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We study the Yamabe flow corresponding to the prescribed scalar curvature problem on compact Riemannian manifolds with negative scalar curvature. The long time existence and convergence of the flow are proved under appropriate conditions on the prescribed scalar curvature function.

1. Introduction

The prescribed scalar curvature problem on a compact Riemannian manifold (M, g_0) of dimension $n \geq 3$, consists of finding a conformal metric g to g_0 whose scalar curvature R_g is equal to a given function $f \in C^\infty(M)$. If we set $g = u^{4/(n-2)} g_0$, where $0 < u \in C^\infty(M)$, then we have

$$R_g = u^{-\frac{n+2}{n-2}}(-c_n \Delta u + R_0 u),$$

where Δ is the Laplace operator associated with g_0, R_0 is the scalar curvature of g_0 and $c_n = 4\frac{n-1}{n-2}$.

Then the prescribed scalar curvature problem,

$$R_g = f,$$

is equivalent to solving the nonlinear PDE

$$-c_n \Delta u + R_0 u = f u^{\frac{n+2}{n-2}} \tag{1-1}$$

on the space of smooth positive functions on M. The solvability of this equation depends on R_0 and the prescribed function f. When f is constant, (1-1) becomes the famous Yamabe equation whose resolution has been a challenging problem in geometric analysis for a long time. See [Aubin 1976; Hebey and Vaugon 1993; Lee and Parker 1987; Schoen 1984; 1991] for more details on the Yamabe problem, and [Ambrosetti and Malchiodi 1999; Bismuth 2000; Escobar and Schoen 1986; Kazdan and Warner 1975; Rauzy 1995; Vázquez and Véron 1991], concerning the prescribed scalar curvature problem.

By changing g_0 conformally if necessary, we may always assume that R_0 satisfies one of the conditions, $R_0 > 0$, $R_0 = 0$ or $R_0 < 0$ everywhere on M. Equation (1-1)

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has a variational structure since there are different functionals whose Euler–Lagrange equations are equivalent to (1-1). When $R_0 < 0$, the following functional seems more appropriate to handle the prescribed scalar curvature problem:

$$\mathcal{E}(g) = \int_M R_g \, dV_g - \frac{n-2}{n} \int_M f \, dV_g, \tag{1-2}$$

where $g = u^{4/(n-2)} g_0$ belongs to the conformal class $[g_0]$ of g_0, R_g is the scalar curvature of g and $dV_g = u^{2n/(n-2)} dV_{g_0}$ is the volume element of g.

Simple computations ([Besse 1987]) show the L^2-gradient of $\mathcal{E}$ is $\frac{n-2}{2n}(R_g - f)g$, and then, after changing time by a constant scale, the associated negative gradient flow equation is

$$\begin{cases} \partial_t g = -(R_g - f)g, \\ g(0) = g^0, \end{cases} \tag{1-3}$$

where $g^0 = u_0^{4/(n-2)} g_0$ is a given metric in the conformal class of g_0.

Since (1-3) preserves the conformal structure of M, then any smooth solution of (1-3) is of the form $g(t) = u(t)^{4/(n-2)} g_0$, where $0 < u(t) \in C^\infty(M)$. For simplicity we have used the notation $u(t) := u(t, \cdot)$, $t \in I$ for any function u defined on $I \times M$, where I is a subset of $\mathbb{R}$. In terms of $u(t)$, the flow (1-3) may be written in the equivalent form:

$$\begin{cases} \partial_t u^N = \frac{n+2}{4}(c_n \Delta u - R_0 u + f u^N), \\ u(0) = u_0 \in C^\infty(M), \ u_0 > 0, \end{cases} \tag{1-4}$$

where $N = \frac{n+2}{n-2}$.

Our aim in this paper is to investigate this gradient flow by proving its longtime existence and analysing its asymptotic behaviour when $t \to +\infty$.

Our first result is the following existence theorem:

Theorem 1.1. *Suppose that $R_0 < 0$ and let $f \in C^\infty(M)$. Then for any $g^0 = u_0^{4/(n-2)} g_0$ with $0 < u_0 \in C^\infty(M)$, there exists a unique solution $g(t) = u(t)^{4/(n-2)} g_0$ of (1-3) defined on $[0, +\infty)$, where $0 < u \in C^\infty([0, +\infty) \times M)$. Moreover, the functional $\mathcal{E}$ is decreasing along the solution $g(t)$, that is,*

$$\frac{d}{dt}\mathcal{E}(g(t)) \leq 0 \quad \text{for all } t \in [0, +\infty).$$

We note here that apart from the smoothness of f, no further assumptions on the function f are needed in Theorem 1.1. However, for the longtime behaviour, it is necessary to assume additional conditions in order to get the convergence of the flow. Indeed, if $f \geq 0$, by applying the maximum principle to (1-4), we can easily check that

$$u(t) \geq \bigl(\min_M u_0^{4/(n-2)} + \min_M |R_0| t\bigr)^{\frac{n-2}{4}} \to +\infty \ \text{ as } t \to +\infty.$$

So if we want to get the convergence of the flow, it is necessary to assume at least that f is negative somewhere on M. We note that this last condition is also necessary for the resolution of (1-1) since it is well known that if the negative gradient flow associated with a functional $\mathcal{F}$ converges (in some sense), then its limit is a critical point of $\mathcal{F}$.

Before giving conditions on f ensuring the convergence of the flow, let us fix some notation: if $\Omega \subset M$ is an open set, we denote by λ_Ω the first eigenvalue of the conformal Laplacian $L = -c_n\Delta + R_0$ on Ω with zero Dirichlet boundary conditions, that is,

$$\lambda_\Omega = \inf_{0 \neq u \in H_0^1(\Omega)} \frac{\int_M (c_n|\nabla u|^2 + R_0 u^2)\, dV_{g_0}}{\int_M u^2\, dV_{g_0}}.$$

We then assume the following conditions on f:

There exists an open set $\Omega \subset M$ such that

(H1) $$\lambda_\Omega > 0 \text{ and } f < 0 \text{ on } M \setminus \Omega$$

and

(H2) $$\sup_{x\in\Omega} f(x) \le C_\Omega \inf_{x \in M\setminus\Omega} |f(x)|,$$

where C_Ω is a positive constant depending only on Ω.

We then have the following result:

Theorem 1.2. *Suppose that $R_0 < 0$ and that $f \in C^\infty(M)$ satisfies conditions* **(H1)** *and* **(H2)**. *Then there exists a function $0 < \bar{u} \in C^\infty(M)$ such that for any smooth metric $g^0 = u_0^{4/(n-2)} g_0$ with $0 < u_0 \le \bar{u}$, the flow $g(t) = u(t)^{4/(n-2)} g_0$ given by Theorem 1.1 converges in the C^∞-topology to a conformal metric $g_\infty = u_\infty^{4/(n-2)} g_0$ whose scalar curvature is f, that is, $R_{g_\infty} = f$.*

A particular interesting case is when the function f satisfies $f(x) < 0$ for almost all $x \in M$. In this case conditions **(H1)** and **(H2)** are automatically satisfied and then we have the following corollary:

Corollary 1.3. *Suppose that $R_0 < 0$ and $f \in C^\infty(M)$ such that $f < 0$ almost everywhere on M. Then there exists a function $0 < \bar{u} \in C^\infty(M)$ such that for any smooth metric $g^0 = u_0^{4/(n-2)} g_0$ with $0 < u_0 \le \bar{u}$, the flow $g(t) = u(t)^{4/(n-2)} g_0$ given by Theorem 1.1 converges in the C^∞-topology to a conformal metric $g_\infty = u_\infty^{4/(n-2)} g_0$ whose scalar curvature is f, that is, $R_{g_\infty} = f$.*

It is natural to ask if conditions **(H1)** and **(H2)** in Theorem 1.2 are necessary. The following theorem gives a partial answer to this question:

Theorem 1.4. *Suppose that $R_0 < 0$ and let $f \in C^\infty(M)$ such that condition* **(H1)** *is not satisfied, that is, for any open set $\Omega \subset M$ such that $f > 0$ on $M \setminus \Omega$, we* *suppose* $\lambda_\Omega \leq 0$. *Then for any* $0 < u_0 \in C^\infty(M)$, *the solution* $u(t)$ *of* (1-4) *satisfies, for some constant* $C > 0$ *depending only on* u_0, g_0, f,

$$\max_{x \in M} u(t,x) \geq C t^{\frac{n-2}{n+2}} \to +\infty \ \ \textit{as}\ t \to +\infty.$$

We note here that condition **(H1)** is conformally invariant. Similar conditions to **(H1)** and **(H2)** were found by many authors to solve (1-1) by the direct method of elliptic PDEs; see [Bismuth 2000; Rauzy 1995; Vázquez and Véron 1991] for more details. To our knowledge, the only known results on Yamabe type flow on dimension $n \geq 3$ concern the case where f is constant or $M = \mathbb{S}^n$. The Yamabe flow was first introduced by Hamilton [1988] and has been the subject of several studies; see [Brendle 2005; 2007; Chow 1992; Schwetlick and Struwe 2003; Ye 1994]. For the case when f is nonconstant, we mention the work of Struwe [2005] about the Nirenberg's problem on the sphere $\mathbb{S}^2$, and the results of Chen and Xu [2012] concerning $\mathbb{S}^n$, $n \geq 3$. A general evolution problem related to the prescribed Gauss curvature on surfaces was studied by Baird, Fardoun and Regbaoui [Baird et al. 2004].

The paper is organized as follows. In Section 2 we prove the global existence of the flow by establishing local C^k-estimates on the solution u of (1-4). In Section 3, we study the asymptotic behaviour of the flow when $t \to +\infty$. In particular we prove uniform C^k-estimates on u which are necessary to get the convergence of the flow.

2. Global existence of the flow

In this section we shall establish some estimates on the solution u of (1-4) which will be an important tool in proving that the flow $g(t)$ is globally defined on $[0, +\infty)$. In this section we suppose that $R_0 < 0$ and $f \in C^\infty(M)$.

As already mentioned in the previous section, (1-3) is equivalent to (1-4), so it suffices to prove the existence of a solution $u(t)$ of (1-4) defined on $[0, +\infty)$ to obtain a metric $g(t)$ solution of (1-3) defined on $[0, +\infty)$. Since (1-4) is a parabolic equation (on the set of smooth positive functions on $[0, T) \times M$, for any $T > 0$), there exists a smooth solution $u(t)$ of (1-4) defined on a maximal interval $[0, T^*)$ satisfying $u(t) > 0$ on $[0, T^*)$. Thus we have a solution $g(t) = u^{4/(n-2)} g_0$ of (1-3) defined on a maximal interval $[0, T^*)$. For simplicity, we shall write u instead of $u(t)$ and g instead of $g(t)$.

Now, we derive some properties of g which will be important later. One can check by using (1-4) that the scalar curvature R_g satisfies the equation

$$\partial_t R_g = (n-1)\Delta_g(R_g - f) + R_g(R_g - f), \tag{2-1}$$

where Δ_g is the Laplacian associated with $g(t)$.

A simple computation using (2-1) gives

$$\frac{d}{dt}\mathcal{E}(g) = -\frac{n-2}{2}\int_M (R_g - f)^2 \, dV_g, \tag{2-2}$$

so the functional $\mathcal{E}$ is decreasing along the flow $g(t)$. If we set

$$E(u) := \mathcal{E}(g) = \mathcal{E}\big(u^{\frac{4}{n-2}} g_0\big) = \int_M \Big(c_n|\nabla u|^2 + R_0 u^2 - \frac{n-2}{n} f u^{\frac{2n}{n-2}}\Big)\, dV_{g_0},$$

then (2-2) can be written in terms of u:

$$\frac{d}{dt}E(u) = -\frac{8}{n-2}\int_M |\partial_t u|^2 u^{\frac{4}{n-2}}\, dV_{g_0} \le 0. \tag{2-3}$$

The following lemma will be very useful to prove integral estimates on the solution g.

Lemma 2.1. *We have for any* $p > 1$,

$$\frac{d}{dt}\int_M |R_g - f|^p\, dV_g = -\frac{4(n-1)(p-1)}{p}\int_M |\nabla_g |R_g - f|^{\frac{p}{2}}|_g^2\, dV_g + \Big(p - \frac{n}{2}\Big)\int_M (R_g - f)|R_g - f|^p\, dV_g + p\int_M f|R_g - f|^p\, dV_g,$$

where ∇_g *is the gradient with respect to the metric* g *and* $|\cdot|_g$ *is the Riemannian norm with respect to* g.

Proof. We have for any $p \ge 1$

$$\frac{d}{dt}\int_M |R_g - f|^p\, dV_g = p\int_M |R_g - f|^{p-2}(R_g - f)\partial_t R_g\, dV_g + \tfrac{1}{2}\int_M R_g\, tr_g(\partial_t g)\, dV_g.$$

Using (1-3) and (2-1), it follows that

$$\begin{aligned}
\frac{d}{dt}&\int_M |R_g - f|^p\, dV_g \\
&= (n-1)p\int_M |R_g - f|^{p-2}(R_g - f)\Delta_g(R_g - f) \\
&\qquad + p\int_M R_g|R_g - f|^p - \frac{n}{2}\int_M |R_g - f|^p(R_g - f)\, dV_g \\
&= -\frac{4(n-1)(p-1)}{p}\int_M |\nabla_g |R_g - f|^{\frac{p}{2}}|_g^2\, dV_g \\
&\qquad + \Big(p - \frac{n}{2}\Big)\int_M (R_g - f)|R_g - f|^p\, dV_g + p\int_M f|R_g - f|^p\, dV_g. \qquad \square
\end{aligned}$$

In order to prove that the solution $g(t) = u(t)^{4/(n-2)} g_0$ is globally defined on $[0, +\infty)$, we need upper and lower bounds on $u(t)$.

Proposition 2.2. *Let* $g(t) = u(t)^{4/(n-2)} g_0$ *be the solution of* (1-3) *defined on a maximal interval* $[0, T^*)$. *Then we have for any* $t \in [0, T^*)$,

$$\min\big(C_0, \min_M u_0\big) \le u(t) \le \max\big(1, \max_M u_0\big)e^{C_1 t}, \tag{2-4}$$

where

$$C_0 = \big(\min_M |R_0| / \max_M |f|\big)^{\frac{n-2}{4}}, \qquad C_1 = \frac{n-2}{4}\big(\max_M |R_0| + \max_M |f|\big).$$

Proof. The proof uses an elementary maximum principal argument. Indeed, fix $t \in [0, T)$ and let $(t_0, x_0) \in [0, t] \times M$ such that $u(t_0, x_0) = \min_{[0,t]\times M} u$. If $t_0 = 0$,

$$\min_{[0,t]\times M} u = \min_M u_0,$$

so the first inequality in (2-4) is proved in this case. Now suppose that $t_0 > 0$. We have then $\partial_t u(t_0, x_0) \le 0$ and $\Delta u(t_0, x_0) \ge 0$. Thus after substituting in (1-4) we obtain that

$$0 \ge -R_0(x_0)u(t_0, x_0) + f(x_0)u^N(t_0, x_0)$$

which implies

$$u(t_0, x_0) \ge \big(\min_M |R_0| / \max_M |f|\big)^{\frac{1}{N-1}},$$

where $N = \frac{n+2}{n-2}$. This proves the first inequality in (2-4). In order to prove the second inequality we set $v = e^{-C_1 t}u$ instead of u, where $C_1 = \frac{4}{n-2}(\max_M |R_0| + \max_M |f|)$. As above, fix $t \in [0, T)$ and let $(t_0, x_0) \in [0, t] \times M$ such that $v(t_0, x_0) = \max_{[0,t]\times M} v$. If $t_0 = 0$, then $\max_{[0,t]\times M} v = \max_M u_0$, which implies

$$\max_{[0,t]\times M} u \le \max_M u_0 \, e^{C_1 t},$$

so the second inequality in (2-4) is proved in this case. Now suppose that $t_0 > 0$. We have then $\partial_t v(t_0, x_0) \ge 0$ and $\Delta v(t_0, x_0) \le 0$, that is, $\partial_t u(t_0, x_0) \ge C_1 u(t_0, x_0)$ and $\Delta u(t_0, x_0) \le 0$. We obtain after substituting in (1-4) that

$$N C_1 u^N(t_0, x_0) \le \frac{n+2}{4}(-R_0(x_0)u(t_0, x_0) + f(x_0)u^N(t_0, x_0))$$

which implies that

$$u(t_0, x_0) \le 1 \tag{2-5}$$

since $NC_1 = \frac{n+2}{4}(\max_M |R_0| + \max_M |f|)$. It is clear that (2-5) implies that

$$\max_{[0,t]\times M} u \le e^{C_1 t}.$$

The proof of Proposition 2.2 is then complete. □

Now we prove integral estimates on R_g which will imply estimates on $\partial_t u$:

Proposition 2.3. *Let $g(t)$ be the solution of* (1.3) *defined on a maximal interval* $[0, T^*)$. *Then we have for any* $t \in [0, T^*)$,

$$\int_M |R_{g(t)} - f|^p \, dV_{g(t)} \le C e^{Ct} \tag{2-6}$$

where $p = \frac{n^2}{2(n-2)}$ and C is a positive constant depending only on f, g_0, u_0.

Proof. In what follows C denotes a positive constant depending on f, g_0, u_0, whose value may change from line to line.

We have by Lemma 2.1 for any $t \in [0, T^*)$

$$\frac{d}{dt}\int_M |R_g - f|^p\, dV_g = -\frac{4(n-1)(p-1)}{p}\int_M \left|\nabla_g |R_g - f|^{\frac{p}{2}}\right|_g^2 dV_g + \left(p - \frac{n}{2}\right)\int_M (R_g - f)|R_g - f|^p\, dV_g + p\int_M f|R_g - f|^p\, dV_g, \tag{2-7}$$

where ∇_g is the gradient with respect to the metric g and $|\cdot|_g$ is the Riemannian norm with respect to g. It follows from (2-7) that

$$\frac{d}{dt}\int_M |R_g - f|^p\, dV_g + \frac{4(n-1)(p-1)}{p}\int_M \left|\nabla_g |R_g - f|^{\frac{p}{2}}\right|_g^2 dV_g \le \left|p - \frac{n}{2}\right| \int_M |R_g - f|^{p+1}\, dV_g + C\int_M |R_g - f|^p\, dV_g. \tag{2-8}$$

By (2-4) we have

$$\int_M \left|\nabla_g |R_g - f|^{\frac{p}{2}}\right|_g^2 dV_g = \int_M \left|\nabla |R_g - f|^{\frac{p}{2}}\right|^2 u^2\, dV_{g_0} \ge C\int_M \left|\nabla |R_g - f|^{\frac{p}{2}}\right|^2 dV_{g_0} \tag{2-9}$$

and

$$\int_M |R_g - f|^p\, dV_g = \int_M |R_g - f|^p u^{\frac{2n}{n-2}}\, dV_{g_0} \ge C\int_M |R_g - f|^p\, dV_{g_0}. \tag{2-10}$$

By Sobolev's inequality we have

$$\left(\int_M |R_g - f|^{\frac{pn}{n-2}}\, dV_{g_0}\right)^{\frac{n-2}{n}} \le C\left(\int_M \left|\nabla |R_g - f|^{\frac{p}{2}}\right|^2 dV_{g_0} + \int_M |R_g - f|^p\, dV_{g_0}\right)$$

which, using (2-4), gives

$$\left(\int_M |R_g - f|^{\frac{pn}{n-2}}\, dV_g\right)^{\frac{n-2}{n}} \le Ce^{Ct}\left(\int_M \left|\nabla |R_g - f|^{\frac{p}{2}}\right|^2 dV_{g_0} + \int_M |R_g - f|^p\, dV_{g_0}\right). \tag{2-11}$$

It follows from (2-8), (2-9) , (2-10) and (2-11) that

$$\frac{d}{dt}\int_M |R_g - f|^p\, dV_g + C^{-1}e^{-Ct}\left(\int_M |R_g - f|^{\frac{pn}{n-2}}\, dV_g\right)^{\frac{n-2}{n}} \le \left(p - \frac{n}{2}\right)\int_M |R_g - f|^{p+1}\, dV_g + C\int_M |R_g - f|^p\, dV_g. \tag{2-12}$$

By taking $p = \frac{n}{2}$ in (2-12) we get

$$\frac{d}{dt}\int_M |R_g - f|^p \, dV_g \le C \int_M |R_g - f|^p \, dV_g,$$

which implies that

$$\int_M |R_g - f|^{\frac{n}{2}} \, dV_g \le Ce^{Ct}. \tag{2-13}$$

Now taking again $p = \frac{n}{2}$ in (2-12) and integrating on $[0, t]$, $t \in [0, T^*)$, by using (2-13) we obtain

$$\int_0^t \left(\int_M |R_{g(s)} - f|^{\frac{n^2}{2(n-2)}} \, dV_{g(s)} \right)^{\frac{n-2}{n}} ds \le Ce^{Ct}. \tag{2-14}$$

We have by Hölder's inequality and Young's inequality, for any $\varepsilon > 0$ and $p > \frac{n}{2}$,

$$\begin{aligned} \int_M |R_g - f|^{p+1} \, dV_g \\ \le \varepsilon \left(\int_M |R_g - f|^{\frac{pn}{n-2}} \, dV_g \right)^{\frac{n-2}{n}} + \varepsilon^{-\frac{n}{2p-n}} \left(\int_M |R_g - f|^p \, dV_g \right)^{\frac{2p-n+2}{2p-n}}. \end{aligned} \tag{2-15}$$

If we combine (2-15) with (2-12) and taking $\varepsilon = (p - \frac{n}{2})^{-1} C^{-1} e^{-Ct}$, we get

$$\frac{d}{dt}\int_M |R_g - f|^p \, dV_g \le Ce^{Ct} \left(\int_M |R_g - f|^p \, dV_g \right)^{\frac{2p-n+2}{2p-n}} + C \int_M |R_g - f|^p \, dV_g,$$

that is,

$$\frac{d}{dt} \log \left(\int_M |R_g - f|^p \, dV_g \right) \le C \left(e^{Ct} \left(\int_M |R_g - f|^p \, dV_g \right)^{\frac{2}{2p-n}} + 1 \right)$$

In particular by choosing $p = \frac{n^2}{2(n-2)}$ and integrating on $[0, t]$, $t \in [0, T^*)$, we obtain

$$\begin{aligned} \log \left(\int_M |R_{g(t)} - f|^{\frac{n^2}{2(n-2)}} \, dV_{g(t)} \right) \le \log \left(\int_M |R_{g(0)} - f|^{\frac{n^2}{2(n-2)}} \, dV_{g(0)} \right) \\ + Ce^{Ct} \int_0^t \left(\int_M |R_{g(s)} - f|^{\frac{n^2}{2(n-2)}} \, dV_{g(s)} \right)^{\frac{n-2}{n}} ds + Ct \end{aligned}$$

which by using (2-14) gives

$$\log \left(\int_M |R_{g(t)} - f|^{\frac{n^2}{2(n-2)}} \, dV_{g(t)} \right) \le Ce^t.$$

This proves Proposition 2.3. □

With the estimates of Proposition 2.2 we would like to apply the classical Schauder estimates for parabolic equations. To this end we need C^α-estimates:

Proposition 2.4. *Let $g(t) = u(t)^{4/(n-2)} g_0$ be the solution of (1-3) defined on a maximal interval $[0, T^*)$. Then we have for some $\alpha \in (0, 1)$ and any $T \in [0, T^*)$*

$$\|u\|_{C^\alpha([0,T]\times M)} \leq Ce^{CT},$$

where C is a positive constant depending only on u_0, g_0 and f.

Proof. By using Propositions 2.2 and 2.3, the proof is identical to that of Proposition 2.6 in Brendle [2005]. □

Proof of Theorem 1.1. Let $g(t) = u(t)^{4/(n-2)} g_0$ be the solution of (1-3) defined on a maximal interval $[0, T^*)$. Assume by contradiction that $T^* < +\infty$. Then by using Propositions 2.2 and 2.4 we have

$$\|u\|_{C^\alpha([0,T^*)\times M)} \leq Ce^{CT^*} \quad \text{and} \quad \min_{[0,T^*)\times M} u \geq \min\big(C_0, \min_M u_0\big)$$

for some $\alpha \in (0, 1)$, where C is a positive constant depending u_0, f, g_0. The classical theory of linear parabolic equations applied to (1-4) implies that u is bounded in $C^k([0, T^*) \times M)$ for any $k \in \mathbb{N}$, that is,

$$\|u\|_{C^k([0,T^*)\times M)} \leq C_k, \tag{2-16}$$

where C_k is a positive constant depending only on u_0, g_0, f and k. It is clear that (2-16) allows us to extend the solution beyond T^* contradicting thus the maximality of T^*. We see from (2-2) that the functional $\mathcal{E}$ is decreasing along the flow. The proof of Theorem 1.1 is then complete. □

3. Long Time behaviour of the flow

In this section we study the asymptotic behaviour of the flow $g(t)$ when $t \to +\infty$. First we prove the following proposition which gives a super solution of (1-1) when conditions **(H1)** and **(H2)** are satisfied.

Proposition 3.1. *Suppose that there exists an open set $\Omega \subset M$ such that conditions* **(H1)** *and* **(H2)** *are satisfied. Then there exists a conformal metric $\bar{g} = \bar{u}^{4/(n-2)} g_0$, $0 < \bar{u} \in C^\infty(M)$, satisfying*

$$R_{\bar{g}} - f \geq 0 \tag{3-1}$$

or equivalently

$$-c_n \Delta \bar{u} + R_0 \bar{u} - f \bar{u}^N \geq 0, \;\; N = \frac{n+2}{n-2}. \tag{3-2}$$

Proof. By hypothesis, there is an open set $\Omega \subset M$ satisfying **(H1)** and **(H2)**, that is,

(H1) $$\lambda_\Omega > 0 \ \text{ and } \ f < 0 \text{ on } M \setminus \Omega$$

and

(H2) $$\sup_{x\in\Omega} f(x) \le C_\Omega \inf_{x\in M\setminus\Omega} |f(x)|,$$

where C_Ω is a positive constant depending only on Ω.

Let $\varepsilon > 0$ and set

$$\Omega_\varepsilon = \{x \in M : d(x, \Omega) < \varepsilon\}.$$

For $\varepsilon > 0$ sufficiently small we have from **(H1)** that $\lambda_{\Omega_\varepsilon} > 0$, where $\lambda_{\Omega_\varepsilon}$ is the first eigenvalue of the operator $-c_n\Delta + R_0$ on Ω_ε with zero Dirichlet boundary conditions. Let $D \subset M$ be an open set of smooth boundary such that $\bar\Omega \subset D \subset \Omega_\varepsilon$. Then we have $\lambda_D \ge \lambda_{\Omega_\varepsilon} > 0$. Let φ_0 an eigenfunction associated with λ_D, that is,

$$-c_n\Delta\varphi_0 + R_0\varphi_0 = \lambda_D\varphi_0.$$

Then we have that $\varphi_0 \in C^\infty(\bar D)$ and using the maximum principle of elliptic equations we have $\varphi_0 > 0$ on D. By normalising if necessary, we may suppose that

(3-3) $$0 < \varphi_0 \le 1 \ \text{ on } D.$$

Let $\chi \in C_0^\infty(D)$ such that $0 \le \chi \le 1$ and $\chi = 1$ on $\bar\Omega$. We define the function $\bar u \in C^\infty(M)$ by setting

$$\bar u = \delta(\chi\varphi_0 + 1 - \chi),$$

where $\delta > 0$ will be chosen later. By (3-3) and the definition of χ it is easy to check

$$m_0 := \inf_M(\chi\varphi_0 + 1 - \chi) > 0,$$

so

(3-4) $$\bar u \ge \delta m_0.$$

Now let us prove that $\bar u$ satisfies (3-2). If we set

$$\mathcal{L}(\bar u) = -c_n\Delta\bar u + R_0\bar u - f\bar u^{\frac{n+2}{n-2}},$$

then (3-2) is equivalent to $\mathcal{L}(\bar u) \ge 0$.

A simple computation shows that we have on Ω (using the fact that $\chi = 1$ on $\bar\Omega$):

$$\mathcal{L}(\bar u) = \lambda_D\delta\varphi_0 - f\delta^N\varphi_0^N = \delta\varphi_0(\lambda_D - \delta^{N-1}f\varphi_0^{N-1})$$

and by using (3-3) it follows that

(3-5) $$\mathcal{L}(\bar u) \ge \delta\varphi_0\big(\lambda_D - \delta^{N-1}\sup_{x\in\Omega} f(x)\big).$$

It follows from (3-5) that if we want $\mathcal{L}(\bar{u}) \geq 0$ on Ω, we have to choose $\delta > 0$ satisfying

$$\delta^{N-1} \sup_{x\in\Omega} f(x) \leq \lambda_D. \tag{3-6}$$

Now we examine the sign of $\mathcal{L}(\bar{u})$ on $M\setminus\Omega$. We have from the definition of $\bar{u}$ that

$$\mathcal{L}(\bar{u}) = \delta(-c_n\Delta + R_0)(\chi\varphi_0 + 1 - \chi) - f\bar{u}^N. \tag{3-7}$$

By using (3-4) and the fact that $f < 0$ on $M \setminus \Omega$, it follows from (3-7) that

$$\mathcal{L}(\bar{u}) \geq -\delta m_1 + \delta^N m_0^N \inf_{x\in M\setminus\Omega} |f(x)|,$$

where

$$m_1 = \sup_M |(-c_n\Delta + R_0)(\chi\varphi_0 + 1 - \chi)|.$$

Thus, if we want $\mathcal{L}(\bar{u}) \geq 0$ on $M \setminus \Omega$, we have to assume

$$-m_1 + \delta^{N-1} m_0^N \inf_{x\in M\setminus\Omega} |f(x)| \geq 0,$$

that is,

$$\delta^{N-1} \inf_{x\in M\setminus\Omega} |f(x)| \geq m_1 m_0^{-N}. \tag{3-8}$$

It is clear that the existence of $\delta > 0$ satisfying both (3-6) and (3-8) is equivalent to condition **(H2)** with $C_\Omega = \lambda_D m_0^N / m_1$. □

Proposition 3.1 allows us to prove uniform L^∞-estimates on the flow.

Proposition 3.2. *Let $0 < u_0 \in C^\infty(M)$ such that $u_0 \leq \bar{u}$ where $\bar{u}$ is given by Proposition 3.1. Then the solution u of (1-4) satisfies, for any $(t, x) \in [0, +\infty) \times M$,*

$$\min\bigl(C_0, \min_M u_0\bigr) \leq u(t, x) \leq \max_M \bar{u}, \tag{3-9}$$

where

$$C_0 = \bigl(\min_M |R_0| / \max_M |f|\bigr)^{\frac{n-2}{4}}.$$

Proof. First observe that the first inequality in (3-9) has already been proved in Proposition 2.2. It remains then to prove the second inequality, that is,

$$u(t, x) \leq \max_M \bar{u}.$$

Let $v = \bar{u} - u$. Since u satisfies (1-4) and $\bar{u}$ satisfies (3-2), we have

$$\partial_t(\bar{u}^N - u^N) \geq \frac{n+2}{4}(c_n\Delta v - R_0 v + f(\bar{u}^N - u^N)). \tag{3-10}$$

We have $\bar{u}^N - u^N = av$, where

$$a(t, x) = N\int_0^1 \bigl(s\bar{u}(t, x) + (1 - s)u(t, x)\bigr)^{N-1} ds,$$

so it follows from (3-10) that

$$\partial_t(av) \geq \frac{n+2}{4}(c_n \Delta v - R_0 v + afv). \tag{3-11}$$

Since $v(0,x) = \bar{u}(x) - u_0(x) \geq 0$, by applying the maximum principle to (3-11) we get $v(t,x) \geq 0$ for any $t \geq 0$, that is,

$$u(t,x) \leq \bar{u}(x).$$

□

Now we prove that the integral estimate (2-6) in Proposition 2.3 can be improved when $t \to +\infty$. More precisely, we have:

Proposition 3.3. *Let $0 < u_0 \in C^\infty(M)$ such that $u_0 \leq \bar{u}$ where $\bar{u}$ is given by Proposition 3.1. Let $g(t)$ be the solution of (1-3) given by Theorem 1.1 such that $g(0) = u_0^{4/(n-2)} g_0$. Then we have, for any $p \geq 1$,*

$$\lim_{t \to +\infty} \int_M |R_{g(t)} - f|^p \, dV_{g(t)} = 0. \tag{3-12}$$

Proof. In what follows C denotes a positive constant depending only on u_0, g_0, f, p, and its value may change from line to line.

We have by (2-2) for any $t \geq 0$,

$$\frac{n-2}{2} \int_0^t \int_M |R_g - f|^2 \, dV_g = \mathcal{E}(g(0)) - \mathcal{E}(g(t)). \tag{3-13}$$

On the other hand, we have

$$\mathcal{E}(g(t)) = \int_M \left(c_n |\nabla u|^2 + R_0 u^2 - \frac{n-2}{n} f u^{\frac{2n}{n-2}} \right) dV_{g_0},$$

and since u is uniformly bounded by Proposition 3.2, we have $\mathcal{E}(g(t)) \geq -C$. So it follows from (3-13) that

$$\int_0^{+\infty} \int_M |R_{g(t)} - f|^2 \, dV_{g(t)} \leq C. \tag{3-14}$$

Since by Proposition 3.2 the volume of $g(t)$ is uniformly bounded, it suffices to prove (3-12) for a sequence $p_k \to +\infty$. We shall prove (3-12) by induction when $p = p_k$, where

$$p_k := \frac{n}{2} \left(\frac{n}{n-2} \right)^k, \quad k \in \mathbb{N}.$$

First we prove (3-12) for $p_0 = \frac{n}{2}$. As in the proof of Proposition 2.3, if we use Lemma 2.1 and the fact that u is uniformly bounded by Proposition 3.2, then we have for any $p > 1$:

$$\frac{d}{dt}\int_M |R_g - f|^p \, dV_g + C^{-1}\left(\int_M |R_g - f|^{\frac{pn}{n-2}} \, dV_g\right)^{\frac{n-2}{n}} \leq C\int_M |R_g - f|^p \, dV_g + \left(p - \frac{n}{2}\right)\int_M |R_g - f|^{p+1} \, dV_g. \tag{3-15}$$

Set

$$\phi_p(t) = \int_M |R_g - f|^p \, dV_g.$$

If $p_0 < 2$, then by using Hölder's inequality and the fact that u is uniformly bounded, we have

$$\phi_{p_0} \leq C\phi_2^{p_0/2}. \tag{3-16}$$

So it follows from (3-15) by taking $p = p_0 = \frac{n}{2}$ that

$$\frac{d}{dt}\phi_{p_0}^{2/p_0} \leq C\phi_2. \tag{3-17}$$

By (3-14) there is a sequence $t_\nu \to +\infty$ such that $\int_{t_\nu}^{+\infty} \phi_2(s)\, ds \to 0$ and $\phi_2(t_\nu) \to 0$. So by integrating (3-17) on $[t_\nu, t]$ and using (3-16) we get

$$\phi_{p_0}^{2/p_0}(t) \leq \phi_{p_0}^{2/p_0}(t_\nu) + C\int_{t_\nu}^{t} \phi_2(s)\, ds \leq C\phi_2(t_\nu) + C\int_{t_\nu}^{t} \phi_2(s)\, ds$$

Letting $t \to +\infty$ and $\nu \to +\infty$ we obtain $\phi_{p_0}(t) \to 0$ as $t \to +\infty$.

If $p_0 \geq 2$, by using Hölder's inequality and Young's inequality we have, for any $\varepsilon > 0$,

$$\int_M |R_g - f|^{p_0} \, dV_g \leq \varepsilon\left(\int_M |R_g - f|^{p_0 n/n-2} \, dV_g\right)^{\frac{n-2}{n}} + \varepsilon^{-n(p_0-2)/4}\left(\int_M |R_g - f|^2 \, dV_g\right)^{p_0/2}.$$

By taking $\varepsilon = \frac{1}{2}C^{-1}$, where C^{-1} is the constant appearing in (3-15), we obtain from (3-15)(where we take $p = p_0 = \frac{n}{2}$),

$$\frac{d}{dt}\phi_{p_0} + C^{-1}\phi_{p_0 n/(n-2)}^{(n-2)/n} \leq C\phi_2^{p_0/2}. \tag{3-18}$$

But by Hölder's inequality, since the volume of g is uniformly bounded, we have

$$\phi_2 \leq C\phi_{p_0}^{2/p_0} \quad \text{and} \quad \phi_{p_0} \leq C\phi_{p_0 n/(n-2)}^{(n-2)/n}.$$

Thus it follows from (3-18) that

$$\frac{d}{dt}\phi_{p_0}^{2/p_0} + C^{-1}\phi_{p_0}^{2/p_0} \leq C\phi_2. \tag{3-19}$$

If we integrate (3-19) on $[0, t]$ and use (3-14) we get

$$\int_0^t \phi_{p_0}^{2/p_0}(s)\,ds \le C,$$

which implies, since $t \ge 0$ is arbitrary,

$$\int_0^{+\infty} \phi_{p_0}^{2/p_0}(s)\,ds \le C.$$

Thus there exists a sequence $t_\nu \to +\infty$ such that $\phi_{p_0}^{2/p_0}(t_\nu) \to 0$ as $\nu \to +\infty$. If we integrate again (3-19) on $[t_\nu, t]$, we obtain

$$\phi_{p_0}^{2/p_0}(t) \le \phi_{p_0}^{2/p_0}(t_\nu) + C\int_{t_\nu}^t \phi_2(s)\,ds.$$

By using (3-14), it follow that $\phi_{p_0}^{2/p_0}(t) \to 0$ as $t \to +\infty$.

Now suppose by induction that

(3-20) $$\lim_{t\to+\infty} \phi_{p_k}(t) = 0.$$

First let us prove that

(3-21) $$\lim_{t\to+\infty} \int_t^{t+1} \phi_{p_{k+1}}^{(n-2)/n}(s)\,ds = 0.$$

We may suppose $k \ge 1$. Indeed, if $k = 0$ (that is, $p_k = p_0 = \frac{n}{2}$), then (3-21) follows directly from (3-15) (with $p = \frac{n}{2}$) by integrating on $[t, t+1]$ and using (3-20). Thus let us prove (3-21) when $k \ge 1$.

By using Hölder's inequality and Young's inequality we have for any $p > \frac{n}{2}$ and $\varepsilon > 0$,

(3-22) $$\int_M |R_g - f|^{p+1}\,dV_g \le \varepsilon\left(\int_M |R_g - f|^{\frac{pn}{n-2}}\,dV_g\right)^{\frac{n-2}{n}} + \varepsilon^{-\frac{n}{2p-n}}\left(\int_M |R_g - f|^p\,dV_g\right)^{1+\frac{2}{2p-n}}$$

By taking $p = p_k$, $\varepsilon = \frac{1}{2}C^{-1}$, where C is the constant appearing in (3-15), from (3-15) we obtain

$$\frac{d}{dt}\phi_{p_k} + \tfrac{1}{2}C^{-1}\phi_{p_{k+1}}^{(n-2)/n} \le C\phi_{p_k}^{1+2/(2p_k-n)} + C\phi_{p_k}.$$

Then (3-21) follows by integrating on $[t, t+1]$ and using (3-20).

Now if we apply (3-22) by taking $p = p_{k+1}$ and $\varepsilon = C^{-1}/(p_{k+1} - n/2)$, where C is the constant appearing in (3-15), we obtain from (3-15) (where we take $p = p_{k+1}$),

$$\frac{d}{dt}\phi_{p_{k+1}} \le C\phi_{p_{k+1}}^{1+\alpha_k} + C\phi_{p_{k+1}},$$

where $\alpha_k = \frac{2}{2p_{k+1}} - n$. The last inequality is equivalent to

$$\frac{d}{dt}\log\phi_{p_{k+1}} \le C\big(\phi_{p_{k+1}}^{\alpha_k} + 1\big). \tag{3-23}$$

By (3-21) there is a sequence $t_\nu \to +\infty$ such that $\nu \le t_\nu \le \nu+1$ satisfying $\phi_{p_{k+1}}(t_\nu) \to 0$ as $\nu \to +\infty$. If we integrate (3-23) on $[t_\nu, t]$ where $t \in [\nu, \nu+1]$, we obtain

$$\log\frac{\phi_{p_{k+1}}(t)}{\phi_{p_{k+1}}(t_\nu)} \le C\bigg(\int_\nu^{\nu+1} \phi_{p_{k+1}}^{\alpha_k}(s)\,ds + 1\bigg). \tag{3-24}$$

We note here that $\alpha_k \le \frac{n-2}{n}$, so by Hölder's inequality we have

$$\int_\nu^{\nu+1} \phi_{p_{k+1}}^{\alpha_k}(s)\,ds \le \bigg(\int_\nu^{\nu+1} \phi_{p_{k+1}}^{(n-2)/n}(s)\,ds\bigg)^{\frac{n\alpha_k}{n-2}} \to 0 \;\text{ as }\; \nu \to +\infty$$

by (3-21). Thus it follows from (3-24) that

$$\log\frac{\phi_{p_{k+1}}(t)}{\phi_{p_{k+1}}(t_\nu)} \le C,$$

which implies that $\phi_{p_{k+1}}(t) \to 0$ as $t \to +\infty$. The proof of Proposition 3.3 is then complete. □

Now we can prove uniform C^α-estimates on the solution.

Proposition 3.4. *Let $0 < u_0 \in C^\infty(M)$ such that $u_0 \le \bar{u}$ where $\bar{u}$ is given by Proposition 3.1. Then the solution u of (1-4) satisfies, for some $\alpha \in (0, 1)$,*

$$\|u\|_{C^\alpha([0,+\infty)\times M)} \le C,$$

where C is a positive constant depending only on u_0, g_0 and f.

Proof. By using Propositions 3.2 and 3.3, the proof is identical to that of Proposition 2.6 in Brendle [2005]. □

Now we are in position to prove Theorem 1.2.

Proof of Theorem 1.2. Let $g = u^{4/(n-2)}g_0$ be the solution of (1-3) given by Theorem 1.1. By Proposition 3.2 we have that u is bounded from below and above uniformly on $[0, +\infty)$. As in the proof of Theorem 1.1, this implies that (1-4) is uniformly parabolic and by Proposition 3.4 we have a uniform C^α-bound on the solution u on $[0, +\infty) \times M$. We then apply the classical regularity theory of linear parabolic equations to obtain uniform C^k-bound for any $k \in \mathbb{N}$, that is,

$$\|u(t)\|_{C^k(M)} \le C_k, \tag{3-25}$$

for some constant C_k independent of t. It follows from (3-25) that there is a sequence $t_\nu \to +\infty$ such that $u(t_\nu)$ converges in $C^k(M)$ for any $k \in \mathbb{N}$, to some function

$u_\infty \in C^\infty(M)$. Since $u(t)$ is uniformly bounded from below by Proposition 3.2, then we have $u_\infty > 0$. By using Proposition 3.3 and passing to the limit when $\nu \to \infty$, we see that $R_{g_\infty} = f$, where $g_\infty = u_\infty^{4/(n-2)} g_0$, that is, f is the scalar curvature of g_∞. By the general result of Simon [1983] on evolution equations, u_∞ is the unique limit of $u(t)$ when $t \to +\infty$. □

Proof of Corollary 1.3. Since $f < 0$ almost everywhere on M, then for $\varepsilon > 0$ small enough, the open set

$$\Omega_\varepsilon = \{x \in M : f(x) > -\varepsilon\}$$

has arbitrary small volume. This implies that the first eigenvalue μ_{Ω_ε} of $-c_n\Delta$ on Ω_ε with zero Dirichlet conditions is arbitrarily large if ε is small enough. But since

$$\lambda_{\Omega_\varepsilon} \geq \mu_{\Omega_\varepsilon} + \min_M R_0,$$

we have $\lambda_{\Omega_\varepsilon} > 0$ if ε is small enough. Thus the condition **(H1)** is satisfied with $\Omega = \Omega_\varepsilon$. Condition **(H2)** is also satisfied since by continuity of f we have $f \leq 0$ everywhere on M. □

Proof of Theorem 1.4. Suppose that condition **(H1)** is not satisfied, that is, for any open set $\Omega \subset M$ such that $f < 0$ on $M \setminus \Omega$, we suppose $\lambda_\Omega \leq 0$. For $\varepsilon > 0$, consider the following family of open sets:

$$\Omega_\varepsilon = \{x \in M : f(x) > -\varepsilon\}.$$

For simplicity of notation we set $\lambda_\varepsilon = \lambda_{\Omega_\varepsilon}$. According to our hypothesis we have

$$\lambda_\varepsilon \leq 0 \quad \text{for all } \varepsilon > 0. \tag{3-26}$$

By using Sard's theorem, there exists a sequence $\varepsilon_n \to 0$ such that ε_n is a regular value of f and then Ω_{ε_n} has a smooth boundary

$$\partial\Omega_{\varepsilon_n} = \{x \in M : f(x) = -\varepsilon_n\}.$$

Let φ_n an eigenfunction of $-c_n\Delta + R_0$ associated with λ_{ε_n}. As already mentioned in the proof of Proposition 3.2, we have by the maximum principle that

$$\varphi_n > 0 \text{ on } \Omega_{\varepsilon_n} \quad \text{and} \quad \frac{\partial \varphi_n}{\partial \nu} \leq 0 \text{ on } \partial\Omega_{\varepsilon_n}, \tag{3-27}$$

where ν is the outer normal vector to $\partial\Omega_{\varepsilon_n}$. By normalising if necessary, we may assume that

$$\int_{\Omega_{\varepsilon_n}} \varphi_n \, dV_{g_0} = 1. \tag{3-28}$$

If we multiply (1-4) by φ_n and integrate on Ω_{ε_n}, we have

$$\frac{d}{dt}\int_{\Omega_{\varepsilon_n}} u^N\varphi_n\,dV_{g_0} = \frac{n+2}{4}\int_{\Omega_{\varepsilon_n}}(c_n\Delta u - R_0u)\varphi_n\,dV_{g_0} + \frac{n+2}{4}\int_{\Omega_{\varepsilon_n}} fu^N\varphi_n\,dV_{g_0}. \tag{3-29}$$

An integration by parts gives

$$\int_{\Omega_{\varepsilon_n}}(c_n\Delta u - R_0u)\varphi_n\,dV_{g_0} = -\lambda_{\varepsilon_n}\int_{\Omega_{\varepsilon_n}} u\varphi_n\,dV_{g_0} - c_n\int_{\partial\Omega_{\varepsilon_n}}\frac{\partial\varphi_n}{\partial\nu}u\,dV_{g_0}.$$

Since $\lambda_{\varepsilon_n}\le 0$, by using (3-27) and (3-28) we then obtain

$$\int_{\Omega_{\varepsilon_n}}(c_n\Delta u - R_0u)\varphi_n\,dV_{g_0} \ge -\lambda_{\varepsilon_n}\inf_M u - c_n\inf_M u\int_{\partial\Omega_{\varepsilon_n}}\frac{\partial\varphi_n}{\partial\nu}\,dV_{g_0}. \tag{3-30}$$

On the other hand we have

$$c_n\int_{\partial\Omega_{\varepsilon_n}}\frac{\partial\varphi_n}{\partial\nu}\,dV_{g_0} = c_n\int_{\Omega_{\varepsilon_n}}\Delta\varphi_n\,dV_{g_0} = \int_{\Omega_{\varepsilon_n}}(-\lambda_{\varepsilon_n}+R_0)\varphi_n\,dV_{g_0}$$

and by using (3-28) we get, since $R_0<0$,

$$-c_n\int_{\partial\Omega_{\varepsilon_n}}\frac{\partial\varphi_n}{\partial\nu}\,dV_{g_0} \ge \lambda_{\varepsilon_n} + \inf_M|R_0|. \tag{3-31}$$

Combining (3-30) and (3-31) we obtain

$$\int_{\Omega_{\varepsilon_n}}(c_n\Delta u - R_0u)\varphi_n\,dV_{g_0} \ge \inf_M|R_0|\inf_M u.$$

If we substitute in (3-29) we get

$$\frac{d}{dt}\int_{\Omega_{\varepsilon_n}} u^N\varphi_n\,dV_{g_0} \ge \frac{n+2}{4}\inf_M|R_0|\inf_M u + \frac{n+2}{4}\int_{\Omega_{\varepsilon_n}} fu^N\varphi_n\,dV_{g_0}. \tag{3-32}$$

By Proposition 3.2 we have $u\ge C_0$, where C_0 is a positive constant depending only on u_0, g_0 and f. Using the fact that $f>-\varepsilon_n$ on Ω_{ε_n}, it follows from (3-32) that

$$\frac{d}{dt}\int_{\Omega_{\varepsilon_n}} u^N\varphi_n\,dV_{g_0} \ge C - \frac{n+2}{4}\,\varepsilon_n\int_{\Omega_{\varepsilon_n}} u^N\varphi_n\,dV_{g_0},$$

where C is a positive constant depending only on u_0, g_0 and f. By integrating this differential inequality on $[0,t]$, we get

$$\int_{\Omega_{\varepsilon_n}} u^N(t)\varphi_n\,dV_{g_0} \ge \int_{\Omega_{\varepsilon_n}} u_0^N\varphi_n\,dV_{g_0} + Ct - \frac{n+2}{4}\,\varepsilon_n\int_0^t\int_{\Omega_{\varepsilon_n}} u^N(s)\varphi_n\,dV_{g_0}\,ds,$$

which, using (3-28), implies

$$\max_{x\in M} u^N(t,x) \geq Ct - \frac{n+2}{4}\,\varepsilon_n \int_0^t \max_{x\in M} u^N(s,x)\,ds.$$

Letting $n \to +\infty$, we obtain

$$\max_{x\in M} u^N(t,x) \geq Ct.$$

The proof of Theorem 1.4 is complete. □

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