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RIGIDITY OF PROPER HOLOMORPHIC MAPPINGS BETWEEN GENERALIZED FOCK–BARGMANN–HARTOGS DOMAINS

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A generalized Fock–Bargmann–Hartogs domain $D_n^{m,p}(\mu)$ is defined as a domain fibered over $\mathbb{C}^n$ with the fiber over $z \in \mathbb{C}^n$ being a generalized complex ellipsoid $\Sigma_z(m, p)$. In general, a generalized Fock–Bargmann–Hartogs domain is an unbounded nonhyperbolic domain without smooth boundary. The main contribution of this paper is as follows. By using the explicit formula of Bergman kernels of the generalized Fock–Bargmann–Hartogs domains, we obtain the rigidity results of proper holomorphic mappings between two equidimensional generalized Fock–Bargmann–Hartogs domains. We therefore exhibit an example of unbounded weakly pseudoconvex domains on which the rigidity results of proper holomorphic mappings can be built.

1. Introduction

A holomorphic map $F : \Omega_1 \to \Omega_2$ between two domains Ω_1, Ω_2 in $\mathbb{C}^n$ is said to be proper if $F^{-1}(K)$ is compact in Ω_1 for every compact subset $K \subset \Omega_2$. In particular, an automorphism $F : \Omega \to \Omega$ of a domain Ω in $\mathbb{C}^n$ is a proper holomorphic mapping of Ω into Ω. There are many works about proper holomorphic mappings between various bounded domains with some requirements of the boundary, e.g., [Bedford and Bell 1982; Diederich and Fornæss 1982; Dini and Selvaggi Primicerio 1997; Tu and Wang 2015]. However, very little seems to be known about proper holomorphic mapping between the unbounded weakly pseudoconvex domains. There are also some works about automorphism groups of hyperbolic domains, e.g., [Isaev 2007; Isaev and Krantz 2001; Kim and Verdiani 2004]. In this paper, we mainly focus our attention on some unbounded nonhyperbolic weakly pseudoconvex domains.

The Fock–Bargmann–Hartogs domain $D_{n,m}(\mu)$ is defined by

$$D_{n,m}(\mu) = \left\{(z, w) \in \mathbb{C}^n \times \mathbb{C}^m : \|w\|^2 < e^{-\mu\|z\|^2}\right\} \quad \text{for } \mu > 0,$$

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where $\|\cdot\|$ is the standard Hermitian norm. The Fock–Bargmann–Hartogs domains $D_{n,m}(\mu)$ are strongly pseudoconvex domains in $\mathbb{C}^{n+m}$ with smooth real-analytic boundary. We note that each $D_{n,m}(\mu)$ contains $\{(z,0)\in\mathbb{C}^n\times\mathbb{C}^m\}\cong\mathbb{C}^n$. Thus each $D_{n,m}(\mu)$ is not hyperbolic in the sense of Kobayashi and $D_{n,m}(\mu)$ can not be biholomorphic to any bounded domain in $\mathbb{C}^{n+m}$. Therefore, each Fock–Bargmann–Hartogs domain $D_{n,m}(\mu)$ is an unbounded nonhyperbolic domain in $\mathbb{C}^{n+m}$.

Yamamori [2013] gave an explicit formula for the Bergman kernels of the Fock–Bargmann–Hartogs domains in terms of the polylogarithm functions. By checking that the Bergman kernel ensures the revised Cartan's theorem, Kim, Ninh and Yamamori [Kim et al. 2014] determined the automorphism group of the Fock–Bargmann–Hartogs domains as follows.

Theorem 1.1 [Kim et al. 2014]. *The automorphism group* $\mathrm{Aut}(D_{n,m}(\mu))$ *is exactly the group generated by all automorphisms of* $D_{n,m}(\mu)$ *as follows*:

$$\begin{aligned}
\varphi_U &: (z,w)\mapsto (Uz,w), && U\in\mathcal{U}(n),\\
\varphi_{U'} &: (z,w)\mapsto (z,U'w), && U'\in\mathcal{U}(m),\\
\varphi_v &: (z,w)\mapsto (z+v, e^{-\mu\langle z,v\rangle-(\mu/2)\|v\|^2}w), && v\in\mathbb{C}^n,
\end{aligned}$$

where $\mathcal{U}(k)$ *is the unitary group of degree* k *and* $\langle\cdot,\cdot\rangle$ *is the standard Hermitian inner product on* $\mathbb{C}^n$.

Recently, [Tu and Wang 2014] has established the rigidity of the proper holomorphic mappings between two equidimensional Fock–Bargmann–Hartogs domains.

Theorem 1.2 [Tu and Wang 2014]. *If* $D_{n,m}(\mu)$ *and* $D_{n',m'}(\mu')$ *are two equidimensional Fock–Bargmann–Hartogs domains with* $m\geq 2$ *and* f *is a proper holomorphic mapping of* $D_{n,m}(\mu)$ *into* $D_{n',m'}(\mu')$, *then* f *is a biholomorphism between* $D_{n,m}(\mu)$ *and* $D_{n',m'}(\mu')$.

A generalized complex ellipsoid (also called generalized pseudoellipsoid) is a domain of the form

$$\Sigma(\boldsymbol{n};\boldsymbol{p})=\left\{(\zeta_1,\dots,\zeta_r)\in\mathbb{C}^{n_1}\times\cdots\times\mathbb{C}^{n_r}:\sum_{k=1}^{r}\|\zeta_k\|^{2p_k}<1\right\},$$

where $\boldsymbol{n}=(n_1,\dots,n_r)\in\mathbb{N}^r$ and $\boldsymbol{p}=(p_1,\dots,p_r)\in(\mathbb{R}_+)^r$. In the special case where all the $p_k=1$, the generalized complex ellipsoid $\Sigma(\boldsymbol{n};\boldsymbol{p})$ reduces to the unit ball in $\mathbb{C}^{n_1+\cdots+n_r}$. Also, it is known that a generalized complex ellipsoid $\Sigma(\boldsymbol{n};\boldsymbol{p})$ is homogeneous if and only if $p_k=1$ for all $1\leq k\leq r$ [Kodama 2014]. In general, a generalized complex ellipsoid is not strongly pseudoconvex and its boundary is not smooth. The automorphism group $\mathrm{Aut}(\Sigma(\boldsymbol{n};\boldsymbol{p}))$ of $\Sigma(\boldsymbol{n};\boldsymbol{p})$ has been studied by Dini and Selvaggi Primicerio [1997], Kodama [2014] and Kodama, Krantz and Ma [Kodama et al. 1992].

Kodama [2014] obtained the result as follows.

Theorem 1.3 [Kodama 2014]. (i) *If* 1 *does not appear in* $p_1, \dots, p_r$, *then any automorphism* $\varphi \in \mathrm{Aut}(\Sigma(\boldsymbol{n}; \boldsymbol{p}))$ *is of the form*

$$\varphi(\zeta_1, \dots, \zeta_r) = (\gamma_1(\zeta_{\sigma(1)}), \dots, \gamma_r(\zeta_{\sigma(r)})), \tag{1-1}$$

where $\sigma \in S_r$ *is a permutation of the* r *numbers* $\{1, \dots, r\}$ *such that* $n_{\sigma(i)} = n_i$, $p_{\sigma(i)} = p_i$ *for* $1 \le i \le r$ *and* $\gamma_1, \dots, \gamma_r$ *are unitary transformations of* $\mathbb{C}^{n_1}, \dots, \mathbb{C}^{n_r}$, *respectively.*

(ii) *If* 1 *appears in* $p_1, \dots, p_r$, *we can assume, without loss of generality, that* $p_1 = 1$, $p_2 \neq 1, \dots, p_r \neq 1$. *Then* $\mathrm{Aut}(\Sigma(\boldsymbol{n}; \boldsymbol{p}))$ *is generated by elements of the form* (1-1) *and automorphisms of the form*

$$\varphi_a(\zeta_1, \zeta_2, \dots, \zeta_r) = \big(T_a(\zeta_1), \zeta_2(\psi_a(\zeta_1))^{1/2p_2}, \dots, \zeta_r(\psi_a(\zeta_1))^{1/2p_r}\big), \tag{1-2}$$

where T_a *is an automorphism of the ball* $\mathbb{B}^{n_1}$ *in* $\mathbb{C}^{n_1}$ *which sends a point* $a \in \mathbb{B}^{n_1}$ *to the origin and*

$$\psi_a(\zeta_1) = \frac{1 - \|a\|^2}{(1 - \langle \zeta_1, a\rangle)^2}.$$

In this paper, we define the generalized Fock–Bargmann–Hartogs domains $D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)$ as

$$D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu) = \Bigg\{(z, w_{(1)}, \dots, w_{(\ell)}) \in \mathbb{C}^{n_0} \times \mathbb{C}^{n_1} \times \cdots \times \mathbb{C}^{n_\ell} : \sum_{j=1}^{\ell} \|w_{(j)}\|^{2p_j} < e^{-\mu\|z\|^2}\Bigg\} \quad (\mu > 0),$$

where $\boldsymbol{p} = (p_1, \dots, p_\ell) \in (\mathbb{R}_+)^\ell$, $\boldsymbol{n} = (n_1, \dots, n_\ell)$ and $w_{(j)} = (w_{j1}, \dots, w_{jn_j}) \in \mathbb{C}^{n_j}$, in which n_j is a positive integer for $1 \le j \le \ell$. Here and henceforth, with no loss of generality, we always assume that $p_i \neq 1$ $(2 \le i \le \ell)$ for $D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)$.

Obviously, each generalized Fock–Bargmann–Hartogs domain $D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)$ is an unbounded nonhyperbolic domain. In general, a generalized Fock–Bargmann–Hartogs domain is not a strongly pseudoconvex domain and its boundary is not smooth.

In this paper, we prove the following results.

Theorem 1.4. *Suppose* $D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)$ *and* $D_{m_0}^{\boldsymbol{m},\boldsymbol{q}}(\nu)$ *are two equidimensional generalized Fock–Bargmann–Hartogs domains. Let*

$$f : D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu) \to D_{m_0}^{\boldsymbol{m},\boldsymbol{q}}(\nu)$$

be a biholomorphic mapping. Then there exists $\phi \in \mathrm{Aut}(D_{m_0}^{\boldsymbol{m},\boldsymbol{q}}(\nu))$ *such that*

$$\phi\circ f(z,w)=(z,w_{(\sigma(1))},\dots,w_{(\sigma(\ell))})\begin{pmatrix}A&&&&\\&\Gamma_1&&&\\&&\Gamma_2&&\\&&&\ddots&\\&&&&\Gamma_\ell\end{pmatrix},\tag{1-3}$$

where $\sigma\in S_\ell$ is a permutation such that $n_{\sigma(j)}=m_j$, $p_{\sigma(j)}=q_j$ $(1\le j\le\ell)$, $\sqrt{\nu/\mu}\,A\in\mathcal{U}(n)$ $(n:=n_0=m_0)$, and $\Gamma_i\in\mathcal{U}(m_i)$ $(1\le i\le\ell)$.

Corollary 1.5. *Let $f:D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)\to D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)$ be a biholomorphic mapping with $f(0)=0$. Then we have*

$$f(z,w)=(z,w_{(\sigma(1))},\dots,w_{(\sigma(\ell))})\begin{pmatrix}A&&&&\\&\Gamma_1&&&\\&&\Gamma_2&&\\&&&\ddots&\\&&&&\Gamma_\ell\end{pmatrix},$$

where $\sigma\in S_\ell$ is a permutation such that $n_{\sigma(j)}=n_j$, $p_{\sigma(j)}=p_j$ $(1\le j\le\ell)$, $A\in\mathcal{U}(n_0)$ and $\Gamma_i\in\mathcal{U}(n_i)$ $(1\le i\le\ell)$.

As a consequence, it is easy for us to prove the following results.

Theorem 1.6. *The automorphism group* $\mathrm{Aut}(D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu))$ *is generated by the following mappings*:

$\varphi_A:(z,w_{(1)},\dots,w_{(\ell)})\mapsto(zA,w_{(1)},\dots,w_{(\ell)})$;

$\varphi_D:(z,w_{(1)},\dots,w_{(\ell)})\mapsto(z,(w_{(\sigma(1))},\dots,w_{(\sigma(\ell))})D)$,

$\varphi_a:(z,w)\mapsto\big(z+a,\,w_{(1)}(e^{-2\mu<z,a>-\mu\|a\|^2})^{1/2p_1},\dots,w_{(\ell)}(e^{-2\mu\langle z,a\rangle-\mu\|a\|^2})^{1/2p_\ell}\big)$,

where $a\in\mathbb{C}^{n_0}$, $A\in\mathcal{U}(n_0)$, $\sigma\in S_\ell$ is a permutation such that $n_{\sigma(j)}=n_j$, $p_{\sigma(j)}=p_j$ $(1\le j\le\ell)$, and

$$D=\begin{pmatrix}\Gamma_1&&&\\&\Gamma_2&&\\&&\ddots&\\&&&\Gamma_\ell\end{pmatrix},$$

in which $\Gamma_i\in\mathcal{U}(n_i)$ $(1\le i\le\ell)$.

Now, for $\boldsymbol{p}$ and $\boldsymbol{q}$, we introduce the notation

$$\epsilon=\begin{cases}1,&p_1=1,\\0,&p_1\ne1,\end{cases}\qquad\delta=\begin{cases}1,&q_1=1\\0,&q_1\ne1.\end{cases}$$

Theorem 1.7. *Suppose $D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)$ and $D_{m_0}^{\boldsymbol{m},\boldsymbol{q}}(\nu)$ are two equidimensional generalized Fock–Bargmann–Hartogs domains with* $\min\{n_{1+\epsilon},n_2,\dots,n_\ell,n_1+\dots+n_\ell\}\ge2$

and $\min\{m_{1+\delta}, m_2, \cdots, m_\ell, m_1 + \cdots + m_\ell\} \geq 2$. *Then any proper holomorphic mapping between* $D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)$ *and* $D_{m_0}^{\boldsymbol{m},\boldsymbol{q}}(\nu)$ *must be a biholomorphism.*

Remark 1.1. The conditions $\min\{n_{1+\epsilon}, n_2, \ldots, n_\ell\} \geq 2$ cannot be removed. For example, $n_1 = 1$ (i.e, $w_{(1)} \in \mathbb{C}$), $p_1 \neq 1$, and

$$F(z, w) : (z, w_{(1)}, \ldots, w_{(\ell)}) \to (z, w_{(1)}^2, w_{(2)}, \ldots, w_{(\ell)}).$$

Then F is a proper holomorphic mapping between $D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)$ and $D_{n_0}^{\boldsymbol{n},\boldsymbol{q}}(\mu)$, where $\boldsymbol{q} = (p_1/2, p_2, \ldots, p_\ell)$. F is not a biholomorphism.

Corollary 1.8. *Suppose* $D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)$ *is a generalized Fock–Bargmann–Hartogs domain with*

$$\min\{n_{1+\epsilon}, n_2, \ldots, n_\ell, n_1 + \cdots + n_\ell\} \geq 2.$$

Then any proper holomorphic self-mapping of $D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)$ *must be an automorphism.*

Remark 1.2. The conditions $n_1 + \cdots + n_\ell \geq 2$ cannot be removed. For instance, with no loss of generality, we can assume $n_1 = 1$ and $n_i = 0$ $(2 \leq i \leq \ell)$. Then

$$F : (z, w_{(1)}) \to \left(\sqrt{2}z, w_{(1)}^2\right)$$

is a proper holomorphic self-mapping of $D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)$ which is not an automorphism.

The paper is organized as follows. In Section 2, using the explicit formula for the Bergman kernels of the generalized Fock–Bargmann–Hartogs domains, we prove that a proper holomorphic mapping between two equidimensional generalized Fock–Bargmann–Hartogs domains extends holomorphically to their closures, and check that Cartan's theorem holds also for the generalized Fock–Bargmann–Hartogs domains. In Section 3, we exploit the boundary structure of generalized Fock–Bargmann–Hartogs domains to prove our results in this paper.

2. Preliminaries

The Bergman kernel of the domain $D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\boldsymbol{\mu})$. For a domain Ω in $\mathbb{C}^n$, let $A^2(\Omega)$ be the Hilbert space of square integrable holomorphic functions on Ω with the inner product

$$\langle f, g\rangle = \int_\Omega f(z)\overline{g(z)}\, dV(z) \quad (f, g \in \mathcal{O}(\Omega)),$$

where dV is the Euclidean volume form. The Bergman kernel $K(z, w)$ of $A^2(\Omega)$ is defined as the reproducing kernel of the Hilbert space $A^2(\Omega)$, that is, for all $f \in A^2(\Omega)$ we have

$$f(z) = \int_\Omega f(w)K(z, w)\, dV(w) \quad (z \in \Omega).$$

For a positive continuous function p on Ω, let $A^2(\Omega, p)$ be the weighted Hilbert space of square integrable holomorphic functions with respect to the weight function p with the inner product

$$\langle f, g\rangle = \int_\Omega f(z)\overline{g(z)}p(z)\,dV(z) \quad (f, g \in \mathcal{O}(\Omega)).$$

Similarly, the weighted Bergman kernel $K_{A^2(\Omega,p)}$ of $A^2(\Omega, p)$ is defined as the reproducing kernel of the Hilbert space $A^2(\Omega, p)$. For a positive integer m, define the Hartogs domain $\Omega_{m,p}$ over Ω by

$$\Omega_{m,p} = \{(z, w) \in \Omega \times \mathbb{C}^m : \|w\|^2 < p(z)\}.$$

Ligocka [1985; 1989] showed that the Bergman kernel of $\Omega_{m,p}$ can be expressed as infinite sum in terms of the weighted Bergman kernel of $A^2(\Omega, p^k)$ $(k = 1, 2, \ldots)$ as follows.

Theorem 2.1 [Ligocka 1989]. *Let K_m be the Bergman kernel of $\Omega_{m,p}$ and let $K_{A^2(\Omega,p^k)}$ be the weighted Bergman kernel of $A^2(\Omega, p^k)$ $(k = 1, 2, \ldots)$. Then*

$$K_m((z, w), (t, s)) = \frac{m!}{\pi^m}\sum_{k=0}^{\infty}\frac{(m+1)_k}{k!}K_{A^2(\Omega,p^{k+m})}(z, t)\langle w, s\rangle^k,$$

where $(a)_k$ denotes the Pochhammer symbol $(a)_k = a(a+1)\cdots(a+k-1)$.

The Fock–Bargmann space is the weighted Hilbert space $A^2(\mathbb{C}^n, e^{-\mu\|z\|^2})$ on $\mathbb{C}^n$ with the Gaussian weight function $e^{-\mu\|z\|^2}$ $(\mu > 0)$. The reproducing kernel of $A^2(\mathbb{C}^n, e^{-\mu\|z\|^2})$, called the Fock–Bargmann kernel, is $\mu^n e^{\mu\langle z,t\rangle}/\pi^n$; see [Bargmann 1967]. Thus, the Fock–Bargmann–Hartogs domain

$$D_{n,m} = \left\{(z, w) \in \mathbb{C}^n \times \mathbb{C}^m : \|w\|^2 < e^{-\mu\|z\|^2}\right\} \quad (\mu > 0)$$

and the Fock–Bargmann space $A^2(\mathbb{C}^n, e^{-\mu\|z\|^2})$ are closely related. Using the above Theorem 2.1 and the expression of the Fock–Bargmann kernel, Yamamori [2013] gave the Bergman kernel of the Fock–Bargmann–Hartogs domain $D_{n,m}$ as follows.

Theorem 2.2 [Yamamori 2013]. *The Bergman kernel of the Fock–Bargmann–Hartogs domain $D_{n,m}$ is given by*

$$K_{D_{n,m}}((z, w), (t, s)) = \frac{m!\mu^n}{\pi^{m+n}}\sum_{k=0}^{\infty}\frac{(m+1)_k(k+m)^n}{k!}e^{\mu(k+m)\langle z,t\rangle}\langle w, s\rangle^k,$$

where $(a)_k$ denotes the Pochhammer symbol $(a)_k = a(a+1)\cdots(a+k-1)$.

Following the idea of Theorem 2.1, we compute the Bergman kernel for the generalized Fock–Bargmann–Hartogs domain $D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)$. In order to compute the Bergman kernel, we first introduce some notation.

Let

$$\alpha = (\alpha_{(1)}, \dots, \alpha_{(\ell)}) \in (\mathbb{R}_+)^{n_1} \times \cdots \times (\mathbb{R}_+)^{n_\ell},$$

where $\alpha_{(i)} = (\alpha_{i1}, \dots, \alpha_{in_i}) \in (\mathbb{R}_+)^{n_i}$ for $1 \le i \le \ell$. For $\alpha \in (\mathbb{R}_+)^n$, we define

$$\beta(\alpha) = \frac{\prod_{i=1}^{n} \Gamma(\alpha_i)}{\Gamma(|\alpha|)};$$

see [D'Angelo 1994]. Here Γ is the usual Euler Gamma function.

Lemma 2.3 [D'Angelo 1994, Lemma 1]. *Suppose $\alpha \in (\mathbb{R}_+)^n$. Then we have*

$$\int_{B_+^n} r^{2\alpha-1}\, dV(r) = \frac{\beta(\alpha)}{2^n |\alpha|},$$

$$\int_{S_+^{n-1}} w^{2\alpha-1}\, d\sigma(w) = \frac{\beta(\alpha)}{2^{n-1}},$$

where dV is the Euclidean n-dimensional volume form, $d\sigma$ is the Euclidean $(n-1)$-dimensional volume form, and the subscript "+" denotes that all the variables are positive, that is, $B_+^n = B^n \cap (\mathbb{R}_+)^n$ and $S_+^{n-1} = S^{n-1} \cap (\mathbb{R}_+)^n$, in which B^n is the unit ball in $\mathbb{R}^n$ and S^{n-1} is the unit sphere in $\mathbb{R}^n$.

Theorem 2.4. *Suppose $\alpha = (\alpha_{(1)}, \dots, \alpha_{(\ell)}) \in (\mathbb{R}_+)^{n_1} \times \cdots \times (\mathbb{R}_+)^{n_\ell}$, with each $\alpha_{(i)} = (\alpha_{i1}, \dots, \alpha_{in_i}) \in (\mathbb{R}_+)^{n_i}$ $(1 \le i \le \ell)$. Then we have the formula*

$$\text{(2-1)}\quad \int_{\sum_{j=1}^{\ell} \|w_{(j)}\|^{2p_j} < t} w^\alpha \bar{w}^\alpha\, dV(w) = (\pi)^{n_1+\cdots+n_\ell} \frac{\prod_{i=1}^{\ell} \prod_{j=1}^{n_i} \Gamma(\alpha_{ij}+1) \prod_{i=1}^{\ell} \Gamma\big((|\alpha_{(i)}|+n_i)/p_i\big)}{\prod_{i=1}^{\ell} p_i \prod_{i=1}^{\ell} \Gamma(|\alpha_{(i)}|+n_i) \Gamma\Big(\sum_{i=1}^{\ell}((|\alpha_{(i)}|+n_i)/p_i)+1\Big)} \times t^{\sum_{i=1}^{\ell}(|\alpha_{(i)}|+n_i)/p_i}.$$

Proof. For the integral

$$\text{(2-2)}\quad \int_{\sum_{j=1}^{\ell} \|w_{(j)}\|^{2p_j} < t} w^\alpha \bar{w}^\alpha\, dV(w),$$

by applying the polar coordinates $w = se^{i\theta}$ (namely, $w_{ij} = s_{ij} e^{i\theta_{ij}}$, $1 \le j \le n_i$, $1 \le i \le \ell$, $s = (s_{(1)}, \dots, s_{(\ell)})$), we have

$$\text{(2-2)} = (2\pi)^{n_1+\cdots+n_\ell} \int_{\substack{\sum_{j=1}^{\ell} \|s_{(j)}\|^{2p_j} < t \\ s_{ji}>0,\ 1\le i\le n_j,\ 1\le j\le \ell}} s^{2\alpha+1}\, dV(s).$$

Using the spherical coordinates in the variables $s_{(1)}, s_{(2)}, \dots, s_{(\ell)}$, we get

$$\begin{aligned}\int_{\substack{\sum_{j=1}^{\ell}\|s_{(j)}\|^{2p_j}<t\\ s_{ji}>0,\,1\le i\le n_j,\,1\le j\le\ell}} & s^{2\alpha+1}\,dV(s)\\ &= \int_{\substack{\sum_{i=1}^{\ell}\rho_i^{2p_i}<t\\ \rho_i>0,\,1\le i\le\ell}} \rho_1^{2|\alpha_{(1)}|+2n_1-1}\cdots\rho_\ell^{2|\alpha_{(\ell)}|+2n_\ell-1}\,d\rho_1\,d\rho_2\cdots d\rho_\ell\\ &\qquad\times\int_{S_+^{n_1-1}}\cdots\int_{S_+^{n_\ell-1}} w_{(1)}^{2\alpha_{(1)}+1}\cdots w_{(\ell)}^{2\alpha_{(\ell)}+1}\,d\sigma(w_{(1)})\cdots d\sigma(w_{(\ell)}).\end{aligned}$$

Let ${\rho_i}^{p_i}=r_i$, $1\le i\le\ell$. Then we have $d\rho_i=\rho_i^{1-p_i}/p_i\,dr_i=r_i^{(1/p_i)-1}/p_i\,dr_i$. Therefore, Lemma 2.3 and the above formulas yield

$$\begin{aligned}\text{(2-2)} = (2\pi)^{n_1+\cdots+n_\ell}\frac{1}{\prod_{i=1}^{\ell}p_i}&\frac{\beta(\alpha_{(1)}+1)}{2^{n_1-1}}\cdots\frac{\beta(\alpha_{(\ell)}+1)}{2^{n_\ell-1}}\\ &\times\int_{\substack{\sum_{i=1}^{\ell}|r_i|^2<t\\ r_i>0,1\le i\le\ell}} r_1^{(2|\alpha_{(1)}|+2n_1)/p_1-1}\cdots r_\ell^{(2|\alpha_{(\ell)}|+2n_\ell)/p_\ell-1}\,dr_1\cdots dr_\ell.\end{aligned}$$

Let $r=(r_1,r_2,\dots,r_\ell)\in(\mathbb{R}_+)^\ell$ and $k:=t^{-1/2}r$. Then $dr=t^{\ell/2}\,dk$. After a straightforward computation, we obtain that

$$\begin{aligned}\text{(2-2)} = (2\pi)^{n_1+\cdots+n_\ell}\frac{1}{\prod_{i=1}^{\ell}p_i}&\frac{\beta(\alpha_{(1)}+1)}{2^{n_1-1}}\cdots\frac{\beta(\alpha_{(\ell)}+1)}{2^{n_\ell-1}}\cdot t^{\sum_{i=1}^{\ell}(|\alpha_{(i)}|+n_i)/p_i}\\ &\times\int_{B_+^{\ell}} k_1^{(2|\alpha_{(1)}|+2n_1)/p_1-1}\cdots k_\ell^{(2|\alpha_{(\ell)}|+2n_\ell)/p_\ell-1}\,dk_1\cdots dk_\ell.\end{aligned}$$

Applying Lemma 2.3 to the above formula, we get

$$\begin{aligned}\text{(2-3)}\quad \text{(2-2)} &= (\pi)^{n_1+\cdots+n_\ell}\beta(\alpha_{(1)}+1)\cdots\beta(\alpha_{(\ell)}+1)\frac{\beta(\alpha')}{|\alpha'|\prod_{i=1}^{\ell}p_i}\cdot t^{\sum_{i=1}^{\ell}(|\alpha_{(i)}|+n_i)/p_i}\\ &= (\pi)^{n_1+\cdots+n_\ell}\frac{1}{\prod_{i=1}^{\ell}p_i}\frac{\prod_{i=1}^{\ell}\prod_{j=1}^{n_i}\Gamma(\alpha_{ij}+1)\prod_{i=1}^{\ell}\Gamma((|\alpha_{(i)}|+n_i)/p_i)}{\prod_{i=1}^{\ell}\Gamma(|\alpha_{(i)}|+n_i)\Gamma\Big(\sum_{i=1}^{\ell}(|\alpha_{(i)}|+n_i)/p_i+1\Big)}\\ &\qquad\qquad\times t^{\sum_{i=1}^{\ell}(|\alpha_{(i)}|+n_i)/p_i},\end{aligned}$$

where $\alpha'=((|\alpha_{(1)}|+n_1)/p_1,\dots,(|\alpha_{(\ell)}|+n_\ell)/p_\ell)\in(\mathbb{R}_+)^\ell$. □

Now we consider the Hilbert space $A^2(D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu))$ of square-integrable holomorphic functions on $D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)$.

Lemma 2.5. *Let* $f \in A^2(D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu))$. *Then*

$$f(z,w) = \sum_{\alpha} f_\alpha(z) w^\alpha,$$

where the series is uniformly convergent on compact subsets of the domain $D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)$, $f_\alpha(z) \in A^2(\mathbb{C}^{n_0}, e^{-\mu\lambda_\alpha\|z\|^2})$ *for any* $\alpha = (\alpha_{(1)}, \dots, \alpha_{(\ell)}) \in \mathbb{N}^{n_1} \times \cdots \times \mathbb{N}^{n_\ell}$ *with* $\alpha_{(i)} = (\alpha_{i1}, \dots, \alpha_{in_i}) \in \mathbb{N}^{n_i}$ $(1 \le i \le \ell)$ *and* $\lambda_\alpha = \sum_{i=1}^{\ell}(|\alpha_{(i)}| + n_i)/p_i$, *in which* $A^2(\mathbb{C}^n, e^{-\mu\lambda_\alpha\|z\|^2})$ *denotes the space of square-integrable holomorphic functions on* $\mathbb{C}^n$ *with respect to the measure* $e^{-\mu\lambda_\alpha\|z\|^2}\, dV_{2n}$.

Proof. Since $D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)$ is a complete Reinhardt domain, each holomorphic function on $D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)$ is the sum of a locally uniformly convergent power series. Thus, for $f \in A^2(D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu))$, we have

$$f(z,w) = \sum_{\alpha} f_\alpha(z) w^\alpha,$$

where the series is uniformly convergent on compact subsets of $D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)$. We choose a sequence of compact subsets D_k $(1 \le k < \infty)$

$$D_k := \left\{ (z, w_{(1)}, \dots, w_{(\ell)}) \in \mathbb{C}^{n_0} \times \mathbb{C}^{n_1} \times \cdots \times \mathbb{C}^{n_\ell} : \sum_{j=1}^{\ell} \|w_{(j)}\|^{2p_j} \le e^{-\mu\|z\|^2} - \frac{1}{k} \right\} \cap \overline{B(0,k)},$$

where $B(0,k)$ is the ball in $\mathbb{C}^{n_0+n_1+\cdots+n_\ell}$ of the radius k. Obviously, $D_k \Subset D_{k+1}$ and $\bigcup_{k=1}^{\infty} D_k = D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)$. Since D_k is a circular domain,

$$f_\alpha(z) w^\alpha \perp f_\beta(z) w^\beta \quad (\alpha \ne \beta)$$

in the Hilbert space $A^2(D_k)$. Hence we have

$$\|f\|^2_{L^2(D_k)} = \sum_{|\alpha|=0}^{\infty} \|f_\alpha(z) w^\alpha\|^2_{L^2(D_k)}.$$

Since $f(z,w) \in A^2(D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu))$, we have

$$\|f_\alpha(z) w^\alpha\|^2_{L^2(D_k)} \le \|f\|^2_{L^2(D_k)} \le \|f\|^2_{L^2(D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu))}.$$

Then $f_\alpha(z) w^\alpha \in A^2(D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu))$. Therefore,

$$\int_{D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)} |f_\alpha(z)|^2 w^\alpha \bar{w}^\alpha \, dV < \infty$$
$$\Longrightarrow \int_{\mathbb{C}^{n_0}} |f_\alpha(z)|^2 \, dV(z) \int_{\sum_{j=1}^{\ell}\|w_{(j)}\|^{2p_j} < e^{-\mu\|z\|^2}} w^\alpha \bar{w}^\alpha \, dV(w) < \infty.$$

By (2-1), it follows that

$$\int_{\mathbb{C}^{n_0}} |f_\alpha(z)|^2 e^{-\mu\lambda_\alpha\|z\|^2}\,dV(z) < \infty.$$

Consequently, $f_\alpha(z) \in A^2(\mathbb{C}^{n_0}, e^{-\mu\lambda_\alpha\|z\|^2})$, where $\lambda_\alpha = \sum_{i=1}^{\ell}(|\alpha_{(i)}|+n_i)/p_i$. □

Lemma 2.5 implies that $\{f(z)w^\alpha : f(z) \in A^2(\mathbb{C}^{n_0}, e^{-\mu\lambda_\alpha\|z\|^2})\}$ forms a linearly dense subset of $A^2(D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu))$. Now we can express the Bergman kernel of $D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)$.

Theorem 2.6. *The Bergman kernel of $D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)$ can be expressed by the form*

$$K_{D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)}[(z,w),(s,t)] = \sum_{|\alpha|=0}^{\infty} c_\alpha \frac{\lambda_\alpha^{n_0}\mu^{n_0}}{\pi^{n_0}} e^{\lambda_\alpha\mu\langle z,s\rangle} w^\alpha \bar{t}^\alpha, \tag{2-4}$$

where $\alpha = (\alpha_{(1)},\dots,\alpha_{(\ell)}) \in \mathbb{N}^{n_1}\times\cdots\times\mathbb{N}^{n_\ell}$, $\alpha_{(i)} = (\alpha_{i1},\dots,\alpha_{in_i}) \in \mathbb{N}^{n_i}$, $1 \le i \le \ell$, and

$$c_\alpha = \frac{\prod_{i=1}^{\ell} p_i \prod_{i=1}^{\ell}\Gamma(|\alpha_{(i)}|+n_i)\Gamma\Big(\sum_{i=1}^{\ell}(|\alpha_{(i)}|+n_i)/p_i+1\Big)}{(\pi)^{n_1+\cdots+n_\ell}\prod_{i=1}^{\ell}\prod_{j=1}^{n_i}\Gamma(\alpha_{ij}+1)\prod_{i=1}^{\ell}\Gamma((|\alpha_{(i)}|+n_i)/p_i)}, \quad \lambda_\alpha = \sum_{i=1}^{\ell}\frac{|\alpha_{(i)}|+n_i}{p_i}.$$

Proof. Since $D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)$ is a complete Reinhardt domain, it follows that

$$K_{D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)}[(z,w),(s,t)] = \sum_{|\beta|=0}^{\infty} c_\beta g_\beta(z,s) w^\beta \bar{t}^\beta,$$

where the sum is locally uniformly convergent, by the invariance of the Bergman kernel $K_{D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)}$ on $D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)$ under the unitary subgroup action

$$(z_1,\dots,z_{n_0+|\boldsymbol{n}|}) \to \big(e^{\sqrt{-1}\theta_1}z_1,\dots,e^{\sqrt{-1}\theta_{n_0+|\boldsymbol{n}|}}z_{n_0+|\boldsymbol{n}|}\big) \quad (\theta_1,\dots,\theta_{n_0+|\boldsymbol{n}|} \in \mathbb{R}).$$

For any $\alpha = (\alpha_{(1)},\dots,\alpha_{(\ell)}) \in \mathbb{N}^{n_1}\times\cdots\times\mathbb{N}^{n_\ell}$ with $\alpha_{(i)} = (\alpha_{i1},\dots,\alpha_{in_i}) \in \mathbb{N}^{n_i}$ $(1 \le i \le \ell)$, and any $f(z) \in A^2(\mathbb{C}^{n_0}, e^{-\mu\lambda_\alpha\|z\|^2})$ for $\lambda_\alpha = \sum_{i=1}^{\ell}(|\alpha_{(i)}|+n_i)/p_i$, we have $f(z)w^\alpha \in A^2(D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu))$. Thus

$$\begin{aligned}
f(z)w^\alpha &= \int_{D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)} f(s)t^\alpha K_{D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)}[(z,w),(s,t)]\,dV \\
&= \int_{\mathbb{C}^{n_0}} f(s)\sum_{\beta=0}^{\infty} c_\beta g_\beta(z,s)w^\beta\,dV(s)\int_{\sum_{j=1}^{\ell}\|t_{(j)}\|^{2p_j}<e^{-\mu\|s\|^2}} t^\alpha\bar{t}^\beta\,dV(t) \\
&= w^\alpha\int_{\mathbb{C}^{n_0}} f(s)g_\alpha(z,s)\big[e^{-\mu\|s\|^2}\big]^{\sum_{i=1}^{\ell}(|\alpha_{(i)}|+n_i)/p_i}\,dV(s) \quad \text{(by (2-1))}.
\end{aligned}$$

By [Bargmann 1967], we get that the Bergman kernel of $A^2(\mathbb{C}^{n_0}, e^{-\mu\lambda_\alpha\|z\|^2})$ can be described by the form

$$K_\alpha(z,w)=\frac{\lambda_\alpha^{n_0}\mu^{n_0}}{\pi^{n_0}}e^{\lambda_\alpha\mu\langle z,w\rangle}. \tag{2-5}$$

Thus we obtain

$$g_\alpha(z,s)=\frac{\lambda_\alpha^{n_0}\mu^{n_0}}{\pi^{n_0}}e^{\lambda_\alpha\mu\langle z,s\rangle}.$$

This completes the proof. □

The transformation rule for Bergman kernels under proper holomorphic mapping (e.g., Theorem 1 in [Bell 1982]) is also valid for unbounded domains (e.g., see Corollary 1 in [Trybuła 2013]). Note that the coordinate functions play a key role in the approach of [Bell 1982] to extend proper holomorphic mapping, but, in general, are no longer square integrable on unbounded domains. In order to overcome this difficulty, by combining the transformation rule for Bergman kernels under proper holomorphic mapping in [Bell 1982] and our explicit form (2-4) of the Bergman kernel function for $D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)$, we prove that a proper holomorphic mapping between two equidimensional generalized Fock–Bargmann–Hartogs domains extends holomorphically to their closures as follows.

Lemma 2.7. *Suppose that $f: D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)\to D_{m_0}^{\boldsymbol{m},\boldsymbol{q}}(\nu)$ is a proper holomorphic mapping between two equidimensional generalized Fock–Bargmann–Hartogs domains. Then f extends holomorphically to a neighborhood of the closure of $D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)$.*

In fact, using the explicit form (2-4) of the Bergman kernel function for $D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)$, we immediately have Lemma 2.7 by a slight modification of the proof of Theorem 2.5 in [Tu and Wang 2014].

Cartan's theorem on $D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)$. Suppose D is a domain in $\mathbb{C}^N$ and let $K_D(z,w)$ be its Bergman kernel. From [Ishi and Kai 2010], we know that if the conditions

(a) $K_D(0,0)>0$,

(b) $T_D(0,0)$ is positive definite,

are satisfied, where T_D is an $N\times N$ matrix

$$T_D(z,w):=\begin{pmatrix}\partial^2\log K_D(z,w)/\partial z_1\partial\overline{w_1} & \cdots & \partial^2\log K_D(z,w)/\partial z_1\partial\overline{w_N}\\ \vdots & \ddots & \vdots\\ \partial^2\log K_D(z,w)/\partial z_N\partial\overline{w_1} & \cdots & \partial^2\log K_D(z,w)/\partial z_N\partial\overline{w_N}\end{pmatrix}.$$

Then Cartan's theorem can also be applied to the case of unbounded circular domains. The above conditions are obviously satisfied by the bounded domain.

Kim, Ninh and Yamamori [Kim et al. 2014] proved the following result.

Lemma 2.8 [Kim et al. 2014, Theorem 4]. *Suppose that D is a circular domain and its Bergman kernel satisfies the above conditions* (a) *and* (b). *If $\varphi\in\mathrm{Aut}(D)$ preserves the origin, then φ is a linear mapping.*

Ishi and Kai [2010] proved the following generalization of Lemma 2.8.

Lemma 2.9 [Ishi and Kai 2010, Proposition 2.1]. *Let D_k be a circular domain (not necessarily bounded) in $\mathbb{C}^N$ with $0 \in D_k$ ($k = 1, 2$), and let $\varphi : D_1 \to D_2$ be a biholomorphism with $\varphi(0) = 0$. If $K_{D_k}(0, 0) > 0$ and $T_{D_k}(0, 0)$ is positive definite ($k = 1, 2$), then φ is linear.*

Therefore, by using the expressions of Bergman kernels of generalized Fock–Bargmann–Hartogs domains, we have the following result.

Theorem 2.10. *Suppose that $\varphi : D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu) \to D_{m_0}^{\boldsymbol{m},\boldsymbol{q}}(\nu)$ be a biholomorphic mapping between two equidimensional generalized Fock–Bargmann–Hartogs domains with $\varphi(0) = 0$. Then φ is linear.*

Proof. By using the expressions (2-4) of Bergman kernels of generalized Fock–Bargmann–Hartogs domains and a straightforward computation, we show that the Bergman kernel of every generalized Fock–Bargmann–Hartogs domain satisfies the above conditions (a) and (b). So we get Theorem 2.10 by Lemma 2.9. □

3. Proof of the main theorem

To begin, we exploit the boundary structure of $D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)$, which is comprised of

$$bD_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu) = b_0 D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu) \cup b_1 D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu) \cup b_2 D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu),$$

where

$$b_0 D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu) := \Big\{(z, w_{(1)}, \dots, w_{(\ell)}) \in \mathbb{C}^{n_0} \times \cdots \times \mathbb{C}^{n_\ell} : \sum_{j=1}^{\ell} \|w_{(j)}\|^{2p_j} = e^{-\mu\|z\|^2},\ \|w_{(j)}\|^2 \neq 0,\ 1+\epsilon \le j \le \ell\Big\},$$

$$b_1 D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu) := \bigcup_{j=1+\epsilon}^{\ell} \Big\{(z, w_{(1)}, \dots, w_{(\ell)}) \in \mathbb{C}^{n_0} \times \cdots \times \mathbb{C}^{n_\ell} : \sum_{j=1}^{\ell} \|w_{(j)}\|^{2p_j} = e^{-\mu\|z\|^2},\ \|w_{(j)}\|^2 = 0,\ p_j > 1\Big\},$$

$$b_2 D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu) := \bigcup_{j=1+\epsilon}^{\ell} \Big\{(z, w_{(1)}, \dots, w_{(\ell)}) \in \mathbb{C}^{n_0} \times \cdots \times \mathbb{C}^{n_\ell} : \sum_{j=1}^{\ell} \|w_{(j)}\|^{2p_j} = e^{-\mu\|z\|^2},\ \|w_{(j)}\|^2 = 0,\ p_j < 1\Big\}.$$

Proposition 3.1. (1) *The boundary $b_0 D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)$ is a real analytic hypersurface in $\mathbb{C}^{n_0+n_1+\cdots+n_\ell}$ and $D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)$ is strongly pseudoconvex at all points of $b_0 D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)$.*

(2) *$D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)$ is weakly pseudoconvex but not strongly pseudoconvex at any point of $b_1 D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)$ and is not smooth at any point of $b_2 D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)$.*

Proof. Let

$$\rho(z, w_{(1)}, \dots, w_{(\ell)}) := \sum_{j=1}^{\ell} \|w_{(j)}\|^{2p_j} - e^{-\mu\|z\|^2}.$$

Then ρ is a real analytic definition function of $b_0 D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)$. Fix a point

$$(z_0, w_{(1)0}, \dots, w_{(\ell)0}) \in b_0 D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)$$

and let $T = (\zeta, \eta_{(1)}, \dots, \eta_{(\ell)}) \in T^{1,0}_{(z_0, w_{(1)0}, \dots, w_{(\ell)0})}(b_0 D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu))$. Then by definition, we know that

$$w_{(j)0} \neq 0 \quad (j = 1+\epsilon, \dots, \ell), \tag{3-1}$$

$$\sum_{k=1}^{\ell} p_k \|w_{(k)0}\|^{2(p_k-1)} \overline{w_{(k)0}} \cdot \eta_{(k)} + \mu e^{-\mu\|z_0\|^2} \overline{z_0} \cdot \zeta = 0, \tag{3-2}$$

$$\sum_{j=1}^{\ell} \|w_{(j)0}\|^{2p_j} - e^{-\mu\|z_0\|^2} = 0. \tag{3-3}$$

Thanks to (3-1), (3-2) and (3-3), the Levi form of ρ at the point $(z_0, w_{(1)0}, \dots, w_{(\ell)0})$ can be computed as follows:

$$\begin{aligned}
&L_\rho(T, T)\\
&\quad := \sum_{i,j=1}^{n_0+n_1+\dots+n_\ell} \frac{\partial^2 \rho}{\partial T_i \partial \overline{T_j}}(z_0, w_{(1)0}, \dots, w_{(\ell)0}) T_i \overline{T_j}\\
&\quad = \sum_{k=1}^{\ell} p_k(p_k-1)\|w_{(k)0}\|^{2(p_k-2)} |\overline{w_{(k)0}} \cdot \eta_{(k)}|^2 + \sum_{k=1}^{\ell} p_k \|w_{(k)0}\|^{2(p_k-1)} \|\eta_{(k)}\|^2\\
&\qquad\qquad + \mu e^{-\mu\|z_0\|^2} \|\zeta\|^2 - \mu^2 e^{-\mu\|z_0\|^2} |\overline{z_0} \cdot \zeta|^2\\
&\quad = \sum_{k=1}^{\ell} {p_k}^2 \|w_{(k)0}\|^{2(p_k-2)} |\overline{w_{(k)0}} \cdot \eta_{(k)}|^2 + \mu e^{-\mu\|z_0\|^2} \|\zeta\|^2 - \mu^2 e^{-\mu\|z_0\|^2} |\overline{z_0} \cdot \zeta|^2\\
&\qquad\qquad + \sum_{k=1}^{\ell} p_k \|w_{(k)0}\|^{2(p_k-2)} \big(\|w_{(k)0}\|^2 \|\eta_{(k)}\|^2 - |\overline{w_{(k)0}} \cdot \eta_{(k)}|^2 \big)\\
&\quad = \Big(\sum_{k=1}^{\ell} \|w_{(k)0}\|^{2p_k} \Big)^{-1} \Big(\sum_{k=1}^{\ell} {p_k}^2 \|w_{(k)0}\|^{2(p_k-2)} |\overline{w_{(k)0}} \cdot \eta_{(k)}|^2 \Big) \Big(\sum_{k=1}^{\ell} \|w_{(k)0}\|^{2p_k} \Big)\\
&\qquad\qquad - \Big(\sum_{k=1}^{\ell} \|w_{(k)0}\|^{2p_k} \Big)^{-1} \Big| \sum_{k=1}^{\ell} p_k \|w_{(k)0}\|^{2(p_k-1)} \overline{w_{(k)0}} \cdot \eta_{(k)} \Big|^2\\
&\qquad + \sum_{k=1}^{\ell} p_k \|w_{(k)0}\|^{2(p_k-2)} \big(\|w_{(k)0}\|^2 \|\eta_{(k)}\|^2 - |\overline{w_{(k)0}} \cdot \eta_{(k)}|^2 \big) + \mu e^{-\mu\|z_0\|^2} \|\zeta\|^2
\end{aligned}$$

$$
\begin{aligned}
&= \Big(\sum_{k=1}^{\ell} \|w_{(k)0}\|^{2p_k}\Big)^{-1} \Big[\Big(\sum_{k=1}^{\ell} {p_k}^2 \|w_{(k)0}\|^{2(p_k-2)} |\overline{w_{(k)0}}\cdot\eta_{(k)}|^2\Big)\Big(\sum_{k=1}^{\ell} \|w_{(k)0}\|^{2p_k}\Big) \\
&\qquad - \Big|\sum_{k=1}^{\ell} p_k \|w_{(k)0}\|^{2(p_k-1)} \overline{w_{(k)0}}\cdot\eta_{(k)}\Big|^2\Big] + \mu e^{-\mu\|z_0\|^2}\|\zeta\|^2 \\
&\qquad\qquad + \sum_{k=1}^{\ell} p_k \|w_{(k)0}\|^{2(p_k-2)} \big(\|w_{(k)0}\|^2\|\eta_{(k)}\|^2 - |\overline{w_{(k)0}}\cdot\eta_{(k)}|^2\big) \\
&\geq \mu e^{-\mu\|z_0\|^2}\|\zeta\|^2 \geq 0.
\end{aligned}
$$

by the Cauchy–Schwarz inequality, for all

$$T = (\zeta, \eta_{(1)}, \ldots, \eta_{(\ell)}) \in T^{1,0}_{(z_0, w_{(1)0}, \ldots, w_{(\ell)0})}(b_0 D^{\boldsymbol{n},\boldsymbol{p}}_{n_0}(\mu)).$$

Obviously, if $\zeta \neq 0$, then $L_\rho(T, T) > 0$.

On the other hand, combining with (3-1), (3-2) and (3-3), we know that the equality holds if and only if

$$\zeta = 0, \tag{3-4}$$

$$\|w_{(k)0}\|^2\|\eta_{(k)}\|^2 - |\overline{w_{(k)0}}\cdot\eta_{(k)}|^2 = 0, \tag{3-5}$$

$$
\begin{aligned}
\Big[\Big(\sum_{k=1}^{\ell} {p_k}^2 \|w_{(k)0}\|^{2(p_k-2)} |\overline{w_{(k)0}}\cdot\eta_{(k)}|^2\Big)\Big(\sum_{k=1}^{\ell} \|w_{(k)0}\|^{2p_k}\Big) \\
- \Big|\sum_{k=1}^{\ell} p_k \|w_{(k)0}\|^{2(p_k-1)} \overline{w_{(k)0}}\cdot\eta_{(k)}\Big|^2\Big] = 0.
\end{aligned} \tag{3-6}
$$

Suppose $\zeta = 0$. Then $T = (\zeta, \eta_{(1)}, \ldots, \eta_{(\ell)}) \neq 0$ implies that there exists $\eta_{(i_0)} \neq 0$. If $L_\rho(T, T) = 0$ for all

$$T \neq 0 \in T^{1,0}_{(z_0, w_{(1)0}, \ldots, w_{(\ell)0})}(b_0 D^{\boldsymbol{n},\boldsymbol{p}}_{n_0}(\mu)),$$

then by (3-1), (3-2), (3-3) and (3-6), we have $\eta_{(k)} = 0$ $(1 \leq k \leq \ell)$. This is a contradiction.

When there exists $j_0 \geq 1 + \epsilon$ such that $\|w_{(j_0)0}\|^2 = 0$ and $p_{j_0} > 1$, then $(z_0, w_{(1)0}, \ldots, w_{(\ell)0}) \in b_1 D^{\boldsymbol{n},\boldsymbol{p}}_{n_0}(\mu)$. Let $T_0 = (0, \ldots, \eta_{(j_0)}, 0, \ldots, 0)$, $\|\eta_{(j_0)}\| \neq 0$. Then $L_\rho(T_0, T_0) = 0$. Hence $D^{\boldsymbol{n},\boldsymbol{p}}_{n_0}(\mu)$ is weakly pseudoconvex but not strongly pseudoconvex on any point of $b_1 D^{\boldsymbol{n},\boldsymbol{p}}_{n_0}(\mu)$.

It is obvious that $D^{\boldsymbol{n},\boldsymbol{p}}_{n_0}(\mu)$ is not smooth at any point of $b_2 D^{\boldsymbol{n},\boldsymbol{p}}_{n_0}(\mu)$. The proof is completed. □

Lemma 3.1 [Tu and Wang 2015]. *Let $\Sigma(\boldsymbol{n}; \boldsymbol{p})$ and $\Sigma(\boldsymbol{m}; \boldsymbol{q})$ be two equidimensional generalized pseudoellipsoids, $\boldsymbol{n}, \boldsymbol{m} \in \mathbb{N}^\ell$, $\boldsymbol{p}, \boldsymbol{q} \in (\mathbb{R}_+)^\ell$ (where $p_k, q_k \neq 1$ for $2 \leq k \leq \ell$). Let $h : \Sigma(\boldsymbol{n}; \boldsymbol{p}) \to \Sigma(\boldsymbol{m}; \boldsymbol{q})$ be a biholomorphic linear isomorphism between $\Sigma(\boldsymbol{n}; \boldsymbol{p})$ and $\Sigma(\boldsymbol{m}; \boldsymbol{q})$. Then there exists a permutation $\sigma \in S_r$ such that*

$n_{\sigma(i)} = m_i$, $p_{\sigma(i)} = q_i$ and

$$h(\zeta_1, \dots, \zeta_r) = (\zeta_{\sigma(1)}, \dots, \zeta_{\sigma(r)}) \begin{pmatrix} U_1 & & & \\ & U_2 & & \\ & & \ddots & \\ & & & U_r \end{pmatrix},$$

where U_i is a unitary transformation of $\mathbb{C}^{m_i}$ $(m_i = n_{\sigma(i)})$ for $1 \le i \le r$.

Define

$$V_1 := \{(z, w_{(1)}, \dots, w_{(\ell)}) \in \mathbb{C}^{n_0} \times \mathbb{C}^{n_1} \times \dots \times \mathbb{C}^{n_\ell} : w_{(1)} = \dots = w_{(\ell)} = 0\},$$
$$V_2 := \{(z, w_{(1)}, \dots, w_{(\ell)}) \in \mathbb{C}^{m_0} \times \mathbb{C}^{m_1} \times \dots \times \mathbb{C}^{m_\ell} : w_{(1)} = \dots = w_{(\ell)} = 0\}$$

(so $V_1 \cong \mathbb{C}^{n_0}$ and $V_2 \cong \mathbb{C}^{m_0}$). Then we have the following lemma.

Lemma 3.2. *Suppose $D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)$ and $D_{m_0}^{\boldsymbol{m},\boldsymbol{q}}(\nu)$ are two equidimensional generalized Fock–Bargmann–Hartogs domains, and $f : D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu) \to D_{m_0}^{\boldsymbol{m},\boldsymbol{q}}(\nu)$ is a biholomorphic mapping. Then we have $f(V_1) \subseteq V_2$ and $f|_{V_1} : V_1 \to V_2$ is biholomorphic, and consequently, $n_0 = m_0$.*

Proof. Let $f(z, 0) = (f_1(z), f_2(z))$. Then we get

$$\sum_{i=1}^{\ell} \| f_{2i} \|^{2q_i} < e^{-\nu \|f_1(z)\|^2} \le 1.$$

Then we obtain that the bounded entire mapping $f_{2i}(z)$ on $\mathbb{C}^{n_0}$ is constant $(1 \le i \le \ell)$ by Liouville's theorem. Since $f(z)$ is biholomorphic, $f_1(z)$ is an unbounded function. Hence there exist $\{z_k\}$ such that $f_1(z_k) \to \infty$ as $k \to \infty$. It implies $f_2(z) \equiv 0$. This proves $f(V_1) \subseteq V_2$. Similarly, by making the same argument for f^{-1}, we have $f^{-1}(V_2) \subseteq V_1$. Namely, $f|_{V_1} : V_1 \to V_2$ is biholomorphic. Hence $n_0 = m_0$. □

Now we give the proof of Theorem 1.4.

Proof of Theorem 1.4. Let $f(0, 0) = (a, b)$ (thus $b = 0$ by Lemma 3.2) and define

$$\phi(z, w_{(1)}, \dots, w_{(\ell)}) := \big(z - a, w_{(1)} \big(e^{2\nu\langle z,a\rangle - \nu\|a\|^2}\big)^{1/2q_1}, \dots, w_{(\ell)} \big(e^{2\nu\langle z,a\rangle - \nu\|a\|^2}\big)^{1/2q_\ell}\big).$$

Obviously, $\phi \in \operatorname{Aut}(D_{m_0}^{\boldsymbol{m},\boldsymbol{q}}(\nu))$ and $\phi \circ f(0, 0) = (0, 0)$. Then $\phi \circ f$ is linear by Theorem 2.10. We describe $\phi \circ f$ as follows:

$$\phi \circ f(z, w) = (z, w) \begin{pmatrix} A & B \\ C & D \end{pmatrix} = (zA + wC, zB + wD).$$

According to Lemma 3.2, we have $f(z, 0) = (f_1(z), 0)$. Thus $B = 0$. Since $g := \phi \circ f$ is biholomorphic, A and D are invertible matrices. We write $g(z, w)$ as

$$g(z,w)=(z,w)\begin{pmatrix} A & 0\\ C & D\end{pmatrix}=(z,w_{(1)},\ldots,w_{(\ell)})\begin{pmatrix} A & 0 & \cdots & 0\\ C_{11} & D_{11} & \cdots & D_{1\ell}\\ \vdots & \vdots & \ddots & \cdots\\ C_{\ell 1} & D_{\ell 1} & \cdots & D_{\ell\ell}\end{pmatrix},$$

which implies that

$$\begin{aligned} g^{-1}(z,w)&=(z,w)\begin{pmatrix} A^{-1} & 0\\ -D^{-1}CA^{-1} & D^{-1}\end{pmatrix}\\ &=(z,w_{(1)},\ldots,w_{(\ell)})\begin{pmatrix} A^{-1} & 0 & \cdots & 0\\ E_{11} & G_{11} & \cdots & G_{1\ell}\\ \vdots & \vdots & \ddots & \vdots\\ E_{\ell 1} & G_{\ell 1} & \cdots & G_{\ell\ell}\end{pmatrix}.\end{aligned}$$

Set

$$\Sigma(\boldsymbol{n};\boldsymbol{p})=\Big\{(w_{(1)},\ldots,w_{(\ell)})\in\mathbb{C}^{n_1}\times\cdots\times\mathbb{C}^{n_\ell}:\sum_{j=1}^{\ell}\|w_{(j)}\|^{2p_j}<1\Big\}.$$

Then, if $\sum_{j=1}^{\ell}\|w_{(j)}\|^{2p_j}<e^{-\mu\|0\|^2}=1$, we obtain

$$\sum_{j=1}^{\ell}\|w_{(1)}D_{1j}+\cdots+w_{(\ell)}D_{\ell j}\|^{2q_j}<e^{-\nu\|wC\|^2}<1,$$

and if $\sum_{j=1}^{\ell}\|w_{(j)}\|^{2q_j}<e^{-\nu\|0\|^2}=1$, we have

$$\sum_{j=1}^{\ell}\|w_{(1)}G_{1j}+\cdots+w_{(\ell)}G_{\ell j}\|^{2p_j}<e^{-\mu\|w(-D^{-1}CA^{-1})\|^2}<1.$$

Therefore, we conclude that the mapping $g_2(w):\Sigma(\boldsymbol{n};\boldsymbol{p})\to\Sigma(\boldsymbol{m};\boldsymbol{q})$ given by

$$g_2(w_{(1)},\ldots,w_{(\ell)})=wD=(w_{(1)},\ldots,w_{(\ell)})\begin{pmatrix} D_{11} & \cdots & D_{1\ell}\\ \vdots & \ddots & \vdots\\ D_{\ell 1} & \cdots & D_{\ell\ell}\end{pmatrix}$$

is a biholomorphic linear mapping. By Lemma 3.1, g_2 can be expressed in the form

$$g_2(w_{(1)},\ldots,w_{(\ell)})=(w_{(\sigma(1))},\ldots,w_{(\sigma(\ell))})\begin{pmatrix}\Gamma_1 & & & \\ & \Gamma_2 & & \\ & & \ddots & \\ & & & \Gamma_\ell\end{pmatrix},$$

where $\sigma\in S_\ell$ is a permutation with $n_{\sigma(j)}=m_j,\ p_{\sigma(j)}=q_j\ (j=1,\ldots,\ell)$ and $\Gamma_i\in\mathcal{U}(m_i)\ (1\le i\le\ell)$. Hence g can be rewritten as follows:

$$g(z,w)=(z,w)\begin{pmatrix} A & 0 \\ C & D \end{pmatrix}=(z,w_{(\sigma(1))},\ldots,w_{(\sigma(\ell))})\begin{pmatrix} A & & & & \\ C_{\sigma(1)1} & \Gamma_1 & & & \\ C_{\sigma(2)1} & & \Gamma_2 & & \\ \vdots & & & \ddots & \\ C_{\sigma(\ell)1} & & & & \Gamma_\ell \end{pmatrix}.$$

Next we prove that $C=0$. The linearity of g yields $g(bD_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu))=bD_{m_0}^{\boldsymbol{m},\boldsymbol{q}}(\nu)$. Let $(0,w)=(0,0,\ldots,w_{(j)},0,\ldots,0)\in bD_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)$, namely, $\|w_{(j)}\|^2=(e^{-\mu\|0\|^2})^{1/p_j}=1$. As Γ_j $(1\le j\le\ell)$ are unitary matrices, moreover, assuming $\sigma(i_0)=j$, we have

$$\|w_{(j)}\|^{2p_j}=\|w_{(\sigma(i_0))}\Gamma_{i_0}\|^{2q_{i_0}}=e^{-\nu\|w_{(\sigma(i_0))}C_{\sigma(i_0)1}\|^2}=1.$$

This implies $w_{(j)}C_{j1}=0$ for all $\|w_{(j)}\|^2=1$. So $C_{j1}=0$ $(1\le j\le\ell)$. Thus we have

$$g(z,w_{(1)},\ldots,w_{(\ell)})=(z,w_{(\sigma(1))},\ldots,w_{(\sigma(\ell))})\begin{pmatrix} A & & & & \\ & \Gamma_1 & & & \\ & & \Gamma_2 & & \\ & & & \ddots & \\ & & & & \Gamma_\ell \end{pmatrix}.$$

Lastly, we show $\sqrt{\nu/\mu}\,A\in\mathcal{U}(n)$ $(n:=n_0=m_0)$. For $z\in\mathbb{C}^{n_0}$, take $(w_{(1)},\ldots,w_{(\ell)})$ such that $e^{-\mu\|z\|^2}=\sum_{j=1}^{\ell}\|w_{(j)}\|^{2p_j}$. By $g(bD_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu))=bD_{m_0}^{\boldsymbol{m},\boldsymbol{q}}(\nu)$, we have

$$\sum_{j=1}^{\ell}\|w_{(\sigma(j))}\Gamma_j\|^{2q_j}=e^{-\nu\|zA\|^2}.$$

Since $\Gamma_j\,(j=1,\ldots,\ell)$ are unitary matrices, we get

$$e^{-\mu\|z\|^2}=\sum_{j=1}^{\ell}\|w_{(\sigma(j))}\|^{2p_{\sigma(j)}}=\sum_{j=1}^{\ell}\|w_{(\sigma(j))}\Gamma_j\|^{2q_j}=e^{-\nu\|zA\|^2}.$$

Therefore, $\nu\|zA\|^2=\mu\|z\|^2$ $(z\in\mathbb{C}^n)$. Then we get $\sqrt{\nu/\mu}\,A\in\mathcal{U}(n)$, and the proof is completed. □

Proof of Corollary 1.5. In fact, the significance of the above ϕ is just to ensure that $\phi\circ f(0)=0$. Then the proof of Theorem 1.4 implies that Corollary 1.5 is obvious. □

Proof of Theorem 1.6. Obviously, φ_A, φ_D and φ_a are biholomorphic self-mappings of $D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)$. On the other hand, for $\varphi\in\operatorname{Aut}(D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu))$, we assume $\varphi(0,0)=(a,b)$ (then $b=0$ by Lemma 3.2). Hence $\varphi_{-a}\circ\varphi$ preserves the origin. Then by Corollary 1.5, we obtain $\varphi_{-a}\circ\varphi=\varphi_D\circ\varphi_A$ for some φ_A,φ_D. Hence $\varphi=\varphi_a\circ\varphi_D\circ\varphi_A$, and the proof is complete. □

Proof of Theorem 1.7. Let f be a proper holomorphic mapping between two equidimensional generalized Fock–Bargmann–Hartogs domains $D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)$ and $D_{m_0}^{\boldsymbol{m},\boldsymbol{q}}(\nu)$. Then by Lemma 2.7, f extends holomorphically to a neighborhood Ω of the closure $\overline{D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)}$ with

$$f(bD_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)) \subset bD_{m_0}^{\boldsymbol{m},\boldsymbol{q}}(\nu).$$

Then by Proposition 3.1 and Lemma 1.3 in [Pinchuk 1975], we have

$$f(M \cap b_0 D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)) \subset b_1 D_{m_0}^{\boldsymbol{m},\boldsymbol{q}}(\nu) \cup b_2 D_{m_0}^{\boldsymbol{m},\boldsymbol{q}}(\nu), \tag{3-7}$$

where $M := \{z \in \Omega : \det(\partial f_i/\partial z_j) = 0\}$ is the zero locus of the complex Jacobian of the holomorphic mapping f on Ω.

If $M \cap bD_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu) \neq \varnothing$, then from

$$\min\{n_{1+\epsilon}, n_2, \dots, n_\ell\} \geq 2,$$

we have $M \cap b_0 D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu) \neq \varnothing$. Take an irreducible component M' of M with $M' \cap b_0 D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu) \neq \varnothing$. Then the intersection $E_{M'}$ of M' with $b_0 D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)$ is a real analytic submanifold of dimensional $2(n_0 + n_1 + \cdots + n_\ell) - 3$ on a dense, open subset of $E_{M'}$. By (3-7), we have $f(E_{M'}) \subset b_1 D_{m_0}^{\boldsymbol{m},\boldsymbol{q}}(\nu) \cup b_2 D_{m_0}^{\boldsymbol{m},\boldsymbol{q}}(\nu)$. Hence

$$f(M' \cap D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)) \subset \bigcup_{j=1+\delta}^{\ell} \mathrm{Pr}_i(D_{m_0}^{\boldsymbol{m},\boldsymbol{q}}(\nu)), \tag{3-8}$$

where $\mathrm{Pr}_i(D_{m_0}^{\boldsymbol{m},\boldsymbol{q}}(\nu)) := \{(z, w_{(1)}, \dots, w_{(\ell)}) \in D_{m_0}^{\boldsymbol{m},\boldsymbol{q}}(\nu) : \|w_{(i)}\| = 0\}$ $(1+\delta \leq i \leq \ell)$ by the uniqueness theorem. Since $\operatorname{codim} M' = 1$,

$$\operatorname{codim}\Biggl[\bigcup_{j=1+\delta}^{\ell} \mathrm{Pr}_i(D_{m_0}^{\boldsymbol{m},\boldsymbol{q}}(\nu))\Biggr] \geq \min\{m_{1+\delta}, \dots, m_\ell, m_1 + \cdots + m_\ell\} \geq 2$$

and $f : D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu) \to D_{m_0}^{\boldsymbol{m},\boldsymbol{q}}(\nu)$ is proper, this contradicts (3-8). Thus we have $M \cap bD_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu) = \varnothing$.

Let $S := M \cap D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu)$. Then we have

$$S \subset D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu), \qquad \bar{S} \cap bD_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu) = \varnothing.$$

If $S \neq \varnothing$, then S is a complex analytic set in $\mathbb{C}^{n_0+n_1+\cdots+n_\ell}$ also. For any $(z, w) \in S$, we have $|w_{\ell n_\ell}|^{2p_\ell} \leq \sum_{j=1}^{\ell} \|w_{(j)}\|^{2p_j} \leq e^{-\mu\|z\|^2} \leq 1$. Thus

$$|w_{\ell n_\ell}|^2 \leq 1 \leq 1 + \|(z, w')\|, \tag{3-9}$$

where $w = (w', w_{\ell n_\ell})$. Then S is an algebraic set of $\mathbb{C}^{n_0+n_1+\cdots+n_\ell}$ by §7.4, Theorem 3 of [Chirka 1989].

Suppose S_1 is an irreducible component of S. Let $\overline{S_1}$ be the closure of S_1 in $\mathbb{P}^{n_0+n_1+\cdots+n_\ell}$. Then by §7.2, Proposition 2 of [Chirka 1989], $\overline{S_1}$ is a projective algebraic set and $\dim \overline{S_1} = n_0 + n_1 + \cdots + n_\ell - 1$. Let $[\xi, z, w]$ be the homogeneous coordinate in $\mathbb{P}^{n_0+n_1+\cdots+n_\ell}$. We embed $\mathbb{C}^{n_0+n_1+\cdots+n_\ell}$ into $\mathbb{P}^{n_0+n_1+\cdots+n_\ell}$ as the affine piece $U_0 = \{[\xi, z, w] \in \mathbb{P}^{n_0+n_1+\cdots+n_\ell} : \xi \neq 0\}$ by $(z, w) \hookrightarrow [1, z, w]$. Then we have

$$D^{\boldsymbol{n},\boldsymbol{p}}_{n_0}(\mu) \cap U_0 = \left\{ [\xi, z, w] : \xi \neq 0,\ \sum_{j=1}^{\ell} \frac{\|w_{(j)}\|^{2p_j}}{|\xi|^{2p_j}} < e^{-\mu\|z\|^2/|\xi|^2} \right\}.$$

Let $H = \{\xi = 0\} \subset \mathbb{P}^{n_0+n_1+\cdots+n_\ell}$. Consider another affine piece

$$U_1 = \{[\xi, z, w] \in \mathbb{P}^{n_0+n_1+\cdots+n_\ell} : z_1 \neq 0\}$$

with affine coordinate $(\zeta, t, s) = (\zeta, t_2, \ldots, t_{n_0}, s_{(1)}, \ldots, s_{(\ell)})$. Let $t' = (1, t_2, \ldots, t_{n_0})$. Since

$$\frac{\|w_{(j)}\|^{2p_j}}{|\xi|^{2p_j}} = \frac{\|w_{(j)}\|^{2p_j}}{|z_1|^{2p_j}} \frac{|z_1|^{2p_j}}{|\xi|^{2p_j}} = \frac{\|s_{(j)}\|^{2p_j}}{|\zeta|^{2p_j}},$$
$$e^{-\mu\|z\|^2/|\xi|^2} = e^{-\mu(\|z\|^2/|z_1|^2)(|z_1|^2/|\xi|^2)} = e^{-\mu(1+|t_2|^2+\cdots+|t_{n_0}|^2)/|\zeta|^2},$$

we obtain

$$D^{\boldsymbol{n},\boldsymbol{p}}_{n_0}(\mu) \cap U_0 \cap U_1 = \left\{ (\zeta, t_2, \ldots, t_{n_0}, s_{(1)}, \ldots, s_{(\ell)}) \in \mathbb{C}^{n_0+n_1+\cdots+n_\ell} : \sum_{j=1}^{\ell} \frac{\|s_{(j)}\|^{2p_j}}{|\zeta|^{2p_j}} < e^{-\mu\|t'\|^2/|\zeta|^2} \right\}. \tag{3-10}$$

Let $S' = \overline{S_1} \cap U_1$ and $H_1 = H \cap U_1 = \{\zeta = 0\}$ (note $\xi = \zeta/z_1$). For every $u \in S' \cap H_1$, there exists a sequence of points $\{u_k\} \subset \overline{S_1} \cap ((U_0 \cap U_1) \backslash H_1)$ such that $u_k \to u$ $(k \to \infty)$. The formula (3-10) implies

$$\|s_{(j)}(u_k)\|^{2p_j} \leq |\zeta(u_k)|^{2p_j} e^{-\mu\|t'\|^2/|\zeta(u_k)|^2} \quad (1 \leq j \leq \ell). \tag{3-11}$$

Since $u \in H_1$, that means $\zeta(u) = 0$ and $\zeta(u_k) \to 0$ $(k \to \infty)$. Therefore we have $\|s_{(j)}(u)\|^{2p_j} \leq 0$ $(1 \leq j \leq \ell)$ as $k \to \infty$. Hence

$$S' \cap H_1 \subset \{\zeta = 0 : s_{(1)} = \cdots = s_{(\ell)} = 0\}.$$

Then $\dim(S' \cap H_1) \leq n_0 - 1$. Theorem 6 in §6.2 of [Shafarevich 1974] implies

$$n_0 - 1 \geq \dim(S' \cap H_1) \geq \dim S' + \dim H_1 - n_0 - n_1 - \cdots - n_\ell \geq \dim S' - 1.$$

This means $\dim S' \leq n_0$, and thus $n_0 + n_1 + \cdots + n_\ell - 1 = \dim S' \leq n_0$. Therefore, we get $n_1 + \cdots + n_\ell \leq 1$, contradicting the assumption that

$$\min\{n_{1+\epsilon}, n_2, \ldots, n_\ell, n_1 + \cdots + n_\ell\} \geq 2.$$

Therefore, $S = \varnothing$ and thus f is unbranched. Since the generalized Fock–Bargmann–Hartogs domain is simply connected, $f : D_{n_0}^{\boldsymbol{n},\boldsymbol{p}}(\mu) \to D_{m_0}^{\boldsymbol{m},\boldsymbol{q}}(\nu)$ is a biholomorphism. The proof is completed. □

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