (m \otimes n) &= -m \cdot (s\alpha) \otimes n + m \otimes (s\alpha) \cdot n \\ &= -(m \cdot \alpha) \cdot s \otimes n + m \cdot \alpha(s) \otimes n + m \otimes s \cdot (\alpha \cdot n) - m \otimes \alpha(s) \cdot n \\ &= s \cdot (\alpha \cdot (m \otimes n)). \end{aligned}$$

(4) It suffices to prove that the given mapping is L-linear. Let $m \in M,\ n \in N,\ \alpha \in L$ and $\varphi \in \mathrm{Hom}_S(N, S)$. Then,

$$\begin{aligned}(\alpha \cdot \theta_{m\otimes n})(\varphi) &= -\theta_{m\otimes n}(\varphi)\cdot\alpha + \theta_{m\otimes n}(\varphi\cdot\alpha) = -(m\cdot\varphi(n))\cdot\alpha + m\cdot(\varphi\cdot\alpha)(n)\\ &= -((m\cdot\alpha)\cdot\varphi(n) - m\cdot\alpha(\varphi(n))) + m\cdot(-\alpha(\varphi(n)) + \varphi(\alpha\cdot n))\\ &= -(m\cdot\alpha)\cdot\varphi(n) + m\cdot\varphi(\alpha\cdot n) = \theta_{\alpha\cdot(m\otimes n)}(\varphi)\,;\end{aligned}$$

thus, $\alpha \cdot \theta_{m\otimes n} = \theta_{\alpha\cdot(m\otimes n)}$. □

Any left $S \rtimes L$-module N may be considered as a symmetric S-bimodule, or equivalently a right S^e-module, such that, for all $n \in N$ and $s, s' \in S$,

$$n \cdot (s \otimes s') = (ss') \cdot n.$$

Accordingly, $N \otimes_{S^e} U^e$ is a right U^e-module in a natural way.

Lemma 3.5.2. *Let N be a compatible left $S \rtimes L$-module.*

(1) *The right U^e-module $N \otimes_{S^e} U^e$ is actually a U-U^e-bimodule such that for all $n \in N,\ u, v \in U$ and $\alpha \in L$,*

$$\alpha\cdot(n\otimes(u\otimes v)) = \alpha\cdot n\otimes(u\otimes v) + n\otimes((\alpha\otimes 1 - 1\otimes\alpha)\cdot(u\otimes v)).$$

(2) *Let M be a right U-module. Then, there exists an isomorphism of left U^e-modules*

$$F(M\otimes_S N) \to M\otimes_U (N\otimes_{S^e} U^e),$$
$$v\otimes(m\otimes n) \mapsto m\otimes(n\otimes(1\otimes v)).$$

Proof. (1) Following part (3) of Lemma 3.5.1, there is a left U-module structure on $U \otimes_S N$ such that, for all $\alpha \in L,\ v \in U$ and $n \in N$,

$$\alpha\cdot(v\otimes n) = -v\alpha\otimes n + v\otimes\alpha\cdot n.$$

Therefore, there is a left U-module structure on $(U \otimes_S N)\otimes_S U$ (see (3-7)) such that, for all $\alpha \in L,\ n \in N$ and $u, v \in U$,

$$\begin{aligned}\alpha\cdot((v\otimes n)\otimes u) &= \alpha\cdot(v\otimes n)\otimes u + (v\otimes n)\otimes\alpha u\\ &= -(v\alpha\otimes n)\otimes u + (v\otimes\alpha\cdot n)\otimes u + (v\otimes n)\otimes\alpha u.\end{aligned}$$

Under the canonical identification

$$N\otimes_{S^e} U^e \to (U\otimes_S N)\otimes_S U,$$
$$n\otimes(u\otimes v) \mapsto (v\otimes n)\otimes u,$$

$N\otimes_{S^e} U^e$ inherits a left U-module structure which is the one claimed in the statement.

Now, $N \otimes_{S^e} U^e$ inherits a right U^e-module structure from U^e. This structure is compatible with the left U-module structure discussed previously so as to yield a left $U \otimes (U^e)^{\mathrm{op}}$-module structure.

(2) Due to (1), there is a right U^e-module structure on $M \otimes_U (N \otimes_{S^e} U^e)$. It is considered here as a left U^e-module structure such that, for all $u, v, u', v' \in U$, $m \in M$ and $n \in N$,

$$(u' \otimes v') \cdot (m \otimes (n \otimes (u \otimes v))) = m \otimes (n \otimes (uv' \otimes u'v)). \tag{3-14}$$

For ease of reading, note that in $F(M \otimes_S N)$,

$$\begin{aligned}(u \otimes 1) \cdot (v \otimes m \otimes n) &= uv \otimes m \otimes n\\ (1 \otimes \alpha) \cdot (v \otimes m \otimes n) &= v\alpha \otimes m \otimes n + v \otimes m \cdot \alpha \otimes n - v \otimes m \otimes \alpha \cdot n,\end{aligned} \tag{3-15}$$

and, in $M \otimes_U (N \otimes_{S^e} U^e)$,

$$m \cdot \alpha \otimes n \otimes u \otimes v = m \otimes \alpha \cdot n \otimes u \otimes v + m \otimes n \otimes \alpha u \otimes v - m \otimes n \otimes u \otimes v\alpha. \tag{3-16}$$

The R-linear mapping from $U \otimes M \otimes N$ to $M \otimes_U (N \otimes_{S^e} U^e)$ given by

$$v \otimes m \otimes n \mapsto m \otimes (n \otimes (1 \otimes v))$$

induces a morphism of S-modules from $U \otimes_S (M \otimes_S N)$ to $M \otimes_U (N \otimes_{S^e} U^e)$ such as in the statement of the lemma. Denote it by Ψ':

$$\Psi' : U \otimes_S (M \otimes_S N) \to M \otimes_U (N \otimes_{S^e} U^e).$$

This is a morphism of left U^e-modules. Indeed, for all $u, v \in U$, $m \in M$, $n \in N$ and $\alpha \in L$,

$$\begin{aligned}\Psi'((u\otimes 1)\cdot(v\otimes m\otimes n)) &= \Psi'(uv\otimes m\otimes n) = m\otimes n\otimes 1\otimes uv\\ &\underset{(3\text{-}14)}{=} (u\otimes 1)\cdot\Psi'(v\otimes m\otimes n),\end{aligned}$$

$$\begin{aligned}\Psi'((1\otimes\alpha)\cdot(v\otimes m\otimes n)) &= \Psi'(v\alpha\otimes m\otimes n + v\otimes m\cdot\alpha\otimes n - v\otimes m\otimes\alpha\cdot n)\\ &= m\otimes n\otimes 1\otimes v\alpha + m\cdot\alpha\otimes n\otimes 1\otimes v - m\otimes\alpha\cdot n\otimes 1\otimes v\\ &\underset{(3\text{-}16)}{=} m\otimes n\otimes\alpha\otimes v\\ &\underset{(3\text{-}14)}{=} (1\otimes\alpha)\cdot\Psi'(v\otimes m\otimes n).\end{aligned}$$

Consider the following morphism of S-modules:

$$\begin{aligned}\phi : M \otimes_S (N \otimes_{S^e} U^e) &\to F(M \otimes_S N),\\ m \otimes (n \otimes (u \otimes v)) &\mapsto (1 \otimes u) \cdot (v \otimes m \otimes n).\end{aligned}$$

Given $m \in M$, $n \in N$, $u, v \in U$ and $\alpha \in L$, then the image under ϕ of the term

$$m \otimes \alpha \cdot n \otimes u \otimes v + m \otimes n \otimes \alpha u \otimes v - m \otimes n \otimes u \otimes v\alpha$$

is equal to

$$(1\otimes u)\cdot(v\otimes m\otimes\alpha\cdot n)+(1\otimes\alpha u)\cdot(v\otimes m\otimes n)-(1\otimes u)\cdot(v\alpha\otimes m\otimes n),$$

which is equal to

$$(1\otimes u)\cdot(v\otimes m\otimes\alpha\cdot n)+(1\otimes u)\cdot(1\otimes\alpha)\cdot(v\otimes m\otimes n)-(1\otimes u)\cdot(v\alpha\otimes m\otimes n).$$

In view of (3-15), this is equal to

$$(1\otimes u)\cdot(v\otimes m\cdot\alpha\otimes n)=\phi(m\cdot\alpha\otimes(n\otimes(u\otimes v))).$$

Thus, ϕ induces a morphism of S-modules

$$\begin{aligned}\Phi' : M\otimes_U (N\otimes_{S^e} U^e) &\to F(M\otimes_S N),\\ m\otimes(n\otimes(u\otimes v)) &\mapsto (1\otimes u)\cdot(v\otimes m\otimes n).\end{aligned}$$

It appears that Φ' is left and right inverse for Ψ'. Indeed,
- $\Phi'\circ\Psi'=\mathrm{Id}_{F(M\otimes_S N)}$, and
- for all $u, v\in U,\ m\in M$ and $n\in N$,

$$\begin{aligned}\Psi'\circ\Phi'(m\otimes n\otimes u\otimes v) &= \Psi'((1\otimes u)\cdot(v\otimes m\otimes n))\\ &= (1\otimes u)\cdot\Psi'(v\otimes m\otimes n) \qquad (\Psi' \text{ is } U^e\text{-linear})\\ &= (1\otimes u)\cdot(m\otimes n\otimes 1\otimes v)\\ &= m\otimes n\otimes u\otimes v.\end{aligned}$$ □

3.6. *Invertible U-bimodules.* The following result is used in Section 5 in order to prove that $\mathrm{Ext}^i_{U^e}(U, U^e)$ is invertible as a U-bimodule, under suitable conditions.

Proposition 3.6.1. *Let R be a commutative ring. Let (S, L) be a Lie–Rinehart algebra over R. Denote by U its enveloping algebra. Let N be a left U-module. Assume that N is invertible as an S-module. Then $F(N)$ is invertible as a U-bimodule.*

This subsection is devoted to the proof of this proposition. Given a left U-module N, then $F(N)=U\otimes_S N$ as left U-modules. Hence, there is a functorial isomorphism

(3-17) $$\theta : \mathrm{Hom}_S(N, U)\to\mathrm{Hom}_U(F(N), U).$$

Note:

- $\mathrm{Hom}_S(N, U)$ is a left U-module (see (3-4)), and it inherits a right U-module structure from U_U; by construction, these two structures form a U-bimodule structure.
- $\mathrm{Hom}_U(F(N), U)$ is a U-bimodule because so are $F(N)$ and U.

- $N \otimes_S \mathrm{Hom}_S(N, U)$ is a left U-module (see (3-7)), and it inherits a right U-module structure from U_U; by construction, these two structures form a U-bimodule structure.

Lemma 3.6.2. *Let N be a left U-module. Then,*

(1) $\theta : \mathrm{Hom}_S(N, U) \to \mathrm{Hom}_U(F(N), U)$ *is an isomorphism in* $\mathrm{Mod}(U^e)$,

(2) *the mapping*

$$\Phi : N \otimes_S \mathrm{Hom}_S(N, U) \to F(N) \otimes_U \mathrm{Hom}_U(F(N), U),$$
$$n \otimes f \mapsto (1 \otimes n) \otimes \theta(f)$$

is an isomorphism in $\mathrm{Mod}(U^e)$, *and*

(3) *the diagram*

$$\begin{array}{ccc} N \otimes_S \mathrm{Hom}_S(N,U) & \longrightarrow & U \\ \Phi \downarrow & & \| \\ F(N) \otimes_U \mathrm{Hom}(F(N),U) & \longrightarrow & U \end{array}$$

with horizontal arrows given by evaluation, is commutative.

Proof. (1) By definition, θ is a morphism of right U-modules. It is also a morphism of left U-modules because, for all $n \in N$, $f \in \mathrm{Hom}_S(N, U)$, $u \in U$ and $\alpha \in L$,

$$\begin{aligned} \theta(\alpha \cdot f)(u \otimes n) &= u(\alpha \cdot f)(n) \\ &= u(\alpha f(n) - f(\alpha \cdot n)) = \theta(f)(u\alpha \otimes n - u \otimes \alpha \cdot n) \\ &= \theta(f)((u \otimes n) \cdot \alpha) = (\alpha \cdot \theta(f))(u \otimes n). \end{aligned}$$

(2) By definition, Φ is a morphism of right U-modules. It is also a morphism of left U-modules because, for all $n \in N$, $f \in \mathrm{Hom}_S(N, U)$ and $\alpha \in L$,

$$\begin{aligned} \Phi(\alpha \cdot (n \otimes f)) &= \Phi(\alpha \cdot n \otimes f + n \otimes \alpha \cdot f) \\ &= (1 \otimes \alpha \cdot n) \otimes \theta(f) + (1 \otimes n) \otimes \underbrace{\theta(\alpha \cdot f)}_{=\alpha \cdot \theta(f)} \\ &= (1 \otimes \alpha \cdot n) \otimes \theta(f) + \underbrace{(1 \otimes n) \cdot \alpha}_{=\alpha \otimes n - 1 \otimes \alpha \cdot n} \otimes \theta(f) \\ &= (\alpha \otimes n) \otimes \theta(f) \\ &= \alpha \cdot \Phi(n \otimes f). \end{aligned}$$

In order to prove that Φ is bijective, consider the linear mapping

$$\psi : F(N) \otimes_S \mathrm{Hom}_U(F(N), U) \to N \otimes_S \mathrm{Hom}_S(N, U),$$
$$(u \otimes n) \otimes g \mapsto u \cdot (n \otimes \theta^{-1}(g)).$$

Note that, for all $u \in U,\ \alpha \in L,\ n \in N$ and $g \in \mathrm{Hom}_U(F(N), U)$,

$$\begin{aligned}\psi((u \otimes n) \cdot \alpha \otimes g) &= \psi((u\alpha \otimes n) \otimes g - (u \otimes \alpha \cdot n) \otimes g) \\ &= u\alpha \cdot (n \otimes \theta^{-1}(g)) - u \cdot (\alpha \cdot n \otimes \theta^{-1}(g)) \\ &= u \cdot (\alpha \cdot n \otimes \theta^{-1}(g) + n \otimes \alpha \cdot \theta^{-1}(g)) - u \cdot (\alpha \cdot n \otimes \theta^{-1}(g)) \\ &= u \cdot (n \otimes \theta^{-1}(\alpha \cdot g)) \qquad \text{(see part (1))} \\ &= \psi((u \otimes n) \otimes \alpha \cdot g).\end{aligned}$$

Hence, ψ induces a linear mapping,

$$\begin{aligned}\Psi : F(N) \otimes_U \mathrm{Hom}_U(F(N), U) &\to N \otimes_S \mathrm{Hom}_S(N, U), \\ (u \otimes n) \otimes g &\mapsto u \cdot (n \otimes \theta^{-1}(g)).\end{aligned}$$

Now, by definition of Φ and Ψ,

$$\Psi \circ \Phi = \mathrm{Id}_{N \otimes_S \mathrm{Hom}_S(N,U)}.$$

Since

- Ψ is a morphism of left U-modules by construction;
- as a left U-module, $F(N) \otimes_U \mathrm{Hom}_U(F(N), U)$ is generated by the image of $(1 \otimes N) \otimes \mathrm{Hom}_U(F(N), U)$; and
- for all $n \in N$ and $g \in \mathrm{Hom}_U(F(N), U)$,

$$\Phi \circ \Psi((1 \otimes n) \otimes g) = (1 \otimes n) \otimes g,$$

the following holds:

$$\Phi \circ \Psi = \mathrm{Id}_{F(N) \otimes_U \mathrm{Hom}_U(F(N),U)}.$$

Altogether, these considerations show that Φ is an isomorphism in $\mathrm{Mod}(U^e)$.

(3) The diagram is commutative by definition of Φ. □

Like in the previous lemma, for all $N \in \mathrm{Mod}(U)$, $\mathrm{Hom}_S(N, U)$ is a U-bimodule, and hence $\mathrm{Hom}_S(N, U) \otimes_S N$ is a U-bimodule by means of (3-7) and the right U-module structure of U.

Lemma 3.6.3. *Let N be a left U-module. Then,*

(1) *the mapping*

$$\begin{aligned}\Phi' : \mathrm{Hom}_S(N, U) \otimes_S N &\to \mathrm{Hom}_U(F(N), U) \otimes_U F(N), \\ f \otimes n &\mapsto \theta(f) \otimes (1 \otimes n)\end{aligned}$$

is an isomorphism in $\mathrm{Mod}(U^e)$; *and*

(2) *the diagram*

$$\begin{array}{ccc} \mathrm{Hom}_S(N,U)\otimes_S N & \longrightarrow & U \\ {\scriptstyle\Phi'}\downarrow & & \| \\ \mathrm{Hom}_U(F(N),U)\otimes_U F(N) & \longrightarrow & U \end{array}$$

with horizontal arrows given by evaluation, is commutative.

Proof. (1) First, since $F(N) = U \otimes_S N$ in $\mathrm{Mod}(U^e)$, then

$$\mathrm{Hom}_U(F(N), U) \otimes_U F(N) \cong \mathrm{Hom}_U(F(N), U) \otimes_S N$$

as left U-modules. Under this identification, Φ' expresses as

$$\Phi' : f \otimes n \mapsto \theta(f) \otimes n.$$

Therefore, Φ' is bijective because so is θ.

Next, Φ' is a morphism of left U-modules because so is θ. And it is a morphism of right U-modules because it is a morphism of right S-modules, and because, for all $f \in \mathrm{Hom}_S(N, U)$, $n \in N$ and $\alpha \in L$,

$$\begin{aligned} \Phi'((f \otimes n) \cdot \alpha) &= \Phi'(f \cdot \alpha \otimes n - f \otimes \alpha \cdot n) \\ &= \underbrace{\theta(f \cdot \alpha)}_{=\theta(f)\cdot\alpha} \otimes (1 \otimes n) - \theta(f) \otimes (1 \otimes \alpha \cdot n) \\ &= \theta(f) \otimes \underbrace{\alpha \cdot (1 \otimes n)}_{=\alpha\otimes n} - \theta(f) \otimes (1 \otimes \alpha \cdot n) \\ &= \theta(f) \otimes ((1 \otimes n) \cdot \alpha) = (\theta(f) \otimes (1 \otimes n)) \cdot \alpha \\ &= \Phi'(f \otimes n) \cdot \alpha. \end{aligned}$$

This proves (1).

(2) The diagram commutes by definition of Φ'. □

It is now possible to prove the result announced at the beginning of the subsection.

Proof of Proposition 3.6.1. Since N is invertible as an S-module, then the following evaluation mappings are bijective

$$N \otimes_S \mathrm{Hom}_S(N, U) \to U \quad \text{and} \quad \mathrm{Hom}_S(N, U) \otimes_S N \to U.$$

According to Lemmas 3.6.2 and 3.6.3, the following evaluation mappings are isomorphisms of U-bimodules

$$F(N) \otimes_U \mathrm{Hom}_U(F(N), U) \to U \quad \text{and} \quad \mathrm{Hom}_U(F(N), U) \otimes_U F(N) \to U.$$

Thus, $F(N)$ is invertible as a U-bimodule. □

4. The action of L on the inverse dualising bimodule of S

This section introduces an action of L on $\mathrm{Ext}^\bullet_{S^e}(S, S^e)$ by means of Lie derivatives, which is used to describe $\mathrm{Ext}^\bullet_{U^e}(U, U^e)$ in the next section. When S is projective in $\mathrm{Mod}(R)$, then $\mathrm{Ext}^\bullet_{S^e}(S, -)$ is the Hochschild cohomology $H^\bullet(S; -)$; in this setting, the Lie derivatives on $H^\bullet(S; S)$ and $H_\bullet(S; S)$ are defined in [Rinehart 1963, Section 9] and have a well-known expression in terms of the Hochschild resolution of S. For the needs of the article, the definition is translated to arbitrary coefficients in terms of any projective resolution of S in $\mathrm{Mod}(S^e)$.

Hence, Section 4.1 introduces preliminary material, Section 4.2 deals with derivations on projective resolutions of S in $\mathrm{Mod}(S^e)$, Section 4.3 defines the Lie derivatives, Section 4.4 presents the action of L on $\mathrm{Ext}^\bullet_{S^e}(S, S^e)$, and Section 4.5 discusses particular situations.

For the section, a projective resolution of S in $\mathrm{Mod}(S^e)$ is considered;

$$(P^\bullet, d) \to S.$$

Denote S by P^1 and the augmentation mapping $P^0 \to S$ by d^0. For all $M \in \mathrm{Mod}(S^e)$ and $s \in S$, denote by λ_s and ρ_s the multiplication mappings

$$\lambda_s : M \to M, \quad m \mapsto (s \otimes 1) \cdot m$$

and

$$\rho_s : M \to M, \quad m \mapsto (1 \otimes s) \cdot m.$$

4.1. ***Data on the projective resolution.*** For all $s \in S$, the mappings λ_s, ρ_s on $P^\bullet$ are morphisms of complexes of left S^e-modules and induce the same mapping

$$\begin{aligned} S &\to S, \\ t &\mapsto st \end{aligned}$$

in cohomology. Hence, there exists a morphism of graded left S^e-modules,

$$k_s : P^\bullet \to P^\bullet[-1], \tag{4-1}$$

such that

$$\lambda_s - \rho_s = k_s \circ d + d \circ k_s. \tag{4-2}$$

Lemma 4.1.1. *Let $\partial : S \to S$ be an R-linear derivation. Let $\psi : P^\bullet \to P^\bullet$ be a morphism of complexes of R-modules such that*

- *$H^0(\psi) : S \to S$ is the zero mapping*;
- *there exists a morphism of graded left S^e-modules*,

$$k : P^\bullet \to P^\bullet[-1],$$

such that, for all $p \in P^\bullet$ and $s, t \in S$,

$$\psi((s \otimes t) \cdot p) = (s \otimes t) \cdot \psi(p) - (1 \otimes \partial)(s \otimes t) \cdot (k \circ d + d \circ k)(p). \tag{4-3}$$

Then, there exists a morphism of graded R-modules,

$$h : P^\bullet \to P^\bullet[-1],$$

such that

- $\psi = h \circ d + d \circ h$; *and*
- *for all $s, t \in S$ and $p \in P^\bullet$,*

$$h((s \otimes t) \cdot p) = (s \otimes t) \cdot h(p) - (1 \otimes \partial)(s \otimes t) \cdot k(p).$$

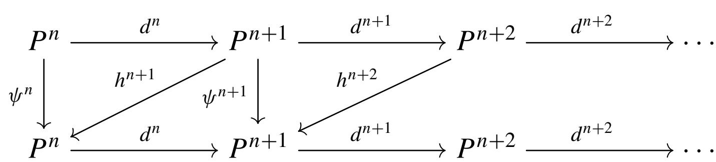
Proof. The proof is an induction on $n \leqslant 1$, taking $h^1 : S \to P^0$ equal to 0. Let $n \leqslant 0$ and assume that there exist linear mappings, for all j such that $n + 1 \leqslant j \leqslant 1$,

$$h^j : P^j \to P^{j-1}$$

such that, for all j satisfying $n + 1 \leqslant j \leqslant 0$, $p \in P^j$ and $s, t \in S$,

$$\begin{aligned} \psi^j &= h^{j+1} \circ d^j + d^{j-1} \circ h^j \\ h^j((s \otimes t) \cdot p) &= (s \otimes t) \cdot h^j(p) - (1 \otimes \partial)(s \otimes t) \cdot k^j(p). \end{aligned} \tag{4-4}$$

This is illustrated in the following diagram:



Let

$$((p_i, \varphi^i))_{i \in I}$$

be a coordinate system of the projective left S^e-module P^n. That is, let $p_i \in P^n$ and $\varphi^i \in \mathrm{Hom}_{S^e}(P^n, S^e)$ for all $i \in I$ such that, for all $p \in P^n$,

$$p = \sum_{i \in I} \varphi^i(p) \cdot p_i,$$

where $\{i \in I \mid \varphi^i(p) \neq 0\}$ is finite. Since $\psi : P^\bullet \to P^\bullet$ is a morphism of complexes, it follows from (4-4) that, for all $i \in I$, there exists $p_i' \in P^{n-1}$ such that

$$\psi^n(p_i) = d^{n-1}(p_i') + h^{n+1} \circ d^n(p_i). \tag{4-5}$$

Denote by h^n the linear mapping from P^n to P^{n-1} such that, for all $p \in P^n$,

$$h^n(p) = \sum_{i \in I} \varphi^i(p) \cdot p_i' - (1 \otimes \partial)(\varphi^i(p)) \cdot k^n(p_i).$$

Then, for all $p \in P^n$ and $s, t \in S$,

$$\begin{aligned}
&h^n((s\otimes t)\cdot p)\\
&=\sum_{i\in I}(s\otimes t)\cdot\varphi^i(p)\cdot p_i'-(s\otimes t)\cdot(1\otimes\partial)(\varphi^i(p))\cdot k^n(p_i)-(1\otimes\partial)(s\otimes t)\cdot\varphi^i(p)\cdot k^n(p_i)\\
&=(s\otimes t)\cdot h^n(p)-(1\otimes\partial)(s\otimes t)\cdot k^n\Big(\sum_{i\in I}\varphi^i(p)\cdot p_i\Big)\\
&=(s\otimes t)\cdot h^n(p)-(1\otimes\partial)(s\otimes t)\cdot k^n(p).
\end{aligned}$$

Moreover,

$$\psi^n = h^{n+1}\circ d^n + d^{n-1}\circ h^n.$$

Indeed, for all $p \in P^n$, $p = \sum_{i\in I} \varphi^i(p)\cdot p_i$, and hence

$$\begin{aligned}
&d^{n-1}\circ h^n(p)+h^{n+1}\circ d^n(p)\\
&= \sum_{i\in I}\varphi^i(p)\cdot d^{n-1}(p_i')-(1\otimes\partial)(\varphi^i(p))\cdot d^{n-1}\circ k^n(p_i)+h^{n+1}\Big(\sum_{i\in I}\varphi^i(p)\cdot d^n(p_i)\Big)\\
&\underset{(4\text{-}4)}{=}\sum_{i\in I}\varphi^i(p)\cdot d^{n-1}(p_i')-(1\otimes\partial)(\varphi^i(p))\cdot d^{n-1}\circ k^n(p_i)\\
&\qquad\qquad+\varphi^i(p)\cdot h^{n+1}\circ d^n(p_i)-(1\otimes\partial)(\varphi^i(p))\cdot k^{n+1}\circ d^n(p_i)\\
&\underset{(4\text{-}3)}{=}\sum_{i\in I}\varphi^i(p)\cdot d^{n-1}(p_i')+\varphi^i(p)\cdot h^{n+1}\circ d^n(p_i)+\psi^n(\varphi^i(p)\cdot p_i)-\varphi^i(p)\cdot\psi^n(p_i)\\
&\underset{(4\text{-}5)}{=}\sum_{i\in I}\psi^n(\varphi^i(p)\cdot p_i)=\psi^n(p).
\end{aligned}$$

□

4.2. ***Derivations on the projective resolution.*** Let $\partial : S \to S$ be an R-linear derivation. It defines an R-linear derivation on S^e denoted by ∂^e,

$$\begin{aligned}
\partial^e : S^e &\to S^e,\\
s\otimes t &\mapsto \partial(s)\otimes t + s\otimes\partial(t).
\end{aligned}$$

For every left S^e-module M, a *derivation* of M relative to ∂ is an R-linear mapping,

$$\partial_M : M \to M,$$

such that, for all $m \in M$ and $s, t \in S$,

$$\partial_M((s\otimes t)\cdot m) = \partial^e(s\otimes t)\cdot m + (s\otimes t)\cdot\partial_M(m).$$

A derivation of $P^\bullet$ relative to ∂ is a morphism of complexes of R-modules,

$$\partial^\bullet : P^\bullet \to P^\bullet,$$

such that $\partial^n : P^n \to P^n$ is a derivation relative to ∂ for all n, and such that

$H^0(\partial^\bullet) = \partial$. Note that a morphism of complexes of R-modules $\partial^\bullet : P^\bullet \to P^\bullet$ such that $H^0(\partial^\bullet) = \partial$ is a derivation relative to ∂ if and only if

$$\begin{cases} \partial^\bullet \circ \lambda_s = \lambda_{\partial(s)} + \lambda_s \circ \partial^\bullet, \\ \partial^\bullet \circ \rho_s = \rho_{\partial(s)} + \rho_s \circ \partial^\bullet. \end{cases} \tag{4-6}$$

Remark. For all derivations $\partial_1^\bullet, \partial_2^\bullet : P^\bullet \to P^\bullet$ relative to ∂, the difference

$$\partial_1^\bullet - \partial_2^\bullet : P^\bullet \to P^\bullet$$

is a null-homotopic morphism of complexes of left S^e-modules.

Lemma 4.2.1. *There exists a mapping, which need not be linear,*

$$\begin{aligned} \mathrm{Der}_R(S) &\to \mathrm{Hom}_R(P^\bullet, P^\bullet), \\ \partial &\mapsto \partial^\bullet \end{aligned} \tag{4-7}$$

such that:

(1) *For all $\partial \in \mathrm{Der}_R(S)$, the mapping $\partial^\bullet$ is a derivation relative to ∂.*

(2) *For all $\partial_1, \partial_2 \in \mathrm{Der}_R(S)$ and $r \in R$, there exist morphisms of graded left S^e-modules,*

$$\ell, \ell' : P^\bullet \to P^\bullet[-1],$$

such that

$$\begin{cases} [\partial_1, \partial_2]^\bullet - [\partial_1^\bullet, \partial_2^\bullet] &= \ell \circ d + d \circ \ell, \\ (\partial_1 + r\partial_2)^\bullet - (\partial_1^\bullet + r\partial_2^\bullet) &= \ell' \circ d + d \circ \ell'. \end{cases} \tag{4-8}$$

(3) *For all $s \in S$ and $\partial \in \mathrm{Der}_R(S)$, there exists a morphism of graded R-modules*

$$h : P^\bullet \to P^\bullet[-1],$$

such that

$$(s\partial)^\bullet - \lambda_s \circ \partial^\bullet = h \circ d + d \circ h \tag{4-9}$$

and, for all $p \in P^\bullet$ and $t_1, t_2 \in S$,

$$h((t_1 \otimes t_2) \cdot p) = (t_1 \otimes t_2) \cdot h(p) - (t_1 \otimes \partial(t_2)) \cdot k_s(p). \tag{4-10}$$

Recall that $k_s : P^\bullet \to P^\bullet[-1]$ is a morphism of graded left S^e-modules such that $\lambda_s - \rho_s = k_s \circ d + d \circ k_s$ (see (4-1) *and* (4-2)*).*

Proof. (1) Let $\partial \in \mathrm{Der}_R(S)$. For convenience, denote ∂ by $\partial^1 : S \to S$. The proof is an induction on $n \leqslant 1$. Let $n \leqslant 0$, and assume that a commutative diagram is given

$$\begin{array}{ccccccccccc} P^n & \xrightarrow{d^n} & P^{n+1} & \longrightarrow & \cdots & \longrightarrow & P^0 & \xrightarrow{d^0} & P^1 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \partial^{n+1}} & & & & \downarrow{\scriptstyle \partial^0} & & \downarrow{\scriptstyle \partial^1} & & \\ P^n & \xrightarrow{d^n} & P^{n+1} & \longrightarrow & \cdots & \longrightarrow & P^0 & \xrightarrow{d^1} & P^1 & \longrightarrow & 0 \end{array}$$

where $\partial^i : P^i \to P^i$ is a derivation relative to ∂ for all $i \in \{n+1, n+2, \cdots, 0\}$. Let

$$((p_i, \varphi^i))_{i\in I}$$

be a coordinate system of the projective left S^e-module P^n (see the proof in 4.1). Then, for all $i \in I$, there exists $p'_i \in P^n$ such that

$$\partial^{n+1} \circ d^n(p_i) = d^n(p'_i).$$

Denote by ∂^n the R-linear mapping from P^n to P^n such that, for all $p \in P^n$,

$$\partial^n(p) = \sum_{i\in I} \partial(\varphi^i(p)) \cdot p_i + \varphi^i(p) \cdot p'_i.$$

Then, for all $p \in P^n$,

$$\begin{aligned} d^n \circ \partial^n(p) &= \sum_{i\in I} \partial(\varphi^i(p)) \cdot d^n(p_i) + \varphi^i(p) \cdot d^n(p'_i) \\ &= \sum_{i\in I} \partial(\varphi^i(p)) \cdot d^n(p_i) + \varphi^i(p) \cdot \partial^{n+1} \circ d^n(p_i) \\ &= \partial^{n+1} \circ d^n\left(\sum_{i\in I} \varphi^i(p) \cdot p_i\right) \\ &= \partial^{n+1} \circ d^n(p). \end{aligned}$$

Thus,

$$d^n \circ \partial^n = \partial^{n+1} \circ d^n.$$

Moreover, ∂^n is a derivation of P^n relative to ∂ because ∂ is a derivation of S^e and $\varphi^i \in \mathrm{Hom}_{S^e}(P^n, S^e)$ for all $i \in I$.

(2) Note that $[\partial_1, \partial_2]^\bullet$ and $[\partial_1^\bullet, \partial_2^\bullet]$ (or, $(\partial_1 + r\partial_2)^\bullet$ and $\partial_1^\bullet + r\partial_2^\bullet$) are derivations of $P^\bullet$ relative to $[\partial_1, \partial_2]$ (or, to $\partial_1 + r\partial_2$, respectively). The conclusion therefore follows from the remark preceding Lemma 4.2.1.

(3) Denote by ψ the mapping $(s\partial)^\bullet - \lambda_s \circ \partial^\bullet$ given by

$$\begin{aligned} &P^\bullet \to P^\bullet, \\ &p \mapsto (s\partial)^\bullet(p) - (s \otimes 1) \cdot \partial^\bullet(p). \end{aligned}$$

Then, for all $p \in P^\bullet$ and $t \in S$,

$$\begin{aligned} &\psi((t\otimes 1)\cdot p) \\ &\quad = (s\partial)^\bullet((t\otimes 1)\cdot p) - (s\otimes 1)\cdot\partial^\bullet((t\otimes 1)\cdot p) \\ &\quad = (s\partial(t)\otimes 1)\cdot p + (t\otimes 1)\cdot(s\partial)^\bullet(p) - (s\otimes 1)\cdot(\partial(t)\otimes 1)\cdot p - (s\otimes 1)\cdot(t\otimes 1)\cdot\partial^\bullet(p) \\ &\quad = (t\otimes 1)\cdot\psi(p) \end{aligned}$$

and

$$
\begin{aligned}
&\psi((1\otimes t)\cdot p)\\
&\quad = (s\partial)^\bullet((1\otimes t)\cdot p)-(s\otimes 1)\cdot\partial^\bullet((1\otimes t)\cdot p)\\
&\quad = (1\otimes s\partial(t))\cdot p+(1\otimes t)\cdot(s\partial)^\bullet(p)-(s\otimes 1)\cdot(1\otimes\partial(t))\cdot p-(s\otimes 1)\cdot(1\otimes t)\cdot\partial^\bullet(p)\\
&\quad = (1\otimes t)\cdot\psi(p)+(1\otimes\partial(t))\cdot(\rho_s-\lambda_s)(p)\\
&\quad \underset{(4\text{-}2)}{=} (1\otimes t)\cdot\psi(p)-(1\otimes\partial(t))\cdot(k_s\circ d+d\circ k_s)(p).
\end{aligned}
$$

Hence, Lemma 4.1.1 may be applied, which yields (3). □

Remark. Using the remark preceding Lemma 4.2.1, it may be checked that, although the mapping $\mathrm{Der}_R(S)\to\mathrm{Hom}_R(P^\bullet, P^\bullet)$ of the lemma is not unique, two such mappings induce the same mapping from $\mathrm{Der}_R(S)$ to $H^0\mathrm{Hom}_R(P^\bullet, P^\bullet)$, which is R-linear.

When S is projective in $\mathrm{Mod}(R)$, it is possible to be more explicit on a possible mapping, $\partial\mapsto\partial^\bullet$. Indeed, the Hochschild complex $B(S)=S^{\otimes\bullet+2}$ is a projective resolution of S. For all $\partial\in\mathrm{Der}_R(S)$, define the following mapping:

$$
\begin{aligned}
L_\partial : B(S) &\to B(S),\\
(s_0|\cdots|s_{n+1}) &\mapsto \sum_{i=0}^{n+1}(s_0|\cdots|s_{i-1}|\partial(s_i)|\cdots|s_{i+1}|\cdots|s_n).
\end{aligned}
$$

This is a derivation of $B(S)$ relative to ∂. It is direct to check that the mapping

$$
\begin{aligned}
\mathrm{Der}_R(S) &\to \mathrm{Hom}_R(B(S), B(S)),\\
\partial &\mapsto L_\partial
\end{aligned}
$$

is a morphism of Lie algebras over R. Now, consider homotopy equivalences of complexes of S^e-modules,

$$
P^\bullet \underset{g}{\overset{f}{\rightleftarrows}} B(S)\,,
$$

and, for all $\partial\in\mathrm{Der}_R(S)$, define $\partial^\bullet$ as

$$
\partial^\bullet = g\circ L_\partial\circ f\,;
$$

this is a derivation relative to ∂ because so is L_∂ and because f and g are morphisms of resolutions of S in $\mathrm{Mod}(S^e)$. The following mapping satisfies the conclusion of the preceding lemma, it is moreover R-linear:

$$
\begin{aligned}
\mathrm{Der}_R(S) &\to \mathrm{Hom}_R(P^\bullet, P^\bullet),\\
\partial &\mapsto \partial^\bullet.
\end{aligned}
$$

4.3. ***Lie derivatives.*** Consider a mapping $\partial \mapsto \partial^\bullet$ such as in Lemma 4.2.1. Let M be an S-bimodule and $\partial : S \to S$ be an R-linear derivation. Let $\partial_M : M \to M$ be a derivation relative to ∂. Given $n \in \mathbb{N}$ and $\psi \in \mathrm{Hom}_{S^e}(P^{-n}, M)$, denote by $\mathcal{L}_\partial(\psi)$ the mapping

$$\mathcal{L}_\partial(\psi) = \partial_M \circ \psi - \psi \circ \partial^{-n}. \tag{4-11}$$

This is a morphism in $\mathrm{Mod}(S^e)$ because so is ψ and because ∂_M and ∂^{-n} are derivations relative to ∂; moreover, it is a cocycle (or a coboundary) as soon as ψ is, because $\partial^\bullet : P^\bullet \to P^\bullet$ is a morphism of complexes. Denote by $\mathcal{L}_\partial$ the resulting mapping in cohomology

$$\mathcal{L}_\partial : \mathrm{Ext}^\bullet_{S^e}(S, M) \to \mathrm{Ext}^\bullet_{S^e}(S, M)$$

such that for all $c \in \mathrm{Ext}^\bullet_{S^e}(S, M)$, say represented by a cocycle ψ, then $\mathcal{L}_\partial(c)$ is represented by the cocycle $\mathcal{L}_\partial(\psi)$. In the situations considered later in the article, there is no ambiguity on ∂_M, whence its omission in the notation.

Following similar considerations denote also by $\mathcal{L}_\partial$ the mapping

$$\mathcal{L}_\partial : \mathrm{Tor}^{S^e}_\bullet(S, M) \to \mathrm{Tor}^{S^e}_\bullet(S, M)$$

such that for all $\omega \in \mathrm{Tor}^{S^e}_\bullet(S, M)$, say represented by a cocycle $m \otimes p \in M \otimes_{S^e} P^\bullet$ with sum sign omitted, $\mathcal{L}_\partial(\omega)$ is represented by the cocycle

$$\mathcal{L}_\partial(m \otimes p) := m \otimes \partial^\bullet(p) + \partial_M(m) \otimes p.$$

When S is projective in $\mathrm{Mod}(R)$, these operations may be written explicitly in terms of the Hochschild resolution. When ψ is a Hochschild cocycle lying in $\mathrm{Hom}_R(S^{\otimes n}, M)$, the mapping $\mathcal{L}_\partial(\psi)$ is given by

$$(s_1|\cdots|s_n) \mapsto \partial_M(f(s_1|\cdots|s_n)) - \sum_{i=1}^{n} f(s_1|\cdots|\partial(s_i)|\cdots|s_n). \tag{4-12}$$

Likewise, the operation in Hochschild homology is induced by the following mapping at the level of Hochschild chains,

$$\begin{aligned} M \otimes S^{\otimes n} &\to M \otimes S^{\otimes n} \\ (m|s_1|\cdots|s_n) &\mapsto (\partial_M(m)|s_1|\cdots|s_n) + \sum_{i=1}^{n} (m|s_1|\cdots|\partial(s_i)|\cdots|s_n). \end{aligned}$$

The operator $\mathcal{L}_\partial$ is of course called the *Lie derivative* of ∂. When $M = S$ and S is projective in $\mathrm{Mod}(R)$, this is nothing else but the classical Lie derivative defined in [Rinehart 1963, Section 9]. In view of the remark following Lemma 4.2.1, these constructions depend only on ∂ and ∂_M and not on the choices of $P^\bullet$ and the mapping $\partial \mapsto \partial^\bullet$.

In the sequel these constructions are considered mainly in the following cases:

- $M = S$ and $\partial_M = \partial$.
- $M = S^e$ and $\partial_M = \partial^e$.
- $M = \mathrm{Ext}^n_{S^e}(S, S^e)$ ($n \in \mathbb{N}$) and $\partial_M = \mathcal{L}_\partial$, which makes sense according to the result below.

In the sequel the following construction is also used. Consider S-bimodules M, N. Let $m, n \in \mathbb{N}$. Let $\partial \in \mathrm{Der}_R(S)$ and let $\partial_M : M \to M$ and $\partial_N : N \to N$ be R-linear derivations relative to ∂. Then, for all $f \in \mathrm{Hom}_R(\mathrm{Ext}^m_{S^e}(S, M), \mathrm{Tor}^{S^e}_n(S, N))$, define $\mathcal{L}_\partial(f)$ as

$$\mathcal{L}_\partial \circ f - f \circ \mathcal{L}_\partial.$$

Recall that for all $M \in \mathrm{Mod}(S^e)$, the spaces $\mathrm{Ext}^\bullet_{S^e}(S, M)$ and $\mathrm{Tor}^{S^e}_\bullet(S, M)$ are left S-modules by means of $\lambda_s : M \to M$ for all $s \in S$; the corresponding multiplication by s on these (co)homology spaces is denoted by λ_s.

Lemma 4.3.1. *Let $M \in \mathrm{Mod}(S^e)$, $n \in \mathbb{N}$ and $s \in S$. Let $\partial, \partial' : S \to S$ be R-linear derivations. Let $\partial_M, \partial'_M : M \to M$ be R-linear derivations relative to ∂ and ∂', respectively. Then, the following hold in* $\mathrm{Ext}^\bullet_{S^e}(S, M)$:

(1) $\mathcal{L}_\partial \circ \lambda_s = \lambda_{\partial(s)} + \lambda_s \circ \mathcal{L}_\partial$.

(2) $\mathcal{L}_{[\partial,\partial']} = [\mathcal{L}_\partial, \mathcal{L}_{\partial'}]$.

(3) *Let $m \in \mathbb{N}$, let N be another S-bimodule and let $\partial_N : N \to N$ be a derivation relative to ∂. Consider the contraction mapping*

$$\begin{aligned}\mathrm{Tor}^{S^e}_m(S, M) &\to \mathrm{Hom}_R(\mathrm{Ext}^n_{S^e}(S, N), \mathrm{Tor}^{S^e}_{m-n}(S, M \otimes_S N)),\\ \omega &\mapsto (c \mapsto \iota_c(\omega)).\end{aligned}$$

If $m = n$, then it is $\mathcal{L}_\partial$-equivariant. When S is projective in $\mathrm{Mod}(R)$, *it is $\mathcal{L}_\partial$ equivariant for all $m, n \in \mathbb{N}$.*

(4) *If M is symmetric as an S-bimodule, $\mathcal{L}_{s\partial} = \lambda_s \circ \mathcal{L}_\partial$.*

(5) *When $M = S^e$ (and $\partial_M = \partial^e$), the following equality holds in* $\mathrm{Ext}^\bullet_{S^e}(S, M)$:

$$\mathcal{L}_{s\partial} = \lambda_s \circ \mathcal{L}_\partial - \lambda_{\partial(s)}.$$

Proof. (1) The equality is checked on cochains. Let $\psi \in \mathrm{Hom}_{S^e}(P^{-n}, M)$. Then,

$$\begin{aligned}\mathcal{L}_\partial \circ \lambda_s(\psi) &= \partial_M \circ \lambda_s \circ \psi - \lambda_s \circ \psi \circ \partial^\bullet\\ &= (\lambda_{\partial(s)} + \lambda_s \circ \partial_M) \circ \psi - \lambda_s \circ \psi \circ \partial^\bullet\\ &= (\lambda_{\partial(s)} + \lambda_s \circ \mathcal{L}_\partial)(\psi).\end{aligned}$$

(2) Note that $\mathcal{L}_{[\partial,\partial']}$ is defined with respect to $[\partial_M, \partial'_M]$, which is a derivation of M relative to $[\partial, \partial']$. Following Lemma 4.2.1, there exists a morphism of graded S^e-modules,

$$\ell : P^\bullet \to P^\bullet[-1],$$

such that

$$[\partial, \partial']^\bullet - [\partial^\bullet, \partial'^\bullet] = \ell \circ d + d \circ \ell.$$

Let $\psi \in \mathrm{Hom}_{S^e}(P^{-n}, M)$. If this is a cocycle, then

$$\begin{aligned}\mathcal{L}_{[\partial,\partial']}(\psi) &= [\partial_M, \partial'_M] \circ \psi - \psi \circ ([\partial^\bullet, \partial'^\bullet] + \ell \circ d + d \circ \ell)\\ &= [\mathcal{L}_\partial, \mathcal{L}_{\partial'}](\psi) - \psi \circ \ell \circ d - \underbrace{\psi \circ d}_{=0} \circ \ell,\end{aligned}$$

which is cohomologous to $[\mathcal{L}_\partial \mathcal{L}_{\partial'}](\psi)$. This proves (2).

(3) Note that the mapping

$$\begin{aligned}\partial_{M\otimes_S N} : M \otimes_S N &\to M \otimes_S N,\\ x \otimes y &\mapsto \partial_M(x) \otimes y + x \otimes \partial_N(y),\end{aligned}$$

is a well-defined derivation relative to ∂, which defines $\mathcal{L}_\partial$ on

$$\mathrm{Tor}^{S^e}_{m-n}(S, M \otimes_S N).$$

Assume first that $m = n$. Let p^0 be any element of the preimage of 1_S under the augmentation mapping $P^0 \to S$. Let $x \otimes p \in M \otimes_{S^e} P^{-m}$ and $\psi \in \mathrm{Hom}_{S^e}(P^{-m}, N)$, and use the notation

$$\iota_\psi(x \otimes p) := (x \otimes \psi(p)) \otimes p^0.$$

Recall that the contraction mapping is induced by the mapping

$$\begin{aligned}M \otimes_{S^e} P^{-m} &\to \mathrm{Hom}_R(\mathrm{Hom}_{S^e}(P^{-m}, N), (M \otimes_S N) \otimes_{S^e} P^0),\\ x \otimes p &\mapsto \iota_?(x \otimes p).\end{aligned}$$

Denote $\mathcal{L}_\partial(\iota_\psi(x \otimes p)) - \iota_{\mathcal{L}_\partial(\psi)}(x \otimes p)$ by δ. Then,

$$\begin{aligned}\delta &= \mathcal{L}_\partial((x \otimes \psi(p)) \otimes p^0) - (x \otimes \mathcal{L}_\partial(\psi)(p)) \otimes p^0\\ &= \partial_M(x) \otimes \psi(p) \otimes p^0 + x \otimes \partial_N(\psi(p)) \otimes p^0 + x \otimes \psi(p) \otimes \partial^0(p^0)\\ &\qquad - x \otimes \partial_N(\psi(p)) \otimes p^0 + x \otimes \psi(\partial^{-m}(p)) \otimes p^0\\ &= \iota_\psi(\mathcal{L}_\partial(x \otimes p)) + x \otimes \psi(p) \otimes \partial^0(p^0).\end{aligned}$$

Note that $\partial^0(p_0)$ lies in the image of $d : P^{-1} \to P^0$ because the image of p^0 under $P^0 \to S$ is 1 and $H^0(\partial^\bullet) = \partial$. These considerations therefore prove (3) when $m = n$.

Now assume that S is projective in $\mathrm{Mod}(R)$. Then, the equivariance may be checked at the level of Hochschild (co)chains. Let $o = (x|s_1|\cdots|s_m) \in S^{\otimes m}$ and $\psi \in \mathrm{Hom}_R(S^{\otimes n}, N)$. Then,

$$\begin{aligned}
&\mathcal{L}_\partial(\iota_\psi(o)) - \iota_{\mathcal{L}_\partial(\psi)}(o) \\
&\quad = \mathcal{L}_\partial(x\otimes\psi(s_1|\cdots|s_n)|s_{n+1}|\cdots|s_m) - (x\otimes\mathcal{L}_\partial(\psi)(s_1|\cdots|s_n)|s_{n+1}|\cdots|s_m) \\
&\quad = (\partial_M(x)\otimes\psi(s_1|\cdots|s_n)|s_{n+1}|\cdots|s_m) + (x\otimes\partial_N(\psi(s_1|\cdots|s_n))|s_{n+1}|\cdots|s_m) \\
&\qquad\qquad + \sum_{j=n+1}^{m}(x\otimes\psi(s_1|\cdots|s_n)|s_{n+1}|\cdots|\partial(s_j)|\cdots|s_m) \\
&\qquad\qquad\qquad - (x\otimes\partial_N(\psi(s_1|\cdots|s_n))|s_{n+1}|\cdots|s_m) \\
&\qquad\qquad\qquad\qquad + \sum_{j=1}^{n}(x\otimes\psi(s_1|\cdots|\partial(s_j)|\cdots|s_n)|s_{n+1}|\cdots|s_m) \\
&\quad = \iota_\psi(\mathcal{L}_\partial(o)),
\end{aligned}$$

which proves (3) for all $m, n \in \mathbb{N}$ when S is projective in $\mathrm{Mod}(R)$.

(4) Note that $\mathcal{L}_{s\partial}$ is defined with respect to the derivation $s\partial_M$ $(= \lambda_s \circ \partial_M)$. Assume that M is symmetric as an S-bimodule. Therefore, the mapping

$$\lambda_s \circ \partial^\bullet : P^\bullet \to P^\bullet$$

is a derivation relative to $s\partial$. Let $\psi \in \mathrm{Hom}_{S^e}(P^\bullet, M)$ be a cocycle with cohomology class denoted by c. Since $\psi \circ \lambda_s = \lambda_s \circ \psi$,

$$\mathcal{L}_{s\partial}(\psi) = (\lambda_s \circ \partial_M) \circ \psi - \psi \circ (\lambda_s \circ \partial^\bullet) = \lambda_s \circ \mathcal{L}_\partial(\psi).$$

Taking cohomology classes yields that $\mathcal{L}_{s\partial}(c) = \lambda_s \circ \mathcal{L}_\partial(c)$.

(5) Recall that, here, ∂_M is taken equal to

$$\begin{aligned}
(s\partial)^e &: S^e \to S^e, \\
s_1 \otimes s_2 &\mapsto s\partial(s_1) \otimes s_2 + s_1 \otimes s\partial(s_2).
\end{aligned}$$

Let $\psi \in \mathrm{Hom}_{S^e}(P^{-n}, M)$ be a cocycle with cohomology class denoted by c. Let h be as in part (3) of Lemma 4.2.1. Then,

$$\begin{aligned}
\mathcal{L}_{s\partial}(\psi) &= (s\partial)^e \circ \psi - \psi \circ (s\partial)^\bullet \\
&= (s\partial \otimes 1 + 1 \otimes s\partial) \circ \psi - \psi \circ (s\partial)^\bullet \\
&= \lambda_s \circ (\partial \otimes 1) \circ \psi + \rho_s \circ (1 \otimes \partial) \circ \psi - \psi \circ (s\partial)^\bullet.
\end{aligned}$$

Using (4-9), the equality becomes

$$\mathcal{L}_{s\partial}(\psi) = \lambda_s \circ (\partial\otimes 1)\circ\psi + \rho_s\circ(1\otimes\partial)\circ\psi - \lambda_s\circ\psi\circ\partial^\bullet - \psi\circ h\circ d - \underbrace{\psi\circ d}_{=0}\circ h.$$

Using $[\partial, \rho_s] = \rho_{\partial(s)}$, it then becomes

$$\begin{aligned}\mathcal{L}_{s\partial}(\psi) &= \lambda_s \circ (\partial \otimes 1) \circ \psi + (1 \otimes \partial) \circ \rho_s \circ \psi - \rho_{\partial(s)} \circ \psi - \lambda_s \circ \psi \circ \partial^\bullet - \psi \circ h \circ d \\ &= \lambda_s \circ (\partial \otimes 1) \circ \psi - \rho_{\partial(s)} \circ \psi + (1 \otimes \partial) \circ \psi \circ (\rho_s - \lambda_s) \\ &\qquad + (1 \otimes \partial) \circ \psi \circ \lambda_s - \lambda_s \circ \psi \circ \partial^\bullet - \psi \circ h \circ d.\end{aligned}$$

Using (4-2), this becomes

$$\begin{aligned}\mathcal{L}_{s\partial}(\psi) &= \lambda_s \circ (\partial\otimes 1)\circ\psi - \rho_{\partial(s)}\circ\psi - (1\otimes\partial)\circ\underbrace{\psi\circ d}_{=0}\circ k_s \\ &\qquad - (1\otimes\partial)\circ\psi\circ k_s\circ d + \underbrace{(1\otimes\partial)\circ\psi\circ\lambda_s}_{=(1\otimes\partial)\circ\lambda_s\circ\psi=\lambda_s\circ(1\otimes\partial)\circ\psi} - \lambda_s\circ\psi\circ\partial^\bullet - \psi\circ h\circ d \\ &= \lambda_s\circ(\partial\otimes 1 + 1\otimes\partial)\circ\psi - \rho_{\partial(s)}\circ\psi - \lambda_s\circ\psi\circ\partial^\bullet - (\psi\circ h + (1\otimes\partial)\circ\psi\circ k_s)\circ d \\ &= \lambda_s\circ(\mathcal{L}_\partial(\psi)) - \rho_{\partial(s)}\circ\psi - (\psi\circ h + (1\otimes\partial)\circ\psi\circ k_s)\circ d.\end{aligned}$$

Now, consider the following R-linear mapping denoted by f:

$$\psi \circ h + (1 \otimes \partial) \circ \psi \circ k_s : P^{-n+1} \to S^e.$$

This is a morphism of S-bimodules. Indeed,

- it is a morphism of left S-modules because so are ψ, $1 \otimes \partial$, k_s and h (see (4-10));
- since ψ and k_s are morphisms of S-bimodules, then, for all $t \in S$,

$$\begin{aligned} f \circ \rho_t &= \psi \circ h \circ \rho_t + (1 \otimes \partial) \circ \rho_t \circ \psi \circ k_s \\ &\underset{(4\text{-}10)}{=} \psi \circ (\rho_t \circ h - \rho_{\partial(t)} \circ k_s) + (1 \otimes \partial) \circ \rho_t \circ \psi \circ k_s \\ &= \rho_t \circ \psi \circ h - \rho_{\partial(t)} \circ \psi \circ k_s + (1 \otimes \partial) \circ \rho_t \circ \psi \circ k_s \\ &= \rho_t \circ \psi \circ h + \rho_t \circ (1 \otimes \partial) \circ \psi \circ k_s \\ &= \rho_t \circ f. \end{aligned}$$

Therefore, $\mathcal{L}_{s\partial}(\psi)$ and $\lambda_s \circ \mathcal{L}_\partial(\psi) - \rho_{\partial(s)} \circ \psi$ are cohomologous. Since so are $\lambda_{\partial(s)} \circ \psi$ and $\rho_{\partial(s)} \circ \psi$ it follows that

$$\mathcal{L}_{s\partial}(c) = \lambda_s \circ \mathcal{L}_\partial(c) - \lambda_{\partial(s)}(c). \qquad \square$$

4.4. ***The action of L on*** $\mathbf{Ext}^\bullet_{S^e}(S, S^e)$**.** According to Lemma 4.3.1, the mapping

$$L \times \mathrm{Ext}^n_{S^e}(S, S^e) \to \mathrm{Ext}^n_{S^e}(S, S^e), \quad (\alpha, e) \mapsto \alpha \cdot e := \mathcal{L}_{\partial_\alpha}(e) \tag{4-13}$$

endows $\mathrm{Ext}^\bullet_{S^e}(S, S^e)$ with a compatible left $S \rtimes L$-module structure in the sense of (3-13), that is, a left $S \rtimes L$-module structure such that, for all $e \in \mathrm{Ext}^\bullet_{S^e}(S, S^e)$, $\alpha \in L$ and $s \in S$,

$$(s\alpha) \cdot e = s \cdot (\alpha \cdot e) - \alpha(s) \cdot e. \tag{4-14}$$

This left $S \rtimes L$-module structure on $\mathrm{Ext}^\bullet_{S^e}(S, S^e)$ does not define a left U-module structure in general. However, Lemma 3.5.1 yields that $\mathrm{Ext}^\bullet_{S^e}(S, S^e)^\vee$ is a right U-module by defining $\theta \cdot \alpha$, for all $\theta \in \mathrm{Ext}^\bullet_{S^e}(S, S^e)^\vee$ and $\alpha \in L$, as

$$\begin{aligned}\theta \cdot \alpha : \mathrm{Ext}^n_{S^e}(S, S^e) &\to S,\\ e &\mapsto -\alpha(\theta(e)) + \theta(\alpha \cdot e).\end{aligned}$$

4.5. ***Particular case of Van den Bergh and Calabi–Yau duality.*** Recall that, whenever $\mathrm{Tor}^{S^e}_n(S, S) \simeq S$ as S-(bi)modules, a *volume form* is a free generator ω_S of $\mathrm{Tor}^{S^e}_n(S, S)$, and the associated *divergence*

$$\mathrm{div} : \mathrm{Der}_R(S) \to S$$

is defined such that, for all $\partial \in \mathrm{Der}_R(S)$,

$$\mathcal{L}_\partial(\omega_S) = \mathrm{div}(\partial)\omega_S. \tag{4-15}$$

When S is Calabi–Yau in dimension n, any free generator e_S of the left S-module $\mathrm{Ext}^n_{S^e}(S, S^e)$ defines an isomorphism of S-bimodules

$$\begin{aligned}\theta : S &\to \mathrm{Ext}^n_{S^e}(S, S^e),\\ s &\mapsto s e_S.\end{aligned}$$

In such a situation, the fundamental class $c_S \in \mathrm{Tor}^{S^e}_n(S, \mathrm{Ext}^n_{S^e}(S, S^e))$ (see 2.1) is a free generator of the left S-module $\mathrm{Tor}^{S^e}_n(S, \mathrm{Ext}^n_{S^e}(S, S^e))$, and hence the preimage ω_S of c_S under the bijective mapping

$$\theta_* : \mathrm{Tor}^{S^e}_n(S, S) \to \mathrm{Tor}^{S^e}_n(S, \mathrm{Ext}^n_{S^e}(S, S^e))$$

is a volume form for S, thus defining a divergence operator.

Proposition 4.5.1. (1) *Assume the following*:

- *R is Noetherian and S is finitely generated as an R-algebra.*
- *S is projective in* $\mathrm{Mod}(R)$.
- *S has Van den Bergh duality with dimension n.*

Then there is an isomorphism of S-modules compatible with Lie derivatives

$$\mathrm{Ext}^n_{S^e}(S, S^e) \simeq \Lambda^n_S \mathrm{Der}_R(S).$$

(2) *Assume that S is Calabi–Yau in dimension n. Let e_S be a free generator of the left S-module* $\mathrm{Ext}^n_{S^e}(S, S^e)$. *Let* div *be the resulting divergence operator. Then, for all* $\partial \in \mathrm{Der}_R(S)$,

$$\mathcal{L}_\partial(e_S) = -\mathrm{div}(\partial) e_S. \tag{4-16}$$

Proof. In both cases, S lies in $\mathrm{per}(S^e)$. Denote the fundamental class of S by c_S. In view of part (3) of Lemma 4.3.1, the definition of c_S gives that

$$\mathcal{L}_\partial(c_S) = 0. \tag{4-17}$$

(1) Denote $\mathrm{Ext}^n_{S^e}(S, S^e)$ by D. In view of Proposition 2.2.1, [Hochschild et al. 1962, Theorem 3.1] applies and yields an isomorphism of S-modules,

$$\mathrm{Tor}^{S^e}_n(S, S) \simeq \Lambda^n_S \Omega_{S/R}. \tag{4-18}$$

Following [Rinehart 1963, Section 9], this isomorphism is compatible with Lie derivatives. Identify D^{-1} with $\mathrm{Hom}_S(D, S)$ and define $\partial_{D^{-1}}$ as follows, for all $\partial \in \mathrm{Der}_R(S)$:

$$\begin{aligned}\partial_{D^{-1}} : \mathrm{Hom}_S(D, S) &\to \mathrm{Hom}_S(D, S),\\ f &\mapsto \partial \circ f - f \circ \mathcal{L}_\partial.\end{aligned}$$

The evaluation isomorphism

$$\mathrm{ev} : D \otimes_S D^{-1} \xrightarrow{\sim} S \tag{4-19}$$

is compatible with Lie derivatives in the following sense, where $\partial \in \mathrm{Der}_R(S)$:

$$\partial \circ \mathrm{ev} = \mathrm{ev} \circ (\mathcal{L}_\partial \otimes \mathrm{Id} + \mathrm{Id} \otimes \partial_{D^{-1}}). \tag{4-20}$$

Besides, the duality isomorphism

$$\iota_?(c_S) : \mathrm{Ext}^0_{S^e}(S, D^{-1}) \to \mathrm{Tor}^{S^e}_n(S, D \otimes_S D^{-1}) \tag{4-21}$$

is compatible with the action of Lie derivatives because of (4-17) (see part (3) of Lemma 4.3.1). Combining (4-18), (4-19), (4-20) and (4-21) yields an isomorphism that is compatible with Lie derivatives

$$D^{-1} \simeq \Lambda^n_S \Omega_{S/R}.$$

This proves (1).

(2) Keep the notation c_S, ω_S, θ, θ_* for the objects defined from e_S before the statement of the proposition. Let $\partial \in \mathrm{Der}_R(S)$. There exists $\lambda \in S$ such that

$$\mathcal{L}_\partial(e_S) = \lambda e_S.$$

Now, for all $s \otimes p \in S \otimes_{S^e} P^{-n}$,

$$\begin{aligned}\mathcal{L}_\partial(\theta_*(s \otimes p)) &= \mathcal{L}_\partial(se_S \otimes p)\\ &= \partial(s)e_S \otimes p + s\mathcal{L}_\partial(e_S) \otimes p + se_S \otimes \partial^\bullet(p)\\ &= \theta_*(\mathcal{L}_\partial(s \otimes p)) + \lambda\theta_*(s \otimes p).\end{aligned}$$

Therefore,

$$0 = \mathcal{L}_\partial(c_S) = \mathcal{L}_\partial(\theta_*(\omega_S)) = \theta_*(\underbrace{\mathcal{L}_\partial(\omega_S)}_{=\operatorname{div}(\partial)\omega_S}) + \lambda\theta_*(\omega_S) = (\lambda + \operatorname{div}(\partial))c_S.$$

Since c_S is regular, $\lambda = -\operatorname{div}(\partial)$. □

5. Proof of the main theorems

The main results of this article are proved in this section. For this purpose, a description of $\operatorname{Ext}^\bullet_{U^e}(U, U^e)$ is made in Section 5.1, the underlying S-module is expressed in terms of $\operatorname{Ext}^\bullet_{S^e}(S, S^e)$ and $\operatorname{Ext}^\bullet_U(S, U)$, and the U-bimodule structure is described using the functor $F : \operatorname{Mod}(U) \to \operatorname{Mod}(U^e)$ and the action of L on $\operatorname{Ext}^\bullet_{S^e}(S, S^e)$ introduced in Section 4. This description is applied in Section 5.2 in order to prove Theorem 1. And Theorem 2 and Corollary 1 are proved in Sections 5.3 and 5.4 by specialising to the situations where $\operatorname{Ext}^{\text{top}}_{S^e}(S, S^e)$ and $\operatorname{Ext}^{\text{top}}_U(S, U)$ are free, and where (S, L) arises from a Poisson bracket on S, respectively.

Throughout the section, $\operatorname{Ext}^\bullet_{S^e}(S, S^e)$ is endowed with its compatible left $S \rtimes L$-module structure introduced in Section 4.4.

5.1. ***The inverse dualising bimodule of U.*** This subsection proves the following result.

Proposition 5.1.1. *Let R be a commutative ring and $d \in \mathbb{N}$. Let (S, L) be a Lie–Rinehart algebra over R. Assume the following*:

(a) *S is flat as an R-module.*

(b) *For all $n \in \mathbb{N}$, the S-module $\operatorname{Ext}^n_{S^e}(S, S^e)$ is projective.*

(c) *$S \in \operatorname{per}(S^e)$.*

(d) *L is finitely generated and projective with constant rank equal to d in* $\operatorname{Mod}(S)$.

Then, $\Lambda^d_S L^\vee \otimes_S \operatorname{Ext}^\bullet_{S^e}(S, S^e)$ is a graded left U-module such that, for all $\alpha \in L$, $c \in \operatorname{Ext}^\bullet_{S^e}(S, S^e)$ and $\varphi \in \Lambda^d_S L^\vee$,

$$\alpha \cdot (\varphi \otimes c) = -\varphi \cdot \alpha \otimes c + \varphi \otimes \alpha \cdot c.$$

Moreover, U is homologically smooth. Finally, there is an isomorphism of graded right U^e-modules,

$$\operatorname{Ext}^\bullet_{U^e}(U, U^e) \simeq F(\Lambda^d_S L^\vee \otimes_S \operatorname{Ext}^{\bullet-d}_{S^e}(S, S^e)).$$

For this subsection, assume (a), (b), (c) and (d) are true, and consider

- a bounded resolution $Q^\bullet \to S$ in $\mathrm{Mod}(U)$ by finitely generated and projective modules (see [Rinehart 1963, Lemma 4.1]),
- a bounded resolution $\pi : P^\bullet \to S$ in $\mathrm{Mod}(S^e)$ by finitely generated and projective modules,
- an injective resolution $j : U^e \to I^\bullet$ in $\mathrm{Mod}(U^e \otimes (U^e)^{\mathrm{op}})$.

Since S is flat over R and L is projective in $\mathrm{Mod}(S)$, part (2) of Lemma 3.0.1 gives that U^e is flat over R. Therefore, the extension-of-scalars functor

$$- \otimes U^e : \mathrm{Mod}(U^e) \to \mathrm{Mod}(U^e \otimes (U^e)^{\mathrm{op}})$$

is exact. Hence, the restriction-of-scalars-functor transforms injective U^e-bimodules into injective left U^e-modules. Thus, $I^\bullet$ is an injective resolution of U^e in $\mathrm{Mod}(U^e)$. Therefore, there is an isomorphism of graded right U^e-modules,

$$\mathrm{Ext}^\bullet_{U^e}(U, U^e) \simeq H^\bullet \mathrm{Hom}_{U^e}(U, I^*). \tag{5-1}$$

The right-hand side is a right U^e-module by means of I^*.

The proof of the above proposition is divided into separate lemmas.

Lemma 5.1.2. *U is homologically smooth.*

Proof. Since U is projective in $\mathrm{Mod}(S)$ (see part (2) of Lemma 3.0.1), the functor

$$F : \mathrm{Mod}(U) \to \mathrm{Mod}(U^e)$$

is exact. Moreover, $F(S) \simeq U$ and $S \in \mathrm{per}(U)$. Therefore, in order to prove that U is homologically smooth, it suffices to prove that $F(U) \in \mathrm{per}(U^e)$, which is equivalent to $F(U)$ being compact in the derived category $\mathcal{D}(U^e)$ of complexes of U-bimodules. Here is a proof of this fact. Let $(M_k)_{k\in K}$ be a family in $\mathcal{D}(U^e)$, denote $\oplus_{k\in K} M_k$ by M, and consider fibrant resolutions of complexes of U-bimodules $M_k \to i(M_k)$, for all $k \in K$, and $M \to i(M)$. Since S is homologically smooth, S is compact in $\mathcal{D}(S^e)$, and hence the following natural mapping is a quasi-isomorphism:

$$\bigoplus_{k\in K} \mathrm{Hom}_{S^e}(P^\bullet, M_k) \to \mathrm{Hom}_{S^e}(P^\bullet, M).$$

Since $P^\bullet$ is a right bounded complex of projective S-bimodules, the functor $\mathrm{Hom}_{S^e}(P^\bullet, -)$ preserves quasi-isomorphisms, and hence the following natural mapping is a quasi-isomorphism:

$$\bigoplus_{k\in K} \mathrm{Hom}_{S^e}(P^\bullet, i(M_k)) \to \mathrm{Hom}_{S^e}(P^\bullet, i(M)).$$

Since U is projective over S on both sides, U^e is projective in $\mathrm{Mod}(S^e)$. Therefore, for all fibrant complexes I of U-bimodules, the functor $\mathrm{Hom}_{S^e}(-, I)$ preserves quasi-isomorphisms. Accordingly, the following natural mapping is a quasi-isomorphism:

$$\bigoplus_{k\in K} \mathrm{Hom}_{S^e}(S, i(M_k)) \to \mathrm{Hom}_{S^e}(S, i(M)) .$$

Since the pair (F, G) is adjoint and G is induced by the functor $\mathrm{Hom}_{S^e}(S, -)$, the following natural mapping is a quasi-isomorphism:

$$\bigoplus_{k\in K} \mathrm{Hom}_{U^e}(F(U), i(M_k)) \to \mathrm{Hom}_{U^e}(F(U), i(M)).$$

Taking cohomology in degree 0 yields that the following natural mapping is bijective:

$$\bigoplus_{k\in K} \mathcal{D}(U^e)(F(U), i(M_k)) \to \mathcal{D}(U^e)(F(U), i(M)) .$$

This proves that $F(U)$ is compact in $\mathcal{D}(U^e)$. Thus, U is homologically smooth. □

The authors thank Bernhard Keller for having pointed out an incorrect argument in a previous version of this proof.

Lemma 5.1.3. *There is an isomorphism of graded right U^e-modules,*

(5-2) $$\mathrm{Ext}^\bullet_{U^e}(U, U^e) \simeq H^\bullet(\mathrm{Hom}_U(Q^*, U) \otimes_U G(I^*)).$$

Proof. Because of the isomorphism $F(S) \simeq U$ in $\mathrm{Mod}(U^e)$ and the adjunction (F, G), there is a functorial isomorphism of complexes of right U^e-modules,

(5-3) $$\mathrm{Hom}_{U^e}(U, I^\bullet) \simeq \mathrm{Hom}_U(S, G(I^\bullet)).$$

Since F is exact and the pair (F, G) is adjoint, $G(I^\bullet)$ is a left bounded complex of injective left U-modules. Hence, $\mathrm{Hom}_U(-, G(I^\bullet))$ preserves quasi-isomorphisms. Thus, the quasi-isomorphism $Q^\bullet \to S$ induces a quasi-isomorphism of complexes of right U^e-modules,

(5-4) $$\mathrm{Hom}_U(S, G(I^\bullet)) \to \mathrm{Hom}_U(Q^\bullet, G(I^\bullet)).$$

Since $Q^\bullet$ is bounded and consists of finitely generated projective left U-modules, the following canonical mapping is a functorial isomorphism:

(5-5) $$\mathrm{Hom}_U(Q^\bullet, U) \otimes_U G(I^\bullet) \to \mathrm{Hom}_U(Q^\bullet, G(I^\bullet)).$$

Note that, whether in (5-3), (5-4), or (5-5), the involved right U^e-module structures are inherited from $I^\bullet$. Thus, the announced isomorphism is proved. □

In order to examine the right-hand side of (5-2) by means of a spectral sequence, the following lemma describes $H^\bullet(G(I^*))$ as a graded $U - U^e$-bimodule.

Lemma 5.1.4. *Consider* $\mathrm{Ext}^\bullet_{S^e}(S, S^e)$ *as a left* $S \rtimes L$*-module as in Section 4.4. Then, there is a* $U - U^e$*-bimodule structure on* $\mathrm{Ext}^\bullet_{S^e}(S, S^e) \otimes_{S^e} U^e$ *such that the right* U^e*-module structure is inherited from* U^e *and for all* $\alpha \in L$, $c \in \mathrm{Ext}^\bullet_{S^e}(S, S^e)$ *and* $u, v \in U$,

$$\alpha \cdot (c \otimes (u \otimes v)) = \alpha \cdot c \otimes (u \otimes v) + c \otimes ((\alpha \otimes 1 - 1 \otimes \alpha) \cdot (u \otimes v)).$$

For this structure, there is an isomorphism of graded $U - U^e$*-bimodules,*

$$H^\bullet(G(I^*)) \simeq \mathrm{Ext}^\bullet_{S^e}(S, S^e) \otimes_{S^e} U^e.$$

Proof. The object $G(I^\bullet)$ is $\mathrm{Hom}_{S^e}(S, I^\bullet)$ as a complex of S-modules, its right U^e-module structure is inherited from $I^\bullet$, and the one of left U-module is given in Section 3.2.

First, since U^e is projective in $\mathrm{Mod}(S^e)$ and $I^\bullet$ consists of injective left U^e-modules, $I^\bullet$ is a left bounded complex of injective left S^e-modules. Hence, $\mathrm{Hom}_{S^e}(-, I^\bullet)$ preserves quasi-isomorphisms. Thus, $\pi : P^\bullet \to S$ induces a quasi-isomorphism of complexes of right S^e-modules,

$$\pi' : \mathrm{Hom}_{S^e}(S, I^\bullet) \to \mathrm{Hom}_{S^e}(P^\bullet, I^\bullet). \tag{5-6}$$

For all $\alpha \in L$, let $\partial^\bullet_\alpha : P^\bullet \to P^\bullet$ be a derivation relative to $\partial_\alpha : S \to S$ (see Section 4.2), and denote by $\delta^\bullet_\alpha$ the mapping from $I^\bullet$ to $I^\bullet$ given by

$$i \mapsto (\alpha \otimes 1 - 1 \otimes \alpha) \cdot i.$$

Then, define $\alpha \cdot f$ and $\alpha \cdot g$, for all $f \in \mathrm{Hom}_{S^e}(S, I^\bullet)$ and $g \in \mathrm{Hom}_{S^e}(P^\bullet, I^\bullet)$, by

$$\alpha \cdot f = \delta^\bullet_\alpha \circ f - f \circ \partial_\alpha$$
$$\alpha \cdot g = \delta^\bullet_\alpha \circ g - g \circ \partial^\bullet_\alpha\, ;$$

since $\pi \circ \partial^\bullet_\alpha = \partial_\alpha \circ \pi$,

$$\pi'(\alpha \cdot f) = \alpha \cdot \pi'(f).$$

The hypotheses on $P^\bullet$ yield an isomorphism of complexes of right U^e-modules,

$$\mathrm{ev} : \mathrm{Hom}_{S^e}(P^\bullet, S^e) \otimes_{S^e} I^\bullet \to \mathrm{Hom}_{S^e}(P^\bullet, I^\bullet). \tag{5-7}$$

Endow the left-hand side term with the following action of L. For all $\alpha \in L$ and $\varphi \otimes i \in \mathrm{Hom}_{S^e}(P^\bullet, S^e) \otimes_{S^e} I^\bullet$, denote by $\alpha \cdot (\varphi \otimes i)$ the (well-defined) element of $\mathrm{Hom}_{S^e}(P^\bullet, S^e) \otimes_{S^e} I^\bullet$,

$$\alpha \cdot \varphi \otimes i + \varphi \otimes (\delta^\bullet_\alpha i).$$

The assignment $\varphi \otimes i \mapsto \alpha \cdot (\varphi \otimes i)$ is a morphism of complexes of R-modules from $\mathrm{Hom}_{S^e}(P^\bullet, S^e) \otimes_{S^e} I^\bullet$ to itself. In view of (4-8) and of the identity

$$(\alpha \otimes 1 - 1 \otimes \alpha) \cdot ((s \otimes t) \cdot j) = \partial_\alpha(s \otimes t) \cdot j + (s \otimes t) \cdot (\alpha \otimes 1 - 1 \otimes \alpha) \cdot j$$

in $I^\bullet$, for all $s, t \in S$ and $j \in I^\bullet$, the following holds:

$$\mathrm{ev}(\alpha \cdot (\varphi \otimes i)) = \alpha \cdot \mathrm{ev}(\varphi \otimes i). \tag{5-8}$$

$\mathrm{Hom}_{S^e}(P^\bullet, S^e)$ is also a bounded complex of projective right S^e-modules. Hence, the functor $\mathrm{Hom}_{S^e}(P^\bullet, S^e) \otimes_{S^e} -$ preserves quasi-isomorphisms. Thus, $j : U^e \to I^\bullet$ induces a quasi-isomorphism of right U^e-modules,

$$\mathrm{Id} \otimes j : \mathrm{Hom}_{S^e}(P^\bullet, S^e) \otimes_{S^e} U^e \to \mathrm{Hom}_{S^e}(P^\bullet, S^e) \otimes_{S^e} I^\bullet. \tag{5-9}$$

Endow the left-hand side term with the following action of L. For all $\alpha \in L$, $\varphi \in \mathrm{Hom}_{S^e}(P^\bullet, S^e)$ and $u, v \in U$, denote by $\alpha \cdot (\varphi \otimes (u \otimes v))$ the following (well-defined) element of $\mathrm{Hom}_{S^e}(P^\bullet, S^e) \otimes_{S^e} U^e$:

$$\alpha \cdot \varphi \otimes (u \otimes v) + \varphi \otimes ((\alpha \otimes 1 - 1 \otimes \alpha) \cdot (u \otimes v)).$$

The assignment $\varphi \otimes (u \otimes v) \mapsto \alpha \cdot (\varphi \otimes (u \otimes v))$ is a morphism of complexes of R-modules from $\mathrm{Hom}_{S^e}(P^\bullet, S^e) \otimes_{S^e} U^e$ to itself, and

$$(\mathrm{Id} \otimes j)(\alpha \cdot (\varphi \otimes (u \otimes v)) = \alpha \cdot ((\mathrm{Id} \otimes j)(\varphi \otimes (u \otimes v)))$$

because $j : U^e \to I^\bullet$ is a morphism of complexes of $U^e - U^e$-bimodules.

Since U^e is projective in $\mathrm{Mod}(S^e)$, there is an isomorphism of right U^e-modules,

$$H^\bullet(\mathrm{Hom}_{S^e}(P^*, S^e) \otimes_{S^e} U^e) \simeq \mathrm{Ext}^\bullet_{S^e}(S, S^e) \otimes_{S^e} U^e. \tag{5-10}$$

For all cocycles $\varphi \in \mathrm{Hom}_{S^e}(P^\bullet, S^e)$, with cohomology class denoted by c, and for all $\alpha \in L$ and $u, v \in U$, the image under (5-10) of the cohomology class of

$$\alpha \cdot (\varphi \otimes (u \otimes v))$$

is

$$\alpha \cdot c \otimes (u \otimes v) + c \otimes ((\alpha \otimes 1 - 1 \otimes \alpha) \cdot (u \otimes v)), \tag{5-11}$$

where $\alpha \cdot c$ is defined in Section 4.4 (see (4-13)).

Combining (5-6), (5-7), (5-9), (5-10) yields an isomorphism of right U^e-modules,

$$\mathrm{Ext}^\bullet_{S^e}(S, S^e) \otimes_{S^e} U^e \xrightarrow{\simeq} H^\bullet(G(I^*)), \tag{5-12}$$

such that, for all $\alpha \in L$, $c \in \mathrm{Ext}^\bullet_{S^e}(S, S^e)$ and $u, v \in U$, if γ denotes the image of $c \otimes (u \otimes v)$ under (5-12), then $\alpha \cdot \gamma$ is the image of (5-11).

Thus, applying part (1) of Lemma 3.5.2 to $N = \mathrm{Ext}^\bullet_{S^e}(S, S^e)$ yields the announced conclusion. □

Proof of Proposition 5.1.1. The statement relative to the left U-module structure on $\Lambda^d_S L^\vee \otimes \mathrm{Ext}^\bullet_{S^e}(S, S^e)$ follows from Lemma 3.5.1, and Lemma 5.1.2 shows that U

is homologically smooth. The (first quadrant, cohomological) spectral sequence of the bicomplex

$$(\mathrm{Hom}_U(Q^p, U) \otimes_U G(I^q))_{p,q} \tag{5-13}$$

converges to $H^\bullet(\mathrm{Hom}_U(Q^*, U) \otimes_U G(I^*))$ and its $E_2^{p,q}$-term is, for all $p, q \in \mathbb{Z}$,

$$H_h^p(H_v^q(\mathrm{Hom}_U(Q^\bullet, U) \otimes_U G(I^\bullet)).$$

Since $\mathrm{Hom}_U(Q^\bullet, U)$ consists of projective right U-modules, there is an isomorphism of right U^e-modules, for all $p, q \in \mathbb{Z}$,

$$H^q(\mathrm{Hom}_U(Q^p, U) \otimes_U G(I^\bullet)) \simeq \mathrm{Hom}_U(Q^p, U) \otimes_U H^q(G(I^\bullet)). \tag{5-14}$$

The description of $H^\bullet(G(I^*))$ made in Lemma 5.1.4 combines with (5-14) into the following isomorphism of right U^e-modules, for all $p, q \in \mathbb{Z}$:

$$H^q(\mathrm{Hom}_U(Q^p, U) \otimes_U G(I^\bullet)) \simeq \mathrm{Hom}_U(Q^p, U) \otimes_U (\mathrm{Ext}^q_{S^e}(S, S^e) \otimes_{S^e} U^e). \tag{5-15}$$

Using Lemma 3.5.2 (part (2)), this isormorphism yields an isomorphism of right U^e-modules, for all $p, q \in \mathbb{Z}$:

$$H^q(\mathrm{Hom}_U(Q^p, U) \otimes_U G(I^\bullet)) \simeq F(\mathrm{Hom}_U(Q^p, U) \otimes_S \mathrm{Ext}^q_{S^e}(S, S^e)). \tag{5-16}$$

Given that F is an exact functor, that $\mathrm{Ext}^q_{S^e}(S, S^e)$ is projective in $\mathrm{Mod}(S)$ for all q and that (S, L) has duality in dimension d, it follows from (5-16) that there is an isomorphism of right U^e-modules, for all $p, q \in \mathbb{Z}$,

$$H_h^p(H_v^q(\mathrm{Hom}_U(Q^\bullet, U) \otimes_U G(I^\bullet))) \simeq \begin{cases} F(\mathrm{Ext}^d_U(S, U) \otimes_S \mathrm{Ext}^q_{S^e}(S, S^e)) & \text{if } p = d, \\ 0 & \text{if } p \neq d. \end{cases}$$

Therefore, the spectral sequence of the bicomplex (5-13) degenerates at E_2. Thus,

$$H^\bullet(\mathrm{Hom}_U(Q^*, U) \otimes_U G(I^*)) \simeq F(\mathrm{Ext}^d_U(S, U) \otimes_S \mathrm{Ext}^{\bullet-d}_{S^e}(S, S^e)) \text{ in } \mathrm{Mod}(S^e).$$

The conclusion follows from (5-2) and from the isomorphism $\mathrm{Ext}^d_U(S, U) \simeq \Lambda^d_S L^\vee$ in $\mathrm{Mod}(U)$ established in [Huebschmann 1999, Theorem 2.10] □

5.2. *Proof of the main theorem.*

Proof of Theorem 1. Following Proposition 5.1.1, U is homologically smooth and there is an isomorphism of graded right U^e-modules,

$$\mathrm{Ext}^\bullet_{U^e}(U, U^e) \simeq F(\Lambda^d_S L^\vee \otimes_S \mathrm{Ext}^{\bullet-d}_{S^e}(S, S^e)).$$

According to Proposition 3.6.1, the functor F transforms left U-modules that are invertible as S-modules into invertible U-bimodules. Note that

- $\Lambda_S^d L^\vee$ is invertible as an S-module because L is projective with constant rank, and
- $\mathrm{Ext}^\bullet_{S^e}(S, S^e)$ is concentrated in degree n and $\mathrm{Ext}^n_{S^e}(S, S^e)$ is invertible as an S-module because S has Van den Bergh duality.

Thus, $\mathrm{Ext}^\bullet_{U^e}(U, U^e)$ is concentrated in degree $n+d$ and $\mathrm{Ext}^{n+d}_{U^e}(U, U^e)$ is invertible as a U-bimodule. Hence, U has Van den Bergh duality in dimension $n+d$. □

5.3. *Proof of Theorem 2.* The hypotheses of Theorem 2 are assumed throughout this subsection. Let φ_L be a free generator of the S-module $\Lambda_S^d L^\vee$. Let e_S be a free generator of the S-module $\mathrm{Ext}^n_{S^e}(S, S^e)$. Therefore, there exist mappings

$$\lambda_L, \lambda_S : L \to S$$

such that, for all $\alpha \in L$,

$$\begin{cases} \alpha \cdot e_S = \lambda_S(\alpha) \cdot e_S, \\ \varphi_L \cdot \alpha = \lambda_L(\alpha) \cdot \varphi_S. \end{cases}$$

Some basic properties of these are summarised below.

Lemma 5.3.1. *Let λ be either one of λ_S or λ_L. Then, for all $\alpha, \beta \in L$ and $s \in S$,*

(1) $\lambda(s\alpha) = s\lambda(\alpha) - \alpha(s)$,

(2) $\lambda([\alpha, \beta]) = \alpha(\lambda(\beta)) - \beta(\lambda(\alpha))$.

Proof. Assume that $\lambda = \lambda_S$. Let $s \in S$ and $\alpha \in L$. Then, using Section 4.4,

$$\begin{aligned}(s\alpha) \cdot e_S &= s \cdot (\alpha \cdot e_S) - \alpha(s) \cdot e_S \\ &= (s\lambda(\alpha) - \alpha(s)) \cdot e_S,\end{aligned}$$

which proves (1), and

$$\begin{aligned}\alpha \cdot (\beta \cdot e_S) &= \alpha \cdot (\lambda(\beta) \cdot e_S) \\ &= \alpha(\lambda(\beta)) \cdot e_S + \lambda(\beta) \cdot (\alpha \cdot e_S) \\ &= (\alpha(\lambda(\beta)) + \lambda(\alpha)\lambda(\beta)) \cdot e_S,\end{aligned}$$

from which (2) may be proved directly. The proof when $\lambda = \lambda_L$ is analogous, using the right U-module structure of $\Lambda_S^d L^\vee$ instead of Section 4.4. □

As proved later, the following automorphism is a Nakayama automorphism for U.

Lemma 5.3.2. *There exists a unique R-algebra homomorphism*,

$$\nu : U \to U,$$

such that, for all $s \in S$ and $\alpha \in L$,

$$\begin{cases} \nu(s) = s, \\ \nu(\alpha) = \alpha + \lambda_L(\alpha) - \lambda_S(\alpha). \end{cases}$$

This is an automorphism of R-algebra.

Proof. The uniqueness is immediate. For all $\alpha \in L$, denote $\alpha + \lambda_L(\alpha) - \lambda_S(\alpha)$ by ν_α. Then, for all $s \in S$ and $\alpha, \beta \in L$,

$$\begin{aligned} [\nu_\alpha, \nu_\beta] &= [\alpha + \lambda_L(\alpha) - \lambda_S(\alpha), \beta + \lambda_L(\beta) - \lambda_S(\beta)] \\ &\underset{\text{Lemma 5.3.1}}{=} [\alpha, \beta] + \lambda_L([\alpha, \beta]) - \lambda_S([\alpha, \beta]) = \nu_{[\alpha,\beta]}, \\ \nu_{s\alpha} &= s\alpha + \lambda_L(s\alpha) - \lambda_S(s\alpha) \\ &\underset{\text{Lemma 5.3.1}}{=} s\alpha + s\lambda_L(\alpha) - s\lambda_S(\alpha) = s\nu_\alpha, \\ [\nu_\alpha, s] &= [\alpha + \lambda_L(\alpha) - \lambda_L(\alpha), s] = \alpha(s). \end{aligned}$$

This proves the existence of ν. Note that ν preserves the filtration of U by the powers of L and that $\mathrm{gr}(\nu)$ is the identity mapping of U. Accordingly, ν is bijective. □

Now it is possible to prove Theorem 2.

Proof of Theorem 2. From Theorem 1, U has Van den Bergh duality in dimension $n+d$ and there is an isomorphism of U-bimodules,

$$\mathrm{Ext}^{n+d}_{U^e}(U, U^e) \simeq F(\Lambda^d_S \Lambda^\vee \otimes_S \mathrm{Ext}^n_{S^e}(S, S^e)), \tag{5-17}$$

where the tensor product inside $F(\bullet)$ is a left U-module by (3-8).

Recall that $\Lambda^d_S L^\vee$ and $\mathrm{Ext}^n_{S^e}(S, S^e)$ are freely generated by φ_L and e_S, respectively. Therefore, the following mapping is an isomorphism of left U-modules (see Section 3.3):

$$\begin{aligned} \Phi : U &\to F(\Lambda^d_S L^\vee \otimes_S \mathrm{Ext}^n_{S^e}(S, S^e)), \\ u &\mapsto u \otimes (\varphi_L \otimes e_S). \end{aligned} \tag{5-18}$$

For all $s \in S$, $\alpha \in L$ and $u \in U$,

$$\Phi(u)s = (u \otimes (\varphi_L \otimes e_S)) \cdot s = us \otimes (\varphi_L \otimes e_S) = \Phi(us),$$

$$\begin{aligned} \Phi(u)\alpha &= (u \otimes (\varphi_L \otimes e_S)) \cdot \alpha \\ &= u\alpha \otimes (\varphi_L \otimes e_S) - u \otimes \alpha \cdot (\varphi_L \otimes e_S) \\ &= u\alpha \otimes (\varphi_L \otimes e_S) - (-u \otimes (\varphi_L \cdot \alpha \otimes e_S) + u \otimes (\varphi_L \otimes \alpha \cdot e_S)) \\ &= (u(\alpha + \lambda_L(\alpha) - \lambda_S(\alpha))) \otimes (\varphi_L \otimes e_S) \\ &= \Phi(u(\alpha + \lambda_L(\alpha) - \lambda_S(\alpha))). \end{aligned}$$

Thus, denoting by ν the automorphism of U considered in Lemma 5.3.2, then, for all $u, v \in U$,

$$\Phi(u) \cdot v = \Phi(u\nu(v)). \tag{5-19}$$

Combining (5-17), (5-18) and (5-19) yields that there is an isomorphism of bimodules,

$$\operatorname{Ext}^{n+d}_{U^e}(U, U^e) \simeq U^\nu.$$

Since $\lambda_S = -\operatorname{div}$ (see Proposition 4.5.1), this proves Theorem 2. □

5.4. ***Case of Poisson algebras.***

Proof of Corollary 1. From Proposition 2.2.1, S has Van den Bergh duality in dimension n. Moreover, Proposition 4.5.1 yields an isomorphism of S-modules $\Lambda^n_S \operatorname{Der}_R(S) \simeq \operatorname{Ext}^n_{S^e}(S, S^e)$ which is compatible with the action of Lie derivatives. Finally, according to (1-3), the dualising module of $(S, \Omega_{S/R})$ is $\Lambda^n_S \operatorname{Der}_R(S)$ with right U-module structure such that, for all $s \in S$ and $\varphi \in \Lambda^n_S \operatorname{Der}_R(S)$,

$$\varphi \cdot ds = -\mathcal{L}_{\{s,-\}}(\varphi).$$

Using these considerations, the corollary follows from Theorems 1 and 2. □

6. Examples

6.1. ***The case where L is free as an S-module.*** In this subsection, it is assumed that L is free as an S-module. Consider a basis $(\alpha_1, \dots, \alpha_d)$ of L over S. Denote the dual basis of $L^\vee$ by $(\alpha_1^*, \dots, \alpha_d^*)$. In particular, $\Lambda^d_S L^\vee$ is free of rank one in $\operatorname{Mod}(S)$, with a generator denoted by φ_L,

$$\varphi_L = \alpha_1^* \wedge \cdots \wedge \alpha_d^*.$$

For all $i \in \{1, \dots, d\}$, consider the matrix of $\operatorname{ad}_{\alpha_i}$, denoted by $(s^i_{j,k})_{j,k} \in M_d(S)$. Hence, for all $i, k \in \{1, \dots, d\}$,

$$[\alpha_i, \alpha_k] = \sum_{j=1}^{d} s^i_{j,k} \alpha_j.$$

In this situation, the action of L on $\Lambda^\bullet_S L$ by Lie derivatives specialises as follows. For all $i, j, k \in \{1, \dots, d\}$,

$$(\lambda_{\alpha_i}(\alpha_j^*))(\alpha_k) = \alpha_i(\alpha_j^*(\alpha_k)) - \alpha_j^*([\alpha_i, \alpha_k]) = -s^i_{j,k}.$$

Hence, for all $i, j \in \{1, \dots, d\}$,

$$\lambda_{\alpha_i}(\alpha_j^*) = -\sum_{k=1}^{d} s^i_{j,k} \alpha_k^*.$$

Thus, the right U-module structure of $\Lambda_S^d L^\vee$ is such that, for all $\alpha \in L$,

$$\varphi_L \cdot \alpha = \mathrm{Tr}(\mathrm{ad}_\alpha)\varphi_L. \tag{6-1}$$

Using this simplified description of $\Lambda_S^d L^\vee$ yields the following corollary of the main theorems of this article.

Corollary 6.1.1. *Let R be a commutative ring. Let (S, L) be a Lie–Rinehart algebra of R. Denote by U its enveloping algebra. Assume that*

- *S is flat as an R-module,*
- *S has Van den Bergh duality in dimension n,*
- *L is free of rank d as an S-module.*

Let $(\alpha_1, \dots, \alpha_d)$ be a basis of L over S as considered previously. Then, U has Van den Bergh duality in dimension $n + d$ and there is an isomorphism of U-bimodules,

$$\mathrm{Ext}_{U^e}^{n+d}(U, U^e) \simeq U \otimes_S \mathrm{Ext}_{S^e}^n(S, S^e),$$

where the left U-module structure on $U \otimes_S \mathrm{Ext}_{S^e}^n(S, S^e)$ is the natural one and the right U-module structure is such that, for all $u \in U$, $e \in \mathrm{Ext}_{S^e}^n(S, S^e)$ and $\alpha \in L$,

$$(u \otimes e) \cdot \alpha = u\alpha \otimes e + u \otimes \mathrm{Tr}(\mathrm{ad}_\alpha)e - u \otimes \mathcal{L}_{\partial_\alpha}(e).$$

If, moreover, S is Calabi–Yau, then U is skew Calabi–Yau and each volume form on S determines a Nakayama automorphism $\nu \in \mathrm{Aut}_R(U)$ such that, for all $s \in S$ and $\alpha \in L$,

$$\begin{cases} \nu(s) = s, \\ \nu(\alpha) = \alpha + \mathrm{Tr}(\mathrm{ad}_\alpha) + \mathrm{div}(\partial_\alpha), \end{cases}$$

where div *denotes the divergence of the chosen volume form.*

Proof. In view of (6-1), there is an isomorphism of right U-modules,

$$\Lambda_S^d L^\vee \simeq S,$$

where the right U-module structure on the right-hand side term is such that, for all $\alpha \in L$,

$$1 \cdot \alpha = \mathrm{Tr}(\mathrm{ad}_\alpha).$$

The corollary therefore follows directly from Theorems 1 and 2. □

The previous corollary applies to any Lie–Rinehart algebra arising from a Poisson structure on $R[x_1, \dots, x_n]$, $n \in \mathbb{N}\backslash\{0, 1\}$.

Example 6.1.2. Let $S = R[x, y]$. Let $P \in S$. This defines a Poisson structure on S such that

$$\{x, y\} = P.$$

Let $L := \Omega_{S/R}$ and consider (S, L) as a Lie–Rinehart algebra over R such that, for all $s, t \in S$,

- $[ds, dt] = d\{s, t\}$;
- $\partial_{ds} = \{s, -\}$.

Then (dx, dy) is a basis of $\Omega_{S/R}$ over S. Note that

$$\begin{cases} \mathrm{Tr}(\mathrm{ad}_{dx}) = \mathrm{div}(\partial_{dx}) = \dfrac{\partial P}{\partial y}, \\ \mathrm{Tr}(\mathrm{ad}_{dy}) = \mathrm{div}(\partial_{dy}) = -\dfrac{\partial P}{\partial x}. \end{cases}$$

From Corollary 6.1.1, U is skew Calabi–Yau in dimension 4 and has a Nakayama automorphism $\nu \in \mathrm{Aut}_R(S)$ such that

$$\begin{cases} \nu(x) = x, \quad \nu(dx) = dx + 2\dfrac{\partial P}{\partial y}, \\ \nu(y) = y, \quad \nu(dy) = dy - 2\dfrac{\partial P}{\partial x}. \end{cases}$$

By considering the filtration of U by the powers of the image of L in U, with associated graded algebra the symmetric algebra of L over S (see [Rinehart 1963, Theorem 3.1]), it appears that $U^\times = S^\times = R^\times$. Accordingly, U has no nontrivial inner automorphism. Consequently, U is Calabi–Yau if and only if $\nu = \mathrm{Id}_U$, that is, if and only if $\mathrm{char}(R) = 2$, or else $P \in R$.

Example 6.1.3. Let $S = R[x, y, z]$. Let $P_x, P_y, P_z \in S$ be such that

$$\overrightarrow{P} \wedge \mathrm{curl}(\overrightarrow{P}) = 0,$$

where $\overrightarrow{P}$ denotes

$$\begin{pmatrix} P_x \\ P_y \\ P_z \end{pmatrix}.$$

Hence, the following defines a Poisson bracket on S,

$$\{x, y\} = P_z, \qquad \{y, z\} = P_x, \qquad \{z, x\} = P_y.$$

As in the previous example, let $(S, L := \Omega_{S/R})$ be the associated Lie–Rinehart algebra over R. As is well-known,

$$\{x, -\} = P_z \frac{\partial}{\partial y} - P_y \frac{\partial}{\partial z}, \qquad \{y, -\} = P_x \frac{\partial}{\partial z} - P_z \frac{\partial}{\partial x}, \qquad \{z, -\} = P_y \frac{\partial}{\partial x} - P_x \frac{\partial}{\partial y}.$$

Therefore, using the basis (dx, dy, dz) of $\Omega_{S/R}$ over S,

$$\begin{pmatrix}\operatorname{div}(\partial_{dx})\\ \operatorname{div}(\partial_{dy})\\ \operatorname{div}(\partial_{dz})\end{pmatrix} = \begin{pmatrix}\operatorname{Tr}(\operatorname{ad}_{dx})\\ \operatorname{Tr}(\operatorname{ad}_{dy})\\ \operatorname{Tr}(\operatorname{ad}_{dz})\end{pmatrix} = \operatorname{curl}(\overrightarrow{P}).$$

Using Corollary 6.1.1, it follows that U is skew Calabi–Yau in dimension 6 and has a Nakayama automorphism $\nu \in \operatorname{Aut}_R(S)$ such that

$$\begin{pmatrix}\nu(x)\\ \nu(y)\\ \nu(z)\end{pmatrix} = \begin{pmatrix}x\\ y\\ z\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}\nu(dx)\\ \nu(dy)\\ \nu(dz)\end{pmatrix} = \begin{pmatrix}dx\\ dy\\ dz\end{pmatrix} + 2\operatorname{curl}(\overrightarrow{P}).$$

As in the previous example, there are no nontrivial inner automorphisms for U. Hence, U is Calabi–Yau if and only if $\operatorname{char}(R) = 2$, or else $\operatorname{curl}(\overrightarrow{P}) = 0$. In particular, when R contains $\mathbb{Q}$ as a subring, then U is Calabi–Yau if and only if the Poisson bracket is Jacobian, that is, there exists $Q \in S$ such that $P = \overrightarrow{\operatorname{grad}}(Q)$.

By the Quillen–Suslin Theorem, when R is a field and $n \in \mathbb{N}$, any $R[x_1, \ldots, x_n]$-module that is finitely generated and projective is free. Hence, Corollary 6.1.1 also applies to all Lie–Rinehart algebras of the shape $(R[x_1, \ldots, x_n], L)$, where R is a field.

6.2. ***On two-dimensional Nambu–Poisson structures.*** Following Corollary 1, U is skew Calabi–Yau when S is flat over R and Calabi–Yau and (S, L) is given by a Poisson bracket on S. Assuming these properties, this section computes a Nakayama automorphism of U for a class of examples of two-dimensional Nambu–Poisson structures (see [Laurent-Gengoux et al. 2013, Section 8.3]).

Let $S = R[x, y, z]/(P)$ where $P = 1 + T$ for some $T \in R[x, y, z]$ which is (p, q, r)-homogeneous in the sense that $p, q, r \in R$ and $t := p\alpha + q\beta + r\gamma$ is a unit in R which does not depend on the monomial $x^\alpha y^\beta z^\gamma$ appearing in T. The hypotheses imply the following equality in S:

$$px\frac{\partial P}{\partial x} + qy\frac{\partial P}{\partial y} + rz\frac{\partial P}{\partial z} = -t \ (\in R^\times). \tag{6-2}$$

Let $Q \in R[x, y, z]$ and endow S with the Poisson structure such that

$$\{x, y\} = Q\frac{\partial P}{\partial z}, \qquad \{y, z\} = Q\frac{\partial P}{\partial x}, \qquad \{z, x\} = Q\frac{\partial P}{\partial y}. \tag{6-3}$$

Consider $(S, L := \Omega_{S/R})$ as a Lie–Rinehart algebra such that, for all $s, t, s' \in S$,

- $[ds, dt] = d\{s, t\}$,
- $(sdt)(s') = s\{t, s'\}$.

Consider the following 2-form on S:

$$\omega_S = px\,dy \wedge dz + qy\,dz \wedge dx + rz\,dx \wedge dy.$$

According to (6-2), $\Omega_{S/R}$ is a projective S-module of rank 2. And the relation

$$\frac{\partial P}{\partial x}dx + \frac{\partial P}{\partial y}dy + \frac{\partial P}{\partial z}dz = 0$$

in $\Omega_{S/R}$ yields the following relations in $\Lambda^2_S\Omega_{S/R}$:

$$\begin{aligned}
\frac{\partial P}{\partial x}\,dx \wedge dy &= \frac{\partial P}{\partial z}\,dy \wedge dz,\\
\frac{\partial P}{\partial y}\,dy \wedge dz &= \frac{\partial P}{\partial x}\,dz \wedge dx,\\
\frac{\partial P}{\partial z}\,dz \wedge dx &= \frac{\partial P}{\partial y}\,dx \wedge dy.
\end{aligned}$$

Combining with (6-2) yields

$$\begin{aligned}
dx \wedge dy &= -t^{-1}\frac{\partial P}{\partial z}\omega_S,\\
dy \wedge dz &= -t^{-1}\frac{\partial P}{\partial x}\omega_S,\\
dz \wedge dx &= -t^{-1}\frac{\partial P}{\partial y}\omega_S.
\end{aligned}$$

Thus, ω_S is a volume form of S.

In order to determine the divergence of ω_S, consider the derivations $\delta_x, \delta_y, \delta_z \in \mathrm{Der}_R(S)$ given by

$$\begin{array}{llllll}
\delta_x : x \mapsto & 0 & \delta_y : x \mapsto & -\dfrac{\partial P}{\partial z} & \delta_z : x \mapsto & \dfrac{\partial P}{\partial y}\\
y \mapsto & \dfrac{\partial P}{\partial z} & y \mapsto & 0 & y \mapsto & -\dfrac{\partial P}{\partial x}\\
z \mapsto & -\dfrac{\partial P}{\partial y} & z \mapsto & \dfrac{\partial P}{\partial x} & z \mapsto & 0.
\end{array}$$

Note that

$$\{x,-\} = Q\delta_x, \qquad \{y,-\} = Q\delta_y \quad \text{and} \quad \{z,-\} = Q\delta_z.$$

Then,

$$\begin{aligned}
\iota_{\delta_x}(\omega_S) &= \iota_{\delta_x}(px\,dy \wedge dz + qy\,dz \wedge dx + rz\,dx \wedge dy)\\
&= px\Big(\frac{\partial P}{\partial z}dz + \frac{\partial P}{\partial y}dy\Big) - qy\frac{\partial P}{\partial y}dx - rz\frac{\partial P}{\partial z}dx\\
&= t\,dx \quad \text{(see (6-2))}.
\end{aligned}$$

Therefore, using the symmetry between x, y and z,

$$\operatorname{div}(\delta_x) = \operatorname{div}(\delta_y) = \operatorname{div}(\delta_z) = 0.$$

Apply Lemma 5.3.1, taking into account that $\lambda_S = -\operatorname{div}$ (see (4-16); then,

$$\operatorname{div}(\{x, -\}) = \operatorname{div}(Q\delta_x) = Q\operatorname{div}(\delta_x) + \delta_x(Q).$$

Therefore,

$$\operatorname{div}(\{x, -\}) = \frac{\partial Q}{\partial y}\frac{\partial P}{\partial z} - \frac{\partial Q}{\partial z}\frac{\partial P}{\partial y}. \tag{6-4}$$

Applying Corollary 1 gives that the enveloping algebra U of $(S, \Omega_{S/R})$ is skew Calabi–Yau. It has a Nakayama automorphism $\nu : U \to U$ such that, for all $s \in S$,

$$\begin{cases} \nu(s) = s, \\ \begin{pmatrix} \nu(dx) \\ \nu(dy) \\ \nu(dz) \end{pmatrix} = \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} + 2\,\overrightarrow{\operatorname{grad}}(Q) \wedge \overrightarrow{\operatorname{grad}}(P). \end{cases}$$

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NONDEGENERACY OF THE GAUSS CURVATURE EQUATION WITH NEGATIVE CONIC SINGULARITY

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We study the Gauss curvature equation with negative singularities. For a local mean field type equation with only one negative index we prove a uniqueness property. For a global equation with one or two negative indexes we prove the nondegeneracy of the linearized equations.

1. Introduction

In this article we study two closely related equations, defined locally and globally in $\mathbb{R}^2$, respectively. The first equation is defined in $\Omega \subset \mathbb{R}^2$, which is simply connected, open and bounded. Throughout the whole article we shall always assume that the boundary of Ω, denoted as $\partial\Omega$, is a rectifiable Jordan curve, and we say Ω is regular. Let $p_0, p_1, \dots, p_m \in \Omega$ be a finite set in Ω. Then we consider v as a solution of

$$\begin{cases} \Delta v + \lambda \frac{e^v}{\int_\Omega e^v} = -4\pi\alpha_0\delta_{p_0} + \sum_{i=1}^m 4\pi\alpha_i\delta_{p_i} \quad \text{in } \Omega, \\ v = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1-1}$$

where $\alpha_0 \in (0, 1)$, $\alpha_1, \dots, \alpha_m > 0$ and $\lambda \in \mathbb{R}$.

The second equation is concerned with the stability of the following global equation, which we suppose has u as a solution:

$$\Delta u + e^u = \sum_{i=1}^N 4\pi\beta_i\delta_{p_i} \quad \text{in } \mathbb{R}^2, \tag{1-2}$$

where $\beta_1, \dots, \beta_n$ are constants greater than -1 and $p_1, \dots, p_n$ are the locations of singular sources in $\mathbb{R}^2$. For this equation we shall prove that under some restrictions of β_i, any bounded solution of the linearized equation has to be the trivial solution.

The background of both equations is incredibly rich not only in mathematics but also in physics. In particular, the study of (1-1) reveals core information on the configuration of vortices in the electroweak theory of Glashow–Salam–Weinberg [Lai 1981] and self-dual Chern–Simons theories [Dunne 1995; Hong et al. 1990;

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Jackiw and Weinberg 1990]. Also in statistical mechanics the behavior of solutions in (1-1) is closely related to Onsager's model of two-dimensional turbulence with vortex sources [Caglioti et al. 1995; Chanillo and Kiessling 1994]. Most of the motivation and applications of both equations come from their connection with conformal geometry. The singular sources represent conic singularities on a surface with constant curvature. There is a large number of interesting works that discuss the qualitative properties of solutions of such equations. We mention [Chang et al. 2003; Bartolucci and Lin 2009; 2014; Bartolucci and Malchiodi 2013; Bartolucci and Tarantello 2002; Chanillo and Kiessling 1994; Chen et al. 2004; Chen and Lin 2010; 2015; Chen and Li 1993; 1995; Li 1999; Lin et al. 2012; Luo and Tian 1992; Malchiodi and Ruiz 2011; 2013; Nolasco and Tarantello 2000; Ohtsuka and Suzuki 2007; Spruck and Yang 1992; Struwe and Tarantello 1998; Tarantello 2010; 2017; Troyanov 1989; 1991; Zhang 2006; 2009]. It is important to observe that it seems there are very few works which discuss singularities with negative strength and even fewer about the comparison between the negative indexes and positive ones. In this article, using an improved version of the Alexandrov–Bol inequality, we discuss the uniqueness property and the nondegeneracy for a local equation and a global equation. Our proof is based on techniques developed in a number of works of Bartolucci, Lin, Chang, Chen and Lin, etc.

To state the main result on the local equation, we first rewrite (1-1) using the following Green's function.

For $p \in \Omega$, let $G_\alpha(x, p)$ satisfy

$$\begin{cases} -\Delta G_\alpha(x, p) = 4\pi\alpha\delta_p & \text{in } \Omega, \\ G_\alpha(x, p) = 0, & x \in \partial\Omega, \end{cases}$$

and

$$u = v - G_{\alpha_0} + \sum_{j=1}^{m} G_{\alpha_j}(x, p_j).$$

Then u satisfies

$$\begin{cases} \Delta u + \lambda(He^u)/\left(\int_\Omega He^u\right) = 0 & \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1-3}$$

where

$$H(x) = e^{G_0(x,p_0)} e^{\sum_{j=1}^m G_{\alpha_j}(x,p)} = e^{h(x)} |x - p_0|^{-2\alpha_0} \prod_{i=1}^{m} |x - p_i|^{2\alpha_i}, \tag{1-4}$$

where h is harmonic in Ω and is continuous up to the boundary.

The first main result is the following theorem:

Theorem 1.1. *Let u be a solution of* (1-3) *and H be defined by* (1-4). *Assume that Ω is regular, then for any $\lambda \le 8\pi(1 - \alpha_0)$ there exists at most one solution to* (1-1).

Here we note that for $\lambda < 8\pi(1-\alpha_0)$, the existence result has been established by Bartolucci and Malchiodi [2013]. The existence result for $\lambda = 8\pi(1-\alpha_0)$ will be discussed in a separate work.

The second main goal of this article is to consider the nondegeneracy of (1-2) when there are exactly two negative indexes:

$$\begin{cases} \Delta u + e^u = -4\pi\alpha_1\delta_{p_1} - 4\pi\alpha_2\delta_{p_2} + \sum_{i=3}^n 4\pi\beta_i\delta_{p_i} \text{ in } \mathbb{R}^2, \\ u(x) = -4\log|x| + \text{ a bounded function near } \infty, \end{cases} \tag{1-5}$$

where $\alpha_1, \alpha_2 \in (0, 1)$ and $\beta_i > 0$ for $i = 3, \ldots, n$ and we assume that $n \geq 3$. The assumption of u at infinity says that ∞ is not a singularity of u when $\mathbb{R}^2$ is identified with $\mathbb{S}^2$.

Let

$$u_1(x) = u(x) + \sum_{i=1}^{2} 2\alpha_i \log|x-p_i| - 2\sum_{i=3}^{n} \beta_i \log|x-p_i|; \tag{1-6}$$

then clearly u_1 satisfies

$$\begin{cases} \Delta u_1 + H_1 e^{u_1} = 0 \quad \text{in } \mathbb{R}^2, \\ u_1(x) = (-4 - 2\alpha_1 - 2\alpha_2 + 2\sum_{i=3}^n \beta_i)\log|x| + O(1), \quad \text{for } |x| > 1, \end{cases}$$

where

$$H_1(x) = \prod_{i=1}^{2} |x-p_i|^{-2\alpha_i} \prod_{i=3}^{n} |x-p_i|^{2\beta_i}, \quad \text{for } x \in \mathbb{R}^2. \tag{1-7}$$

Our second main result is the following theorem:

Theorem 1.2. *Let u, u_1 and H_1 be defined as in* (1-5), (1-6) *and* (1-7), *respectively. Suppose ϕ is a classical solution of*

$$\Delta\phi + H_1(x)e^{u_1}\phi = 0 \quad \text{in } \mathbb{R}^2. \tag{1-8}$$

If $\lim_{x\to\infty}|\phi(x)|/\log|x| = 0$ *and $\alpha_1, \alpha_2, \beta_i$ satisfy the condition*

$$-\max\{\alpha_1, \alpha_2\} + \min\{\alpha_1, \alpha_2\} + \sum_{i=3}^{n} \beta_i \leq 0, \tag{1-9}$$

then $\phi \equiv 0$.

Here we recall that the total angles at singularities are $2\pi(1-\alpha_1)$, $2\pi(1-\alpha_2)$, $2\pi(1+\beta_i)$ $(i = 3, \ldots, n)$. For a surface S with conic singularities, let

$$\chi(S, \theta) = \chi(S) + \sum_i \left(\frac{\theta_i}{2\pi} - 1\right),$$

where θ_i is the total angle at a conic singularity, and $\chi(S)$ is the Euler characteristic of S. The purpose of introducing $\chi(S,\theta)$ is to put all surfaces with conic singularities into three cases:

(i) The subcritical case if $\chi(S,\theta) < \min_i\{2, \theta_i/\pi\}$,

(ii) The critical case if $\chi(S,\theta) = \min_i\{2, \theta_i/\pi\}$,

(iii) The supercritical case if $\chi(S,\theta) > \min_i\{2, \theta_i/\pi\}$.

In our case $\chi(S) = 2$ because S is the standard sphere. It is easy to see that (1-9) refers to the supercritical case. For the subcritical case, Troyanov's well-known result [1991] states that every conic singular metric is pointwise conformal to a metric with constant curvature.

Finally, if there is only one negative singular source, a similar result still holds: Let u satisfy

$$\begin{cases} \Delta u + e^u = -4\pi\alpha\delta_{p_1} + \sum_{i=2}^n 4\pi\beta_i\delta_{p_i} & \text{in } \mathbb{R}^2, \\ u(x) = -4\log|x| + \text{a bounded function near } \infty, \end{cases} \tag{1-10}$$

where $\alpha \in (0, 1)$ and $\beta_i > 0$ for $i = 2, \dots, n$ and we assume that $n \geq 3$.

Let

$$u_1(x) = u(x) + 2\alpha\log|x - p_1| - 2\sum_{i=2}^n \beta_i \log|x - p_i|;$$

then clearly u_1 satisfies

$$\begin{cases} \Delta u_1 + H_2 e^{u_1} = 0 & \text{in } \mathbb{R}^2, \\ u_1(x) = (-4 - 2\alpha + 2\sum_{i=2}^n \beta_i)\log|x| + O(1), & \text{for } |x| > 1, \end{cases} \tag{1-11}$$

where

$$H_2(x) = |x - p_1|^{-2\alpha} \prod_{i=2}^n |x - p_i|^{2\beta_i}, \quad \text{for } x \in \mathbb{R}^2. \tag{1-12}$$

Our third main result is:

Theorem 1.3. *Let u_1 be a solution of* (1-11) *with H_2 defined in* (1-12). *Let ϕ be a classical solution of*

$$\Delta\phi + H_2(x)e^{u_1}\phi = 0 \quad \text{in } \mathbb{R}^2. \tag{1-13}$$

If $\lim_{x\to\infty}|\phi(x)|/\log|x| = 0$ *and α, β_i satisfy*

$$-\alpha + \sum_{i=2}^n \beta_i \leq 0, \tag{1-14}$$

then $\phi \equiv 0$.

The organization of this article is as follows. In Section 2 we derive a Bol's inequality with one negative singular source. Then in Section 3 the first two eigenvalues of the linearized local equation are discussed. The proofs of the major theorems are arranged in Sections 4 and 5. The main approach of this article follows closely from previous works of Bartolucci, Chang, Chen and Lin, etc.

2. On Bol's inequality and the first eigenvalues of the local equation

One of the major tools we shall use is Bol's inequality:

Proposition 2.1. *Let $\Omega \Subset \mathbb{R}^2$ be a simply connected, open and bounded domain in $\mathbb{R}^2$. Let u be a solution of*

$$\Delta u + Ve^u = 0 \quad \text{in } \Omega$$

for

$$V = |x-p_1|^{-2\alpha_0} \prod_{i=2}^{n} |x-p_i|^{2\beta_i} e^g \tag{2-1}$$

and $\Delta g \geq 0$ in Ω. Here $p_1, \dots, p_n$ ($n \geq 2$) are distinct points in Ω. Let $\omega \subset \Omega$ be an open subset of Ω such that $\partial\omega$ is a finite union of rectifiable Jordan curves. Let

$$L_{\alpha_0}(\partial\omega) = \int_{\partial\omega} (Ve^u)^{\frac{1}{2}}\, ds, \quad M_{\alpha_0}(\omega) = \int_{\omega} Ve^u\, dx.$$

Then

$$2L_{\alpha_0}^2(\partial\omega) \geq (8\pi(1-\alpha_0) - M_{\alpha_0}(\omega))M_{\alpha_0}(\omega). \tag{2-2}$$

The strict inequality holds if ω contains more than one singular source or is multiply connected.

Our proof of Proposition 2.1 is motivated by the argument in [Bartolucci and Castorina 2016] and [Bartolucci and Lin 2009; 2014]. For the case where $\alpha_0 = 0$, the proposition was established in [Bartolucci and Lin 2014], and for the case where V has only a singular source at 0, it was established by Bartolucci and Castorina. It all starts from an inequality of Huber:

Theorem A [Huber 1954]. *Let ω be an open, bounded, simply connected domain with $\partial\omega$ being a rectifiable Jordan curve, $\tilde{V} = |x|^{-2\alpha_0}e^g$, for some $\Delta g \geq 0$ in ω. Then*

$$\left(\int_{\partial\omega} \tilde{V}^{\frac{1}{2}}\, ds\right)^2 \geq 4\pi(1-\alpha_0) \int_{\omega} \tilde{V}\, dx \quad \text{if } 0 \in \omega,$$

$$\left(\int_{\partial\omega} \tilde{V}^{\frac{1}{2}}\, ds\right)^2 \geq 4\pi \int_{\omega} \tilde{V} dx \quad \text{if } 0 \notin \omega.$$

Huber's theorem can be adjusted to the following version:

Theorem B (Bartolucci–Castorina). *Let $\omega \subset \mathbb{R}^2$ be an open bounded domain such that $\partial\omega$ is a rectifiable Jordan curve. Suppose $\bar{\omega}_B$ is the closure of a possibly disconnected bounded component of $\mathbb{R}^2 \setminus \omega$ and ω_B is the interior of $\bar{\omega}_B$. Let $\tilde{V} = |x|^{-2\alpha_0} e^g$ for some g satisfying $\Delta g \geq 0$ in the interior of $\bar{\omega} \cup \bar{\omega}_B$. Then*

$$\left(\int_{\partial\omega} \tilde{V}^{\frac{1}{2}}\, ds\right)^2 \geq 4\pi(1-\alpha_0)\int_\omega \tilde{V}\, dx,$$

if 0 is in the interior of $\bar{\omega} \cup \bar{\omega}_B$, and

$$\left(\int_{\partial\omega} \tilde{V}^{\frac{1}{2}}\, ds\right)^2 \geq 4\pi\int_\omega \tilde{V}\, dx,$$

if 0 is not in the interior of $\bar{\omega} \cup \bar{\omega}_B$.

Proof of Proposition 2.1. We shall only consider the first case mentioned in Theorem B because the other case corresponds to $\alpha_0 = 0$. Let

$$\begin{cases} \Delta q = 0 & \text{in } \omega, \\ q = u & \text{on } \partial\omega. \end{cases}$$

and let $\eta = u - q$. Then the equation for η is

$$\begin{cases} \Delta\eta + Ve^q e^\eta = 0 & \text{in } \omega, \\ \eta = 0 & \text{on } \partial\omega, \end{cases} \tag{2-3}$$

and we use

$$t_m = \max_{\bar{\omega}} \eta.$$

Then we set

$$\Omega(t) = \{x \in \omega;\ \eta(x) > t\}, \quad \Gamma(t) = \partial\Omega(t), \quad \mu(t) = \int_{\Omega(t)} Ve^q\, dx.$$

Clearly $\Omega(0) = \omega$, $\mu(0) = \int_\omega Ve^q\, dx$, and $\mu(t_m) = \lim_{t \to t_m-} \mu(t) = 0$. Since μ is continuous and strictly decreasing, it is easy to see that

$$\frac{d\mu(t)}{dt} = -\int_{\Gamma(t)} \frac{Ve^q}{|\nabla\eta|}\, ds, \quad \text{for almost every } t \in [0, t_m]. \tag{2-4}$$

For all $s \in [0, \mu(0)]$, set

$$\eta^*(s) = |\{t \in [0, t_m], \quad \mu(t) > s\}|,$$

where $|E|$ is the Lebesgue measure of the measurable set $E \in \mathbb{R}$. It is easy to see that η^* is the inverse of μ on $[0, t_m]$ and is continuous, strictly monotone and differentiable almost everywhere. By (2-4) we have, for almost all $s \in [0, \mu(0)]$,

$$\frac{d\eta^*}{ds} = -\left(\int_{\Gamma(\eta^*(s))} \frac{Ve^q}{|\nabla\eta|}\, dt\right)^{-1}. \tag{2-5}$$

Let

$$F(s) = \int_{\Omega(\eta^*(s))} e^{\eta} V e^{q}\, dx, \quad \text{for almost every } s \in [0, \mu(0)].$$

Then by the definition of $\Omega(t)$ we see that

$$F(s) = \int_{\eta^*(s)}^{t_m} e^t \left(\int_{\Gamma_t} \frac{V e^q}{|\nabla \eta|}\, ds \right) dt.$$

Using $\beta = \mu(t)$, we further have

$$F(s) = \int_0^s e^{\eta^*(\beta)}\, d\beta, \tag{2-6}$$

where $\eta^* = \mu^{-1}$ and (2-4) are used. The definition of F also gives

$$F(0) = \int_{\Omega(\eta^*(0))} e^{\eta} V e^{q} = \int_{\Omega(t_m)} e^{\eta} V e^{q} = 0$$

and $F(\mu(0)) = \int_\omega e^\eta d\tau = M(\omega)$. Consequently, from (2-6) we obtain

$$\frac{dF}{ds} = e^{\eta^*(s)}, \quad \frac{d^2F}{ds^2} = \frac{d\eta^*}{ds} e^{\eta^*(s)} = \frac{d\eta^*}{ds}\frac{dF}{ds}, \quad \text{for almost every } s. \tag{2-7}$$

Here we use the argument from [Bartolucci and Castorina 2016] to show that η^* is locally Lipschitz in $(0, \mu(0))$:

Lemma 2.1. *For any* $0 < \bar{a} \le a < b \le \bar{b} < \bar{u}(0)$, *there exists* $C(\bar{a}, \bar{b}, \beta_1, \dots, \beta_k) > 0$ *such that*

$$\eta^*(a) - \eta^*(b) \le C(b-a).$$

Proof of Lemma 2.1. First we find $\Omega_{a,b}$ that satisfies

$$\{x \in \omega;\ \ \eta^*(b) \le \eta(x) \le \eta^*(a)\} \Subset \Omega_{a,b} \Subset \omega.$$

Using Green's representation formula we have

$$|\nabla \eta(x)| \le C + C \int_{\Omega_{a,b}} \frac{1}{|x-y|} |y - p_1|^{-2\alpha_0}\, dy.$$

A standard estimate gives

$$|\nabla \eta(x)| \le C + C|x - p_0|^{1-2\alpha_0}. \tag{2-8}$$

Recall that $d\eta = Ve^q dx$. Thus

$$\begin{aligned} b-a &= \mu(\eta^*(b)) - \mu(\eta^*(a)) \\ &= \int_{\eta>\eta^*(b)} d\tau - \int_{\eta>\eta^*(a)} d\tau \geq \int_{\eta^*(b)<\eta<\eta^*(a)} d\tau \\ &= \int_{\eta^*(b)}^{\eta^*(a)} \biggl(\int_{\Gamma(t)} \frac{Ve^q}{|\nabla\eta|}\, ds \biggr) dt. \end{aligned}$$

Using the expression of V in (2-1) and (2-8) we further have

$$\begin{aligned} b-a &\geq \frac{1}{C} \int_{\eta^*(b)}^{\eta^*(a)} \biggl(\int_{\Gamma(t)} \frac{1}{|x-p_0|^{2\alpha_0} + |x-p_0|} \biggr) dt \\ &\geq \frac{1}{C} \int_{\eta^*(b)}^{\eta^*(a)} L_1(\Gamma(t))\, dt \\ &\geq \min_{\eta^*(b)\leq t\leq \eta^*(a)} L_1(\Gamma(t)) \int_{\eta^*(b)}^{\eta^*(a)} dt \\ &\geq C(\eta^*(a) - \eta^*(b)), \end{aligned}$$

where the estimate of $\nabla\eta$ was used, $L_1(\Gamma(t))$ stands for the Lebesgue measure of Γ and in the last inequality, the standard isoperimetric inequality

$$L_1(\Gamma(t)) \geq 4\pi |\Omega(t)| \geq 4\pi |\Omega(\eta^*(\bar{a})| > 0$$

is used. Lemma 2.1 is established. □

Now we go back to the proof of Proposition 2.1. By Cauchy's inequality

$$\begin{aligned} \text{(2-9)}\quad & \biggl(\int_{\Gamma(\eta^*(s))} (Ve^q)^{\frac{1}{2}}\, ds \biggr)^2 \\ &\leq \biggl(\int_{\Gamma(\eta^*(s))} \frac{Ve^q}{|\nabla\eta|}\, ds \biggr) \biggl(\int_{\Gamma(\eta^*(s))} |\nabla\eta|\, ds \biggr) \\ &= \biggl(-\frac{d\eta^*}{ds} \biggr)^{-1} \biggl(\int_{\Gamma(\eta^*(s))} \biggl(-\frac{\partial\eta}{\partial\nu} \biggr) ds \biggr), \qquad \text{for almost every } s \in [0, \mu(0)], \end{aligned}$$

where $\nu = \nabla\eta/|\nabla\eta|$. Moreover from (2-3)

(2-10)
$$\int_{\Gamma(\eta^*(s))} \biggl(-\frac{\partial\eta}{\partial\nu} \biggr) ds = \int_{\Omega(\eta^*(s))} Ve^q e^\eta\, dx = F(s), \quad \text{for almost every } s \in [0, \mu(0)].$$

By Theorem A, the following inequality holds for almost all $s \in [0, \mu(0)]$:

$$\text{(2-11)}\qquad \biggl(\int_{\Gamma(\eta^*(s))} (Ve^q)^{\frac{1}{2}} \biggr)^2 \geq 4\pi(1-\alpha_0)\mu(\eta^*(s)) = 4\pi(1-\alpha_0)s.$$

Putting (2-10) into (2-9) yields

$$\left(\int_{\Gamma(\eta^*(s))} (Ve^q)^{\frac{1}{2}}\, ds\right)^2 \le \left(-\frac{d\eta^*}{ds}\right)^{-1} F(s). \tag{2-12}$$

Using (2-11) in (2-12), we have

$$4\pi(1-\alpha_0)s \le \left(-\frac{d\eta^*}{ds}\right)^{-1} F(s), \quad \text{for almost every } s \in [0, \mu(0)],$$

which is equivalent to

$$4\pi(1-\alpha_0)s\frac{d\eta^*}{ds} + F(s) \ge 0, \quad \text{for almost every } s \in [0, \mu(0)]. \tag{2-13}$$

By (2-7) and (2-13), we obtain

$$\frac{d}{ds}\left[4\pi(1-\alpha_0)\left(s\frac{dF}{ds} - F(s)\right) + \tfrac{1}{2}F^2(s)\right] \ge 0, \quad \text{for almost every } s \in [0, \mu(0)].$$

Let $P(s)$ denote the function in the brackets, then P is well defined, continuous, nondecreasing on $[0, \mu(0)]$. By the Lipschitz property of η^*, P is absolutely continuous on $[0, \mu(0)]$;

$$P(\mu(0)) - P(0) = \lim_{b\to\mu(0)^-} \lim_{a\to 0^+} \int_a^b \frac{dP}{ds}\, ds.$$

Using $F(0) = 0$, $F(\mu(0)) = M(\omega)$, and $\frac{dF}{ds}|_{s=\mu(0)} = e^0 = 1$, we have

$$8\pi(1-\alpha_0)(\mu(0) - M(\omega)) + M(\omega)^2 \ge 0.$$

Then Huber's inequality and $\Gamma(0) = \partial\omega$ further yield

$$\begin{aligned} 2l^2(\partial\omega) = 2\left(\int_{\partial\omega} (Ve^v)^{\frac{1}{2}}\, ds\right)^2 &= 2\left(\int_{\partial\omega} (Ve^q)^{\frac{1}{2}}\, ds\right)^2 \\ &\ge 8\pi(1-\alpha_0)\mu(0) \ge M(\omega)(8\pi(1-\alpha_0) - M(\omega)), \end{aligned}$$

where we have used the fact that $v = q$ on $\partial\omega$. The Bol's inequality is established. The equality holds if $Ve^q = |x - p_0|^{-2\alpha_0}|\Phi_t'|^2 e^k$ on $\Omega(t)$ for almost all $t \in (0, t_m)$ where k is a constant. In particular for $t = 0$, Φ_0 maps Ω to a ball. In this case g must be harmonic. On the other hand from the equality of Cauchy's inequality we have

$$Ve^q = c_t|\nabla\eta|^2 \quad \text{on } \Gamma(t), \quad \text{for almost every } t \in (0, t_m),$$

for some $c_t > 0$. Putting $w = \Phi_0(z)$ and $\xi(w) = \eta(\Phi_0^{-1}(w)) + k$, we see that ξ satisfies

$$\Delta\xi + |x|^{-2\alpha_0}e^{\xi} = 0,$$

and ξ is radial. This ξ is a scaling of

$$\log \frac{8(1-\alpha_0)^2}{1+|x|^{2(1-\alpha_0))^2}}.$$

Thus we have strict inequality in Bol's inequality if at least one of the following situations occurs:

(1) $p_1 \notin \omega$,

(2) ω has at least two singular sources

(3) ω is not simply connected. □

3. The first eigenvalues of the linearized local equation

Proposition 3.1. *Let Ω be an open, bounded domain of $\mathbb{R}^2$ with rectifiable boundary $\partial\Omega$, $V = |x|^{-\alpha_0}\prod_{i=1}^k |x-p_i|^{2\beta_i} e^g$ for some subharmonic and smooth function g, $\alpha_0 \in (0,1)$, $\beta_1,\dots,\beta_k > 0$, and assume that all the singular points are in Ω: 0, $p_1,\dots,p_k \in \Omega$. Let w be a classical solution of*

$$\Delta w + Ve^w = 0 \quad \text{in } \Omega.$$

Suppose $\hat{\nu}_1$ is the first eigenvalue of

$$\begin{cases} -\Delta\phi - Ve^w\phi = \hat{\nu}_1 Ve^w\phi & \text{in } \Omega, \\ \phi = 0 & \text{on } \partial\Omega. \end{cases} \tag{3-1}$$

Then if $\int_\Omega Ve^w \le 4\pi(1-\alpha_0)$ we have $\hat{\nu}_1 > 0$. Moreover if $\int_\Omega Ve^w \le 8\pi(1-\alpha_0)$ we have $\hat{\nu}_2 > 0$.

Proof. Let $\nu_1 = \hat{\nu}_1 + 1$ and ϕ be the eigenfunction corresponding to $\hat{\nu}_1$, then we have $\phi > 0$ and

$$\begin{cases} -\Delta\phi = \nu_1 Ve^w\phi & \text{in } \Omega, \\ \phi = 0 & \text{on } \partial\Omega. \end{cases}$$

Let

$$U_0(x) = (-2)\log(1+|x|^{2(1-\alpha_0)}) + \log(8(1-\alpha_0)^2).$$

Then clearly U_0 solves

$$\Delta U_0 + |x|^{-2\alpha_0} e^{U_0} = 0 \quad \text{in } \mathbb{R}^2.$$

For $t \in (0, t_+)$ where $t_+ = \max_{\bar{\Omega}} \phi$, we set $\Omega(t) = \{x \in \Omega,\ \phi(x) > t\}$ and we set $R(t)$ to satisfy

$$\int_{\Omega(t)} Ve^w = \int_{B_{R(t)}} e^{U_0}|x|^{-2\alpha_0}.$$

Clearly $\Omega(0)=\Omega$, $R_0=\lim_{t\to 0+}R(t)$, $\lim_{t\to t_+-}R(t)=0$. Let ϕ^* be a radial function from $B_{R_0}\to\mathbb{R}$. For $y\in B_{R_0}$ and $|y|=r$, set

$$\phi^*(r)=\sup\{t\in(0,t_+)\mid R(t)>r\}.$$

Then $\phi^*(R_0)=\lim_{r\to R_0-}\phi^*(r)=0$, and the definition implies

$$B_{R(t)}=\{y\in\mathbb{R}^2,\ \phi^*(y)>t\}.$$

$$\int_{\phi^*>t}e^{U_0}|x|^{-2\alpha_0}=\int_{\Omega(t)}Ve^w,\quad t\in[0,t_+].$$

$$\int_{B_{R_0}}|x|^{-2\alpha_0}e^{U_0}|\phi^*|^2=\int_{\Omega}Ve^w\phi^2.$$

Then for almost all t

$$
\begin{aligned}
(3\text{-}2)\quad -\frac{d}{dt}\int_{\Omega(t)}|\nabla\phi|^2&=\int_{\phi=t}|\nabla\phi|\\
&\geq\left(\int_{\phi=t}(Ve^w)^{\frac12}\,ds\right)^2\left(\int_{\phi=t}\frac{Ve^w}{|\nabla\phi|}\,ds\right)^{-1},\\
&=\left(-\frac{d}{dt}\int_{\Omega(t)}Ve^w\right)^{-1}\left(\int_{\phi=t}(Ve^w)^{\frac12}\,ds\right)^2\\
&\geq\frac12\left(8\pi(1-\alpha_0)-\int_{\Omega(t)}Ve^w\right)\left(\int_{\Omega_t}Ve^w\right)\left(-\frac{d}{dt}\int_{\Omega(t)}Ve^w\right)^{-1},\\
&=\frac12\left(8\pi(1-\alpha_0)-\int_{\phi^*>t}e^{U_0}|x|^{-2\alpha_0}\right)\\
&\qquad\times\left(\int_{\phi^*>t}e^{U_0}|x|^{-2\alpha_0}\right)\left(-\frac{d}{dt}\int_{\phi^*>t}e^{U_0}|x|^{-2\alpha_0}\right)^{-1}.
\end{aligned}
$$

Applying the same computation to ϕ^* we see that for almost all t, since ϕ^* is radial, we have

$$
\begin{aligned}
-\frac{d}{dt}\int_{\Omega(t)}|\nabla\phi^*|^2&=\int_{\phi^*=t}|\nabla\phi^*|\\
&=\left(\int_{\phi^*=t}|x|^{-\alpha_0}e^{U_0/2}\,ds\right)^2\left(\int_{\phi^*=t}\frac{|x|^{-2\alpha_0}e^{U_0}}{|\nabla\phi^*|}\,ds\right)^{-1}\\
&=\left(-\frac{d}{dt}\int_{\Omega(t)}|x|^{-2\alpha_0}e^{U_0}\right)^{-1}\left(\int_{\phi^*=t}|x|^{-\alpha_0}e^{U_0/2}\,ds\right)^2.
\end{aligned}
$$

Direct computation on U_0 gives

$$\left(-\frac{d}{dt}\int_{\Omega(t)}|x|^{-2\alpha_0}e^{U_0}\right)^{-1}=\frac{1}{2}\left(8\pi(1-\alpha_0)-\int_{\phi^*>t}e^{U_0}|x|^{-2\alpha_0}\right)\left(\int_{\phi^*>t}e^{U_0}|x|^{-2\alpha_0}\right).$$

Thus the combination of the two equations above gives

$$-\frac{d}{dt}\int_{\Omega(t)}|\nabla\phi^*|^2=\frac{1}{2}\left(8\pi(1-\alpha_0)-\int_{\phi^*>t}e^{U_0}|x|^{-2\alpha_0}\right) \times\left(\int_{\phi^*>t}e^{U_0}|x|^{-2\alpha_0}\right)\left(-\frac{d}{dt}\int_{\phi^*>t}e^{U_0}|x|^{-2\alpha_0}\right)^{-1} \tag{3-3}$$

for almost all $t\in(0,t_+)$.

Integrating (3-2) and (3-3) for $t\in(0,t_+)$ we have

$$\int_{B_{R_0}}|\nabla\phi^*|^2\le\int_\Omega|\nabla\phi|^2.$$

If $\nu_1\le 1$, we obtain from (3-1) that

$$\begin{aligned}0\ge(\nu_1-1)\int_\Omega Ve^w|\phi|^2&=\int_\Omega|\nabla\phi|^2-\int_\Omega Ve^w|\phi|^2\\&\ge\int_{B_{R_0}}|\nabla\phi^*|^2-\int_{B_{R_0}}e^{U_0}|x|^{-2\alpha_0}|\phi^*|^2.\end{aligned}$$

Thus the first eigenvalue of

$$-\Delta-|x|^{-2\alpha_0}e^{U_0}$$

on B_{R_0} with Dirichlet boundary condition is nonpositive. Since

$$\psi=2(1-\alpha_0)\frac{1-|x|^{2(1-\alpha_0)}}{1+|x|^{2(1-\alpha_0)}}$$

satisfies

$$-\Delta\psi-|x|^{-2\alpha_0}e^{U_0}\psi=0\quad\text{in }\mathbb{R}^2,$$

we see that $R_0\ge 1$. But

$$\int_{B_1}|x|^{-2\alpha_0}e^{U_0}=4\pi(1-\alpha_0),$$

so we clearly have $\hat{\nu}\ge 0$. From the proof of Bol's inequality we see that the strict inequality holds because Ω has more than one singular point in its interior.

The proof of $\hat{\nu}_2>0$ for a higher threshold of $\int_\Omega Ve^w$ is very similar. If we consider Ω_+ and Ω_-, which are the set of points where ϕ is positive or negative, respectively, the integral of Ve^w on at least one of them is less than or equal to $4\pi(1-\alpha_0)$. The argument of redistribution of mass can be applied to at least one of them. Then we see that either one of them has the integral of Ve^w strictly less than $4\pi(1-\alpha_0)$, which leads to a contradiction, or both regions have their integral equal to $4\pi(1-\alpha_0)$. In the latter case, the equality cannot hold because 0 can only be in the interior of at most one region. Then at least one region either does not contain 0 in its interior, or is not simply connected. The strict inequality holds in at least one region. Thus $\hat{\nu}_2 > 0$ if $\int_\Omega Ve^w \le 8\pi(1-\alpha_0)$. □

4. The proof of Theorem 1.2

First we claim that ϕ in the linearized equation is actually bounded. Recall that u_1 satisfies

$$\Delta u_1 + H_1 e^{u_1} = 0 \quad \text{in } \mathbb{R}^2,$$
$$u_1(x) = \left(-4 + 2\alpha_1 + 2\alpha_2 - 2\sum_{i=3}^{n}\beta_i\right)\log|x| + O(1) \quad \text{at } \infty.$$

By the equation for ϕ and the mild growth rate of ϕ at infinity, we have

$$\phi(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2}\log|x-y|H_1(y)e^{u_1(y)}\phi(y)\,dy + c, \qquad x\in\mathbb{R}^2,$$

for some $c\in\mathbb{R}$.

Differentiating the equation above, we have

$$\partial_i\phi(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{x_i-y_i}{|x-y|^2}H_1e^{u_1}\phi(y)\,dy, \qquad i=1,2,\;\; x\in\mathbb{R}^2.$$

By standard estimates in different regions of $\mathbb{R}^2$, it is easy to see that

$$\partial_i\phi(x) = A\frac{x_i}{|x|^2} + O(|x|^{-1-\delta}), \qquad |x|>1,\;\; i=1,2,$$

for $A = \frac{1}{2\pi}\int_{\mathbb{R}^2}H_1e^{u_1}\phi$ and some $\delta>0$. Thus the assumption $\phi(x) = o(\log|x|)$ actually implies

$$\int_{\mathbb{R}^2}H_1e^{u_1}\phi = 0. \tag{4-1}$$

and

$$\phi(x) = C + O(|x|^{-\delta}), \qquad |x|>1, \tag{4-2}$$

for some $\delta>0$.

Next we make a transformation on the equation for u_1. Without loss of generality we assume $p_1 = 0$ and we write H_1 as

$$H_1(x) = |x|^{-2\alpha_1}V_1.$$

Let

$$u_2(x) = u_1\left(\frac{x}{|x|^2}\right) - (4 - 2\alpha_1)\log|x|,$$

then direct computation shows that

$$\Delta u_2 + V_2 e^{u_2} = 0 \quad \text{in } \mathbb{R}^2$$

and

$$u_2(x) = (-4 + 2\alpha_1)\log|x| + O(1) \quad \text{at } \infty,$$

where $V_2(x) = V_1(x/|x|^2)$. It is also easy to verify that

$$\int_{\mathbb{R}^2} H_1 e^{u_1} = \int_{\mathbb{R}^2} V_2 e^{u_2}. \tag{4-3}$$

Setting $\phi_1(x) = \phi(x/|x|^2)$, we see that

$$\Delta\phi_1 + V_2 e^{u_2}\phi_1 = 0 \quad \text{in } \mathbb{R}^2.$$

Here we note that by the bound of ϕ_1 near the origin, the equation above holds in the whole $\mathbb{R}^2$.

First, by the asymptotic behavior of u_1 at infinity, integration of the equation for u_1 gives

$$\frac{1}{2\pi}\int_{\mathbb{R}^2} H_1 e^{u_1} = 4 - 2(\alpha_1 + \alpha_2) + 2\sum_{i=3}^{n}\beta_i \le 4(1 - \alpha_2). \tag{4-4}$$

From the definition of ϕ we have $\phi_1(x) \to c_0$ as $x \to \infty$ for some $c_0 \in \mathbb{R}$. Without loss of generality we assume $c_0 \le 0$. By the same estimate for ϕ we have

$$\int_{\mathbb{R}^2} V_2 e^{u_2}\phi_1 = 0. \tag{4-5}$$

By (4-3) and (4-4) we have

$$\int_{\mathbb{R}^2} V_2 e^{u_2} \le 8\pi(1 - \alpha_2).$$

Let ϕ_2 be an eigenfunction corresponding to eigenvalue $\hat{\nu}$:

$$\begin{cases} -\Delta\phi_2 - V_2 e^{u_2}\phi_2 = \hat{\nu} V_2 e^{u_2}\phi_2 \quad \text{in } \mathbb{R}^2, \\ \lim_{x\to\infty}\phi_2(x) = c_0 \le 0, \\ \int_{\mathbb{R}^2} V_2 e^{u_2}\phi_2 = 0. \end{cases}$$

We claim that $\hat{\nu} > 0$.

By way of contradiction we assume that $\hat{\nu} \le 0$. By setting $\nu = 1 + \hat{\nu}$ we clearly have $\nu \le 1$ and

$$\Delta\phi_2 + \nu V_2 e^{u_2}\phi_2 = 0 \quad \text{in } \mathbb{R}^2.$$

Let $\Omega^+ = \{x;\ \phi_2(x) > c_0\}$, then by the same argument as in the proof of the previous proposition we must have

$$\int_{\Omega^+} V_2 e^{u_2} = c_2(c_0) \geq 4\pi(1-\alpha_2)$$

and if the equality holds, we have $c_0 = 0$. Then there is one singular source with negative index $-4\pi\alpha_2$ in the interior of Ω_+, which has to be simply connected at the same time. All other singular sources (which have positive indexes) are not in the interior of Ω_+.

Let ϕ^* be the rearrangement of ϕ_2 in Ω_+. By the previous argument we have

$$\int_{\Omega_+} |\nabla \phi_2|^2 \leq \int_{B_{R_1}} |\nabla \phi^*|^2$$

and $c_2(c_0) = \int_{B_{R_1}} |x|^{-2\alpha_2} e^{U_0}$. Let

$$c_1 = \min_{\mathbb{R}^2} \phi_2$$

and we set R_2 to make

$$\int_{B_{R_2}\setminus B_{R_1}} |x|^{-2\alpha_2} e^{U_0} = \int_{\mathbb{R}^2 \setminus \Omega_+} V_1 e^{u_2}.$$

Note that R_2 could be ∞. Then we define a radial function ϕ^{**} from $B_{R_2} \setminus B_{R_1} \to \mathbb{R}$: for any $y \in B_{R_2} \setminus B_{R_1},\ |y| = r$,

$$\phi^{**}(r) = \inf\{t \in (c_1, c_0) \mid R^{(-)}(t) < r\},$$

where $R^{(-)}(t)$ is defined by

$$\int_{B_{R_2}\setminus B_{R^{(-)}(t)}} |x|^{-2\alpha_2} e^{U_0} = \int_{\phi_2 < t} V_2 e^{u_2}, \quad \text{for all } t \in (c_1, c_0).$$

The definition of ϕ^{**} implies

$$\int_{B_{R_2}\setminus B_{R_1}} |x|^{-2\alpha_2} e^{U_0} |\phi^{(**)}|^2 = \int_{\Omega^-} V_2 e^{u_2} |\phi_2|^2, \quad \Omega^- = \mathbb{R}^2 \setminus \Omega_+,$$

and

$$\int_{B_{R_2}\setminus B_{R_1}} |x|^{-2\alpha_2} e^{U_0} \phi^{(**)} = \int_{\Omega^-} V_2 e^{u_2} \phi_2, \quad \Omega^- = \mathbb{R}^2 \setminus \Omega_+.$$

The symmetrization also gives

$$\int_{B_{R_2}\setminus B_{R_1}} |\nabla \phi^{**}|^2 \leq \int_{\Omega^-} |\nabla \phi_2|^2.$$

Now we set

$$\phi_*: B_{R_2} \to \mathbb{R}, \quad \phi_* \text{ radial } \phi_*(r) = \begin{cases} \phi^*(r) & \text{for } r \in [0, R_1], \\ \phi^{**}(r) & \text{for } r \in [R_1, R_2). \end{cases}$$

Since ϕ_* is continuous, monotone, we have

$$\int_{B_{R_2}} |\nabla \phi_*|^2 \le \int_{\mathbb{R}^2} |\nabla \phi_2|^2 = \int_{\mathbb{R}^2} V_2 e^{u_2} |\phi_2|^2 = \int_{B_{R_2}} |x|^{-2\alpha_2} e^{U_0} |\phi_*|^2.$$

From the definition of ϕ_* we also have

$$\int_{B_{R_2}} |x|^{-2\alpha_2} e^{U_0} \phi_* = 0.$$

Let

$$K^* = \inf\Big\{ \textstyle\int_{\mathbb{R}^2} |\nabla \psi|^2 \, dx, \ \psi \text{ is radial}, \ \int_{\mathbb{R}^2} |x|^{-2\alpha_2} e^{U_0} \psi = 0, \ \int_{\mathbb{R}^2} |x|^{-2\alpha_2} e^{U_0} \psi^2 = 1 \Big\}.$$

By Hölder's inequality we have

$$\left| \int_{\mathbb{R}^2} |x|^{-2\alpha_2} e^{U_0} \psi \, dx \right| \le \left(\int_{\mathbb{R}^2} |x|^{-2\alpha_2} e^{U_0} \psi^2 \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^2} |x|^{-2\alpha_2} e^{U_0} \right)^{\frac{1}{2}},$$

which implies that the minimizer (say ψ^*) also satisfies

$$\int_{\mathbb{R}^2} |x|^{-2\alpha_2} e^{U_0} \psi^* = 0.$$

Clearly the minimizer ψ^* satisfies

$$\Delta \psi^* + K^* |x|^{-2\alpha_2} e^{U_0} \psi^* = 0 \quad \text{in } \mathbb{R}^2.$$

From ϕ_* and the definition of K^* we already know $K^* \in (0, 1)$. Our goal is to show that $K^* = 1$ by an argument of Chang, Chen and Lin [Chang et al. 2003]. The minimizer ψ^* should only change sign once. Let ξ_0 be the zero of ψ^*.

Integrating the equation for ψ^*, we have

$$r \frac{d}{dr} \psi^*(r) = -K^* \int_0^r |s|^{1-2\alpha_2} e^{U_0(s)} \psi^*(s) \, ds = K^* \int_r^\infty s^{1-2\alpha_2} e^{U_0} \psi^*(s) \, ds < 0,$$

for $r > \xi_0$. Thus ψ^* is decreasing for $r \ge \xi_0$ and $r \frac{d}{dr} \psi^*(r) \to 0$ as $r \to \infty$. The equation for ψ^* also gives

$$\left| r \frac{d}{dr} \psi^*(r) \right| \le K^* \left(\int_r^\infty |s|^{1-2\alpha_2} e^{U_0} (\psi^*(s))^2 \, ds \right)^{\frac{1}{2}} \left(\int_r^\infty s^{1-2\alpha_2} e^{U_0(s)} \, ds \right)^{\frac{1}{2}} \le C r^{-1},$$

for large r. Therefore $\lim_{r \to \infty} \psi^*(r)$ exists and is a negative constant.

Let

$$\psi(r) = 2(1-\alpha_2) \frac{1 - r^{2(1-\alpha_2)}}{1 + r^{2(1-\alpha_2)}}.$$

Then ψ satisfies

$$\Delta\psi + r^{-2\alpha_2} e^{U_0}\psi = 0 \quad \text{in } \mathbb{R}^2.$$

It is easy to obtain the following from the equations for ψ and for ψ^*:

$$r\left(\frac{\psi^*}{\psi(r)}\right)' = \frac{1-K^*}{\psi^2(r)} \int_0^r s^{1-2\alpha_2} e^{U_0(s)} \psi^*(s)\psi(s)\,ds.$$

If $\xi_0 < 1$, $\frac{\psi^*(r)}{\psi(r)}$ is increasing from $r \in (0, \xi_0]$. Clearly this is not possible because otherwise this could happen:

$$0 < \frac{\psi^*(0)}{\psi(0)} < \frac{\psi^*(\xi_0)}{\psi(\xi_0)} = 0.$$

On other hand, we observe that it is also absurd to have $\xi_0 > 1$, indeed, had this happened, we would start from

$$\lim_{R\to\infty} R\left(\frac{\psi^*}{\psi}\right)'(R)\psi^2(R) - r\left(\frac{\psi^*}{\psi}\right)'(r)\psi^2(r) = (1-K^*)\int_r^\infty s^{1-2\alpha_2} e^{U_0}\psi^*(s)\psi(s)\,ds.$$

Since

$$\lim_{R\to\infty} R\Big(\frac{d}{dr}\psi^*(R)\psi(R) - \psi'(R)\psi^*(R)\Big) = 0,$$

we have

$$-r\left(\frac{\psi^*}{\psi}\right)'\psi^2(r) = (1-K^*)\int_r^\infty s^{1-2\alpha_2} e^{U_0(s)}\psi^*(s)\psi(s)\,ds.$$

If $\xi_0 > 1$, $(\psi^*(r))/(\psi(r))$ is decreasing for $r > 1$, which yields

$$0 = \frac{\psi^*(\xi_0)}{\psi(\xi_0)} > \lim_{r\to\infty} \frac{\psi^*(r)}{\psi(r)} = -\frac{1}{2(1-\alpha_2)} \lim_{r\to\infty} \psi^*(r) > 0.$$

This contradiction proves that $\xi_0 = 1$ and $\psi^*(r)\psi(r) > 0$ for all $r \neq 1$. Furthermore

$$\begin{aligned} 0 &= \lim_{r\to\infty}\left(\frac{d}{dr}\psi^*(r)\psi(r) - \frac{d}{dr}\psi(r)\psi^*(r)\right)r \\ &= (1-K^*)\int_0^\infty s^{1-2\alpha_2} e^{U_0}\psi^*(s)\psi(s)\,ds. \end{aligned}$$

Thus we have proved that $K^* = 1$ and the desired contradiction. Theorem 1.2 is established. □

The proof of Theorem 1.3 is very similar, we just use Kelvin transformation to move the negative singularity to infinity, then use the same argument with the standard Bol's inequality for nonnegative indexes.

5. The proof of Theorem 1.1.

Our argument follows from a previous result of Bartolucci and Lin [2009] for nonnegative indexed singularities. We prove by way of contradiction. Suppose u is a solution of (1-3) and a nonzero function $\tilde{\phi} \in H_0^1(\Omega)$ is a solution of

$$\begin{cases} -\Delta\tilde{\phi} - \lambda(He^u)/\big(\int_\Omega He^u\,dx\big)\tilde{\phi} + \lambda\big(\int_\Omega He^u\tilde{\phi}\big)(He^u)/\big(\int_\Omega He^u\big)^2 = 0 & \text{in } \Omega, \\ \tilde{\phi} = 0 & \text{on } \partial\Omega. \end{cases}$$

Let $w = u + \log\lambda - \log\big(\int_\Omega He^u\,dx\big)$ and

$$\phi = \tilde{\phi} - \frac{\int_\Omega He^u\tilde{\phi}}{\int_\Omega He^u};$$

we have

$$\begin{cases} \Delta\phi + He^w\phi = 0 & \text{in } \Omega, \\ \phi = c_0 & \text{on } \partial\Omega, \\ \int_\Omega He^w\phi = 0, \\ \lambda = \int_\Omega He^w \le 8\pi(1-\alpha_0). \end{cases} \tag{5-1}$$

Without loss of generality we assume $c_0 \le 0$. Our goal is to show that $\phi \equiv c_0$, which further leads to $c_0 = 0$, obviously. If $c_0 = 0$, ϕ must change sign if not identically equal to 0. But this situation is ruled out by Proposition 3.1 that $\nu_2 > 0$. So we only consider $c_0 < 0$. Let

$$\Omega_+ = \{x \in \Omega,\ \phi(x) > 0\}, \quad \Omega_- = \{x \in \Omega,\ \phi(x) < 0\}.$$

Clearly $\text{dist}(\Omega_+, \partial\Omega) > 0$. Then if $\int_{\Omega_+} He^w \le 4\pi(1-\alpha_0)$ there is no way for ϕ to satisfy (5-1) on Ω_+ without being identically zero. Then using the same rearrangement argument as in the proof of Theorem 1.2 we can also reach the following conclusion: if ϕ_2 is a solution of

$$\begin{cases} -\Delta\phi_2 - \lambda e^u w\phi_2 = \nu e^u w\phi_2 & \text{in } \Omega, \\ \phi_2 = c_0 & \text{on } \partial\Omega, \end{cases}$$

then $\nu > 0$. The remaining part of the proof of Theorem 1.1 follows by standard argument in [Chang et al. 2003] and [Bartolucci and Lin 2009]. We include it with necessary modification.

If we use L_λ to denote the linearized operator of (1-3), we know that all eigenvalues of L_λ are strictly positive for $\lambda \in [0, 8\pi(1-\alpha_0)]$. By using the improved Moser–Trudinger inequality [Malchiodi and Ruiz 2011], one can easily find a solution of (1-3) by the direct minimization method. By the uniform estimate of the linearized equation and standard elliptic estimate we have: for any $\epsilon \in (0, 8\pi(1-\alpha_0))$,

$$\|u_\lambda\|_\infty \le \lambda C_\epsilon, \tag{5-2}$$

for some $C_\epsilon > 0$, $\lambda \in [0, 8\pi(1-\alpha_0)]$ and u_λ as a solution of (1-3). Let S_λ be the solution's branch for (1-3) bifurcating from $(u, \lambda) = (0, 0)$. The standard bifurcation theory of Crandall and Rabinowitz [1975] gives that S_λ is a simple branch near $\lambda = 0$. This means that for $\lambda > 0$ small there exists one and only solution for (1-3) and S_λ is smooth in $C^2(\Omega) \times \mathbb{R}$. By the implicit function theorem (because L_λ has positive first eigenvalue) S_λ can be extended uniquely for $\lambda \in (0, 8\pi(1-\alpha_0))$. If for any given $\lambda \in (0, 8\pi(1-\alpha_0)$ there is another solution, it implies the other solution's branch does not bend in $[0, 8\pi(1-\alpha_0))$. By the uniform estimate (5-2), this second branch intersects S_λ at $(u, \lambda) = (0, 0)$. This contradiction proves the uniqueness for $\lambda \in [0, 8\pi(1-\alpha_0))$. If a solution exists for $\lambda = 8\pi(1-\alpha_0)$, the implicit function theorem and the uniqueness result can be combined to prove the uniqueness in this case as well. Theorem 1.1 is established. □

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EXTENSIONS OF ALMOST FAITHFUL PRIME IDEALS IN VIRTUALLY NILPOTENT MOD-p IWASAWA ALGEBRAS

WILLIAM WOODS

Let G be a nilpotent-by-finite compact p-adic analytic group for some $p > 2$, and $H = \mathbf{FN}_p(G)$ be its finite-by-(nilpotent p-valuable) radical. Fix a finite field k of characteristic p, and write kG for the completed group ring of G over k. We show that almost faithful G-stable prime ideals P of kH extend to prime ideals PkG of kG. This is a crucial building block in understanding the prime spectrum of kG.

Introduction

Let G be a compact p-adic analytic group and k a pseudocompact ring. Denote by kG (or $k[\![G]\!]$) the completed group ring (sometimes known as the *Iwasawa algebra*) of G over k.

In a survey paper, Ardakov and Brown posed the following (imprecise) question in the cases when $k = \mathbb{F}_p$ or $\mathbb{Z}_p$:

Question [Ardakov and Brown 2007, Question G]. Is there a mechanism for constructing ideals of kG which involves neither central elements nor closed normal subgroups?

A negative answer was given in [Ardakov 2012] for prime ideals in the case when G is a nilpotent p-valuable group and k is a field of characteristic p. Explicitly: Let P be a prime ideal of kG. Then there is a closed normal subgroup $P^\dagger \le G$ such that P contains the augmentation ideal $\ker(kG \to k[\![G/P^\dagger]\!])$ [Ardakov 2012, §1.3], with the following properties. Firstly, $\overline{G} := G/P^\dagger$ is torsion-free [Ardakov 2012, Lemma 5.3(c)], and hence remains p-valuable [Lazard 1965, III, 3.1.7.6; IV, 3.4.2]. Secondly, and writing $\overline{(\cdot)}$ for the natural ring homomorphism $kG \to k\overline{G}$

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given by quotienting by the above augmentation ideal, we have that $\overline{P}$ is *controlled* by $Z := Z(\overline{G})$ [Ardakov 2012, Theorem 8.4], that is, $(\overline{P} \cap kZ)k\overline{G} = \overline{P}$. Hence P can be completely understood in terms of the augmentation ideal of the closed normal subgroup $P^\dagger$ and the ideal $\overline{P} \cap kZ$ of the central subring $kZ \subseteq k\overline{G}$.

In this paper, we wish to take some steps toward extending this result.

Henceforth, we will take G to be a nilpotent-by-finite compact p-adic analytic group, and k a finite field of characteristic p. Recall the characteristic open subgroup $H = \mathbf{FN}_p(G)$, the *finite-by-(nilpotent p-valuable) radical* of G, defined in [Woods 2018, Theorem C]. This plays an important role in the structure of the group G; for instance, see the structure theorem [Woods 2018, Theorem D].

In this paper, we demonstrate a connection between certain prime ideals of kH and those of kG. Recall [Ardakov 2012, §1.3] that a prime ideal P of kH is *faithful* if the natural map $H \to (kH/P)^\times$ is injective. The main result of this paper is:

Theorem A. *Fix some prime $p > 2$. Let G be a nilpotent-by-finite compact p-adic analytic group, $H = \mathbf{FN}_p(G)$, and k a finite field of characteristic p. Let P be an almost faithful, G-stable prime ideal of kH. Then PkG is a prime ideal of kG.*

Anticipating a future application, we actually prove something slightly stronger: not just that PkG is prime, but that $P(kG)_\alpha$ is prime for certain *central 2-cocycle twists* $(kG)_\alpha$ of kG; see [Woods 2016, §4.2] for details of where these arise. However, for the purposes of this paper, this is just a formality; the proofs are identical, and the reader would lose nothing by assuming $(kG)_\alpha = kG$ throughout.

Recall the results of [Woods 2016], which relate the structure of the ring kH to that of $k'N$, where $N = H/\Delta^+(H)$ is the largest nilpotent p-valued quotient of H and k'/k is a finite field extension. We will combine these results with Theorem A in future work to shed light on the prime spectrum of kG by relating it to that of $k'N$, which, by the results of Ardakov [2012] above, is already well understood. As the arguments required to make these deductions are rather involved and of a different nature to those required to prove Theorem A, we will say no more about them here, and refer the reader to [Woods 2017] instead.

The proof of Theorem A, given in Propositions 3.7 and 3.8, comprises several distinct technical elements, which we outline below.

Firstly, some notation: as in [Woods 2018], we will write throughout this paper

$$\Delta(G) = \{x \in G \mid [G : \mathbf{C}_G(x)] < \infty\},$$
$$\Delta^+(G) = \{x \in \Delta \mid o(x) < \infty\},$$

where $o(x)$ denotes the order of x. We will also often simply write Δ and Δ^+ to denote $\Delta(G)$ and $\Delta^+(G)$. For the basic properties of these closed, characteristic subgroups, see [Woods 2018, Lemma 1.3 and Theorem D].

Suppose until otherwise stated that G has no nontrivial finite normal subgroups, and let $H = \mathbf{FN}_p(G)$. Note that H is p-valuable [Lazard 1965, III, 2.1.2], and that G acts on the set of p-valuations of H as follows: if α is a p-valuation on H and $g \in G$, then we may define a new p-valuation $g \cdot \alpha$ on H by

$$g \cdot \alpha(x) = \alpha(g^{-1}xg).$$

Recall the definition of an *isolated* orbital (closed) subgroup L of H from [Woods 2018, Definition 1.4], and that normal subgroups are automatically orbital. We show in Definition 1.2 that, if ω is a p-valuation on H and L is a closed isolated normal subgroup of H, then ω induces a *quotient* p-valuation Ω on H/L. We also define the (t, p)*-filtration* (actually a p-valuation) on a free abelian pro-p group A of finite rank in Definition 1.4; this is a particularly "uniform" p-valuation on A, analogous to the p-adic valuation v_p on $\mathbb{Z}_p$.

It is now easy to show the following.

Theorem B. *With the above notation, let L be a proper closed isolated normal subgroup of H containing the commutator subgroup $[H, H]$. Then there exists a p-valuation ω on H with the following properties*:

(i) *ω is G-invariant.*

(ii) *There exists a real number $t > (p-1)^{-1}$ such that $\omega|_L > t$, and the quotient p-valuation induced by ω on H/L is the (t, p)-filtration.*

Now continue with the above notation, and let P be a faithful, G-stable prime ideal of kH. We may consider the ring kG/PkG, and fix a crossed product decomposition $(kH/P) * F$ of this ring, where $F = G/H$. We would like to study automorphisms σ of the ring kH/P induced by inner automorphisms of G, in order to determine whether the extension of P to kG remains prime; cf. [McConnell and Robson 1987, Proposition 8.12]. In order to do this, we will "approximate" the ring kH/P by imposing a filtration on it and studying the induced action of σ on a related graded ring.

A filtration, which we will denote f_1, is defined on kH/P in [Ardakov 2012, proof of Theorem 8.6]. It is partly constructed out of a p-valuation ω on H, and we will choose ω to be the one given by Theorem B; this will allow us to retain some control over the filtration f_1. Indeed, if $\omega(\sigma(h)) = \omega(h)$ for all $h \in H$, then we will be able to deduce that $f_1(\sigma(x)) = f_1(x)$ for certain $x \in kH/P$, which will be crucial to our calculations later.

The filtration f_1 is known to have nice properties, as follows. Ardakov [2012, Theorem 8.4] shows that $P = \mathfrak{p}kH$ for some faithful prime ideal $\mathfrak{p}$ of kZ, where Z is the centre of H; and, furthermore, in [2012, proof of Theorem 8.6], that there exists an integer e such that

$$(\dagger) \qquad \mathrm{gr}_{f_1}(kH/P) \cong (\mathrm{gr}_{v_1}(kZ/\mathfrak{p}))[Y_1, \ldots, Y_e],$$

where $v_1 = f_1|_{kZ/\mathfrak{p}}$ is a valuation, and $\mathrm{gr}_{f_1}(kZ/\mathfrak{p})$ is a commutative domain.

Unfortunately, even though we will have $f_1(\sigma(x)) = f_1(x)$ for many $x \in kH/P$, this will not be true for all $x \in kH/P$, so σ does not induce a well-defined action on $\mathrm{gr}_{f_1}(kH/P)$. Our next theorem plugs this gap.

Write Q' for the classical ring of quotients of $kZ/\mathfrak{p}$. We will also write Q for a certain partial ring of quotients of kH/P containing Q', and we may naturally form $Q * F$ as an overring of $(kH/P) * F$. Now, v_1 extends uniquely to Q', and f_1 extends uniquely to Q, and we will continue to call these extensions v_1 and f_1, respectively. The conjugation action of G on H induces an action of F on the sets of filtrations of Q' and Q. Suppose v_1 has some orbit $\{v_1, \ldots, v_s\}$; then, due to our judicious earlier choice of ω by Theorem B(i), f_1 will turn out to have orbit $\{f_1, \ldots, f_s\}$, where $v_i = f_i|_{kZ/\mathfrak{p}}$ for each $1 \le i \le s$.

Theorem C. *In the above notation; there exists a G-stable filtration $\hat{f}$ on $Q * F$ such that*

(i) $\mathrm{gr}_{\hat{f}}(Q * F) \cong \mathrm{gr}_{\hat{f}}(Q) * F$, *where the right-hand side is some crossed product*,

(ii) $\mathrm{gr}_{\hat{f}}(Q) \cong \bigoplus_{i=1}^{s} \mathrm{gr}_{f_i}(Q)$,

(iii) $\mathrm{gr}_{f_i}(Q) \cong (\mathrm{gr}_{v_i} Q')[Y_1, \ldots, Y_e]$ *for all* $1 \le i \le s$,

where s and e are determined as in (†), *and the action of F in the crossed product of* (*i*) *permutes the s summands in the decomposition of* (*ii*) *transitively by conjugation.*

We combine Theorems B and C as follows.

Theorem C, of course, only invokes (†) in the case when $Q \neq Q'$, so we suppose that we are in this case, which occurs precisely when H is nonabelian. Take L to be the smallest closed isolated normal subgroup of H containing both the isolated derived subgroup H' [Woods 2018, Theorem B] and the centre Z of H. Now L is a proper subgroup by Lemma 3.5, and we will choose ω for L as in Theorem B. We may arrange it in (†) and Theorem C so that, for some $l \le e$, the elements $Y_1, \ldots, Y_l$ correspond to a set of elements $y_1, \ldots, y_l \in H$, whose images $x_1, \ldots, x_l$ form a $\mathbb{Z}_p$-module basis for H/L. By the construction of $\hat{f}$ and the choice of ω in Theorem B(ii), we can directly compute that $Y_i \in Q_t/Q_{t^+}$, corresponding to the fact that $\omega(y_i) = t$.

Some calculations allow us to show the following.

Theorem D. *Take an automorphism σ of H. Suppose that the induced automorphism on $\mathrm{gr}_{\hat{f}}(Q * F)$ fixes each of the valuations $v_1, \ldots, v_s$ and fixes each of the elements $Y_1, \ldots, Y_l$. Then the induced automorphism on H/L* (*which we view as a matrix $M_\sigma \in \mathrm{GL}_l(\mathbb{Z}_p)$*) *lies in the first congruence subgroup of $\mathrm{GL}_l(\mathbb{Z}_p)$, i.e., it takes the form $M_\sigma \in 1 + pX$ for some $X \in M_l(\mathbb{Z}_p)$. In particular, when $p > 2$, σ has finite order if and only if it is the identity automorphism.*

Using Theorem D, a long but elementary argument allows us to answer the question of when the automorphism $\sigma \in \mathrm{Inn}(G)$ induces an *X-inner* automorphism of $\mathrm{gr}(Q)$; see Definition 3.1 and Proposition 3.7 for details. As promised, we can derive from this a special case of Theorem A, in which $\Delta^+ = 1$.

Finally, in Proposition 3.8, we revert to the case when $\Delta^+ \neq 1$. The "untwisting" results of [Woods 2016, Theorems B–E] allow us to understand the prime ideals of kH, along with the conjugation action of G, in terms of the corresponding information for $k'[\![H/\Delta^+]\!]$ for various finite field extensions k'/k. But now, as $\Delta^+(G/\Delta^+) = 1$ and $H/\Delta^+ = \mathbf{FN}_p(G/\Delta^+)$, we may replace G by G/Δ^+, and we are already back in the previous case.

1. *p*-valuations and crossed products

1A. *Preliminaries on p-valuations.*

Definition 1.1. Recall from [Lazard 1965, III, 2.1.2] that a *p-valuation* on a group G is a function $\omega : G \to \mathbb{R} \cup \{\infty\}$ satisfying:

- $\omega(xy^{-1}) \geq \min\{\omega(x), \omega(y)\}$ for all $x, y \in G$.
- $\omega([x, y]) \geq \omega(x) + \omega(y)$ for all $x, y \in G$.
- $\omega(x) = \infty$ if and only if $x = 1$.
- $\omega(x) > \frac{1}{p-1}$ for all $x \in G$.
- $\omega(x^p) = \omega(x) + 1$ for all $x \in G$.

Throughout this paper we will often be considering several p-valuations admitted by a group G, so to clarify we may refer to G together with a p-valuation ω as the p-valued group (G, ω). However, when the p-valuation in question is clear from context, we will simply write G.

Given a p-valuation ω on a group G, we may write

$$G_\lambda := G_{\omega,\lambda} := \omega^{-1}([\lambda, \infty]),$$
$$G_{\lambda^+} := G_{\omega,\lambda^+} := \omega^{-1}((\lambda, \infty]),$$

and define the *graded group*

$$\mathrm{gr}_\omega G := \bigoplus_{\lambda \in \mathbb{R}} G_\lambda / G_{\lambda^+}.$$

Then each element $1 \neq x \in G$ has a *principal symbol*

$$\mathrm{gr}_\omega(x) := xG_{\mu^+} \in G_\mu / G_{\mu^+} \leq \mathrm{gr}_\omega G,$$

where μ is defined such that $\mu = \omega(x)$.

Remark. Let (G, ω) be a p-valued group, and N be an arbitrary subgroup of G. Then $(N, \omega|_N)$ is p-valued. Moreover, if G has finite rank [Lazard 1965, III, 2.1.3], then so does N; and if G is complete with respect to ω and N is a closed subgroup of G, then N is complete with respect to $\omega|_N$.

Definition 1.2. Given an arbitrary complete p-valued group (G, ω) of finite rank, and a closed isolated normal subgroup K (i.e., a closed normal subgroup K such that G/K is torsion-free), we may define the *quotient p-valuation* Ω induced by ω on G/K as follows:

$$\Omega(gK) = \sup_{k \in K}\{\omega(gk)\}.$$

This is defined by Lazard, but the definition is spread across several results, so we collate them here for convenience. The definition in the case of filtered modules is [Lazard 1965, I, 2.1.7], and is modified to the case of filtered groups in [loc. cit, the remark after II, 1.1.4.1]. The specialisation from filtered groups to p-saturable groups is done in [loc. cit, III, 3.3.2.4], where it is proved that Ω is indeed still a p-valuation on G/K; and the general case is stated in [loc. cit, III, 3.1.7.6], and eventually proved in [loc. cit, IV, 3.4.2].

As a partial inverse to the above process of passing to a quotient p-valuation, we prove the following general result about "lifting" p-valuations from torsion-free quotients.

Theorem 1.3. *Let G be a complete p-valued group of finite rank, and N a closed isolated orbital (hence normal) subgroup of G. Suppose we are given two functions*

$$\alpha, \beta : G \to \mathbb{R} \cup \{\infty\},$$

such that α is a p-valuation on G, and β descends to a p-valuation on G/N, i.e., β factors as

$$G \xrightarrow{nat.} G/N \xrightarrow{\bar{\beta}} \mathbb{R} \cup \{\infty\}.$$

Then $\omega = \inf\{\alpha, \beta\}$ is a p-valuation on G.

Proof. Since α and β are both filtrations on G (in the sense of [loc. cit, II, 1.1.1]), by [loc. cit, II, 1.2.10], ω is also a filtration. Following [loc. cit, III, 2.1.2], for ω to be a p-valuation, we need to check the following three conditions:

(i) $\omega(x) < \infty$ for all $x \in G$, $x \neq 1$. This follows from the fact that α is a p-valuation, and hence $\alpha(x) < \infty$ for all $x \in G$, $x \neq 1$.

(ii) $\omega(x) > (p-1)^{-1}$ for all $x \in G$. This follows from the fact that $\alpha(x) > (p-1)^{-1}$ and $\beta(x) > (p-1)^{-1}$ for all $x \in G$ by definition.

(iii) $\omega(x^p) = \omega(x) + 1$ for all $x \in G$. Take any $x \in G$. As α is a p-valuation, we have by definition that $\alpha(x^p) = \alpha(x) + 1$.

If $x \in N$, this alone is enough to establish the condition, as $\omega|_N = \alpha|_N$ (since $\beta(x) = \infty$).

Suppose instead that $x \in G \setminus N$. Then, as N is assumed *isolated* orbital in G, we also have $x^p \in G \setminus N$, so by definition of β we have

$$\beta(x^p) = \bar{\beta}((xN)^p) = \bar{\beta}(xN) + 1 = \beta(x) + 1,$$

with the middle equality coming from the fact that $\bar{\beta}$ is a p-valuation. Now it is clear that $\omega(x^p) = \omega(x) + 1$ by definition of ω. □

Finally, the following function will be crucial. As we will remark in Lemma 1.6, it is easily seen to be a p-valuation.

Definition 1.4. Let A be a free abelian pro-p group of rank $d > 0$ (here written multiplicatively). Choose a real number $t > (p-1)^{-1}$. Then the *(t, p)-filtration* on A is the function $\omega : A \to \mathbb{R} \cup \{\infty\}$ defined by

$$\omega(x) = t + n,$$

where n is the nonnegative integer such that $x \in A^{p^n} \setminus A^{p^{n-1}}$. (By convention, $\omega(1) = \infty$.)

1B. *Ordered bases.*

Definition 1.5. Recall from [Ardakov 2012, §4.2] that an *ordered basis* for a p-valued group (G, ω) is a finite totally ordered set $\{g_1, \dots, g_e\}$ of elements of G such that every element $x \in G$ can be uniquely written as the ordered product

$$x = \prod_{1 \le i \le e} g_i^{\lambda_i}$$

for some $\lambda_i \in \mathbb{Z}_p$, and

$$\omega(x) = \inf_{1 \le i \le e} \{\omega(g_i) + v_p(\lambda_i)\},$$

where v_p is the usual p-adic valuation on $\mathbb{Z}_p$.

Remark. An ordered basis for (G, ω) need not be an ordered basis for (G, ω') for another p-valuation ω'.

As in [Ardakov 2012], we will often write

$$\boldsymbol{g}^{\lambda} := \prod_{1 \le i \le e} g_i^{\lambda_i}$$

as shorthand, where $\lambda = (\lambda_1, \dots, \lambda_e) \in \mathbb{Z}_p^e$.

We now demonstrate some properties of the function ω given in Definition 1.4.

Lemma 1.6. *Let A, t and ω be as in Definition 1.4.*

(i) *The (t, p)-filtration ω is a p-valuation on A.*

(ii) *Suppose we are given a $\mathbb{Z}_p$-module basis $B = \{a_1, \dots, a_d\}$ for A, and a p-valuation α on A satisfying $\alpha(a_1) = \cdots = \alpha(a_d) = t$. Then α is the (t, p)-filtration on A, and B is an ordered basis for (A, α).*

(iii) *The* (t, p)*-filtration* ω *is completely invariant under automorphisms of* A, *i.e.*, *the subgroups* $A_{\omega,\lambda}$ *and* A_{ω,λ^+} *are characteristic in* A.

Proof. (i) This is a trivial check from Definitions 1.1 and 1.4.

By [Lazard 1965, III, 2.2.4], we see that

$$\alpha(a_1^{\lambda_1} \cdots a_d^{\lambda_d}) = t + \inf_{1 \le i \le d} \{v_p(\lambda_i)\},$$

which is precisely the (t, p)-filtration.

(ii) The subgroups A^{p^n} are clearly characteristic in A. □

Remark. A notion of (t, p)-filtration is defined in [Lazard 1965, II, 3.2.1] for a much larger class of compact p-adic analytic groups. In the special case of a free abelian pro-p group of finite rank, the definition given there is equivalent to ours.

Recall from [Woods 2018, Definitions 1.1 and 1.4] that a closed subgroup H of a profinite group G is (G-)*orbital* if it has finitely many G-conjugates, and *isolated orbital* if any G-orbital $H' \gneq H$ satisfies $[H' : H] = \infty$.

The following is a general property of ordered bases.

Lemma 1.7. *Let* (G, ω) *be a complete* p*-valued group of finite rank*, *and* N *a closed isolated normal subgroup of* G. *Then there exist sets* $B_N \subseteq B_G$ *such that* B_N *is an ordered basis for* $(N, \omega|_N)$ *and* B_G *is an ordered basis for* (G, ω).

Proof. This was established in [Ardakov 2012, proof of Lemma 8.5(a)]. □

Remark. It may be helpful to think of this as follows:

$$B_G = \{ \underbrace{x_1, \dots, x_r}_{B_{G/N}}, \underbrace{x_{r+1}, \dots, x_s}_{B_N} \},$$

where $B_{G/N} = B_G \setminus B_N$ is in fact some appropriate preimage in G of any ordered basis for $(G/N, \Omega)$, where Ω is the quotient p-valuation.

Lemma 1.8. *Let* (G, α) *be a complete* p*-valued group of finite rank*, *and* N *a closed isolated orbital* (*hence normal* [*Woods 2018, Definition* 1.4, *Lemma* 1.10; *Ardakov 2012, Proposition* 5.9]) *subgroup of* G. *Take also a* p*-valuation* $\bar{\beta}$ *on* G/N. *Suppose we are given sets*

$$B_G = \{ \underbrace{x_1, \dots, x_r}_{B_{G/N}}, \underbrace{x_{r+1}, \dots, x_s}_{B_N} \},$$

such that

- B_N *is an ordered basis for* $(N, \alpha|_N)$,
- B_G *is an ordered basis for* (G, α), *and*
- *the image in* G/N *of* $B_{G/N}$ *is an ordered basis for* $(G/N, \bar{\beta})$.

In the notation of Theorem 1.3, write β for the composite of $G \to G/N$ with $\bar{\beta}$, and form the p-valuation $\omega = \inf\{\alpha, \beta\}$ for G.

Then B_G is an ordered basis for (G, ω).

Proof. We need only check that

$$\omega(\boldsymbol{x}^{\lambda}) = \inf_{1 \le i \le s} \{\omega(x_i) + v_p(\lambda_i)\}$$

for any $\lambda \in \mathbb{Z}_p^s$. But we have by definition that

$$\alpha(\boldsymbol{x}^{\lambda}) = \inf_{1 \le i \le s} \{\alpha(x_i) + v_p(\lambda_i)\}, \qquad \beta(\boldsymbol{x}^{\lambda}) = \inf_{1 \le i \le r} \{\beta(x_i) + v_p(\lambda_i)\},$$

and the result follows trivially. □

1C. *Separating a free abelian quotient.* Results in later sections will require the existence of a p-valuation on an appropriate group G satisfying a certain technical property, which we can now finally state:

Definition 1.9. Let (G, ω) be a complete p-valued group of finite rank, and L a closed isolated normal subgroup containing $[G, G]$ (and hence containing the isolated derived subgroup G', which was defined and written as $G^{(1)}$ in [Woods 2018, Theorem B]). We will say that ω satisfies property (A_L) if there is an ordered basis $\{g_{d+1}, \dots, g_e\}$ for $(L, \omega|_L)$, contained in an ordered basis $\{g_1, \dots, g_e\}$ for (G, ω) (e.g., constructed by Lemma 1.7), such that $\omega(g_1) = \cdots = \omega(g_d)$ and, for all $\ell \in L$, $\omega(g_1) < \omega(\ell)$.

Remark. In the notation of the above definition, suppose ω satisfies $\omega(g_1) < \omega(\ell)$ for all $\ell \in L$. Then, by our earlier remarks, we note that the condition (A_L) is equivalent to the statement that the quotient p-valuation induced by ω on G/L is the (t, p)-filtration for $t := \omega(g_1)$. In particular, whether or not ω satisfies (A_L) is independent of the ordered basis chosen for (G, ω).

Definition 1.10. Following [Lazard 1965, III, 2.1.2], we will say that a group G is *p-valuable* if there exists a p-valuation ω for G, and G is complete with respect to ω and has finite rank.

Lemma 1.11. *Let G be a nilpotent p-valuable group, and L be a closed isolated normal subgroup containing G'. Then there exists some p-valuation ω for G satisfying (A_L).*

Proof. Let α be a p-valuation on G. Take an ordered basis $\{g_{d+1}, \dots, g_e\}$ for $(L, \alpha|_L)$ and extend it to an ordered basis $\{g_1, \dots, g_e\}$ for (G, α) by Lemma 1.7. Fix a number t satisfying

$$(p-1)^{-1} < t \le \inf_{1 \le i \le e} \alpha(g_i).$$

Applying Theorem 1.3 with $N = L$ and $\bar{\beta}$ the (t, p)-filtration on G/L, we see that $\omega = \inf\{\alpha, \beta\}$ is a p-valuation for G; and by Lemma 1.8, $\{g_1, \dots, g_e\}$ is still an ordered basis for (G, ω), so we can check easily that ω satisfies (A_L) by construction. □

Remark. Suppose ω satisfies (A_L). Write $t := \omega(g_1)$. Then, by construction, we have $G_t = G$ and $G_{t^+} = G^p \cdot L$, an open normal subgroup. If we further assume that L is characteristic, then G_{t^+} is also characteristic, by Lemma 1.6(iii); and so, given any automorphism σ of G and any $1 \le i \le d$, we have $\omega(\sigma(g_i)) = t$.

Now let G be a p-valuable group with fixed p-valuation ω, and let $\sigma \in \mathrm{Aut}(G)$. In this subsection and the next, we seek to establish conditions under which a given automorphism σ of G will preserve the "dominant" part of certain elements $x \in G$ (with respect to ω). That is, we are looking for a condition under which

$$\mathrm{gr}_\omega(\sigma(x)) = \mathrm{gr}_\omega(x).$$

Clearly it is necessary and sufficient that the following holds:

$$\omega(\sigma(x)x^{-1}) > \omega(x). \tag{1-1}$$

The results of this paper rely on our ability to invoke the following technical result.

Theorem 1.12. *Let G be a p-valuable group, and let L be a proper closed isolated orbital (hence normal) subgroup containing $[G, G]$, so that we have an isomorphism $\varphi : G/L \to \mathbb{Z}_p^d$ for some $d \ge 1$. Fix a $\mathbb{Z}_p$-basis $\{e_1, \dots, e_d\}$ for $\mathbb{Z}_p^d$. Write $q : G \to G/L$ for the natural quotient map.*

For each $1 \le i \le d$, fix an element $g_i \in G$ with $\varphi \circ q(g_i) = e_i$. Fix an automorphism σ of G preserving L, so that σ induces an automorphism $\bar{\sigma}$ of G/L, and hence an automorphism $\hat{\sigma} = \varphi \circ \bar{\sigma} \circ \varphi^{-1}$ of $\mathbb{Z}_p^d$. Let M_σ be the matrix of $\hat{\sigma}$ with respect to the basis $\{e_1, \dots, e_d\}$.

Suppose there exist some p-valuation ω on G and some real number t with the following properties:

(i) (1-1) *holds for all $x \in \{g_1, \dots, g_d\}$.*

(ii) *$\omega(g_1) = \cdots = \omega(g_d) (= t$, say).*

(iii) *$\omega(\ell) > t$ for all $\ell \in L$.*

Then $M_\sigma - 1 \in pM_d(\mathbb{Z}_p)$.

Remark. The conditions (i)–(iii) imply that ω satisfies (A_L). Conversely, suppose that the hypotheses of the first paragraph of this theorem are satisfied, and suppose moreover that we have a p-valuation ω on G satisfying (A_L). Then any ordered basis $\{g_{d+1}, \dots, g_e\}$ for $(L, \omega|_L)$ has the following properties. Firstly, for any choice of elements $g_1, \dots, g_d \in G$ such that $\varphi \circ q(g_i) = e_i$ for $1 \le i \le d$, we have that $\{g_1, \dots, g_e\}$ is an ordered basis for (G, ω); secondly, with respect to any such ordered basis for G, ω satisfies conditions (i)–(iii) with $t = \omega(g_1)$.

Proof. Define the function $\Omega : \mathbb{Z}_p^d \to \mathbb{R} \cup \{\infty\}$ by

$$\Omega \circ \varphi(gL) = \sup_{\ell \in L}\{\omega(g\ell)\}.$$

By the remarks made in Definition 1.2, Ω is in fact a p-valuation.

By assumption (iii), we see that, for each $1 \le i \le d$ and any $\ell \in L$, we have $\omega(g_i) = \omega(g_i\ell)$, so that

$$\Omega(e_i) = \Omega \circ \varphi(g_i L) = \sup_{\ell \in L}\{\omega(g_i\ell)\} = \omega(g_i),$$

so by assumption (ii), $\Omega(e_i) = t$. Hence, by Lemma 1.6(ii), Ω must be the (t, p)-filtration on $\mathbb{Z}_p^d$. Now, by assumption (i), we have

$$\Omega(\hat{\sigma}(x) - x) > t$$

for all $x \in \{e_1, \dots, e_d\}$, and hence, as $\Omega - t$ takes integer values (by Definition 1.4),

$$\Omega(\hat{\sigma}(x) - x) \ge t + 1,$$

and so $\hat{\sigma}(x) - x \in p\mathbb{Z}_p^d$ for each $x \in \{e_1, \dots, e_d\}$, which is what we wanted to prove. □

1D. *Invariance under the action of a crossed product.*

Definition 1.13. Let R be a ring, and fix a subgroup $G \le R^\times$; let F be a group. Fix a crossed product

$$S = R \underset{\langle\sigma,\tau\rangle}{*} F.$$

Here we are using the notation of [Woods 2016, Definition 4.7 and Notation 4.9], and in particular we write

- $\overline{F}$ for the fixed generating set of S as a free R-module, with elements $\bar{g}$ for each $g \in F$,
- $\sigma : F \to \operatorname{Aut}(R)$ for the *action*, defined by $r\bar{g} = \bar{g}r^{\sigma(g)}$ for all $r \in R$ and $g \in F$,
- $\tau : F \times F \to R^\times$ for the *twisting*, defined by $\bar{g}\bar{h} = \overline{gh}\tau(g, h)$ for all $g, h \in F$,
- $\eta(g, h)$ for the automorphism of R given by conjugation (on the right) by $\tau(g, h)$, so that $\sigma(g)\sigma(h) = \sigma(gh)\eta(g, h)$.

See [Passman 1989, §1] or [Woods 2016, Definition 4.7] for more details on crossed products.

The structural maps in this crossed product may satisfy certain properties, which we will now give names.

(N_G): We will say that σ satisfies (N_G) if the image $\sigma(F)$ normalises G, i.e., $x^{\sigma(f)} \in G$ for all $x \in G$, $f \in F$.

(P_G): We will say that τ satisfies (P_G) if the image $\tau(F, F)$ is a subset of G.

In the case when G is p-valuable, consider the set of p-valuations of G. Then $\mathrm{Aut}(G)$ acts on this set as follows: $\varphi \in \mathrm{Aut}(G)$ sends ω to $\varphi \cdot \omega$, defined by $(\varphi \cdot \omega)(x) = \omega(x^{\varphi})$ for $x \in G$. When S satisfies (P_G), $\tau(F, F) \subseteq G$, so we get $\eta(F, F) \subseteq \mathrm{Inn}(G) \subseteq \mathrm{Aut}(G)$, with elements of G acting by conjugation, so we also make the following definition.

(Q_G): We will say that τ satisfies (Q_G) if every p-valuation ω of G is invariant under all elements of $\eta(F, F)$.

Lemma 1.14. *Any p-valuation ω on G is invariant under inner automorphisms of G. In particular, if τ satisfies* (P_G), *then τ satisfies* (Q_G).

Proof. Fix some $t \in G$. Then, for any $x \in G$, we have

$$\omega(t^{-1}xt) = \omega(x \cdot [x, t]) \geq \min\{\omega(x), \omega([x, t])\} = \omega(x).$$

Hence, by symmetry, $\omega(t^{-1}xt) = \omega(x)$. □

Definition 1.15. Recall, from [Woods 2016, Definition 4.11], that if we have a fixed crossed product

$$S = R \underset{\langle\sigma,\tau\rangle}{*} F \tag{1-2}$$

and a 2-cocycle

$$\alpha \in Z^2_\sigma(F, Z(R^\times)),$$

then we may define the ring

$$S_\alpha = R \underset{\langle\sigma,\tau\alpha\rangle}{*} F,$$

the *2-cocycle twist* (*of R, by α, with respect to the decomposition* (1-2)).

Lemma 1.16. *The automorphism τ satisfies* (Q_G) *if and only if $\tau\alpha$ satisfies* (Q_G).

Proof. As $\alpha(F, F) \subseteq Z(R)^\times$, conjugation by α is the identity automorphism on G, and hence certainly preserves all p-valuations of G. □

These properties will be interesting to us later as they will allow us to invoke the following lemma:

Lemma 1.17. *If σ satisfies* (N_G), *then, given any $g \in F$ and p-valuation ω on G, the function $g \cdot \omega$ given by*

$$(g \cdot \omega)(x) = \omega(x^{\sigma(g)})$$

*is well defined, and is a p-valuation on G. If, further, τ satisfies (Q_G), then the function $g \mapsto (\omega \mapsto g \cdot \omega)$ defines a **group** action of F on the set of p-valuations of G.*

Proof. To see that $g\cdot\omega$ is well defined, we need to check that $\omega(x^{\sigma(g)})$ is meaningful, i.e., $x^{\sigma(g)} \in G$. But this is implied by the hypothesis that the σ satisfy (N_G).

Suppose τ satisfies (Q_G). Then, for any $g, h \in F$, $x \in G$, and p-valuation ω on G, we have

$$\begin{aligned}(g\cdot(h\cdot\omega))(x) &= h\cdot\omega(x^{\sigma(g)})\\ &= \omega(x^{\sigma(g)\sigma(h)})\\ &= \omega(x^{\sigma(gh)\eta(g,h)})\\ &= \omega(x^{\sigma(gh)}) \qquad \text{(by } (\mathsf{Q}_G))\\ &= (gh\cdot\omega)(x).\end{aligned}$$ ☐

The following lemma will allow us to prove the existence of a sufficiently "nice" p-valuation.

Lemma 1.18. *Suppose that the group K acts on the set of p-valuations on G. Let ω be a p-valuation on G with finite K-orbit. Then $\omega'(x) = \inf_{g\in K}(g\cdot\omega)(x)$ defines a K-invariant p-valuation on G.*

Furthermore, if L is a closed isolated characteristic subgroup of G containing G', and ω satisfies (A_L) *(as in Definition 1.9), then ω' satisfies* (A_L).

Proof. The function ω' satisfies the condition in [Lazard 1965, III, 2.1.2.2], since the K-orbit of ω is finite, and is hence a p-valuation that is K-stable by the remark in [Lazard 1965, III, 2.1.2].

Suppose ω satisfies (A_L). That is, for some $t > (p-1)^{-1}$, ω induces the (t, p)-filtration on G/L, and $\omega(\ell) > t$ for all $\ell \in L$. But, given any $g \in K$, clearly $g\cdot\omega$ still induces the (t, p)-filtration on G/L by Lemma 1.6(iii), and

$$(g\cdot\omega)(\ell) = \omega(\ell^{\sigma(g)}) > t,$$

since $\ell^{\sigma(g)} \in L$ as L is characteristic. Taking the infimum over the finitely many distinct $g\cdot\omega$, $g \in K$, shows that ω' also satisfies (A_L). ☐

Recall the *finite radical* $\Delta^+ = \Delta^+(G)$ from [Woods 2018, Definition 1.2].

Definition 1.19. Let G be an arbitrary compact p-adic analytic group with $\Delta^+ = 1$, H an open normal subgroup of G, $F = G/H$, and P a faithful G-stable ideal of kH. Recall from [Woods 2016, Definition 4.20] that the crossed product decomposition

$$kG/PkG = kH/P \underset{\langle\sigma,\tau\rangle}{*} F$$

is *standard* if the generating set $\overline{F}$ is a subset of the image of the map $G \hookrightarrow (kG/PkG)^\times$.

Lemma 1.20. *Suppose that*

$$kG/PkG = kH/P \underset{\langle\sigma,\tau\rangle}{*} F$$

is a standard crossed product decomposition. Take any

$$\alpha \in Z^2_\sigma(F, Z((kH/P)^\times)),$$

and form the central 2*-cocycle twist*

$$(kG/PkG)_\alpha := kH/P \underset{\langle\sigma,\tau\alpha\rangle}{*} F$$

with respect to this decomposition [Woods 2016, Definition 4.11].

Consider H as a subgroup of $(kH/P)^\times$, then conjugation by elements of $\overline{G}$ inside $((kG/PkG)_\alpha)^\times$ induces a group action of F on the set of p-valuations of H.

Remark. As the crossed product notation suggests, this lemma simply says that the action of F on H, via σ, is unchanged after applying $(-)_\alpha$.

Proof. As the decomposition is standard, it is trivial to see that σ satisfies (N_H) (as H is normal in G) and τ satisfies (P_H). By Lemma 1.14(iii), τ also satisfies (Q_H), and now Lemma 1.16 shows that $\tau\alpha$ also satisfies (Q_H). Hence σ induces a group action of F on the p-valuations of H inside $(kG/PkG)_\alpha$ by Lemma 1.17. □

Let L be a closed isolated characteristic subgroup of H containing $[H, H]$.

Corollary 1.21. *With notation as above, we can find an F-stable p-valuation ω on H satisfying (A_L).*

Proof. This now follows immediately from Lemmas 1.11, 1.18 and 1.20 (as F is finite). □

Proof of Theorem B. This follows from Corollary 1.21. □

2. A graded ring

2A. *Generalities on ring filtrations.*

Definition 2.1. Recall that a *filtration* v on the ring R is a function $v : R \to \mathbb{R} \cup \{\infty\}$ satisfying, for all $x, y \in R$,

- $v(x + y) \geq \min\{v(x), v(y)\}$,
- $v(xy) \geq v(x) + v(y)$,
- $v(0) = \infty$, $v(1) = 0$.

If in addition we have $v(xy) = v(x) + v(y)$ for all $x, y \in R$, then v is a *valuation* on R.

First, a basic property of ring filtrations.

Lemma 2.2. *Suppose v is a filtration on R which takes nonnegative values, i.e., $v(R) \subseteq [0, \infty]$, and let $u \in R^\times$. Then $v(ux) = v(xu) = v(x)$ for all $x \in R$.*

Proof. By the definition of v, we have

$$0 = v(1) = v(uu^{-1}) \geq v(u) + v(u^{-1}).$$

As $v(u) \geq 0$ and $v(u^{-1}) \geq 0$, we must have $v(u) = 0 = v(u^{-1})$. Then

$$v(x) = v(u^{-1}ux) \geq v(u^{-1}) + v(ux) = v(ux) \geq v(u) + v(x) = v(x),$$

from which we see that $v(x) = v(ux)$, and by a symmetric argument, we also have $v(xu) = v(x)$. □

We will fix the following notation for this subsection.

Notation. Let G be a p-valuable group equipped with the fixed p-valuation ω, and k a field of characteristic p. Take an ordered basis $\{g_1, \dots, g_d\}$ for G, and write $b_i = g_1 - 1 \in kG$ for all $1 \leq i \leq d$. As in [Ardakov 2012], we make the following definitions:

- For each $\alpha \in \mathbb{N}^d$, $\boldsymbol{b}^\alpha$ means the (ordered) product $b_1^{\alpha_1} \cdots b_d^{\alpha_d} \in kG$.
- For each $\alpha \in \mathbb{Z}_p^d$, $\boldsymbol{g}^\alpha$ means the (ordered) product $g_1^{\alpha_1} \cdots g_d^{\alpha_d} \in G$.
- For each $\alpha \in \mathbb{N}^d$, $\langle \alpha, \omega(\boldsymbol{g}) \rangle$ means $\sum_{i=1}^d \alpha_i \omega(g_i)$.
- The canonical ring homomorphism $\mathbb{Z}_p \to k$ will sometimes be left implicit, but will be denoted by ι when necessary for clarity.

Definition 2.3. Let w be the valuation on kG defined in [Ardakov 2012, §6.2], given by

$$\sum_{\alpha \in \mathbb{N}^d} \lambda_\alpha \boldsymbol{b}^\alpha \mapsto \inf_{\alpha \in \mathbb{N}^d} \{\langle \alpha, \omega(\boldsymbol{g}) \rangle \mid \lambda_\alpha \neq 0\}.$$

Note that, in light of this formula [Ardakov 2012, Corollary 6.2(b)], and by the construction [Lazard 1965, III, 2.3.3] of w, it is clear that the value of w is in fact independent of the ordered basis chosen. In particular, if φ is an automorphism of G, then $\{g_1^\varphi, \dots, g_d^\varphi\}$ is another ordered basis of G. Hence, if ω is φ-stable, then w is φ-stable, in the following sense: if $\omega(g^\varphi) = \omega(g)$ for all $g \in G$, then $w(x^{\hat{\varphi}}) = w(x)$ for all $x \in kG$, where $\hat{\varphi}$ here denotes the natural extension of φ to kG, obtained by the universal property [Woods 2016, Lemma 1.3].

We will need the following result:

Lemma 2.4. *Let*

$$b = b_0 + b_1 p + b_2 p^2 + \cdots \in \mathbb{Z}_p,$$
$$n = n_0 + n_1 p + n_2 p^2 + \cdots + n_s p^s \in \mathbb{N},$$

where all $b_i, n_i \in \{0, 1, \dots, p-1\}$. *Then*

$$\binom{b}{n} \equiv \prod_{i=0}^{s} \binom{b_i}{n_i} \mod p.$$

Proof. See, e.g., [Alperin 1985, Theorem]. □

Corollary 2.5. *Let* $b \in \mathbb{Z}_p$, $n \in \mathbb{N}$. *If*

$$v_p\left(\binom{b}{n}\right) = 0, \tag{2-1}$$

then $v_p(b) \le v_p(n)$. *Further, for fixed* $b \in \mathbb{Z}_p$,

$$\inf\left\{n \in \mathbb{N} \mid v_p\left(\binom{b}{n}\right) = 0\right\} = p^{v_p(b)}.$$

Proof. From Lemma 2.4 above, we can see that

$$\binom{b}{n} \equiv 0 \mod p$$

if and only if, for some $0 \le i \le s$,

$$\binom{b_i}{n_i} = 0,$$

which happens if and only if one of the pairs (b_i, n_i) for $0 \le i \le s$ has $b_i < n_i$. In particular, to ensure that this does not happen, we cannot have $b_i = 0 \ne n_i$, which implies that we must have $v_p(b) \le v_p(n)$. But now it is clear that $p^{v_p(b)}$ is the least $m \in \mathbb{N}$ with $v_p(b) \le v_p(m)$, and we see from Lemma 2.4 that it satisfies (2-1). □

Theorem 2.6. *Take any* $x \in G$, *and* $t = \inf \omega(G)$. *Then* $w(x-1) > t$ *implies* $\omega(x) > t$.

Proof. Write $x = \boldsymbol{g}^\alpha$. In order to show that $\omega(\boldsymbol{g}^\alpha) > t$, it suffices to show that

$$\omega(g_j) + v_p(\alpha_j) > t$$

for each $1 \le j \le d$ by Definition 1.5, and hence that $v_p(\alpha_j) \ge 1$ for all j such that $\omega(g_j) = t$.

For now, let $1 \le j \le d$ be arbitrary. Let $\beta^{(j)}$ be the d-tuple with i-th entry $\delta_{ij} p^{v_p(\alpha_j)}$. Then, of course,

$$\langle \beta^{(j)}, \omega(\boldsymbol{g}) \rangle = p^{v_p(\alpha_j)} \omega(g_j),$$

and by Corollary 2.5, we have

$$\binom{\alpha}{\beta^{(j)}} \not\equiv 0 \mod p.$$

Now suppose $w(\boldsymbol{g}^\alpha - 1) > t$. We perform binomial expansion in kG to see that

$$\begin{aligned}\boldsymbol{g}^\alpha - 1 &= \prod_{1\le i\le d} (1+b_i)^{\alpha_i} - 1 \qquad \text{(ordered product)}\\ &= \sum_{\beta\in\mathbb{N}^d} \iota\tbinom{\alpha}{\beta}\boldsymbol{b}^\beta - 1\\ &= \sum_{\beta\neq 0} \iota\tbinom{\alpha}{\beta}\boldsymbol{b}^\beta,\end{aligned}$$

so that

$$w(\boldsymbol{g}^\alpha - 1) = \inf\bigl\{\langle\beta, \omega(\boldsymbol{g})\rangle \bigm| \beta\neq 0, \tbinom{\alpha}{\beta}\not\equiv 0 \mod p\bigr\}.$$

So in particular, we have

$$t < w(\boldsymbol{g}^\alpha - 1) \le \langle\beta^{(j)}, \omega(\boldsymbol{g})\rangle = p^{v_p(\alpha_j)}\omega(g_j).$$

Now assume that $\omega(g_j) = t$. We have seen that $p^{v_p(\alpha_j)}t = p^{v_p(\alpha_j)}\omega(g_j) > t$, i.e., $p^{v_p(\alpha_j)} > 1$, which is equivalent to $v_p(\alpha_j) \ge 1$. This is what we wanted to prove. □

2B. ***Constructing a suitable valuation.*** Let H be a nilpotent p-valuable group with centre Z. If k is a field of characteristic p, and $\mathfrak{p}$ is a faithful prime ideal of kZ, then by [Ardakov 2012, Theorem 8.4], the ideal $P := \mathfrak{p}kH$ is again a faithful prime ideal of kH.

We will fix the following notation for this subsection.

Notation. Let G be a nilpotent-by-finite compact p-adic analytic group, with $\Delta^+ = 1$, and let $H = \mathbf{FN}_p(G)$ [Woods 2018, Definition 5.3], here a *nilpotent p-valuable radical*, so that $\Delta = Z := Z(H)$ [Woods 2018, proof of Lemma 1.2.3(iii)]. We will also write $F = G/H$.

Define $Q' = \mathbf{Q}(kZ/\mathfrak{p})$, the classical field of fractions of the commutative domain $kZ/\mathfrak{p}$, and

$$Q = Q' \underset{kZ}{\otimes} kH,$$

a tensor product of kZ-algebras, which (as $P = \mathfrak{p}kH$) we may naturally identify with the right localisation of kH/P with respect to $(kZ/\mathfrak{p})\setminus\{0\}$ — a subring of the Goldie ring of quotients $\mathbf{Q}(kH/P)$.

Suppose further that the prime ideal $\mathfrak{p} \lhd kZ$ is invariant under conjugation by elements of G.

Choose a crossed product decomposition

$$kG/PkG = kH/P \underset{\langle\sigma,\tau\rangle}{*} F$$

which is *standard* in the sense of the notation of Definition 1.19. Choose also any

$\alpha \in Z^2_\sigma(F, Z((kH/P)^\times))$, and form as in [Woods 2016, Definition 4.11] the central 2-cocycle twist

$$(kG/PkG)_\alpha = kH/P \underset{\langle\sigma,\tau\alpha\rangle}{*} F.$$

Now the right divisor set $(kZ/\mathfrak{p}) \setminus \{0\}$ is G-stable by assumption, so by [Passman 1989, Lemma 37.7], we may define the partial quotient ring

$$R := Q \underset{\langle\sigma,\tau\alpha\rangle}{*} F. \tag{2-2}$$

Our aim in this subsection is to construct an appropriate filtration f on the ring R. We will build this up in stages, following [Ardakov 2012]. First, we define a finite set of valuations on Q'.

Definition 2.7. In [Ardakov 2012, Theorem 7.3], a valuation on $\mathbf{Q}(kH/P)$ is defined; let v_1 be the restriction of this valuation to Q', so that $v_1(x+\mathfrak{p}) \geq w(x)$ for all $x \in kZ$ (where w is as in Definition 2.3).

Lemma 2.8. *The automorphism σ induces a group action of F on the set of valuations of Q'.*

Proof. Let u be a valuation of Q'. G acts on the set of valuations of Q' as follows:

$$(g\cdot u)(x) = u(g^{-1}xg).$$

Clearly, if $g \in H$, then $g^{-1}xg = x$ (as $x \in \mathbf{Q}(kZ/\mathfrak{p})$ where Z is the centre of H). Hence H lies in the kernel of this action, and we get an action of F on the set of valuations. By our choice of $\overline{F}$ as a subset of the image of G, this is the same as σ. □

Write $\{v_1, \dots, v_s\}$ for the F-orbit of v_1.

Lemma 2.9. *The valuations $v_1, \dots, v_s$ are independent.*

Proof. The v_i are all nontrivial valuations with value groups equal to subgroups of $\mathbb{R}$ by definition. Hence, by [Bourbaki 1972, VI.4, Proposition 7], they have height 1.

They are also pairwise inequivalent. Indeed, suppose v_i is equivalent to $g\cdot v_i$ for some $g \in F$. Then by [Bourbaki 1972, VI.3, Proposition 3], there exists a positive real number λ with $v_i = \lambda(g\cdot v_i)$, and so $v_i = \lambda^n(g^n \cdot v_i)$ for all n, as the actions of λ and g commute. But F is a finite group, so, taking $n = o(g)$, we get $v_i = \lambda^n v_i$. As v_i is nontrivial, we must have that $\lambda^n = 1$, and so $\lambda = 1$. So we may conclude, from [Bourbaki 1972, VI.4, Proposition 6(c)], that the valuations $v_1, \dots, v_s$ are independent. □

Definition 2.10. Let v be the filtration of Q' defined by

$$v(x) = \inf_{1\leq i\leq s} v_i(x)$$

for each $x \in Q'$.

Lemma 2.11.
$$\operatorname{gr}_v Q' \cong \bigoplus_{i=1}^{s} \operatorname{gr}_{v_i} Q'.$$

Proof. The natural map
$$Q'_{v,\lambda} \to \bigoplus_{i=1}^{s} Q'_{v_i,\lambda}/Q'_{v_i,\lambda^+}$$
clearly has kernel $\bigcap_{i=1}^{s} Q'_{v_i,\lambda^+} = Q'_{v,\lambda^+}$, giving an injective map
$$\operatorname{gr}_v Q' \to \bigoplus_{i=1}^{s} \operatorname{gr}_{v_i} Q'.$$
The surjectivity of this map now follows from the approximation theorem [Bourbaki 1972, VI.7.2, Théorème 1], as the v_i are independent by Lemma 2.9. □

Next, we will extend the v_i and v from Q' to Q, as in the proof of [Ardakov 2012, Theorem 8.6].

Notation. Continue with the notation above. Now, H is p-valuable, and by Lemma 1.20, F acts on the set of p-valuations of H; hence, by Lemma 1.18 (or Corollary 1.21), we may choose a p-valuation ω which is F-stable. Fix such an ω, and construct the valuation w on kH from it as defined in Definition 2.3.

Let $\{y_{e+1}, \dots, y_d\}$ be an ordered basis for Z, and extend it to an ordered basis $\{y_1, \dots, y_d\}$ for H as in Lemma 1.7 (noting that Z is a closed isolated normal subgroup of H by [Ardakov 2012, Lemma 8.4(a)]). For each $1 \le j \le e$, set $c_j = y_j - 1$ inside the ring kH/P.

Recall from [Ardakov 2012, §8.5] that elements of Q may be written uniquely as
$$\sum_{\gamma\in\mathbb{N}^e} r_\gamma \boldsymbol{c}^\gamma,$$
where $r_\gamma \in Q'$ and $\boldsymbol{c}^\gamma := c_1^{\gamma_1} \cdots c_e^{\gamma_e}$, so that $Q \subseteq Q'[\![c_1, \dots, c_e]\!]$ as a left Q'-module.

Definition 2.12. For each $1 \le i \le s$, as in [Ardakov 2012, proof of Theorem 8.6], we will define the valuation $f_i : Q \to \mathbb{R} \cup \{\infty\}$ by
$$f_i\left(\sum_{\gamma\in\mathbb{N}^e} r_\gamma \boldsymbol{c}^\gamma\right) = \inf_{\gamma\in\mathbb{N}^e} \{v_i(r_\gamma) + w(\boldsymbol{c}^\gamma)\}.$$

Remark. We point out here a slight abuse of notation: the domain of w is kH, and so $w(\boldsymbol{c}^\gamma)$ must be understood to mean $w(\boldsymbol{b}^\gamma)$, where $b_j = y_j - 1$ inside the ring kH for each $1 \le j \le e$. That is, b_j is the "obvious" lift of c_j from kH/P to kH. Our ability to do this relies on the assumption that P is faithful.

Note in particular that $f_i|_{Q'} = v_i$, and $\operatorname{gr}_{f_i} Q$ is a commutative domain, again by [Ardakov 2012, proof of Theorem 8.6].

Lemma 2.13. *The automorphism σ induces a group action of F on the set of valuations of Q.*

Proof. Let u be a valuation of Q. Again, G acts on the set of valuations of Q by $(g \cdot u)(x) = u(g^{-1}xg)$. Now, any $n \in H$ can be considered as an element of $Q^\times$, so that

$$(n \cdot u)(x) = u(n^{-1}xn) = u(n^{-1}) + u(x) + u(n) = u(x). \qquad \square$$

In the following lemma, we crucially use the fact that ω has been chosen to be F-stable.

Lemma 2.14. *The filtrations $f_1, \dots, f_s$ give the F-orbit of f_1.*

Proof. Take some $g \in F$ and some $1 \le i, j \le s$ such that $v_j = g \cdot v_i$. We will first show that, for all $x \in Q$, we have $f_j(x) \le g \cdot f_i(x)$. Indeed, as

$$f_j|_{Q'} = v_j = g \cdot v_i = g \cdot f_i|_{Q'},$$

and both f_j and $g \cdot f_i$ are valuations, it will suffice to show that

$$(w(c_k) =) f_j(c_k) \le g \cdot f_i(c_k)$$

for each $1 \le k \le e$.

Fix some $1 \le k \le e$. Write $y_k^g = z\boldsymbol{y}^\alpha$ for some $\alpha \in \mathbb{Z}_p^e$ and $z \in Z$, so that

$$\begin{aligned} c_k^g = y_k^g - 1 = z\boldsymbol{y}^\alpha - 1 \\ &= (z-1) + z\left(\prod_{i=1}^{e}(1+c_i)^{\alpha_i} - 1\right) \qquad \text{(ordered product)} \\ &= (z-1) + z\left(\sum_{\beta \ne 0} \iota\binom{\alpha}{\beta}\boldsymbol{c}^\beta\right), \end{aligned}$$

and hence

$$\begin{aligned} (g \cdot f_i)(c_k) &= \inf\left\{v_i(z-1),\, w(\boldsymbol{c}^\beta) \;\middle|\; \iota\binom{\alpha}{\beta} \ne 0\right\} \qquad \text{(by Definition 2.12)} \\ &\ge \inf\left\{w(z-1),\, w(\boldsymbol{c}^\beta) \;\middle|\; \iota\binom{\alpha}{\beta} \ne 0\right\} \qquad \text{(by Definition 2.7)} \\ &= w(c_k^g), \end{aligned}$$

with this final equality following from [Ardakov 2012, Lemma 8.5(b)]. But now, as ω has been chosen to be F-stable, w is also F-stable; see the remark in Definition 2.3. In particular, $w(c_k^g) = w(c_k)$.

Now, we have shown that, if $v_j = g \cdot v_i$ on Q', then $f_j \le g \cdot f_i$ on Q.

Similarly, we have $v_i = g^{-1} \cdot v_j$ on Q', so $f_i \le g^{-1} \cdot f_j$ on Q. But $f_i(x) \le f_j(x^{g^{-1}})$ for all $x \in Q$ is equivalent to $f_i(y^g) \le f_j(y)$ for all $y \in Q$, by setting $x = y^g$. Hence we have $f_i = g \cdot f_j$ on Q, and we are done. $\square$

As in Definition 2.10:

Definition 2.15. Let f be the filtration of Q defined by

$$f(x) = \inf_{1 \le i \le s} f_i(x)$$

for each $x \in Q$.

We now verify that the relationship between f and v is the same as that between the f_i and the v_i (Definition 2.12).

Lemma 2.16. *Take any $x \in Q$, and write it in standard form as*

$$x = \sum_{\gamma \in \mathbb{N}^e} r_\gamma \boldsymbol{c}^\gamma.$$

Then we have

$$f(x) = \inf_{\gamma \in \mathbb{N}^e} \{v(r_\gamma) + w(\boldsymbol{c}^\gamma)\}.$$

Proof. This is immediate from Definitions 2.10, 2.12 and 2.15. □

Now we can extend Lemma 2.11 to Q:

Lemma 2.17.
$$\operatorname{gr}_f Q \cong \bigoplus_{i=1}^{s} \operatorname{gr}_{f_i} Q.$$

Proof. As in the proof of Lemma 2.11, we get an injective map

$$\operatorname{gr}_f Q \to \bigoplus_{i=1}^{s} \operatorname{gr}_{f_i} Q.$$

The proof of [Ardakov 2012, Theorem 8.6] gives a map

$$(\operatorname{gr}_v(kZ/\mathfrak{p}))[Y_1, \dots, Y_e] \to \operatorname{gr}_f(kH/P)$$

and isomorphisms

$$(\operatorname{gr}_{v_i}(kZ/\mathfrak{p}))[Y_1, \dots, Y_e] \cong \operatorname{gr}_{f_i}(kH/P)$$

for each $1 \le i \le s$, in each case mapping Y_j to $\operatorname{gr}(c_j)$ for each $1 \le j \le e$.

Now, $\operatorname{gr} kH$ is a gr-*free* [Li and van Oystaeyen 1996, §I.4.1, p. 28] $\operatorname{gr} kZ$-module with respect to f and each f_i, and each of these filtrations is discrete on kH by construction; see [Ardakov 2012, Corollary 6.2 and proof of Theorem 7.3]. So, by [Li and van Oystaeyen 1996, I.6.2(3)], kH is a filt-free kZ-module with respect to f and each f_i; and by [Li and van Oystaeyen 1996, I.6.14], these maps extend to a map $(\operatorname{gr}_v Q')[Y_1, \dots, Y_e] \to \operatorname{gr}_f Q$ and isomorphisms $(\operatorname{gr}_{v_i} Q')[Y_1, \dots, Y_e] \cong \operatorname{gr}_{f_i} Q$ for each i.

Applying Lemma 2.14 to each $1 \le i \le s$, we get isomorphisms

$$(\operatorname{gr}_{v_i} Q')[Y_1, \dots, Y_e] \to \operatorname{gr}_{f_i} Q,$$

which give the following commutative diagram:

$$\begin{array}{ccc} (\mathrm{gr}_v Q')[Y_1, \dots, Y_e] & \xrightarrow{\cong} & \bigoplus_{i=1}^{s} (\mathrm{gr}_{v_i} Q')[Y_1, \dots, Y_e] \\ \downarrow & & \downarrow \cong \\ \mathrm{gr}_f Q & \hookrightarrow & \bigoplus_{i=1}^{s} \mathrm{gr}_{f_i} Q \end{array}$$

Hence clearly all maps in this diagram are isomorphisms. □

Now we return to the ring $R = Q * F$ defined in (3-2).

Definition 2.18. We can extend the filtration f on Q to an F-stable filtration on R by giving elements of the basis $\overline{F}$ value 0. That is, writing $\overline{F} = \{\bar{g}_1, \dots, \bar{g}_m\}$, any element of $Q * F$ can be expressed uniquely as $\sum_{r=1}^{m} \bar{g}_r x_r$ for some $x_r \in Q$; the assignment

$$Q * F \to \mathbb{R} \cup \{\infty\}, \qquad \sum_{r=1}^{m} \bar{g}_r x_r \mapsto \inf_{1 \le r \le m} \{f(x_r)\}$$

is clearly a filtration on $Q * F$ whose restriction to Q is just f. We will temporarily refer to this filtration as $\hat{f}$, though later we will drop the hat and simply call it f.

Note that, for any real number λ,

$$(Q * F)_{\hat{f},\lambda} = \bigoplus_{i=1}^{m} \overline{g_i}(Q_{f,\lambda}), \qquad (Q * F)_{\hat{f},\lambda^+} = \bigoplus_{i=1}^{m} \overline{g_i}(Q_{f,\lambda^+}),$$

so that

$$\begin{aligned} \mathrm{gr}_{\hat{f}}(Q * F) &= \bigoplus_{\lambda \in \mathbb{R}} \left(\bigoplus_{i=1}^{m} \overline{g_i}(Q_{f,\lambda}/Q_{f,\lambda^+}) \right) \\ &= \bigoplus_{i=1}^{m} \overline{g_i} \left(\bigoplus_{\lambda \in \mathbb{R}} (Q_{f,\lambda}/Q_{f,\lambda^+}) \right) = \bigoplus_{i=1}^{m} \overline{g_i} \Big(\mathrm{gr}_f(Q) \Big). \end{aligned}$$

That is, given the data of a crossed product $Q * F$ as in (3-2), we may view $\mathrm{gr}_{\hat{f}}(Q * F)$ as $\mathrm{gr}_f(Q) * F$ in a natural way.

We will finally record this as:

Lemma 2.19. *We have*

$$\begin{aligned}\mathrm{gr}_f(Q*F)=\mathrm{gr}_f(Q)*F &\cong \left(\bigoplus_{i=1}^{s}\mathrm{gr}_{f_i}Q\right)*F\\ &\cong \left(\bigoplus_{i=1}^{s}(\mathrm{gr}_{v_i}Q')[Y_1,\dots,Y_e]\right)*F,\end{aligned}$$

where each $\mathrm{gr}_{f_i}Q$ *(or equivalently each* $\mathrm{gr}_{v_i}Q'$*) is a domain*; *see Definition 2.12. F permutes the f_i (or equivalently the v_i) transitively by conjugation.*

Proof of Theorem C. This is Lemma 2.19. □

2C. ***Automorphisms trivial on a free abelian quotient.*** We will fix the following notation for this subsection.

Notation. Let H be a nilpotent but nonabelian p-valuable group with centre Z. Write H' for its isolated derived subgroup [Woods 2018, Theorem B]. Suppose we are given a closed isolated proper characteristic subgroup L of H which contains H' and Z. (We will show that such an L always exists in Lemma 3.5.) Fix a p-valuation ω on H satisfying (A_L) — this is possible by Corollary 1.21.

Let $\{g_{m+1},\dots,g_n\}$ be an ordered basis for Z. Using Lemma 1.7 twice, extend this to an ordered basis $\{g_{l+1},\dots,g_n\}$ for L, and then extend this to an ordered basis $\{g_1,\dots,g_n\}$ for H. Diagrammatically,

$$B_H=\{\underbrace{g_1,\dots,g_l}_{B_{H/L}},\ \underbrace{g_{l+1},\dots,g_m}_{B_{L/Z}},\ \underbrace{g_{m+1},\dots,g_n}_{B_Z}\},$$

extending the notation of the remark after Lemma 1.7 in the obvious way. Here, $0<l\le m<n$, corresponding to the chain of subgroups $1\lneq Z\le L\lneq H$.

Let k be a field of characteristic p. As before, let $\mathfrak{p}$ be a faithful prime ideal of kZ, so that $P:=\mathfrak{p}kH$ is a faithful prime ideal of kH, and write $b_j=g_j-1\in kH/P$ for all $1\le j\le m$. (As before, since P is faithful, we are abusing notation to identify H with its image under the natural map $H\to(kH/P)^\times$.)

In this subsection, we will write:

- For each $\alpha\in\mathbb{N}^m$, $\boldsymbol{b}^\alpha$ means the (ordered) product $b_1^{\alpha_1}\cdots b_m^{\alpha_m}\in kH/P$.
- For each $\alpha\in\mathbb{Z}_p^m$, $\boldsymbol{g}^\alpha$ means the (ordered) product $g_1^{\alpha_1}\cdots g_m^{\alpha_m}\in H$.
- For each $\alpha\in\mathbb{N}^m$, $\langle\alpha,\omega(\boldsymbol{g})\rangle$ means $\sum_{i=1}^m\alpha_i\omega(g_i)$.

Note the use of m rather than n in each case. This means, by [Ardakov 2012, §8.5], that every element $x\in H$ may be written uniquely as

$$x=z\boldsymbol{g}^\alpha$$

for some $\alpha\in\mathbb{Z}_p^m$ and $z\in Z$, and every element $y\in kH/P$ may be written uniquely as

$$y = \sum_{\gamma \in \mathbb{N}^m} r_\gamma \boldsymbol{b}^\gamma$$

for some elements $r_\gamma \in kZ/\mathfrak{p}$.

Recall the definitions of the filtrations w on kH (Definition 2.3), v on $kZ/\mathfrak{p}$ (Definition 2.10) and f on kH/P (Definition 2.15). We will continue to abuse notation slightly for w, as in Definition 2.12.

Recall also that, as we have chosen ω to satisfy (A_L), we have that

$$w(b_1) = \cdots = w(b_l) < w(b_r)$$

for all $r > l$.

Let σ be an automorphism of H, and suppose that, when naturally extended to an automorphism of kH, it satisfies $\sigma(P) = P$. Hence we will consider σ as an automorphism of kH/P, preserving the subgroup $H \subseteq (kH/P)^\times$.

Corollary 2.20. *With the above notation, fix* $1 \leq i \leq l$. *If* $f(\sigma(b_i) - b_i) > f(b_i)$, *then* $w(\sigma(b_i) - b_i) > w(b_i)$.

Proof. Write in standard form

$$\sigma(b_i) - b_i = \sum_{\gamma \in \mathbb{N}^m} r_\gamma \boldsymbol{b}^\gamma,$$

for some $r_\gamma \in kZ$, and suppose that $f(\sigma(b_i) - b_i) > f(b_i)$. That is, by Lemma 2.16,

$$v(r_\gamma) + w(\boldsymbol{b}^\gamma) > w(b_i)$$

for each fixed $\gamma \in \mathbb{N}^m$.

We will show that $w(r_\gamma) + w(\boldsymbol{b}^\gamma) > w(b_i)$ for each γ. We deal with two cases.

Case 1: $w(\boldsymbol{b}^\gamma) > w(b_i)$. Here, as w takes nonnegative values on kH, we are already done.

Case 2: $w(\boldsymbol{b}^\gamma) \leq w(b_i)$. Here, by ($\mathsf{A}_L$), we have either $w(r_\gamma) > w(b_i)$ or $w(r_\gamma) = 0$. In the former case, we are done automatically, so assume we are in the latter case and $w(r_\gamma) = 0$. Then, by [Ardakov 2012, §6.2], r_γ must be a unit in kZ, and so $f(r_\gamma) = 0$ by Lemma 2.2, a contradiction.

Hence $w(r_\gamma) + w(\boldsymbol{b}^\gamma) > w(b_i)$ for all $\gamma \in \mathbb{N}^m$. But, as w is discrete by [Ardakov 2012, §6.2], we may now take the infimum over all $\gamma \in \mathbb{N}^m$, and the inequality remains strict. □

Let σ be an automorphism of H, and recall that H/L is a free abelian pro-p group of rank l. Choose a basis $e_1, \ldots, e_l$ for $\mathbb{Z}_p^l$, then the map $g_i L \mapsto e_i$ for $1 \leq i \leq l$ is an isomorphism $j : H/L \to \mathbb{Z}_p^l$. As L is characteristic in H by assumption, σ induces an automorphism of H/L, which gives a matrix $M_\sigma \in \mathrm{GL}_l(\mathbb{Z}_p)$ under this isomorphism.

Write

$$\overline{\omega} : H/L \to \mathbb{R} \cup \{\infty\}$$

for the quotient p-valuation on H/L induced by ω, i.e.,

$$\overline{\omega}(xL) = \sup_{\ell \in L}\{\omega(x\ell)\};$$

note that this is just the (t, p)-filtration (Definition 1.4), as we have chosen ω to satisfy (A_L). Then write

$$\Omega : \mathbb{Z}_p^l \to \mathbb{R} \cup \{\infty\}$$

for the map $\Omega = \overline{\omega} \circ j^{-1}$, the (t, p)-filtration on $\mathbb{Z}_p^l$ corresponding to $\overline{\omega}$ under the isomorphism j.

Remark. If $x \in \mathbb{Z}_p^l$ has $\Omega(x) \geq t + 1$, then $x \in p\mathbb{Z}_p^l$, by the definition of the (t, p)-filtration.

We will write $\Gamma(1) := 1 + p\mathrm{GL}_l(\mathbb{Z}_p)$ for the *first congruence subgroup* of $\mathrm{GL}_l(\mathbb{Z}_p)$, the open subgroup of $\mathrm{GL}_l(\mathbb{Z}_p)$ whose elements are congruent to the identity element modulo p.

Corollary 2.21. *With the above notation, if $f(\sigma(b_i) - b_i) > f(b_i)$ for all $1 \leq i \leq l$, then $M_\sigma \in \Gamma(1)$.*

Proof. We have, for all $1 \leq i \leq l$,

$$\begin{aligned} f(\sigma(b_i) - b_i) > f(b_i) &\Longrightarrow w(\sigma(b_i) - b_i) > w(b_i) && \text{by Corollary 2.20,} \\ &\Longrightarrow \omega(\sigma(g_i)g_i^{-1}) > \omega(g_i) && \text{by Theorem 2.6;} \end{aligned}$$

which is condition (1-1). Now we may invoke Theorem 1.12. □

Corollary 2.22. *Suppose further that σ is an automorphism of H of **finite order**. If $p > 2$ and $f(\sigma(b_i) - b_i) > f(b_i)$ for all $1 \leq i \leq l$, then σ induces the identity automorphism on H/L.*

Proof. We have shown that $M_\sigma \in \Gamma(1)$, which is a p-valuable (hence torsion-free) group for $p > 2$ by [Dixon et al. 1999, Theorem 5.2]; and if σ has finite order, then M_σ must have finite order. So M_σ is the identity map. □

Proof of Theorem D. This now follows from Corollaries 2.21 and 2.22. □

Remark. When $p = 2$, $\Gamma(1)$ is no longer p-valuable.

Example 2.23. Let $p = 2$, and let

$$H = \overline{\langle x, y, z \mid [x, y] = z, [x, z] = 1, [y, z] = 1\rangle}$$

be the (2-valuable) $\mathbb{Z}_2$-Heisenberg group. Let σ be the automorphism sending x to x^{-1}, y to y^{-1} and z to z. Take $L = \overline{\langle z\rangle}$, and $P = 0$.

Write $X = x - 1 \in kH/P$, and likewise $Y = y - 1$ and $Z = z - 1$. Now,

$$\sigma(X) = \sigma(x) - 1 = x^{-1} - 1 = (1+X)^{-1} - 1 = -X + X^2 - X^3 + \cdots,$$

and so $\sigma(X) - X = X^2 - X^3 + \cdots$ (as $\operatorname{char} k = 2$). Hence

$$f(\sigma(X) - X) = f(X^2) > f(X),$$

but

$$M_\sigma = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in \Gamma(1, \mathrm{GL}_2(\mathbb{Z}_2)),$$

and in particular $M_\sigma \neq 1$.

3. Extending prime ideals from $\mathbf{FN}_p(G)$

3A. *X-inner automorphisms.*

Definition 3.1. We recall the notation of [Woods 2016, §4.2]: given R a ring, G a group and a fixed crossed product S of R by G, we will sometimes write the structure explicitly as

$$S = R \underset{\langle\sigma,\tau\rangle}{*} G,$$

where $\sigma : G \to \mathrm{Aut}(R)$ is the *action* and $\tau : G \times G \to R^\times$ the *twisting*.

Furthermore, we say that an automorphism $\varphi \in \mathrm{Aut}(R)$ is *X-inner* if there exist nonzero elements $a, b, c, d \in R$ such that, for all $x \in R$,

$$axb = cx^\varphi d,$$

where x^φ denotes the image of x under φ. Write $\mathrm{Xinn}(R)$ for the subgroup of $\mathrm{Aut}(R)$ consisting of X-inner automorphisms; and, given a crossed product as in the previous paragraph, we will write $\mathrm{Xinn}_S(R; G) = \sigma^{-1}(\sigma(G) \cap \mathrm{Xinn}(R))$.

Lemma 3.2. *Let R be a prime ring and $R * G$ be a crossed product. Let $G_{\mathrm{inn}} := \mathrm{Xinn}_{R*G}(R; G)$.*

(i) *If $\sigma \in \mathrm{Aut}(R)$ is X-inner, then σ is trivial on the centre of R.*

(ii) *If H is a subgroup of G containing G_{inn}, and $R * H$ is a prime ring, then $R * G$ is a prime ring.*

Proof. (i) This follows from the description of X-inner automorphisms of R as restrictions of inner automorphisms of the Martindale symmetric ring of quotients $\mathrm{Q}_s(R)$, and the fact that $Z(R)$ stays central in $\mathrm{Q}_s(R)$; see [Passman 1989, §12] for details.

(ii) This follows from [Passman 1989, Corollary 12.6]: if I is a nonzero ideal of $R * G$, then $I \cap R * G_{\mathrm{inn}}$ is nonzero, and hence $I \cap R * H$ is nonzero. □

3B. ***Properties of*** $\mathbf{FN}_p(G)$**.** We prove here some important facts about the group $\mathbf{FN}_p(G)$ (defined in [Woods 2018, Theorem C]).

Lemma 3.3. *Let G be a nilpotent-by-finite compact p-adic analytic group with $\Delta^+ = 1$. Let $H = \mathbf{FN}_p(G)$, and write*

$$K := K_G(H) = \{x \in G \mid [H, x] \leq H'\},$$

where H' denotes the isolated derived subgroup of H [Woods 2018, Theorem B]. *Then $K = H$.*

Proof. Firstly, note that K clearly contains H, by definition of H'.

Secondly, suppose that H is p-saturable. By the same argument as in [Woods 2018, Lemma 4.3], K acts nilpotently on H, and so K acts nilpotently on the Lie algebra $\mathfrak{h}$ associated to H under Lazard's isomorphism of categories [1965, IV, 3.2.6]. That is, we get a group representation $\mathrm{Ad}: K \to \mathrm{Aut}(\mathfrak{h})$, and $(\mathrm{Ad}(k)-1)(\mathfrak{h}_i) \subseteq \mathfrak{h}_{i+1}$ for each $k \in K$ and each i. (Here, $\mathfrak{h}_i$ denotes the i-th term in the lower central series for $\mathfrak{h}$.)

Choosing a basis for $\mathfrak{h}$ adapted to the flag

$$\mathfrak{h} \supsetneq \mathfrak{h}_2 \supsetneq \cdots \supsetneq \mathfrak{h}_r = 0,$$

we see that Ad is a representation of K for which $\mathrm{Ad}(k)-1$ is strictly upper triangular for each $k \in K$; in other words, $\mathrm{Ad} : K \to \mathcal{U}$, where $\mathcal{U}$ is a closed subgroup of some $\mathrm{GL}_n(\mathbb{Z}_p)$ consisting of unipotent upper-triangular matrices. Hence the image $\mathrm{Ad}(K)$ is nilpotent and torsion-free.

Furthermore, $\ker \mathrm{Ad}$ is the subgroup of K consisting of those elements k which centralise $\mathfrak{h}$, and therefore centralise H. This clearly contains $Z(H)$. On the other hand, if k centralises H, then k is centralised by H, an open subgroup of G, and so k must be contained in Δ. But $\Delta = Z(H)$ by [Woods 2018, Lemma 5.1(ii)]

Hence K is a central extension of two nilpotent, torsion-free compact p-adic analytic groups of finite rank, and so is such a group itself; hence K is nilpotent p-valuable by [Woods 2018, Lemma 2.3], and so must be contained in H by definition of $\mathbf{FN}_p(G)$.

Now suppose H is not p-saturable, and fix a p-valuation on H. Conjugation by $k \in K$ induces the trivial automorphism on H/H', so by [Lazard 1965, IV, 3.3.2.4] it does also on $\mathrm{Sat}\,(H/H')$, which is naturally isomorphic to $\mathrm{Sat}\,H/(\mathrm{Sat}\,H)'$ by [Woods 2018, Lemma 3.2]. This shows that $K \subseteq K_G(\mathrm{Sat}\,H)$. But now, writing $\mathfrak{h}$ for the Lie algebra associated to $\mathrm{Sat}\,H$, the same argument as above, mutatis mutandis, shows that $K_G(\mathrm{Sat}\,H) = H$. □

Lemma 3.4. *Let G be a compact p-adic analytic group with $\Delta^+ = 1$, and write $H = \mathbf{FN}_p(G)$. If H is not abelian, then $H/Z = \mathbf{FN}_p(G/Z)$.*

Proof. H/Z is a nilpotent p-valuable open normal subgroup of G/Z, so must be contained within $\mathbf{FN}_p(G/Z)$. Conversely, the preimage in G of $\mathbf{FN}_p(G/Z)$ is a central extension of Z by $\mathbf{FN}_p(G/Z)$, two nilpotent and torsion-free groups, and hence is nilpotent and torsion-free, so must be p-valuable by [Woods 2018, Lemma 2.3], which shows that it must be contained within H. □

Recall that, if J is a closed isolated subgroup of H, then there exists a unique smallest isolated orbital subgroup of H containing J, which we call its *isolator*, and denote $\mathrm{i}_H(J)$, as in [Woods 2018, Definition 1.6].

The closed, isolated orbital, characteristic subgroup $\mathrm{i}_H(H'Z)$ of $H = \mathbf{FN}_p(G)$ will be crucial throughout this section, so we record some results.

Lemma 3.5. *Let H be a nilpotent p-valuable group. If H is not abelian, then $H \neq \mathrm{i}_H(H'Z)$.*

Proof. Suppose first that H is p-saturated, and write $\mathfrak{h}$ and $\mathfrak{z}$ for the Lie algebras of H and Z respectively under Lazard's correspondence [1965, IV, 3.2.6]. If $\mathfrak{h} = \mathfrak{h}_2\mathfrak{z}$ (writing $\mathfrak{h}_2$ for the second term in the lower central series of $\mathfrak{h}$), then by applying $[\mathfrak{h}, -]$ to both sides, we see that $\mathfrak{h}_2 = \mathfrak{h}_3$. But as $\mathfrak{h}$ is nilpotent, this implies that $\mathfrak{h}_2 = 0$, so that $\mathfrak{h}$ is abelian, a contradiction.

When H is not p-saturated, note that $\mathrm{i}_H(H'Z) = \mathrm{Sat}(H'Z) \cap H$, by [Woods 2018, Lemma 3.1], and so $\mathrm{Sat}(H/\mathrm{i}_H(H'Z)) \cong \mathrm{Sat}(H)/\mathrm{Sat}(H'Z)$ by [Woods 2018, Lemma 3.2]. Hence $H/\mathrm{i}_H(H'Z)$ has the same rank as $\mathrm{Sat}(H)/\mathrm{Sat}(H'Z)$, and in particular this rank is nonzero. □

Lemma 3.6. *Let G be a nilpotent-by-finite compact p-adic analytic group with $\Delta^+ = 1$. Let $H = \mathbf{FN}_p(G)$, and assume that H is not abelian. Write*

$$M := M_G(H) = \{x \in G \mid [H, x] \leq \mathrm{i}_H(H'Z)\},$$

where H' denotes the isolated derived subgroup of H, and Z the centre of H. Then $M = H$.

Proof. Clearly $Z \leq M$. We will calculate M/Z.

First, note that $\mathrm{i}_H(H'Z)/Z$ is an isolated normal subgroup of H/Z, as the quotient is isomorphic to $H/\mathrm{i}_H(H'Z)$, which is torsion-free. Also, as $\mathrm{i}_H(H'Z)$ contains $H'Z$ and hence $\overline{[H, H]}Z$ as an open subgroup, clearly $\mathrm{i}_H(H'Z)/Z$ contains $\overline{[H, H]}Z/Z$ as an open subgroup, so that $\mathrm{i}_H(H'Z)/Z \leq \mathrm{i}_{H/Z}(\overline{[H, H]}Z/Z)$.

Now, $[H/Z, H/Z] = [H, H]Z/Z$ as abstract groups, so by taking their closures followed by their (H/Z)-isolators, we see that

$$(H/Z)' = \mathrm{i}_{H/Z}\big(\overline{[H, H]Z/Z}\big) = \mathrm{i}_{H/Z}\big(\overline{[H, H]}Z/Z\big),$$

so that

$$\mathrm{i}_H(H'Z)/Z = (H/Z)'.$$

But $x \in M$ if and only if $[H, x] \leq \mathrm{i}_H(H'Z)$, which is equivalent to

$$[H/Z, xZ] \leq (H/Z)',$$

or in other words $xZ \in K_{G/Z}(H/Z) = H/Z$ by Lemma 3.3. So $M/Z = H/Z$, and hence $M = H$. □

3C. *The extension theorem.*

Proposition 3.7. *Fix a prime $p > 2$ and a finite field k of characteristic p. Let G be a nilpotent-by-finite compact p-adic analytic group with $\Delta^+ = 1$. Suppose $H = \mathbf{FN}_p(G)$, and write $F = G/H$. Let P be a G-stable, faithful prime ideal of kH. Let $(kG)_\alpha$ be a central 2-cocycle twist of kG with respect to a standard decomposition* (Definition 1.19)

$$kG = kH \underset{\langle\sigma,\tau\rangle}{*} F,$$

for some $\alpha \in Z^2_\sigma(F, Z((kH)^\times))$, as in [*Woods 2016, Theorem* 4.21]. *Then $P(kG)_\alpha$ is a prime ideal of $(kG)_\alpha$.*

Proof. First, we note that the claim that $P(kG)_\alpha$ is a prime ideal of $(kG)_\alpha$ is equivalent to the claim that

$$(kG)_\alpha/P(kG)_\alpha = kH/P \underset{\langle\sigma,\tau\alpha\rangle}{*} F$$

is a prime ring.

We will sometimes write $Z := \Delta$, to emphasise that we are thinking of it as the centre of H [Woods 2018, Lemma 5.1(ii)].

Case 1. Suppose that G centralises Z. If H is abelian, so that $H = Z$, then every $g \in G$ is centralised by Z, an open subgroup of G. Hence $g \in \Delta$, and as $g \in G$ was arbitrary, we deduce that $G = \Delta$. But, by [Woods 2018, Theorem D], $\Delta \leq H$, and so we have $G = H$ and there is nothing to prove.

So suppose henceforth that $Z \lneq H$, and write $L := \mathrm{i}_H(H'Z)$, so that, by Lemma 3.5, we have $L \lneq H$. As the decomposition of kG is standard, we may view F as a subset of G.

The idea behind the proof is as follows. We will construct a crossed product $R * F'$, where R is a certain commutative domain and F' is a certain subgroup of F, with the following property: if $R * F'$ is a prime ring, then $P(kG)_\alpha$ is a prime ideal. Then, by using the well-understood structure of R, we will show that the action of F' on R is X-outer (in the sense of Definition 3.1), so that $R * F'$ is a prime ring.

By Corollary 1.21, we can see that H admits an F-stable p-valuation ω satisfying (A_L). Hence, in the notation of Section 2A, we may define the filtration w

from ω as in Definition 2.3. Furthermore, we write

$$Q' = \mathbf{Q}(kZ/P \cap kZ), \qquad Q = Q' \underset{kZ}{\otimes} kN,$$

as in Section 2B; and we endow Q with the F-orbit of filtrations f_i $(1 \le i \le s)$ and the filtration f of Definitions 2.12 and 2.15, defined in terms of the filtration w above.

By [McConnell and Robson 1987, 2.1.16(vii)], in order to show that the crossed product

$$kH/P \underset{\langle\sigma,\tau\alpha\rangle}{*} F \tag{3-1}$$

is a prime ring, it suffices to show that the related crossed product

$$Q \underset{\langle\sigma,\tau\alpha\rangle}{*} F \tag{3-2}$$

is prime, where this crossed product is defined in Section 2B. Then, by [Li and van Oystaeyen 1996, II.3.2.7], it suffices to show that

$$\operatorname{gr}_f(Q * F) \tag{3-3}$$

is prime. Details of this graded ring are given in Lemma 2.19; in particular, note that

$$\operatorname{gr}_f(Q * F) \cong \left(\bigoplus_{i=1}^{s} \operatorname{gr}_{f_i} Q\right) * F.$$

Now, as noted in Definition 2.12, each $\operatorname{gr}_{f_i} Q$ is a commutative domain, and by construction, F permutes the summands $\operatorname{gr}_{f_i} Q$ transitively. So by [Passman 1989, Corollary 14.8] it suffices to show that

$$\operatorname{gr}_{f_1} Q * F' \tag{3-4}$$

is prime, where $F' = \operatorname{Stab}_F(f_1)$.

Notation. We set up notation in order to be able to apply the results of Section 2C. Let $\{y_{m+1}, \dots, y_n\}$ be an ordered basis for Z, which we extend to an ordered basis $\{y_{l+1}, \dots, y_n\}$ for L, which we extend to an ordered basis $\{y_1, \dots, y_n\}$ for H. Set $b_i = y_i - 1 \in kH/P$, and let $Y_i = \operatorname{gr}_{f_1}(b_i)$ for all $1 \le i \le m$. Then

$$\operatorname{gr}_{f_1} Q \cong (\operatorname{gr}_{v_1} Q')[Y_1, \dots, Y_m].$$

The ring on the right-hand side inherits a crossed product structure

$$(\operatorname{gr}_{v_1} Q')[Y_1, \dots, Y_m] * F'. \tag{3-5}$$

from (3-4). Writing $R := (\operatorname{gr}_{v_1} Q')[Y_1, \dots, Y_m]$, we have now shown, by passing along the chain

$$(3\text{-}5) \to (3\text{-}4) \to (3\text{-}3) \to (3\text{-}2) \to (3\text{-}1),$$

that we need only show that $R * F'$ is prime.

Write F'_{inn} for the subgroup of F' acting on R by X-inner automorphisms in the crossed product (3-5), i.e.,

$$F'_{\mathrm{inn}} = \mathrm{Xinn}_{R*F'}(R;\, F')$$

in the notation of Definition 3.1. By the obvious abuse of notation, we will denote this action as the map of sets $\mathrm{gr}\,\sigma : F' \to \mathrm{Aut}(R)$.

Take some $g \in F'$. If $\mathrm{gr}\,\sigma(g)$ acts nontrivially on R, then as R is commutative, we have $g \notin F'_{\mathrm{inn}}$. Hence, as by Lemma 3.2(ii) we need only show that $R * F'_{\mathrm{inn}}$ is prime, we may restrict our attention to those $g \in F'$ that act trivially on R. In particular, such a $g \in F'$ must centralise each Y_i. But

$$\mathrm{gr}\,\sigma(g)(Y_i) = Y_i \Leftrightarrow f(\sigma(g)(b_i) - b_i) > f(b_i).$$

Now we see (as $p > 2$) from Corollary 2.22 that $\sigma(g)$ induces the identity automorphism on H/L, and hence from Lemma 3.6 that $g \in H$. That is, F'_{inn} is the trivial group, so that $R * F'_{\mathrm{inn}} = R$ is automatically prime.

Case 2. Suppose some $x \in F$ does not centralise Z. Write F_{inn} for the subgroup of F acting by X-inner automorphisms on kH/P in the crossed product (3-1), i.e.,

$$F_{\mathrm{inn}} := \mathrm{Xinn}_{(kG)_\alpha/P(kG)_\alpha}(kH/P;\, F).$$

Then, by Lemma 3.2(i), $x \notin F_{\mathrm{inn}}$, so F_{inn} is contained in $\mathbf{C}_F(Z)$, and we need only prove that the subcrossed product $(kH/P) * \mathbf{C}_F(Z)$ is prime by Lemma 3.2(ii). This reduces the problem to Case 1. □

Proposition 3.8. *Let G be a nilpotent-by-finite compact p-adic analytic group, and k be a finite field of characteristic $p > 2$. Let $H = \mathbf{FN}_p(G)$, and write $F = G/H$. Let P be a G-stable, almost faithful prime ideal of kH. Then PkG is prime.*

Proof. We assume familiarity with [Woods 2016, Lemma 1.6], and adopt the notation of [Woods 2016, Notation 1.10] for this proof.

Let $e \in \mathrm{cpi}^{\overline{k\Delta^+}}(P)$, and write $f_H = e|^H$, $f = e|^G$. Then PkG is a prime ideal of kG if and only if $f \cdot \overline{PkG}$ is prime in $f \cdot \overline{kG}$.

Write $H_1 = \mathrm{Stab}_H(e)$ and $G_1 = \mathrm{Stab}_G(e)$. Then, by the matrix units lemma [Woods 2016, Lemma 5.1], we get an isomorphism

$$f \cdot \overline{kG} \cong M_s(e \cdot \overline{kG_1})$$

for some s, under which the ideal $f \cdot \overline{PkG}$ is mapped to $M_s(e \cdot \overline{P_1 kG_1})$, where P_1 is the preimage in kH_1 of $e \cdot \overline{P} \cdot e$. See [Woods 2016, Theorems D and E] for details of this isomorphism. It is easy to see that P_1 is prime in kH_1; indeed, applying the matrix units lemma to kH, we get

$$f_H \cdot \overline{kH} \cong M_{s'}(e \cdot \overline{kH_1}),$$

under which $f_H \cdot \overline{P} \mapsto M_{s'}(e \cdot \overline{P_1})$, so that P_1 is prime by Morita equivalence (see, e.g., [McConnell and Robson 1987, 3.5.5 and Proposition 3.5.10]). We also know from [Woods 2016, Proposition 5.9] that

$$P^\dagger = \bigcap_{h \in H} (P_1^\dagger)^h.$$

Now, writing q to denote the natural map $G \to G/\Delta^+$,

$$q((P_1^\dagger \cap \Delta)^h) = q(P_1^\dagger \cap \Delta)$$

for all $h \in H$, since $q(\Delta) = Z(q(H))$ by definition of H (see [Woods 2018, Lemma 5.1(ii)]), and so

$$q(P^\dagger \cap \Delta) = q(P_1^\dagger \cap \Delta) = q(1).$$

But $q(P_1^\dagger)$ is a normal subgroup of the nilpotent group $q(H_1)$. Hence, as the intersection of $q(P_1^\dagger)$ with the centre $q(\Delta)$ of $q(H)$ is trivial, we must have that $q(P_1^\dagger)$ is trivial also [Robinson 1982, 5.2.1]. That is, $P_1^\dagger \leq \Delta^+(H_1) = \Delta^+$.

Now, in order to show that $M_s(e \cdot \overline{P_1 kG_1})$ is prime, we may equivalently (by Morita equivalence) show that $e \cdot \overline{P_1 kG_1}$ is prime. By [Woods 2016, Theorem B], we get an isomorphism

$$e \cdot \overline{kG_1} \cong M_t((k'[\![G_1/\Delta^+]\!])_\alpha),$$

for some integer t, some finite field extension k'/k, and a central 2-cocycle twist (see above or [Woods 2016, Definition 4.11]) of $k'[\![G_1/\Delta^+]\!]$ with respect to a standard crossed product decomposition

$$k'[\![G_1/\Delta^+]\!] = k'[\![H_1/\Delta^+]\!] \underset{\langle \sigma, \tau \rangle}{*} (G_1/H_1)$$

given by some

$$\alpha \in Z_\sigma^2\big(G_1/H_1, Z((k'[\![H_1/\Delta^+]\!])^\times)\big).$$

Writing the image of $e \cdot \overline{P_1}$ as $M_t(\mathfrak{p})$ for some ideal $\mathfrak{p} \in k'[\![H_1/\Delta^+]\!]$, we see by above or [Woods 2016, Theorem C] that $\mathfrak{p}$ is a faithful, (G_1/Δ^+)-stable prime ideal of $k'[\![H_1/\Delta^+]\!]$. It now remains only to show that the extension of $\mathfrak{p}$ to $k'[\![G_1/\Delta^+]\!]$ is prime; but this now follows from Proposition 3.7. □

Proof of Theorem A. This follows from Proposition 3.8. □

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