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**DUALITY FOR DIFFERENTIAL OPERATORS OF  
LIE-RINEHART ALGEBRAS**

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## DUALITY FOR DIFFERENTIAL OPERATORS OF LIE–RINEHART ALGEBRAS

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Let  $(S, L)$  be a Lie–Rinehart algebra over a commutative ring  $R$ . This article proves that, if  $S$  is flat as an  $R$ -module and has Van den Bergh duality in dimension  $n$ , and if  $L$  is finitely generated and projective with constant rank  $d$  as an  $S$ -module, then the enveloping algebra of  $(S, L)$  has Van den Bergh duality in dimension  $n + d$ . When, moreover,  $S$  is Calabi–Yau and the  $d$ -th exterior power of  $L$  is free over  $S$ , the article proves that the enveloping algebra is skew Calabi–Yau, and it describes a Nakayama automorphism of it. These considerations are specialised to Poisson enveloping algebras. They are also illustrated on Poisson structures over two- and three-dimensional polynomial algebras and on Nambu–Poisson structures on certain two-dimensional hypersurfaces.

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### Introduction

Rinehart [1963] introduced the concept of Lie–Rinehart algebra  $(S, L)$  over a commutative ring  $R$  and defined its enveloping algebra  $U$ . This generalises both constructions of universal enveloping algebras of  $R$ -Lie algebras and algebras of differential operators of commutative  $R$ -algebras. Huebschmann [1999] investigated Poincaré duality on the (co)homology groups of  $(S, L)$ . This duality is defined by

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the existence of a right  $U$ -module  $C$ , called the *dualising* module of  $(S, L)$  such that, for all left  $U$ -modules  $M$  and  $k \in \mathbb{N}$ ,

$$(0-1) \quad \text{Ext}_U^k(S, M) \cong \text{Tor}_{d-k}^U(C, M).$$

Chemla [1999] proved that for Lie–Rinehart algebras arising from affine complex Lie algebroids, the algebra  $U$  has a rigid dualising complex, which she determined, and has Van den Bergh duality [1998]. Having Van den Bergh duality in dimension  $n$  for an  $R$ -algebra  $A$  means that

- $A$  is homologically smooth, that is,  $A$  lies in the perfect derived category  $\text{per}(A^e)$  of the algebra  $A^e := A \otimes_R A^{\text{op}}$ ; and
- $\text{Ext}_{A^e}^\bullet(A, A^e)$  is zero for  $\bullet \neq 0$  and invertible as an  $A$ -bimodule if  $\bullet = n$ .

When this occurs, there is a functorial isomorphism, for all  $A$ -bimodules  $M$  and integers  $i$  (see [Van den Bergh 1998]),

$$\text{Ext}_{A^e}^i(A, M) \cong \text{Tor}_{n-i}^{A^e}(A, \text{Ext}_{A^e}^n(A, A^e) \otimes_A M);$$

and  $\text{Ext}_{A^e}^n(A, A^e)$  is called the inverse dualising bimodule of  $A$ . Two classes of algebras with Van den Bergh duality are of particular interest, namely,

- *Calabi–Yau* algebras, for which  $\text{Ext}_{A^e}^n(A, A^e)$  is required to be isomorphic to  $A$  as an  $A$ -bimodule (see [Ginzburg 2006]); and
- *skew Calabi–Yau* algebras, for which there exists an automorphism

$$\nu \in \text{Aut}_{R\text{-alg}}(A)$$

such that  $\text{Ext}_{A^e}^n(A, A^e) \simeq A^\nu$  as  $A$ -bimodules (see [Reyes et al. 2014]); here  $A^\nu$  denotes the  $A$ -bimodule obtained from  $A$  by twisting the action of  $A$  on the right by  $\nu$ .

The relevance of these algebras comes from their role in the noncommutative geometry initiated in [Artin and Schelter 1987] and in the investigation of Calabi–Yau categories, and also from the specificities of their Hochschild cohomology when  $R$  is a field. For instance, it is proved in [Ginzburg 2006; Lambre 2010] that the Gerstenhaber bracket of the Hochschild cohomology of Calabi–Yau algebras have a BV generator.

This article investigates when the enveloping algebra  $U$  of a general Lie–Rinehart algebra  $(S, L)$  over a commutative ring  $R$  has Van den Bergh duality.

It considers Lie–Rinehart algebras  $(S, L)$  such that  $S$  has Van den Bergh duality and is flat as an  $R$ -module, and  $L$  is finitely generated and projective with constant rank  $d$  as an  $S$ -module. Under these conditions, it is proved that  $U$  has Van den Bergh duality. Note that, when  $R$  is a perfect field, the former condition amounts to saying that  $S$  is a smooth affine  $R$ -algebra [Krämer 2007]. Note also that,

under the latter condition, it is proved in [Huebschmann 1999, Theorem 2.10] that  $(S, L)$  has duality in the sense of (0-1). Under the additional assumption that  $S$  is Calabi–Yau and  $\Lambda^d L$  is free as an  $S$ -module, it appears as a corollary that  $U$  is skew Calabi–Yau, and a Nakayama automorphism may be described explicitly. These considerations are specialised to the situation where the Lie–Rinehart algebra  $(S, L)$  arises from a Poisson structure on  $S$ . Also they are illustrated by detailed examples in the following cases:

- For Poisson brackets on polynomial algebras in two or three variables.
- For Nambu–Poisson structures on two-dimensional hypersurfaces of the shape  $1 + T(x, y, z) = 0$ , where  $T$  is a weight homogeneous polynomial.

Throughout the article,  $R$  denotes a commutative ring,  $(S, L)$  denotes a Lie–Rinehart algebra over  $R$  and  $U$  denotes its enveloping algebra. Given an  $R$ -Lie algebra  $\mathfrak{g}$ , its universal enveloping algebra is denoted by  $\mathcal{U}_R(\mathfrak{g})$ . For an  $R$ -algebra  $A$ , the category of left  $A$ -modules is denoted by  $\text{Mod}(A)$  and  $\text{Mod}(A^{\text{op}})$  is identified with the category of right  $A$ -modules. For simplicity, the piece of notation  $\otimes$  is used for  $\otimes_R$ . All complexes have differential of degree  $+1$ .

### 1. Main results

A Lie–Rinehart algebra over a commutative ring  $R$  is a pair  $(S, L)$  where  $S$  is a commutative  $R$ -algebra and  $L$  is a Lie  $R$ -algebra which is also a left  $S$ -module, endowed with a homomorphism of  $R$ -Lie algebras,

$$(1-1) \quad \begin{aligned} L &\rightarrow \text{Der}_R(S), \\ \alpha &\mapsto \partial_\alpha := \alpha(-), \end{aligned}$$

such that, for all  $\alpha, \beta \in L$  and  $s \in S$ ,

$$[\alpha, s\beta] = s[\alpha, \beta] + \alpha(s)\beta.$$

Following [Huebschmann 1999], the enveloping algebra  $U$  of  $(S, L)$  is identified with the algebra

$$(S \rtimes L)/I,$$

where  $S \rtimes L$  is the smash-product algebra of  $S$  by the action of  $L$  on  $S$  by derivations and  $I$  is the two-sided ideal of  $S \rtimes L$  generated by

$$\{s \otimes \alpha - 1 \otimes s\alpha \mid s \in S, \alpha \in L\}$$

(see Lemma 3.0.1); it is proved in [Huebschmann 1999] that this set generates  $I$  as a right ideal.

As mentioned in the introduction, when  $L$  is a finitely generated  $S$ -module with constant rank  $d$ , the Lie–Rinehart algebra  $(S, L)$  has duality in the sense of (0-1)

with  $C = \Lambda_S^d L^\vee$ . Here  $-^\vee$  is the duality  $\text{Hom}_S(-, S)$  and  $\Lambda_S^d L^\vee$  is considered as a right  $U$ -module using the Lie derivative  $\lambda_\alpha$ , for  $\alpha \in L$  (see [Huebschmann 1999, Section 2]),

$$\lambda_\alpha : \Lambda_S^\bullet L^\vee \rightarrow \Lambda_S^\bullet L^\vee ;$$

this is the derivation of  $\Lambda_S^\bullet L^\vee$  such that, for all  $s \in S$ ,  $\varphi \in L^\vee$  and  $\beta \in L$ ,

$$\lambda_\alpha(s) = \alpha(s) \quad \text{and} \quad \lambda_\alpha(\varphi)(\beta) = \alpha(\varphi(\beta)) - \varphi([\alpha, \beta]).$$

The right  $U$ -module structure of  $\Lambda_S^d L^\vee$  is such that, for all  $\varphi \in \Lambda_S^d L^\vee$  and  $\alpha \in L$ ,

$$(1-2) \quad \varphi \cdot \alpha = -\lambda_\alpha(\varphi).$$

The first main result of the article gives sufficient conditions for  $U$  to have Van den Bergh duality. It also describes the inverse dualising bimodule. Here are some explanations on this description. On one hand,  $R$ -linear derivations  $\partial \in \text{Der}_R(S)$  act on  $\text{Ext}_{S^e}^n(S, S^e)$ ,  $n \in \mathbb{N}$ , by Lie derivatives (see Section 4),

$$\mathcal{L}_\partial : \text{Ext}_{S^e}^n(S, S^e) \rightarrow \text{Ext}_{S^e}^n(S, S^e).$$

Combining with the action of  $L$  on  $S$  yields an action  $\alpha \otimes e \mapsto \alpha \cdot e$  of  $L$  on  $\text{Ext}_{S^e}^n(S, S^e)$  such that, for all  $\alpha \in L$  and  $e \in \text{Ext}_{S^e}^n(S, S^e)$ ,

$$\alpha \cdot e = \mathcal{L}_{\partial_\alpha}(e).$$

Although this is not a  $U$ -module structure on  $\text{Ext}_{S^e}^n(S, S^e)$ , it defines a left  $U$ -module structure on  $\Lambda_S^d L^\vee \otimes_S \text{Ext}_{S^e}^n(S, S^e)$ ,  $d \in \mathbb{N}$ , such that, for all  $\alpha \in L$ ,  $\varphi \in \Lambda_S^d L^\vee$  and  $e \in \text{Ext}_{S^e}^n(S, S^e)$ ,

$$\alpha \cdot (\varphi \otimes e) = -\varphi \cdot \alpha \otimes e + \varphi \otimes \alpha \cdot e.$$

On the other hand, consider the functor

$$F : \text{Mod}(U) \rightarrow \text{Mod}(U^e)$$

(see Section 3.3) such that, if  $N \in \text{Mod}(U)$ , then  $F(N)$  equals  $U \otimes_S N$  in  $\text{Mod}(U)$  and has a right  $U$ -module structure defined by the following formula, for all  $\alpha \in L$ ,  $u \in U$  and  $n \in N$ :

$$(u \otimes n) \cdot \alpha = u\alpha \otimes n - u \otimes \alpha \cdot n.$$

This functor takes left  $U$ -modules which are invertible as  $S$ -modules to invertible  $U$ -bimodules (see Section 3.6). The main result of this article is the following.

**Theorem 1.** *Let  $R$  be a commutative ring. Let  $(S, L)$  be a Lie–Rinehart algebra over  $R$ . Denote by  $U$  the enveloping algebra of  $(S, L)$ . Assume that*

- $S$  is flat as an  $R$ -module,
- $S$  has Van den Bergh duality in dimension  $n$ ,
- $L$  is finitely generated and projective with constant rank  $d$  as an  $S$ -module.

Then,  $U$  has Van den Bergh duality in dimension  $n + d$  and there is an isomorphism of  $U$ -bimodules,

$$\text{Ext}_{U^e}^{n+d}(U, U^e) \simeq F(\Lambda_S^d L^\vee \otimes_S \text{Ext}_{S^e}^n(S, S^e)).$$

Note that when  $R$  is Noetherian and  $S$  is finitely generated as an  $R$ -algebra and projective as an  $R$ -module, then there is an isomorphism of  $S$ -(bi)modules,

$$\text{Ext}_{S^e}^n(S, S^e) \simeq \Lambda_S^n \text{Der}_R(S);$$

this isomorphism is compatible with the actions by Lie derivatives (see Section 4.5). The above theorem was proved in [Chemla 1999, Theorem 4.4.1] when  $R = \mathbb{C}$  and  $S$  is finitely generated as a  $\mathbb{C}$ -algebra.

The preceding theorem specialises to the situation where the involved invertible  $S$ -modules are free. On one hand, when  $(\Lambda_S^d L)^\vee$  is free as an  $S$ -module with free generator  $\varphi_L$ , there is an associated *trace* mapping

$$\lambda_L : L \rightarrow S,$$

such that, for all  $\alpha \in L$ ,

$$\varphi_L \cdot \alpha = \lambda_L(\alpha) \cdot \varphi_L,$$

where the action on the left-hand side is given by (1-2) and that on the right-hand side is just given by the  $S$ -module structure. On the other hand, when  $S$  is Calabi–Yau in dimension  $n$ , each generator of the free of rank one  $S$ -module  $\text{Ext}_{S^e}^n(S, S^e)$  determines a volume form  $\omega_S \in \Lambda_S^n \Omega_{S/R}$ , and the *divergence*

$$\text{div} : \text{Der}_R(S) \rightarrow S$$

associated with  $\omega_S$  is defined by the following equality, for all  $\partial \in \text{Der}_R(S)$ :

$$\mathcal{L}_\partial(\omega_S) = \text{div}(\partial)\omega_S;$$

(see 4.5 for details). The second main result of the article then reads as follows.

**Theorem 2.** *Let  $R$  be a commutative ring. Let  $(S, L)$  be a Lie–Rinehart algebra over  $R$ . Denote by  $U$  the enveloping algebra of  $(S, L)$ . Assume that*

- $S$  is flat as an  $R$ -module,
- $S$  is Calabi–Yau in dimension  $n$ ,
- $L$  is finitely generated and projective with constant rank  $d$  and  $\Lambda_S^d L$  is free as  $S$ -modules.

*Then,  $U$  is skew Calabi–Yau with a Nakayama automorphism  $\nu \in \text{Aut}_R(U)$  such that, for all  $s \in S$ , and  $\alpha \in L$ ,*

$$\begin{cases} \nu(s) = s, \\ \nu(\alpha) = \alpha + \lambda_L(\alpha) + \text{div}(\partial_\alpha), \end{cases}$$

*where  $\lambda_L$  is any trace mapping on  $\Lambda_S^d L^\vee$  and  $\text{div}$  is any divergence.*

Among all Lie–Rinehart algebras, those arising from Poisson structures on  $S$  play a special role because of the connection to Poisson (co)homology. Recall that any  $R$ -bilinear Poisson bracket  $\{-, -\}$  on  $S$  defines a Lie–Rinehart algebra structure on  $(S, L) = (S, \Omega_{S/R})$  such that, for all  $s, t \in S$ ,

- $\partial_{ds} = \{s, -\}$ ;
- $[ds, dt] = d\{s, t\}$ .

In this case, the formulations of Theorems 1 and 2 simplify because, when  $\Omega_{S/R}$  is projective with constant rank  $n$  as an  $S$ -module, the right  $U$ -module structure of  $\Lambda_S^n \Omega_{S/R}^\vee$  (see (1-2)) is given by classical Lie derivatives; that is, for all  $s \in S$ ,

$$(1-3) \quad \lambda_{ds}(\varphi) = \mathcal{L}_{\{s, -\}}(\varphi).$$

More precisely, these theorems specialise as follows.

**Corollary 1.** *Let  $R$  be a Noetherian ring. Let  $(S, \{-, -\})$  be a finitely generated Poisson algebra over  $R$ . Denote by  $U$  the enveloping algebra of the associated Lie Rinehart algebra  $(S, \Omega_{S/R})$ . Assume that*

- $S$  is projective in  $\text{Mod}(R)$ ;
- $S \in \text{per}(S^e)$ ;
- $\Omega_{S/R}$ , which is then projective in  $\text{Mod}(S)$ , has constant rank  $n$ .

*Then,  $U$  has Van den Bergh duality in dimension  $2n$  and there is an isomorphism of  $U$ -bimodules,*

$$\text{Ext}_{U^e}^{2n}(U, U^e) \simeq U \otimes_S \Lambda_S^n \text{Der}_R(S) \otimes_S \Lambda_S^n \text{Der}_R(S),$$

*where the right-hand side term is a left  $U$ -module in a natural way and a right  $U$ -module such that, for all  $u \in U$ ,  $\varphi, \varphi' \in \Lambda_S^n \text{Der}_R(S)$  and  $s \in S$ ,*

$$(u \otimes \varphi \otimes \varphi') \cdot ds = u ds \otimes \varphi \otimes \varphi' - u \otimes (\mathcal{L}_{\{s, -\}}(\varphi) \otimes \varphi' + \varphi \otimes \mathcal{L}_{\{s, -\}}(\varphi')).$$

*In particular, if  $S$  has a volume form, then  $U$  is skew Calabi–Yau with a Nakayama automorphism  $\nu : U \rightarrow U$  such that, for all  $s \in S$ ,*

$$\begin{cases} \nu(s) = s, \\ \nu(ds) = ds + 2 \text{div}(\{s, -\}), \end{cases}$$

*where  $\text{div}$  is the divergence of the chosen volume form.*

For the case where  $R = \mathbb{C}$  and  $S$  is finitely generated as a  $\mathbb{C}$ -algebra, the above corollary is announced in [Lü et al. 2017, Theorem 0.7, Corollary 0.8] using the main results of [Chemla 1999].

This article is structured as follows. Section 2 presents useful information on the case where  $S$  has Van den Bergh duality. Section 3 is devoted to technical lemmas on



$U$ -(bi)modules. In particular, it presents the above mentioned functor  $F$  and its right adjoint  $G$ , which play an essential role in the proof of the main results. Section 4 introduces the action of  $L$  on  $\text{Ext}_{S^e}^*(S, S^e)$  by Lie derivatives. This structure is used in Section 5 in order to describe  $\text{Ext}_{U^e}^*(U, U^e)$  and prove Theorem 1, Theorem 2 and Corollary 1. Finally, Section 6 applies this corollary to a class of examples of Nambu–Poisson surfaces.

## 2. Poincaré duality for $S$

As proved in [Van den Bergh 1998] when  $R$  is a field, if  $S$  has Van den Bergh duality in dimension  $n$ , then there is a functorial isomorphism, for all  $S$ -bimodules  $N$ ,

$$\text{Ext}_{S^e}^*(S, N) \simeq \text{Tor}_{n-\bullet}^{S^e}(S, \text{Ext}_{S^e}^n(S, S^e) \otimes_S N).$$

It is direct to check that this is still the case without assuming that  $R$  is a field. In view of the proof of the main results of the article, Section 2.1 relates the above mentioned isomorphism to the fundamental class of  $S$ , following [Lambre 2010], and Section 2.2 relates Van den Bergh duality to the regularity of commutative algebras, following [Krämer 2007].

**2.1. Fundamental class and contraction.** Consider a projective resolution  $P^\bullet$  in  $\text{Mod}(S^e)$ ,

$$\dots \rightarrow P^{-2} \rightarrow P^{-1} \rightarrow P^0 \xrightarrow{\epsilon} S,$$

and let  $p^0 \in P^0$  be such that  $\epsilon(p^0) = 1_S$ . For all  $M, N \in \text{Mod}(S^e)$  and  $n \in \mathbb{N}$ , define the contraction

$$\begin{aligned} \text{Tor}_n^{S^e}(S, M) \times \text{Ext}_{S^e}^n(S, N) &\rightarrow \text{Tor}_0^{S^e}(S, M \otimes_S N), \\ (\omega, e) &\mapsto \iota_e(\omega) \end{aligned}$$

as the mapping induced by the following one:

$$\begin{aligned} M \otimes_{S^e} P^{-n} &\rightarrow \text{Hom}_R(\text{Hom}_{S^e}(P^{-n}, N), (M \otimes_S N) \otimes_{S^e} P^0), \\ x \otimes p &\mapsto (\varphi \mapsto (x \otimes \varphi(p)) \otimes p^0). \end{aligned}$$

This makes sense because  $P^\bullet$  is concentrated in nonpositive degrees. The construction depends neither on the choice of  $p^0$  nor on that of  $P^\bullet$ .

Following the proof of [Lambre 2010, Proposition 3.3], when  $S \in \text{per}(S^e)$  and  $n$  is taken equal to  $\text{pd}_{S^e}(S)$ , the contraction induces an isomorphism for all  $N \in \text{Mod}(S^e)$ ,

$$\begin{aligned} \text{Tor}_n^{S^e}(S, \text{Ext}_{S^e}^n(S, S^e)) &\rightarrow \text{Hom}_{S^e}(\text{Ext}_{S^e}^n(S, N), \text{Tor}_0^{S^e}(S, \text{Ext}_{S^e}^n(S, S^e) \otimes_S N)), \\ \omega &\mapsto \iota_\omega(\omega). \end{aligned}$$

In the particular case  $N = S^e$ , the *fundamental class* of  $S$  is the element  $c_S \in \text{Tor}_n^{S^e}(S, \text{Ext}_{S^e}^n(S, S^e))$  such that

$$(\iota_?(c_S))|_{\text{Ext}_{S^e}^n(S, S^e)} = \text{Id}_{\text{Ext}_{S^e}^n(S, S^e)}.$$

Following the arguments in the proof of [Lambre 2010, Théorème 4.2], when  $S$  has Van den Bergh duality in dimension  $n$ , which gives that  $n = \text{pd}_{S^e}(S)$ , the contraction with  $c_S$  induces an isomorphism, for all  $N \in \text{Mod}(S^e)$ ,

$$(2-1) \quad \iota_?(c_S) : \text{Ext}_{S^e}^n(S, N) \xrightarrow{\sim} \text{Tor}_0^{S^e}(S, \text{Ext}_{S^e}^n(S, S^e) \otimes_S N).$$

When  $S$  is projective in  $\text{Mod}(R)$ , the Hochschild complex  $S^{\otimes \bullet + 2}$  is a resolution of  $S$  in  $\text{Mod}(S^e)$  and the contraction

$$\begin{aligned} \text{Tor}_n^{S^e}(S, M) \times \text{Ext}_{S^e}^m(S, N) &\rightarrow \text{Tor}_{n-m}^{S^e}(S, M \otimes_S N), \\ (\omega, e) &\mapsto \iota_e(\omega) \end{aligned}$$

may be defined for all  $M, N \in \text{Mod}(S^e)$  and  $m, n \in \mathbb{N}$ , as the mapping induced at the level of Hochschild (co)chains by

$$\begin{aligned} M \otimes S^{\otimes n} \times \text{Hom}_R(S^{\otimes m}, N) &\rightarrow (M \otimes_S N) \otimes S^{\otimes(m-n)}, \\ ((x|s_1|\cdots|s_n), \psi) &\mapsto (x \otimes \psi(s_1|\cdots|s_m)|s_{m+1}|\cdots|s_n). \end{aligned}$$

When, in addition,  $S$  has Van den Bergh duality in dimension  $n$ , then [Lambre 2010, Théorème 4.2] asserts that the following mapping given by contraction with  $c_S$  is an isomorphism, for all  $N \in \text{Mod}(S^e)$  and  $m \in \mathbb{N}$ ,

$$\iota_?(c_S) : \text{Ext}_{S^e}^m(S, N) \rightarrow \text{Tor}_{n-m}^{S^e}(S, \text{Ext}_{S^e}^n(S, S^e) \otimes_S N).$$

**2.2. Relationship to regularity.** The main results of this article assume that  $S$  has Van den Bergh duality. For commutative algebras, this property is related to smoothness and regularity. The relationship is detailed in [Krähmer 2007] for the case where  $R$  is a perfect field, and is summarised below in the present setting.

**Proposition 2.2.1** [Krähmer 2007]. *Let  $R$  be a Noetherian commutative ring. Let  $S$  be a finitely generated commutative  $R$ -algebra and projective as an  $R$ -module. Let  $n \in \mathbb{N}$ . The following properties are equivalent.*

- (i)  $S$  has Van den Bergh duality in dimension  $n$ .
- (ii)  $\text{gl. dim}(S^e) < \infty$  and  $\Omega_{S/R}$ , which is then projective in  $\text{Mod}(S)$ , has constant rank  $n$ .

When these properties are true,  $\text{gl. dim}(S) < \infty$  and  $\text{Ext}_{S^e}^n(S, S^e) \simeq \Lambda_S^n \text{Der}_R(S)$  as  $S$ -modules.

*Proof.* See [Krähmer 2007] for full details. Since  $S$  is projective over  $R$ , then  $\text{pd}_{(S^e)^e}(S^e) \leq 2 \text{pd}_{S^e}(S)$  [Cartan and Eilenberg 1956, Chap. IX, Proposition 7.4]; besides, using the Hochschild resolution of  $S$  in  $\text{Mod}(S^e)$  yields that

$$\text{gl. dim}(S) \leq \text{pd}_{S^e}(S) \leq \text{gl. dim}(S^e);$$

thus

$$(2-2) \quad \begin{aligned} S \in \text{per}(S^e) &\Leftrightarrow \text{gl. dim}(S^e) < \infty \\ &\Rightarrow \text{gl. dim}(S) < \infty. \end{aligned}$$

Note also that, following [Hochschild et al. 1962, Theorem 3.1],

$$(2-3) \quad \text{gl. dim}(S^e) < \infty \quad \Rightarrow \quad \Omega_{S/R} \text{ is projective in } \text{Mod}(S).$$

Denote by  $\mu$  the multiplication mapping  $S \otimes S \rightarrow S$ . Assume  $\text{gl. dim}(S^e) < \infty$ , let  $\mathfrak{p} \in \text{Spec}(S) (\subseteq \text{Spec}(S^e))$  and denote by  $d$  the rank of  $(\Omega_{S/R})_{\mathfrak{p}}$ . Since  $\Omega_{S/R} \simeq \text{Ker}(\mu)/\text{Ker}(\mu)^2$  as modules over  $S (\simeq S^e/\text{Ker}(\mu))$  and  $\text{gl. dim}(S^e) < \infty$ , the  $(S^e)_{\mathfrak{p}}$ -module  $\text{Ker}(\mu)_{\mathfrak{p}}$  is generated by a regular sequence having  $d$  elements. There results a Koszul resolution of  $S_{\mathfrak{p}}$  in  $\text{Mod}((S^e)_{\mathfrak{p}})$ . Using this resolution and the isomorphism  $\text{Ext}_{S^e}^*(S, S^e)_{\mathfrak{p}} \simeq \text{Ext}_{(S^e)_{\mathfrak{p}}}^*(S_{\mathfrak{p}}, (S^e)_{\mathfrak{p}})$  in  $\text{Mod}((S^e)_{\mathfrak{p}})$  yields isomorphisms of  $(S^e)_{\mathfrak{p}}$ -modules,

$$(2-4) \quad \text{Ext}_{S^e}^*(S, S^e)_{\mathfrak{p}} \simeq \begin{cases} 0 & \text{if } \bullet \neq d, \\ S_{\mathfrak{p}} & \text{if } \bullet = d. \end{cases}$$

Now assume (i). Then,  $\text{gl. dim}(S^e) < \infty$  (see (2-2)),  $\Omega_{S/R}$  is projective (see (2-3)) and has constant rank  $n$  (see (2-4)). Conversely, assume that  $\text{gl. dim}(S^e) < \infty$  and  $\Omega_{S/R}$  has constant rank  $n$ . Then,  $S \in \text{per}(S^e)$  (see (2-2)) and the  $S$ -module (equivalently, the symmetric  $S$ -bimodule)  $\text{Ext}_{S^e}^*(S, S^e)$  is zero if  $\bullet \neq n$  and is invertible if  $\bullet = n$  (see (2-4)). Thus,

$$(i) \Leftrightarrow (ii).$$

Finally, assume that both (i) and (ii) are true. Then,  $\text{gl. dim}(S) < \infty$  (see (2-2)). Moreover, Van den Bergh duality [1998, Theorem 1] does apply here and provides an isomorphism of  $S$ -modules,

$$\text{Ext}_{S^e}^0(S, \text{Ext}_{S^e}^n(S; S^e)^{-1}) \simeq \text{Tor}_n^{S^e}(S, S),$$

whereas [Hochschild et al. 1962, Theorem 3.1] yields an isomorphism of  $S$ -modules,

$$\text{Tor}_n^{S^e}(S, S) \simeq \Lambda_S^n \Omega_{S/R}.$$

Thus,  $\text{Ext}_{S^e}^n(S, S^e) \simeq \Lambda_S^n \text{Der}_R(S)$  in  $\text{Mod}(S)$ . □

### 3. Material on $U$ -(bi)modules

The purpose of this section is to introduce an adjoint pair of functors  $(F, G)$  between  $\text{Mod}(U)$  and  $\text{Mod}(U^e)$ . In the proof of Theorem 1, the  $U$ -bimodule  $\text{Ext}_{U^e}^*(U, U^e)$  is described as the image under  $F$  of a certain left  $U$ -module which is invertible as an  $S$ -module. This section develops the needed properties of  $F$ . Hence, Section 3.1 recalls the basic constructions of  $U$ -modules; Sections 3.2 and 3.3 introduce the functors  $G$  and  $F$ , respectively; Section 3.4 proves that  $(F, G)$  is adjoint; Section 3.5 introduces and collects basic properties of compatible left  $S \rtimes L$ -modules, these are applied in Section 4 to the action of  $L$  on  $\text{Ext}_{S^e}^*(S, S^e)$  by Lie derivatives; and Section 3.6 proves that the functor  $F$  transforms left  $U$ -modules that are invertible as  $S$ -modules into invertible  $U$ -bimodules. These results are based on the description of  $U$  as a quotient of the smash-product  $S \rtimes L$  given in the following lemma. This description is established in [Lambre et al. 2017, Proposition 2.10] in the case of Lie–Rinehart algebras arising from Poisson algebras.

**Lemma 3.0.1.** (1) *The identity mappings  $\text{Id}_S : S \rightarrow S$  and  $\text{Id}_L : L \rightarrow L$  induce an isomorphism of  $R$ -algebras*

$$(3-1) \quad (S \rtimes L)/I \rightarrow U,$$

where  $I$  is the two-sided ideal of the smash-product algebra  $S \rtimes L$  generated by

$$\{s \otimes \alpha - 1 \otimes s\alpha \mid s \in S, \alpha \in L\}.$$

(2) *If  $L$  is projective as a left  $S$ -module, then  $U$  is projective both as a left and as a right  $S$ -module.*

*Proof.* (1) Recall (see [Rinehart 1963]) that  $U$  is defined as follows: Endow  $S \oplus L$  with an  $R$ -Lie algebra structure such that, for all  $s, t \in S$  and  $\alpha, \beta \in L$ ,

$$[s + \alpha, t + \beta] = \alpha(t) - \beta(s) + [\alpha, \beta].$$

Then,  $U$  is the factor  $R$ -algebra of the subalgebra of the universal enveloping algebra  $\mathcal{U}_R(S \oplus L)$  generated by the image of  $S \oplus L$  by the two-sided ideal generated by the classes in  $\mathcal{U}_R(S \oplus L)$  of the following elements, for  $s, t \in S$  and  $\alpha \in L$ :

$$s \otimes t - st, \quad s \otimes \alpha - s\alpha.$$

Recall also that  $S \rtimes L$  is the  $R$ -algebra with underlying  $R$ -module

$$S \otimes \mathcal{U}_R(L),$$

such that the images of  $S \otimes 1$  and  $1 \otimes \mathcal{U}_R(L)$  are subalgebras, and the following hold, for all  $s, t \in S$  and  $\alpha, \beta \in S$ :

$$\begin{cases} (s \otimes 1) \cdot (1 \otimes \alpha) = s \otimes \alpha, \\ (1 \otimes \alpha) \cdot (s \otimes 1) = \alpha(s) \otimes 1 + s \otimes \alpha. \end{cases}$$

Therefore, the natural mappings  $S \rightarrow U$  and  $L \rightarrow U$  induce an  $R$ -algebra homomorphism from  $S \rtimes L$  to  $U$ . This homomorphism vanishes on  $I$  whence the  $R$ -algebra homomorphism (3-1).

Besides, the universal property of  $U$  stated in [Huebschmann 1999, Section 2, p. 110] yields an  $R$ -algebra homomorphism,

$$(3-2) \quad U \rightarrow (S \rtimes L)/I,$$

induced by the natural mappings  $S \rightarrow (S \rtimes L)/I$  and  $L \rightarrow (S \rtimes L)/I$ . In view of the behaviour of (3-1) and (3-2) on the respective images of  $S \cup L$ , these algebra homomorphisms are inverse to each other.

(2) It is proved in [Rinehart 1963, Lemma 4.1] that  $U$  is projective as a left  $S$ -module. Consider the increasing filtration of  $U$  by the left  $S$ -submodules

$$0 \subseteq F_0U \subseteq F_1U \subseteq \dots,$$

where  $F_pU$  is the image of  $\bigoplus_{i=0}^p S \otimes L^{\otimes i}$  in  $U$ , for all  $p \in \mathbb{N}$ . In view of the equality

$$\alpha s = s\alpha + \alpha(s)$$

in  $U$  for all  $s \in S$  and  $\alpha \in L$ , the left  $S$ -module  $F_pU$  is also a right  $S$ -submodule of  $U$ , and  $F_pU/F_{p-1}U$  is a symmetric  $S$ -bimodule for all  $p \in \mathbb{N}$ . Therefore, the considerations used in the proof of [Rinehart 1963, Lemma 4.1] may be adapted in order to prove that  $U$  is projective as a right  $S$ -module. □

**3.1. Basic constructions of  $U$ -modules.** Left  $S \rtimes L$ -modules are identified with  $R$ -modules  $N$  endowed with a left  $S$ -module structure, and a left  $L$ -module structure such that, for all  $n \in N$ ,  $\alpha \in L$  and  $s \in S$ ,

$$\alpha \cdot (s \cdot n) = \alpha(s) \cdot n + s \cdot (\alpha \cdot n).$$

Left  $U$ -modules are identified with left  $S \rtimes L$ -modules  $N$  such that, for all  $n \in N$ ,  $\alpha \in L$  and  $s \in S$ ,

$$s \cdot (\alpha \cdot n) = (s\alpha) \cdot n.$$

Recall that the action of  $L$  endows  $S$  with a left  $U$ -module structure.

Right  $S \rtimes L$ -modules are identified with the  $R$ -modules  $M$  endowed with a right  $S$ -module structure and a right  $L$ -module structure such that, for all  $m \in M$ ,  $\alpha \in L$  and  $s \in S$ ,

$$(m \cdot \alpha) \cdot s = m \cdot \alpha(s) + (m \cdot s) \cdot \alpha.$$

Right  $U$ -modules are identified with right  $S \rtimes L$ -modules  $M$  such that, for all  $m \in M$ ,  $s \in S$  and  $\alpha \in L$ ,

$$(m \cdot s) \cdot \alpha = m \cdot (s\alpha).$$

The following constructions are classical. The corresponding  $U$ -module structures are introduced in [Huebschmann 1999, Section 2].

Let  $M, M'$  be right  $S \rtimes L$ -modules. Let  $N, N'$  be a left  $S \rtimes L$ -module. Then:

- $N$  is a right  $S \rtimes L$ -module for the right  $L$ -module structure such that, for all  $n \in N, s \in S$  and  $\alpha \in L$ ,

$$(3-6) \quad n \cdot s = s \cdot n \quad \text{and} \quad n \cdot \alpha = -\alpha \cdot n.$$

- $\text{Hom}_S(N, N')$  is a left  $S \rtimes L$ -module for the left  $L$ -module structure such that, for all  $f \in \text{Hom}_S(N, N'), n \in N$  and  $\alpha \in L$ ,

$$(3-4) \quad (\alpha \cdot f)(n) = \alpha \cdot f(n) - f(\alpha \cdot n);$$

moreover, this is a left  $U$ -module structure if  $N$  and  $N'$  are left  $U$ -modules.

- $\text{Hom}_S(M, M')$  is a left  $S \rtimes L$ -module for the left  $L$ -module structure such that, for all  $f \in \text{Hom}_S(M, M'), m \in M$  and  $\alpha \in L$ ,

$$(3-5) \quad (\alpha \cdot f)(m) = -f(m) \cdot \alpha + f(m \cdot \alpha).$$

- $\text{Hom}_S(N, S)$  is a right  $S \rtimes L$ -module for the right  $L$ -module structure such that, for all  $f \in \text{Hom}_S(N, S), n \in N$  and  $\alpha \in L$ ,

$$(3-6) \quad (f \cdot \alpha)(n) = -\alpha(f(n)) + f(\alpha \cdot n).$$

- $N \otimes_S N'$  is a left  $S \rtimes L$ -module for the left  $L$ -module structure such that, for all  $n \in N, n' \in N$  and  $\alpha \in L$ ,

$$(3-7) \quad \alpha \cdot (n \otimes n') = \alpha \cdot n \otimes n' + n \otimes \alpha \cdot n';$$

moreover, this is a left  $U$ -module structure if  $N$  and  $N'$  are left  $U$ -modules.

- $M \otimes_S N$  is a left  $S \rtimes L$ -module for the left  $L$ -module structure such that, for all  $m \in M, n \in N$  and  $\alpha \in L$ ,

$$(3-8) \quad \alpha \cdot (m \otimes n) = -m \cdot \alpha \otimes n + m \otimes \alpha \cdot n.$$

**3.2. The functor  $G = \text{Hom}_{S^e}(S, -) : \mathbf{Mod}(U^e) \rightarrow \mathbf{Mod}(U)$ .** Given  $M \in \mathbf{Mod}(U^e)$ , recall that

$$M^S = \{m \in M \mid (\text{for all } s \in S) (s \otimes 1 - 1 \otimes s) \cdot m = 0\}.$$

This is a symmetric  $S^e$ -submodule of  $M$ . Recall also the canonical isomorphisms that are inverse to each other:

$$(3-9) \quad \begin{aligned} M^S &\leftrightarrow \text{Hom}_{S^e}(S, M) \\ m &\mapsto (s \mapsto (s \otimes 1) \cdot m) \\ \varphi(1) &\leftarrow \varphi. \end{aligned}$$

**Lemma 3.2.1.** *Let  $M \in \text{Mod}(U^e)$ . Then,*

(1)  $M^S$  is a left  $U$ -module such that, for all  $m \in M^S$  and  $\alpha \in L$ ,

$$(3-10) \quad \alpha \cdot m := (\alpha \otimes 1 - 1 \otimes \alpha) \cdot m;$$

(2) the corresponding left  $U$ -module structure on  $\text{Hom}_{S^e}(S, M)$  (under the identification (3-9)) is such that, for all  $\varphi \in \text{Hom}_{S^e}(S, M)$ ,  $\alpha \in L$  and  $s \in S$ ,

$$(\alpha \cdot \varphi)(s) = (\alpha \otimes 1 - 1 \otimes \alpha) \cdot \varphi(s) - \varphi(\alpha(s)).$$

*Proof.* (1) Given all  $s \in S$  and  $\alpha \in L$ , denote

$$s \otimes 1 - 1 \otimes s \in U^e \quad \text{and} \quad \alpha \otimes 1 - 1 \otimes \alpha \in U^e$$

by  $ds$  and  $d\alpha$ , respectively; in particular

$$d\alpha \cdot ds = ds \cdot d\alpha + d(\alpha(s)),$$

and, for all  $m \in M^S$ ,

$$ds \cdot (d\alpha \cdot m) = d\alpha \cdot (ds \cdot m) - d(\alpha(s)) \cdot m = 0,$$

which proves that  $d\alpha \cdot m \in M^S$ . Therefore, (3-10) defines a left  $L$ -module structure on  $M^S$ . Now, for all  $m \in M^S$ ,  $s \in S$  and  $\alpha \in L$ ,

$$\begin{aligned} \alpha \cdot (s \otimes 1) \cdot m &= d\alpha \cdot (s \otimes 1) \cdot m = (\alpha(s) \otimes 1 + s\alpha \otimes 1 - s \otimes \alpha) \cdot m \\ &= (\alpha(s) \otimes 1) \cdot m + (s \otimes 1)(\alpha \otimes 1 - 1 \otimes \alpha) \cdot m \\ &= (\alpha(s) \otimes 1) \cdot m + (s \otimes 1) \cdot (\alpha \cdot m), \end{aligned}$$

$$\begin{aligned} (s \otimes 1) \cdot (\alpha \cdot m) &= (s \otimes 1) \cdot (\alpha \otimes 1 - 1 \otimes \alpha) \cdot m = (s\alpha \otimes 1) \cdot m - (s \otimes 1) \cdot (1 \otimes \alpha) \cdot m \\ &= (s\alpha \otimes 1) \cdot m - (1 \otimes \alpha) \cdot (s \otimes 1) \cdot m = (s\alpha \otimes 1) \cdot m - (1 \otimes \alpha) \cdot (1 \otimes s) \cdot m \\ &= (s\alpha \otimes 1 - 1 \otimes s\alpha) \cdot m = (s\alpha) \cdot m. \end{aligned}$$

Hence, this left  $L$ -module structure on  $M^S$  is a left  $U$ -module structure.

(2) By definition,  $\text{Hom}_{S^e}(S, M)$  is endowed with the left  $U$ -module structure such that (3-9) is an isomorphism in  $\text{Mod}(U)$ . Let  $\varphi \in \text{Hom}_{S^e}(S, M)$ ,  $\alpha \in L$  and  $s \in S$ . Then,

$$\begin{aligned} (\alpha \cdot \varphi)(s) &= (1 \otimes s) \cdot (\alpha \cdot \varphi(1)) = ((1 \otimes s)(\alpha \otimes 1 - 1 \otimes \alpha)) \cdot \varphi(1) \\ &= (\alpha \otimes s - 1 \otimes s\alpha - 1 \otimes \alpha(s)) \cdot \varphi(1) \\ &= ((\alpha \otimes 1 - 1 \otimes \alpha)(1 \otimes s) - 1 \otimes \alpha(s)) \cdot \varphi(1) \\ &= \alpha \cdot (1 \otimes s) \cdot \varphi(1) - (1 \otimes \alpha(s)) \cdot \varphi(1) = \alpha \cdot \varphi(s) - \varphi(\alpha(s)). \quad \square \end{aligned}$$

Thus, the assignment  $M \mapsto M^S$  defines a functor

$$(3-11) \quad \begin{aligned} G : \text{Mod}(U^\epsilon) &\rightarrow \text{Mod}(U), \\ M &\mapsto M^S. \end{aligned}$$

**3.3. The functor  $F = U \otimes_S - : \mathbf{Mod}(U) \rightarrow \mathbf{Mod}(U^\epsilon)$ .** Let  $N \in \text{Mod}(U)$ . In view of [Huebschmann 1999, (2.4)],  $U \otimes_S N$  is a right  $U$ -module such that, for all  $u \in U$ ,  $n \in N$ ,  $s \in S$  and  $\alpha \in L$ ,

$$(u \otimes n) \cdot s = u \otimes sn = us \otimes n \quad \text{and} \quad (u \otimes n) \cdot \alpha = u\alpha \otimes n - u \otimes \alpha \cdot n.$$

Besides,  $U \otimes_S N$  is a left  $U$ -module such that, for all  $u, u' \in U$  and  $n \in N$ ,

$$u' \cdot (u \otimes n) = u'u \otimes n.$$

Therefore,  $U \otimes_S N$  is a  $U$ -bimodule, and hence a left  $U^\epsilon$ -module. These considerations define a functor,

$$(3-12) \quad \begin{aligned} F : \text{Mod}(U) &\rightarrow \text{Mod}(U^\epsilon), \\ N &\mapsto U \otimes_S N. \end{aligned}$$

### 3.4. The adjunction between $F$ and $G$ .

**Proposition 3.4.1.** *The functors  $F = U \otimes_S -$  and  $G = \text{Hom}_{S^\epsilon}(S, -)$  introduced in Section 3.2 and Section 3.3 form an adjoint pair,*

$$\begin{array}{c} \text{Mod } U \\ \begin{array}{c} \downarrow F \\ \uparrow G \end{array} \\ \text{Mod } U^\epsilon \end{array}$$

*In particular, there is a functorial isomorphism, for all  $M \in \text{Mod}(U^\epsilon)$  and  $N \in \text{Mod}(U)$ ,*

$$\text{Hom}_U(N, G(M)) \xrightarrow{\sim} \text{Hom}_{U^\epsilon}(F(N), M).$$

*Proof.* Given  $f \in \text{Hom}_U(N, G(M))$ , denote by  $\Phi(f)$  the well-defined mapping

$$\begin{aligned} U \otimes_S N &\rightarrow M, \\ u \otimes n &\mapsto (u \otimes 1) \cdot f(n). \end{aligned}$$

Consider  $F(N)$  ( $= U \otimes_S N$ ) as a  $U$ -bimodule. Then, for all  $u, u' \in U$ ,  $n \in N$ ,  $s \in S$  and  $\alpha \in L$ ,

$$\begin{aligned} \Phi(f)(u' \cdot (u \otimes n)) &= \Phi(f)(u'u \otimes n) = (u'u \otimes 1) \cdot f(n) \\ &= (u' \otimes 1) \cdot \Phi(f)(u \otimes n), \end{aligned}$$



$$\begin{aligned}
 \Phi(f)((u \otimes n) \cdot s) &= \Phi(f)(u \otimes s \cdot n) = (u \otimes 1) \cdot f(s \cdot n) \\
 &= (u \otimes 1) \cdot ((1 \otimes s) \cdot f(n)) = ((1 \otimes s) \cdot (u \otimes 1)) \cdot f(n) \\
 &= (1 \otimes s) \cdot \Phi(f)(u \otimes n) = (\Phi(f)(u \otimes n)) \cdot s, \\
 \Phi(f)((u \otimes n) \cdot \alpha) &= \Phi(f)(u\alpha \otimes n - u \otimes \alpha \cdot n) \\
 &= (u\alpha \otimes 1) \cdot f(n) - (u \otimes 1) \cdot f(\alpha \cdot n) \\
 &= (u\alpha \otimes 1) \cdot f(n) - (u \otimes 1) \cdot (\alpha \otimes 1 - 1 \otimes \alpha) \cdot f(n) \\
 &= (u \otimes \alpha) \cdot f(n) = (1 \otimes \alpha) \cdot \Phi(f)(u \otimes n) \\
 &= (\Phi(f)(u \otimes n)) \cdot \alpha.
 \end{aligned}$$

In other words,

$$\Phi(f) \in \text{Hom}_{U^e}(F(N), M).$$

Given  $g \in \text{Hom}_{U^e}(F(N), M)$ , then, for all  $n \in N$  and  $s \in S$ ,

$$(s \otimes 1 - 1 \otimes s) \cdot g(1 \otimes n) = g(s \otimes_S n - 1 \otimes_S s \cdot n) = 0;$$

hence, denote by  $\Psi(g)$  the well-defined mapping

$$\begin{aligned}
 N &\rightarrow M^S, \\
 n &\mapsto g(1 \otimes n).
 \end{aligned}$$

Therefore, for all  $n \in N$ ,  $s \in S$  and  $\alpha \in L$ ,

$$\begin{aligned}
 \Psi(g)(s \cdot n) &= g(1 \otimes s \cdot n) = g(s \otimes n) = g((s \otimes 1) \cdot (1 \otimes n)) \\
 &= (s \otimes 1) \cdot g(1 \otimes n) = (s \otimes 1) \cdot \Psi(g)(n), \\
 \Psi(g)(\alpha \cdot n) &= g(1 \otimes \alpha \cdot n) = g(\alpha \otimes n - (1 \otimes \alpha) \cdot (1 \otimes n)) \\
 &= (\alpha \otimes 1) \cdot g(1 \otimes n) - (1 \otimes \alpha) \cdot g(1 \otimes n) = \alpha \cdot \Psi(g)(n);
 \end{aligned}$$

in other words,

$$\Psi(g) \in \text{Hom}_U(N, G(M)).$$

By construction,  $\Psi$  and  $\Phi$  are inverse to each other. □

**3.5. Compatible left  $S \rtimes L$ -modules.** As explained in Section 1, the main results of this article are expressed in terms of the action of  $L$  on  $\text{Ext}_{S^e}^*(S, S^e)$  by Lie derivatives and will be presented in Section 4. Although this action does not define a  $U$ -module structure on  $\text{Ext}_{S^e}^*(S, S^e)$ , it satisfies some compatibility with the  $S$ -module structure. The actions of  $L$  satisfying such a compatibility have specific properties that are used in the rest of the article and which are summarised below.

Define a *compatible* left  $S \rtimes L$ -module as a left  $S \rtimes L$ -module  $N$  such that, for all  $n \in N$ ,  $\alpha \in L$  and  $s \in S$ , the elements  $s\alpha \in L$  and  $\alpha(s) \in S$  satisfy

$$(3-13) \quad (s\alpha) \cdot n = s \cdot (\alpha \cdot n) - \alpha(s) \cdot n.$$

Note that a left  $S \rtimes L$ -module is both compatible and a left  $U$ -module if and only if  $L$  acts trivially, that is, by the zero action.

The two following lemmas present the properties of compatible left  $S \rtimes L$ -modules used in the rest of the article.

**Lemma 3.5.1.** *Let  $M$  be a right  $U$ -module. Let  $N$  be a compatible left  $S \rtimes L$ -module. Then:*

- (1) *The right  $S \rtimes L$ -module  $N^\vee = \text{Hom}_S(N, S)$  is a right  $U$ -module.*
- (2) *The left  $S \rtimes L$ -module  $\text{Hom}_S(N^\vee, M)$  is a left  $U$ -module.*
- (3) *The left  $S \rtimes L$ -module  $M \otimes_S N$  is a left  $U$ -module.*
- (4) *The following canonical mapping is a morphism of left  $U$ -modules:*

$$\begin{aligned} \theta : M \otimes_S N &\rightarrow \text{Hom}_S(N^\vee, M), \\ m \otimes n &\mapsto (\theta_{m \otimes n} : \varphi \mapsto m \cdot \varphi(n)). \end{aligned}$$

*Proof.* (1) Given  $\varphi \in N^\vee$ ,  $s \in S$  and  $\alpha \in L$ , then

$$\varphi \cdot (s\alpha) = (\varphi \cdot s) \cdot \alpha.$$

Indeed, for all  $n \in N$ ,

$$\begin{aligned} (\varphi \cdot (s\alpha))(n) &= -(s\alpha)(\varphi(n)) + \varphi((s\alpha) \cdot n) \\ &= -s\alpha(\varphi(n)) + \varphi(s \cdot (\alpha \cdot n) - \alpha(s) \cdot n) \\ &= -s\alpha(\varphi(n)) + s\varphi(\alpha \cdot n) - \alpha(s)\varphi(n) \\ &= ((\varphi \cdot \alpha) \cdot s)(n) - (\varphi \cdot \alpha(s))(n) \\ &= ((\varphi \cdot s) \cdot \alpha)(n). \end{aligned}$$

(2) This is precisely [Huebschmann 1999, (2.3)].

(3) The  $S \rtimes L$ -module structure of  $M \otimes_S N$  is described in (3-8). Given  $m \in M$ ,  $n \in N$ ,  $s \in S$  and  $\alpha \in L$ , then

$$\begin{aligned} (s\alpha) \cdot (m \otimes n) &= -m \cdot (s\alpha) \otimes n + m \otimes (s\alpha) \cdot n \\ &= -(m \cdot \alpha) \cdot s \otimes n + m \cdot \alpha(s) \otimes n + m \otimes s \cdot (\alpha \cdot n) - m \otimes \alpha(s) \cdot n \\ &= s \cdot (\alpha \cdot (m \otimes n)). \end{aligned}$$

(4) It suffices to prove that the given mapping is  $L$ -linear. Let  $m \in M$ ,  $n \in N$ ,  $\alpha \in L$  and  $\varphi \in \text{Hom}_S(N, S)$ . Then,

$$\begin{aligned} (\alpha \cdot \theta_{m \otimes n})(\varphi) &= -\theta_{m \otimes n}(\varphi) \cdot \alpha + \theta_{m \otimes n}(\varphi \cdot \alpha) = -(m \cdot \varphi(n)) \cdot \alpha + m \cdot (\varphi \cdot \alpha)(n) \\ &= -((m \cdot \alpha) \cdot \varphi(n) - m \cdot \alpha(\varphi(n))) + m \cdot (-\alpha(\varphi(n)) + \varphi(\alpha \cdot n)) \\ &= -(m \cdot \alpha) \cdot \varphi(n) + m \cdot \varphi(\alpha \cdot n) = \theta_{\alpha \cdot (m \otimes n)}(\varphi); \end{aligned}$$

thus,  $\alpha \cdot \theta_{m \otimes n} = \theta_{\alpha \cdot (m \otimes n)}$ . □

Any left  $S \rtimes L$ -module  $N$  may be considered as a symmetric  $S$ -bimodule, or equivalently a right  $S^e$ -module, such that, for all  $n \in N$  and  $s, s' \in S$ ,

$$n \cdot (s \otimes s') = (ss') \cdot n.$$

Accordingly,  $N \otimes_{S^e} U^e$  is a right  $U^e$ -module in a natural way.

**Lemma 3.5.2.** *Let  $N$  be a compatible left  $S \rtimes L$ -module.*

(1) *The right  $U^e$ -module  $N \otimes_{S^e} U^e$  is actually a  $U$ - $U^e$ -bimodule such that for all  $n \in N$ ,  $u, v \in U$  and  $\alpha \in L$ ,*

$$\alpha \cdot (n \otimes (u \otimes v)) = \alpha \cdot n \otimes (u \otimes v) + n \otimes ((\alpha \otimes 1 - 1 \otimes \alpha) \cdot (u \otimes v)).$$

(2) *Let  $M$  be a right  $U$ -module. Then, there exists an isomorphism of left  $U^e$ -modules*

$$\begin{aligned} F(M \otimes_S N) &\rightarrow M \otimes_U (N \otimes_{S^e} U^e), \\ v \otimes (m \otimes n) &\mapsto m \otimes (n \otimes (1 \otimes v)). \end{aligned}$$

*Proof.* (1) Following part (3) of Lemma 3.5.1, there is a left  $U$ -module structure on  $U \otimes_S N$  such that, for all  $\alpha \in L$ ,  $v \in U$  and  $n \in N$ ,

$$\alpha \cdot (v \otimes n) = -v\alpha \otimes n + v \otimes \alpha \cdot n.$$

Therefore, there is a left  $U$ -module structure on  $(U \otimes_S N) \otimes_S U$  (see (3-7)) such that, for all  $\alpha \in L$ ,  $n \in N$  and  $u, v \in U$ ,

$$\begin{aligned} \alpha \cdot ((v \otimes n) \otimes u) &= \alpha \cdot (v \otimes n) \otimes u + (v \otimes n) \otimes \alpha u \\ &= -(v\alpha \otimes n) \otimes u + (v \otimes \alpha \cdot n) \otimes u + (v \otimes n) \otimes \alpha u. \end{aligned}$$

Under the canonical identification

$$\begin{aligned} N \otimes_{S^e} U^e &\rightarrow (U \otimes_S N) \otimes_S U, \\ n \otimes (u \otimes v) &\mapsto (v \otimes n) \otimes u, \end{aligned}$$

$N \otimes_{S^e} U^e$  inherits a left  $U$ -module structure which is the one claimed in the statement.

Now,  $N \otimes_{S^e} U^e$  inherits a right  $U^e$ -module structure from  $U^e$ . This structure is compatible with the left  $U$ -module structure discussed previously so as to yield a left  $U \otimes (U^e)^{\text{op}}$ -module structure.

(2) Due to (1), there is a right  $U^e$ -module structure on  $M \otimes_U (N \otimes_{S^e} U^e)$ . It is considered here as a left  $U^e$ -module structure such that, for all  $u, v, u', v' \in U$ ,  $m \in M$  and  $n \in N$ ,

$$(3-14) \quad (u' \otimes v') \cdot (m \otimes (n \otimes (u \otimes v))) = m \otimes (n \otimes (uv' \otimes u'v)).$$

For ease of reading, note that in  $F(M \otimes_S N)$ ,

$$(3-15) \quad \begin{aligned} (u \otimes 1) \cdot (v \otimes m \otimes n) &= uv \otimes m \otimes n \\ (1 \otimes \alpha) \cdot (v \otimes m \otimes n) &= v\alpha \otimes m \otimes n + v \otimes m \cdot \alpha \otimes n - v \otimes m \otimes \alpha \cdot n, \end{aligned}$$

and, in  $M \otimes_U (N \otimes_{S^e} U^e)$ ,

$$(3-16) \quad m \cdot \alpha \otimes n \otimes u \otimes v = m \otimes \alpha \cdot n \otimes u \otimes v + m \otimes n \otimes \alpha u \otimes v - m \otimes n \otimes u \otimes v\alpha.$$

The  $R$ -linear mapping from  $U \otimes M \otimes N$  to  $M \otimes_U (N \otimes_{S^e} U^e)$  given by

$$v \otimes m \otimes n \mapsto m \otimes (n \otimes (1 \otimes v))$$

induces a morphism of  $S$ -modules from  $U \otimes_S (M \otimes_S N)$  to  $M \otimes_U (N \otimes_{S^e} U^e)$  such as in the statement of the lemma. Denote it by  $\Psi'$ :

$$\Psi' : U \otimes_S (M \otimes_S N) \rightarrow M \otimes_U (N \otimes_{S^e} U^e).$$

This is a morphism of left  $U^e$ -modules. Indeed, for all  $u, v \in U$ ,  $m \in M$ ,  $n \in N$  and  $\alpha \in L$ ,

$$\begin{aligned} \Psi'((u \otimes 1) \cdot (v \otimes m \otimes n)) &= \Psi'(uv \otimes m \otimes n) = m \otimes n \otimes 1 \otimes uv \\ &= (u \otimes 1) \cdot \Psi'(v \otimes m \otimes n), \end{aligned} \tag{3-14}$$

$$\begin{aligned} \Psi'((1 \otimes \alpha) \cdot (v \otimes m \otimes n)) &= \Psi'(v\alpha \otimes m \otimes n + v \otimes m \cdot \alpha \otimes n - v \otimes m \otimes \alpha \cdot n) \\ &= m \otimes n \otimes 1 \otimes v\alpha + m \cdot \alpha \otimes n \otimes 1 \otimes v - m \otimes \alpha \cdot n \otimes 1 \otimes v \\ &= m \otimes n \otimes \alpha \otimes v \\ &= (1 \otimes \alpha) \cdot \Psi'(v \otimes m \otimes n). \end{aligned} \tag{3-16}$$

Consider the following morphism of  $S$ -modules:

$$\begin{aligned} \phi : M \otimes_S (N \otimes_{S^e} U^e) &\rightarrow F(M \otimes_S N), \\ m \otimes (n \otimes (u \otimes v)) &\mapsto (1 \otimes u) \cdot (v \otimes m \otimes n). \end{aligned}$$

Given  $m \in M$ ,  $n \in N$ ,  $u, v \in U$  and  $\alpha \in L$ , then the image under  $\phi$  of the term

$$m \otimes \alpha \cdot n \otimes u \otimes v + m \otimes n \otimes \alpha u \otimes v - m \otimes n \otimes u \otimes v\alpha$$

is equal to

$$(1 \otimes u) \cdot (v \otimes m \otimes \alpha \cdot n) + (1 \otimes \alpha u) \cdot (v \otimes m \otimes n) - (1 \otimes u) \cdot (v \alpha \otimes m \otimes n),$$

which is equal to

$$(1 \otimes u) \cdot (v \otimes m \otimes \alpha \cdot n) + (1 \otimes u) \cdot (1 \otimes \alpha) \cdot (v \otimes m \otimes n) - (1 \otimes u) \cdot (v \alpha \otimes m \otimes n).$$

In view of (3-15), this is equal to

$$(1 \otimes u) \cdot (v \otimes m \cdot \alpha \otimes n) = \phi(m \cdot \alpha \otimes (n \otimes (u \otimes v))).$$

Thus,  $\phi$  induces a morphism of  $S$ -modules

$$\begin{aligned} \Phi' : M \otimes_U (N \otimes_{S^e} U^e) &\rightarrow F(M \otimes_S N), \\ m \otimes (n \otimes (u \otimes v)) &\mapsto (1 \otimes u) \cdot (v \otimes m \otimes n). \end{aligned}$$

It appears that  $\Phi'$  is left and right inverse for  $\Psi'$ . Indeed,

- $\Phi' \circ \Psi' = \text{Id}_{F(M \otimes_S N)}$ , and
- for all  $u, v \in U, m \in M$  and  $n \in N$ ,

$$\begin{aligned} \Psi' \circ \Phi'(m \otimes n \otimes u \otimes v) &= \Psi'((1 \otimes u) \cdot (v \otimes m \otimes n)) \\ &= (1 \otimes u) \cdot \Psi'(v \otimes m \otimes n) && (\Psi' \text{ is } U^e\text{-linear}) \\ &= (1 \otimes u) \cdot (m \otimes n \otimes 1 \otimes v) \\ &= m \otimes n \otimes u \otimes v. \end{aligned} \quad \square$$

**3.6. Invertible  $U$ -bimodules.** The following result is used in Section 5 in order to prove that  $\text{Ext}_{U^e}^i(U, U^e)$  is invertible as a  $U$ -bimodule, under suitable conditions.

**Proposition 3.6.1.** *Let  $R$  be a commutative ring. Let  $(S, L)$  be a Lie–Rinehart algebra over  $R$ . Denote by  $U$  its enveloping algebra. Let  $N$  be a left  $U$ -module. Assume that  $N$  is invertible as an  $S$ -module. Then  $F(N)$  is invertible as a  $U$ -bimodule.*

This subsection is devoted to the proof of this proposition. Given a left  $U$ -module  $N$ , then  $F(N) = U \otimes_S N$  as left  $U$ -modules. Hence, there is a functorial isomorphism

$$(3-17) \quad \theta : \text{Hom}_S(N, U) \rightarrow \text{Hom}_U(F(N), U).$$

Note:

- $\text{Hom}_S(N, U)$  is a left  $U$ -module (see (3-4)), and it inherits a right  $U$ -module structure from  $U_U$ ; by construction, these two structures form a  $U$ -bimodule structure.
- $\text{Hom}_U(F(N), U)$  is a  $U$ -bimodule because so are  $F(N)$  and  $U$ .

- $N \otimes_S \text{Hom}_S(N, U)$  is a left  $U$ -module (see (3-7)), and it inherits a right  $U$ -module structure from  $U_U$ ; by construction, these two structures form a  $U$ -bimodule structure.

**Lemma 3.6.2.** *Let  $N$  be a left  $U$ -module. Then,*

- (1)  $\theta : \text{Hom}_S(N, U) \rightarrow \text{Hom}_U(F(N), U)$  is an isomorphism in  $\text{Mod}(U^e)$ ,
- (2) the mapping

$$\begin{aligned} \Phi : N \otimes_S \text{Hom}_S(N, U) &\rightarrow F(N) \otimes_U \text{Hom}_U(F(N), U), \\ n \otimes f &\mapsto (1 \otimes n) \otimes \theta(f) \end{aligned}$$

is an isomorphism in  $\text{Mod}(U^e)$ , and

- (3) the diagram

$$\begin{array}{ccc} N \otimes_S \text{Hom}_S(N, U) & \longrightarrow & U \\ \Phi \downarrow & & \parallel \\ F(N) \otimes_U \text{Hom}(F(N), U) & \longrightarrow & U \end{array}$$

with horizontal arrows given by evaluation, is commutative.

*Proof.* (1) By definition,  $\theta$  is a morphism of right  $U$ -modules. It is also a morphism of left  $U$ -modules because, for all  $n \in N$ ,  $f \in \text{Hom}_S(N, U)$ ,  $u \in U$  and  $\alpha \in L$ ,

$$\begin{aligned} \theta(\alpha \cdot f)(u \otimes n) &= u(\alpha \cdot f)(n) \\ &= u(\alpha f(n) - f(\alpha \cdot n)) = \theta(f)(u\alpha \otimes n - u \otimes \alpha \cdot n) \\ &= \theta(f)((u \otimes n) \cdot \alpha) = (\alpha \cdot \theta(f))(u \otimes n). \end{aligned}$$

(2) By definition,  $\Phi$  is a morphism of right  $U$ -modules. It is also a morphism of left  $U$ -modules because, for all  $n \in N$ ,  $f \in \text{Hom}_S(N, U)$  and  $\alpha \in L$ ,

$$\begin{aligned} \Phi(\alpha \cdot (n \otimes f)) &= \Phi(\alpha \cdot n \otimes f + n \otimes \alpha \cdot f) \\ &= (1 \otimes \alpha \cdot n) \otimes \theta(f) + (1 \otimes n) \otimes \underbrace{\theta(\alpha \cdot f)}_{=\alpha \cdot \theta(f)} \\ &= (1 \otimes \alpha \cdot n) \otimes \theta(f) + \underbrace{(1 \otimes n) \cdot \alpha}_{=\alpha \otimes n - 1 \otimes \alpha \cdot n} \otimes \theta(f) \\ &= (\alpha \otimes n) \otimes \theta(f) \\ &= \alpha \cdot \Phi(n \otimes f). \end{aligned}$$

In order to prove that  $\Phi$  is bijective, consider the linear mapping

$$\begin{aligned} \psi : F(N) \otimes_S \text{Hom}_U(F(N), U) &\rightarrow N \otimes_S \text{Hom}_S(N, U), \\ (u \otimes n) \otimes g &\mapsto u \cdot (n \otimes \theta^{-1}(g)). \end{aligned}$$

Note that, for all  $u \in U$ ,  $\alpha \in L$ ,  $n \in N$  and  $g \in \text{Hom}_U(F(N), U)$ ,

$$\begin{aligned} \psi((u \otimes n) \cdot \alpha \otimes g) &= \psi((u\alpha \otimes n) \otimes g - (u \otimes \alpha \cdot n) \otimes g) \\ &= u\alpha \cdot (n \otimes \theta^{-1}(g)) - u \cdot (\alpha \cdot n \otimes \theta^{-1}(g)) \\ &= u \cdot (\alpha \cdot n \otimes \theta^{-1}(g)) + n \otimes \alpha \cdot \theta^{-1}(g) - u \cdot (\alpha \cdot n \otimes \theta^{-1}(g)) \\ &= u \cdot (n \otimes \theta^{-1}(\alpha \cdot g)) \quad (\text{see part (1)}) \\ &= \psi((u \otimes n) \otimes \alpha \cdot g). \end{aligned}$$

Hence,  $\psi$  induces a linear mapping,

$$\begin{aligned} \Psi : F(N) \otimes_U \text{Hom}_U(F(N), U) &\rightarrow N \otimes_S \text{Hom}_S(N, U), \\ (u \otimes n) \otimes g &\mapsto u \cdot (n \otimes \theta^{-1}(g)). \end{aligned}$$

Now, by definition of  $\Phi$  and  $\Psi$ ,

$$\Psi \circ \Phi = \text{Id}_{N \otimes_S \text{Hom}_S(N, U)}.$$

Since

- $\Psi$  is a morphism of left  $U$ -modules by construction;
- as a left  $U$ -module,  $F(N) \otimes_U \text{Hom}_U(F(N), U)$  is generated by the image of  $(1 \otimes N) \otimes \text{Hom}_U(F(N), U)$ ; and
- for all  $n \in N$  and  $g \in \text{Hom}_U(F(N), U)$ ,

$$\Phi \circ \Psi((1 \otimes n) \otimes g) = (1 \otimes n) \otimes g,$$

the following holds:

$$\Phi \circ \Psi = \text{Id}_{F(N) \otimes_U \text{Hom}_U(F(N), U)}.$$

Altogether, these considerations show that  $\Phi$  is an isomorphism in  $\text{Mod}(U^e)$ .

(3) The diagram is commutative by definition of  $\Phi$ . □

Like in the previous lemma, for all  $N \in \text{Mod}(U)$ ,  $\text{Hom}_S(N, U)$  is a  $U$ -bimodule, and hence  $\text{Hom}_S(N, U) \otimes_S N$  is a  $U$ -bimodule by means of (3-7) and the right  $U$ -module structure of  $U$ .

**Lemma 3.6.3.** *Let  $N$  be a left  $U$ -module. Then,*

(1) *the mapping*

$$\begin{aligned} \Phi' : \text{Hom}_S(N, U) \otimes_S N &\rightarrow \text{Hom}_U(F(N), U) \otimes_U F(N), \\ f \otimes n &\mapsto \theta(f) \otimes (1 \otimes n) \end{aligned}$$

*is an isomorphism in  $\text{Mod}(U^e)$ ; and*

(2) *the diagram*

$$\begin{array}{ccc} \mathrm{Hom}_S(N, U) \otimes_S N & \longrightarrow & U \\ \Phi' \downarrow & & \parallel \\ \mathrm{Hom}_U(F(N), U) \otimes_U F(N) & \longrightarrow & U \end{array}$$

with horizontal arrows given by evaluation, is commutative.

*Proof.* (1) First, since  $F(N) = U \otimes_S N$  in  $\mathrm{Mod}(U^e)$ , then

$$\mathrm{Hom}_U(F(N), U) \otimes_U F(N) \cong \mathrm{Hom}_U(F(N), U) \otimes_S N$$

as left  $U$ -modules. Under this identification,  $\Phi'$  expresses as

$$\Phi' : f \otimes n \mapsto \theta(f) \otimes n.$$

Therefore,  $\Phi'$  is bijective because so is  $\theta$ .

Next,  $\Phi'$  is a morphism of left  $U$ -modules because so is  $\theta$ . And it is a morphism of right  $U$ -modules because it is a morphism of right  $S$ -modules, and because, for all  $f \in \mathrm{Hom}_S(N, U)$ ,  $n \in N$  and  $\alpha \in L$ ,

$$\begin{aligned} \Phi'((f \otimes n) \cdot \alpha) &= \Phi'(f \cdot \alpha \otimes n - f \otimes \alpha \cdot n) \\ &= \underbrace{\theta(f \cdot \alpha)}_{=\theta(f) \cdot \alpha} \otimes (1 \otimes n) - \theta(f) \otimes (1 \otimes \alpha \cdot n) \\ &= \theta(f) \otimes \underbrace{\alpha \cdot (1 \otimes n)}_{=\alpha \otimes n} - \theta(f) \otimes (1 \otimes \alpha \cdot n) \\ &= \theta(f) \otimes ((1 \otimes n) \cdot \alpha) = (\theta(f) \otimes (1 \otimes n)) \cdot \alpha \\ &= \Phi'(f \otimes n) \cdot \alpha. \end{aligned}$$

This proves (1).

(2) The diagram commutes by definition of  $\Phi'$ . □

It is now possible to prove the result announced at the beginning of the subsection.

*Proof of Proposition 3.6.1.* Since  $N$  is invertible as an  $S$ -module, then the following evaluation mappings are bijective

$$N \otimes_S \mathrm{Hom}_S(N, U) \rightarrow U \quad \text{and} \quad \mathrm{Hom}_S(N, U) \otimes_S N \rightarrow U.$$

According to Lemmas 3.6.2 and 3.6.3, the following evaluation mappings are isomorphisms of  $U$ -bimodules

$$F(N) \otimes_U \mathrm{Hom}_U(F(N), U) \rightarrow U \quad \text{and} \quad \mathrm{Hom}_U(F(N), U) \otimes_U F(N) \rightarrow U.$$

Thus,  $F(N)$  is invertible as a  $U$ -bimodule. □



**4. The action of  $L$  on the inverse dualising bimodule of  $S$**

This section introduces an action of  $L$  on  $\text{Ext}_{S^e}^*(S, S^e)$  by means of Lie derivatives, which is used to describe  $\text{Ext}_{U^e}^*(U, U^e)$  in the next section. When  $S$  is projective in  $\text{Mod}(R)$ , then  $\text{Ext}_{S^e}^*(S, -)$  is the Hochschild cohomology  $H^\bullet(S; -)$ ; in this setting, the Lie derivatives on  $H^\bullet(S; S)$  and  $H_\bullet(S; S)$  are defined in [Rinehart 1963, Section 9] and have a well-known expression in terms of the Hochschild resolution of  $S$ . For the needs of the article, the definition is translated to arbitrary coefficients in terms of any projective resolution of  $S$  in  $\text{Mod}(S^e)$ .

Hence, Section 4.1 introduces preliminary material, Section 4.2 deals with derivations on projective resolutions of  $S$  in  $\text{Mod}(S^e)$ , Section 4.3 defines the Lie derivatives, Section 4.4 presents the action of  $L$  on  $\text{Ext}_{S^e}^*(S, S^e)$ , and Section 4.5 discusses particular situations.

For the section, a projective resolution of  $S$  in  $\text{Mod}(S^e)$  is considered;

$$(P^\bullet, d) \rightarrow S.$$

Denote  $S$  by  $P^1$  and the augmentation mapping  $P^0 \rightarrow S$  by  $d^0$ . For all  $M \in \text{Mod}(S^e)$  and  $s \in S$ , denote by  $\lambda_s$  and  $\rho_s$  the multiplication mappings

$$\lambda_s : M \rightarrow M, \quad m \mapsto (s \otimes 1) \cdot m$$

and

$$\rho_s : M \rightarrow M, \quad m \mapsto (1 \otimes s) \cdot m.$$

**4.1. Data on the projective resolution.** For all  $s \in S$ , the mappings  $\lambda_s, \rho_s$  on  $P^\bullet$  are morphisms of complexes of left  $S^e$ -modules and induce the same mapping

$$\begin{aligned} S &\rightarrow S, \\ t &\mapsto st \end{aligned}$$

in cohomology. Hence, there exists a morphism of graded left  $S^e$ -modules,

$$(4-1) \quad k_s : P^\bullet \rightarrow P^\bullet[-1],$$

such that

$$(4-2) \quad \lambda_s - \rho_s = k_s \circ d + d \circ k_s.$$

**Lemma 4.1.1.** *Let  $\partial : S \rightarrow S$  be an  $R$ -linear derivation. Let  $\psi : P^\bullet \rightarrow P^\bullet$  be a morphism of complexes of  $R$ -modules such that*

- $H^0(\psi) : S \rightarrow S$  is the zero mapping;
- there exists a morphism of graded left  $S^e$ -modules,

$$k : P^\bullet \rightarrow P^\bullet[-1],$$

such that, for all  $p \in P^\bullet$  and  $s, t \in S$ ,

$$(4-3) \quad \psi((s \otimes t) \cdot p) = (s \otimes t) \cdot \psi(p) - (1 \otimes \partial)(s \otimes t) \cdot (k \circ d + d \circ k)(p).$$

Then, there exists a morphism of graded  $R$ -modules,

$$h : P^\bullet \rightarrow P^\bullet[-1],$$

such that

- $\psi = h \circ d + d \circ h$ ; and
- for all  $s, t \in S$  and  $p \in P^\bullet$ ,

$$h((s \otimes t) \cdot p) = (s \otimes t) \cdot h(p) - (1 \otimes \partial)(s \otimes t) \cdot k(p).$$

*Proof.* The proof is an induction on  $n \leq 1$ , taking  $h^1 : S \rightarrow P^0$  equal to 0. Let  $n \leq 0$  and assume that there exist linear mappings, for all  $j$  such that  $n + 1 \leq j \leq 1$ ,

$$h^j : P^j \rightarrow P^{j-1}$$

such that, for all  $j$  satisfying  $n + 1 \leq j \leq 0$ ,  $p \in P^j$  and  $s, t \in S$ ,

$$(4-4) \quad \begin{aligned} \psi^j &= h^{j+1} \circ d^j + d^{j-1} \circ h^j \\ h^j((s \otimes t) \cdot p) &= (s \otimes t) \cdot h^j(p) - (1 \otimes \partial)(s \otimes t) \cdot k^j(p). \end{aligned}$$

This is illustrated in the following diagram:

$$\begin{array}{ccccccc} P^n & \xrightarrow{d^n} & P^{n+1} & \xrightarrow{d^{n+1}} & P^{n+2} & \xrightarrow{d^{n+2}} & \dots \\ \psi^n \downarrow & \nearrow h^{n+1} & \downarrow \psi^{n+1} & \nearrow h^{n+2} & & & \\ P^n & \xrightarrow{d^n} & P^{n+1} & \xrightarrow{d^{n+1}} & P^{n+2} & \xrightarrow{d^{n+2}} & \dots \end{array}$$

Let

$$((p_i, \varphi^i))_{i \in I}$$

be a coordinate system of the projective left  $S^e$ -module  $P^n$ . That is, let  $p_i \in P^n$  and  $\varphi^i \in \text{Hom}_{S^e}(P^n, S^e)$  for all  $i \in I$  such that, for all  $p \in P^n$ ,

$$p = \sum_{i \in I} \varphi^i(p) \cdot p_i,$$

where  $\{i \in I \mid \varphi^i(p) \neq 0\}$  is finite. Since  $\psi : P^\bullet \rightarrow P^\bullet$  is a morphism of complexes, it follows from (4-4) that, for all  $i \in I$ , there exists  $p'_i \in P^{n-1}$  such that

$$(4-5) \quad \psi^n(p_i) = d^{n-1}(p'_i) + h^{n+1} \circ d^n(p_i).$$

Denote by  $h^n$  the linear mapping from  $P^n$  to  $P^{n-1}$  such that, for all  $p \in P^n$ ,

$$h^n(p) = \sum_{i \in I} \varphi^i(p) \cdot p'_i - (1 \otimes \partial)(\varphi^i(p)) \cdot k^n(p_i).$$

Then, for all  $p \in P^n$  and  $s, t \in S$ ,

$$\begin{aligned} & h^n((s \otimes t) \cdot p) \\ &= \sum_{i \in I} (s \otimes t) \cdot \varphi^i(p) \cdot p'_i - (s \otimes t) \cdot (1 \otimes \partial)(\varphi^i(p)) \cdot k^n(p_i) - (1 \otimes \partial)(s \otimes t) \cdot \varphi^i(p) \cdot k^n(p_i) \\ &= (s \otimes t) \cdot h^n(p) - (1 \otimes \partial)(s \otimes t) \cdot k^n \left( \sum_{i \in I} \varphi^i(p) \cdot p_i \right) \\ &= (s \otimes t) \cdot h^n(p) - (1 \otimes \partial)(s \otimes t) \cdot k^n(p). \end{aligned}$$

Moreover,

$$\psi^n = h^{n+1} \circ d^n + d^{n-1} \circ h^n.$$

Indeed, for all  $p \in P^n$ ,  $p = \sum_{i \in I} \varphi^i(p) \cdot p_i$ , and hence

$$\begin{aligned} & d^{n-1} \circ h^n(p) + h^{n+1} \circ d^n(p) \\ &= \sum_{i \in I} \varphi^i(p) \cdot d^{n-1}(p'_i) - (1 \otimes \partial)(\varphi^i(p)) \cdot d^{n-1} \circ k^n(p_i) + h^{n+1} \left( \sum_{i \in I} \varphi^i(p) \cdot d^n(p_i) \right) \\ &\stackrel{(4.4)}{=} \sum_{i \in I} \varphi^i(p) \cdot d^{n-1}(p'_i) - (1 \otimes \partial)(\varphi^i(p)) \cdot d^{n-1} \circ k^n(p_i) \\ &\quad + \varphi^i(p) \cdot h^{n+1} \circ d^n(p_i) - (1 \otimes \partial)(\varphi^i(p)) \cdot k^{n+1} \circ d^n(p_i) \\ &\stackrel{(4.3)}{=} \sum_{i \in I} \varphi^i(p) \cdot d^{n-1}(p'_i) + \varphi^i(p) \cdot h^{n+1} \circ d^n(p_i) + \psi^n(\varphi^i(p) \cdot p_i) - \varphi^i(p) \cdot \psi^n(p_i) \\ &\stackrel{(4.5)}{=} \sum_{i \in I} \psi^n(\varphi^i(p) \cdot p_i) = \psi^n(p). \quad \square \end{aligned}$$

**4.2. Derivations on the projective resolution.** Let  $\partial : S \rightarrow S$  be an  $R$ -linear derivation. It defines an  $R$ -linear derivation on  $S^e$  denoted by  $\partial^e$ ,

$$\begin{aligned} \partial^e : S^e &\rightarrow S^e, \\ s \otimes t &\mapsto \partial(s) \otimes t + s \otimes \partial(t). \end{aligned}$$

For every left  $S^e$ -module  $M$ , a *derivation* of  $M$  relative to  $\partial$  is an  $R$ -linear mapping,

$$\partial_M : M \rightarrow M,$$

such that, for all  $m \in M$  and  $s, t \in S$ ,

$$\partial_M((s \otimes t) \cdot m) = \partial^e(s \otimes t) \cdot m + (s \otimes t) \cdot \partial_M(m).$$

A derivation of  $P^\bullet$  relative to  $\partial$  is a morphism of complexes of  $R$ -modules,

$$\partial^\bullet : P^\bullet \rightarrow P^\bullet,$$

such that  $\partial^n : P^n \rightarrow P^n$  is a derivation relative to  $\partial$  for all  $n$ , and such that

$H^0(\partial^\bullet) = \partial$ . Note that a morphism of complexes of  $R$ -modules  $\partial^\bullet : P^\bullet \rightarrow P^\bullet$  such that  $H^0(\partial^\bullet) = \partial$  is a derivation relative to  $\partial$  if and only if

$$(4-6) \quad \begin{cases} \partial^\bullet \circ \lambda_s = \lambda_{\partial(s)} + \lambda_s \circ \partial^\bullet, \\ \partial^\bullet \circ \rho_s = \rho_{\partial(s)} + \rho_s \circ \partial^\bullet. \end{cases}$$

**Remark.** For all derivations  $\partial_1^\bullet, \partial_2^\bullet : P^\bullet \rightarrow P^\bullet$  relative to  $\partial$ , the difference

$$\partial_1^\bullet - \partial_2^\bullet : P^\bullet \rightarrow P^\bullet$$

is a null-homotopic morphism of complexes of left  $S^e$ -modules.

**Lemma 4.2.1.** *There exists a mapping, which need not be linear,*

$$(4-7) \quad \begin{aligned} \text{Der}_R(S) &\rightarrow \text{Hom}_R(P^\bullet, P^\bullet), \\ \partial &\mapsto \partial^\bullet \end{aligned}$$

such that:

- (1) For all  $\partial \in \text{Der}_R(S)$ , the mapping  $\partial^\bullet$  is a derivation relative to  $\partial$ .
- (2) For all  $\partial_1, \partial_2 \in \text{Der}_R(S)$  and  $r \in R$ , there exist morphisms of graded left  $S^e$ -modules,

$$\ell, \ell' : P^\bullet \rightarrow P^\bullet[-1],$$

such that

$$(4-8) \quad \begin{cases} [\partial_1, \partial_2]^\bullet - [\partial_1^\bullet, \partial_2^\bullet] &= \ell \circ d + d \circ \ell, \\ (\partial_1 + r\partial_2)^\bullet - (\partial_1^\bullet + r\partial_2^\bullet) &= \ell' \circ d + d \circ \ell'. \end{cases}$$

- (3) For all  $s \in S$  and  $\partial \in \text{Der}_R(S)$ , there exists a morphism of graded  $R$ -modules

$$h : P^\bullet \rightarrow P^\bullet[-1],$$

such that

$$(4-9) \quad (s\partial)^\bullet - \lambda_s \circ \partial^\bullet = h \circ d + d \circ h$$

and, for all  $p \in P^\bullet$  and  $t_1, t_2 \in S$ ,

$$(4-10) \quad h((t_1 \otimes t_2) \cdot p) = (t_1 \otimes t_2) \cdot h(p) - (t_1 \otimes \partial(t_2)) \cdot k_s(p).$$

Recall that  $k_s : P^\bullet \rightarrow P^\bullet[-1]$  is a morphism of graded left  $S^e$ -modules such that  $\lambda_s - \rho_s = k_s \circ d + d \circ k_s$  (see (4-1) and (4-2)).

*Proof.* (1) Let  $\partial \in \text{Der}_R(S)$ . For convenience, denote  $\partial$  by  $\partial^1 : S \rightarrow S$ . The proof is an induction on  $n \leq 1$ . Let  $n \leq 0$ , and assume that a commutative diagram is given

$$\begin{array}{ccccccc} P^n & \xrightarrow{d^n} & P^{n+1} & \longrightarrow & \dots & \longrightarrow & P^0 \xrightarrow{d^0} P^1 \longrightarrow 0 \\ & & \downarrow \partial^{n+1} & & & & \downarrow \partial^0 & \downarrow \partial^1 \\ P^n & \xrightarrow{d^n} & P^{n+1} & \longrightarrow & \dots & \longrightarrow & P^0 \xrightarrow{d^1} P^1 \longrightarrow 0 \end{array}$$

where  $\partial^i : P^i \rightarrow P^i$  is a derivation relative to  $\partial$  for all  $i \in \{n+1, n+2, \dots, 0\}$ . Let

$$((p_i, \varphi^i))_{i \in I}$$

be a coordinate system of the projective left  $S^e$ -module  $P^n$  (see the proof in 4.1). Then, for all  $i \in I$ , there exists  $p'_i \in P^n$  such that

$$\partial^{n+1} \circ d^n(p_i) = d^n(p'_i).$$

Denote by  $\partial^n$  the  $R$ -linear mapping from  $P^n$  to  $P^n$  such that, for all  $p \in P^n$ ,

$$\partial^n(p) = \sum_{i \in I} \partial(\varphi^i(p)) \cdot p_i + \varphi^i(p) \cdot p'_i.$$

Then, for all  $p \in P^n$ ,

$$\begin{aligned} d^n \circ \partial^n(p) &= \sum_{i \in I} \partial(\varphi^i(p)) \cdot d^n(p_i) + \varphi^i(p) \cdot d^n(p'_i) \\ &= \sum_{i \in I} \partial(\varphi^i(p)) \cdot d^n(p_i) + \varphi^i(p) \cdot \partial^{n+1} \circ d^n(p_i) \\ &= \partial^{n+1} \circ d^n \left( \sum_{i \in I} \varphi^i(p) \cdot p_i \right) \\ &= \partial^{n+1} \circ d^n(p). \end{aligned}$$

Thus,

$$d^n \circ \partial^n = \partial^{n+1} \circ d^n.$$

Moreover,  $\partial^n$  is a derivation of  $P^n$  relative to  $\partial$  because  $\partial$  is a derivation of  $S^e$  and  $\varphi^i \in \text{Hom}_{S^e}(P^n, S^e)$  for all  $i \in I$ .

(2) Note that  $[\partial_1, \partial_2]^\bullet$  and  $[\partial_1^\bullet, \partial_2^\bullet]$  (or,  $(\partial_1 + r\partial_2)^\bullet$  and  $\partial_1^\bullet + r\partial_2^\bullet$ ) are derivations of  $P^\bullet$  relative to  $[\partial_1, \partial_2]$  (or, to  $\partial_1 + r\partial_2$ , respectively). The conclusion therefore follows from the remark preceding Lemma 4.2.1.

(3) Denote by  $\psi$  the mapping  $(s\partial)^\bullet - \lambda_s \circ \partial^\bullet$  given by

$$\begin{aligned} P^\bullet &\rightarrow P^\bullet, \\ p &\mapsto (s\partial)^\bullet(p) - (s \otimes 1) \cdot \partial^\bullet(p). \end{aligned}$$

Then, for all  $p \in P^\bullet$  and  $t \in S$ ,

$$\begin{aligned} \psi((t \otimes 1) \cdot p) &= (s\partial)^\bullet((t \otimes 1) \cdot p) - (s \otimes 1) \cdot \partial^\bullet((t \otimes 1) \cdot p) \\ &= (s\partial(t \otimes 1) \cdot p) + (t \otimes 1) \cdot (s\partial)^\bullet(p) - (s \otimes 1) \cdot (\partial(t \otimes 1) \cdot p) - (s \otimes 1) \cdot (t \otimes 1) \cdot \partial^\bullet(p) \\ &= (t \otimes 1) \cdot \psi(p) \end{aligned}$$

and

$$\begin{aligned}
 \psi((1 \otimes t) \cdot p) &= (s\partial)^*((1 \otimes t) \cdot p) - (s \otimes 1) \cdot \partial^*((1 \otimes t) \cdot p) \\
 &= (1 \otimes s\partial(t)) \cdot p + (1 \otimes t) \cdot (s\partial)^*(p) - (s \otimes 1) \cdot (1 \otimes \partial(t)) \cdot p - (s \otimes 1) \cdot (1 \otimes t) \cdot \partial^*(p) \\
 &= (1 \otimes t) \cdot \psi(p) + (1 \otimes \partial(t)) \cdot (\rho_s - \lambda_s)(p) \\
 &\stackrel{(4-2)}{=} (1 \otimes t) \cdot \psi(p) - (1 \otimes \partial(t)) \cdot (k_s \circ d + d \circ k_s)(p).
 \end{aligned}$$

Hence, Lemma 4.1.1 may be applied, which yields (3). □

**Remark.** Using the remark preceding Lemma 4.2.1, it may be checked that, although the mapping  $\text{Der}_R(S) \rightarrow \text{Hom}_R(P^\bullet, P^\bullet)$  of the lemma is not unique, two such mappings induce the same mapping from  $\text{Der}_R(S)$  to  $H^0 \text{Hom}_R(P^\bullet, P^\bullet)$ , which is  $R$ -linear.

When  $S$  is projective in  $\text{Mod}(R)$ , it is possible to be more explicit on a possible mapping,  $\partial \mapsto \partial^*$ . Indeed, the Hochschild complex  $B(S) = S^{\otimes \bullet + 2}$  is a projective resolution of  $S$ . For all  $\partial \in \text{Der}_R(S)$ , define the following mapping:

$$\begin{aligned}
 L_\partial : B(S) &\rightarrow B(S), \\
 (s_0 | \cdots | s_{n+1}) &\mapsto \sum_{i=0}^{n+1} (s_0 | \cdots | s_{i-1} | \partial(s_i) | \cdots | s_{i+1} | \cdots | s_n).
 \end{aligned}$$

This is a derivation of  $B(S)$  relative to  $\partial$ . It is direct to check that the mapping

$$\begin{aligned}
 \text{Der}_R(S) &\rightarrow \text{Hom}_R(B(S), B(S)), \\
 \partial &\mapsto L_\partial
 \end{aligned}$$

is a morphism of Lie algebras over  $R$ . Now, consider homotopy equivalences of complexes of  $S^e$ -modules,

$$P^\bullet \begin{array}{c} \xrightarrow{f} \\ \xleftarrow{g} \end{array} B(S),$$

and, for all  $\partial \in \text{Der}_R(S)$ , define  $\partial^*$  as

$$\partial^* = g \circ L_\partial \circ f;$$

this is a derivation relative to  $\partial$  because so is  $L_\partial$  and because  $f$  and  $g$  are morphisms of resolutions of  $S$  in  $\text{Mod}(S^e)$ . The following mapping satisfies the conclusion of the preceding lemma, it is moreover  $R$ -linear:

$$\begin{aligned}
 \text{Der}_R(S) &\rightarrow \text{Hom}_R(P^\bullet, P^\bullet), \\
 \partial &\mapsto \partial^*.
 \end{aligned}$$

**4.3. Lie derivatives.** Consider a mapping  $\partial \mapsto \partial^\bullet$  such as in Lemma 4.2.1. Let  $M$  be an  $S$ -bimodule and  $\partial : S \rightarrow S$  be an  $R$ -linear derivation. Let  $\partial_M : M \rightarrow M$  be a derivation relative to  $\partial$ . Given  $n \in \mathbb{N}$  and  $\psi \in \text{Hom}_{S^e}(P^{-n}, M)$ , denote by  $\mathcal{L}_\partial(\psi)$  the mapping

$$(4-11) \quad \mathcal{L}_\partial(\psi) = \partial_M \circ \psi - \psi \circ \partial^{-n}.$$

This is a morphism in  $\text{Mod}(S^e)$  because so is  $\psi$  and because  $\partial_M$  and  $\partial^{-n}$  are derivations relative to  $\partial$ ; moreover, it is a cocycle (or a coboundary) as soon as  $\psi$  is, because  $\partial^\bullet : P^\bullet \rightarrow P^\bullet$  is a morphism of complexes. Denote by  $\mathcal{L}_\partial$  the resulting mapping in cohomology

$$\mathcal{L}_\partial : \text{Ext}_{S^e}^\bullet(S, M) \rightarrow \text{Ext}_{S^e}^\bullet(S, M)$$

such that for all  $c \in \text{Ext}_{S^e}^\bullet(S, M)$ , say represented by a cocycle  $\psi$ , then  $\mathcal{L}_\partial(c)$  is represented by the cocycle  $\mathcal{L}_\partial(\psi)$ . In the situations considered later in the article, there is no ambiguity on  $\partial_M$ , whence its omission in the notation.

Following similar considerations denote also by  $\mathcal{L}_\partial$  the mapping

$$\mathcal{L}_\partial : \text{Tor}_{S^e}^\bullet(S, M) \rightarrow \text{Tor}_{S^e}^\bullet(S, M)$$

such that for all  $\omega \in \text{Tor}_{S^e}^\bullet(S, M)$ , say represented by a cocycle  $m \otimes p \in M \otimes_{S^e} P^\bullet$  with sum sign omitted,  $\mathcal{L}_\partial(\omega)$  is represented by the cocycle

$$\mathcal{L}_\partial(m \otimes p) := m \otimes \partial^\bullet(p) + \partial_M(m) \otimes p.$$

When  $S$  is projective in  $\text{Mod}(R)$ , these operations may be written explicitly in terms of the Hochschild resolution. When  $\psi$  is a Hochschild cocycle lying in  $\text{Hom}_R(S^{\otimes n}, M)$ , the mapping  $\mathcal{L}_\partial(\psi)$  is given by

$$(4-12) \quad (s_1 | \cdots | s_n) \mapsto \partial_M(f(s_1 | \cdots | s_n)) - \sum_{i=1}^n f(s_1 | \cdots | \partial(s_i) | \cdots | s_n).$$

Likewise, the operation in Hochschild homology is induced by the following mapping at the level of Hochschild chains,

$$M \otimes S^{\otimes n} \rightarrow M \otimes S^{\otimes n}$$

$$(m | s_1 | \cdots | s_n) \mapsto (\partial_M(m) | s_1 | \cdots | s_n) + \sum_{i=1}^n (m | s_1 | \cdots | \partial(s_i) | \cdots | s_n).$$

The operator  $\mathcal{L}_\partial$  is of course called the *Lie derivative* of  $\partial$ . When  $M = S$  and  $S$  is projective in  $\text{Mod}(R)$ , this is nothing else but the classical Lie derivative defined in [Rinehart 1963, Section 9]. In view of the remark following Lemma 4.2.1, these constructions depend only on  $\partial$  and  $\partial_M$  and not on the choices of  $P^\bullet$  and the mapping  $\partial \mapsto \partial^\bullet$ .

In the sequel these constructions are considered mainly in the following cases:

- $M = S$  and  $\partial_M = \partial$ .
- $M = S^e$  and  $\partial_M = \partial^e$ .
- $M = \text{Ext}_{S^e}^n(S, S^e)$  ( $n \in \mathbb{N}$ ) and  $\partial_M = \mathcal{L}_\partial$ , which makes sense according to the result below.

In the sequel the following construction is also used. Consider  $S$ -bimodules  $M, N$ . Let  $m, n \in \mathbb{N}$ . Let  $\partial \in \text{Der}_R(S)$  and let  $\partial_M : M \rightarrow M$  and  $\partial_N : N \rightarrow N$  be  $R$ -linear derivations relative to  $\partial$ . Then, for all  $f \in \text{Hom}_R(\text{Ext}_{S^e}^m(S, M), \text{Tor}_n^{S^e}(S, N))$ , define  $\mathcal{L}_\partial(f)$  as

$$\mathcal{L}_\partial \circ f - f \circ \mathcal{L}_\partial.$$

Recall that for all  $M \in \text{Mod}(S^e)$ , the spaces  $\text{Ext}_{S^e}^\bullet(S, M)$  and  $\text{Tor}_{S^e}^\bullet(S, M)$  are left  $S$ -modules by means of  $\lambda_s : M \rightarrow M$  for all  $s \in S$ ; the corresponding multiplication by  $s$  on these (co)homology spaces is denoted by  $\lambda_s$ .

**Lemma 4.3.1.** *Let  $M \in \text{Mod}(S^e)$ ,  $n \in \mathbb{N}$  and  $s \in S$ . Let  $\partial, \partial' : S \rightarrow S$  be  $R$ -linear derivations. Let  $\partial_M, \partial'_M : M \rightarrow M$  be  $R$ -linear derivations relative to  $\partial$  and  $\partial'$ , respectively. Then, the following hold in  $\text{Ext}_{S^e}^\bullet(S, M)$ :*

- (1)  $\mathcal{L}_\partial \circ \lambda_s = \lambda_{\partial(s)} + \lambda_s \circ \mathcal{L}_\partial$ .
- (2)  $\mathcal{L}_{[\partial, \partial']} = [\mathcal{L}_\partial, \mathcal{L}_{\partial'}]$ .
- (3) *Let  $m \in \mathbb{N}$ , let  $N$  be another  $S$ -bimodule and let  $\partial_N : N \rightarrow N$  be a derivation relative to  $\partial$ . Consider the contraction mapping*

$$\begin{aligned} \text{Tor}_m^{S^e}(S, M) &\rightarrow \text{Hom}_R(\text{Ext}_{S^e}^n(S, N), \text{Tor}_{m-n}^{S^e}(S, M \otimes_S N)), \\ \omega &\mapsto (c \mapsto \iota_c(\omega)). \end{aligned}$$

*If  $m = n$ , then it is  $\mathcal{L}_\partial$ -equivariant. When  $S$  is projective in  $\text{Mod}(R)$ , it is  $\mathcal{L}_\partial$  equivariant for all  $m, n \in \mathbb{N}$ .*

- (4) *If  $M$  is symmetric as an  $S$ -bimodule,  $\mathcal{L}_{s\partial} = \lambda_s \circ \mathcal{L}_\partial$ .*
- (5) *When  $M = S^e$  (and  $\partial_M = \partial^e$ ), the following equality holds in  $\text{Ext}_{S^e}^\bullet(S, M)$ :*

$$\mathcal{L}_{s\partial} = \lambda_s \circ \mathcal{L}_\partial - \lambda_{\partial(s)}.$$

*Proof.* (1) The equality is checked on cochains. Let  $\psi \in \text{Hom}_{S^e}(P^{-n}, M)$ . Then,

$$\begin{aligned} \mathcal{L}_\partial \circ \lambda_s(\psi) &= \partial_M \circ \lambda_s \circ \psi - \lambda_s \circ \psi \circ \partial^\bullet \\ &= (\lambda_{\partial(s)} + \lambda_s \circ \partial_M) \circ \psi - \lambda_s \circ \psi \circ \partial^\bullet \\ &= (\lambda_{\partial(s)} + \lambda_s \circ \mathcal{L}_\partial)(\psi). \end{aligned}$$



(2) Note that  $\mathcal{L}_{[\partial, \partial']}$  is defined with respect to  $[\partial_M, \partial'_M]$ , which is a derivation of  $M$  relative to  $[\partial, \partial']$ . Following Lemma 4.2.1, there exists a morphism of graded  $S^e$ -modules,

$$\ell : P^\bullet \rightarrow P^\bullet[-1],$$

such that

$$[\partial, \partial']^\bullet - [\partial^\bullet, \partial'^\bullet] = \ell \circ d + d \circ \ell.$$

Let  $\psi \in \text{Hom}_{S^e}(P^{-n}, M)$ . If this is a cocycle, then

$$\begin{aligned} \mathcal{L}_{[\partial, \partial']}(\psi) &= [\partial_M, \partial'_M] \circ \psi - \psi \circ ([\partial^\bullet, \partial'^\bullet] + \ell \circ d + d \circ \ell) \\ &= [\mathcal{L}_\partial, \mathcal{L}_{\partial'}](\psi) - \psi \circ \ell \circ d - \underbrace{\psi \circ d \circ \ell}_{=0}, \end{aligned}$$

which is cohomologous to  $[\mathcal{L}_\partial \mathcal{L}_{\partial'}](\psi)$ . This proves (2).

(3) Note that the mapping

$$\begin{aligned} \partial_{M \otimes_S N} : M \otimes_S N &\rightarrow M \otimes_S N, \\ x \otimes y &\mapsto \partial_M(x) \otimes y + x \otimes \partial_N(y), \end{aligned}$$

is a well-defined derivation relative to  $\partial$ , which defines  $\mathcal{L}_\partial$  on

$$\text{Tor}_{m-n}^{S^e}(S, M \otimes_S N).$$

Assume first that  $m = n$ . Let  $p^0$  be any element of the preimage of  $1_S$  under the augmentation mapping  $P^0 \rightarrow S$ . Let  $x \otimes p \in M \otimes_{S^e} P^{-m}$  and  $\psi \in \text{Hom}_{S^e}(P^{-m}, N)$ , and use the notation

$$\iota_\psi(x \otimes p) := (x \otimes \psi(p)) \otimes p^0.$$

Recall that the contraction mapping is induced by the mapping

$$\begin{aligned} M \otimes_{S^e} P^{-m} &\rightarrow \text{Hom}_R(\text{Hom}_{S^e}(P^{-m}, N), (M \otimes_S N) \otimes_{S^e} P^0), \\ x \otimes p &\mapsto \iota_\psi(x \otimes p). \end{aligned}$$

Denote  $\mathcal{L}_\partial(\iota_\psi(x \otimes p)) - \iota_{\mathcal{L}_\partial(\psi)}(x \otimes p)$  by  $\delta$ . Then,

$$\begin{aligned} \delta &= \mathcal{L}_\partial((x \otimes \psi(p)) \otimes p^0) - (x \otimes \mathcal{L}_\partial(\psi)(p)) \otimes p^0 \\ &= \partial_M(x) \otimes \psi(p) \otimes p^0 + x \otimes \partial_N(\psi(p)) \otimes p^0 + x \otimes \psi(p) \otimes \partial^0(p^0) \\ &\quad - x \otimes \partial_N(\psi(p)) \otimes p^0 + x \otimes \psi(\partial^{-m}(p)) \otimes p^0 \\ &= \iota_\psi(\mathcal{L}_\partial(x \otimes p)) + x \otimes \psi(p) \otimes \partial^0(p^0). \end{aligned}$$

Note that  $\partial^0(p_0)$  lies in the image of  $d : P^{-1} \rightarrow P^0$  because the image of  $p^0$  under  $P^0 \rightarrow S$  is 1 and  $H^0(\partial^\bullet) = \partial$ . These considerations therefore prove (3) when  $m = n$ .

Now assume that  $S$  is projective in  $\text{Mod}(R)$ . Then, the equivariance may be checked at the level of Hochschild (co)chains. Let  $o = (x|s_1|\cdots|s_m) \in S^{\otimes m}$  and  $\psi \in \text{Hom}_R(S^{\otimes n}, N)$ . Then,

$$\begin{aligned} \mathcal{L}_\partial(\iota_\psi(o)) - \iota_{\mathcal{L}_\partial(\psi)}(o) &= \mathcal{L}_\partial(x \otimes \psi(s_1|\cdots|s_n)|s_{n+1}|\cdots|s_m) - (x \otimes \mathcal{L}_\partial(\psi)(s_1|\cdots|s_n)|s_{n+1}|\cdots|s_m) \\ &= (\partial_M(x) \otimes \psi(s_1|\cdots|s_n)|s_{n+1}|\cdots|s_m) + (x \otimes \partial_N(\psi(s_1|\cdots|s_n))|s_{n+1}|\cdots|s_m) \\ &\quad + \sum_{j=n+1}^m (x \otimes \psi(s_1|\cdots|s_n)|s_{n+1}|\cdots|\partial(s_j)|\cdots|s_m) \\ &\quad - (x \otimes \partial_N(\psi(s_1|\cdots|s_n))|s_{n+1}|\cdots|s_m) \\ &\quad + \sum_{j=1}^n (x \otimes \psi(s_1|\cdots|\partial(s_j)|\cdots|s_n)|s_{n+1}|\cdots|s_m) \\ &= \iota_\psi(\mathcal{L}_\partial(o)), \end{aligned}$$

which proves (3) for all  $m, n \in \mathbb{N}$  when  $S$  is projective in  $\text{Mod}(R)$ .

(4) Note that  $\mathcal{L}_{s\partial}$  is defined with respect to the derivation  $s\partial_M (= \lambda_s \circ \partial_M)$ . Assume that  $M$  is symmetric as an  $S$ -bimodule. Therefore, the mapping

$$\lambda_s \circ \partial^\bullet : P^\bullet \rightarrow P^\bullet$$

is a derivation relative to  $s\partial$ . Let  $\psi \in \text{Hom}_{S^e}(P^\bullet, M)$  be a cocycle with cohomology class denoted by  $c$ . Since  $\psi \circ \lambda_s = \lambda_s \circ \psi$ ,

$$\mathcal{L}_{s\partial}(\psi) = (\lambda_s \circ \partial_M) \circ \psi - \psi \circ (\lambda_s \circ \partial^\bullet) = \lambda_s \circ \mathcal{L}_\partial(\psi).$$

Taking cohomology classes yields that  $\mathcal{L}_{s\partial}(c) = \lambda_s \circ \mathcal{L}_\partial(c)$ .

(5) Recall that, here,  $\partial_M$  is taken equal to

$$\begin{aligned} (s\partial)^e : S^e &\rightarrow S^e, \\ s_1 \otimes s_2 &\mapsto s\partial(s_1) \otimes s_2 + s_1 \otimes s\partial(s_2). \end{aligned}$$

Let  $\psi \in \text{Hom}_{S^e}(P^{-n}, M)$  be a cocycle with cohomology class denoted by  $c$ . Let  $h$  be as in part (3) of Lemma 4.2.1. Then,

$$\begin{aligned} \mathcal{L}_{s\partial}(\psi) &= (s\partial)^e \circ \psi - \psi \circ (s\partial)^\bullet \\ &= (s\partial \otimes 1 + 1 \otimes s\partial) \circ \psi - \psi \circ (s\partial)^\bullet \\ &= \lambda_s \circ (\partial \otimes 1) \circ \psi + \rho_s \circ (1 \otimes \partial) \circ \psi - \psi \circ (s\partial)^\bullet. \end{aligned}$$

Using (4-9), the equality becomes

$$\mathcal{L}_{s\partial}(\psi) = \lambda_s \circ (\partial \otimes 1) \circ \psi + \rho_s \circ (1 \otimes \partial) \circ \psi - \lambda_s \circ \psi \circ \partial^\bullet - \underbrace{\psi \circ h \circ d - \psi \circ d \circ h}_{=0}.$$

Using  $[\partial, \rho_s] = \rho_{\partial(s)}$ , it then becomes

$$\begin{aligned} \mathcal{L}_{s\partial}(\psi) &= \lambda_s \circ (\partial \otimes 1) \circ \psi + (1 \otimes \partial) \circ \rho_s \circ \psi - \rho_{\partial(s)} \circ \psi - \lambda_s \circ \psi \circ \partial^\bullet - \psi \circ h \circ d \\ &= \lambda_s \circ (\partial \otimes 1) \circ \psi - \rho_{\partial(s)} \circ \psi + (1 \otimes \partial) \circ \psi \circ (\rho_s - \lambda_s) \\ &\quad + (1 \otimes \partial) \circ \psi \circ \lambda_s - \lambda_s \circ \psi \circ \partial^\bullet - \psi \circ h \circ d. \end{aligned}$$

Using (4-2), this becomes

$$\begin{aligned} \mathcal{L}_{s\partial}(\psi) &= \lambda_s \circ (\partial \otimes 1) \circ \psi - \rho_{\partial(s)} \circ \psi - \underbrace{(1 \otimes \partial) \circ \psi \circ d \circ k_s}_{=0} \\ &\quad - (1 \otimes \partial) \circ \psi \circ k_s \circ d + \underbrace{(1 \otimes \partial) \circ \psi \circ \lambda_s}_{=(1 \otimes \partial) \circ \lambda_s \circ \psi = \lambda_s \circ (1 \otimes \partial) \circ \psi} - \lambda_s \circ \psi \circ \partial^\bullet - \psi \circ h \circ d \\ &= \lambda_s \circ (\partial \otimes 1 + 1 \otimes \partial) \circ \psi - \rho_{\partial(s)} \circ \psi - \lambda_s \circ \psi \circ \partial^\bullet - (\psi \circ h + (1 \otimes \partial) \circ \psi \circ k_s) \circ d \\ &= \lambda_s \circ (\mathcal{L}_\partial(\psi)) - \rho_{\partial(s)} \circ \psi - (\psi \circ h + (1 \otimes \partial) \circ \psi \circ k_s) \circ d. \end{aligned}$$

Now, consider the following  $R$ -linear mapping denoted by  $f$ :

$$\psi \circ h + (1 \otimes \partial) \circ \psi \circ k_s : P^{-n+1} \rightarrow S^e.$$

This is a morphism of  $S$ -bimodules. Indeed,

- it is a morphism of left  $S$ -modules because so are  $\psi$ ,  $1 \otimes \partial$ ,  $k_s$  and  $h$  (see (4-10));
- since  $\psi$  and  $k_s$  are morphisms of  $S$ -bimodules, then, for all  $t \in S$ ,

$$\begin{aligned} f \circ \rho_t &= \psi \circ h \circ \rho_t + (1 \otimes \partial) \circ \rho_t \circ \psi \circ k_s \\ &\stackrel{(4-10)}{=} \psi \circ (\rho_t \circ h - \rho_{\partial(t)} \circ k_s) + (1 \otimes \partial) \circ \rho_t \circ \psi \circ k_s \\ &= \rho_t \circ \psi \circ h - \rho_{\partial(t)} \circ \psi \circ k_s + (1 \otimes \partial) \circ \rho_t \circ \psi \circ k_s \\ &= \rho_t \circ \psi \circ h + \rho_t \circ (1 \otimes \partial) \circ \psi \circ k_s \\ &= \rho_t \circ f. \end{aligned}$$

Therefore,  $\mathcal{L}_{s\partial}(\psi)$  and  $\lambda_s \circ \mathcal{L}_\partial(\psi) - \rho_{\partial(s)} \circ \psi$  are cohomologous. Since so are  $\lambda_{\partial(s)} \circ \psi$  and  $\rho_{\partial(s)} \circ \psi$  it follows that

$$\mathcal{L}_{s\partial}(c) = \lambda_s \circ \mathcal{L}_\partial(c) - \lambda_{\partial(s)}(c). \quad \square$$

**4.4. The action of  $L$  on  $\text{Ext}_{S^e}^*(S, S^e)$ .** According to Lemma 4.3.1, the mapping

$$(4-13) \quad \begin{aligned} L \times \text{Ext}_{S^e}^n(S, S^e) &\rightarrow \text{Ext}_{S^e}^n(S, S^e), \\ (\alpha, e) &\mapsto \alpha \cdot e := \mathcal{L}_{\partial_\alpha}(e) \end{aligned}$$

endows  $\text{Ext}_{S^e}^*(S, S^e)$  with a compatible left  $S \rtimes L$ -module structure in the sense of (3-13), that is, a left  $S \rtimes L$ -module structure such that, for all  $e \in \text{Ext}_{S^e}^*(S, S^e)$ ,  $\alpha \in L$  and  $s \in S$ ,

$$(4-14) \quad (s\alpha) \cdot e = s \cdot (\alpha \cdot e) - \alpha(s) \cdot e.$$

This left  $S \rtimes L$ -module structure on  $\text{Ext}_{S^e}^*(S, S^e)$  does not define a left  $U$ -module structure in general. However, Lemma 3.5.1 yields that  $\text{Ext}_{S^e}^*(S, S^e)^\vee$  is a right  $U$ -module by defining  $\theta \cdot \alpha$ , for all  $\theta \in \text{Ext}_{S^e}^*(S, S^e)^\vee$  and  $\alpha \in L$ , as

$$\begin{aligned} \theta \cdot \alpha : \text{Ext}_{S^e}^n(S, S^e) &\rightarrow S, \\ e &\mapsto -\alpha(\theta(e)) + \theta(\alpha \cdot e). \end{aligned}$$

**4.5. Particular case of Van den Bergh and Calabi–Yau duality.** Recall that, whenever  $\text{Tor}_n^{S^e}(S, S) \simeq S$  as  $S$ -(bi)modules, a *volume form* is a free generator  $\omega_S$  of  $\text{Tor}_n^{S^e}(S, S)$ , and the associated *divergence*

$$\text{div} : \text{Der}_R(S) \rightarrow S$$

is defined such that, for all  $\partial \in \text{Der}_R(S)$ ,

$$(4-15) \quad \mathcal{L}_\partial(\omega_S) = \text{div}(\partial)\omega_S.$$

When  $S$  is Calabi–Yau in dimension  $n$ , any free generator  $e_S$  of the left  $S$ -module  $\text{Ext}_{S^e}^n(S, S^e)$  defines an isomorphism of  $S$ -bimodules

$$\begin{aligned} \theta : S &\rightarrow \text{Ext}_{S^e}^n(S, S^e), \\ s &\mapsto se_S. \end{aligned}$$

In such a situation, the fundamental class  $c_S \in \text{Tor}_n^{S^e}(S, \text{Ext}_{S^e}^n(S, S^e))$  (see 2.1) is a free generator of the left  $S$ -module  $\text{Tor}_n^{S^e}(S, \text{Ext}_{S^e}^n(S, S^e))$ , and hence the preimage  $\omega_S$  of  $c_S$  under the bijective mapping

$$\theta_* : \text{Tor}_n^{S^e}(S, S) \rightarrow \text{Tor}_n^{S^e}(S, \text{Ext}_{S^e}^n(S, S^e))$$

is a volume form for  $S$ , thus defining a divergence operator.

**Proposition 4.5.1.** (1) *Assume the following:*

- $R$  is Noetherian and  $S$  is finitely generated as an  $R$ -algebra.
- $S$  is projective in  $\text{Mod}(R)$ .
- $S$  has Van den Bergh duality with dimension  $n$ .

*Then there is an isomorphism of  $S$ -modules compatible with Lie derivatives*

$$\text{Ext}_{S^e}^n(S, S^e) \simeq \Lambda_S^n \text{Der}_R(S).$$

(2) Assume that  $S$  is Calabi–Yau in dimension  $n$ . Let  $e_S$  be a free generator of the left  $S$ -module  $\text{Ext}_{S^e}^n(S, S^e)$ . Let  $\text{div}$  be the resulting divergence operator. Then, for all  $\partial \in \text{Der}_R(S)$ ,

$$(4-16) \quad \mathcal{L}_\partial(e_S) = -\text{div}(\partial)e_S.$$

*Proof.* In both cases,  $S$  lies in  $\text{per}(S^e)$ . Denote the fundamental class of  $S$  by  $c_S$ . In view of part (3) of Lemma 4.3.1, the definition of  $c_S$  gives that

$$(4-17) \quad \mathcal{L}_\partial(c_S) = 0.$$

(1) Denote  $\text{Ext}_{S^e}^n(S, S^e)$  by  $D$ . In view of Proposition 2.2.1, [Hochschild et al. 1962, Theorem 3.1] applies and yields an isomorphism of  $S$ -modules,

$$(4-18) \quad \text{Tor}_n^{S^e}(S, S) \simeq \Lambda_S^n \Omega_{S/R}.$$

Following [Rinehart 1963, Section 9], this isomorphism is compatible with Lie derivatives. Identify  $D^{-1}$  with  $\text{Hom}_S(D, S)$  and define  $\partial_{D^{-1}}$  as follows, for all  $\partial \in \text{Der}_R(S)$ :

$$\begin{aligned} \partial_{D^{-1}} : \text{Hom}_S(D, S) &\rightarrow \text{Hom}_S(D, S), \\ f &\mapsto \partial \circ f - f \circ \mathcal{L}_\partial. \end{aligned}$$

The evaluation isomorphism

$$(4-19) \quad \text{ev} : D \otimes_S D^{-1} \xrightarrow{\sim} S$$

is compatible with Lie derivatives in the following sense, where  $\partial \in \text{Der}_R(S)$ :

$$(4-20) \quad \partial \circ \text{ev} = \text{ev} \circ (\mathcal{L}_\partial \otimes \text{Id} + \text{Id} \otimes \partial_{D^{-1}}).$$

Besides, the duality isomorphism

$$(4-21) \quad \iota_?(c_S) : \text{Ext}_{S^e}^0(S, D^{-1}) \rightarrow \text{Tor}_n^{S^e}(S, D \otimes_S D^{-1})$$

is compatible with the action of Lie derivatives because of (4-17) (see part (3) of Lemma 4.3.1). Combining (4-18), (4-19), (4-20) and (4-21) yields an isomorphism that is compatible with Lie derivatives

$$D^{-1} \simeq \Lambda_S^n \Omega_{S/R}.$$

This proves (1).

(2) Keep the notation  $c_S$ ,  $\omega_S$ ,  $\theta$ ,  $\theta_*$  for the objects defined from  $e_S$  before the statement of the proposition. Let  $\partial \in \text{Der}_R(S)$ . There exists  $\lambda \in S$  such that

$$\mathcal{L}_\partial(e_S) = \lambda e_S.$$

Now, for all  $s \otimes p \in S \otimes_{S^e} P^{-n}$ ,

$$\begin{aligned} \mathcal{L}_\partial(\theta_*(s \otimes p)) &= \mathcal{L}_\partial(se_S \otimes p) \\ &= \partial(s)e_S \otimes p + s\mathcal{L}_\partial(e_S) \otimes p + se_S \otimes \partial^*(p) \\ &= \theta_*(\mathcal{L}_\partial(s \otimes p)) + \lambda\theta_*(s \otimes p). \end{aligned}$$

Therefore,

$$0 = \mathcal{L}_\partial(c_S) = \mathcal{L}_\partial(\theta_*(\omega_S)) = \theta_*(\underbrace{\mathcal{L}_\partial(\omega_S)}_{=\text{div}(\partial)\omega_S}) + \lambda\theta_*(\omega_S) = (\lambda + \text{div}(\partial))c_S.$$

Since  $c_S$  is regular,  $\lambda = -\text{div}(\partial)$ . □

## 5. Proof of the main theorems

The main results of this article are proved in this section. For this purpose, a description of  $\text{Ext}_{U^e}^*(U, U^e)$  is made in Section 5.1, the underlying  $S$ -module is expressed in terms of  $\text{Ext}_{S^e}^*(S, S^e)$  and  $\text{Ext}_U^*(S, U)$ , and the  $U$ -bimodule structure is described using the functor  $F : \text{Mod}(U) \rightarrow \text{Mod}(U^e)$  and the action of  $L$  on  $\text{Ext}_{S^e}^*(S, S^e)$  introduced in Section 4. This description is applied in Section 5.2 in order to prove Theorem 1. And Theorem 2 and Corollary 1 are proved in Sections 5.3 and 5.4 by specialising to the situations where  $\text{Ext}_{S^e}^{\text{top}}(S, S^e)$  and  $\text{Ext}_U^{\text{top}}(S, U)$  are free, and where  $(S, L)$  arises from a Poisson bracket on  $S$ , respectively.

Throughout the section,  $\text{Ext}_{S^e}^*(S, S^e)$  is endowed with its compatible left  $S \rtimes L$ -module structure introduced in Section 4.4.

**5.1. The inverse dualising bimodule of  $U$ .** This subsection proves the following result.

**Proposition 5.1.1.** *Let  $R$  be a commutative ring and  $d \in \mathbb{N}$ . Let  $(S, L)$  be a Lie–Rinehart algebra over  $R$ . Assume the following:*

- (a)  $S$  is flat as an  $R$ -module.
  - (b) For all  $n \in \mathbb{N}$ , the  $S$ -module  $\text{Ext}_{S^e}^n(S, S^e)$  is projective.
  - (c)  $S \in \text{per}(S^e)$ .
  - (d)  $L$  is finitely generated and projective with constant rank equal to  $d$  in  $\text{Mod}(S)$ .
- Then,  $\Lambda_S^d L^\vee \otimes_S \text{Ext}_{S^e}^*(S, S^e)$  is a graded left  $U$ -module such that, for all  $\alpha \in L$ ,  $c \in \text{Ext}_{S^e}^*(S, S^e)$  and  $\varphi \in \Lambda_S^d L^\vee$ ,

$$\alpha \cdot (\varphi \otimes c) = -\varphi \cdot \alpha \otimes c + \varphi \otimes \alpha \cdot c.$$

Moreover,  $U$  is homologically smooth. Finally, there is an isomorphism of graded right  $U^e$ -modules,

$$\text{Ext}_{U^e}^*(U, U^e) \simeq F(\Lambda_S^d L^\vee \otimes_S \text{Ext}_{S^e}^{*-d}(S, S^e)).$$

For this subsection, assume (a), (b), (c) and (d) are true, and consider

- a bounded resolution  $Q^\bullet \rightarrow S$  in  $\text{Mod}(U)$  by finitely generated and projective modules (see [Rinehart 1963, Lemma 4.1]),
- a bounded resolution  $\pi : P^\bullet \rightarrow S$  in  $\text{Mod}(S^e)$  by finitely generated and projective modules,
- an injective resolution  $j : U^e \rightarrow I^\bullet$  in  $\text{Mod}(U^e \otimes (U^e)^{\text{op}})$ .

Since  $S$  is flat over  $R$  and  $L$  is projective in  $\text{Mod}(S)$ , part (2) of Lemma 3.0.1 gives that  $U^e$  is flat over  $R$ . Therefore, the extension-of-scalars functor

$$- \otimes U^e : \text{Mod}(U^e) \rightarrow \text{Mod}(U^e \otimes (U^e)^{\text{op}})$$

is exact. Hence, the restriction-of-scalars-functor transforms injective  $U^e$ -bimodules into injective left  $U^e$ -modules. Thus,  $I^\bullet$  is an injective resolution of  $U^e$  in  $\text{Mod}(U^e)$ . Therefore, there is an isomorphism of graded right  $U^e$ -modules,

$$(5-1) \quad \text{Ext}_{U^e}^\bullet(U, U^e) \simeq H^\bullet \text{Hom}_{U^e}(U, I^*).$$

The right-hand side is a right  $U^e$ -module by means of  $I^*$ .

The proof of the above proposition is divided into separate lemmas.

**Lemma 5.1.2.**  *$U$  is homologically smooth.*

*Proof.* Since  $U$  is projective in  $\text{Mod}(S)$  (see part (2) of Lemma 3.0.1), the functor

$$F : \text{Mod}(U) \rightarrow \text{Mod}(U^e)$$

is exact. Moreover,  $F(S) \simeq U$  and  $S \in \text{per}(U)$ . Therefore, in order to prove that  $U$  is homologically smooth, it suffices to prove that  $F(U) \in \text{per}(U^e)$ , which is equivalent to  $F(U)$  being compact in the derived category  $\mathcal{D}(U^e)$  of complexes of  $U$ -bimodules. Here is a proof of this fact. Let  $(M_k)_{k \in K}$  be a family in  $\mathcal{D}(U^e)$ , denote  $\bigoplus_{k \in K} M_k$  by  $M$ , and consider fibrant resolutions of complexes of  $U$ -bimodules  $M_k \rightarrow i(M_k)$ , for all  $k \in K$ , and  $M \rightarrow i(M)$ . Since  $S$  is homologically smooth,  $S$  is compact in  $\mathcal{D}(S^e)$ , and hence the following natural mapping is a quasi-isomorphism:

$$\bigoplus_{k \in K} \text{Hom}_{S^e}(P^\bullet, M_k) \rightarrow \text{Hom}_{S^e}(P^\bullet, M).$$

Since  $P^\bullet$  is a right bounded complex of projective  $S$ -bimodules, the functor  $\text{Hom}_{S^e}(P^\bullet, -)$  preserves quasi-isomorphisms, and hence the following natural mapping is a quasi-isomorphism:

$$\bigoplus_{k \in K} \text{Hom}_{S^e}(P^\bullet, i(M_k)) \rightarrow \text{Hom}_{S^e}(P^\bullet, i(M)).$$

Since  $U$  is projective over  $S$  on both sides,  $U^e$  is projective in  $\text{Mod}(S^e)$ . Therefore, for all fibrant complexes  $I$  of  $U$ -bimodules, the functor  $\text{Hom}_{S^e}(-, I)$  preserves quasi-isomorphisms. Accordingly, the following natural mapping is a quasi-isomorphism:

$$\bigoplus_{k \in K} \text{Hom}_{S^e}(S, i(M_k)) \rightarrow \text{Hom}_{S^e}(S, i(M)).$$

Since the pair  $(F, G)$  is adjoint and  $G$  is induced by the functor  $\text{Hom}_{S^e}(S, -)$ , the following natural mapping is a quasi-isomorphism:

$$\bigoplus_{k \in K} \text{Hom}_{U^e}(F(U), i(M_k)) \rightarrow \text{Hom}_{U^e}(F(U), i(M)).$$

Taking cohomology in degree 0 yields that the following natural mapping is bijective:

$$\bigoplus_{k \in K} \mathcal{D}(U^e)(F(U), i(M_k)) \rightarrow \mathcal{D}(U^e)(F(U), i(M)).$$

This proves that  $F(U)$  is compact in  $\mathcal{D}(U^e)$ . Thus,  $U$  is homologically smooth.  $\square$

The authors thank Bernhard Keller for having pointed out an incorrect argument in a previous version of this proof.

**Lemma 5.1.3.** *There is an isomorphism of graded right  $U^e$ -modules,*

$$(5-2) \quad \text{Ext}_{U^e}^\bullet(U, U^e) \simeq H^\bullet(\text{Hom}_U(Q^*, U) \otimes_U G(I^*)).$$

*Proof.* Because of the isomorphism  $F(S) \simeq U$  in  $\text{Mod}(U^e)$  and the adjunction  $(F, G)$ , there is a functorial isomorphism of complexes of right  $U^e$ -modules,

$$(5-3) \quad \text{Hom}_{U^e}(U, I^\bullet) \simeq \text{Hom}_U(S, G(I^\bullet)).$$

Since  $F$  is exact and the pair  $(F, G)$  is adjoint,  $G(I^\bullet)$  is a left bounded complex of injective left  $U$ -modules. Hence,  $\text{Hom}_U(-, G(I^\bullet))$  preserves quasi-isomorphisms. Thus, the quasi-isomorphism  $Q^\bullet \rightarrow S$  induces a quasi-isomorphism of complexes of right  $U^e$ -modules,

$$(5-4) \quad \text{Hom}_U(S, G(I^\bullet)) \rightarrow \text{Hom}_U(Q^\bullet, G(I^\bullet)).$$

Since  $Q^\bullet$  is bounded and consists of finitely generated projective left  $U$ -modules, the following canonical mapping is a functorial isomorphism:

$$(5-5) \quad \text{Hom}_U(Q^\bullet, U) \otimes_U G(I^\bullet) \rightarrow \text{Hom}_U(Q^\bullet, G(I^\bullet)).$$

Note that, whether in (5-3), (5-4), or (5-5), the involved right  $U^e$ -module structures are inherited from  $I^\bullet$ . Thus, the announced isomorphism is proved.  $\square$

In order to examine the right-hand side of (5-2) by means of a spectral sequence, the following lemma describes  $H^\bullet(G(I^*))$  as a graded  $U - U^e$ -bimodule.



**Lemma 5.1.4.** *Consider  $\text{Ext}_{S^e}^*(S, S^e)$  as a left  $S \rtimes L$ -module as in Section 4.4. Then, there is a  $U - U^e$ -bimodule structure on  $\text{Ext}_{S^e}^*(S, S^e) \otimes_{S^e} U^e$  such that the right  $U^e$ -module structure is inherited from  $U^e$  and for all  $\alpha \in L$ ,  $c \in \text{Ext}_{S^e}^*(S, S^e)$  and  $u, v \in U$ ,*

$$\alpha \cdot (c \otimes (u \otimes v)) = \alpha \cdot c \otimes (u \otimes v) + c \otimes ((\alpha \otimes 1 - 1 \otimes \alpha) \cdot (u \otimes v)).$$

*For this structure, there is an isomorphism of graded  $U - U^e$ -bimodules,*

$$H^*(G(I^*)) \simeq \text{Ext}_{S^e}^*(S, S^e) \otimes_{S^e} U^e.$$

*Proof.* The object  $G(I^*)$  is  $\text{Hom}_{S^e}(S, I^*)$  as a complex of  $S$ -modules, its right  $U^e$ -module structure is inherited from  $I^*$ , and the one of left  $U$ -module is given in Section 3.2.

First, since  $U^e$  is projective in  $\text{Mod}(S^e)$  and  $I^*$  consists of injective left  $U^e$ -modules,  $I^*$  is a left bounded complex of injective left  $S^e$ -modules. Hence,  $\text{Hom}_{S^e}(-, I^*)$  preserves quasi-isomorphisms. Thus,  $\pi : P^\bullet \rightarrow S$  induces a quasi-isomorphism of complexes of right  $S^e$ -modules,

$$(5-6) \quad \pi' : \text{Hom}_{S^e}(S, I^*) \rightarrow \text{Hom}_{S^e}(P^\bullet, I^*).$$

For all  $\alpha \in L$ , let  $\partial_\alpha^\bullet : P^\bullet \rightarrow P^\bullet$  be a derivation relative to  $\partial_\alpha : S \rightarrow S$  (see Section 4.2), and denote by  $\delta_\alpha^\bullet$  the mapping from  $I^*$  to  $I^*$  given by

$$i \mapsto (\alpha \otimes 1 - 1 \otimes \alpha) \cdot i.$$

Then, define  $\alpha \cdot f$  and  $\alpha \cdot g$ , for all  $f \in \text{Hom}_{S^e}(S, I^*)$  and  $g \in \text{Hom}_{S^e}(P^\bullet, I^*)$ , by

$$\begin{aligned} \alpha \cdot f &= \delta_\alpha^\bullet \circ f - f \circ \partial_\alpha \\ \alpha \cdot g &= \delta_\alpha^\bullet \circ g - g \circ \partial_\alpha^\bullet; \end{aligned}$$

since  $\pi \circ \partial_\alpha^\bullet = \partial_\alpha \circ \pi$ ,

$$\pi'(\alpha \cdot f) = \alpha \cdot \pi'(f).$$

The hypotheses on  $P^\bullet$  yield an isomorphism of complexes of right  $U^e$ -modules,

$$(5-7) \quad \text{ev} : \text{Hom}_{S^e}(P^\bullet, S^e) \otimes_{S^e} I^* \rightarrow \text{Hom}_{S^e}(P^\bullet, I^*).$$

Endow the left-hand side term with the following action of  $L$ . For all  $\alpha \in L$  and  $\varphi \otimes i \in \text{Hom}_{S^e}(P^\bullet, S^e) \otimes_{S^e} I^*$ , denote by  $\alpha \cdot (\varphi \otimes i)$  the (well-defined) element of  $\text{Hom}_{S^e}(P^\bullet, S^e) \otimes_{S^e} I^*$ ,

$$\alpha \cdot \varphi \otimes i + \varphi \otimes (\delta_\alpha^\bullet i).$$

The assignment  $\varphi \otimes i \mapsto \alpha \cdot (\varphi \otimes i)$  is a morphism of complexes of  $R$ -modules from  $\text{Hom}_{S^e}(P^\bullet, S^e) \otimes_{S^e} I^*$  to itself. In view of (4-8) and of the identity

$$(\alpha \otimes 1 - 1 \otimes \alpha) \cdot ((s \otimes t) \cdot j) = \partial_\alpha(s \otimes t) \cdot j + (s \otimes t) \cdot (\alpha \otimes 1 - 1 \otimes \alpha) \cdot j$$

in  $I^\bullet$ , for all  $s, t \in S$  and  $j \in I^\bullet$ , the following holds:

$$(5-8) \quad \text{ev}(\alpha \cdot (\varphi \otimes i)) = \alpha \cdot \text{ev}(\varphi \otimes i).$$

$\text{Hom}_{S^e}(P^\bullet, S^e)$  is also a bounded complex of projective right  $S^e$ -modules. Hence, the functor  $\text{Hom}_{S^e}(P^\bullet, S^e) \otimes_{S^e} -$  preserves quasi-isomorphisms. Thus,  $j : U^e \rightarrow I^\bullet$  induces a quasi-isomorphism of right  $U^e$ -modules,

$$(5-9) \quad \text{Id} \otimes j : \text{Hom}_{S^e}(P^\bullet, S^e) \otimes_{S^e} U^e \rightarrow \text{Hom}_{S^e}(P^\bullet, S^e) \otimes_{S^e} I^\bullet.$$

Endow the left-hand side term with the following action of  $L$ . For all  $\alpha \in L$ ,  $\varphi \in \text{Hom}_{S^e}(P^\bullet, S^e)$  and  $u, v \in U$ , denote by  $\alpha \cdot (\varphi \otimes (u \otimes v))$  the following (well-defined) element of  $\text{Hom}_{S^e}(P^\bullet, S^e) \otimes_{S^e} U^e$ :

$$\alpha \cdot \varphi \otimes (u \otimes v) + \varphi \otimes ((\alpha \otimes 1 - 1 \otimes \alpha) \cdot (u \otimes v)).$$

The assignment  $\varphi \otimes (u \otimes v) \mapsto \alpha \cdot (\varphi \otimes (u \otimes v))$  is a morphism of complexes of  $R$ -modules from  $\text{Hom}_{S^e}(P^\bullet, S^e) \otimes_{S^e} U^e$  to itself, and

$$(\text{Id} \otimes j)(\alpha \cdot (\varphi \otimes (u \otimes v))) = \alpha \cdot ((\text{Id} \otimes j)(\varphi \otimes (u \otimes v)))$$

because  $j : U^e \rightarrow I^\bullet$  is a morphism of complexes of  $U^e - U^e$ -bimodules.

Since  $U^e$  is projective in  $\text{Mod}(S^e)$ , there is an isomorphism of right  $U^e$ -modules,

$$(5-10) \quad H^\bullet(\text{Hom}_{S^e}(P^\bullet, S^e) \otimes_{S^e} U^e) \simeq \text{Ext}_{S^e}^\bullet(S, S^e) \otimes_{S^e} U^e.$$

For all cocycles  $\varphi \in \text{Hom}_{S^e}(P^\bullet, S^e)$ , with cohomology class denoted by  $c$ , and for all  $\alpha \in L$  and  $u, v \in U$ , the image under (5-10) of the cohomology class of

$$\alpha \cdot (\varphi \otimes (u \otimes v))$$

is

$$(5-11) \quad \alpha \cdot c \otimes (u \otimes v) + c \otimes ((\alpha \otimes 1 - 1 \otimes \alpha) \cdot (u \otimes v)),$$

where  $\alpha \cdot c$  is defined in Section 4.4 (see (4-13)).

Combining (5-6), (5-7), (5-9), (5-10) yields an isomorphism of right  $U^e$ -modules,

$$(5-12) \quad \text{Ext}_{S^e}^\bullet(S, S^e) \otimes_{S^e} U^e \xrightarrow{\sim} H^\bullet(G(I^*)),$$

such that, for all  $\alpha \in L$ ,  $c \in \text{Ext}_{S^e}^\bullet(S, S^e)$  and  $u, v \in U$ , if  $\gamma$  denotes the image of  $c \otimes (u \otimes v)$  under (5-12), then  $\alpha \cdot \gamma$  is the image of (5-11).

Thus, applying part (1) of Lemma 3.5.2 to  $N = \text{Ext}_{S^e}^\bullet(S, S^e)$  yields the announced conclusion.  $\square$

*Proof of Proposition 5.1.1.* The statement relative to the left  $U$ -module structure on  $\Lambda_S^d L^\vee \otimes \text{Ext}_{S^e}^\bullet(S, S^e)$  follows from Lemma 3.5.1, and Lemma 5.1.2 shows that  $U$

is homologically smooth. The (first quadrant, cohomological) spectral sequence of the bicomplex

$$(5-13) \quad (\text{Hom}_U(Q^p, U) \otimes_U G(I^q))_{p,q}$$

converges to  $H^\bullet(\text{Hom}_U(Q^*, U) \otimes_U G(I^*))$  and its  $E_2^{p,q}$ -term is, for all  $p, q \in \mathbb{Z}$ ,

$$H_h^p(H_v^q(\text{Hom}_U(Q^\bullet, U) \otimes_U G(I^\bullet))).$$

Since  $\text{Hom}_U(Q^\bullet, U)$  consists of projective right  $U$ -modules, there is an isomorphism of right  $U^e$ -modules, for all  $p, q \in \mathbb{Z}$ ,

$$(5-14) \quad H^q(\text{Hom}_U(Q^p, U) \otimes_U G(I^\bullet)) \simeq \text{Hom}_U(Q^p, U) \otimes_U H^q(G(I^\bullet)).$$

The description of  $H^\bullet(G(I^*))$  made in Lemma 5.1.4 combines with (5-14) into the following isomorphism of right  $U^e$ -modules, for all  $p, q \in \mathbb{Z}$ :

$$(5-15) \quad H^q(\text{Hom}_U(Q^p, U) \otimes_U G(I^\bullet)) \simeq \text{Hom}_U(Q^p, U) \otimes_U (\text{Ext}_{S^e}^q(S, S^e) \otimes_{S^e} U^e).$$

Using Lemma 3.5.2 (part (2)), this isomorphism yields an isomorphism of right  $U^e$ -modules, for all  $p, q \in \mathbb{Z}$ :

$$(5-16) \quad H^q(\text{Hom}_U(Q^p, U) \otimes_U G(I^\bullet)) \simeq F(\text{Hom}_U(Q^p, U) \otimes_S \text{Ext}_{S^e}^q(S, S^e)).$$

Given that  $F$  is an exact functor, that  $\text{Ext}_{S^e}^q(S, S^e)$  is projective in  $\text{Mod}(S)$  for all  $q$  and that  $(S, L)$  has duality in dimension  $d$ , it follows from (5-16) that there is an isomorphism of right  $U^e$ -modules, for all  $p, q \in \mathbb{Z}$ ,

$$H_h^p(H_v^q(\text{Hom}_U(Q^\bullet, U) \otimes_U G(I^\bullet))) \simeq \begin{cases} F(\text{Ext}_U^d(S, U) \otimes_S \text{Ext}_{S^e}^q(S, S^e)) & \text{if } p = d, \\ 0 & \text{if } p \neq d. \end{cases}$$

Therefore, the spectral sequence of the bicomplex (5-13) degenerates at  $E_2$ . Thus,

$$H^\bullet(\text{Hom}_U(Q^*, U) \otimes_U G(I^*)) \simeq F(\text{Ext}_U^d(S, U) \otimes_S \text{Ext}_{S^e}^{\bullet-d}(S, S^e)) \text{ in } \text{Mod}(S^e).$$

The conclusion follows from (5-2) and from the isomorphism  $\text{Ext}_U^d(S, U) \simeq \Lambda_S^d L^\vee$  in  $\text{Mod}(U)$  established in [Huebschmann 1999, Theorem 2.10]  $\square$

### 5.2. Proof of the main theorem.

*Proof of Theorem 1.* Following Proposition 5.1.1,  $U$  is homologically smooth and there is an isomorphism of graded right  $U^e$ -modules,

$$\text{Ext}_{U^e}^\bullet(U, U^e) \simeq F(\Lambda_S^d L^\vee \otimes_S \text{Ext}_{S^e}^{\bullet-d}(S, S^e)).$$

According to Proposition 3.6.1, the functor  $F$  transforms left  $U$ -modules that are

invertible as  $S$ -modules into invertible  $U$ -bimodules. Note that

- $\Lambda_S^d L^\vee$  is invertible as an  $S$ -module because  $L$  is projective with constant rank, and
- $\text{Ext}_{S^e}^\bullet(S, S^e)$  is concentrated in degree  $n$  and  $\text{Ext}_{S^e}^n(S, S^e)$  is invertible as an  $S$ -module because  $S$  has Van den Bergh duality.

Thus,  $\text{Ext}_{U^e}^\bullet(U, U^e)$  is concentrated in degree  $n+d$  and  $\text{Ext}_{U^e}^{n+d}(U, U^e)$  is invertible as a  $U$ -bimodule. Hence,  $U$  has Van den Bergh duality in dimension  $n+d$ .  $\square$

**5.3. Proof of Theorem 2.** The hypotheses of Theorem 2 are assumed throughout this subsection. Let  $\varphi_L$  be a free generator of the  $S$ -module  $\Lambda_S^d L^\vee$ . Let  $e_S$  be a free generator of the  $S$ -module  $\text{Ext}_{S^e}^n(S, S^e)$ . Therefore, there exist mappings

$$\lambda_L, \lambda_S : L \rightarrow S$$

such that, for all  $\alpha \in L$ ,

$$\begin{cases} \alpha \cdot e_S = \lambda_S(\alpha) \cdot e_S, \\ \varphi_L \cdot \alpha = \lambda_L(\alpha) \cdot \varphi_S. \end{cases}$$

Some basic properties of these are summarised below.

**Lemma 5.3.1.** *Let  $\lambda$  be either one of  $\lambda_S$  or  $\lambda_L$ . Then, for all  $\alpha, \beta \in L$  and  $s \in S$ ,*

- (1)  $\lambda(s\alpha) = s\lambda(\alpha) - \alpha(s)$ ,
- (2)  $\lambda([\alpha, \beta]) = \alpha(\lambda(\beta)) - \beta(\lambda(\alpha))$ .

*Proof.* Assume that  $\lambda = \lambda_S$ . Let  $s \in S$  and  $\alpha \in L$ . Then, using Section 4.4,

$$\begin{aligned} (s\alpha) \cdot e_S &= s \cdot (\alpha \cdot e_S) - \alpha(s) \cdot e_S \\ &= (s\lambda(\alpha) - \alpha(s)) \cdot e_S, \end{aligned}$$

which proves (1), and

$$\begin{aligned} \alpha \cdot (\beta \cdot e_S) &= \alpha \cdot (\lambda(\beta) \cdot e_S) \\ &= \alpha(\lambda(\beta)) \cdot e_S + \lambda(\beta) \cdot (\alpha \cdot e_S) \\ &= (\alpha(\lambda(\beta)) + \lambda(\alpha)\lambda(\beta)) \cdot e_S, \end{aligned}$$

from which (2) may be proved directly. The proof when  $\lambda = \lambda_L$  is analogous, using the right  $U$ -module structure of  $\Lambda_S^d L^\vee$  instead of Section 4.4.  $\square$

As proved later, the following automorphism is a Nakayama automorphism for  $U$ .

**Lemma 5.3.2.** *There exists a unique  $R$ -algebra homomorphism,*

$$v : U \rightarrow U,$$

such that, for all  $s \in S$  and  $\alpha \in L$ ,

$$\begin{cases} v(s) = s, \\ v(\alpha) = \alpha + \lambda_L(\alpha) - \lambda_S(\alpha). \end{cases}$$

This is an automorphism of  $R$ -algebra.

*Proof.* The uniqueness is immediate. For all  $\alpha \in L$ , denote  $\alpha + \lambda_L(\alpha) - \lambda_S(\alpha)$  by  $v_\alpha$ . Then, for all  $s \in S$  and  $\alpha, \beta \in L$ ,

$$\begin{aligned} [v_\alpha, v_\beta] &= [\alpha + \lambda_L(\alpha) - \lambda_S(\alpha), \beta + \lambda_L(\beta) - \lambda_S(\beta)] \\ &\stackrel{\text{Lemma 5.3.1}}{=} [\alpha, \beta] + \lambda_L([\alpha, \beta]) - \lambda_S([\alpha, \beta]) = v_{[\alpha, \beta]}, \\ v_{s\alpha} &= s\alpha + \lambda_L(s\alpha) - \lambda_S(s\alpha) \\ &\stackrel{\text{Lemma 5.3.1}}{=} s\alpha + s\lambda_L(\alpha) - s\lambda_S(\alpha) = sv_\alpha, \\ [v_\alpha, s] &= [\alpha + \lambda_L(\alpha) - \lambda_S(\alpha), s] = \alpha(s). \end{aligned}$$

This proves the existence of  $v$ . Note that  $v$  preserves the filtration of  $U$  by the powers of  $L$  and that  $\text{gr}(v)$  is the identity mapping of  $U$ . Accordingly,  $v$  is bijective.  $\square$

Now it is possible to prove Theorem 2.

*Proof of Theorem 2.* From Theorem 1,  $U$  has Van den Bergh duality in dimension  $n + d$  and there is an isomorphism of  $U$ -bimodules,

$$(5-17) \quad \text{Ext}_{U^e}^{n+d}(U, U^e) \simeq F(\Lambda_S^d \Lambda^\vee \otimes_S \text{Ext}_{S^e}^n(S, S^e)),$$

where the tensor product inside  $F(\bullet)$  is a left  $U$ -module by (3-8).

Recall that  $\Lambda_S^d L^\vee$  and  $\text{Ext}_{S^e}^n(S, S^e)$  are freely generated by  $\varphi_L$  and  $e_S$ , respectively. Therefore, the following mapping is an isomorphism of left  $U$ -modules (see Section 3.3):

$$(5-18) \quad \begin{aligned} \Phi : U &\rightarrow F(\Lambda_S^d L^\vee \otimes_S \text{Ext}_{S^e}^n(S, S^e)), \\ u &\mapsto u \otimes (\varphi_L \otimes e_S). \end{aligned}$$

For all  $s \in S$ ,  $\alpha \in L$  and  $u \in U$ ,

$$\Phi(u)s = (u \otimes (\varphi_L \otimes e_S)) \cdot s = us \otimes (\varphi_L \otimes e_S) = \Phi(us),$$

$$\begin{aligned} \Phi(u)\alpha &= (u \otimes (\varphi_L \otimes e_S)) \cdot \alpha \\ &= u\alpha \otimes (\varphi_L \otimes e_S) - u \otimes \alpha \cdot (\varphi_L \otimes e_S) \\ &= u\alpha \otimes (\varphi_L \otimes e_S) - (-u \otimes (\varphi_L \cdot \alpha \otimes e_S) + u \otimes (\varphi_L \otimes \alpha \cdot e_S)) \\ &= (u(\alpha + \lambda_L(\alpha) - \lambda_S(\alpha))) \otimes (\varphi_L \otimes e_S) \\ &= \Phi(u(\alpha + \lambda_L(\alpha) - \lambda_S(\alpha))). \end{aligned}$$

Thus, denoting by  $\nu$  the automorphism of  $U$  considered in Lemma 5.3.2, then, for all  $u, v \in U$ ,

$$(5-19) \quad \Phi(u) \cdot v = \Phi(u\nu(v)).$$

Combining (5-17), (5-18) and (5-19) yields that there is an isomorphism of bimodules,

$$\text{Ext}_{U^e}^{n+d}(U, U^e) \simeq U^\nu.$$

Since  $\lambda_S = -\text{div}$  (see Proposition 4.5.1), this proves Theorem 2. □

**5.4. Case of Poisson algebras.**

*Proof of Corollary 1.* From Proposition 2.2.1,  $S$  has Van den Bergh duality in dimension  $n$ . Moreover, Proposition 4.5.1 yields an isomorphism of  $S$ -modules  $\Lambda_S^n \text{Der}_R(S) \simeq \text{Ext}_{S^e}^n(S, S^e)$  which is compatible with the action of Lie derivatives. Finally, according to (1-3), the dualising module of  $(S, \Omega_{S/R})$  is  $\Lambda_S^n \text{Der}_R(S)$  with right  $U$ -module structure such that, for all  $s \in S$  and  $\varphi \in \Lambda_S^n \text{Der}_R(S)$ ,

$$\varphi \cdot ds = -\mathcal{L}_{\{s, -\}}(\varphi).$$

Using these considerations, the corollary follows from Theorems 1 and 2. □

**6. Examples**

**6.1. The case where  $L$  is free as an  $S$ -module.** In this subsection, it is assumed that  $L$  is free as an  $S$ -module. Consider a basis  $(\alpha_1, \dots, \alpha_d)$  of  $L$  over  $S$ . Denote the dual basis of  $L^\vee$  by  $(\alpha_1^*, \dots, \alpha_d^*)$ . In particular,  $\Lambda_S^d L^\vee$  is free of rank one in  $\text{Mod}(S)$ , with a generator denoted by  $\varphi_L$ ,

$$\varphi_L = \alpha_1^* \wedge \dots \wedge \alpha_d^*.$$

For all  $i \in \{1, \dots, d\}$ , consider the matrix of  $\text{ad}_{\alpha_i}$ , denoted by  $(s_{j,k}^i)_{j,k} \in M_d(S)$ . Hence, for all  $i, k \in \{1, \dots, d\}$ ,

$$[\alpha_i, \alpha_k] = \sum_{j=1}^d s_{j,k}^i \alpha_j.$$

In this situation, the action of  $L$  on  $\Lambda_S^* L$  by Lie derivatives specialises as follows. For all  $i, j, k \in \{1, \dots, d\}$ ,

$$(\lambda_{\alpha_i}(\alpha_j^*))(\alpha_k) = \alpha_i(\alpha_j^*(\alpha_k)) - \alpha_j^*([\alpha_i, \alpha_k]) = -s_{j,k}^i.$$

Hence, for all  $i, j \in \{1, \dots, d\}$ ,

$$\lambda_{\alpha_i}(\alpha_j^*) = - \sum_{k=1}^d s_{j,k}^i \alpha_k^*.$$

Thus, the right  $U$ -module structure of  $\Lambda_S^d L^\vee$  is such that, for all  $\alpha \in L$ ,

$$(6-1) \quad \varphi_L \cdot \alpha = \text{Tr}(\text{ad}_\alpha)\varphi_L.$$

Using this simplified description of  $\Lambda_S^d L^\vee$  yields the following corollary of the main theorems of this article.

**Corollary 6.1.1.** *Let  $R$  be a commutative ring. Let  $(S, L)$  be a Lie–Rinehart algebra of  $R$ . Denote by  $U$  its enveloping algebra. Assume that*

- $S$  is flat as an  $R$ -module,
- $S$  has Van den Bergh duality in dimension  $n$ ,
- $L$  is free of rank  $d$  as an  $S$ -module.

*Let  $(\alpha_1, \dots, \alpha_d)$  be a basis of  $L$  over  $S$  as considered previously. Then,  $U$  has Van den Bergh duality in dimension  $n + d$  and there is an isomorphism of  $U$ -bimodules,*

$$\text{Ext}_{U^e}^{n+d}(U, U^e) \simeq U \otimes_S \text{Ext}_{S^e}^n(S, S^e),$$

*where the left  $U$ -module structure on  $U \otimes_S \text{Ext}_{S^e}^n(S, S^e)$  is the natural one and the right  $U$ -module structure is such that, for all  $u \in U$ ,  $e \in \text{Ext}_{S^e}^n(S, S^e)$  and  $\alpha \in L$ ,*

$$(u \otimes e) \cdot \alpha = u\alpha \otimes e + u \otimes \text{Tr}(\text{ad}_\alpha)e - u \otimes \mathcal{L}_{\partial_\alpha}(e).$$

*If, moreover,  $S$  is Calabi–Yau, then  $U$  is skew Calabi–Yau and each volume form on  $S$  determines a Nakayama automorphism  $\nu \in \text{Aut}_R(U)$  such that, for all  $s \in S$  and  $\alpha \in L$ ,*

$$\begin{cases} \nu(s) = s, \\ \nu(\alpha) = \alpha + \text{Tr}(\text{ad}_\alpha) + \text{div}(\partial_\alpha), \end{cases}$$

*where  $\text{div}$  denotes the divergence of the chosen volume form.*

*Proof.* In view of (6-1), there is an isomorphism of right  $U$ -modules,

$$\Lambda_S^d L^\vee \simeq S,$$

where the right  $U$ -module structure on the right-hand side term is such that, for all  $\alpha \in L$ ,

$$1 \cdot \alpha = \text{Tr}(\text{ad}_\alpha).$$

The corollary therefore follows directly from Theorems 1 and 2. □

The previous corollary applies to any Lie–Rinehart algebra arising from a Poisson structure on  $R[x_1, \dots, x_n]$ ,  $n \in \mathbb{N} \setminus \{0, 1\}$ .

**Example 6.1.2.** Let  $S = R[x, y]$ . Let  $P \in S$ . This defines a Poisson structure on  $S$  such that

$$\{x, y\} = P.$$

Let  $L := \Omega_{S/R}$  and consider  $(S, L)$  as a Lie–Rinehart algebra over  $R$  such that, for all  $s, t \in S$ ,

- $[ds, dt] = d\{s, t\}$ ;
- $\partial_{ds} = \{s, -\}$ .

Then  $(dx, dy)$  is a basis of  $\Omega_{S/R}$  over  $S$ . Note that

$$\begin{cases} \text{Tr}(\text{ad}_{dx}) = \text{div}(\partial_{dx}) = \frac{\partial P}{\partial y}, \\ \text{Tr}(\text{ad}_{dy}) = \text{div}(\partial_{dy}) = -\frac{\partial P}{\partial x}. \end{cases}$$

From Corollary 6.1.1,  $U$  is skew Calabi–Yau in dimension 4 and has a Nakayama automorphism  $\nu \in \text{Aut}_R(S)$  such that

$$\begin{cases} \nu(x) = x, & \nu(dx) = dx + 2\frac{\partial P}{\partial y}, \\ \nu(y) = y, & \nu(dy) = dy - 2\frac{\partial P}{\partial x}. \end{cases}$$

By considering the filtration of  $U$  by the powers of the image of  $L$  in  $U$ , with associated graded algebra the symmetric algebra of  $L$  over  $S$  (see [Rinehart 1963, Theorem 3.1]), it appears that  $U^\times = S^\times = R^\times$ . Accordingly,  $U$  has no nontrivial inner automorphism. Consequently,  $U$  is Calabi–Yau if and only if  $\nu = \text{Id}_U$ , that is, if and only if  $\text{char}(R) = 2$ , or else  $P \in R$ .

**Example 6.1.3.** Let  $S = R[x, y, z]$ . Let  $P_x, P_y, P_z \in S$  be such that

$$\vec{P} \wedge \text{curl}(\vec{P}) = 0,$$

where  $\vec{P}$  denotes

$$\begin{pmatrix} P_x \\ P_y \\ P_z \end{pmatrix}.$$

Hence, the following defines a Poisson bracket on  $S$ ,

$$\{x, y\} = P_z, \quad \{y, z\} = P_x, \quad \{z, x\} = P_y.$$

As in the previous example, let  $(S, L := \Omega_{S/R})$  be the associated Lie–Rinehart algebra over  $R$ . As is well-known,

$$\{x, -\} = P_z \frac{\partial}{\partial y} - P_y \frac{\partial}{\partial z}, \quad \{y, -\} = P_x \frac{\partial}{\partial z} - P_z \frac{\partial}{\partial x}, \quad \{z, -\} = P_y \frac{\partial}{\partial x} - P_x \frac{\partial}{\partial y}.$$



Therefore, using the basis  $(dx, dy, dz)$  of  $\Omega_{S/R}$  over  $S$ ,

$$\begin{pmatrix} \operatorname{div}(\partial_{dx}) \\ \operatorname{div}(\partial_{dy}) \\ \operatorname{div}(\partial_{dz}) \end{pmatrix} = \begin{pmatrix} \operatorname{Tr}(\operatorname{ad}_{dx}) \\ \operatorname{Tr}(\operatorname{ad}_{dy}) \\ \operatorname{Tr}(\operatorname{ad}_{dz}) \end{pmatrix} = \operatorname{curl}(\vec{P}).$$

Using Corollary 6.1.1, it follows that  $U$  is skew Calabi–Yau in dimension 6 and has a Nakayama automorphism  $\nu \in \operatorname{Aut}_R(S)$  such that

$$\begin{pmatrix} \nu(x) \\ \nu(y) \\ \nu(z) \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \nu(dx) \\ \nu(dy) \\ \nu(dz) \end{pmatrix} = \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} + 2 \operatorname{curl}(\vec{P}).$$

As in the previous example, there are no nontrivial inner automorphisms for  $U$ . Hence,  $U$  is Calabi–Yau if and only if  $\operatorname{char}(R) = 2$ , or else  $\operatorname{curl}(\vec{P}) = 0$ . In particular, when  $R$  contains  $\mathbb{Q}$  as a subring, then  $U$  is Calabi–Yau if and only if the Poisson bracket is Jacobian, that is, there exists  $Q \in S$  such that  $P = \overrightarrow{\operatorname{grad}}(Q)$ .

By the Quillen–Suslin Theorem, when  $R$  is a field and  $n \in \mathbb{N}$ , any  $R[x_1, \dots, x_n]$ -module that is finitely generated and projective is free. Hence, Corollary 6.1.1 also applies to all Lie–Rinehart algebras of the shape  $(R[x_1, \dots, x_n], L)$ , where  $R$  is a field.

**6.2. On two-dimensional Nambu–Poisson structures.** Following Corollary 1,  $U$  is skew Calabi–Yau when  $S$  is flat over  $R$  and Calabi–Yau and  $(S, L)$  is given by a Poisson bracket on  $S$ . Assuming these properties, this section computes a Nakayama automorphism of  $U$  for a class of examples of two-dimensional Nambu–Poisson structures (see [Laurent-Gengoux et al. 2013, Section 8.3]).

Let  $S = R[x, y, z]/(P)$  where  $P = 1 + T$  for some  $T \in R[x, y, z]$  which is  $(p, q, r)$ -homogeneous in the sense that  $p, q, r \in R$  and  $t := p\alpha + q\beta + r\gamma$  is a unit in  $R$  which does not depend on the monomial  $x^\alpha y^\beta z^\gamma$  appearing in  $T$ . The hypotheses imply the following equality in  $S$ :

$$(6-2) \quad px \frac{\partial P}{\partial x} + qy \frac{\partial P}{\partial y} + rz \frac{\partial P}{\partial z} = -t \ (\in R^\times).$$

Let  $Q \in R[x, y, z]$  and endow  $S$  with the Poisson structure such that

$$(6-3) \quad \{x, y\} = Q \frac{\partial P}{\partial z}, \quad \{y, z\} = Q \frac{\partial P}{\partial x}, \quad \{z, x\} = Q \frac{\partial P}{\partial y}.$$

Consider  $(S, L := \Omega_{S/R})$  as a Lie–Rinehart algebra such that, for all  $s, t, s' \in S$ ,

- $[ds, dt] = d\{s, t\}$ ,
- $(sdt)(s') = s\{t, s'\}$ .

Consider the following 2-form on  $S$ :

$$\omega_S = px dy \wedge dz + qy dz \wedge dx + rz dx \wedge dy.$$

According to (6-2),  $\Omega_{S/R}$  is a projective  $S$ -module of rank 2. And the relation

$$\frac{\partial P}{\partial x} dx + \frac{\partial P}{\partial y} dy + \frac{\partial P}{\partial z} dz = 0$$

in  $\Omega_{S/R}$  yields the following relations in  $\Lambda_S^2 \Omega_{S/R}$ :

$$\begin{aligned} \frac{\partial P}{\partial x} dx \wedge dy &= \frac{\partial P}{\partial z} dy \wedge dz, \\ \frac{\partial P}{\partial y} dy \wedge dz &= \frac{\partial P}{\partial x} dz \wedge dx, \\ \frac{\partial P}{\partial z} dz \wedge dx &= \frac{\partial P}{\partial y} dx \wedge dy. \end{aligned}$$

Combining with (6-2) yields

$$\begin{aligned} dx \wedge dy &= -t^{-1} \frac{\partial P}{\partial z} \omega_S, \\ dy \wedge dz &= -t^{-1} \frac{\partial P}{\partial x} \omega_S, \\ dz \wedge dx &= -t^{-1} \frac{\partial P}{\partial y} \omega_S. \end{aligned}$$

Thus,  $\omega_S$  is a volume form of  $S$ .

In order to determine the divergence of  $\omega_S$ , consider the derivations  $\delta_x, \delta_y, \delta_z \in \text{Der}_R(S)$  given by

$$\begin{array}{lll} \delta_x : x \mapsto 0 & \delta_y : x \mapsto -\frac{\partial P}{\partial z} & \delta_z : x \mapsto \frac{\partial P}{\partial y} \\ y \mapsto \frac{\partial P}{\partial z} & y \mapsto 0 & y \mapsto -\frac{\partial P}{\partial x} \\ z \mapsto -\frac{\partial P}{\partial y} & z \mapsto \frac{\partial P}{\partial x} & z \mapsto 0. \end{array}$$

Note that

$$\{x, -\} = Q\delta_x, \quad \{y, -\} = Q\delta_y \quad \text{and} \quad \{z, -\} = Q\delta_z.$$

Then,

$$\begin{aligned} \iota_{\delta_x}(\omega_S) &= \iota_{\delta_x}(px dy \wedge dz + qy dz \wedge dx + rz dx \wedge dy) \\ &= px \left( \frac{\partial P}{\partial z} dz + \frac{\partial P}{\partial y} dy \right) - qy \frac{\partial P}{\partial y} dx - rz \frac{\partial P}{\partial z} dx \\ &= t dx \quad (\text{see (6-2)}). \end{aligned}$$

Therefore, using the symmetry between  $x$ ,  $y$  and  $z$ ,

$$\operatorname{div}(\delta_x) = \operatorname{div}(\delta_y) = \operatorname{div}(\delta_z) = 0.$$

Apply Lemma 5.3.1, taking into account that  $\lambda_S = -\operatorname{div}$  (see (4-16)); then,

$$\operatorname{div}(\{x, -\}) = \operatorname{div}(Q\delta_x) = Q\operatorname{div}(\delta_x) + \delta_x(Q).$$

Therefore,

$$(6-4) \quad \operatorname{div}(\{x, -\}) = \frac{\partial Q}{\partial y} \frac{\partial P}{\partial z} - \frac{\partial Q}{\partial z} \frac{\partial P}{\partial y}.$$

Applying Corollary 1 gives that the enveloping algebra  $U$  of  $(S, \Omega_{S/R})$  is skew Calabi–Yau. It has a Nakayama automorphism  $\nu : U \rightarrow U$  such that, for all  $s \in S$ ,

$$\left\{ \begin{array}{l} \nu(s) = s, \\ \begin{pmatrix} \nu(dx) \\ \nu(dy) \\ \nu(dz) \end{pmatrix} = \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} + 2 \overrightarrow{\operatorname{grad}}(Q) \wedge \overrightarrow{\operatorname{grad}}(P). \end{array} \right.$$

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
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