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GALOISIAN METHODS FOR TESTING IRREDUCIBILITY OF ORDER TWO NONLINEAR DIFFERENTIAL EQUATIONS

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We provide a criterion to compute the Malgrange pseudogroup, the nonlinear analog of the differential Galois group, for classes of second order differential equations. Let G_k be the differential Galois groups of their k-th variational equations along an algebraic solution Γ. We show that if the dimension of one of the G_k is large enough, then the Malgrange pseudogroup is known. This in turn proves the irreducibility of the original nonlinear differential equation. To make the criterion applicable, we give a method to compute the dimensions of the variational Galois groups G_k via constructive reduced form theory. As an application, we reprove the irreducibility of the second and third Painlevé equations for special values of their parameter. In the appendices, we recast the various notions of variational equations found in the literature and prove their equivalences.

Introduction

The Malgrange pseudogroup of a vector field may be seen as a nonlinear analog of the Galois group of linear differential equations. Our aim in this work is to provide a criterion to compute Malgrange pseudogroups using an approach initiated by Casale: we study variational equations along a given algebraic solution curve Γ and use the fact that their Galois groups lie, in a certain sense, in $\mathrm{Mal}(X)$. Our main

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theorem below shows that if the dimension of these Galois groups is large enough, then $\mathrm{Mal}(X)$ is large and known.

In previous works, Casale applied this to integrability. We apply it to a stronger notion; the irreducibility of nonlinear differential equations.

Irreducibility of differential equations. The first formalized definition of reducibility appears in the Stockholm lessons of Paul Painlevé [1897]. A complete algebraization of this definition was given by K. Nishioka [1988] and H. Umemura [1988]. Note that Nishioka's concept of decomposable extension may be more general than reducibility. The first application was the proof of the irreducibility of the first Painlevé equation [Painlevé 1900; Nishioka 1988; Umemura 1988; 1990]. Umemura gave a simple criterion to prove irreducibility and the Japanese school applied it to all Painlevé equations [Noumi and Okamoto 1997; Umemura and Watanabe 1997; 1998; Watanabe 1995; 1998]. These papers deal with reducibility of *solutions*; in this paper, we will emphasize the (stronger) notion of *reducibility of an equation* (see next section for proper definitions).

Painlevé [1902] suggested that irreducibility of a differential equation can be proved by the computation of its (hypothetical) "rationality group", as (incorrectly) defined by J. Drach [1898]. Such a group-like object was finally defined in [Umemura 1996] (where it is a group functor) and in [Malgrange 2001] (where it is an algebraic pseudogroup); see also [Pommaret 1983].

The Malgrange pseudogroup. Let X denote a vector field on a manifold M. The smallest algebraic pseudogroup containing the flows of X is the Malgrange pseudogroup, denoted by $\mathrm{Mal}(X)$ (see Appendix C2, and references therein for a more precise definition).

The computation of the Malgrange pseudogroup of a differential equation is a difficult (and currently wide open) problem. In this paper, we use differential Galois groups of the variational equations along an algebraic solution of equations of the form $y'' = f(x, y)$ to determine their Malgrange pseudogroup.

The study of an equation through its linearization is ancient. Applications to integrability of differential equations were revived by S. L. Ziglin [1982], followed by many authors, notably J. J. Morales-Ruiz and J.-P. Ramis [2001a; 2001b] and then jointly with C. Simó [Morales-Ruiz et al. 2007] using the differential Galois group of the variational equations along a solution.

Casale [2009] proved that these Galois groups provide a lower bound for the Malgrange pseudogroup in the following way. This pseudogroup acts on the phase space and the algebraic solution (along which we linearize) parametrizes a curve $\mathscr{C}$ in this space. Then the group of k-jets of elements fixing a point in $\mathscr{C}$ contains the Galois group of the k-th order variational equation along $\mathscr{C}$.

Using techniques developed in [Morales-Ruiz and Ramis 2001a; 2001b; 2007] and the Malgrange pseudogroup following [Casale 2009], we will prove the following theorem, which is the main result of this work.

Theorem 1. *Let M be a smooth irreducible algebraic 3-fold over $\mathbb{C}$ and X be a rational vector field on M such that there exist a closed rational 1-form α with $\alpha(X) = 1$ and a closed rational 2-form γ with $\iota_X\gamma = 0$.*

Assume $\mathscr{C}$ is an algebraic X-invariant curve with $X_{\mathscr{C}} \not\equiv 0$. Assume that the following two conditions are satisfied:

- (i) *The differential Galois group of the first variational equation of X along $\mathscr{C}$ is not virtually solvable*;
- (ii) *There exists an integer k such that the dimension of the differential Galois group of the k-th variational equation is at least 6.*

Then, the Malgrange pseudogroup is

$$\mathrm{Mal}(X) = \{\varphi \mid \varphi^*\alpha = \alpha,\ \varphi^*\gamma = \gamma\}.$$

Moreover, if there exist rational coordinates x, y, z on M such that

$$X = \frac{\partial}{\partial x} + z\frac{\partial}{\partial y} + f(x, y, z)\frac{\partial}{\partial z}$$

then the equation $y'' = f(x, y, y')$ is irreducible.

The proof will be given in Appendix C, essentially because it requires a number of definitions and clarifications which we give there.

Another way to express the conclusion of the theorem is that the singular holomorphic foliation $\mathscr{F}_X$ of M defined by trajectories of X has no transversal rational geometric structure except the transversal rational volume form given by γ.

This theorem can be applied to compute the Malgrange pseudogroup of equations of the form $y'' = f(x, y)$. Solutions $x \mapsto (x, y(x), y'(x))$ of such an equation are trajectories of the vector field $\frac{\partial}{\partial x} + z\frac{\partial}{\partial y} + f(x, y)\frac{\partial}{\partial z}$ on the phase space. The forms $\alpha = dx$ and $\gamma = \iota_X(dx \wedge dy \wedge dz)$ are closed and $\alpha(X) = 1,\ \iota_X\gamma = 0$. To apply the theorem, a particular solution is needed.

Irreducibility and Malgrange pseudogroup. After Umemura, the Japanese school proved irreducibility of solutions of Painlevé equations using the so-called J condition.

Theorem [Umemura 1990]. *If $E \in \mathbb{C}[x, y, y', y'']$ is a second order differential equation and φ is a nonalgebraic reducible solution then there exists a differential field extension $\mathbb{C}(x) \subset L$ and a first order differential equation $F \in L[y, y']$ such that $F(\varphi, \varphi') = 0$.*

Casale improved this condition under some assumptions:

Theorem [Casale 2009]. *If $E \in \mathbb{C}[x, y, y', y'']$ is a second order differential equation and φ is a nonalgebraic reducible solution and the Malgrange pseudogroup of E is big enough then there exists a first order differential equation $F \in \mathbb{C}[x, y, y']$ such that $F(\varphi, \varphi') = 0$.*

This theorem can be rephrased as follows:

> *If the Malgrange pseudogroup of an equation is big enough then a reducible solution cannot be a general solution.*

This leads us to define reducibility of an equation as the existence of a reducible general solution. A link with the Malgrange pseudogroup is given by the following.

Theorem 2 [Casale 2009]. *Let M be a smooth irreducible algebraic 3-fold over $\mathbb{C}$ and X be a rational vector field on M such that there exist a closed rational 1-form α with $\alpha(X) = 1$ and a closed rational 2-form γ with $\iota_X \gamma = 0$.*

If the Malgrange pseudogroup is

$$\mathrm{Mal}(X) = \{\varphi \mid \varphi^*\alpha = \alpha,\ \varphi^*\gamma = \gamma\}$$

and there exist rational coordinates x, y, z on M such that

$$X = \frac{\partial}{\partial x} + z\frac{\partial}{\partial y} + f(x, y, z)\frac{\partial}{\partial z},$$

then the equation $y'' = f(x, y, y')$ is irreducible.

Casale [2008] applied this philosophy to reprove the irreducibility of the first Painlevé equation. S. Cantat and F. Loray [2009] used this to reprove the irreducibility of the sixth Painlevé equation.

The strongest notion of solvability used in differential equations is Liouville integrability of Hamiltonian systems (and derived notions such as Bogoyavlenskij integrability [1998]). In this case, $\mathrm{Mal}(X)$ is commutative. In the case when the system is integrable by quadratures, $\mathrm{Mal}(X)$ is solvable. In the reducible case, the consequence on $\mathrm{Mal}(X)$ is more technical (see Appendix C and the proof of the main theorem in [Casale 2009]).

Applications. The second Painlevé equation with parameter a is

$$(P_{\mathrm{II}}(a)) \qquad y'' = xy + 2y^3 + a.$$

There is a Bäcklund transformation ([Noumi and Okamoto 1997]) linking $(P_{\mathrm{II}}(a))$ and $(P_{\mathrm{II}}(a + 1))$. Hence, determining the Malgrange pseudogroup for $(P_{\mathrm{II}}(a))$ determines it for all $(P_{\mathrm{II}}(a + n))$, $n \in \mathbb{Z}$.

M. Noumi and K. Okamoto [1997] proved that, apart from the rational solutions when $\alpha \in \mathbb{Z}$ and hypergeometric solutions when $\alpha \in 1/2 + \mathbb{Z}$, the solutions of this equation are irreducible in the sense of Nishioka and Umemura.

Painlevé equations can be presented as Hamiltonian systems with two degrees of freedom. Morales-Ruiz applied Morales–Ramis theory to the Hamiltonian form of $(P_{\mathrm{II}}(n))$ in order to prove its non-Liouville-integrability. His work has been continued in [Stoyanova and Christov 2007; Horozov and Stoyanova 2007; Żołądek and Filipuk 2015]. These computations can be reinterpreted following [Casale 2009]: the nonsolvability of the Galois group of the first variational equation implies the nonsolvability of the Malgrange pseudogroup, and hence the nonintegrability by quadratures of $(P_{\mathrm{II}}(n))$.

For an ordinary differential equation, reducibility is much more general than integrability by quadratures and the corresponding property of its Malgrange pseudogroup is less easy to formulate precisely. The seminal work of Morales-Ruiz has to be continued further and Galois groups of higher order variational equations must be computed.

The approach presented here uses the Malgrange pseudogroup of the rational vector field X on $M = \mathbb{C}^3$ given by

$$X_2 = \frac{\partial}{\partial x} + z\frac{\partial}{\partial y} + (xy + 2y^3)\frac{\partial}{\partial z},$$

whose trajectories are parametrized by solutions of $(P_{\mathrm{II}}(n))$. Using the notation of the theorem, $\alpha = dx,\ \gamma = \iota_X(dx \wedge dy \wedge dz)$ and $\mathscr{C} = \{y = z = 0\}$, we prove that

$$\mathrm{Mal}(X) = \{\varphi \mid \varphi^*\alpha = \alpha,\ \varphi^*\gamma = \gamma\}.$$

This equality implies the irreducibility of $(P_{\mathrm{II}}(n))$. Note that this property of the Malgrange pseudogroup is stronger than irreducibility in the sense of Nishioka and Umemura. However, it is not a purely algebraic property: it is formulated for the differential field $\mathbb{C}(x)$ and seems to be specific to differential fields which are finitely generated over the constants, whereas the definition of irreducibility can be stated over any differential field.

The application of our theorem to prove the irreducibility of the second Painlevé equation requires two steps.

First, one needs to check whether the Galois group of the first variational equation is solvable after an algebraic extension (or virtually solvable). This differential equation reduces to the Airy equation $y'' = xy$ and it is easy, for example by using the Kovacic algorithm [1986], to show that its differential Galois group is $\mathrm{SL}(2, \mathbb{C})$.

Then, to check the dimension condition seems more hazardous at first sight. We would need to compute Galois groups of higher order variational equations until we found a Galois group of dimension at least 6. Until now, no bound is known on the order of the required variational equation that one would have to study to prove this. Moreover, the size of the (linearized) variational equations grows quickly and, even though there are theoretical methods to compute differential Galois groups in [Hrushovski 2002], the computation of differential Galois groups of such big systems is still unfeasible in general.

In our case, the situation is better because the methods of P. H. Berman and M. F. Singer [Berman and Singer 1999; Berman 2002] could allow us to determine the differential Galois group. We choose another approach, following the works of A. Aparicio, E. Compoint, T. Dreyfus and J.-A. Weil on reduced forms of linear differential systems (see [Aparicio and Weil 2012; Aparicio et al. 2013]), notably [Aparicio and Weil 2011; Aparicio et al. 2016] where new effective techniques allow the computation of the Lie algebra of the differential Galois group of a variational equation of order k when the variational equation of order $k-1$ has an abelian differential Galois group. We show how to extend their method to our situation.

These computations can be reused to compute the Malgrange pseudogroup and prove irreducibility of a larger class of differential equations: $y'' = xy + y^n P(x, y)$. We will then show how this technique can be used to compute the Malgrange pseudogroup of a family of Painlevé III equations and prove their irreducibility.

Organization of the paper. Section 1 contains the definitions of reducibility, variational equations and their differential Galois groups in order to state the main theorem. The proof of the main theorem is postponed until Appendix C. In Section 2, we elaborate a simple irreducibility criterion for equations of the form $y'' = xy + y^n P(x, y)$ and give two irreducibility proofs for a Painlevé II equation. In Section 3, we apply a similar scheme to prove the irreducibility of a Painlevé III equation from statistical physics.

In the appendices, we detail the constructions needed to prove the main theorem. In Appendix A, we recast the Galois groups in the context of G-principal connections. In Appendix B, we describe and compare various notions of variational equations (arc space and frame bundle viewpoints), as the literature is occasionally hazy on this point. In Appendix C, we give the definition of the Malgrange pseudogroup of a vector field and give some of its properties regarding the reducibility and the variational equations. Together with the Cartan classification of pseudogroups in dimension 2 (in a neighborhood of a generic point), this allows us to finally prove Theorem 1.

1. Definitions

1.1. ***Irreducibility.*** In the 21st of his Stockholm lessons, Painlevé [1897] defined different classes of transcendental functions and gave the definition of second order differential equations reducible to first order differential equations. Then he proved that the so-called Picard–Painlevé equation, a special case of Painlevé's sixth equation discovered by E. Picard, is irreducible. This proof relies on the fact that this equation has no moving singularities and that its flow gives bimeromorphic transformations of the plane $\mathbb{C}^2$. In this situation, reducible equations have a flow sending a foliation by algebraic curves onto another algebraic one. This is not the case for the Picard–Painlevé equation.

Later, Painlevé claimed without proof that the computation of Drach's rationality group [Drach 1898] would prove the irreducibility of an equation. He tried to compute it for the first Painlevé equation in [Painlevé 1902].

Definition 3 [Painlevé 1897; Nishioka 1988; Umemura 1988]. Let (K, δ) be an ordinary differential field, y be a differential indeterminate and

$$(E) : \delta^2 y = F(y, \delta y) \in K(y, \delta y)$$

be a second order differential equation defined on K. A solution of the equation (E) is called a *reducible solution* if it lies in a differential extension L of K built in the following way:

$$K = K_0 \subset K_1 \subset \cdots \subset K_m = L$$

with one of the following elementary extensions for any i. Either

- $K_i \subset K_{i+1}$ is an algebraic extension, or
- $K_i \subset K_{i+1}$ is a linear extension, i.e., $K_{i+1} = K_i(f_j^p;\ 1 \le p, j \le n)$ with $\delta f_j^p = \sum_k A_j^k f_k^p,\ A_j^k \in K_i$, or
- $K_i \subset K_{i+1}$ is an abelian extension, i.e., $K_{i+1} = K_i(\varphi_j(a_1, \dots, a_n);\ 1 \le j \le n)$ with φ's a basis of periodic functions on $\mathbb{C}^n$ given by the field of rational functions on an abelian variety over $\mathbb{C}$ and a's in K_i, or
- $K_i \subset K_{i+1}$ has transcendence degree 1, i.e., $K_{i+1} = K_i(z, \delta z)$ with $P(z, \delta z) = 0$, $P \in K_i[X, Y] - \{0\}$.

Note that Nishioka's definition of decomposable extension seems more general than reducibility. We do not know any example of a decomposable irreducible extension nor any proof of the equivalence of the two notions. In the articles of Umemura, the notion of reducibility appears together with the notion of classical functions. The latter is similar except that the last kind of elementary extension is not allowed.

This definition may not be the most relevant to understand the geometry of the differential equation: a second order differential equation may have two functionally independent first integrals in a Picard–Vessiot extension of $\mathbb{C}(x, y, z)$ without being reducible. This is the case for the Picard–Painlevé equation as it is explained in the 21st lesson of Painlevé [1897]; see also [Casale 2007] and [Watanabe 1998].

The above definition is a property of individual solutions; however, the equation may have an exceptional solution which is reducible whereas the others are not. For example, any equation $\delta^2 y = yF(y, \delta y) + \delta y G(y, \delta y) \in K[y, \delta y]$ admits $y = 0$ as a solution. Therefore we will introduce a notion of *reducibility of the equation* which translates, in algebraic terms, the idea that the general solution of the equation is reducible.

Definition 4. Let (K, δ) be an ordinary differential field and

$$(E) : \delta^2 y = F(y, \delta y) \in K(y, \delta y)$$

be a second order differential equation defined on K. The equation (E) is called a *reducible differential equation over* K if there exists a reducible solution f such that the transcendence degree of $K(f, \delta f)/K$ equals 2 (i.e., the general solution of the equation is reducible).

Example 1. Consider the equation $\delta^2 y = 0$. We want to show that it is reducible over $\big(\mathbb{C}(x), \delta = \partial/\partial x\big)$. Its general solution is $f = ax + b$ for arbitrary (i.e., transcendental) constants a and b. Here, $K = \mathbb{C}(x)$ and $K(f, \delta f) = \mathbb{C}(a, b)(x)$ (with a and b transcendental over $\mathbb{C}$) so that the transcendence degree of $K(f, \delta f)/K$ does indeed equal 2. This is why, in the second condition for reducibility of solutions (in Definition 3 above), we allow *linear* extensions with possibly new constants (and not only Picard–Vessiot extensions).

Remark 5. Solutions of a reducible second order differential equation are reducible. Reducibility of the equation means that one can choose a geometric model (M, X) for the differential field K_m and a dominant rational map π from M to $\mathbb{A}^2_K$ such that the rational vector field X is π-projectable on $\partial + y'\frac{\partial}{\partial y} + F(y, y')\frac{\partial}{\partial y'}$. A solution is an integral curve of this vector field. Now the image of a rational map is constructible so that either the solution is algebraic or it is in the image of π. In each case, the solution is reducible.

Using the Malgrange pseudogroup of a vector field and É. Cartan's classification of pseudogroups, Casale proved the following theorem.

Theorem 6 [Casale 2008, Annexe A]. *Let X be a rational vector field on M, a smooth irreducible algebraic 3-fold. Assume there exist a rational closed 1-form α such that $\alpha(X) = 1$ and a rational closed 2-form γ such that $\iota_X \gamma = 0$. Then one of the following holds.*

- *There exists a 1-form ω with coefficients in the algebraic closure $\overline{\mathbb{C}(M)}^{\mathrm{alg}}$ such that $\omega(X) = 0$ and for any local determination of algebraic functions $\omega \wedge d\omega = 0$.*
- *There exist θ_1 and θ_2, two rational 1-forms vanishing on X, and a traceless 2×2 matrix (θ_i^j) of rational 1-forms such that $\theta_i(X) = 0$, $d\theta_i = \sum_k \theta_i^k \wedge \theta_k$ and $d\theta_i^j = \sum_k \theta_k^j \wedge \theta_i^k$, for all $(i, j) \in \{1, 2\}^2$.*
- *The Malgrange pseudogroup is* $\mathrm{Mal}(X) = \{\varphi \mid \varphi^*\alpha = \alpha,\ \varphi^*(\gamma) = \gamma\}$.

The systems of PDE given in the first two items of the statement are the analog of the resolvent equations in classical Galois theory. The existence of a rational solution to the resolvent equations would imply that the Malgrange pseudogroup is small. Then in [Casale 2009], the claim of Painlevé is proved.

Theorem 7 [Casale 2009]. *Let E be a rational equation of order two,*

$$y'' = F(x, y) \in \mathbb{C}(x, y),$$

and $X = \partial/\partial x + z\,\partial/\partial y + F(x, y)\,\partial/\partial z$ be the rational vector field on $\mathbb{C}^3$ associated to E. If $\mathrm{Mal}(X) = \{\varphi \mid \varphi^* dx = dx,\ \varphi^*(\iota_X dx \wedge dy \wedge dz) = \iota_X dx \wedge dy \wedge dz\}$ *then E is irreducible.*

1.2. ***Variational equations.*** Let X be a vector field on an algebraic manifold M and $\mathscr{C} \subset M$ an algebraic X-invariant curve such that $X_{\mathscr{C}} \not\equiv 0$. Variational equations can be written easily in local coordinates. Intrinsic versions will be given in the appendices. In local coordinates $(x_1, \ldots, x_n)$ on M, the flow equations of $X = \sum a_i(x)\partial/\partial x_i$ are

$$\frac{d}{dt} x_i = a_i(x), \quad i = 1, \ldots, n.$$

This flow can be used to move germs of analytic curves on M pointwise. Let $\epsilon \mapsto x(\epsilon)$ be such a germ defined on $(\mathbb{C}, 0)$. For any ϵ small enough, one has

$$\frac{d}{dt} x_i(\epsilon) = a_i(x(\epsilon)) \quad i = 1, \ldots, n.$$

Analyticity allow us to expand this equality. Let

$$x(\epsilon) = \left(\sum_k x_1^{(k)} \frac{\epsilon^k}{k!}, \ldots, \sum_k x_n^{(k)} \frac{\epsilon^k}{k!}\right),$$

then

$$(\mathrm{VE}_k) \qquad \begin{cases} \dfrac{d}{dt} x_i^0 = a_i(x^0), \\ \dfrac{d}{dt} x_i^{(1)} = \sum_j \dfrac{\partial a_i}{\partial x_j}(x^0) x_j^{(1)}, \\ \dfrac{d}{dt} x_i^{(2)} = \sum_j \dfrac{\partial a_i}{\partial x_j}(x^0) x_j^{(2)} + \sum_{j,\ell} \dfrac{\partial^2 a_i}{\partial x_j \partial x_\ell}(x^0) x_j^{(1)} x_\ell^{(1)}, \\ \dfrac{d}{dt} x_i^{(3)} = \sum_j \dfrac{\partial a_i}{\partial x_j}(x^0) x_j^{(3)} + \sum_{j,\ell} 3 \dfrac{\partial^2 a_i}{\partial x_j \partial x_\ell}(x^0) x_j^{(2)} x_\ell^{(1)}, \\ \qquad\qquad + \sum_{j,\ell,m} \dfrac{\partial^3 a_i}{\partial x_j \partial x_\ell \partial x_m}(x^0) x_j^{(1)} x_\ell^{(1)} x_m^{(1)}, \\ \quad \vdots \\ \dfrac{d}{dt} x_i^{(k)} = F_k(\partial^\beta a_i(x_0), x_i^{(\ell)} \mid i = 1, \ldots, n, |\beta| \leq k,\ \ell \leq k), \end{cases}$$

where the F's are given by Faa di Bruno formulas (see formula (14) on page 860 in [Morales-Ruiz et al. 2007]). The k-th order variational equation is the differential system on k-th order jets of parametrized curves on M obtained in this way. Because $\mathscr{C}$ is an algebraic X-invariant curve, the space of parametrized curves with $x^0 \in \mathscr{C}$ is an algebraic subvariety invariant by the variational equation. The variational equation gives a nonlinear connection on the bundle over $\mathscr{C}$ of parametrized curves pointed on $\mathscr{C}$. This restriction is the variational equation along $\mathscr{C}$.

The system (VE_k) is a rank nk nonlinear system but it can be linearized. For instance the third order variational equation is linearized using new unknowns $z_{\ell,k,j} = x_\ell^{(1)} x_k^{(1)} x_j^{(1)}$, $z_{k,j} = x_k^{(2)} x_j^{(1)}$ and $z_k = x_k^{(3)}$, which amounts to performing some tensor constructions on lower order linearized variational equations (such as symmetric powers of the first variational equation); see [Simon 2014; Aparicio and Weil 2011; Morales-Ruiz et al. 2007]. The linear system obtained is

$$(\mathrm{LVE}_3)\quad \begin{cases} \dfrac{d}{dt} x_i^0 &= a_i(x^0), \\ \dfrac{d}{dt} z_{\ell,k,j} &= \sum_{b,c,d} \Big(\dfrac{\partial a_\ell}{\partial x_b} + \dfrac{\partial a_k}{\partial x_c} + \dfrac{\partial a_j}{\partial x_d} \Big)(x^0) z_{b,c,d}, \\ \dfrac{d}{dt} z_{k,j} &= \sum_{b,c} \Big(\dfrac{\partial a_\ell}{\partial x_b} + \dfrac{\partial a_k}{\partial x_c} \Big)(x^0) z_{b,c} + \sum_{c,d} \dfrac{\partial^2 a_k}{\partial x_c \partial x_d}(x^0) z_{c,d,j}, \\ \dfrac{d}{dt} z_i &= \sum_j \dfrac{\partial a_i}{\partial x_j}(x^0) z_j + \sum_{j,\ell} 3 \dfrac{\partial^2 a_i}{\partial x_j \partial x_\ell}(x^0) z_{j,\ell} \\ & \qquad + \sum_{j,\ell,m} \dfrac{\partial^3 a_i}{\partial x_j \partial x_\ell \partial x_m}(x^0) z_{j,\ell,m}. \end{cases}$$

When X preserves a transversal fibration $\pi : M \to B$, the parametrized curves $\epsilon \to x(\epsilon)$ included in fibers of π give a subset of curves invariant by X. The restriction of the variational equation to this subset is called the π-normal variational equation. The main case of interest is the normal variational equation of an ODE. Such a differential equation gives a vector field $\partial/\partial x_1 + \cdots$ where x_1 is the independent coordinate. The normal variational equation (with respect to the projection on the curve of the independent coordinate) is obtained from the variational equation by setting $x_1^{(k)} = 0$ when $k \geq 1$.

The k-th order linearized normal variational equation is obtained from the k-th order linearized variational equation by setting $z_\alpha = 0$ when a coordinate of $\alpha \in \mathbb{N}^k$ is equal to 1. The induced system will be denoted by NLVE_k.

1.3. ***The Galois group and the main theorem.*** Following E. Picard and E. Vessiot, the differential Galois group of a linear differential system $\frac{d}{dt} Y = AY$ with $A \in \mathrm{GL}(n, \mathbb{C}(t))$ can be defined in the following way.

Select a regular point t_0 of the differential system and a fundamental matrix $F(t) \in \mathrm{GL}(\mathbb{C}\{t - t_0\})$ of holomorphic solutions at this point. Then the splitting field, called the Picard–Vessiot extension, is $L = \mathbb{C}(t, F_i^j(t) \mid 1 \le i, j \le n)$ and the differential Galois group G is the group of $\mathbb{C}(t)$-automorphisms of L commuting with d/dt.

Picard proved that this group G is a linear algebraic subgroup of $\mathrm{GL}(n, \mathbb{C})$ and Vessiot proved the Galois correspondence. In our context, the linearized normal variational equation is a subsystem of the linearized variational equation so the Galois correspondence implies that its Galois group is a quotient of the Galois group of the variational equation.

Introductions to this theory may be found in [Magid 1994] or the reference book [van der Put and Singer 2003]. Other variations on that theme can be found, for example, in [Katz 1990; Kolchin 1973; Bertrand 1996]. We propose an overview of the theory from the "principal bundle" point of view in the appendices.

The statement of our main theorem involves the Malgrange pseudogroup of a vector field. We recall its definition in Appendix C2.

Theorem 1. *Let M be a smooth irreducible algebraic 3-fold over $\mathbb{C}$ and X be a rational vector field on M such that there exist a closed rational 1-form α with $\alpha(X) = 1$ and a closed rational 2-form γ with $\iota_X \gamma = 0$.*

Assume $\mathscr{C}$ is an algebraic X-invariant curve with $X_{\mathscr{C}} \not\equiv 0$. Assume that the following two conditions are satisfied:

(i) *The differential Galois group of the first variational equation of X along $\mathscr{C}$ is not virtually solvable*;

(ii) *There exists an integer k such that the dimension of the differential Galois group of the k-th variational equation is at least 6.*

Then, the Malgrange pseudogroup is

$$\mathrm{Mal}(X) = \{\varphi \mid \varphi^*\alpha = \alpha,\ \varphi^*\gamma = \gamma\}.$$

Moreover, if there exist rational coordinates x, y, z on M such that

$$X = \frac{\partial}{\partial x} + z\frac{\partial}{\partial y} + f(x, y, z)\frac{\partial}{\partial z}$$

then the equation $y'' = f(x, y, y')$ is irreducible.

In the application, we will compute the Galois group of the normal variational equation. As this group is a quotient of the group used in the theorem, one can replace (VE_1) by (NVE_1) and (LVE_k) by (NLVE_k) without changing the conclusion of the theorem. We postpone the proof of the theorem to the appendices because it requires additional technology which is recalled there. In the next two sections, we show applications of this theorem to the irreducibility of second order equations such as the Painlevé equations (P_{II}) and (P_{III}).

2. Irreducibility of $d^2y/dx^2 = f(x, y)$ and the Painlevé II equation

We will compute the differential Galois group of some normal variational equation along the solution $y = 0$ of differential equations of the form

$$\frac{d^2y}{dx^2} = xy + y^n P(x, y) \text{ with } P \in \mathbb{C}(x, y) \text{ without poles along } y = 0 \text{ and } n \geq 2.$$

The vector field of our equation is

$$X = \frac{\partial}{\partial x} + z\frac{\partial}{\partial y} + (xy + y^n P(x, y))\frac{\partial}{\partial z}.$$

This equation has a solution $y = z = 0$. The first normal variational equation along this curve is

$$\frac{\partial}{\partial x} + z^{(1)}\frac{\partial}{\partial y^{(1)}} + xy^{(1)}\frac{\partial}{\partial z^{(1)}}$$

Using a parametrization $x = t$ of this curve, we get a linear system,

$$\frac{d}{dt}Y = AY, \qquad \text{with } A = \begin{pmatrix} 0 & 1 \\ t & 0 \end{pmatrix}.$$

It is easily seen, from the form of the equation, that the variational equations of order less than n bring no new information, because of the term in y^n. Letting

$$y = \sum_{i=1}^{n} y^{(i)}\epsilon^i/(i!) \quad \text{and} \quad p(x) = n!P(x, 0),$$

we have

$$xy + y^n P(x, y) = \sum_{i=1}^{n-1} xy^{(i)}\epsilon^i + (xy^{(n)} + (y^{(1)})^n p(x))\epsilon^n + o(\epsilon^n)$$

and the n-th order normal variational equation along the solution $y = z = 0$ is

$$\frac{\partial}{\partial x} + \left(\sum_{k=1}^{n-1} z^{(k)}\frac{\partial}{\partial y^{(k)}} + xy^{(k)}\frac{\partial}{\partial z^{(k)}}\right) + z^{(n)}\frac{\partial}{\partial y^{(n)}} + (xy^{(n)} + p(x)(y^{(1)})^n)\frac{\partial}{\partial z^{(n)}}.$$

The linearized normal variational system can be reduced to

$$(\mathrm{NVE}_n) \quad \frac{d}{dt}\left(\begin{array}{c} \vdots \\ \vdots \\ \binom{n}{k}(y^{(1)})^{n-k}(z^{(1)})^k \\ \vdots \\ \hline y^{(n)} \\ z^{(n)} \end{array}\right) = \left(\begin{array}{ccc|cc} & & & \vdots & \\ & \mathtt{sym}^n\begin{pmatrix} 0 & 1 \\ t & 0 \end{pmatrix} & & 0 & \\ & & & \vdots & \\ \hline 0 & \cdots & 0 & 0 & 1 \\ p(t) & \cdots & 0 & t & 0 \end{array}\right)\left(\begin{array}{c} \vdots \\ \vdots \\ \binom{n}{k}(y^{(1)})^{n-k}(z^{(1)})^k \\ \vdots \\ \hline y^{(n)} \\ z^{(n)} \end{array}\right).$$

Example 2. For example, in the case of the second Painlevé equation with $a = 0$ or (P_{II}), we have $n = 3$ and the linearized variational system is

$$(\mathrm{PNVE}_3) \qquad \frac{d}{dt}Y = \mathcal{A}\cdot Y, \quad \text{with } \mathcal{A} = \left(\begin{array}{cccc|cc} 0 & 1 & 0 & 0 & 0 & 0 \\ 3t & 0 & 2 & 0 & 0 & 0 \\ 0 & 2t & 0 & 3 & 0 & 0 \\ 0 & 0 & t & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \\ 12 & 0 & 0 & 0 & t & 0 \end{array}\right).$$

2.1. ***Reduced forms and a first irreducibility proof of*** (P_{II})**.** We introduce material from [Aparicio et al. 2013; Aparicio and Weil 2012] concerning the Kolchin–Kovacic reduced forms of linear differential systems.

Consider a differential system $[A]:\ Y' = AY$ with $A \in \mathrm{Mat}(n, k)$. Let G denote its differential Galois group and $\mathfrak{g}$ its Lie algebra. Given a matrix $P \in \mathrm{GL}(n, \bar{k})$, the change of variable $Y = P.Z$ transforms $[A]$ into a system $Z' = B.Z$, where

$$B = PAP^{-1} - P'P^{-1}.$$

The standard notation is $B = P[A]$. The systems $[A]$ and $[P[A]]$ are called *equivalent over* $\bar{k}$. The Galois group may change but its Lie algebra $\mathfrak{g}$ is preserved under this transformation.

We say that $[A]$ is *in reduced form* if $A \in \mathfrak{g}(k)$. When this is not the case, we say that a matrix $B \in \mathrm{Mat}(n, \bar{k})$ is a *reduced form* of $[A]$ if there exists $P \in \mathrm{GL}(n, \bar{k})$ such that $B = P[A]$ and $B \in \mathfrak{g}(\bar{k})$. Our technique to find $\mathfrak{g}$, for the variational equations, will be to transform them into reduced form.

Example 3. The first variational equation of Painlevé II has matrix

$$A_1 = \begin{pmatrix} 0 & 1 \\ t & 0 \end{pmatrix}.$$

This corresponds to the Airy equation and its Galois group is known to be $\mathrm{SL}(2, \mathbb{C})$. Obviously, $A_1 \in \mathfrak{sl}(2, \mathbb{C}(t))$ so the first variational equation is in reduced form.

Let $a_1(x), \ldots, a_r(x) \in k$ be a basis of the $\mathbb{C}$-vector space generated by the coefficients of A. We may decompose A as $A = \sum_{i=1}^{r} a_i(x) M_i$, where the M_i are constant matrices. The *Lie algebra associated to* A, denoted $\mathrm{Lie}(A)$, is the *algebraic Lie algebra generated by the* M_i: it is the smallest Lie algebra which contains the M_i and is also the Lie algebra of some (connected) linear algebraic group $\mathcal{H}$; see [Aparicio et al. 2013].

Example 4. We compute the Lie algebra Lie($\mathcal{A}$) associated to $\mathcal{A}$ in the system (PNVE_3). Let

$$X := \left(\begin{array}{cccc|cc} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) = \left(\begin{array}{c|c} \mathtt{sym}^n \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} & 0 \\ \hline 0 & \begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix} \end{array}\right),$$

$$Y := \left(\begin{array}{cccc|cc} 0 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right) = \left(\begin{array}{c|c} \mathtt{sym}^n \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} & 0 \\ \hline 0 & \begin{matrix} 0 & 0 \\ 1 & 0 \end{matrix} \end{array}\right),$$

and

$$H := [X, Y] = \left(\begin{array}{cccc|cc} 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{array}\right)$$

this is the standard $\mathfrak{sl}_2$ triplet with $[H, X] = 2X$, $[H, Y] = -2Y$), and we introduce the off-diagonal matrices

$$E_i = \left(\begin{array}{c|c} \begin{matrix} \ddots & & \\ & 0 & \\ & & \ddots \end{matrix} & \begin{matrix} \vdots \\ 0 \\ \vdots \end{matrix} \\ \hline B_i & 0 \end{array}\right),$$

where

$$B_0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$B_3 = \begin{pmatrix} 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad B_4 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We have $[X, E_i]=(i+1)E_{i+1}$, $[Y, E_i]=(5-i)E_{i-1}$, $[H, E_i]=(-4+2i)E_i$ and $[E_i, E_j]=0$ (with $E_{-1} = E_5 = 0$). We now show that Lie($\mathcal{A}$) is generated (as a Lie algebra) by X, Y and E_1. Indeed, Lie($\mathcal{A}$) is generated (as a Lie algebra) by

$M_1 := X + E_1$ and $M_2 := Y$; thus $[M_1, M_2] = H$ and $[M_1, H] = -2X - 4E_1$ so $[M_1, H] + 2M_1 = -2E_1$ and so $E_1 \in \mathrm{Lie}(\mathcal{A})$.

The above calculations then show that $\mathrm{Lie}(\mathcal{A})$ has dimension 8 and that it has $\{X, Y, H, E_0, \dots, E_4\}$ as a basis. We admit, as is proved later, that this Lie algebra is actually an algebraic Lie algebra: there exists an algebraic group $\mathcal{H}$ whose Lie algebra $\mathfrak{h}$ is equal to $\mathrm{Lie}(\mathcal{A})$.

A theorem of Kolchin ([van der Put and Singer 2003, Proposition 1.31]), shows that $\mathfrak{g} \subseteq \mathrm{Lie}(\mathcal{A})$ (and that $G \subseteq H$). A reduced form is obtained when we achieve equality in that inclusion. Moreover, when G is connected (which will be the case in this paper), the reduction theorem of Kolchin and Kovacic ([van der Put and Singer 2003, Corollaire 1.32]), shows that a reduced form exists and that the reduction matrix P may be chosen in $\mathcal{H}(k)$.

Let us now continue the above examples with the third variational equation of Painlevé II. Denote by $\mathfrak{h}_{\mathrm{diag}}$ the Lie algebra generated by the block-diagonal elements X, Y, H. Similarly, let $\mathfrak{h}_{\mathrm{sub}}$ be the Lie algebra generated by the off-diagonal matrices E_i (closed under conjugation by $\mathfrak{h}_{\mathrm{diag}}$). Of course, $\mathfrak{h}_{\mathrm{diag}}$ is $\mathfrak{sl}_2$ in its representation on a direct sum $\mathrm{Sym}^n(\mathbb{C}^2) \oplus \mathbb{C}^2$.

We see that $\mathfrak{h} = \mathfrak{h}_{\mathrm{diag}} \oplus \mathfrak{h}_{\mathrm{sub}}$. It is easily seen that $\mathfrak{h}_{\mathrm{diag}}$ is a Lie subalgebra of $\mathfrak{h}$ and that $\mathfrak{h}_{\mathrm{sub}}$ is an ideal in $\mathfrak{h}$.

We have seen that $\mathfrak{g} \subset \mathrm{Lie}(\mathcal{A})$. Furthermore, $\mathfrak{h}_{\mathrm{diag}} \subset \mathfrak{g}$ (because VE_1 has Galois group $\mathrm{SL}_2(\mathbb{C})$). It follows that $\mathfrak{g} = \mathfrak{h}_{\mathrm{diag}} \oplus \tilde{\mathfrak{g}}$, where $\tilde{\mathfrak{g}} \subset \mathfrak{h}_{\mathrm{sub}}$ is an ideal in $\mathfrak{g}$; in particular, it is closed under the bracket with elements of $\mathfrak{h}_{\mathrm{diag}}$ (adjoint action of $\mathfrak{h}_{\mathrm{diag}}$ on $\mathfrak{h}_{\mathrm{sub}}$).

Now the only invariant subsets of $\mathfrak{h}_{\mathrm{sub}}$ under this adjoint action are seen to be $\{0\}$ and $\mathfrak{h}_{\mathrm{sub}}$ (this is reproved and generalized in Proposition 9 below and its lemmas). So the Lie algebra $\mathfrak{g}$ is either $\mathfrak{sl}_2$ (of dimension 3) or $\mathfrak{h}$ (of dimension 8).

As the Galois group of the block-diagonal part is connected, the differential Galois group G of $[\mathcal{A}]$ is connected. Hence we know (by the reduction theorem of Kolchin and Kovacic cited above) that there exists a reduction matrix $P \in H(k)$. Furthermore, as the block-diagonal part of $\mathcal{A}$ is already in reduced form, the block-diagonal part of the reduction matrix P may be chosen to be the identity. So there exists a reduction matrix of the form

$$P = \mathrm{Id} + \sum_{i=1}^{5} f_i(t) E_i, \quad \text{with } \; f_i(t) \in \mathbb{C}(t).$$

A simple calculation shows that $P\mathcal{A}P^{-1} = \mathcal{A} + \sum_{i=1}^{5} f_i(t)[E_i, X + tY]$ and so

$$P[\mathcal{A}] = X + tY + E_1 + \sum_{i=1}^{5} f_i(t)[E_i, X + tY] - \sum_{i=1}^{5} f_i'(t) E_i.$$

We see that the case $\mathfrak{g} = \mathfrak{sl}_2$ happens if and only if we can find $f_i \in \mathbb{C}(t)$ such that $\sum_{i=1}^5 f_i'(t) E_i = \sum_{i=1}^5 f_i(t)[X + tY, E_i] + E_1$. Let Ψ denote the matrix of the adjoint action $[X + tY, \bullet]$ of $X + tY$ on $\mathfrak{h}_{\text{sub}}$. We see that $\mathfrak{g} = \mathfrak{sl}_2$ if and only if we can find an $F \in \mathbb{C}(t)^5$ solution of the differential system

$$F' = \Psi \cdot F + \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

We now gather the properties of (P_{II}) elaborated in this sequence of examples.

Proposition 8. *The Painlevé II equation is irreducible when the parameter $a = 0$.*

Proof. Using the Barkatou algorithm and its Maple implementation [Barkatou 1999; Barkatou et al. 2012], one easily sees that the above differential system does not have a rational solution. If follows that, using the notations of the above examples, we have $\mathfrak{g} = \mathfrak{h}$ of dimension 8. So, for (P_{II}), we have: the Galois group of the first variational equation is $\text{SL}(2, \mathbb{C})$ which is not virtually solvable; the Galois group of the third variational equation has dimension $8 > 5$. Theorem 1 thus shows that the Painlevé II equation is irreducible. □

2.2. ***The Galois group of the n-th variational equation.*** We will now generalize this process to all equations of the form $\frac{d^2y}{dx^2} = xy + y^n P(x, y)$. We will elaborate a much easier irreducibility criterion, which will allow to reprove the above proposition without having to trust a computer.

The aim of this subsection is to prove the following:

Proposition 9. *The Galois group of the n-th variational equation* (LNVE_n) *is either* $\text{SL}_2(\mathbb{C})$ *or its dimension is $n + 5$ and then the differential equation $y'' = xy + y^n P(x, y)$ is irreducible.*

2.2.1. *Adjoint action.*

Lemma 10. *Let A be a 2×2 matrix of rational function of the variable t such that the Galois group G_1 of the differential system $\frac{dY}{dt} = AY$ has Lie algebra $\mathfrak{sl}_2$. Consider a system*

$$\frac{d}{dt}\begin{pmatrix} Z \\ Y \end{pmatrix} = \left(\begin{array}{c|c} \mathtt{sym}^n A & 0 \\ \hline B & A \end{array}\right)\begin{pmatrix} Z \\ Y \end{pmatrix}$$

with differential Galois group G. Then G has dimension 3 or $n + 3$ or $n + 5$ or $2n + 5$.

Proof. Let $\mathfrak{g}$ be the Lie algebra of G. It has a block lower triangular form shaped by the form of the system, i.e., $\mathfrak{g} \subset \mathfrak{h} \subset \mathfrak{gl}_{n+3}$ where

$$\mathfrak{h} = \left\{ \left(\begin{array}{c|c} \mathtt{sym}^n a & 0 \\ \hline b & a \end{array} \right), \; a \in \mathfrak{sl}_2, b \in (\mathbb{C}^2)^\vee \otimes \mathrm{Sym}^n(\mathbb{C}^2) \right\}.$$

The south-east block A defines a subsystem, thus G contains a subgroup isomorphic to SL_2. The north-west block defines a quotient system so there is a surjective group morphism from G onto SL_2. The kernel of this map is a commutative ideal (the off-diagonal matrices E_i, in our examples) and inherits the structure of an $\mathfrak{sl}_2$-module for the inclusion of $\mathfrak{sl}_2$ in $\mathfrak{h}$ via $\mathfrak{g}$. As a representation, $\mathfrak{g} \cap ((\mathbb{C}^2)^\vee \otimes \mathrm{Sym}^n(\mathbb{C}^2))$ is a subspace of $(\mathbb{C}^2)^\vee \otimes \mathrm{Sym}^n(\mathbb{C}^2)$. This representation is nothing but the adjoint representation. The decomposition in irreducible representations is

$$(\mathbb{C}^2)^\vee \otimes \mathrm{Sym}^n(\mathbb{C}^2) = \mathrm{Sym}^{n-1}(\mathbb{C}^2) \oplus \mathrm{Sym}^{n+1}(\mathbb{C}^2)$$

(see [Fulton and Harris 1991, Example 11.11]). So the Lie algebra $\mathfrak{g}$ is either $\mathfrak{sl}_2$, or $\mathfrak{sl}_2 \ltimes \mathrm{Sym}^{n-1}(\mathbb{C}^2)$, or $\mathfrak{sl}_2 \ltimes \mathrm{Sym}^{n+1}(\mathbb{C}^2)$ or $\mathfrak{sl}_2 \ltimes (\mathrm{Sym}^{n-1}(\mathbb{C}^2) \oplus \mathrm{Sym}^{n-1}(\mathbb{C}^2))$. Its dimension is then 3 or $n+3$ or $n+5$ or $2n+5$. □

2.2.2. *Vector field interpretation.* To simplify our computations, we will use the following identification. The Lie algebra $\mathfrak{sl}_2$ may be viewed as a Lie algebra of linear vector fields on $\mathbb{C}^2$, namely,

$$\mathbb{C}X + \mathbb{C}H + \mathbb{C}Y$$

with $X = x\partial/\partial y$, $H = x\,\partial/\partial x - y\,\partial/\partial y$ and $Y = y\partial/\partial x$. These are the same standard X, Y and H as the matrices of Example 4.

The dual representation $\mathbb{C}^2 \otimes \mathrm{Sym}^n((\mathbb{C}^2)^\vee)$ is the space of vector fields on $\mathbb{C}^2$ whose coefficients are homogeneous polynomials of degree n. The decomposition in irreducible representation is the decomposition of any vector field in $\mathbb{C}^2 \otimes \mathrm{Sym}^n((\mathbb{C}^2)^\vee)$ as

$$A\frac{\partial}{\partial x} + B\frac{\partial}{\partial y} = G(x,y)\Big(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\Big) + \frac{\partial K}{\partial y}\frac{\partial}{\partial x} - \frac{\partial K}{\partial x}\frac{\partial}{\partial y}$$

with $G \in \mathrm{Sym}^{n-1}((\mathbb{C}^2)^\vee)$ and $K \in \mathrm{Sym}^{n+1}((\mathbb{C}^2)^\vee)$.[1]

The *symplectic gradient* of a polynomial K will be denoted by

$$J\nabla K := \frac{\partial K}{\partial y}\frac{\partial}{\partial x} - \frac{\partial K}{\partial x}\frac{\partial}{\partial y}.$$

If we define

$$K_i := \left(\binom{n+1}{i}\right) x^{n+1-i} y^i \quad \text{and} \quad E_i := \frac{1}{n+1} J\nabla(K_i),$$

[1]Using $G = \frac{1}{n+1}\big(\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y}\big)$ and $K = \frac{1}{n+1}(yA - xB)$.

then calculation shows that, as in the previous section,

$$[X, E_i]=(i+1)E_{i+1}, \quad [Y, E_i]=(n+2-i)E_{i-1} \quad \text{and} \quad [H, E_i]=(2i-n-1)E_i.$$

Lemma 11. *Let* $\mathfrak{h} := \mathrm{Lie}(\mathcal{A})$ *be the Lie algebra associated to the matrix* $\mathcal{A}$ *of* (LNVE$_n$). *Let G denote the differential Galois group of* (LNVE$_n$) *and* $\mathfrak{g}$ *be its Lie algebra. With the standard notation of Example 4, we have*:

(1) $\mathfrak{h}$ *is generated, as a Lie algebra, by X, Y and* E_0 *and* $\mathfrak{h} = \mathfrak{sl}_2 \rtimes \mathrm{Sym}^{n+1}(\mathbb{C}^2)$.

(2) *Either* $\mathfrak{g} = \mathfrak{sl}(2)$ (*of dimension* 3) *or* $\mathfrak{g} = \mathfrak{h}$ (*of dimension* $n+5$).

Proof. With the matrices of Example 4, we have $\mathcal{A} = X + tY + p(t)E_0$. As $[X, E_i] = (n+1-i)E_{i+1}$, the Lie algebra generated by X, Y and E_0 has dimension $n+5$ and may be identified with $\mathfrak{sl}_2 \rtimes \mathrm{Sym}^{n+1}(\mathbb{C}^2)$. Moreover, a Lie algebra containing X, Y and any of the E_i contains $\mathfrak{sl}_2 \rtimes \mathrm{Sym}^{n+1}(\mathbb{C}^2)$ (because $[Y, E_i] = (n+2-i)E_{i-1}$).

If 1, t and $p(t)$ are linearly independent over $\mathbb{C}$ then $\mathrm{Lie}(\mathcal{A})$ is the algebraic envelope of the Lie algebra generated by X, Y and E_0; because the latter is algebraic (it is $\mathfrak{sl}_2 \rtimes \mathrm{Sym}^{n+1}(\mathbb{C}^2)$), we have $\mathrm{Lie}(\mathcal{A}) = \mathfrak{sl}_2 \rtimes \mathrm{Sym}^{n+1}(\mathbb{C}^2)$. Now, we have seen that $\mathfrak{g}$ is either $\mathfrak{sl}_2$, or $\mathfrak{sl}_2 \rtimes \mathrm{Sym}^{n-1}(\mathbb{C}^2)$, or $\mathfrak{sl}_2 \rtimes \mathrm{Sym}^{n+1}(\mathbb{C}^2)$ or $\mathfrak{sl}_2 \rtimes (\mathrm{Sym}^{n-1}(\mathbb{C}^2) \oplus \mathrm{Sym}^{n-1}(\mathbb{C}^2))$. Among these, only $\mathfrak{sl}_2$ and $\mathfrak{sl}_2 \rtimes \mathrm{Sym}^{n+1}(\mathbb{C}^2)$ are in $\mathrm{Lie}(\mathcal{A})$, which proves the lemma in that case.

We are left with the case $p(t) = a + bt$ with $(a, b) \in \mathbb{C}^2$. Then $\mathrm{Lie}(\mathcal{A})$ is the algebraic envelope of the Lie algebra generated by $M_1 := X + aE_0$ and $M_2 := Y + bE_0$. If $b = 0$, then $[M_1, M_2] = H$ and $[M_1, H] = 2X + a(n+1)E_0$ so $[M_1, H] - 2M_1 = a(n-1)E_0$. So $\mathrm{Lie}(\mathcal{A})$ contains E_0 and we are done. If $b \neq 0$ then let $M_3 := [M_1, M_2] = H + (n+1)^2 bE_1$; then $[M_3, Y] = -2Y - (n+1)bE_0$ so $2M_2 - [M_3, Y]$ is a multiple of E_0 and the result is again true. □

Proof of Proposition 9. This follows from the above two lemmas and Theorem 1. □

2.3. ***Irreducibility criteria.*** Thanks to Proposition 9, to show the irreducibility of $y'' = xy + y^n P(x, y)/(Q(x, y))$, it is enough to show that (LNVE)$_n$ has a Lie algebra not isomorphic to $\mathfrak{sl}(2)$. Using the Kolchin–Kovacic reduction theory, we achieve this by proving (as in our first proof of irreducibility of Painlevé II) that there is no reduction matrix that transforms our system to one with Lie algebra $\mathfrak{sl}(2)$. This gives us the following simple irreducibility criterion.

Theorem 12. *We consider the equation* $(E): \ y'' = xy + y^n P(x, y)$. *Let* $p(t) := n!P(t, 0)$. *Let* $L_{n+1} := \mathtt{sym}^{n+1}(\partial_t^2 - t)$ *denote the* $(n+1)$*-th symmetric power of the Airy equation. Assume the equation* $L_{n+1}(f) = p(t)$ *does not admit a rational solution. Then, if X is the vector fields* $\partial/\partial x + z\,\partial/\partial y + (xy + y^n P(x, y))\,\partial/\partial z$, $\alpha = dx$ *and* $\gamma \iota_X dx \wedge dy \wedge dz$, *we have*

$$\mathrm{Mal}(X) = \{\varphi \mid \varphi^*\alpha = \alpha,\ \varphi^*\gamma = \gamma\}.$$

Corollary 13. *Under the assumption of Theorem 12, the equation* (E) *is irreducible.*

Lemma 14. *Let* $A_1 = \left(\begin{smallmatrix} 0 & 1 \\ t & 0 \end{smallmatrix}\right)$ *denote the companion matrix of the Airy equation. Let* $A_1^\vee := -A_1^T$ *denote the matrix of the dual system. In a convenient basis, the matrix* Ψ *of the adjoint action* $[\mathcal{A}_{\mathrm{diag}}, \bullet]$ *of* $\mathcal{A}_{\mathrm{diag}}$ *on* $\mathfrak{h}_{\mathrm{sub}}$ *is* $\Psi = \mathtt{sym}^{n+1}(A_1)^\vee$.

Proof. We have

$$\mathtt{sym}^{n+1}(A)^\vee = \begin{pmatrix} 0 & -(n+1)t & \ddots & & \\ -1 & \ddots & -nt & 0 & \\ \ddots & -2 & \ddots & \ddots & \ddots \\ & 0 & \ddots & \ddots & -t \\ & & \ddots & -n & 0 \end{pmatrix}.$$

We choose the following basis of $\mathfrak{h}_{\mathrm{sub}}$, using again the vector field representation. Start from the same matrix $F_0 := E_0$ and set $F_{i+1} := -\frac{1}{i+1}[X, F_i]$. Then, one can check that $[Y, F_i] = -(n+2-i)F_{i-1}$. So the matrix Ψ of the map $[X + tY, \bullet]$ on the basis (F_i) is naturally $\mathtt{sym}^{n+1}(A)^\vee$. □

Proof of Theorem 12. Let us go backwards: assume that the equation $y'' = xy + y^n P(x, y)$ is reducible. Then we must have $\mathfrak{g}_3 = \mathfrak{sl}_2$ (otherwise, the dimension of the Lie Algebra $\mathfrak{g}_3$ would be $n+5$, thus exceeding the bound of Theorem 1). Reducing to $\mathfrak{sl}_2$ implies that we can find a rational solution to $Y' = \Psi Y + \vec{b}$, where $\vec{b} = (p(t), 0, \ldots, 0)^T$. Transforming the latter to an operator, via the cyclic vector $(0, \ldots, 0, 1)$ reduces the system to the equation $\mathtt{sym}^{n+1}(\partial_t^2 - t)^\vee = p(t)$. But $\partial^2 - t$ is selfadjoint, hence the result. □

The proof of the corollary is a direct result of Theorem 2.

Corollary 15. *The equation* $(P_{\mathrm{II}}) : y'' = xy + 2y^3$ *is irreducible.*

Proof. In this case, $n = 3$ and $L_4 = \partial^5 - 20t\partial^3 - 30\partial^2 + 64t^2\partial + 64t$. A solution of $L_4(y) = 12$ would be a polynomial (because L_4 has no finite singularity); now the image of a polynomial of degree N by L_4 is a polynomial of degree $N+1$ so 12 cannot be in the image of L_4. As equation $L_4(y) = 12$ has no rational solution, Theorem 12 shows that (P_{II}) is irreducible. □

Corollary 16. *Assume that* $p(t)$ *has a pole of order* k, $1 \le k \le n+2$. *Then the equation* $y'' = xy + y^n P(x, y)$ *is irreducible.*

Proof. As Airy has no finite singularity, neither does L_{n+1}. Thus, if a function $f \in \mathbb{C}(t)$ has a pole of order $d > 0$, then $L(f)$ has this pole of order $d + n + 2$. So if $p(t)$ is in the image of f by L then all its poles have order at least $n + 3$. □

3. Irreducibility of Painlevé III equations

The third Painlevé equation is

$$(P_{\mathrm{III}}) \qquad \frac{d^2y}{dx^2} = \frac{1}{y}\left(\frac{dy}{dx}\right)^2 - \frac{1}{x}\frac{dy}{dx} + \frac{1}{x}(\alpha y^2 + \beta) + \gamma y^3 + \delta\frac{1}{y}$$

with $(\alpha, \beta, \gamma, \delta) \in \mathbb{C}^4$. For special values $(\alpha, \beta, \gamma, \delta) = (2\mu - 1, -2\mu + 1, 1, -1)$, $\mu \in \mathbb{C}$, this equation has a solution: $y = 1$. For $\mu = \frac{1}{2}$, this equation is related to the 2D Ising model in statistical physics; see [McCoy et al. 1977; Tracy and Widom 2011]. We will show that the latter equation is irreducible (in fact, we prove its irreducibility for $\mu \notin \mathbb{Z}$).

This equation has a time-dependent Hamiltonian form (see, e.g., [Clarkson 2006; 2010]). Letting

$$xH(x, y, z) = 2y^2z^2 - (xy^2 - 2\mu y - x)z - \mu xy,$$

we may consider the time-dependent Hamiltonian system

$$\left\{\frac{dy}{dx} = \frac{\partial H}{\partial z}, \quad \frac{dz}{dx} = -\frac{\partial H}{\partial y}\right\}.$$

Eliminating z between these equations shows that y satisfies (P_{III}). It also means that solutions of P_{III} parametrize curves

$$x \mapsto \left(x,\ y(x),\ \frac{xy'(x) + xy(x)^2 - 2\mu y(x) - x}{4y(x)^2}\right)$$

which are integral curves of the vector field $X = \partial/\partial x + \partial H/\partial z\, \partial/\partial y - \partial H/\partial y\, \partial/\partial z$.

Proposition 17. *Let* $(\alpha, \beta, \gamma, \delta) := (2\mu - 1, -2\mu + 1, 1, -1)$, *where* $\mu \notin \mathbb{Z}$. *The third Painlevé equation* (P_{III}) *with parameters* $(\alpha, \beta, \gamma, \delta)$ *is irreducible.*

Before we prove the theorem, we remark that it includes the case $\mu = 1/2$: the third Painlevé equation (P_{III}), as it appears in the study of the 2D Ising model in statistical physics in [McCoy et al. 1977; Tracy and Widom 2011], is irreducible.

Proof. This vector field X satisfies the hypothesis of our theorem with the forms $\alpha = dx$, $\gamma = dy \wedge dz + dH \wedge dx$ and the algebraic invariant curve (Γ) given by $y = 1, z = -\frac{\mu}{2}$.

The first variational equation along Γ has matrix

$$A_1 = \begin{pmatrix} -2 - 2\,\dfrac{\mu}{x} & \dfrac{4}{x} \\ -\mu - \dfrac{\mu^2}{x} & 2 + 2\,\dfrac{\mu}{x} \end{pmatrix}.$$

Conjugation by

$$Q_1 := \begin{pmatrix} -2\mu & 1 \\ -\mu^2 & 0 \end{pmatrix}$$

puts it in Jordan normal formal at 0, giving us

$$\tilde{A}_1 := Q_1^{-1}.A_1.Q_1 = \begin{pmatrix} 0 & \frac{1}{\mu} + \frac{1}{x} \\ 4\,\mu & 0 \end{pmatrix} = \frac{1}{x}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \frac{1}{\mu} \\ 4\,\mu & 0 \end{pmatrix}.$$

We have $\mathrm{Trace}(\tilde{A}_1) = 0$ so $\mathrm{Gal}(VE_1) \subset \mathrm{SL}(2,\mathbb{C})$. This first variational equation is equivalent to the differential operator

$$L_2 := \left(\frac{d}{dx}\right)^2 - 4 - 4\,\frac{\mu}{x}.$$

This L_2 is reducible for integer μ (it then has an exponential solution $e^{\pm 2x} P_\mu(x)$, where P_μ is a polynomial of degree $|\mu|$) and it is irreducible otherwise. Moreover, it admits a log in its local solution at 0, as shown by the Jordan form structure of the local matrix at 0. So, for $\mu \notin \mathbb{Z}$, the criterion of Boucher and Weil [2003] shows $\mathrm{Gal}(VE_1) = \mathrm{SL}(2,\mathbb{C})$ and that the first variational equation is in reduced form.

Let A_2 be the matrix of the second variational equation. As A_1 is in reduced form, we let

$$Q_2 := \left(\begin{array}{c|c} \mathrm{Sym}^2(Q_1) & 0_{3\times 2} \\ \hline 0_{2\times 3} & Q_1 \end{array}\right)$$

and $\tilde{A}_2 := Q_2^{-1} \cdot A_2 \cdot Q_2$. We obtain $\tilde{A}_2 = C_\infty + \frac{1}{x} C_0$, where C_i are constant matrices. Indeed, setting $M_1 := C_0$ and $M_2 := C_\infty - \frac{1}{\mu} C_0$, we have

$$M_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -4\,\mu^2 & 2\,\mu & 0 & 0 & 1 \\ 0 & 4\,\mu^2 & -2\,\mu & 0 & 0 \end{pmatrix} \quad \text{and} \quad M_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 8\,\mu & 0 & 0 & 0 & 0 \\ 0 & 4\,\mu & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ -12\,\mu^2 & 0 & 1 & 4\,\mu & 0 \end{pmatrix}.$$

Now, letting $M_3 := \frac{1}{8\mu}[M_1, M_2]$, a simple calculation shows $[M_1, M3] = -M_1$ and $[M_2, M_3] = M_2$. It follows that the Lie algebra $\mathrm{Lie}(\tilde{A}_2)$ is equal to $\mathrm{SL}(2,\mathbb{C})$ (in a 5-dimensional representation). It follows that $\mathrm{Gal}(VE_2) \subseteq \mathrm{SL}(2,\mathbb{C})$. However, we know that $\mathrm{SL}(2,\mathbb{C}) \subseteq \mathrm{Gal}(VE_2)$ (because $\mathrm{Gal}(VE_1) = \mathrm{SL}(2,\mathbb{C})$) so $\mathrm{Gal}(VE_2) = \mathrm{SL}(2,\mathbb{C})$ and $\tilde{A}_2$ is in reduced form.

We thus need to go to the third variational equation. Its matrix has the form

$$A_3 = \left(\begin{array}{c|c|c} \mathtt{sym}^3(A_1) & & \\ \hline B_2^{(3)} & \mathtt{sym}^2(A_1) & \\ \hline B_3^{(3)} & B_2^{(2)} & A_1 \end{array}\right), \quad \text{where } A_2 = \left(\begin{array}{c|c} \mathtt{sym}^2(A_1) & \\ \hline B_2^{(2)} & A_1 \end{array}\right)$$

and $B_2^{(3)}$ comes from $B_2^{(2)}$ so the really new part is the south-west block $B_3^{(3)}$.

The situation is strikingly similar to the $(P_{\rm II})$ case from the previous section. Let

$$N_1 = \left(\begin{array}{cccc|c} \multicolumn{4}{c|}{0_{7\times 7}} & 0_{7\times 2} \\ \hline 0 & 0 & 0 & 0 & \\ 1 & 0 & 0 & 0 & 0_{2\times 2} \end{array}\right), \; N_2 = \left(\begin{array}{cccc|c} \multicolumn{4}{c|}{0_{7\times 7}} & 0_{7\times 2} \\ \hline 1 & 0 & 0 & 0 & \\ 0 & -1 & 0 & 0 & 0_{2\times 2} \end{array}\right),$$

$$N_3 = \left(\begin{array}{cccc|c} \multicolumn{4}{c|}{0_{7\times 7}} & 0_{7\times 2} \\ \hline 0 & 1 & 0 & 0 & \\ 0 & 0 & -1 & 0 & 0_{2\times 2} \end{array}\right), \; N_4 = \left(\begin{array}{cccc|c} \multicolumn{4}{c|}{0_{7\times 7}} & 0_{7\times 2} \\ \hline 0 & 0 & 1 & 0 & \\ 0 & 0 & 0 & -1 & 0_{2\times 2} \end{array}\right),$$

$$N_5 = \left(\begin{array}{cccc|c} \multicolumn{4}{c|}{0_{7\times 7}} & 0_{7\times 2} \\ \hline 0 & 0 & 0 & 1 & \\ 0 & 0 & 0 & 0 & 0_{2\times 2} \end{array}\right).$$

As in the study of the variational equation, we form a block-diagonal partial reduction matrix Q_3 with blocks $\mathtt{sym}^3(Q_1)$, $\mathtt{sym}^2(Q_1)$, Q_1 and let $\tilde{A}_3 = Q_3^{(-1)} \cdot A_3 \cdot Q_3$. Again, we obtain $\tilde{A}_3 = C_\infty + \frac{1}{x} C_0$, where C_i are constant matrices. We set $M_1 := C_0$ and $M_2 := \frac{1}{4\mu} C_\infty - \frac{1}{\mu} C_0$ and $M_3 := [M_1, M_2]$. Then, direct inspection shows that $\mathrm{Lie}(\tilde{A}_3)$ is equal to $\mathrm{vect}_{\mathbb{C}}(M_1, M_2, M_3, N_1, \ldots, N_5)$ and has dimension 8. Using the results of the previous section, it follows that we have either $\mathfrak{g}_3 = \mathfrak{sl}(2)$ of dimension 3 or $\mathfrak{g}_3 = \mathrm{Lie}(\tilde{A}_3)$ of dimension 8.

The adjoint maps $\mathrm{Ad}_{M_i} := [M_i, \bullet]$ acting on $\mathrm{vect}_{\mathbb{C}}(N_1, \ldots, N_5)$ have respective matrices

$$\Psi_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 \end{pmatrix} \quad \text{and} \quad \Psi_2 = \begin{pmatrix} 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and $\Psi_3 = [\Psi_1, \Psi_2]$ (this follows from the Jacobi identities on Lie brackets). The matrix of the adjoint action of $\tilde{A}_3$ on $\mathrm{vect}_k(N_1, \ldots, N_5)$ is $\Psi := \left(\frac{1}{\mu} + \frac{1}{x}\right)\Psi_1 + 4\mu\Psi_2$.

In order to have $\mathfrak{g}_3 = \mathfrak{sl}(2)$, we would need to find a find a gauge transformation matrix $P = \mathrm{Id}_{9\times 9} + \sum_{i=1}^{5} f_i N_i$ (with $f_i \in \bar{k}$) such that $\mathrm{Lie}(P[\tilde{A}_3]) = \mathrm{vect}_C(M_1, M_2, M_3)$. Let $\vec{b} = (b_1, \ldots, b_5)^T$ be defined by $\tilde{B}_3^{(3)} = \sum_{i=1}^{5} b_i N_i$, namely

$$\vec{b} = \left(-32\,\frac{\mu^4}{x},\; -8\,\frac{\mu^3}{x},\; 4/3\,\frac{\mu^2}{x},\; 0,\; 0\right)^T.$$

Then, letting $\vec{F} = (f_1, \ldots, f_5)^T$, the method developed for $(P_{\rm II})$ in the previous section shows $\mathrm{Lie}(P[\tilde{A}_3]) = \mathrm{vect}_C(M_1, M_2, M_3)$ if and only if the 5×5 system $\vec{F}' = \Psi \cdot \vec{F} + \vec{b}$ has an algebraic solution.

It is easily seen that the latter is impossible. For example, the above system converts to $L(f_1) = g$ where

$$g = 8192\,\frac{\mu^4}{x} + 5120\,\frac{(4\,\mu+1)\mu^4}{x^2} + 512\,\frac{(24\,\mu^2+16\,\mu-7)\mu^4}{x^3} - 256\,\frac{(31\,\mu+3)\mu^4}{x^4} + 768\,\frac{\mu^4}{x^5}$$

and $L = \mathtt{sym}^4(L_2)$ where $L_2 = \left(\frac{d}{dx}\right)^2 - 4 - 4\,\frac{\mu}{x}$. When $\mu \notin \mathbb{Z}$ (as assumed here), the differential Galois group of L_2 (and hence of L) is $\mathrm{SL}(2, \mathbb{C})$). So the equation $L(f_1) = g$ has an algebraic solution if and only if it has a rational solution. Let us prove that the latter is impossible.

The exponents of L at zero are positive integers; it follows that, if f_1 had a pole of order $n \geq 1$ at zero, $L(f_1)$ would have a pole of order $n+5 \geq 6$ at zero. As g only has a pole of order 5 at zero, f_1 must be a polynomial. But then $L(f_1)$ would have a pole of order at most 4 at zero, contradicting the relation $L(f_1) = g$.

Reasoning as in Section 2, it follows that the Lie algebra of the Galois group of the third variational equation is $\mathfrak{g}_3 = \mathrm{Lie}(\tilde{A}_3)\ \mathrm{vect}_{\mathbb{C}}(M_1, M_2, M_3, N_1, \ldots, N_5)$ and has dimension 8. Our Theorem 1 thus implies that (P_{III}) is irreducible for these values of its parameters. □

Appendix A: Review on principal connections

The G-principal connections are the version of linear differential systems in fundamental form for an algebraic group G that may not be a linear group or not be canonically embedded in a GL_n. They are a geometric version of Kolchin's strongly normal extensions [1973]. The differential systems in vector form appear as a quotient of this fundamental (or principal) form.

A1. ***G-principal partial connection.*** Consider an algebraic group G and a smooth algebraic manifold M. A *principal G-bundle* is a bundle $P \xrightarrow{\pi} M$ over M such that G acts on P and the map $P \times G \to P \times_M P$ given by $(p, g) \mapsto (p, pg)$ is an isomorphism.

Let $\mathscr{F}$ be an algebraic singular foliation on M. A *connection along $\mathscr{F}$* (or a *partial connection*) on a bundle $P \xrightarrow{\pi} M$ is a lift of vector field tangent to $\mathscr{F}$ on P. If $0 \to T(P/M) \to TP \xrightarrow{\pi_*} TM \times_M P \to 0$ is the tangential exact sequence then a connection along $\mathscr{F}$ is a splitting above $\mathscr{F}$ given by $\nabla : T\mathscr{F} \times_M P \to TP$. Such a partial connection is called a *rational partial connection* when the splitting is rational.

We are interested in the case where $\mathscr{F}$ is defined by a rational vector field X on M. In this situation, it is enough to lift X to P by a rational vector field ∇_X such that $\pi_* \nabla_X = X$. Then ∇ is defined on a vector collinear to X by linearity.

A *G-principal connection along $\mathscr{F}$* is a G-equivariant splitting $\nabla : T\mathscr{F} \times_M P \to TP$ such that $\nabla(X)(pg) = g_* \nabla(X)(p)$ where $g_* : TP \to TP$ is the map induced by the action of g on P.

If $G \subset H$ is an inclusion of algebraic groups and P is a G-principal bundle then one defines an H-principal bundle $HP = (H \times P)/G$ where $(h, p)g = (hg, pg)$. A partial G-connection $\nabla : T\mathscr{F} \times_M P \to TP$ can be composed with the inclusion $H \times TP \subset T(H \times P)$ and we obtain a map $T\mathscr{F} \times_M (H \times P) \to T(H \times P)$. This map is G-equivariant. By taking quotients, we get the induced H-principal connection along $\mathscr{F}$, given by $H\nabla : T\mathscr{F} \times_M (HP) \to T(HP)$. It is the extension of ∇ to H. In particular, if G is a linear group then the extension of a partial G-principal connection to $\mathrm{GL}(n, \mathbb{C})$ is a usual linear connection in fundamental form with respect to variables tangent to $\mathscr{F}$.

A2. *G-connections and their Galois groups.* In this paper, a G-bundle $E \to M$ is given by: the typical fiber E_* (an affine variety with an action of G), a G-principal bundle $P \to M$ and a quotient $E = (P \times E_*)/G$ for the diagonal action of G. A principal connection along $\mathscr{F}$ on P will induce a connection along $\mathscr{F}$ on E. Such a connection is called a *partial G-connection on E*.

A connection ∇ on a bundle $E \to M$ may be viewed as a G-connection for many different groups (and maybe for no group). If we know that such a group G exists, we denote by GE the principal bundle and $G\nabla$ the G-principal connection. The Galois group of the G-connection will be a good candidate for such a group.

If $\mathbb{C}(M)^{\mathscr{F}} = \mathbb{C}$, i.e., when the foliation has no rational first integrals, then there exists a smallest algebraic group $\mathrm{Gal}\,\nabla \subset G$ such that ∇ is birational to a $\mathrm{Gal}\,\nabla$-connection. This group is well defined up to conjugation in G and is called the *Galois group of* ∇. Its existence is proved following the classical Picard–Vessiot theory in the following way. A $\mathrm{Gal}\,\nabla$-principal bundle is obtained as a minimal $G\nabla$-invariant algebraic subvariety $Q \subset GE$ dominating M and $\mathrm{Gal}\,\nabla$ is the stabilizer of Q in G. It is easy to prove that this group is a well-defined subgroup of G up to conjugacy.

When a connection ∇ is given, it is not easy to find a group G which would endow ∇ with a structure of G-connection. If such groups exist, we have to prove that our result does not depend on the choice of one of these groups. In the case of linear connections, there is a canonical choice (up to the choice of a point on M).

Given a vector bundle E, we say that ∇ is a *linear connection* when, for any $X \in \mathcal{F}$, $\nabla(X)$ preserves the module $\mathcal{E}$ of functions on E which are linear on the fibers. Then there is a canonical way to obtain a principal connection. Following Picard and Vessiot, if E_m is the fiber of E at $m \in M$ then the tensor product $E \otimes E_m^*$ of our vector bundle with the dual of the trivial vector bundle $M \times E_m$ is endowed with

- a connection given by the connection on the first factor,

- an action of $\mathrm{GL}(E_*)$ on the second factor, thus preserving the connection,
- a canonical point $id \in E_m \otimes E_m^*$ in the fiber at m.

Then the space $\max(E \otimes E_m^*)$ of tensors of maximal rank is a $\mathrm{GL}(E_m)$-principal bundle endowed with a principal partial connection. From the action of $\mathrm{GL}(E_m)$ on E_m, we see that linear connections are GL-connections.

The Galois group obtained from a linear connection using this principal bundle and the minimal invariant subvariety Q containing id is called $\mathrm{Gal}_m \nabla$.

A3. ***Principal bundle and groupoids.*** Given $P \xrightarrow{\pi} M$, a G-principal bundle over M, one obtains a groupoid $\mathcal{G}$ by taking the quotient $\mathcal{G} := (P \times P)/G$ of the cartesian product by the diagonal action of G (see [Mackenzie 1987] for more details; the main example is described in Appendices B2 and C2). The identity is the subvariety quotient of the diagonal in $P \times P$. From a G-principal connection ∇ on P, one derives a connection $\nabla \oplus \nabla$ on the product $P \times P$ defined in an obvious way from the decomposition $T(M \times M) \times_{M \times M} (P \times P) = TM \times_M P \oplus_P TM \times_M P$. This connection is the so-called *flows matrix equation*.

Let $G \subset H$ be an inclusion of algebraic groups and $HP \to M$ and $H\nabla$ be an extension of the principal connection to H. One gets a groupoid inclusion $\mathcal{G} \to \mathcal{H}$ such that $(H\nabla \oplus H\nabla)|_{\mathcal{G}} = \nabla \oplus \nabla$.

Remark 18. The following claims are not used in this paper. They may help the reader to understand the links between the various definitions of differential Galois groups appearing in the literature [Bertrand 1996; Katz 1990; Pillay 2004; Cartier 2009].

- The smallest algebraic subvariety of $\mathcal{G}$ which contains the identity and is $\nabla \oplus \nabla$-invariant is the Galois groupoid of ∇.
- Its restriction above $\{x\} \times M \subset M \times M$ is the Picard–Vessiot extension pointed at $x \in M$.
- Its restriction over the diagonal $M \subset M \times M$ is a $\mathcal{D}_M$-group bundle called the intrinsic Galois group of ∇ in the sense of Pillay [2004].

Appendix B: Variational equations

Various types of variational equations appear in the literature. Morales-Ruiz, Ramis and Simó discuss three of them in [Morales-Ruiz et al. 2007]. More precisely, there are various ways to obtain a linear system from the variational equation seen as an equation on germs of curves. In this paper, for the theoretical result, we consider the *frame variational equation* (see below) as the principal connection coming from the variational equation. However, for practical calculations, one generally linearizes the variational equation.

In this appendix, we give the definitions and the proofs needed to compare these different approaches. Some of these results can be found in [Morales-Ruiz et al. 2007, Propositions 8 to 12].

B1. ***Arc bundles and the variational equation.*** This variational equation does not appear in [Morales-Ruiz et al. 2007]. It has been used by several authors as a perturbative variational equation; see [Boucher and Weil 2003] for references.

The set of all parametrized curves on M is denoted by $CM = \{c : \widehat{(\mathbb{C}, 0)} \to M\}$. It has a natural structure of proalgebraic variety. Let $\mathbb{C}[M]$ be the coordinate ring of M and $\mathbb{C}[\delta]$ be the $\mathbb{C}$-vector space of linear ordinary differential operators with constant coefficients. The coordinate ring of CM is $\mathrm{Sym}(\mathbb{C}[M] \otimes \mathbb{C}[\delta])/L$, where

- the tensor product is a tensor product of $\mathbb{C}$-vector spaces,
- $\mathrm{Sym}(V)$ is the $\mathbb{C}$-algebra generated by the vector space V,
- $\mathrm{Sym}(\mathbb{C}[M] \otimes \mathbb{C}[\delta])$ has a structure of a δ-differential algebra via the right composition of differential operators,
- the Leibniz ideal L is the δ-ideal generated by $fg \otimes 1 - (f \otimes 1)(g \otimes 1)$ for all $(f, g) \in \mathbb{C}[M]^2$.

Local coordinates $(x_1, \ldots, x_n)$ on M induce local coordinates on CM via the Taylor expansion of curves c at 0:

$$c(\epsilon) = \left(\sum c_1^{(k)} \frac{\epsilon^k}{k!}, \ldots, \sum c_n^{(k)} \frac{\epsilon^k}{k!} \right).$$

Let $x_i^{(k)} : CM \to \mathbb{C}$ be defined by $x_i^{(k)}(c) = c_i^{(k)}$. This function is the element $x_i \otimes \delta^k$ in $\mathbb{C}[CM]$ and we have the following facts.

(1) $\mathbb{C}[CM]$ is the δ-algebra generated by $\mathbb{C}[M]$. The action of $\delta : \mathbb{C}[CM] \to \mathbb{C}[CM]$ can be written in local coordinates and gives the total derivative operator $\sum_{i,k} x_i^{(k+1)} \partial/\partial x_i^{(k)}$.

(2) Morphisms and derivations of $\mathbb{C}[M]$ act on $\mathbb{C}[CM]$ as morphisms and derivations, respectively, via the first factor (it can be easily checked that the Leibniz ideal is preserved).

(3) The vector space $\mathbb{C}[\delta]$ is filtered by the spaces $\mathbb{C}[\delta]^{\leq k}$ of operators of order less than k. This gives a filtration of $\mathbb{C}[CM]$ by $\mathbb{C}$-algebras of finite type.

(4) These algebras are coordinate rings of the space of k-jets of parametrized curves $C_k M = \{j_k c \mid c \in CM\}$.

(5) The action of δ has degree $+1$ with respect to the filtration.

(6) Prolongations of morphisms and derivations of $\mathbb{C}[M]$ on $\mathbb{C}[CM]$ have degree 0.

Set theoretically, the prolongations are obtained in the following way. Any holomorphic map $\varphi : U \to V$ between open subsets of M can be prolonged on open sets $C_k U$ of $C_k M$ of curves through points in U by $C_k\varphi : C_k U \to C_k V$; $j_k c \mapsto j_k(\varphi \circ c)$. One easily checks that $C_k(\varphi_1 \circ \varphi_2) = C_k\varphi_1 \circ C_k\varphi_2$. This equality can be used to prolong holomorphic vector fields defined on open subsets $U \subset M$ by the infinitesimal generator of the local 1-parameter group obtained by prolongation of the flow of X, i.e., $C_k(\exp(tX)) = \exp(tC_k X)$.

When X is a rational vector field on M, its prolongation $C_k X$ on $C_k M$ is also rational. Let $X = \sum a_i(x)\partial/\partial x_i$ be a vector field on M in local coordinates. One gets $C_k X = \sum_{i,\ell\leq k} \delta^\ell(a_i)\partial/\partial x_i^\ell$.

If $\mathscr{F}$ is the foliation by integral curves of X on M then $C_k X$ defines a rational connection along $\mathscr{F}$ on $C_k M$: for a vector V tangent to $\mathscr{F}$ at $x \in M$ with $X(x) \neq 0$ or ∞, one defines $\nabla_V(j_k c) = V/(X(x)) R_k X(j_k c)$. It is the *k-th order variational connection/equation* of X.

Usually, the variational equation is studied along a given integral curve of X: if $\mathscr{C}$ is an invariant curve and if $C_k M_{\mathscr{C}}$ is the subspace of $C_k M$ of curves through points in $\mathscr{C}$, the vector field $C_k X$ preserves $C_k M_{\mathscr{C}}$. Its restriction to $C_k M_{\mathscr{C}}$ is called *the k-th order variational connection/equation along* $\mathscr{C}$.

B2. ***Frame bundles and the frame variational equation.*** This variational equation is the one used in the theoretical part of [Morales-Ruiz et al. 2007, Section 3.4] as well as in [Casale 2009].

The set of all formal frames on M is denoted by

$$RM = \{r : \widehat{(\mathbb{C}^n, 0)} \to M \mid \det(\mathrm{Jac}(r)) \neq 0\}.$$

Like the arc spaces, this set has a natural structure of proalgebraic variety. Let $\mathbb{C}[\partial_1, \dots, \partial_n]$ be the $\mathbb{C}$-vector space of linear partial differential operators with constant coefficients. The coordinate ring of RM is

$$(\mathrm{Sym}(\mathbb{C}[M] \otimes \mathbb{C}[\partial_1, \dots, \partial_n])/L)[1/\mathrm{Jac}],$$

where

- the tensor product is a tensor product of $\mathbb{C}$-vector spaces;
- $\mathrm{Sym}(V)$ is the $\mathbb{C}$-algebra generated by the vector space V;
- $\mathrm{Sym}(\mathbb{C}[M] \otimes \mathbb{C}[\partial_1, \dots, \partial_n])$ has a structure of a $\mathbb{C}[\partial_1, \dots, \partial_n]$-differential algebra via the right composition of differential operators;
- the Leibniz ideal L is the $\mathbb{C}[\partial_1, \dots, \partial_n]$-ideal generated by

  $$fg \otimes 1 - (f \otimes 1)(g \otimes 1)$$

  for all $(f, g) \in \mathbb{C}[M]^2$;

- the quotient is then localized by Jac, the sheaf of ideals (not differential!), generated by $\det([x_i \otimes \partial_j]_{i,j})$ for a transcendental basis $(x_1, \ldots, x_n)$ of $\mathbb{C}(M)$ on a Zariski open subset of M where such a basis is defined.

Local coordinates $(x_1, \ldots, x_n)$ on M induce local coordinates on RM via the Taylor expansion of maps r at 0:

$$r(\epsilon_1 \ldots, \epsilon_n) = \left(\sum r_1^\alpha \frac{\epsilon^\alpha}{\alpha!}, \ldots, \sum r_n^\alpha \frac{\epsilon^\alpha}{\alpha!}\right).$$

Let $x_i^\alpha : RM \to \mathbb{C}$ be defined by $x_i^\alpha(r) = r_i^\alpha$. This function is the element $x_i \otimes \partial^\alpha$ in $\mathbb{C}[RM]$.

(1) The action of $\partial_j : \mathbb{C}[RM] \to \mathbb{C}[RM]$ can be written in local coordinates and gives the total derivative operator $\sum_{i,\alpha} x_i^{\alpha+1_j} \partial/(\partial x_i^\alpha)$.

(2) We leave to the reader the translation of the properties from Appendix B1 in this multivariate situation.

All the remarks we have made about arc spaces extend *mutatis mutandis* to the frame bundle. There is one important difference: RM is a principal bundle over M. Let us describe this structure here.

The proalgebraic group

$$\Gamma = \{\gamma : \widehat{(\mathbb{C}^n, 0)} \xrightarrow{\sim} \widehat{(\mathbb{C}^n, 0)}, \text{ where } \gamma \text{ is formal invertible}\}$$

is the projective limit of groups

$$\Gamma_k = \{j_k\gamma \mid \gamma : (\mathbb{C}^n, 0) \xrightarrow{\sim} (\mathbb{C}^n, 0), \text{ where } \gamma \text{ is holomorphic invertible}\}.$$

It acts on RM and the map $RM \times \Gamma \to RM \times_M RM$ sending (r, γ) to $(r, r \circ \gamma)$ is an isomorphism. The action of $\gamma \in \Gamma$ on RM is denoted by $S\gamma : RM \to RM$ as it acts as a change of source coordinates of frames. At the coordinate ring level, this action is given by the action of formal change of coordinates on $\mathbb{C}[\partial_1, \ldots, \partial_n]$ followed by the evaluation at 0 in order to get an operator with constant coefficients. This action has degree 0 with respect to the filtration induced by the order of differential operators. For any k, this means that the bundle of order k frames $R_k M$ is a principal bundle over M for the group Γ_k.

When X is a rational vector field on M, its prolongation $R_k X$ on $R_k M$ is also rational. Let $X = \sum a_i(x)\partial/\partial x_i$ be a vector field on M in local coordinates. One gets $R_k X = \sum_{i,|\alpha|\leq k} \partial^\alpha(a_i)\partial/\partial x_i^\alpha$.

If $\mathscr{F}$ is the foliation by integral curves of X on M then $R_k X$ defines a rational connection along $\mathscr{F}$ on $R_k M$. Moreover the prolongation is defined by an action of the first factor on a tensor product whereas Γ_k acts on the other factor. These two actions commute, meaning that $R_k X$ is a Γ_k-principal connection along $\mathscr{F}$. It is the *k-th order frame variational connection/equation* of X.

As for variational equations, one can restrict this connection above an integral curve $\mathscr{C}$ of X: one gets *the k-th order frame variational connection/equation along $\mathscr{C}$*. This connection is a principal connection of a bundle on $\mathscr{C}$. After choosing a point $m \in \mathscr{C}$ where X is defined, we obtain a Galois group $\mathrm{Gal}_m(R_k X|_{\mathscr{C}}) \subset \Gamma_k$.

From a frame $r : \widehat{(\mathbb{C}^n, 0)} \to M$, one derives many parametrized curves such that CM is a Γ-bundle. More precisely: if V_k denotes the vector space of k-jets of maps $\widehat{(\mathbb{C}, 0)} \to \widehat{(\mathbb{C}^n, 0)}$ then $C_k M = (R_k M \times V_k)/\Gamma$. The k-th order variational connection is a Γ_k-connection.

B3. *The linearized variational equations.* The variational equations are usually given in the linearized form described in Section 1.2. In [Morales-Ruiz et al. 2007], another linear variational equation is introduced, using a faithful linear representation of Γ. Let us recall these constructions and their relations with the frame variational equations above.

B3.1 *The Morales–Ramis–Simó linearization.* The theoretically easier linearization of variational equation is done through linearization of frame variational equations. This is the approach followed by Morales-Ruiz, Ramis and Simó. It is based on the fact that Γ_k is a linear group. Let V_k be the set of k-th order jets of map form $(\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ without invertibility condition. Using coordinates on $(\mathbb{C}^n, 0)$ one can check that V_k is a vector space (the addition depends on the choice of coordinates) and, using Faa di Bruno formulas, one can check that $(j_k s, j_k\gamma) \mapsto j_k(s \circ \gamma)$ defines a faithful representation of Γ_k on V_k. Then, from this inclusion $\Gamma_k \subset \mathrm{GL}(V_k)$, one gets an extension of the principal variational equation to the *jet-linearized principal variational connection*.

B3.2 *The geometric explanation of the linearization.* The second linearization (see Section 1) is done in the following way. The coordinate ring of the arc space has a natural degree from the filtration. It is defined on generators of the algebra by $d^\circ(f \otimes \delta^i) = i$. The Leibniz rule implies that the Leibniz ideal L (see Appendix B2) is generated by homogeneous elements and then the degree is well defined on $\mathbb{C}[CM]$.

This degree gives a decomposition $\mathbb{C}[CM] = \oplus_k \mathcal{E}_k$ in subspaces of homogeneous functions of degree k called jet differentials of degree k [Green and Griffiths 1980]. It is a straightforward to verify the following properties:

- $\mathcal{E}_k$ is a locally free $\mathbb{C}[M]$-module of finite rank.
- If $\varphi : U \to V$ is a biholomorphism on open sets of M then $C\varphi$, sending $\mathcal{O}_V \otimes_{\mathbb{C}[M]} \mathbb{C}[CM]$ to $\mathcal{O}_U \otimes_{\mathbb{C}[M]} \mathbb{C}[CM]$ preserves $\mathcal{O}_M \otimes \mathcal{E}_k$.
- If X is an holomorphic vector field on a open set U of M then $\mathcal{E}_k$ is CX-invariant.

Now X is a rational vector field on M and E_k is the dual vector bundle of $\mathcal{E}_k$. From these properties, we find that E_k, endowed with the action of X through CX, is a linear $\mathscr{F}$-connection. It is the *degree-linearized k-th order variational equation*.

The choice of an invariant curve $\mathscr{C}$ and a point $c \in \mathscr{C}$, such that X_c is defined and not zero, will give the Galois group of the degree-linearized variational equation along $\mathscr{C}$ at c denoted by $\mathrm{Gal}_c(LV_k\mathscr{C})$ (even though it depends on X).

The right composition with ∂ gives an inclusion $\mathcal{E}_k \to \mathcal{E}_{k+1}$ and thus a projection $\mathrm{Gal}_c(LV_{k+1}\mathscr{C})$ onto $\mathrm{Gal}_c(LV_k\mathscr{C})$. The inductive limit of differential systems is denoted by $LV_{\mathscr{C}}$, it is the *degree-linearized variational equation*. The projective limit of groups is denoted by $\mathrm{Gal}_c(LV\mathscr{C})$.

Proposition 19. *The Galois group of the degree-linearized variational equation is isomorphic to the Galois group of the frame variational equation.*

The proof will be given in Appendix C.

B4. ***The covariational equations.*** This variational equation is the one used in [Morales-Ruiz et al. 2007, p. 861] to linearize the variational equation.

The set of all formal functions on M is denoted by

$$FM = \{f : \widehat{(M, m)} \to \widehat{(\mathbb{C}, 0)}\}.$$

Its structural ring is $\mathbb{C}[FM] = \mathrm{Sym}(\mathcal{D}_M^{\geq 1})$, the $\mathbb{C}[M]$-algebra generated by the module of differential operators on M generated as an operator algebra by derivations. From this definition, FM is the vector bundle over M dual of $\mathcal{D}_M^{\geq 1}$. It is a projective limit of F_kM, the bundle of k-jets of functions (the dual of operators of order less than k).

A vector field X on M acts on $\mathcal{D}_M^{\geq 1}$ by the commutator $P \mapsto [X, P]$ and this action preserves the order. This gives a linear $\mathscr{F}$-connection on each F_kM. This is the linearized variational equation of [Morales-Ruiz et al. 2007]. In this paper, it is called the *covariational equation*. The following comparison result is proved in Appendix C1 below:

Proposition 20. *The covariational equation of order k and the variational equation of order k have the same Galois group.*

B5. ***Normal variational and normal covariational equations.*** When the vector field X preserves a foliation $\mathscr{G}$ on M then its prolongation C_kX on the space C_kM of k-jets of parametrized curves on M preserves the subspace $C_k\mathscr{G}$ of curves contained in leaves of $\mathscr{G}$. The subspace $C_k\mathscr{G}$ is an algebraic subvariety of C_kM and the restriction of C_kX on $C_k\mathscr{G}$ is the k-th order variational equation tangent to $\mathscr{G}$. When $\mathscr{G}$ is generically transversal to the trajectories of X, this equation is called the *k-th order normal variational equation*. We don't know how this equation depends on the choice of such a foliation $\mathscr{G}$. However, in our situation of a vector field given by a differential equation, there is a canonical transversal foliation given by the levels of the independent variable.

Let B be the curve with local coordinate x and $\pi : M \dashrightarrow B$ be the phase space of a differential equation with independent variable x. The foliation $\mathscr{G}$ is given by the level subsets of π. Using local coordinates $x_1, \dots, x_n$ on M such that $x_1 = x$, the subvariety $C_k\mathscr{G} \subset C_kM$ is described by the equations $x_1^\ell = 0$, $1 \le \ell \le k$. The variational equation in local coordinates is the system (VE_k) page 307. By setting $x_1^\ell = 0$, $1 \le \ell \le k$ into this system, one gets the differential system for the normal variational equation.

The linearization of the normal variational equation is done by the linearization of the variational equation. Let $I \subset \mathbb{C}[C_kM]$ be the ideal defining the subvariety $C_k\mathscr{G}$. Then $\mathscr{E}_k \cap I \subset \mathscr{E}_k \subset \mathbb{C}[C_kM]$ are finite rank linear spaces invariant under the action of C_kX. The induced action on the quotient $\mathscr{E}_k/(\mathscr{E}_k \cap I)$ is the *linearized k-th order normal variational equation*.

The normal covariational equation is more intrinsic. Let $F_k^X M \subset F_kM$ be the space of k-jets of first integrals f of X,

$$f : \widehat{(M, x)} \to \widehat{(\mathbb{C}, 0)},$$

such that $X.f = 0$. It is a linear subspace defined by its annihilator

$$\mathcal{D}_M^{\ge 1} \cdot X \subset \mathcal{D}_M^{\ge 1}.$$

The commutator $P \to [X, P]$ preserves $\mathcal{D}_M^{\ge 1} \cdot X$. So, it defines a linear connection on $F_k^X M$: this is the *normal covariational equation*.

Appendix C: The proofs

We recall the definitions and results of Casale [2009] using the frame bundle RM of M. It has a central place in the theory. In this appendix, it is used to present the Malgrange pseudogroup and in the previous one it was used to have the variational equation in fundamental form.

Because it is a principal bundle, it has an associated groupoid: $\mathrm{Aut}(M) = (RM \times RM)/\Gamma$. The Γ-orbit of a couple of frames (r, s) is the set of all $(r \circ \gamma, s \circ \gamma)$ for $\gamma \in \Gamma$. It is characterized by the formal map $r \circ s^{-1} : \widehat{(M, s(0))} \to \widehat{(M, r(0))}$. The quotient $\mathrm{Aut}(M)$ is the space of formal selfmaps on M with its natural structure of groupoid. For an $m \in M$, we define $\mathrm{Aut}(M)_{m,M}$ to be the space of maps with source at m and target anywhere on M. The choice of a frame $r : \widehat{(\mathbb{C}^n, 0)} \to \widehat{(M, m)}$ gives an isomorphism between $\mathrm{Aut}(M)_{m,M}$ and RM.

C1. *Proofs of the comparison propositions.*

Proof of Proposition 19. We will first compare these variational equations for a fixed order, then study their projective limits.

In order to compare all the variational equations, we will need to look more carefully at the frame bundle. The proof is then just another way to write the properties of $\mathscr{E}_k$. Its second property says that we have a group inclusion $\mathrm{Aut}_k(M)_{c,c} \to \mathrm{GL}(E_k(c))$ and a compatible inclusion of principal bundles,

$$\mathrm{Aut}_k(M)_{c,\mathscr{C}} \to E_k(\mathscr{C}) \otimes (E_k(c))^*.$$

This inclusion is compatible with the action of the vector field X. This means that the fundamental form of the k-th order degree-linearized variational equation (i.e., $C_k X$ action on $\max(E_k(\mathscr{C}) \otimes (E_k(c))^*)$) is an extension of the frame variational equation. Thus, their Galois groups are the same.

The comparison of limit groups is not direct because the family $(\mathrm{GL}(E_k(c))_k$ is not a projective system. The module $\mathcal{E}_k$ is filtered by

$$\mathcal{E}_0 \circ \delta^k \subset \mathcal{E}_1 \circ \delta^{k-1} \subset \cdots \subset \mathcal{E}_k.$$

Let $T_k \subset \mathrm{GL}(E_k(c))$ denote the subgroup preserving this filtration. Now,

- there is a natural projection $T_k \to T_{k-1}$,
- the Galois group of the k-th order degree-linearized variational equation is a subgroup of T_k, and
- the inclusion $\mathrm{Aut}_k(M)_{c,c} \to T_k$ is compatible with the projections.

This proves the proposition. □

Proof of Proposition 20. There is a direct way to see that the variational equation and the covariational equation will have the same Galois group. Instead of using the Picard–Vessiot principal bundle for the covariational equation, one can build a better principal bundle. Consider the bundle of coframes

$$R^{-1}M = \{f : \widehat{(M, m)} \to \widehat{(\mathbb{C}^n, 0)}, \text{ where } f \text{ is formal and invertible}\}$$

whose coordinate ring is $\mathrm{Sym}(\mathcal{D}_M^{\geq 1} \otimes \mathbb{C}^n)[1/\mathrm{Jac}]$. This is a Γ-principal bundle. The action of X by the commutator defines a Γ-principal connection. This connection is called the *coframe variational equation*. The map $R \to R^{-1}$ sending a frame r to its inverse r^{-1} is an isomorphism of principal bundles (up to changing the side of the group action) conjugating the frame and the coframe variational equations.

Now let F_k be the vector space of k-jets of formal maps $\widehat{(\mathbb{C}^n, 0)} \to \widehat{(\mathbb{C}, 0)}$ and C_k be the vector space of k-jets of formal maps $\widehat{(\mathbb{C}, 0)} \to \widehat{(\mathbb{C}^n, 0)}$; then one has

$$F_k M = (R_k^{-1} M \times F_k) / \Gamma_k$$

and $C_k M = (R_k M \times C_k) / \Gamma_k$. Moreover these two isomorphisms are compatible with the various variational equations. So, the Galois group of the covariational equation equals the one of the variational equation. □

C2. ***The Malgrange pseudogroup.*** The *Malgrange pseudogroup of a vector field X on M* is a subgroupoid of $\mathrm{Aut}(M)$. We choose here to call it a pseudogroup as its elements are formal diffeomorphisms between the formal neighborhoods of points of M satisfying the definition of a pseudogroup; see [Casale 2009].

It is defined by means of differential invariants of X, i.e., rational functions $H \in \mathbb{C}(RM)$ such that $RX \cdot H = 0$. Let $\mathrm{Inv}(X) \subset \mathbb{C}(RM)$ be the subfield of differential invariants of X. Let W be a model for Inv and $\pi : RM \dashrightarrow W$ be the dominant map from the inclusion $\mathrm{Inv} \subset \mathbb{C}(RM)$. Let $\mathrm{Mal}(X)$ be $(RM \times_W RM)/\Gamma \subset \mathrm{Aut}\, M$. To define properly this fiber product, one needs to restrict $\pi : (RM)^o \to W$ on its domain of definition. Then, $RM \times_W RM$ is defined to be the Zariski closure of $(RM)^o \times_W (RM)^o$ in $RM \times RM$. By construction, any Taylor expansion of the flow of X belongs to $\mathrm{Mal}(X)$.

Malgrange [2001] showed that there exists a Zariski open subset M^o of M such that the restriction on $\mathrm{Mal}(X)$ to $\mathrm{Aut}(M^o)$ is a subgroupoid. This result was extended by Casale [2009] and allows us to view the Malgrange pseudogroup as a set-theoretical subgroupoid of $\mathrm{Aut}(M)$.

From the Cartan classification of pseudogroups [1908], one gets the following theorem for rank two differential system; see the appendix of [Casale 2008] for a proof.

Theorem 21 [Cartan 1908; Casale 2008]. *Let M be a smooth irreducible algebraic 3-fold over $\mathbb{C}$ and X be a rational vector field on M such that there exist a closed rational 1-form α with $\alpha(X) = 1$ and a closed rational 2-form γ with $\iota_X \gamma = 0$. One of the following statements holds.*

- *On a covering $\widetilde{M} \xrightarrow{\pi} M$ of a Zariski open subset of M, there exists a rational 1-form ω such that $\omega \wedge d\omega = 0$ and $\omega(\pi^* X) = 0$. Then*

$$\mathrm{Mal}(X) \subset \{\varphi \mid \varphi^*\alpha = \alpha,\ (\widetilde{\varphi}^*\omega) \wedge \omega = 0\},$$

  where $\widetilde{\varphi}$ stands for any lift of φ to $\widetilde{M}$. The vector field is said to be ***transversally imprimitive****.*

- *There exists a vector of rational 1-forms $\Theta = \binom{\theta_1}{\theta_2}$ such that $\theta_1 \wedge \theta_2 = \gamma$ and a trace free matrix of 1-forms Ω such that*

$$d\Theta = \Omega \wedge \Theta \quad \text{and} \quad d\Omega = -\Omega \wedge \Omega.$$

  One has $\mathrm{Mal}(X) \subset \{\varphi \mid \varphi^*\alpha = \alpha,\ \varphi^*\Theta = D\Theta,\ \text{and}\ dD = [D, \Omega]\}$. *The vector field is said to be* ***transversally affine****.*

- $\mathrm{Mal}(X) = \{\varphi \mid \varphi^*\alpha = \alpha,\ \varphi^*\gamma = \gamma\}$.

In order to compute dimensions of Malgrange pseudogroups and of Galois groups of variational equations, it will be easier to work with the Lie algebra of the Malgrange pseudogroup. Roughly speaking, $\mathfrak{mal}(X)$ is the sheaf of Lie algebra of vector fields whose flows belongs to $\mathrm{Mal}(X)$. The reader is invited to read [Malgrange 2001] for a formal definition.

C3. *Proof of our main theorem.*

Proof. The assumptions made on X, α and γ ensure that the first variational equation is reducible to a block-diagonal matrix with a first block in $\mathrm{SL}_2(\mathbb{C})$ and a second equal to the identity matrix.

Using theorems above, the proof of this theorem is reduced to Lemmas 22 and 26.

Lemma 22. *If there exists a finite map $\pi : \widetilde{M} \to M$ and an integrable 1-form on $\widetilde{M}$ vanishing on $\pi^* X$ then the Lie algebra of the Galois group of the first variational equation along any solution is solvable.*

Proof. This lemma is proved in the spirit of Ziglin and Morales and Ramis (see, for example, [Audin 2001]).

First we have to descend from the covering to M. Remark that ω is a 1-form on M whose coefficients are algebraic functions. Letting ω_i, $i = 1, \dots, \ell$, be the conjugates of ω, the product $\bar{\omega} = \prod \omega_i \in \mathrm{Sym}^\ell \Omega^1(M)$ is a well-defined rational symmetric form on M. For any holomorphic function f on open subsets of M, $f\bar{\omega}$ satisfies all hypotheses so that one can assume that the rational symmetric form $\bar{\omega}$ is holomorphic at the generic point of $\mathscr{C}$.

At generic $p \in \mathscr{C}$, the section $\bar{\omega} : M \dashrightarrow \mathrm{Sym}^\ell T^* M$ vanishes at order k, thus one can write $\bar{\omega} = \bar{\omega}_k(p) + \cdots$, where $\bar{\omega}_k(p)$ is lowest order homogeneous part of $\bar{\omega}$. It is a well-defined symmetric form ℓ on the tangent space $T_p M$, that is, $\bar{\omega}_k(p) : T_p M \to \mathrm{Sym}^\ell T^*(T_p M)$; note that $\bar{\omega}_k(p)(\lambda v)(\mu w) = \lambda^k \mu^\ell \bar{\omega}_k(p)(v)$ for any $v \in TpM$ and any $w \in T_v(T_p M)$. We may say that $\bar{\omega}$ is a symmetric ℓ-form on the space $T_p M$, homogeneous of degree k.

Sublemma 23. *The vanishing order of $\bar{\omega}$ is constant on a Zariski open subset of $\mathscr{C}$.*

Proof. Let $p \in \mathscr{C}$ be such that $X(p) \neq 0$. One can choose rectifying coordinates $x_1, \dots, x_n$ such that $x(p) = 0$, $X = \partial/\partial x_1$ and

$$\bar{\omega} = \sum_{\substack{\alpha \in \times \mathbb{N}^{n-1} \\ |\alpha| = \ell}} w_\alpha(x) \prod_{i=2}^{n} dx_i^{\alpha_i}.$$

Since $\mathscr{L}_X \bar{\omega} \wedge \bar{\omega} = 0$, one has that, for any $c \in \mathbb{C}$ small enough, $w_\alpha(x_1 + c, \dots, x_n) = f_c(x) w_\alpha(x_1, \dots, x_n)$ where f_c is a holomorphic function depending on c not on α. Now $f_0 = 1$ thus $f_c(0) \neq 0$ for c small enough and the vanishing order of ω at 0 equals the one at $(c, 0, \dots, 0)$. □

This sublemma enables us to define a rational section

$$\bar{\omega}_k : T_{\mathscr{C}} M \dashrightarrow \mathrm{Sym}^\ell V^*(T_{\mathscr{C}} M),$$

where $V^*(T_{\mathscr{C}} M) = T^*(T_{\mathscr{C}} M)/T^*\mathscr{C}$.

Remember that from X, we get a vector field C_1X on $T_{\mathscr{C}}M$ called the first order variational equation along $\mathscr{C}$.

Sublemma 24. $$\mathscr{L}_{C_1X}\bar{\omega}_k \wedge \bar{\omega}_k = 0.$$

Proof. Here again, we will prove the sublemma in local analytic coordinates. Let $p \in \mathscr{C}$ be such that $X(p) \neq 0$. One can choose rectifying coordinates $x_1, \dots, x_n$ such that $x(p) = 0$, $X = \partial/\partial x_1$ and

$$\bar{\omega} = \sum_{\substack{\alpha \in \times \mathbb{N}^{n-1} \\ |\alpha| = \ell}} w_\alpha(x) \prod_{i=2}^{n} dx_i^{\alpha_i}.$$

For any $c \in \mathbb{C}$ small enough, $w_\alpha(x_1 + c, \dots, x_n) = f_c(x) w_\alpha(x_1, \dots, x_n)$ so the zero set of $\bar{\omega}$ in $T_{\mathscr{C}}M$ is a subvariety invariant under translations collinear to X. One can get local equations for this zero set in the form

$$\eta = \sum_{\substack{\alpha \in \times \mathbb{N}^{n-1} \\ |\alpha| = \ell}} n_\alpha(x_2, \dots, x_n) \prod_{i=2}^{n} dx_i^{\alpha_i}$$

and there exists a holomorphic h such that $\bar{\omega} = h\eta$. Now, by taking the lowest order homogeneous parts, one gets $\bar{\omega}_k = h_{k_1}\eta_{k_2}$. Since η is x_1-independent so is η_{k_2}. In local coordinates induced on $T_{\mathscr{C}}M$, $C_1X = \partial/\partial x_1$ so $\mathscr{L}_{C_1X}\eta_{k_2} = 0$ and a direct computation proves that $\mathscr{L}_{C_1X}\bar{\omega}_k \wedge \bar{\omega}_k = 0$. □

Sublemma 25. $\mathrm{Gal}(C_1X)$ *is virtually solvable.*

Proof. The rational form $\bar{\omega}_k$ defines in each fiber of $T_{\mathscr{C}}M$ a homogeneous ℓ-web. This fiberwise rational web is C_1X-invariant. This implies that the action of the Galois group on a fiber T_pM must preserve this web. In other words, the Galois group at p preserves the set of symmetric forms on T_pM which are rational multiples of $\bar{\omega}_k(p)$.

The form η given in the previous sublemma shows that the web is a pull-back of a web defined on the normal bundle of $\mathscr{C}$ in M. The group $\mathrm{Gal}(C_1X)$ is included in a block diagonal group with a block (1) and a 2×2 block given by a subgroup of $\mathrm{SL}(2, \mathbb{C})$. As $\mathrm{SL}(2, \mathbb{C})$ does not preserve a web on $\mathbb{C}^2$, the 2×2 block is a proper subgroup of $\mathrm{SL}(2, \mathbb{C})$. This proves the sublemma, and thus concludes the proof of Lemma 22. □

Lemma 26. *If X is transversally affine then the Galois group of the formal variational equation along any solution has dimension smaller than* 5.

Proof. We will see that this lemma is a consequence of theorems and lemmas from [Casale 2009]. From Theorem 2.4 there, the Galois group of the formal variational equation along $\mathscr{C}$ is a subgroup of

$$\mathrm{Mal}(X)_p = \{\varphi : (M, p) \to (M, p) \mid \varphi \in \mathrm{Mal}(X)\}$$

for a generic $p \in \mathscr{C}$. Its Lie algebra is included in

$$\mathfrak{mal}(X)^0_p = \{Y \text{ a vector field on } (M, p) \mid Y(p) = 0,\, Y \in \mathfrak{mal}(X)\}.$$

From Lemma 3.8 of [Casale 2009], the dimension of

$$\mathfrak{mal}(X)_p = \{Y \text{a vector field on } (M, p) \mid Y \in \mathfrak{mal}(X)\}$$

for $p \in \mathscr{C}$ is smaller than the dimension of the same Lie algebra for generic $p \in M$.

Assume X is transversally affine and choose a point $p \in M$ such that the 1-form α and the forms Θ^0 and Θ^1 from the definition are holomorphic and $\alpha \wedge \theta^0_1 \wedge \theta^0_2 \neq 0$. Then we choose local analytic coordinates such that $\alpha = dx_1$ and $\begin{bmatrix} dx_2 \\ dx_3 \end{bmatrix} = F\Theta^0$ with $dF + F\Theta^1 = 0$. In these coordinates, $\varphi \in \text{Mal}$ satisfies $\varphi^*\alpha = \alpha$, $\varphi^*\Theta^0 = D\Theta^0$ and $dD = [D, \Theta^1]$ if and only if

$$\varphi(x_1, x_2, x_3) = (x_1 + c_0,\, c_1x_2 + c_2x_3 + c_3,\, c_4x_1 + c_5x_2 + c_6)$$

with $c \in \mathbb{C}^7$ such that $\det\begin{bmatrix} c_1 & c_2 \\ c_4 & c_5 \end{bmatrix} = 1$. The infinitesimal version of these calculations shows that the dimension of $\mathfrak{mal}(X)_p$ is smaller than 6. The Lie algebra $\mathfrak{mal}(X)^0_p$ is strictly smaller than $\mathfrak{mal}(X)_p$ as it does not contain $X = \partial/\partial x_1$ so the dimension of the Galois group of the formal variational equation is smaller than or equal to 5. □

Combining these two lemmas, we see Theorem 1 follows from Theorem 21. □

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