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We investigate the cases for which the Rankin–Cohen brackets of two quasimodular eigenforms give rise to eigenforms. More precisely, we characterise all the cases in a subspace of the space of quasimodular forms for which Rankin–Cohen brackets of two quasimodular eigenforms are again eigenforms. In the process, we obtain some new polynomial identities among quasimodular eigenforms. To prove the results on quasimodular forms, we prove several results in the theory of nearly holomorphic modular forms. These new results in the theory of nearly holomorphic modular forms are of independent interest.

1. Introduction

For an even positive integer k, let M_k and S_k be the respective spaces of modular forms and cusp forms of weight k for the full modular group $\mathrm{SL}_2(\mathbb{Z})$. For an even positive integer k, the Eisenstein series of weight k is defined by

$$E_k(z) = 1 - \frac{2k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(n)e^{2\pi i n z},$$

where B_k is the k-th Bernoulli number, $\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}$ and z is in the complex upper-half plane $\mathcal{H}$. We know that for $k \geq 4$, $E_k \in M_k$, but E_2 is not a modular form, rather it is a quasimodular form of weight 2 for $\mathrm{SL}_2(\mathbb{Z})$. There are numerous identities among modular forms. A direct implication of these identities are nice relations among Fourier coefficients of various modular forms. For example, it is well known that $E_4^2 = E_8$ and by comparing the Fourier coefficients of both sides of this identity, we obtain

$$\sigma_7(n) = \sigma_3(n) + 120\sum_{m=1}^{n-1}\sigma_3(m)\sigma_3(n-m) \tag{1}$$

for $n \geq 1$. Since E_k is an eigenform for any even integer $k \geq 4$, the identity

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$E_4^2 = E_8$ can be interpreted as an identity where the product of two eigenforms results in an eigenform. So it is natural to look for other identities which can be obtained in this way, i.e., those identities in which the product of two eigenforms is an eigenform. The investigation for such identities in the space of modular forms for the full modular group $\mathrm{SL}_2(\mathbb{Z})$ has been done by Duke [1999] and Ghate [2000] independently. They explicitly provided all of the cases in which the product of two eigenforms for the full modular group $\mathrm{SL}_2(\mathbb{Z})$ gives an eigenform. The phenomenon of the product of two eigenforms giving rise to an eigenform can be generalized in two different ways. One generalization is, instead of taking two eigenforms, one may take products of an arbitrary number of eigenforms. Another generalization is by taking the Rankin–Cohen brackets of two eigenforms. Here we note that the product of two modular forms is a particular case of a Rankin–Cohen bracket of two modular forms. Both of these generalizations have been well studied, and we have satisfactory answers for them. Products of arbitrary numbers of eigenforms giving eigenforms have been classified by Emmons and Lanphier [2007], and Rankin–Cohen brackets of eigenforms have been studied by Lanphier and Takloo-Bighash [2004]. In this paper we study the Rankin–Cohen brackets of quasimodular eigenforms. We note that quasimodular forms are a generalization of modular forms. The motivation for studying Rankin–Cohen brackets of quasimodular eigenforms is the well-known identity $E_2\Delta = D\Delta$, where $D = \frac{1}{2\pi i}\frac{d}{dz}$ is the differential operator and

$$\Delta(z) = e^{2\pi i z}\prod_{n\geq 1}(1-e^{2\pi i n z})^{24} = \sum_{n=1}^{\infty}\tau(n)e^{2\pi i n z}$$

is the Ramanujan delta function. The identity $E_2\Delta = D\Delta$ is an identity between quasimodular forms for the group $\mathrm{SL}_2(\mathbb{Z})$ in which the product of two quasimodular eigenforms gives rise to an eigenform. The phenomenon of the product of two quasimodular eigenforms giving an eigenform has been studied in [Meher 2012] and [Das and Meher 2015]. The phenomenon of products of arbitrary numbers of quasimodular eigenforms giving eigenforms has been studied in [Kumar and Meher 2016]. To state our main result we first recall the notion of Rankin–Cohen brackets on quasimodular forms.

Rankin–Cohen brackets for quasimodular forms have been defined by Martin and Royer [2009]. Let f and g be two quasimodular forms of weights k and l and depths s and t respectively. Then for any integer $\nu \geq 0$, the ν-th Rankin–Cohen bracket of f and g is defined by

$$[f, g]_\nu := \sum_{\alpha=0}^{\nu}(-1)^\alpha\binom{k-s+\nu-1}{\nu-\alpha}\binom{l-t+\nu-1}{\alpha}D^\alpha f D^{\nu-\alpha} g. \tag{2}$$

Let $\widetilde{M}_k^{\leq s}$ be the space of quasimodular forms of weight k and depth at most s for the full modular group $\mathrm{SL}_2(\mathbb{Z})$. Note that the differential operator D maps

$\widetilde{M}_k^{\leq s}$ into $\widetilde{M}_{k+2}^{\leq s+1}$. It is known from [Martin and Royer 2009] that if $f \in \widetilde{M}_k^{\leq s}$ and $g \in \widetilde{M}_l^{\leq t}$, then $[f, g]_\nu \in \widetilde{M}_{k+l+2\nu}^{\leq s+t}$. We now state the main result of this paper:

Theorem 1.1. *Let f and g be two quasimodular eigenforms such that the depth of each of the forms f and g is strictly less than the half of the weight of the form. Then there are only finitely many triples (f, g, ν) with the property that f and g are quasimodular eigenforms and $[f, g]_\nu$ is again an eigenform. All the possible cases* (*up to some constant multiple*) *are the following*:

- $[E_4, E_4]_0 = E_8$, $\quad [E_4, E_6]_0 = E_{10}$,
  $[E_4, E_{10}]_0 = [E_6, E_8]_0 = E_{14}$, $\quad [E_4, DE_4]_0 = \frac{1}{2} DE_8$.
- *If $k, l \in \{4, 6, 8, 10, 14\}$ and $\nu \geq 1$ with $k + l + 2\nu \in \{12, 16, 18, 20, 22, 26\}$, then*
  $$[E_k, E_l]_\nu = c_\nu(k, l)\Delta_{k+l+2\nu},$$
  where
  $$c_\nu(k, l) = -\frac{2l}{B_l}\binom{\nu+l-1}{\nu} + (-1)^{\nu+1}\frac{2k}{B_k}\binom{\nu+k-1}{\nu}.$$
- *If $k \in \{4, 6, 8, 10, 14\}$ and $\nu \geq 0$ with $l, k + l + 2\nu \in \{12, 16, 18, 20, 22, 26\}$, then*
  $$[E_k, \Delta_l]_\nu = c_\nu(l)\Delta_{k+l+2\nu},$$
  where
  $$c_\nu(l) = \binom{\nu+l-1}{\nu}.$$
- $[E_4, DE_4]_1 = 960\Delta_{12}$, $\quad [E_4, DE_8]_1 = [E_8, DE_4]_1 = 1920\Delta_{16}$,
  $[E_6, DE_6]_1 = -3024\Delta_{16}$, $\quad [E_4, DE_6]_2 = -5040\Delta_{16}$,
  $[E_6, DE_4]_2 = 5040\Delta_{16}$, $\quad [E_4, DE_4]_3 = 4800\Delta_{16}$,
  $[E_8, DE_8]_1 = 3840\Delta_{20}$, $\quad [E_6, DE_6]_3 = -28224\Delta_{20}$,
  $[E_4, DE_4]_5 = 13440\Delta_{20}$.
- $[E_4, D^2E_4]_1 = 960D\Delta_{12}$, $\quad [E_4, DE_6]_1 = -2016D\Delta_{12}$,
  $[E_6, DE_4]_1 = 1440D\Delta_{12}$, $\quad [E_4, DE_4]_2 = 2400D\Delta_{12}$,
  $[E_6, D^2E_6]_1 = -3024D\Delta_{16}$, $\quad [E_6, DE_6]_2 = -10584D\Delta_{16}$,
  $[E_4, D^2E_4]_3 = 4800D\Delta_{16}$, $\quad [E_4, DE_4]_4 = 8400D\Delta_{16}$,
  $[E_8, D^2E_8]_1 = 3840D\Delta_{20}$ $\quad [E_8, DE_8]_2 = 17280D\Delta_{20}$,
  $[E_6, D^2E_6]_3 = -28224D\Delta_{20}$, $\quad [E_6, DE_6]_4 = -63504D\Delta_{20}$,
  $[E_4, D^2E_4]_5 = 13440D\Delta_{20}$, $\quad [E_4, DE_4]_6 = 20160D\Delta_{20}$.

We see from the list of identities in the above theorem that there are some new identities. These identities give new relations among the Fourier coefficients of modular forms. From the list we also see that in some cases the Rankin–Cohen brackets of two quasimodular forms give rise to modular forms. It would be interesting to further investigate the cases for which Rankin–Cohen brackets of two quasimodular forms give rise to modular forms.

The idea of the proof of the above theorem is to prove a similar result in the space of nearly holomorphic modular forms in certain cases and then use the isomorphism between the space of nearly holomorphic modular forms and the space of quasimodular forms to prove the result in the space of quasimodular forms. The advantage of using the space of nearly holomorphic modular forms is the existence of the Petersson inner product. To prove Theorem 1.1, we define the Rankin–Cohen brackets on nearly holomorphic modular forms and prove various results involving certain operators on nearly holomorphic modular forms. Rankin–Cohen brackets and properties of various operators on nearly holomorphic modular forms are of independent interest.

The article is organized as follows. In Section 2, we recall some basic results and prove some new results in the theory of nearly holomorphic modular forms. In Section 3, we state some basic results in the theory of quasimodular forms. In Section 4, we define the Rankin–Cohen brackets on nearly holomorphic modular forms and prove some basic results which will be useful for our purpose. In Section 5, we prove some results which are generalizations of a result of Shimura [1976] to the case of Rankin–Cohen brackets of nearly holomorphic modular forms. These results are the main ingredients in the proof of Theorem 1.1. In Section 6, we prove Theorem 1.1.

2. Nearly holomorphic modular forms

Notations and basic results.

Definition 2.1. A nearly holomorphic modular form f of weight k and depth $\leq p$ for $\mathrm{SL}_2(\mathbb{Z})$ is a polynomial in $1/\mathrm{Im}(z)$ of degree $\leq p$ whose coefficients are holomorphic functions on $\mathcal{H}$ with moderate growth such that

$$(cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right) = f(z),$$

for any $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and $z \in \mathcal{H}$, where $\mathrm{Im}(z)$ is the imaginary part of z.

We denote by $\widehat{M}_k^{\leq p}$ the space of all nearly holomorphic modular forms of weight k and depth $\leq p$ for $\mathrm{SL}_2(\mathbb{Z})$. We also denote by $\widehat{M}_k = \bigcup_p \widehat{M}_k^{\leq p}$ the space of all nearly holomorphic modular forms of weight k.

Definition 2.2. The Maass–Shimura operator R_k on $f \in \widehat{M}_k$ is defined by

$$R_k f(z) = \frac{1}{2\pi i}\Big(\frac{k}{2i\mathrm{Im}(z)} + \frac{\partial}{\partial z}\Big) f(z).$$

The operator R_k takes $\widehat{M}_k$ into $\widehat{M}_{k+2}$. Thus it is also called the Maass-raising operator. We write $R_k^m := R_{k+2m-2} \circ \cdots \circ R_{k+2} \circ R_k$ with $R_k^0 = \mathrm{id}$ and $R_k^1 = R_k$, where id is the identity map. We state the following decomposition theorem of the space of nearly holomorphic modular forms [Shimura 2012, Theorem 5.2].

Theorem 2.3. *Let $k \geq 2$ be even. If $f \in \widehat{M}_k^{\leq p}$ and $p < k/2$ then*

$$\widehat{M}_k^{\leq p} = \bigoplus_{r=0}^{p} R_{k-2r}^r M_{k-2r},$$

and if $p \geq k/2$ then

$$\widehat{M}_k^{\leq p} = \bigoplus_{r=0}^{\frac{k}{2}-1} R_{k-2r}^r M_{k-2r} \oplus \mathbb{C} R_2^{\frac{k}{2}-1} E_2^*,$$

where $E_2^*(z) := E_2(z) - \frac{3}{\pi \mathrm{Im}(z)}$ is a nearly holomorphic modular form of weight 2 and depth 1 for the group $\mathrm{SL}_2(\mathbb{Z})$.

Following Shimura [2012, pp. 32], we define the slowly increasing and rapidly decreasing functions in $\widehat{M}_k$. Shimura has defined slowly increasing and rapidly decreasing functions in a broader space than $\widehat{M}_k$. Here we define those for $\widehat{M}_k$.

Definition 2.4. Let $f \in \widehat{M}_k$. Then f is called a

- **slowly increasing function** if for each $\alpha \in \mathrm{SL}_2(\mathbb{Q})$ there exist positive constants A, B and c depending on f and α such that

$$|\mathrm{Im}(\alpha z)^{k/2} f(\alpha z)| < A y^c \quad \text{if } y = \mathrm{Im}(z) > B;$$

- **rapidly decreasing function** if for each $\alpha \in \mathrm{SL}_2(\mathbb{Q})$ and a positive real number c, there exist positive constants A and B depending on f, α and c such that

$$|\mathrm{Im}(\alpha z)^{k/2} f(\alpha z)| < A y^{-c} \quad \text{if } y = \mathrm{Im}(z) > B.$$

Remark 2.5. If $f \in M_k$, then f is a slowly increasing function. In addition, if $f \in S_k$ then f is a rapidly decreasing function. From the above definitions we observe that the product of a rapidly decreasing function with any nearly holomorphic modular form gives a rapidly decreasing function.

We state the following result [Shimura 2012, Lemma 6.10].

Lemma 2.6. *If $f \in M_k$, then $R_k^m f$ is a slowly increasing function for any integer $m \geq 0$. Moreover, it is a rapidly decreasing function if $f \in S_k$.*

If $f, g \in \widehat{M}_k$ are such that the product fg is a rapidly decreasing function, then the Petersson inner product of f and g is defined by

$$\langle f, g\rangle := \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}} f(z)\overline{g(z)}y^k \frac{dxdy}{y^2}, \tag{3}$$

where $z = x + iy$. The above integral is convergent since fg is a rapidly decreasing function.

The Maass-lowering operator is the differential operator defined by

$$L := -y^2 \frac{\partial}{\partial \bar{z}}\cdot$$

The operator L maps $\widehat{M}_{k+2}$ to $\widehat{M}_k$. From the definition of L, it is clear that L annihilates any holomorphic function. We state the following result [Shimura 2012, Theorem 6.8] which shows that under certain conditions, the operators L and R_k are adjoint to each other with respect to the Petersson inner product.

Lemma 2.7. *Let $f \in \widehat{M}_k$ and $g \in \widehat{M}_{k-2}$ be such that fg, $f(R_{k-2}g)$ and $(Lf)g$ are rapidly decreasing functions. Then we have*

$$\langle f, R_{k-2}g\rangle = \langle Lf, g\rangle.$$

In a particular case of the above result, we obtain the following result which plays a crucial role in the proof of our main result.

Lemma 2.8. *Let $f \in S_k$. Then $\langle f, R_{k-2}g\rangle = 0$ for every $g \in \widehat{M}_{k-2}$ such that both g and $R_{k-2}g$ are slowly increasing functions.*

Eisenstein series. Let

$$\Gamma_\infty = \left\{\pm\begin{pmatrix}1 & m\\ 0 & 1\end{pmatrix} \,\middle|\, m \in \mathbb{Z}\right\}.$$

For any integer $k \geq 0$, $z \in \mathcal{H}$ and $s \in \mathbb{C}$, the Eisenstein series $E_k(z, s)$ is defined by

$$E_k(z, s) = \sum_{\gamma\in\Gamma_\infty\backslash \mathrm{SL}_2(\mathbb{Z})} j(\gamma, z)^{-k}|j(\gamma, z)|^{-2s},$$

where $j(\gamma, z) = (cz + d)$ for $\gamma = \left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)$. The series of $E_k(z, s)$ is absolutely convergent for $\mathrm{Re}(2s) > 2 - k$. It is well known that

$$R_k^r E_k(z) = (-4\pi y)^{-r}\frac{\Gamma(k + r)}{\Gamma(k)} E_{k+2r}(z, -r), \tag{4}$$

where $y = \mathrm{Im}(z)$.

Following [Diamantis and O'Sullivan 2010], we also recall the completed normalized nonholomorphic Eisenstein series, defined by

$$E_k^*(z,s) = \theta_k(s) \sum_{\gamma \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} (\mathrm{Im}(\gamma z))^s \left(\frac{j(\gamma,z)}{|j(\gamma,z)|} \right)^{-k}, \tag{5}$$

where

$$\theta_k(s) = \pi^{-s}\Gamma(s+k/2)\zeta(2s) \quad \text{and} \quad \gamma z := \frac{az+b}{cz+d} \quad \text{for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

We observe that

$$E_k(z,s) = \frac{y^{-s-k/2}}{\theta_k(s+k/2)} E_k^*(z, s+k/2). \tag{6}$$

Hecke operators. For $f \in \widehat{M}_k$ and any integer $n \geq 1$, the action of the n-th Hecke operator on f is defined by

$$(T_n f)(z) = n^{k-1} \sum_{d|n} d^{-k} \sum_{b=0}^{d-1} f\left(\frac{nz+bd}{d^2} \right). \tag{7}$$

For each integer $n \geq 1$, T_n maps $\widehat{M}_k$ to $\widehat{M}_k$. A nearly holomorphic modular form is called an eigenform if it is an eigenvector for each Hecke operator T_n ($n \geq 1$). We recall the following commuting relation between Maass–Shimura operators and Hecke operators [Beyerl et al. 2012, Propositions 2.4 and 2.5].

Proposition 2.9. *Let $f \in \widehat{M}_k$. Then*

$$(R_k^m(T_n f))(z) = \frac{1}{n^m}(T_n(R_k^m f))(z)$$

for any integer $m \geq 0$. Moreover, $R_k^m f$ is an eigenvector for T_n if and only if f is. In this case, if λ_n is the eigenvalue of T_n corresponding to f then the eigenvalue of T_n corresponding to $R_k^m f$ is $n^m \lambda_n$.

The following result characterizes all nearly holomorphic eigenforms for the full modular group $\mathrm{SL}_2(\mathbb{Z})$. The result has been proved in [Kumar and Meher 2016, Theorem 1.1].

Proposition 2.10. *Let f be a nearly holomorphic eigenform of weight k and depth p for the full modular group $\mathrm{SL}_2(\mathbb{Z})$. If $p < k/2$ then $f = R_{k-2p}^p f_p$, where $f_p \in M_{k-2p}$ is an eigenform, and if $p = k/2$ then $f \in \mathbb{C} R_2^{k/2-1} E_2^*$.*

Properties of raising and lowering operators. We first recall the following relation [Shimura 2012, 6.13c, pp. 34].

$$L R_k = R_{k-2} L + \frac{k}{4}. \tag{8}$$

Using the above identity we prove the following lemma.

Lemma 2.11. *Let m and r be two positive integers. Then*:

- $L^r R_k = R_{k-2r} L^r + \frac{1}{4} r(k-r+1) L^{r-1}$.
- $L R_k^m = R_{k-2}^m L + \frac{1}{4} m(k+m-1) R_k^{m-1}$.
- *For $m \le r$ we have*
$$L^r R_k^m = R_{k-2r}^m L^r + c_1 R_{k-2r+2}^{m-1} L^{r-1} + c_2 R_{k-2r+4}^{m-2} L^{r-2} + \cdots + c_m L^{r-m},$$
where
$$c_i = \frac{1}{4^i} r(r-1)\cdots(r-i+1)(k+i-r)(k+i-r+1)\cdots(k+2i-r-1).$$
- *For any positive integer r and any nonnegative integer m, we have*
$$L^r R_k^{r+m} = R_{k+2m-2r}^r L^r R_k^m + c_1 R_{k+2m-2r+2}^{r-1} L^{r-1} R_k^m + \cdots + c_{r-1} R_{k+2m-2} L R_k^m + c_r R_k^m,$$
where c_i is as defined in the previous identity.

Proof. For $r = 1$ the first identity is true by (8). Then the first identity can be proved by using induction on r. Similarly for $m = 1$, the second identity is true by (8), and then the second identity can be proved by using induction on m. The third identity can be proved by using the first identity and induction on m. The fourth identity is a direct application of the third identity. □

Using the above lemma, we now prove the following result which is of independent interest and is also useful for our purposes.

Theorem 2.12. *Let $f \in S_k$ and $g \in \widehat{M}_l$. Assume that r and s are positive integers such that $k + 2r = l + 2s$. Then*
$$\langle R_k^r f, R_l^s g\rangle = \begin{cases} c_r \langle f, g\rangle & \text{if } r = s, \\ 0 & \text{if } r \ne s, \end{cases}$$
where
$$c_r = \frac{r!}{4^r} k(k+1)\cdots(k+r-1).$$

Proof. If $r = s$ then $k = l$ and by using Lemma 2.7, we obtain
$$\langle R_k^r f, R_k^r g\rangle = \langle f, L_k^r R_k^r g\rangle.$$
Using the fourth identity of Lemma 2.11 in the above expression we obtain
$$\langle R_k^r f, R_k^r g\rangle = \langle f, R_{k+2m-2r}^r L^r g + c_1 R_{k+2m-2r+2}^{r-1} L^{r-1} g + \cdots + c_{r-1} R_{k+2m-2} L g + c_r g\rangle.$$

Now applying Lemma 2.8 to the right-hand side of the above expression, we obtain the required result in this case. If $r \neq s$, without loss of any generality we may assume that $r < s$. Let $r = s + m$ for some positive integer m. Then again by the fourth identity of Lemma 2.11 we get

$$\langle R_k^r f, R_k^s g\rangle = \big\langle f, R_{l+2m-2r}^r L^r R_l^m g + c_1 R_{l+2m-2r+2}^{r-1} L^{r-1} R_l^m g + \cdots + c_{r-1} R_{l+2m-2} L R_l^m g + c_r R_l^m g\big\rangle.$$

Applying Lemma 2.8 to the right-hand side of the above expression, we deduce that

$$\langle R_k^r f, R_k^s g\rangle = 0. \qquad \square$$

Let $\widehat{S}_k$ be the subspace of $\widehat{M}_k$ consisting of rapidly decreasing functions. As an application of the above theorem, we have the following result.

Corollary 2.13. *There exists an orthogonal basis of $\widehat{S}_k$ consisting of Hecke eigenforms with respect to the Petersson inner product.*

Proof. Using the property of rapidly decreasing functions and the decomposition theorem for the space of nearly holomorphic modular forms, given in Theorem 2.3, it follows that

$$\widehat{S}_k = \bigoplus_{r=0}^{k/2-1} R_{k-2r}^r S_{k-2r}.$$

Since S_k has an orthogonal basis consisting of Hecke eigenforms with respect to the Petersson inner product, the result follows from Proposition 2.10 and Theorem 2.12. $\square$

3. Quasimodular forms

Definition 3.1. A holomorphic function f on $\mathcal{H}$ is called a quasimodular form of weight k and depth p for $\mathrm{SL}_2(\mathbb{Z})$ if there exist holomorphic functions f_0, f_1, $f_2, \ldots, f_p$ on $\mathcal{H}$ with moderate growth such that

$$(cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right) = \sum_{j=0}^{p} f_j(z)\left(\frac{c}{cz+d}\right)^j$$

for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, and f_p is not identically vanishing.

We denote by $\widetilde{M}_k^{\leq p}$ the space of all quasimodular forms of weight k and depth $\leq p$ for the full modular group $\mathrm{SL}_2(\mathbb{Z})$. We also denote by $\widetilde{M}_k = \bigcup_p \widetilde{M}_k^{\leq p}$ the space of all quasimodular forms of weight k. Any quasimodular form f of weight k and depth p for $\mathrm{SL}_2(\mathbb{Z})$ can be written as

$$f(z) = g_0(z) + g_1(z)E_2(z) + \cdots + g_p(z)E_2^p(z), \tag{9}$$

where $g_i \in M_{k-2i}$ for $0 \le i \le p$ and $g_p \not\equiv 0$. For any integer $n \ge 1$, the action of the Hecke operator T_n on a quasimodular form is the same as the action on a nearly holomorphic modular form as given in (7). For each integer $n \ge 1$, T_n maps $\widetilde{M}_k$ to itself. A quasimodular form is called an eigenform if it is an eigenvector for each Hecke operator T_n $(n \ge 1)$. We state the following result [Das and Meher 2015, Proposition 3.1] which characterises all quasimodular eigenforms for the full modular group $\mathrm{SL}_2(\mathbb{Z})$.

Proposition 3.2. *Let f be a quasimodular eigenform of weight k and depth p for $\mathrm{SL}_2(\mathbb{Z})$. If $p < k/2$ then $f = D^p f_p$, where $f_p \in M_{k-2p}$ is an eigenform, and if $p = k/2$ then $f \in \mathbb{C}D^{k/2-1}E_2$.*

We also recall the following results on quasimodular eigenforms [Kumar and Meher 2016, Lemma 4.3, 4.4].

Lemma 3.3. *If $f = \sum_{n=0}^{\infty} a(n)e^{2\pi inz} \in \widetilde{M}_k$ is a nonzero eigenform then $a(1) \neq 0$.*

Lemma 3.4. *A quasimodular eigenform $f \in \widetilde{M}_k$ with nonzero constant Fourier coefficient is an eigenform if and only if $f \in \mathbb{C}E_k$.*

Let $\widehat{M}_*^{\le p}$ be the space of all nearly holomorphic modular forms of depth at most p for the group $\mathrm{SL}_2(\mathbb{Z})$, and let $\widetilde{M}_*^{\le p}$ be the space of all quasimodular forms of depth at most p for the group $\mathrm{SL}_2(\mathbb{Z})$. Then there is an isomorphism between $\widehat{M}_*^{\le p}$ and $\widetilde{M}_*^{\le p}$ given in the next theorem [Ouled Azaiez 2008, Theorem 1].

Theorem 3.5. *The map*

$$f(z) = \sum_{j=0}^{p} \frac{f_j(z)}{\mathrm{Im}(z)^j} \mapsto f_0(z)$$

from $\widehat{M}_*^{\le p}$ to $\widetilde{M}_*^{\le p}$ is an isomorphism.

The map above induces a ring isomorphism between $\widehat{M}_*$ and $\widetilde{M}_*$. Also if $f \in M_k$, then the above isomorphism from $\widehat{M}_*$ and $\widetilde{M}_*$ maps $R_k^m f$ to $D^m f$ and $R_2^m E_2^*$ to $D^m E_2$ for any integer $m \ge 0$. Thus from Propositions 2.10 and 3.2 we have the following result.

Proposition 3.6. *A polynomial relation among eigenforms in $\widehat{M}_*$ gives rise to a corresponding polynomial relation in $\widetilde{M}_*$ and vice versa.*

4. Rankin–Cohen brackets and Rankin–Selberg L-functions

Rankin–Cohen brackets. Let F and G be two nearly holomorphic modular forms of weights k and l, and depths s and t, respectively, for the group $\mathrm{SL}_2(\mathbb{Z})$. Analogous to the Rankin–Cohen brackets defined for quasimodular forms in (2), we define the

Rankin–Cohen brackets for nearly holomorphic modular forms. For any integer $\nu \geq 0$, the ν-th Rankin–Cohen bracket of F and G is defined by

$$[F, G]_\nu := \sum_{\alpha=0}^{\nu} (-1)^\alpha \binom{k-s+\nu-1}{\nu-\alpha} \binom{l-t+\nu-1}{\alpha} (R_k^\alpha F)(R_l^{\nu-\alpha} G). \tag{10}$$

By abuse of notation, the ν-th Rankin–Cohen bracket of two nearly holomorphic modular forms is denoted by the same notation as the ν-th Rankin–Cohen bracket of two quasimodular forms.

Theorem 4.1. *Let F and G be as above. Then for any integer $\nu \geq 0$ we have $[F, G]_\nu \in \widehat{M}_{k+l+2\nu}^{\leq s+t}$.*

Proof. From the definition of Rankin–Cohen brackets in (10), it is easy to see that $[F, G]_\nu \in \widehat{M}_{k+l+2\nu}^{\leq s+t+\nu}$. Thus it remains to show that the depth of $[F, G]_\nu$ is in fact at most $s+t$. Let f and g be the respective constant coefficients of F and G when we write both F and G as polynomials in $1/\mathrm{Im}(z)$. Then we know that f and g are quasimodular forms of weights k and l and depths s and t, respectively. From (10) and (2) we see that if we write $[F, G]_\nu$ as a polynomial in $1/\mathrm{Im}(z)$, then the constant coefficient of $[F, G]_\nu$ is $[f, g]_\nu$. But we know that the depth of the quasimodular form $[f, g]_\nu$ is at most $s+t$. Hence by Theorem 3.5, the depth of $[F, G]_\nu$ is at most $s+t$. □

Rankin–Selberg L-functions. Let $f = \sum_{m=0}^{\infty} a(m) e^{2\pi i m z} \in M_k$. The L-function attached to f is defined by

$$L(f, s) = \sum_{m=1}^{\infty} \frac{a(m)}{m^s}.$$

If $f \in S_k$, then $L(f, s)$ is analytically continued to the whole complex plane and it satisfies the functional equation

$$L^*(f, s) := (2\pi)^{-s} \Gamma(s) L(f, s) = (-1)^{k/2} L^*(f, k-s).$$

If $f(z) = \sum_{m=0}^{\infty} a(m) e^{2\pi i m z}$ and $g(z) = \sum_{m=0}^{\infty} b(m) e^{2\pi i m z}$ are modular forms of weights k and l for $\mathrm{SL}_2(\mathbb{Z})$, respectively, then the Rankin–Selberg L-function associated with f and g is defined by

$$L(f \times g, s) := \sum_{m=1}^{\infty} \frac{a(m)\overline{b(m)}}{m^s}.$$

We recall a result of Zagier [1977, Proposition 6].

Theorem 4.2. *Let k_1, k_2, k and n be integers satisfying $k_2 \geq k_1 + 2 > 2$ and $k = k_1 + k_2 + 2n$. Let $f(z) = \sum_{m=1}^{\infty} a(m) e^{2\pi i m z} \in S_k$ and $g(z) = \sum_{m=0}^{\infty} b(m) e^{2\pi i m z} \in M_{k_1}$.*

Then

$$\langle f, [g, E_{k_2}]_n\rangle = (-1)^n \frac{\Gamma(k-1)\Gamma(k_2+n)}{(4\pi)^{k-1} n!\Gamma(k_2)} L(f \times g, k_1+k_2+n-1).$$

Remark 4.3. In the hypothesis of Theorem 4.2, the condition $k_2 \geq k_1+2 > 2$ can be removed if g is a cusp form.

If $g = E_{k_1}$, then we have the following result; see [Lanphier and Takloo-Bighash 2004, Theorem 2.2].

Theorem 4.4. *Let $k_1, k_2 \geq 4$ be even integers and let n be a nonnegative integer and $k = k_1+k_2+2n$. Suppose that $f \in S_k$ is a normalized eigenform. Then*

$$\langle f, [E_{k_1}, E_{k_2}]_n\rangle = (-1)^{k_2/2+n} \frac{2k_1}{B_{k_1}} \frac{2k_2}{B_{k_2}} \frac{\Gamma(k-1)}{n! 2^{k-1}\Gamma(k-n-1)} L^*(f, k-n-1) L^*(f, k_2+n).$$

Remark 4.5. Note that we have an extra $(-1)^n$ appearing in the right-hand sides of both the expressions given in Theorems 4.2 and 4.4. This is because an extra $(-1)^n$ appears in the definition of Rankin–Cohen brackets given in [Zagier 1977].

We now recall an interesting nonvanishing result of the L-function $L(f, s)$ associated with the cusp form f [Lanphier and Takloo-Bighash 2004, Corollary 3.2] at the center critical point.

Lemma 4.6. *Suppose that $k > 20$ and $k \equiv 0 \pmod 4$. Then there are two eigenforms $f, g \in S_k$ with $L^*(f, k/2) \neq 0$ and $L^*(g, k/2) \neq 0$.*

Remark 4.7. We know that $[E_4, E_4]_2 = 4800\Delta_{12}$ for some nonzero constant $c \in \mathbb{R}$. Also we have

$$\langle \Delta_{12}, \Delta_{12}\rangle = 4800\langle \Delta_{12}, [E_4, E_4]_2\rangle \neq 0.$$

Thus by Theorem 4.4, $L^*(\Delta_{12}, 6) \neq 0$. Similarly one proves that $L^*(\Delta_{16}, 8) \neq 0$ and $L^*(\Delta_{20}, 10) \neq 0$. Therefore by Lemma 4.6 we deduce that for each integer $k \geq 12$ with $k \equiv 0 \pmod 4$, there exists a nonzero eigenform $f \in S_k$ such that $L^*(f, k/2) \neq 0$.

5. Preparatory results

We start the section with the following result of Shimura [1976, Theorem 2], who has proved the result for modular forms of higher level with characters. Here we state the result for the group $\mathrm{SL}_2(\mathbb{Z})$ for our purpose.

Theorem 5.1. *Suppose $f \in S_k$, $g \in M_{k_1}$, and $k_1 + 2r_2 < k$ with a nonnegative integer r_2. Then*

$$\langle f, g \cdot R_{k_2}^{r_2} E_{k_2}\rangle = cL(f \times g, k-1-r_2),$$

where $k_2 = k - k_1 - 2r_2$, and $c = \Gamma(k-1-r_2)\Gamma(k-k_1-r_2)/\Gamma(k-k_1-2r_2)$.

The following result generalizes Theorem 5.1 and may be of independent interest. We follow the idea of Shimura to prove the result. We obtain Theorem 4.2 as a special case of the following result.

Theorem 5.2. *Let $k_1, k_2, k, r_1, r_2, \nu$ be nonnegative integers such that $k_2 \geq 4$, $k + 2r = k_1 + k_2 + 2r_1 + 2r_2 + 2\nu$. Suppose that $f = \sum_{n=1}^{\infty} a(n)e^{2\pi inz} \in S_k$ and $g = \sum_{n=0}^{\infty} b(n)e^{2\pi inz} \in M_{k_1}$. Assume that either g is a cusp form or $k_2 \geq k_1+2$. Then we have*

$$\langle R_k^r f, [R_{k_1}^{r_1} g, R_{k_2}^{r_2} E_{k_2}]_\nu \rangle = c(k, r; k_1, r_1, k_2, r_2) \cdot L\Big(f \times g, \frac{k}{2} + \frac{k_1}{2} + \frac{k_2}{2} - 1\Big), \tag{11}$$

where

$$c(k,r;k_1,r_1,k_2,r_2) = \frac{(-1)^{r_2+\nu}}{(4\pi)^{k+2r-1}} \sum_{\alpha=0}^{\nu} A_\alpha \sum_{u=0}^{r} \sum_{v=0}^{r_1+\alpha} (-1)^{-u-v} P_{u,k}^{(r)} P_{v,k_1}^{(r_1+\alpha)} \Gamma(k+2r-r_2-\nu+\alpha-u-v-1),$$

with

$$A_\alpha = \binom{k_1+r_1+\nu-1}{\nu-\alpha} \binom{k_2+r_2+\nu-1}{\alpha} \frac{\Gamma(k_2 + r_2 + \nu - \alpha)}{\Gamma(k_2)}$$

and

$$P_{u,k}^{(r)} = \binom{r}{u} \frac{\Gamma(k+r)}{\Gamma(k+r-u)}.$$

Moreover, for $r = 0$ we have

$$\begin{aligned} &c(k, 0; k_1, r_1, k_2, r_2) \\ &\quad = \frac{(-1)^{r_2+\nu}}{(4\pi)^{k-1}} \frac{\Gamma(k_2 + r_1 + r_2 + \nu)\Gamma(k_1 + k_2 + r_1 + r_2 + 2\nu - 1)}{\Gamma(k_2)\Gamma(\nu + 1)} \neq 0. \end{aligned} \tag{12}$$

Proof. Using the definitions of Rankin–Cohen brackets and the Petersson inner product we have

$$\begin{aligned} \langle R_k^r f, [R_{k_1}^{r_1} g, R_{k_2}^{r_2} E_{k_2}]_\nu \rangle &= \sum_{\alpha=0}^{\nu} (-1)^\alpha \binom{k_1+r_1+\nu-1}{\nu-\alpha} \binom{k_2+r_2+\nu-1}{\alpha} \\ &\quad \times \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}} R_k^r f \; \overline{R_{k_1}^{r_1+\alpha} g} \; \overline{R_{k_2}^{r_2+\nu-\alpha} E_{k_2}} y^{k+2r} \frac{dx\, dy}{y^2}. \end{aligned}$$

Using the identity (4) for $R_{k_2}^{r_2+\nu-\alpha} E_{k_2}$ in the above expression we obtain

$$\begin{aligned} &\langle R_k^r f, [R_{k_1}^{r_1} g, R_{k_2}^{r_2} E_{k_2}]_\nu \rangle = \\ &\sum_{\alpha=0}^{\nu} (-1)^{r_2+\nu} \frac{A_\alpha}{(4\pi)^{r_2+\nu-\alpha}} \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}} \sum_{\gamma \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} R_k^r f \, \overline{R_{k_1}^{r_1+\alpha} g} \\ &\qquad \times \overline{j(\gamma, z)^{-k_2-2r_2-2\nu+2\alpha} |j(\gamma, z)^{-2(-r_2-\nu+\alpha)}|} y^{k+2r-r_2-\nu+\alpha} \frac{dx\, dy}{y^2}. \end{aligned} \tag{13}$$

To interchange the sum and integral in (13), we observe that

$$\int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}} \sum_{\gamma\in\Gamma_\infty\backslash \mathrm{SL}_2(\mathbb{Z})} \left| R_k^r f \ \overline{R_{k_1}^{r_1+\alpha} g} \ \overline{j(\gamma,z)^{-k_2-2r_2-2\nu+2\alpha} |j(\gamma,z)|^{2(r_2+\nu-\alpha)}} \right| \times y^{k+2r-r_2-\nu+\alpha} \frac{dx\, dy}{y^2}$$

$$\leq \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}} \left| y^{\frac{1}{2}(k+2r+k_1+2r_1+2\alpha)} R_k^r f R_{k_1}^{r_1+\alpha} g \right| \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} |j(\gamma,z)|^{-k_2} \ y^{k_2/2} \frac{dx\, dy}{y^2}.$$

For $k_2 \geq 4$ note that

$$\sum_{\gamma\in\Gamma_\infty\backslash \mathrm{SL}_2(\mathbb{Z})} |j(\gamma,z)|^{-k_2} \leq \zeta(k_2-1).$$

Also, since f is a cusp form, $R_k^r f$ is a rapidly decreasing function and so is $(R_k^r f)(R_{k_1}^{r_1+\alpha} g)$. Hence, the integral in the right-hand side of above inequality is a finite quantity. Interchanging the sum and integral in (13), we obtain

$$\langle R_k^r f, [R_{k_1}^{r_1} g, R_{k_2}^{r_2} E_{k_2}]_\nu\rangle = (-1)^{r_2+\nu} \sum_{\alpha=0}^{\nu} \frac{A_\alpha}{(4\pi)^{r_2+\nu-\alpha}} \sum_{\gamma\in\Gamma_\infty\backslash \mathrm{SL}_2(\mathbb{Z})} \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathcal{H}} R_k^r f(z) \ \overline{R_{k_1}^{r_1+\alpha} g(z)} \times \overline{j(\gamma,z)^{-k_2-2r_2-2\nu+2\alpha} |j(\gamma,z)|^{-2(-r_2-\nu+\alpha)}} \, y^{k+2r-r_2-\nu+\alpha} \frac{dx\, dy}{y^2}.$$

Changing the variable $z \mapsto \gamma^{-1} z$ in the above expression and unfolding we obtain

$$\langle R_k^r f, [R_{k_1}^{r_1} g, R_{k_2}^{r_2} E_{k_2}]_\nu\rangle = (-1)^{r_2+\nu} \sum_{\alpha=0}^{\nu} \frac{A_\alpha}{(4\pi)^{r_2+\nu-\alpha}} \int_0^\infty \int_0^1 R_k^r f(z) \overline{R_{k_1}^{r_1+\alpha} g(z)} \ y^{k+2r-r_2-\nu+\alpha-2} dx\, dy.$$

From the Fourier expansions of f and g we get

$$\text{(14)} \quad \langle R_k^r f, [R_{k_1}^{r_1} g, R_{k_2}^{r_2} E_{k_2}]_\nu\rangle = (-1)^{r_2+\nu} \sum_{\alpha=0}^{\nu} \frac{A_\alpha}{(4\pi)^{r_2+\nu-\alpha}} \sum_{u=0}^{r} P_{u,k}^{(r)} \sum_{v=0}^{r_1+\alpha} P_{v,k_1}^{(r_1+\alpha)} (-4\pi)^{-u-v} \times \int_0^\infty \int_0^1 \sum_{m\geq 1,\, n\geq 0} a(m)\overline{b(n)} m^{r-u} n^{r_1+\alpha-v} e^{2\pi i x(m-n)} e^{-2\pi y(m+n)} \times y^{k+2r-r_2-\nu+\alpha-2-u-v}\, dx\, dy.$$

Since either g is a cusp form or $k_2 \geq k_1+2$, by using the bounds of Fourier coefficients we can interchange the sum and integration of the above expression, and using the fact that the integral over the variable x will be nonzero only when $m=n$, we obtain

$$\begin{aligned}\langle R_k^r f, [R_{k_1}^{r_1} g, R_{k_2}^{r_2} E_{k_2}]_\nu\rangle &= (-1)^{r_2+\nu}\sum_{\alpha=0}^{\nu}\frac{A_\alpha}{(4\pi)^{r_2+\nu-\alpha}}\sum_{u=0}^{r}P_{u,k}^{(r)}\sum_{v=0}^{r_1+\alpha}P_{v,k_1}^{(r_1+\alpha)}(-4\pi)^{-u-v}\\ &\quad \sum_{m\ge 1}a(m)\overline{b(m)}m^{r+r_1+\alpha-u-v}\int_0^\infty e^{-4\pi y m}\, y^{k+2r-r_2-\nu+\alpha-2-u-v}\,dy.\end{aligned}$$

Using the definition of the Gamma function above gives (11). It remains to simplify the constant for $r=0$. The proof is straightforward and purely combinatorial. We use the following two binomial identities, which hold for nonnegative integers x, j, n with $x, j \ge n$. The first identity (see [Quaintance and Gould 2016, pp. 74]) is

$$\sum_{i=0}^{n}(-1)^i\binom{n}{i}\binom{x-i}{j}=\binom{x-n}{j-n} \tag{15}$$

and second is the well-known Vandermonde's identity, given by

$$\sum_{k=0}^{n}\binom{x}{k}\binom{j}{n-k}=\binom{x+j}{n}. \tag{16}$$

In the expression of $c(k,0;k_1,r_1,k_2,r_2)$, we first apply (15) to the sum over the variable v, simplify the obtained expression and then use (16) to obtain its required form as given in (12). □

Next we prove a result similar to Theorem 5.2 in the case when g is the Eisenstein series E_{k_1} and $k_1, k_2 \ge 4$ are any even integers. We first recall a result of Diamantis and O'Sullivan [2010, Proposition 2.1].

Lemma 5.3. *Let k_1, k_2 be even and nonnegative with $k=k_1+k_2$. Then for any normalized eigenform $f\in S_k$ and for all $s, w\in\mathbb{C}$, we have*

$$\langle f, y^{-k/2}E_{k_1}^*(z,\bar u)E_{k_2}^*(z,\bar v)\rangle = (-1)^{k_2/2}2\pi^{k/2}L^*(f,s)L^*(f,w),$$

where $2u=s+w-k+1$ and $2v=-s+w+1$.

We show the following result.

Theorem 5.4. *Let k_1, k_2 be even nonnegative integers and $r_1, r_2\ge 0$ be integers. For any integer $\nu\ge 0$, let $k=k_1+k_2+2r_1+2r_2+2\nu$. Then for any normalized eigenform $f\in S_k$, we have*

$$\begin{aligned}\langle f, [R_{k_1}^{r_1}E_{k_1}, R_{k_2}^{r_2}E_{k_2}]_\nu\rangle &= c(k;k_1,r_1,k_2,r_2)\cdot L^*\Big(f,\frac{k}{2}+\frac{k_1}{2}-\frac{k_2}{2}\Big)L^*\Big(f,\frac{k}{2}+\frac{k_1}{2}+\frac{k_2}{2}-1\Big),\end{aligned}$$

where

$$c(k; k_1, r_1, k_2, r_2) = \frac{(-1)^{k_1/2-r_1}}{2^{k-1}} \frac{2k_1}{B_{k_1}} \frac{2k_2}{B_{k_2}} \binom{k_1+k_2+r_1+r_2+2\nu-2}{\nu}.$$

Proof. Using the definition of Rankin–Cohen brackets

$$\begin{aligned}(17)\quad &\langle f, [R_{k_1}^{r_1} E_{k_1}, R_{k_2}^{r_2} E_{k_2}]_\nu \rangle \\ &= \sum_{\alpha=0}^{\nu} (-1)^\alpha \binom{k_1+r_1+\nu-1}{\nu-\alpha} \binom{k_2+r_2+\nu-1}{\alpha} \langle f, (R_{k_1}^{r_1+\alpha} E_{k_1})(R_{k_2}^{r_2+\nu-\alpha} E_{k_2}) \rangle.\end{aligned}$$

For any $0 \le \alpha \le \nu$, by (4) we write

$$\begin{aligned}&\langle f, (R_{k_1}^{r_1+\alpha} E_{k_1})(R_{k_2}^{r_2+\nu-\alpha} E_{k_2}) \rangle \\ &\quad = (-4\pi)^{-r_1-r_2-\nu} \frac{\Gamma(k_1+r_1+\alpha)\Gamma(k_2+r_2+\nu-\alpha)}{\Gamma(k_1)\Gamma(k_2)} \\ &\qquad \times \langle f, y^{-r_1-r_2-\nu} E_{k_1+2r_1+2\alpha}(z, -r_1-\alpha) E_{k_2+2r_2+2\nu-2\alpha}(z, -r_2-\nu+\alpha) \rangle.\end{aligned}$$

Using the relation given in (6) between the Eisenstein series $E_k(z, s)$ and the completed Eisenstein series $E^*(z, s)$, the above identity can be rewritten as

$$\begin{aligned}&\langle f, (R_{k_1}^{r_1+\alpha} E_{k_1})(R_{k_2}^{r_2+\nu-\alpha} E_{k_2}) \rangle \\ &\qquad = \frac{(-4\pi)^{-r_1-r_2-\nu}\Gamma(k_1+r_1+\alpha)\Gamma(k_2+r_2+\nu-\alpha)}{\Gamma(k_1)\Gamma(k_2)\theta_{k_1+2r_1+2\alpha}(k_1/2)\theta_{k_2+2r_2+2\nu-2\alpha}(k_2/2)} \\ &\qquad\qquad \times \langle f, y^{-k/2} E^*_{k_1+2r_1+2\alpha}(z, k_1/2) E^*_{k_2+2r_2+2\nu-2\alpha}(z, k_2/2) \rangle.\end{aligned}$$

Now using Lemma 5.3 in the above identity and substituting it into (17), we obtain

$$\begin{aligned}&\langle f, [R_{k_1}^{r_1} E_{k_1}, R_{k_2}^{r_2} E_{k_2}]_\nu \rangle \\ &\qquad = c(k; k_1, r_1, k_2, r_2) \cdot L^*\Big(f, \frac{k}{2} + \frac{k_1}{2} - \frac{k_2}{2}\Big) L^*\Big(f, \frac{k}{2} + \frac{k_1}{2} + \frac{k_2}{2} - 1\Big),\end{aligned}$$

where

$$\begin{aligned}&c(k; k_1, r_1, k_2, r_2) \\ &\quad = \sum_{\alpha=0}^{\nu} (-1)^\alpha \binom{k_1+r_1+\nu-1}{\nu-\alpha} \binom{k_2+r_2+\nu-1}{\alpha} \frac{(-1)^{k_2/2+r_1+\alpha} 2\pi^{k_1+k_2}}{4^{r+r_2+\nu}\zeta(k_1)\zeta(k_2)\Gamma(k_1)\Gamma(k_2)}.\end{aligned}$$

Using (16) we further simplify the above expression and deduce that

$$c(k; k_1, r_1, k_2, r_2) = \frac{2(-1)^{k_2/2-r_1}\pi^{k_1+k_2}}{4^{r_1+r_2+\nu}\Gamma(k_1)\Gamma(k_2)\zeta(k_1)\zeta(k_2)} \binom{k_1+k_2+r_1+r_2+2\nu-2}{\nu}.$$

For any positive even integer m, we get the required result by using the well-known relation $\zeta(m) = -\frac{1}{2}(2\pi i)^m B_m/m!$, in the above expression. □

We also need the following result which we will use in the proof of Theorem 1.1.

Theorem 5.5. *Let k, k_1, k_2, r_1, r_2 and ν be as in the Theorem 5.4. Then for any normalized eigenform $f \in S_{k-2}$ we have*

$$\langle R_{k-2} f, [R_{k_1}^{r_1} E_{k_1}, R_{k_2}^{r_2} E_{k_2}]_\nu\rangle = c^1(k; k_1, r_1, k_2, r_2) \cdot L^*\Big(f, \frac{k}{2} + \frac{k_1}{2} - \frac{k_2}{2} - 1\Big) L^*\Big(f, \frac{k}{2} + \frac{k_1}{2} + \frac{k_2}{2} - 1\Big),$$

where

$$c^1(k; k_1, r_1, k_2, r_2) = \frac{(-1)^{k_1/2}}{2^{k-1}} \frac{2k_1}{B_{k_1}} \frac{2k_2}{B_{k_2}} \sum_{\alpha=0}^{\nu} A_\alpha t_\alpha,$$

with $t_\alpha = (-1)^{r_1-1}(r_1+\alpha)(k_1+r_1+\alpha-1) + (-1)^{r_2+\nu-1}(r_2+\nu-\alpha)(k_2+r_2+\nu-\alpha-1)$ *and*

$$A_\alpha = \binom{k_1+r_1+\nu-1}{\nu-\alpha}\binom{k_2+r_2+\nu-1}{\alpha}.$$

Furthermore, we have

$$c^1(k; k_1, 0, k_2, r_2) \neq 0 \quad \textit{for } r_2 \neq 0. \tag{18}$$

Proof. Using the definition of Rankin–Cohen bracket and Lemma 2.7 we have

$$\langle R_{k-2} f, [R_{k_1}^{r_1} E_{k_1}, R_{k_2}^{r_2} E_{k_2}]_\nu\rangle = \sum_{\alpha=0}^{\nu} (-1)^\alpha A_\alpha \big\{\langle f, (L R_{k_1}^{r_1+\alpha} E_{k_1})(R_{k_2}^{r_2+\nu-\alpha} E_{k_2})\rangle + \langle f, (R_{k_1}^{r_1+\alpha} E_{k_1})(L R_{k_2}^{r_2+\nu-\alpha} E_{k_2})\rangle\big\}.$$

Now using the second identity of Lemma 2.11 and then Lemma 2.8 in the last two inner products, we obtain

$$\langle R_{k-2} f, [R_{k_1}^{r_1} E_{k_1}, R_{k_2}^{r_2} E_{k_2}]_\nu\rangle = \sum_{\alpha=0}^{\nu} \frac{(-1)^\alpha A_\alpha}{4} \Big\{(r_1+\alpha)(k_1+r_1+\alpha-1)\langle f, (R_{k_1}^{r_1+\alpha-1} E_{k_1})(R_{k_2}^{r_2+\nu-\alpha} E_{k_2})\rangle + (r_2+\nu-\alpha)(k_2+r_2+\nu-\alpha-1)\langle f, (R_{k_1}^{r_1+\alpha} E_{k_1})(R_{k_2}^{r_2+\nu-\alpha-1} E_{k_2})\rangle\Big\}.$$

Applying the same method as used in the proof of Theorem 5.4 for both the terms on the right-hand side of the above identity, we get the main result. To complete the proof, we prove (18) by using simple combinatorial methods. Let $r_1 = 0$ and $r_2 \neq 0$. If $r_2 + \nu$ is even or $\nu = 0$, the result follows trivially and hence we assume that $r_2 + \nu$ is odd and $\nu \geq 1$. After simplifying the expression for t_α, we see that

$$\sum_{\alpha=0}^{\nu} A_\alpha t_\alpha = -(k-2) \sum_{\alpha=0}^{\nu} A_\alpha \alpha + (r_2+\nu)(k_2+r_2+\nu-1) \sum_{\alpha=0}^{\nu} A_\alpha.$$

Using Vandermonde's identity (16) for both the sums in the above expression, a simple calculation gives

$$\sum_{\alpha=0}^{\nu} A_\alpha t_\alpha = \frac{r_2(k_2+r_2+\nu-1)(k-\nu-r_2-2)}{\nu}\binom{k_1+k_2+r_2+2\nu-3}{\nu-1},$$

which is nonzero. This completes the proof. □

6. Proof of Theorem 1.1

By Proposition 3.2, Lemma 3.3 and Lemma 3.4, to prove Theorem 1.1 we need to check only whether the following cases give eigenforms:

- $[E_{k_1}, D^{r_2}E_{k_2}]_\nu$ for $k_1 \neq k_2$.
- $[E_{k_1}, D^{r_1}E_{k_1}]_\nu$.
- $[D^{r_1}f, E_{k_2}]_\nu$, where $f \in S_{k_1}$ is an eigenform.

By Theorem 3.5 it is equivalent to check the following cases of Rankin–Cohen brackets of nearly holomorphic modular forms are eigenforms:

- $[E_{k_1}, R_{k_2}^{r_2}E_{k_2}]_\nu$ for $k_1 \neq k_2$.
- $[E_{k_1}, R_{k_1}^{r_1}E_{k_1}]_\nu$.
- $[R_{k_1}^{r_1}f, E_{k_2}]_\nu$, where $f \in S_{k_1}$ is an eigenform.

Consider the first case. Let $[E_{k_1}, R_{k_2}^{r_2}E_{k_2}]_\nu$ be an eigenform, where $k_1 \neq k_2$. Put $k = k_1+k_2+2r_2+2\nu$ and $a = k_1+k_2+r_2+\nu$. By Proposition 2.10 we have

(19) $$[E_{k_1}, R_{k_2}^{r_2}E_{k_2}]_\nu = R_{k-2r'}^{r'}g,$$

where $g \in M_{k-2r'}$ is an eigenform and $r' \geq 0$ is an integer.

If $k = 24$ or $k \geq 28$, then the dimension of S_k is at least 2. Therefore if $r' = 0$ and $k = 24$ or $k \geq 28$ in (19), there exists a nonzero eigenform $h \in S_k$ such that

$$\langle h, g\rangle = \langle h, [E_{k_1}, R_{k_2}^{r_2}E_{k_2}]_\nu\rangle = 0.$$

Then by Theorem 5.4 we deduce that

(20) $$L(h, a-1)L(h, a-k_1) = 0.$$

Since $L(h,s)$ has an Euler product in the region $\mathrm{Re}(s) > \frac{k+1}{2}$, $L(h,s)$ does not vanish for $\mathrm{Re}(s) > \frac{k+1}{2}$. We see that $a-1 > \frac{k+1}{2}$ and therefore $L(h, a-1) \neq 0$. We will also prove that $L(h, a-k_1) \neq 0$. First assume that $k_2 > k_1$. Since $k_2 > k_1$ and k_1 and k_2 are even positive numbers, we have $k_1 < k_2+1$. Thus $L(h, a-k_1) \neq 0$ as $a-k_1 > \frac{k+1}{2}$. If $k_1 > k_2$, then $k-a+k_1 > \frac{k+1}{2}$, and by the functional equation of L-functions, we deduce that $L(h, a-k_1) \neq 0$. Therefore, the above discussion gives a contradiction to (20). Thus if $r' = 0$ and $k = 24$ or $k \geq 28$, $[E_{k_1}, R_{k_2}^{r_2}E_{k_2}]_\nu$ is not an eigenform whenever $k_1 \neq k_2$.

For $r' = 0$, $k \neq 24$ and $k < 28$, we have finitely many cases to verify. We find that there are the following cases for which Rankin–Cohen brackets of nearly holomorphic eigenforms give rise to eigenforms:

- The holomorphic modular cases listed in Theorem 1.1 for which $k_1 \neq k_2$.
- The nonholomorphic cases given by $[E_4, R_8 E_8]_1 = [E_8, R_4 E_4]_1 = 1920\Delta_{16}$, $[E_4, R_6 E_6]_2 = -5040\Delta_{16}$, $[E_6, R_4 E_4]_2 = 5040\Delta_{16}$.

By Theorem 3.5, we get the corresponding cases for quasimodular eigenforms.

If $r' \geq 1$ and $k \neq 14$, by employing Lemma 2.8 as done in the case when $r' = 0$ and $k = 24$ or $k \geq 28$, we deduce that the Rankin–Cohen brackets of eigenforms do not result in eigenforms. If $r' \geq 1$ and $k = 14$, we get the following cases for which we get eigenforms:

$$[E_6, DE_4]_1 = 1440 D\Delta_{12}, \quad [E_4, DE_6]_1 = -2016 D\Delta_{12}.$$

Now consider the second case. Assume that

$$[E_{k_1}, R_{k_1}^{r_1} E_{k_1}]_\nu = R_{k-2r'}^{r'} f, \tag{21}$$

where $f \in M_{k-2r'}$ is an eigenform and $r' \geq 0$ is an integer. Put $k = 2k_1 + 2r_1 + 2\nu$.

If $\nu = 0$, the Rankin–Cohen bracket reduces to the product of two nearly holomorphic eigenforms. This has been done in [Kumar and Meher 2016]. By Theorem 3.5 we see that the only case for which the product of quasimodular eigenforms is an eigenform, is

$$E_4(DE_4) = \tfrac{1}{2} DE_8.$$

Assume that $\nu \geq 1$. If $r' = 0$, comparing the Fourier expansion of both sides of (21), we deduce that f has to be a cusp form. Since $\langle f, f \rangle \neq 0$, we have

$$\langle f, [E_{k_1}, R_{k_1}^{r_1} E_{k_1}]_\nu \rangle \neq 0.$$

Thus by Theorem 5.4 we have

$$L^*(f, k/2 + k_1 - 1) L^*(f, k/2) \neq 0.$$

Since $k/2 + k_1 - 1$ lies in the region in which $L(f, s)$ has an Euler product, we have $L(f, k/2 + k_1 - 1) \neq 0$. Thus $L^*(f, k/2) \neq 0$. From the functional equation of $L(f, s)$ we see that $L^*(f, k/2) = 0$ if $k \equiv 2 \pmod 4$. Therefore $k \equiv 0 \pmod 4$. If $k > 20$ and $k \equiv 0 \pmod 4$, by Lemma 4.6, there exist two eigenforms $g, h \in S_k$ such that

$$\langle g, f \rangle \neq 0 \quad \text{and} \quad \langle h, f \rangle \neq 0.$$

This contradicts the fact that f is an eigenform. If $k \leq 20$, there are only finitely many cases to verify, and we obtain the following cases for which the Rankin–Cohen brackets of two nearly holomorphic eigenforms give rise to eigenforms:

- The holomorphic modular cases listed in Theorem 1.1.

- The nonholomorphic modular cases

$$[E_4, R_4E_4]_1 = 960\Delta_{12}, \quad [E_6, R_6E_6]_1 = -3024\Delta_{16},$$
$$[E_4, R_4E_4]_3 = 4800\Delta_{16}, \quad [E_8, R_8E_8]_1 = 3840\Delta_{20},$$
$$[E_6, R_6E_6]_3 = -28224\Delta_{20}, \quad [E_4, R_4E_4]_5 = 13440\Delta_{20}.$$

Then by Theorem 3.5 we get the corresponding result for quasimodular eigenforms.

Let $r' \geq 1$. By (21) and Lemma 2.8, for any eigenform $g \in S_k$ we have

$$\langle g, [E_{k_1}, R_{k_1}^{r_1} E_{k_1}]_\nu \rangle = 0.$$

As done in the case when $r' = 0$, we deduce that $L^*(g, k/2) = 0$. By Remark 4.7 this implies that $k \equiv 2 \pmod 4$. If $r' = 1$ then $[E_{k_1}, R_{k_1}^{r_1} E_{k_1}]_\nu = R_{k-2} f$ and $k-2 \equiv 0 \pmod 4$. Also if $k-2 > 20$ and $k-2 \equiv 0 \pmod 4$, by Lemma 4.6 there exist two normalized eigenforms f_1 and f_2 in S_{k-2} such that

$$L^*\Big(f_1, \frac{k-2}{2}\Big) \neq 0 \quad \text{and} \quad L^*\Big(f_2, \frac{k-2}{2}\Big) \neq 0.$$

Then by Theorem 2.12 we have

$$\langle f_1, f \rangle = \frac{1}{c_1} \langle R_{k-2} f_1, R_{k-2} f \rangle = \frac{1}{c_1} \langle R_{k-2} f_1, [E_{k_1}, R_{k_1}^{r_1} E_{k_1}]_\nu \rangle,$$

and

$$\langle f_2, f \rangle = \frac{1}{c_1} \langle R_{k-2} f_2, R_{k-2} f \rangle = \frac{1}{c_1} \langle R_{k-2} f_1, [E_{k_1}, R_{k_1}^{r_1} E_{k_1}]_\nu \rangle.$$

Thus by applying Theorem 5.5 (in view of (18)) we deduce that there are two normalized eigenforms f_1 and f_2 in S_{k-2} such that

$$\langle f_1, f \rangle \neq 0 \quad \text{and} \quad \langle f_2, f \rangle \neq 0.$$

This gives a contradiction.

If $k-2 \leq 20$, we verify the finitely many remaining cases and deduce that if $r' = 1$ and $\nu \geq 1$, we get the following cases for which Rankin–Cohen brackets of two nearly holomorphic modular forms are again eigenforms:

- The holomorphic modular cases listed in Theorem 1.1.
- The nonholomorphic cases:

$$[E_4, R_4^2 E_4]_1 = 960 R_{12}\Delta_{12}, \quad [E_4, R_4E_4]_2 = 2400 R_{12}\Delta_{12},$$
$$[E_6, R_6^2 E_6]_1 = -3024 R_{16}\Delta_{16}, \quad [E_6, R_6E_6]_2 = -10584 R_{16}\Delta_{16},$$
$$[E_4, R_4^2 E_4]_3 = 4800 R_{16}\Delta_{16}, \quad [E_4, R_4E_4]_4 = 8400 R_{16}\Delta_{16},$$

$$[E_8, R_8^2 E_8]_1 = 3840 R_{20}\Delta_{20}, \qquad [E_8, R_8 E_8]_2 = 17280 R_{20}\Delta_{20},$$
$$[E_6, R_6^2 E_6]_3 = -28224 R_{20}\Delta_{20}, \quad [E_6, R_6 E_6]_4 = -63504 R_{20}\Delta_{20},$$
$$[E_4, R_4^2 E_4]_5 = 13440 R_{20}\Delta_{20}, \qquad [E_4, R_4 E_4]_6 = 20160 R_{20}\Delta_{20}.$$

By Theorem 3.5 we have the corresponding cases for quasimodular forms.

Let $r' \geq 2$. If $g \in S_{k-2}$ is any eigenform, then Theorem 2.12 implies that

$$\langle R_{k-2} g, [E_{k_1}, R_{k_1}^{r_1} E_{k_1}]_\nu \rangle = \langle R_{k-2} g, R_{k-2r'}^{r'} f \rangle = 0.$$

Thus by Theorem 5.5 (in view of (18)), the above identity implies $L^*\big(g, \frac{k-2}{2}\big) = 0$. We have already proved that if $r' \geq 1$, then $k \equiv 2 \pmod 4$. Since $k-2 \equiv 0 \pmod 4$ and g is an arbitrary eigenform, if $k-2 > 20$, Lemma 4.6 gives a contradiction. If $k-2 \leq 20$, by checking the remaining finitely many cases, we deduce that if $r' \geq 2$, we do not get any case where Rankin–Cohen brackets of two nearly holomorphic eigenforms give rise to eigenforms. Thus by Theorem 3.5, we get the corresponding result in the case of quasimodular forms.

Now consider the third case. Let $[R_{k_1}^{r_1} f, E_{k_2}]_\nu$ be an eigenform, where $f \in S_k$ is an eigenform. Let $k = k_1 + k_2 + 2r_1 + 2\nu$. By Proposition 2.10 we have

$$[R_{k_1}^{r_1} f, E_{k_2}]_\nu = R_{k-2r'}^{r'} g,$$

where r' is a nonnegative integer and $g \in M_{k-2r'}$ is an eigenform. If either $k = 24$ or $k \geq 28$, the dimension of S_k is at least 2. Therefore if $r' = 0$ and either $k = 24$ or $k \geq 28$, there exists a nonzero eigenform $h \in S_k$ such that

$$\langle h, g \rangle = \langle h, [R_{k_1}^{r_1} f, E_{k_2}]_\nu \rangle = 0.$$

Applying Theorem 5.2 (in view of (12)), we get

$$L\Big(h \times f, \frac{k}{2} + \frac{k_1}{2} + \frac{k_2}{2} - 1\Big) = 0. \tag{22}$$

Since h and f are both eigenforms of weight k and k_1, respectively, $L(h \times f, s)$ has an Euler product in the region $\mathrm{Re}(s) > \frac{k}{2} + \frac{k_1}{2}$ and hence $L\big(h \times f, \frac{k}{2} + \frac{k_1}{2} + \frac{k_2}{2} - 1\big) \neq 0$. This contradicts (22). Therefore, when $r' = 0$ and either $k = 24$ or $k \geq 28$, the Rankin–Cohen brackets do not result in eigenforms. If $r' = 0$, $k \neq 24$ and $k < 28$, we verify these finitely many cases and deduce that we obtain only the modular cases listed in Theorem 1.1. If $r' \geq 1$, by employing Lemma 2.8 as done in the case when $r' = 0$ and either $k = 24$ or $k \geq 28$, we deduce that the Rankin–Cohen brackets of eigenforms do not result in eigenforms. This proves Theorem 1.1.

7. Further remarks

Although Theorem 1.1 is a result about quasimodular eigenforms, it is clear from the proof of Theorem 1.1 that one can state a similar result in the case of nearly holomorphic eigenforms. The result in the case of nearly holomorphic eigenforms is a generalization of the main result of [Beyerl et al. 2012] to the case of Rankin–Cohen brackets. Thus in this way we give a different proof of the main result of [Beyerl et al. 2012]. Our proof has the same flavor as the proofs of the main results given in [Duke 1999; Ghate 2000; Lanphier and Takloo-Bighash 2004].

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