Pacific Journal of Mathematics

DUALITY FOR DIFFERENTIAL OPERATORS OF LIE–RINEHART ALGEBRAS

THIERRY LAMBRE AND PATRICK LE MEUR

Volume 297 No. 2 December 2018

PACIFIC JOURNAL OF MATHEMATICS
Vol. 297, No. 2, 2018
dx.doi.org/10.2140/pjm.2018.297.405

DUALITY FOR DIFFERENTIAL OPERATORS OF LIE–RINEHART ALGEBRAS

THIERRY LAMBRE AND PATRICK LE MEUR

Let (S, L) be a Lie–Rinehart algebra over a commutative ring R. This article proves that, if S is flat as an R-module and has Van den Bergh duality in dimension n, and if L is finitely generated and projective with constant rank d as an S-module, then the enveloping algebra of (S, L) has Van den Bergh duality in dimension $n + d$. When, moreover, S is Calabi–Yau and the d-th exterior power of L is free over S, the article proves that the enveloping algebra is skew Calabi–Yau, and it describes a Nakayama automorphism of it. These considerations are specialised to Poisson enveloping algebras. They are also illustrated on Poisson structures over two- and three-dimensional polynomial algebras and on Nambu–Poisson structures on certain two-dimensional hypersurfaces.

Introduction

Rinehart [1963] introduced the concept of Lie–Rinehart algebra (S, L) over a commutative ring R and defined its enveloping algebra U. This generalises both constructions of universal enveloping algebras of R-Lie algebras and algebras of differential operators of commutative R-algebras. Huebschmann [1999] investigated Poincaré duality on the (co)homology groups of (S, L). This duality is defined by

MSC2010: primary 16E35, 16E40, 16S32, 16W25; secondary 17B63, 17B66.
Keywords: Lie–Rinehart algebra, enveloping algebra, Calabi–Yau algebra, skew Calabi–Yau algebra, Van den Bergh duality.

the existence of a right U-module C, called the *dualising* module of (S, L) such that, for all left U-modules M and $k \in \mathbb{N}$,

$$\mathrm{Ext}^k_U(S, M) \cong \mathrm{Tor}^U_{d-k}(C, M). \tag{0-1}$$

Chemla [1999] proved that for Lie–Rinehart algebras arising from affine complex Lie algebroids, the algebra U has a rigid dualising complex, which she determined, and has Van den Bergh duality [1998]. Having Van den Bergh duality in dimension n for an R-algebra A means that

- A is homologically smooth, that is, A lies in the perfect derived category $\mathrm{per}(A^e)$ of the algebra $A^e := A \otimes_R A^{\mathrm{op}}$; and
- $\mathrm{Ext}^{\bullet}_{A^e}(A, A^e)$ is zero for $\bullet \neq 0$ and invertible as an A-bimodule if $\bullet = n$.

When this occurs, there is a functorial isomorphism, for all A-bimodules M and integers i (see [Van den Bergh 1998]),

$$\mathrm{Ext}^i_{A^e}(A, M) \cong \mathrm{Tor}^{A^e}_{n-i}(A, \mathrm{Ext}^n_{A^e}(A, A^e) \otimes_A M);$$

and $\mathrm{Ext}^n_{A^e}(A, A^e)$ is called the inverse dualising bimodule of A. Two classes of algebras with Van den Bergh duality are of particular interest, namely,

- *Calabi–Yau* algebras, for which $\mathrm{Ext}^n_{A^e}(A, A^e)$ is required to be isomorphic to A as an A-bimodule (see [Ginzburg 2006]); and
- *skew Calabi–Yau* algebras, for which there exists an automorphism

  $$\nu \in \mathrm{Aut}_{R-\mathrm{alg}}(A)$$

  such that $\mathrm{Ext}^n_{A^e}(A, A^e) \simeq A^{\nu}$ as A-bimodules (see [Reyes et al. 2014]); here A^{ν} denotes the A-bimodule obtained from A by twisting the action of A on the right by ν.

The relevance of these algebras comes from their role in the noncommutative geometry initiated in [Artin and Schelter 1987] and in the investigation of Calabi–Yau categories, and also from the specificities of their Hochschild cohomology when R is a field. For instance, it is proved in [Ginzburg 2006; Lambre 2010] that the Gerstenhaber bracket of the Hochschild cohomology of Calabi–Yau algebras have a BV generator.

This article investigates when the enveloping algebra U of a general Lie–Rinehart algebra (S, L) over a commutative ring R has Van den Bergh duality.

It considers Lie–Rinehart algebras (S, L) such that S has Van den Bergh duality and is flat as an R-module, and L is finitely generated and projective with constant rank d as an S-module. Under these conditions, it is proved that U has Van den Bergh duality. Note that, when R is a perfect field, the former condition amounts to saying that S is a smooth affine R-algebra [Krähmer 2007]. Note also that, under the latter condition, it is proved in [Huebschmann 1999, Theorem 2.10] that (S, L) has duality in the sense of (0-1). Under the additional assumption that S is Calabi–Yau and $\Lambda^d L$ is free as an S-module, it appears as a corollary that U is skew Calabi–Yau, and a Nakayama automorphism may be described explicitly. These considerations are specialised to the situation where the Lie–Rinehart algebra (S, L) arises from a Poisson structure on S. Also they are illustrated by detailed examples in the following cases:

- For Poisson brackets on polynomial algebras in two or three variables.
- For Nambu–Poisson structures on two-dimensional hypersurfaces of the shape $1 + T(x, y, z) = 0$, where T is a weight homogeneous polynomial.

Throughout the article, R denotes a commutative ring, (S, L) denotes a Lie–Rinehart algebra over R and U denotes its enveloping algebra. Given an R-Lie algebra $\mathfrak{g}$, its universal enveloping algebra is denoted by $\mathcal{U}_R(\mathfrak{g})$. For an R-algebra A, the category of left A-modules is denoted by $\mathrm{Mod}(A)$ and $\mathrm{Mod}(A^{\mathrm{op}})$ is identified with the category of right A-modules. For simplicity, the piece of notation $\otimes$ is used for $\otimes_R$. All complexes have differential of degree $+1$.

1. Main results

A *Lie–Rinehart* algebra over a commutative ring R is a pair (S, L) where S is a commutative R-algebra and L is a Lie R-algebra which is also a left S-module, endowed with a homomorphism of R-Lie algebras,

$$\begin{aligned} L &\to \mathrm{Der}_R(S), \\ \alpha &\mapsto \partial_\alpha := \alpha(-), \end{aligned} \tag{1-1}$$

such that, for all $\alpha, \beta \in L$ and $s \in S$,

$$[\alpha, s\beta] = s[\alpha, \beta] + \alpha(s)\beta.$$

Following [Huebschmann 1999], the *enveloping* algebra U of (S, L) is identified with the algebra

$$(S \rtimes L)/I,$$

where $S \rtimes L$ is the smash-product algebra of S by the action of L on S by derivations and I is the two-sided ideal of $S \rtimes L$ generated by

$$\{s \otimes \alpha - 1 \otimes s\alpha \mid s \in S,\ \alpha \in L\}$$

(see Lemma 3.0.1); it is proved in [Huebschmann 1999] that this set generates I as a right ideal.

As mentioned in the introduction, when L is a finitely generated S-module with constant rank d, the Lie–Rinehart algebra (S, L) has duality in the sense of (0-1) with $C = \Lambda_S^d L^\vee$. Here $-^\vee$ is the duality $\mathrm{Hom}_S(-, S)$ and $\Lambda_S^d L^\vee$ is considered as a right U-module using the Lie derivative λ_α, for $\alpha \in L$ (see [Huebschmann 1999, Section 2]),

$$\lambda_\alpha : \Lambda_S^\bullet L^\vee \to \Lambda_S^\bullet L^\vee ;$$

this is the derivation of $\Lambda_S^\bullet L^\vee$ such that, for all $s \in S$, $\varphi \in L^\vee$ and $\beta \in L$,

$$\lambda_\alpha(s) = \alpha(s) \quad \text{and} \quad \lambda_\alpha(\varphi)(\beta) = \alpha(\varphi(\beta)) - \varphi([\alpha, \beta]).$$

The right U-module structure of $\Lambda_S^d L^\vee$ is such that, for all $\varphi \in \Lambda_S^d L^\vee$ and $\alpha \in L$,

$$\varphi \cdot \alpha = -\lambda_\alpha(\varphi). \tag{1-2}$$

The first main result of the article gives sufficient conditions for U to have Van den Bergh duality. It also describes the inverse dualising bimodule. Here are some explanations on this description. On one hand, R-linear derivations $\partial \in \mathrm{Der}_R(S)$ act on $\mathrm{Ext}^n_{S^e}(S, S^e)$, $n \in \mathbb{N}$, by Lie derivatives (see Section 4),

$$\mathcal{L}_\partial : \mathrm{Ext}^n_{S^e}(S, S^e) \to \mathrm{Ext}^n_{S^e}(S, S^e).$$

Combining with the action of L on S yields an action $\alpha \otimes e \mapsto \alpha \cdot e$ of L on $\mathrm{Ext}^n_{S^e}(S, S^e)$ such that, for all $\alpha \in L$ and $e \in \mathrm{Ext}^n_{S^e}(S, S^e)$,

$$\alpha \cdot e = \mathcal{L}_{\partial_\alpha}(e).$$

Although this is not a U-module structure on $\mathrm{Ext}^n_{S^e}(S, S^e)$, it defines a left U-module structure on $\Lambda_S^d L^\vee \otimes_S \mathrm{Ext}^n_{S^e}(S, S^e)$, $d \in \mathbb{N}$, such that, for all $\alpha \in L$, $\varphi \in \Lambda_S^d L^\vee$ and $e \in \mathrm{Ext}^n_{S^e}(S, S^e)$,

$$\alpha \cdot (\varphi \otimes e) = -\varphi \cdot \alpha \otimes e + \varphi \otimes \alpha \cdot e.$$

On the other hand, consider the functor

$$F : \mathrm{Mod}(U) \to \mathrm{Mod}(U^e)$$

(see Section 3.3) such that, if $N \in \mathrm{Mod}(U)$, then $F(N)$ equals $U \otimes_S N$ in $\mathrm{Mod}(U)$ and has a right U-module structure defined by the following formula, for all $\alpha \in L$, $u \in U$ and $n \in N$:

$$(u \otimes n) \cdot \alpha = u\alpha \otimes n - u \otimes \alpha \cdot n.$$

This functor takes left U-modules which are invertible as S-modules to invertible U-bimodules (see Section 3.6). The main result of this article is the following.

Theorem 1. *Let R be a commutative ring. Let (S, L) be a Lie–Rinehart algebra over R. Denote by U the enveloping algebra of (S, L). Assume that*

- *S is flat as an R-module,*
- *S has Van den Bergh duality in dimension n,*
- *L is finitely generated and projective with constant rank d as an S-module.*

Then, U has Van den Bergh duality in dimension $n+d$ and there is an isomorphism of U-bimodules,

$$\operatorname{Ext}^{n+d}_{U^e}(U, U^e) \simeq F(\Lambda^d_S L^\vee \otimes_S \operatorname{Ext}^n_{S^e}(S, S^e)).$$

Note that when R is Noetherian and S is finitely generated as an R-algebra and projective as an R-module, then there is an isomorphism of S-(bi)modules,

$$\operatorname{Ext}^n_{S^e}(S, S^e) \simeq \Lambda^n_S \operatorname{Der}_R(S)\,;$$

this isomorphism is compatible with the actions by Lie derivatives (see Section 4.5). The above theorem was proved in [Chemla 1999, Theorem 4.4.1] when $R = \mathbb{C}$ and S is finitely generated as a $\mathbb{C}$-algebra.

The preceding theorem specialises to the situation where the involved invertible S-modules are free. On one hand, when $(\Lambda^d_S L)^\vee$ is free as an S-module with free generator φ_L, there is an associated *trace* mapping

$$\lambda_L : L \to S,$$

such that, for all $\alpha \in L$,

$$\varphi_L \cdot \alpha = \lambda_L(\alpha) \cdot \varphi_L,$$

where the action on the left-hand side is given by (1-2) and that on the right-hand side is just given by the S-module structure. On the other hand, when S is Calabi–Yau in dimension n, each generator of the free of rank one S-module $\operatorname{Ext}^n_{S^e}(S, S^e)$ determines a volume form $\omega_S \in \Lambda^n_S \Omega_{S/R}$, and the *divergence*

$$\operatorname{div} : \operatorname{Der}_R(S) \to S$$

associated with ω_S is defined by the following equality, for all $\partial \in \operatorname{Der}_R(S)$:

$$\mathcal{L}_\partial(\omega_S) = \operatorname{div}(\partial)\omega_S;$$

(see 4.5 for details). The second main result of the article then reads as follows.

Theorem 2. *Let R be a commutative ring. Let (S, L) be a Lie–Rinehart algebra over R. Denote by U the enveloping algebra of (S, L). Assume that*

- *S is flat as an R-module,*
- *S is Calabi–Yau in dimension n,*
- *L is finitely generated and projective with constant rank d and $\Lambda^d_S L$ is free as S-modules.*

Then, U is skew Calabi–Yau with a Nakayama automorphism $\nu \in \operatorname{Aut}_R(U)$ such that, for all $s \in S$, and $\alpha \in L$,

$$\begin{cases} \nu(s) = s, \\ \nu(\alpha) = \alpha + \lambda_L(\alpha) + \operatorname{div}(\partial_\alpha), \end{cases}$$

where λ_L is any trace mapping on $\Lambda^d_S L^\vee$ and div *is any divergence.*

Among all Lie–Rinehart algebras, those arising from Poisson structures on S play a special role because of the connection to Poisson (co)homology. Recall that any R-bilinear Poisson bracket $\{-,-\}$ on S defines a Lie–Rinehart algebra structure on $(S, L) = (S, \Omega_{S/R})$ such that, for all $s, t \in S$,

- $\partial_{ds} = \{s, -\}$;
- $[ds, dt] = d\{s, t\}$.

In this case, the formulations of Theorems 1 and 2 simplify because, when $\Omega_{S/R}$ is projective with constant rank n as an S-module, the right U-module structure of $\Lambda_S^n \Omega_{S/R}^\vee$ (see (1-2)) is given by classical Lie derivatives; that is, for all $s \in S$,

$$\lambda_{ds}(\varphi) = \mathcal{L}_{\{s,-\}}(\varphi). \tag{1-3}$$

More precisely, these theorems specialise as follows.

Corollary 1. *Let R be a Noetherian ring. Let $(S, \{-,-\})$ be a finitely generated Poisson algebra over R. Denote by U the enveloping algebra of the associated Lie Rinehart algebra $(S, \Omega_{S/R})$. Assume that*

- *S is projective in* $\mathrm{Mod}(R)$;
- $S \in \mathrm{per}(S^e)$;
- *$\Omega_{S/R}$, which is then projective in* $\mathrm{Mod}(S)$, *has constant rank n.*

Then, U has Van den Bergh duality in dimension $2n$ and there is an isomorphism of U-bimodules,

$$\mathrm{Ext}^{2n}_{U^e}(U, U^e) \simeq U \otimes_S \Lambda_S^n \mathrm{Der}_R(S) \otimes_S \Lambda_S^n \mathrm{Der}_R(S),$$

where the right-hand side term is a left U-module in a natural way and a right U-module such that, for all $u \in U$, $\varphi, \varphi' \in \Lambda_S^n \mathrm{Der}_R(S)$ and $s \in S$,

$$(u \otimes \varphi \otimes \varphi') \cdot ds = u\, ds \otimes \varphi \otimes \varphi' - u \otimes (\mathcal{L}_{\{s,-\}}(\varphi) \otimes \varphi' + \varphi \otimes \mathcal{L}_{\{s,-\}}(\varphi')).$$

In particular, if S has a volume form, then U is skew Calabi–Yau with a Nakayama automorphism $\nu : U \to U$ such that, for all $s \in S$,

$$\begin{cases} \nu(s) = s, \\ \nu(ds) = ds + 2\,\mathrm{div}(\{s, -\}), \end{cases}$$

where div *is the divergence of the chosen volume form.*

For the case where $R = \mathbb{C}$ and S is finitely generated as a $\mathbb{C}$-algebra, the above corollary is announced in [Lü et al. 2017, Theorem 0.7, Corollary 0.8] using the main results of [Chemla 1999].

This article is structured as follows. Section 2 presents useful information on the case where S has Van den Bergh duality. Section 3 is devoted to technical lemmas on

U-(bi)modules. In particular, it presents the above mentioned functor F and its right adjoint G, which play an essential role in the proof of the main results. Section 4 introduces the action of L on $\mathrm{Ext}^\bullet_{S^e}(S, S^e)$ by Lie derivatives. This structure is used in Section 5 in order to describe $\mathrm{Ext}^\bullet_{U^e}(U, U^e)$ and prove Theorem 1, Theorem 2 and Corollary 1. Finally, Section 6 applies this corollary to a class of examples of Nambu–Poisson surfaces.

2. Poincaré duality for S

As proved in [Van den Bergh 1998] when R is a field, if S has Van den Bergh duality in dimension n, then there is a functorial isomorphism, for all S-bimodules N,

$$\mathrm{Ext}^\bullet_{S^e}(S, N) \simeq \mathrm{Tor}^{S^e}_{n-\bullet}(S, \mathrm{Ext}^n_{S^e}(S, S^e) \otimes_S N).$$

It is direct to check that this is still the case without assuming that R is a field. In view of the proof of the main results of the article, Section 2.1 relates the above mentioned isomorphism to the fundamental class of S, following [Lambre 2010], and Section 2.2 relates Van den Bergh duality to the regularity of commutative algebras, following [Krähmer 2007].

2.1. ***Fundamental class and contraction.*** Consider a projective resolution $P^\bullet$ in $\mathrm{Mod}(S^e)$,

$$\cdots \to P^{-2} \to P^{-1} \to P^0 \xrightarrow{\epsilon} S,$$

and let $p^0 \in P^0$ be such that $\epsilon(p_0) = 1_S$. For all $M, N \in \mathrm{Mod}(S^e)$ and $n \in \mathbb{N}$, define the contraction

$$\begin{aligned} \mathrm{Tor}^{S^e}_n(S, M) \times \mathrm{Ext}^n_{S^e}(S, N) &\to \mathrm{Tor}^{S^e}_0(S, M \otimes_S N), \\ (\omega, e) &\mapsto \iota_e(\omega) \end{aligned}$$

as the mapping induced by the following one:

$$\begin{aligned} M \otimes_{S^e} P^{-n} &\to \mathrm{Hom}_R\big(\mathrm{Hom}_{S^e}(P^{-n}, N), (M \otimes_S N) \otimes_{S^e} P^0\big), \\ x \otimes p &\mapsto (\varphi \mapsto (x \otimes \varphi(p)) \otimes p^0). \end{aligned}$$

This makes sense because $P^\bullet$ is concentrated in nonpositive degrees. The construction depends neither on the choice of p^0 nor on that of $P^\bullet$.

Following the proof of [Lambre 2010, Proposition 3.3], when $S \in \mathrm{per}(S^e)$ and n is taken equal to $\mathrm{pd}_{S^e}(S)$, the contraction induces an isomorphism for all $N \in \mathrm{Mod}(S^e)$,

$$\begin{aligned} \mathrm{Tor}^{S^e}_n(S, \mathrm{Ext}^n_{S^e}(S, S^e)) &\to \mathrm{Hom}_{S^e}\big(\mathrm{Ext}^n_{S^e}(S, N), \mathrm{Tor}^{S^e}_0(S, \mathrm{Ext}^n_{S^e}(S, S^e) \otimes_S N)\big), \\ \omega &\mapsto \iota_?(\omega). \end{aligned}$$

In the particular case $N = S^e$, the *fundamental class* of S is the element $c_S \in \operatorname{Tor}_n^{S^e}(S, \operatorname{Ext}_{S^e}^n(S, S^e))$ such that

$$(\iota_?(c_S))_{|\operatorname{Ext}^n_{S^e}(S,S^e)} = \operatorname{Id}_{\operatorname{Ext}^n_{S^e}(S,S^e)}.$$

Following the arguments in the proof of [Lambre 2010, Théorème 4.2], when S has Van den Bergh duality in dimension n, which gives that $n = \operatorname{pd}_{S^e}(S)$, the contraction with c_S induces an isomorphism, for all $N \in \operatorname{Mod}(S^e)$,

$$\iota_?(c_S) : \operatorname{Ext}^n_{S^e}(S, N) \xrightarrow{\sim} \operatorname{Tor}_0^{S^e}(S, \operatorname{Ext}^n_{S^e}(S, S^e) \otimes_S N). \tag{2-1}$$

When S is projective in $\operatorname{Mod}(R)$, the Hochschild complex $S^{\otimes \bullet+2}$ is a resolution of S in $\operatorname{Mod}(S^e)$ and the contraction

$$\operatorname{Tor}_n^{S^e}(S, M) \times \operatorname{Ext}^m_{S^e}(S, N) \to \operatorname{Tor}^{S^e}_{n-m}(S, M \otimes_S N),$$
$$(\omega, e) \mapsto \iota_e(\omega)$$

may be defined for all $M, N \in \operatorname{Mod}(S^e)$ and $m, n \in \mathbb{N}$, as the mapping induced at the level of Hochschild (co)chains by

$$M \otimes S^{\otimes n} \times \operatorname{Hom}_R(S^{\otimes m}, N) \to (M \otimes_S N) \otimes S^{\otimes(m-n)},$$
$$((x|s_1|\cdots|s_n), \psi) \mapsto (x \otimes \psi(s_1|\cdots|s_m)|s_{m+1}|\cdots|s_n).$$

When, in addition, S has Van den Bergh duality in dimension n, then [Lambre 2010, Théorème 4.2] asserts that the following mapping given by contraction with c_S is an isomorphism, for all $N \in \operatorname{Mod}(S^e)$ and $m \in \mathbb{N}$,

$$\iota_?(c_S) : \operatorname{Ext}^m_{S^e}(S, N) \to \operatorname{Tor}^{S^e}_{n-m}(S, \operatorname{Ext}^n_{S^e}(S, S^e) \otimes_S N).$$

2.2. ***Relationship to regularity.*** The main results of this article assume that S has Van den Bergh duality. For commutative algebras, this property is related to smoothness and regularity. The relationship is detailed in [Krähmer 2007] for the case where R is a perfect field, and is summarised below in the present setting.

Proposition 2.2.1 [Krähmer 2007]. *Let R be a Noetherian commutative ring. Let S be a finitely generated commutative R-algebra and projective as an R-module. Let $n \in \mathbb{N}$. The following properties are equivalent.*

(i) *S has Van den Bergh duality in dimension n.*

(ii) $\operatorname{gl.dim}(S^e) < \infty$ *and $\Omega_{S/R}$, which is then projective in* $\operatorname{Mod}(S)$, *has constant rank n.*

When these properties are true, $\operatorname{gl.dim}(S) < \infty$ *and* $\operatorname{Ext}^n_{S^e}(S, S^e) \simeq \Lambda^n_S \operatorname{Der}_R(S)$ *as S-modules.*

Proof. See [Krähmer 2007] for full details. Since S is projective over R, then $\mathrm{pd}_{(S^e)^e}(S^e) \leqslant 2\,\mathrm{pd}_{S^e}(S)$ [Cartan and Eilenberg 1956, Chap. IX, Proposition 7.4]; besides, using the Hochschild resolution of S in $\mathrm{Mod}(S^e)$ yields that

$$\mathrm{gl.\,dim}(S) \leqslant \mathrm{pd}_{S^e}(S) \leqslant \mathrm{gl.\,dim}(S^e);$$

thus

$$\begin{aligned} S \in \mathrm{per}(S^e) &\Leftrightarrow \mathrm{gl.\,dim}(S^e) < \infty \\ &\Rightarrow \mathrm{gl.\,dim}(S) < \infty. \end{aligned} \tag{2-2}$$

Note also that, following [Hochschild et al. 1962, Theorem 3.1],

$$\mathrm{gl.\,dim}(S^e) < \infty \quad \Rightarrow \quad \Omega_{S/R} \text{ is projective in } \mathrm{Mod}(S). \tag{2-3}$$

Denote by μ the multiplication mapping $S \otimes S \to S$. Assume $\mathrm{gl.\,dim}(S^e) < \infty$, let $\mathfrak{p} \in \mathrm{Spec}(S)$ ($\subseteq \mathrm{Spec}(S^e)$) and denote by d the rank of $(\Omega_{S/R})_{\mathfrak{p}}$. Since $\Omega_{S/R} \simeq \mathrm{Ker}(\mu)/\mathrm{Ker}(\mu)^2$ as modules over S ($\simeq S^e/\mathrm{Ker}(\mu)$) and $\mathrm{gl.\,dim}(S^e) < \infty$, the $(S^e)_{\mathfrak{p}}$-module $\mathrm{Ker}(\mu)_{\mathfrak{p}}$ is generated by a regular sequence having d elements. There results a Koszul resolution of $S_{\mathfrak{p}}$ in $\mathrm{Mod}((S^e)_{\mathfrak{p}})$. Using this resolution and the isomorphism $\mathrm{Ext}^{\bullet}_{S^e}(S, S^e)_{\mathfrak{p}} \simeq \mathrm{Ext}^{\bullet}_{(S^e)_{\mathfrak{p}}}(S_{\mathfrak{p}}, (S^e)_{\mathfrak{p}})$ in $\mathrm{Mod}((S^e)_{\mathfrak{p}})$ yields isomorphisms of $(S^e)_{\mathfrak{p}}$-modules,

$$\mathrm{Ext}^{\bullet}_{S^e}(S, S^e)_{\mathfrak{p}} \simeq \begin{cases} 0 & \text{if } \bullet \neq d, \\ S_{\mathfrak{p}} & \text{if } \bullet = d. \end{cases} \tag{2-4}$$

Now assume (i). Then, $\mathrm{gl.\,dim}(S^e) < \infty$ (see (2-2)), $\Omega_{S/R}$ is projective (see (2-3)) and has constant rank n (see (2-4)). Conversely, assume that $\mathrm{gl.\,dim}(S^e) < \infty$ and $\Omega_{S/R}$ has constant rank n. Then, $S \in \mathrm{per}(S^e)$ (see (2-2)) and the S-module (equivalently, the symmetric S-bimodule) $\mathrm{Ext}^{\bullet}_{S^e}(S, S^e)$ is zero if $\bullet \neq n$ and is invertible if $\bullet = n$ (see (2-4)). Thus,

$$\text{(i)} \Leftrightarrow \text{(ii)}.$$

Finally, assume that both (i) and (ii) are true. Then, $\mathrm{gl.\,dim}(S) < \infty$ (see (2-2)). Moreover, Van den Bergh duality [1998, Theorem 1] does apply here and provides an isomorphism of S-modules,

$$\mathrm{Ext}^0_{S^e}(S, \mathrm{Ext}^n_{S^e}(S; S^e)^{-1}) \simeq \mathrm{Tor}^{S^e}_n(S, S),$$

whereas [Hochschild et al. 1962, Theorem 3.1] yields an isomorphism of S-modules,

$$\mathrm{Tor}^{S^e}_n(S, S) \simeq \Lambda^n_S \Omega_{S/R}.$$

Thus, $\mathrm{Ext}^n_{S^e}(S, S^e) \simeq \Lambda^n_S \mathrm{Der}_R(S)$ in $\mathrm{Mod}(S)$. □

3. Material on U-(bi)modules

The purpose of this section is to introduce an adjoint pair of functors (F, G) between $\mathrm{Mod}(U)$ and $\mathrm{Mod}(U^e)$. In the proof of Theorem 1, the U-bimodule $\mathrm{Ext}^\bullet_{U^e}(U, U^e)$ is described as the image under F of a certain left U-module which is invertible as an S-module. This section develops the needed properties of F. Hence, Section 3.1 recalls the basic constructions of U-modules; Sections 3.2 and 3.3 introduce the functors G and F, respectively; Section 3.4 proves that (F, G) is adjoint; Section 3.5 introduces and collects basic properties of compatible left $S \rtimes L$-modules, these are applied in Section 4 to the action of L on $\mathrm{Ext}^\bullet_{S^e}(S, S^e)$ by Lie derivatives; and Section 3.6 proves that the functor F transforms left U-modules that are invertible as S-modules into invertible U-bimodules. These results are based on the description of U as a quotient of the smash-product $S \rtimes L$ given in the following lemma. This description is established in [Lambre et al. 2017, Proposition 2.10] in the case of Lie–Rinehart algebras arising from Poisson algebras.

Lemma 3.0.1. (1) *The identity mappings* $\mathrm{Id}_S : S \to S$ *and* $\mathrm{Id}_L : L \to L$ *induce an isomorphism of* R*-algebras*

$$(S \rtimes L)/I \to U, \tag{3-1}$$

where I *is the two-sided ideal of the smash-product algebra* $S \rtimes L$ *generated by*

$$\{s \otimes \alpha - 1 \otimes s\alpha \mid s \in S, \alpha \in L\}.$$

(2) *If* L *is projective as a left* S*-module, then* U *is projective both as a left and as a right* S*-module.*

Proof. (1) Recall (see [Rinehart 1963]) that U is defined as follows: Endow $S \oplus L$ with an R-Lie algebra structure such that, for all $s, t \in S$ and $\alpha, \beta \in L$,

$$[s + \alpha, t + \beta] = \alpha(t) - \beta(s) + [\alpha, \beta].$$

Then, U is the factor R-algebra of the subalgebra of the universal enveloping algebra $\mathcal{U}_R(S \oplus L)$ generated by the image of $S \oplus L$ by the two-sided ideal generated by the classes in $\mathcal{U}_R(S \oplus L)$ of the following elements, for $s, t \in S$ and $\alpha \in L$:

$$s \otimes t - st, \qquad s \otimes \alpha - s\alpha.$$

Recall also that $S \rtimes L$ is the R-algebra with underlying R-module

$$S \otimes \mathcal{U}_R(L),$$

such that the images of $S \otimes 1$ and $1 \otimes \mathcal{U}_R(L)$ are subalgebras, and the following hold, for all $s, t \in S$ and $\alpha, \beta \in S$:

$$\begin{cases} (s \otimes 1) \cdot (1 \otimes \alpha) = s \otimes \alpha, \\ (1 \otimes \alpha) \cdot (s \otimes 1) = \alpha(s) \otimes 1 + s \otimes \alpha. \end{cases}$$

Therefore, the natural mappings $S \to U$ and $L \to U$ induce an R-algebra homomorphism from $S \rtimes L$ to U. This homomorphism vanishes on I whence the R-algebra homomorphism (3-1).

Besides, the universal property of U stated in [Huebschmann 1999, Section 2, p. 110] yields an R-algebra homomorphism,

$$U \to (S \rtimes L)/I, \tag{3-2}$$

induced by the natural mappings $S \to (S \rtimes L/I)$ and $L \to (S \rtimes L)/I$. In view of the behaviour of (3-1) and (3-2) on the respective images of $S \cup L$, these algebra homomorphisms are inverse to each other.

(2) It is proved in [Rinehart 1963, Lemma 4.1] that U is projective as a left S-module. Consider the increasing filtration of U by the left S-submodules

$$0 \subseteq F_0U \subseteq F_1U \subseteq \cdots ,$$

where F_pU is the image of $\oplus_{i=0}^{p} S \otimes L^{\otimes i}$ in U, for all $p \in \mathbb{N}$. In view of the equality

$$\alpha s = s\alpha + \alpha(s)$$

in U for all $s \in S$ and $\alpha \in L$, the left S-module F_pU is also a right S-submodule of U, and $F_pU/F_{p-1}U$ is a symmetric S-bimodule for all $p \in \mathbb{N}$. Therefore, the considerations used in the proof of [Rinehart 1963, Lemma 4.1] may be adapted in order to prove that U is projective as a right S-module. □

3.1. ***Basic constructions of U-modules.*** Left $S \rtimes L$-modules are identified with R-modules N endowed with a left S-module structure, and a left L-module structure such that, for all $n \in N$, $\alpha \in L$ and $s \in S$,

$$\alpha \cdot (s \cdot n) = \alpha(s) \cdot n + s \cdot (\alpha \cdot n).$$

Left U-modules are identified with left $S \rtimes L$-modules N such that, for all $n \in N$, $\alpha \in L$ and $s \in S$,

$$s \cdot (\alpha \cdot n) = (s\alpha) \cdot n.$$

Recall that the action of L endows S with a left U-module structure.

Right $S \rtimes L$-modules are identified with the R-modules M endowed with a right S-module structure and a right L-module structure such that, for all $m \in M$, $\alpha \in L$ and $s \in S$,

$$(m \cdot \alpha) \cdot s = m \cdot \alpha(s) + (m \cdot s) \cdot \alpha.$$

Right U-modules are identified with right $S \rtimes L$-modules M such that, for all $m \in M$, $s \in S$ and $\alpha \in L$,

$$(m \cdot s) \cdot \alpha = m \cdot (s\alpha).$$

The following constructions are classical. The corresponding U-module structures are introduced in [Huebschmann 1999, Section 2].

Let M, M' be right $S \rtimes L$-modules. Let N, N' be a left $S \rtimes L$-module. Then:

- N is a right $S \rtimes L$-module for the right L-module structure such that, for all $n \in N$, $s \in S$ and $\alpha \in L$,

$$n \cdot s = s \cdot n \quad \text{and} \quad n \cdot \alpha = -\alpha \cdot n. \tag{3-3}$$

- $\mathrm{Hom}_S(N, N')$ is a left $S \rtimes L$-module for the left L-module structure such that, for all $f \in \mathrm{Hom}_S(N, N')$, $n \in N$ and $\alpha \in L$,

$$(\alpha \cdot f)(n) = \alpha \cdot f(n) - f(\alpha \cdot n); \tag{3-4}$$

  moreover, this is a left U-module structure if N and N' are left U-modules.

- $\mathrm{Hom}_S(M, M')$ is a left $S \rtimes L$-module for the left L-module structure such that, for all $f \in \mathrm{Hom}_S(M, M')$, $m \in M$ and $\alpha \in L$,

$$(\alpha \cdot f)(m) = -f(m) \cdot \alpha + f(m \cdot \alpha). \tag{3-5}$$

- $\mathrm{Hom}_S(N, S)$ is a right $S \rtimes L$-module for the right L-module structure such that, for all $f \in \mathrm{Hom}_S(N, S)$, $n \in N$ and $\alpha \in L$,

$$(f \cdot \alpha)(n) = -\alpha(f(n)) + f(\alpha \cdot n). \tag{3-6}$$

- $N \otimes_S N'$ is a left $S \rtimes L$-module for the left L-module structure such that, for all $n \in N$, $n' \in N$ and $\alpha \in L$,

$$\alpha \cdot (n \otimes n') = \alpha \cdot n \otimes n' + n \otimes \alpha \cdot n'; \tag{3-7}$$

  moreover, this is a left U-module structure if N and N' are left U-modules.

- $M \otimes_S N$ is a left $S \rtimes L$-module for the left L-module structure such that, for all $m \in M$, $n \in N$ and $\alpha \in L$,

$$\alpha \cdot (m \otimes n) = -m \cdot \alpha \otimes n + m \otimes \alpha \cdot n. \tag{3-8}$$

3.2. ***The functor $G = \mathrm{Hom}_{S^e}(S, -) : \mathrm{Mod}(U^e) \to \mathrm{Mod}(U)$.*** Given $M \in \mathrm{Mod}(U^e)$, recall that

$$M^S = \{m \in M \mid (\text{for all } s \in S)\ (s \otimes 1 - 1 \otimes s) \cdot m = 0\}.$$

This is a symmetric S^e-submodule of M. Recall also the canonical isomorphisms that are inverse to each other:

$$\begin{aligned} M^S &\leftrightarrow \mathrm{Hom}_{S^e}(S, M) \\ m &\mapsto (s \mapsto (s \otimes 1) \cdot m) \\ \varphi(1) &\leftarrow\!\shortmid \varphi. \end{aligned} \tag{3-9}$$

Lemma 3.2.1. *Let $M \in \mathrm{Mod}(U^e)$. Then,*

(1) *M^S is a left U-module such that, for all $m \in M^S$ and $\alpha \in L$,*

$$\alpha \cdot m := (\alpha \otimes 1 - 1 \otimes \alpha) \cdot m; \tag{3-10}$$

(2) *the corresponding left U-module structure on $\mathrm{Hom}_{S^e}(S, M)$ (under the identification (3-9)) is such that, for all $\varphi \in \mathrm{Hom}_{S^e}(S, M)$, $\alpha \in L$ and $s \in S$,*

$$(\alpha \cdot \varphi)(s) = (\alpha \otimes 1 - 1 \otimes \alpha) \cdot \varphi(s) - \varphi(\alpha(s)).$$

Proof. (1) Given all $s \in S$ and $\alpha \in L$, denote

$$s \otimes 1 - 1 \otimes s \in U^e \quad \text{and} \quad \alpha \otimes 1 - 1 \otimes \alpha \in U^e$$

by ds and $d\alpha$, respectively; in particular

$$d\alpha \cdot ds = ds \cdot d\alpha + d(\alpha(s)),$$

and, for all $m \in M^S$,

$$ds \cdot (d\alpha \cdot m) = d\alpha \cdot (ds \cdot m) - d(\alpha(s)) \cdot m = 0,$$

which proves that $d\alpha \cdot m \in M^S$. Therefore, (3-10) defines a left L-module structure on M^S. Now, for all $m \in M^S$, $s \in S$ and $\alpha \in L$,

$$\begin{aligned}\alpha \cdot (s \otimes 1) \cdot m &= d\alpha \cdot (s \otimes 1) \cdot m = (\alpha(s) \otimes 1 + s\alpha \otimes 1 - s \otimes \alpha) \cdot m \\ &= (\alpha(s) \otimes 1) \cdot m + (s \otimes 1)(\alpha \otimes 1 - 1 \otimes \alpha) \cdot m \\ &= (\alpha(s) \otimes 1) \cdot m + (s \otimes 1) \cdot (\alpha \cdot m),\end{aligned}$$

$$\begin{aligned}(s \otimes 1) \cdot (\alpha \cdot m) &= (s \otimes 1) \cdot (\alpha \otimes 1 - 1 \otimes \alpha) \cdot m = (s\alpha \otimes 1) \cdot m - (s \otimes 1) \cdot (1 \otimes \alpha) \cdot m \\ &= (s\alpha \otimes 1) \cdot m - (1 \otimes \alpha) \cdot (s \otimes 1) \cdot m = (s\alpha \otimes 1) \cdot m - (1 \otimes \alpha) \cdot (1 \otimes s) \cdot m \\ &= (s\alpha \otimes 1 - 1 \otimes s\alpha) \cdot m = (s\alpha) \cdot m.\end{aligned}$$

Hence, this left L-module structure on M^S is a left U-module structure.

(2) By definition, $\mathrm{Hom}_{S^e}(S, M)$ is endowed with the left U-module structure such that (3-9) is an isomorphism in $\mathrm{Mod}(U)$. Let $\varphi \in \mathrm{Hom}_{S^e}(S, M)$, $\alpha \in L$ and $s \in S$. Then,

$$\begin{aligned}(\alpha \cdot \varphi)(s) &= (1 \otimes s) \cdot (\alpha \cdot \varphi(1)) = ((1 \otimes s)(\alpha \otimes 1 - 1 \otimes \alpha)) \cdot \varphi(1) \\ &= (\alpha \otimes s - 1 \otimes s\alpha - 1 \otimes \alpha(s)) \cdot \varphi(1) \\ &= ((\alpha \otimes 1 - 1 \otimes \alpha)(1 \otimes s) - 1 \otimes \alpha(s)) \cdot \varphi(1) \\ &= \alpha \cdot (1 \otimes s) \cdot \varphi(1) - (1 \otimes \alpha(s)) \cdot \varphi(1) = \alpha \cdot \varphi(s) - \varphi(\alpha(s)).\end{aligned}$$

□

Thus, the assignment $M \mapsto M^S$ defines a functor

$$G : \mathrm{Mod}(U^e) \to \mathrm{Mod}(U), \quad M \mapsto M^S. \tag{3-11}$$

3.3. *The functor $F = U \otimes_S - : \mathrm{Mod}(U) \to \mathrm{Mod}(U^e)$.* Let $N \in \mathrm{Mod}(U)$. In view of [Huebschmann 1999, (2.4)], $U_U \otimes_S N$ is a right U-module such that, for all $u \in U,\ n \in N,\ s \in S$ and $\alpha \in L$,

$$(u \otimes n) \cdot s = u \otimes sn = us \otimes n \quad \text{and} \quad (u \otimes n) \cdot \alpha = u\alpha \otimes n - u \otimes \alpha \cdot n.$$

Besides, $U \otimes_S N$ is a left U-module such that, for all $u, u' \in U$ and $n \in N$,

$$u' \cdot (u \otimes n) = u'u \otimes n.$$

Therefore, $U \otimes_S N$ is a U-bimodule, and hence a left U^e-module. These considerations define a functor,

$$F : \mathrm{Mod}(U) \to \mathrm{Mod}(U^e), \quad N \mapsto U \otimes_S N. \tag{3-12}$$

3.4. *The adjunction between F and G.*

Proposition 3.4.1. *The functors $F = U \otimes_S -$ and $G = \mathrm{Hom}_{S^e}(S, -)$ introduced in Section 3.2 and Section 3.3 form an adjoint pair,*

$$\begin{array}{c}\mathrm{Mod}\,U \\ F\big\downarrow\big\uparrow G \\ \mathrm{Mod}\,U^e\end{array}$$

In particular, there is a functorial isomorphism, for all $M \in \mathrm{Mod}(U^e)$ and $N \in \mathrm{Mod}(U)$,

$$\mathrm{Hom}_U(N, G(M)) \xrightarrow{\sim} \mathrm{Hom}_{U^e}(F(N), M).$$

Proof. Given $f \in \mathrm{Hom}_U(N, G(M))$, denote by $\Phi(f)$ the well-defined mapping

$$U \otimes_S N \to M, \quad u \otimes n \mapsto (u \otimes 1) \cdot f(n).$$

Consider $F(N)$ $(= U \otimes_S N)$ as a U-bimodule. Then, for all $u, u' \in U,\ n \in N,\ s \in S$ and $\alpha \in L$,

$$\begin{aligned}\Phi(f)(u' \cdot (u \otimes n)) = \Phi(f)(u'u \otimes n) &= (u'u \otimes 1) \cdot f(n) \\ &= (u' \otimes 1) \cdot \Phi(f)(u \otimes n),\end{aligned}$$

$$
\begin{aligned}
\Phi(f)((u\otimes n)\cdot s) &= \Phi(f)(u\otimes s\cdot n) = (u\otimes 1)\cdot f(s\cdot n)\\
&= (u\otimes 1)\cdot((1\otimes s)\cdot f(n)) = ((1\otimes s)\cdot(u\otimes 1))\cdot f(n)\\
&= (1\otimes s)\cdot\Phi(f)(u\otimes n) = (\Phi(f)(u\otimes n))\cdot s,
\end{aligned}
$$

$$
\begin{aligned}
\Phi(f)((u\otimes n)\cdot\alpha) &= \Phi(f)(u\alpha\otimes n - u\otimes\alpha\cdot n)\\
&= (u\alpha\otimes 1)\cdot f(n) - (u\otimes 1)\cdot f(\alpha\cdot n)\\
&= (u\alpha\otimes 1)\cdot f(n) - (u\otimes 1)\cdot(\alpha\otimes 1 - 1\otimes\alpha)\cdot f(n)\\
&= (u\otimes\alpha)\cdot f(n) = (1\otimes\alpha)\cdot\Phi(f)(u\otimes n)\\
&= (\Phi(f)(u\otimes n))\cdot\alpha.
\end{aligned}
$$

In other words,

$$\Phi(f)\in \mathrm{Hom}_{U^e}(F(N), M).$$

Given $g\in \mathrm{Hom}_{U^e}(F(N), M)$, then, for all $n\in N$ and $s\in S$,

$$(s\otimes 1 - 1\otimes s)\cdot g(1\otimes n) = g(s\otimes_S n - 1\otimes_S s\cdot n) = 0\,;$$

hence, denote by $\Psi(g)$ the well-defined mapping

$$
\begin{aligned}
N &\to M^S,\\
n &\mapsto g(1\otimes n).
\end{aligned}
$$

Therefore, for all $n\in N,\ s\in S$ and $\alpha\in L$,

$$
\begin{aligned}
\Psi(g)(s\cdot n) &= g(1\otimes s\cdot n) = g(s\otimes n) = g((s\otimes 1)\cdot(1\otimes n))\\
&= (s\otimes 1)\cdot g(1\otimes n) = (s\otimes 1)\cdot\Psi(g)(n),
\end{aligned}
$$

$$
\begin{aligned}
\Psi(g)(\alpha\cdot n) &= g(1\otimes\alpha\cdot n) = g(\alpha\otimes n - (1\otimes\alpha)\cdot(1\otimes n))\\
&= (\alpha\otimes 1)\cdot g(1\otimes n) - (1\otimes\alpha)\cdot g(1\otimes n) = \alpha\cdot\Psi(g)(n);
\end{aligned}
$$

in other words,

$$\Psi(g)\in \mathrm{Hom}_U(N, G(M)).$$

By construction, Ψ and Φ are inverse to each other. □

3.5. ***Compatible left $S\rtimes L$-modules.*** As explained in Section 1, the main results of this article are expressed in terms of the action of L on $\mathrm{Ext}^\bullet_{S^e}(S, S^e)$ by Lie derivatives and will be presented in Section 4. Although this action does not define a U-module structure on $\mathrm{Ext}^\bullet_{S^e}(S, S^e)$, it satisfies some compatibility with the S-module structure. The actions of L satisfying such a compatibility have specific properties that are used in the rest of the article and which are summarised below.

Define a *compatible* left $S \rtimes L$-module as a left $S \rtimes L$-module N such that, for all $n \in N$, $\alpha \in L$ and $s \in S$, the elements $s\alpha \in L$ and $\alpha(s) \in S$ satisfy

$$(s\alpha) \cdot n = s \cdot (\alpha \cdot n) - \alpha(s) \cdot n. \tag{3-13}$$

Note that a left $S \rtimes L$-module is both compatible and a left U-module if and only if L acts trivially, that is, by the zero action.

The two following lemmas present the properties of compatible left $S \rtimes L$-modules used in the rest of the article.

Lemma 3.5.1. *Let M be a right U-module. Let N be a compatible left $S \rtimes L$-module. Then*:

(1) *The right $S \rtimes L$-module $N^\vee = \mathrm{Hom}_S(N, S)$ is a right U-module.*

(2) *The left $S \rtimes L$-module $\mathrm{Hom}_S(N^\vee, M)$ is a left U-module.*

(3) *The left $S \rtimes L$-module $M \otimes_S N$ is a left U-module.*

(4) *The following canonical mapping is a morphism of left U-modules*:

$$\begin{aligned} \theta : M \otimes_S N &\to \mathrm{Hom}_S(N^\vee, M), \\ m \otimes n &\mapsto (\theta_{m\otimes n} : \varphi \mapsto m \cdot \varphi(n)). \end{aligned}$$

Proof. (1) Given $\varphi \in N^\vee$, $s \in S$ and $\alpha \in L$, then

$$\varphi \cdot (s\alpha) = (\varphi \cdot s) \cdot \alpha.$$

Indeed, for all $n \in N$,

$$\begin{aligned} (\varphi\cdot(s\alpha))(n) &= -(s\alpha)(\varphi(n)) + \varphi((s\alpha)\cdot n) \\ &= -s\,\alpha(\varphi(n)) + \varphi(s\cdot(\alpha\cdot n) - \alpha(s)\cdot n) \\ &= -s\,\alpha(\varphi(n)) + s\,\varphi(\alpha\cdot n) - \alpha(s)\varphi(n) \\ &= ((\varphi\cdot\alpha)\cdot s)(n) - (\varphi\cdot\alpha(s))(n) \\ &= ((\varphi\cdot s)\cdot\alpha)(n). \end{aligned}$$

(2) This is precisely [Huebschmann 1999, (2.3)].

(3) The $S \rtimes L$-module structure of $M \otimes_S N$ is described in (3-8). Given $m \in M$, $n \in N$, $s \in S$ and $\alpha \in L$, then

$$\begin{aligned} (s\alpha) \cdot (m \otimes n) &= -m \cdot (s\alpha) \otimes n + m \otimes (s\alpha) \cdot n \\ &= -(m \cdot \alpha) \cdot s \otimes n + m \cdot \alpha(s) \otimes n + m \otimes s \cdot (\alpha \cdot n) - m \otimes \alpha(s) \cdot n \\ &= s \cdot (\alpha \cdot (m \otimes n)). \end{aligned}$$

(4) It suffices to prove that the given mapping is L-linear. Let $m \in M$, $n \in N$, $\alpha \in L$ and $\varphi \in \mathrm{Hom}_S(N, S)$. Then,

$$\begin{aligned}(\alpha \cdot \theta_{m\otimes n})(\varphi) &= -\theta_{m\otimes n}(\varphi)\cdot\alpha + \theta_{m\otimes n}(\varphi\cdot\alpha) = -(m\cdot\varphi(n))\cdot\alpha + m\cdot(\varphi\cdot\alpha)(n)\\ &= -((m\cdot\alpha)\cdot\varphi(n) - m\cdot\alpha(\varphi(n))) + m\cdot(-\alpha(\varphi(n)) + \varphi(\alpha\cdot n))\\ &= -(m\cdot\alpha)\cdot\varphi(n) + m\cdot\varphi(\alpha\cdot n) = \theta_{\alpha\cdot(m\otimes n)}(\varphi)\,;\end{aligned}$$

thus, $\alpha\cdot\theta_{m\otimes n} = \theta_{\alpha\cdot(m\otimes n)}$. □

Any left $S \rtimes L$-module N may be considered as a symmetric S-bimodule, or equivalently a right S^e-module, such that, for all $n \in N$ and $s, s' \in S$,

$$n\cdot(s\otimes s') = (ss')\cdot n.$$

Accordingly, $N \otimes_{S^e} U^e$ is a right U^e-module in a natural way.

Lemma 3.5.2. *Let N be a compatible left $S \rtimes L$-module.*

(1) *The right U^e-module $N \otimes_{S^e} U^e$ is actually a U-U^e-bimodule such that for all $n \in N$, $u, v \in U$ and $\alpha \in L$,*

$$\alpha\cdot(n\otimes(u\otimes v)) = \alpha\cdot n\otimes(u\otimes v) + n\otimes((\alpha\otimes 1 - 1\otimes\alpha)\cdot(u\otimes v)).$$

(2) *Let M be a right U-module. Then, there exists an isomorphism of left U^e-modules*

$$\begin{aligned}F(M\otimes_S N) &\to M\otimes_U(N\otimes_{S^e} U^e),\\ v\otimes(m\otimes n) &\mapsto m\otimes(n\otimes(1\otimes v)).\end{aligned}$$

Proof. (1) Following part (3) of Lemma 3.5.1, there is a left U-module structure on $U \otimes_S N$ such that, for all $\alpha \in L$, $v \in U$ and $n \in N$,

$$\alpha\cdot(v\otimes n) = -v\alpha\otimes n + v\otimes\alpha\cdot n.$$

Therefore, there is a left U-module structure on $(U \otimes_S N) \otimes_S U$ (see (3-7)) such that, for all $\alpha \in L$, $n \in N$ and $u, v \in U$,

$$\begin{aligned}\alpha\cdot((v\otimes n)\otimes u) &= \alpha\cdot(v\otimes n)\otimes u + (v\otimes n)\otimes\alpha u\\ &= -(v\alpha\otimes n)\otimes u + (v\otimes\alpha\cdot n)\otimes u + (v\otimes n)\otimes\alpha u.\end{aligned}$$

Under the canonical identification

$$\begin{aligned}N\otimes_{S^e} U^e &\to (U\otimes_S N)\otimes_S U,\\ n\otimes(u\otimes v) &\mapsto (v\otimes n)\otimes u,\end{aligned}$$

$N \otimes_{S^e} U^e$ inherits a left U-module structure which is the one claimed in the statement.

Now, $N \otimes_{S^e} U^e$ inherits a right U^e-module structure from U^e. This structure is compatible with the left U-module structure discussed previously so as to yield a left $U \otimes (U^e)^{\mathrm{op}}$-module structure.

(2) Due to (1), there is a right U^e-module structure on $M \otimes_U (N \otimes_{S^e} U^e)$. It is considered here as a left U^e-module structure such that, for all $u, v, u', v' \in U$, $m \in M$ and $n \in N$,

$$(u' \otimes v') \cdot (m \otimes (n \otimes (u \otimes v))) = m \otimes (n \otimes (uv' \otimes u'v)). \tag{3-14}$$

For ease of reading, note that in $F(M \otimes_S N)$,

$$\begin{aligned} &(u \otimes 1) \cdot (v \otimes m \otimes n) = uv \otimes m \otimes n \\ &(1 \otimes \alpha) \cdot (v \otimes m \otimes n) = v\alpha \otimes m \otimes n + v \otimes m \cdot \alpha \otimes n - v \otimes m \otimes \alpha \cdot n, \end{aligned} \tag{3-15}$$

and, in $M \otimes_U (N \otimes_{S^e} U^e)$,

$$m \cdot \alpha \otimes n \otimes u \otimes v = m \otimes \alpha \cdot n \otimes u \otimes v + m \otimes n \otimes \alpha u \otimes v - m \otimes n \otimes u \otimes v\alpha. \tag{3-16}$$

The R-linear mapping from $U \otimes M \otimes N$ to $M \otimes_U (N \otimes_{S^e} U^e)$ given by

$$v \otimes m \otimes n \mapsto m \otimes (n \otimes (1 \otimes v))$$

induces a morphism of S-modules from $U \otimes_S (M \otimes_S N)$ to $M \otimes_U (N \otimes_{S^e} U^e)$ such as in the statement of the lemma. Denote it by Ψ':

$$\Psi' : U \otimes_S (M \otimes_S N) \to M \otimes_U (N \otimes_{S^e} U^e).$$

This is a morphism of left U^e-modules. Indeed, for all $u, v \in U$, $m \in M$, $n \in N$ and $\alpha \in L$,

$$\begin{aligned} \Psi'((u \otimes 1) \cdot (v \otimes m \otimes n)) &= \Psi'(uv \otimes m \otimes n) = m \otimes n \otimes 1 \otimes uv \\ &\underset{(3\text{-}14)}{=} (u \otimes 1) \cdot \Psi'(v \otimes m \otimes n), \end{aligned}$$

$$\begin{aligned} \Psi'((1 \otimes \alpha) \cdot (v \otimes m \otimes n)) &= \Psi'(v\alpha \otimes m \otimes n + v \otimes m \cdot \alpha \otimes n - v \otimes m \otimes \alpha \cdot n) \\ &= m \otimes n \otimes 1 \otimes v\alpha + m \cdot \alpha \otimes n \otimes 1 \otimes v - m \otimes \alpha \cdot n \otimes 1 \otimes v \\ &\underset{(3\text{-}16)}{=} m \otimes n \otimes \alpha \otimes v \\ &\underset{(3\text{-}14)}{=} (1 \otimes \alpha) \cdot \Psi'(v \otimes m \otimes n). \end{aligned}$$

Consider the following morphism of S-modules:

$$\begin{aligned} \phi : M \otimes_S (N \otimes_{S^e} U^e) &\to F(M \otimes_S N), \\ m \otimes (n \otimes (u \otimes v)) &\mapsto (1 \otimes u) \cdot (v \otimes m \otimes n). \end{aligned}$$

Given $m \in M$, $n \in N$, $u, v \in U$ and $\alpha \in L$, then the image under ϕ of the term

$$m \otimes \alpha \cdot n \otimes u \otimes v + m \otimes n \otimes \alpha u \otimes v - m \otimes n \otimes u \otimes v\alpha$$

is equal to

$$(1 \otimes u) \cdot (v \otimes m \otimes \alpha \cdot n) + (1 \otimes \alpha u) \cdot (v \otimes m \otimes n) - (1 \otimes u) \cdot (v\alpha \otimes m \otimes n),$$

which is equal to

$$(1 \otimes u) \cdot (v \otimes m \otimes \alpha \cdot n) + (1 \otimes u) \cdot (1 \otimes \alpha) \cdot (v \otimes m \otimes n) - (1 \otimes u) \cdot (v\alpha \otimes m \otimes n).$$

In view of (3-15), this is equal to

$$(1 \otimes u) \cdot (v \otimes m \cdot \alpha \otimes n) = \phi(m \cdot \alpha \otimes (n \otimes (u \otimes v))).$$

Thus, ϕ induces a morphism of S-modules

$$\begin{aligned} \Phi' : M \otimes_U (N \otimes_{S^e} U^e) &\to F(M \otimes_S N), \\ m \otimes (n \otimes (u \otimes v)) &\mapsto (1 \otimes u) \cdot (v \otimes m \otimes n). \end{aligned}$$

It appears that Φ' is left and right inverse for Ψ'. Indeed,

- $\Phi' \circ \Psi' = \mathrm{Id}_{F(M \otimes_S N)}$, and
- for all $u, v \in U$, $m \in M$ and $n \in N$,

$$\begin{aligned} \Psi' \circ \Phi'(m \otimes n \otimes u \otimes v) &= \Psi'((1 \otimes u) \cdot (v \otimes m \otimes n)) \\ &= (1 \otimes u) \cdot \Psi'(v \otimes m \otimes n) \qquad (\Psi' \text{ is } U^e\text{-linear}) \\ &= (1 \otimes u) \cdot (m \otimes n \otimes 1 \otimes v) \\ &= m \otimes n \otimes u \otimes v. \end{aligned}$$

□

3.6. ***Invertible U-bimodules.*** The following result is used in Section 5 in order to prove that $\mathrm{Ext}^i_{U^e}(U, U^e)$ is invertible as a U-bimodule, under suitable conditions.

Proposition 3.6.1. *Let R be a commutative ring. Let (S, L) be a Lie–Rinehart algebra over R. Denote by U its enveloping algebra. Let N be a left U-module. Assume that N is invertible as an S-module. Then $F(N)$ is invertible as a U-bimodule.*

This subsection is devoted to the proof of this proposition. Given a left U-module N, then $F(N) = U \otimes_S N$ as left U-modules. Hence, there is a functorial isomorphism

$$\theta : \mathrm{Hom}_S(N, U) \to \mathrm{Hom}_U(F(N), U). \tag{3-17}$$

Note:

- $\mathrm{Hom}_S(N, U)$ is a left U-module (see (3-4)), and it inherits a right U-module structure from U_U; by construction, these two structures form a U-bimodule structure.
- $\mathrm{Hom}_U(F(N), U)$ is a U-bimodule because so are $F(N)$ and U.

- $N \otimes_S \operatorname{Hom}_S(N, U)$ is a left U-module (see (3-7)), and it inherits a right U-module structure from U_U; by construction, these two structures form a U-bimodule structure.

Lemma 3.6.2. *Let N be a left U-module. Then,*

(1) $\theta : \operatorname{Hom}_S(N, U) \to \operatorname{Hom}_U(F(N), U)$ *is an isomorphism in* $\operatorname{Mod}(U^e)$,

(2) *the mapping*

$$\Phi : N \otimes_S \operatorname{Hom}_S(N, U) \to F(N) \otimes_U \operatorname{Hom}_U(F(N), U),$$
$$n \otimes f \mapsto (1 \otimes n) \otimes \theta(f)$$

is an isomorphism in $\operatorname{Mod}(U^e)$, *and*

(3) *the diagram*

$$\begin{array}{ccc} N\otimes_S \operatorname{Hom}_S(N,U) & \longrightarrow & U \\ {\scriptstyle\Phi}\downarrow & & \| \\ F(N)\otimes_U \operatorname{Hom}(F(N),U) & \longrightarrow & U \end{array}$$

with horizontal arrows given by evaluation, is commutative.

Proof. (1) By definition, θ is a morphism of right U-modules. It is also a morphism of left U-modules because, for all $n \in N$, $f \in \operatorname{Hom}_S(N, U)$, $u \in U$ and $\alpha \in L$,

$$\begin{aligned} \theta(\alpha \cdot f)(u \otimes n) &= u(\alpha \cdot f)(n) \\ &= u(\alpha f(n) - f(\alpha \cdot n)) = \theta(f)(u\alpha \otimes n - u \otimes \alpha \cdot n) \\ &= \theta(f)((u \otimes n) \cdot \alpha) = (\alpha \cdot \theta(f))(u \otimes n). \end{aligned}$$

(2) By definition, Φ is a morphism of right U-modules. It is also a morphism of left U-modules because, for all $n \in N$, $f \in \operatorname{Hom}_S(N, U)$ and $\alpha \in L$,

$$\begin{aligned} \Phi(\alpha \cdot (n \otimes f)) &= \Phi(\alpha \cdot n \otimes f + n \otimes \alpha \cdot f) \\ &= (1 \otimes \alpha \cdot n) \otimes \theta(f) + (1 \otimes n) \otimes \underbrace{\theta(\alpha \cdot f)}_{=\alpha\cdot\theta(f)} \\ &= (1 \otimes \alpha \cdot n) \otimes \theta(f) + \underbrace{(1 \otimes n) \cdot \alpha}_{=\alpha\otimes n - 1\otimes \alpha\cdot n} \otimes \theta(f) \\ &= (\alpha \otimes n) \otimes \theta(f) \\ &= \alpha \cdot \Phi(n \otimes f). \end{aligned}$$

In order to prove that Φ is bijective, consider the linear mapping

$$\psi : F(N) \otimes_S \operatorname{Hom}_U(F(N), U) \to N \otimes_S \operatorname{Hom}_S(N, U),$$
$$(u \otimes n) \otimes g \mapsto u \cdot (n \otimes \theta^{-1}(g)).$$

Note that, for all $u \in U,\ \alpha \in L,\ n \in N$ and $g \in \mathrm{Hom}_U(F(N), U)$,

$$\begin{aligned}\psi((u\otimes n)\cdot\alpha\otimes g) &= \psi((u\alpha\otimes n)\otimes g-(u\otimes\alpha\cdot n)\otimes g)\\ &= u\alpha\cdot(n\otimes\theta^{-1}(g))-u\cdot(\alpha\cdot n\otimes\theta^{-1}(g))\\ &= u\cdot(\alpha\cdot n\otimes\theta^{-1}(g)+n\otimes\alpha\cdot\theta^{-1}(g))-u\cdot(\alpha\cdot n\otimes\theta^{-1}(g))\\ &= u\cdot(n\otimes\theta^{-1}(\alpha\cdot g)) \qquad \text{(see part (1))}\\ &= \psi((u\otimes n)\otimes\alpha\cdot g).\end{aligned}$$

Hence, ψ induces a linear mapping,

$$\begin{aligned}\Psi : F(N)\otimes_U \mathrm{Hom}_U(F(N), U) &\to N\otimes_S \mathrm{Hom}_S(N, U),\\ (u\otimes n)\otimes g &\mapsto u\cdot(n\otimes\theta^{-1}(g)).\end{aligned}$$

Now, by definition of Φ and Ψ,

$$\Psi\circ\Phi = \mathrm{Id}_{N\otimes_S \mathrm{Hom}_S(N,U)}.$$

Since

- Ψ is a morphism of left U-modules by construction;
- as a left U-module, $F(N)\otimes_U \mathrm{Hom}_U(F(N), U)$ is generated by the image of $(1\otimes N)\otimes \mathrm{Hom}_U(F(N), U)$; and
- for all $n \in N$ and $g \in \mathrm{Hom}_U(F(N), U)$,

$$\Phi\circ\Psi((1\otimes n)\otimes g) = (1\otimes n)\otimes g,$$

the following holds:

$$\Phi\circ\Psi = \mathrm{Id}_{F(N)\otimes_U \mathrm{Hom}_U(F(N),U)}.$$

Altogether, these considerations show that Φ is an isomorphism in $\mathrm{Mod}(U^e)$.

(3) The diagram is commutative by definition of Φ. □

Like in the previous lemma, for all $N \in \mathrm{Mod}(U)$, $\mathrm{Hom}_S(N, U)$ is a U-bimodule, and hence $\mathrm{Hom}_S(N, U)\otimes_S N$ is a U-bimodule by means of (3-7) and the right U-module structure of U.

Lemma 3.6.3. *Let N be a left U-module. Then,*

(1) *the mapping*

$$\begin{aligned}\Phi' : \mathrm{Hom}_S(N, U)\otimes_S N &\to \mathrm{Hom}_U(F(N), U)\otimes_U F(N),\\ f\otimes n &\mapsto \theta(f)\otimes(1\otimes n)\end{aligned}$$

is an isomorphism in $\mathrm{Mod}(U^e)$; *and*

(2) *the diagram*

$$\begin{array}{ccc} \mathrm{Hom}_S(N,U)\otimes_S N & \longrightarrow & U \\ {\scriptstyle \Phi'}\downarrow & & \| \\ \mathrm{Hom}_U(F(N),U)\otimes_U F(N) & \longrightarrow & U \end{array}$$

with horizontal arrows given by evaluation, *is commutative.*

Proof. (1) First, since $F(N) = U \otimes_S N$ in $\mathrm{Mod}(U^e)$, then

$$\mathrm{Hom}_U(F(N), U) \otimes_U F(N) \cong \mathrm{Hom}_U(F(N), U) \otimes_S N$$

as left U-modules. Under this identification, Φ' expresses as

$$\Phi' : f \otimes n \mapsto \theta(f) \otimes n.$$

Therefore, Φ' is bijective because so is θ.

Next, Φ' is a morphism of left U-modules because so is θ. And it is a morphism of right U-modules because it is a morphism of right S-modules, and because, for all $f \in \mathrm{Hom}_S(N, U)$, $n \in N$ and $\alpha \in L$,

$$\begin{aligned} \Phi'((f \otimes n)\cdot\alpha) &= \Phi'(f\cdot\alpha \otimes n - f \otimes \alpha\cdot n) \\ &= \underbrace{\theta(f\cdot\alpha)}_{=\theta(f)\cdot\alpha} \otimes (1\otimes n) - \theta(f) \otimes (1 \otimes \alpha\cdot n) \\ &= \theta(f) \otimes \underbrace{\alpha\cdot(1\otimes n)}_{=\alpha\otimes n} - \theta(f)\otimes(1\otimes\alpha\cdot n) \\ &= \theta(f)\otimes((1\otimes n)\cdot\alpha) = (\theta(f)\otimes(1\otimes n))\cdot\alpha \\ &= \Phi'(f\otimes n)\cdot\alpha. \end{aligned}$$

This proves (1).

(2) The diagram commutes by definition of Φ'. □

It is now possible to prove the result announced at the beginning of the subsection.

Proof of Proposition 3.6.1. Since N is invertible as an S-module, then the following evaluation mappings are bijective

$$N \otimes_S \mathrm{Hom}_S(N, U) \to U \quad \text{and} \quad \mathrm{Hom}_S(N, U) \otimes_S N \to U.$$

According to Lemmas 3.6.2 and 3.6.3, the following evaluation mappings are isomorphisms of U-bimodules

$$F(N) \otimes_U \mathrm{Hom}_U(F(N), U) \to U \quad \text{and} \quad \mathrm{Hom}_U(F(N), U) \otimes_U F(N) \to U.$$

Thus, $F(N)$ is invertible as a U-bimodule. □

4. The action of L on the inverse dualising bimodule of S

This section introduces an action of L on $\mathrm{Ext}^{\bullet}_{S^e}(S, S^e)$ by means of Lie derivatives, which is used to describe $\mathrm{Ext}^{\bullet}_{U^e}(U, U^e)$ in the next section. When S is projective in $\mathrm{Mod}(R)$, then $\mathrm{Ext}^{\bullet}_{S^e}(S, -)$ is the Hochschild cohomology $H^{\bullet}(S; -)$; in this setting, the Lie derivatives on $H^{\bullet}(S; S)$ and $H_{\bullet}(S; S)$ are defined in [Rinehart 1963, Section 9] and have a well-known expression in terms of the Hochschild resolution of S. For the needs of the article, the definition is translated to arbitrary coefficients in terms of any projective resolution of S in $\mathrm{Mod}(S^e)$.

Hence, Section 4.1 introduces preliminary material, Section 4.2 deals with derivations on projective resolutions of S in $\mathrm{Mod}(S^e)$, Section 4.3 defines the Lie derivatives, Section 4.4 presents the action of L on $\mathrm{Ext}^{\bullet}_{S^e}(S, S^e)$, and Section 4.5 discusses particular situations.

For the section, a projective resolution of S in $\mathrm{Mod}(S^e)$ is considered;

$$(P^{\bullet}, d) \to S.$$

Denote S by P^1 and the augmentation mapping $P^0 \to S$ by d^0. For all $M \in \mathrm{Mod}(S^e)$ and $s \in S$, denote by λ_s and ρ_s the multiplication mappings

$$\lambda_s : M \to M, \quad m \mapsto (s \otimes 1) \cdot m$$

and

$$\rho_s : M \to M, \quad m \mapsto (1 \otimes s) \cdot m.$$

4.1. ***Data on the projective resolution.*** For all $s \in S$, the mappings λ_s, ρ_s on $P^{\bullet}$ are morphisms of complexes of left S^e-modules and induce the same mapping

$$S \to S,$$
$$t \mapsto st$$

in cohomology. Hence, there exists a morphism of graded left S^e-modules,

$$k_s : P^{\bullet} \to P^{\bullet}[-1], \tag{4-1}$$

such that

$$\lambda_s - \rho_s = k_s \circ d + d \circ k_s. \tag{4-2}$$

Lemma 4.1.1. *Let $\partial : S \to S$ be an R-linear derivation. Let $\psi : P^{\bullet} \to P^{\bullet}$ be a morphism of complexes of R-modules such that*

- *$H^0(\psi) : S \to S$ is the zero mapping;*
- *there exists a morphism of graded left S^e-modules,*

$$k : P^{\bullet} \to P^{\bullet}[-1],$$

such that, for all $p \in P^\bullet$ and $s, t \in S$,

$$\psi((s \otimes t) \cdot p) = (s \otimes t) \cdot \psi(p) - (1 \otimes \partial)(s \otimes t) \cdot (k \circ d + d \circ k)(p). \tag{4-3}$$

Then, there exists a morphism of graded R-modules,

$$h : P^\bullet \to P^\bullet[-1],$$

such that

- $\psi = h \circ d + d \circ h$; *and*
- *for all $s, t \in S$ and $p \in P^\bullet$,*

$$h((s \otimes t) \cdot p) = (s \otimes t) \cdot h(p) - (1 \otimes \partial)(s \otimes t) \cdot k(p).$$

Proof. The proof is an induction on $n \leqslant 1$, taking $h^1 : S \to P^0$ equal to 0. Let $n \leqslant 0$ and assume that there exist linear mappings, for all j such that $n + 1 \leqslant j \leqslant 1$,

$$h^j : P^j \to P^{j-1}$$

such that, for all j satisfying $n + 1 \leqslant j \leqslant 0$, $p \in P^j$ and $s, t \in S$,

$$\begin{aligned} \psi^j &= h^{j+1} \circ d^j + d^{j-1} \circ h^j \\ h^j((s \otimes t) \cdot p) &= (s \otimes t) \cdot h^j(p) - (1 \otimes \partial)(s \otimes t) \cdot k^j(p). \end{aligned} \tag{4-4}$$

This is illustrated in the following diagram:

$$\begin{array}{ccccccc} P^n & \xrightarrow{d^n} & P^{n+1} & \xrightarrow{d^{n+1}} & P^{n+2} & \xrightarrow{d^{n+2}} & \cdots \\ {\scriptstyle \psi^n}\downarrow & \swarrow{\scriptstyle h^{n+1}} & \downarrow{\scriptstyle \psi^{n+1}} & \swarrow{\scriptstyle h^{n+2}} & & & \\ P^n & \xrightarrow{d^n} & P^{n+1} & \xrightarrow{d^{n+1}} & P^{n+2} & \xrightarrow{d^{n+2}} & \cdots \end{array}$$

Let

$$((p_i, \varphi^i))_{i \in I}$$

be a coordinate system of the projective left S^e-module P^n. That is, let $p_i \in P^n$ and $\varphi^i \in \operatorname{Hom}_{S^e}(P^n, S^e)$ for all $i \in I$ such that, for all $p \in P^n$,

$$p = \sum_{i \in I} \varphi^i(p) \cdot p_i,$$

where $\{i \in I \mid \varphi^i(p) \neq 0\}$ is finite. Since $\psi : P^\bullet \to P^\bullet$ is a morphism of complexes, it follows from (4-4) that, for all $i \in I$, there exists $p_i' \in P^{n-1}$ such that

$$\psi^n(p_i) = d^{n-1}(p_i') + h^{n+1} \circ d^n(p_i). \tag{4-5}$$

Denote by h^n the linear mapping from P^n to P^{n-1} such that, for all $p \in P^n$,

$$h^n(p) = \sum_{i \in I} \varphi^i(p) \cdot p_i' - (1 \otimes \partial)(\varphi^i(p)) \cdot k^n(p_i).$$

Then, for all $p \in P^n$ and $s, t \in S$,

$$\begin{aligned}
&h^n((s\otimes t)\cdot p)\\
&=\sum_{i\in I}(s\otimes t)\cdot\varphi^i(p)\cdot p_i' - (s\otimes t)\cdot(1\otimes\partial)(\varphi^i(p))\cdot k^n(p_i) - (1\otimes\partial)(s\otimes t)\cdot\varphi^i(p)\cdot k^n(p_i)\\
&=(s\otimes t)\cdot h^n(p) - (1\otimes\partial)(s\otimes t)\cdot k^n\Big(\sum_{i\in I}\varphi^i(p)\cdot p_i\Big)\\
&=(s\otimes t)\cdot h^n(p) - (1\otimes\partial)(s\otimes t)\cdot k^n(p).
\end{aligned}$$

Moreover,

$$\psi^n = h^{n+1}\circ d^n + d^{n-1}\circ h^n.$$

Indeed, for all $p \in P^n$, $p = \sum_{i\in I}\varphi^i(p)\cdot p_i$, and hence

$$\begin{aligned}
&d^{n-1}\circ h^n(p) + h^{n+1}\circ d^n(p)\\
&= \sum_{i\in I}\varphi^i(p)\cdot d^{n-1}(p_i') - (1\otimes\partial)(\varphi^i(p))\cdot d^{n-1}\circ k^n(p_i) + h^{n+1}\Big(\sum_{i\in I}\varphi^i(p)\cdot d^n(p_i)\Big)\\
&\underset{(4\text{-}4)}{=} \sum_{i\in I}\varphi^i(p)\cdot d^{n-1}(p_i') - (1\otimes\partial)(\varphi^i(p))\cdot d^{n-1}\circ k^n(p_i)\\
&\qquad\qquad + \varphi^i(p)\cdot h^{n+1}\circ d^n(p_i) - (1\otimes\partial)(\varphi^i(p))\cdot k^{n+1}\circ d^n(p_i)\\
&\underset{(4\text{-}3)}{=} \sum_{i\in I}\varphi^i(p)\cdot d^{n-1}(p_i') + \varphi^i(p)\cdot h^{n+1}\circ d^n(p_i) + \psi^n(\varphi^i(p)\cdot p_i) - \varphi^i(p)\cdot\psi^n(p_i)\\
&\underset{(4\text{-}5)}{=} \sum_{i\in I}\psi^n(\varphi^i(p)\cdot p_i) = \psi^n(p).
\end{aligned}$$

□

4.2. ***Derivations on the projective resolution.*** Let $\partial : S \to S$ be an R-linear derivation. It defines an R-linear derivation on S^e denoted by ∂^e,

$$\begin{aligned}
\partial^e : S^e &\to S^e,\\
s\otimes t &\mapsto \partial(s)\otimes t + s\otimes\partial(t).
\end{aligned}$$

For every left S^e-module M, a *derivation* of M relative to ∂ is an R-linear mapping,

$$\partial_M : M \to M,$$

such that, for all $m \in M$ and $s, t \in S$,

$$\partial_M((s\otimes t)\cdot m) = \partial^e(s\otimes t)\cdot m + (s\otimes t)\cdot\partial_M(m).$$

A derivation of $P^\bullet$ relative to ∂ is a morphism of complexes of R-modules,

$$\partial^\bullet : P^\bullet \to P^\bullet,$$

such that $\partial^n : P^n \to P^n$ is a derivation relative to ∂ for all n, and such that

$H^0(\partial^\bullet) = \partial$. Note that a morphism of complexes of R-modules $\partial^\bullet : P^\bullet \to P^\bullet$ such that $H^0(\partial^\bullet) = \partial$ is a derivation relative to ∂ if and only if

$$\begin{cases} \partial^\bullet \circ \lambda_s = \lambda_{\partial(s)} + \lambda_s \circ \partial^\bullet, \\ \partial^\bullet \circ \rho_s = \rho_{\partial(s)} + \rho_s \circ \partial^\bullet. \end{cases} \tag{4-6}$$

Remark. For all derivations $\partial_1^\bullet, \partial_2^\bullet : P^\bullet \to P^\bullet$ relative to ∂, the difference

$$\partial_1^\bullet - \partial_2^\bullet : P^\bullet \to P^\bullet$$

is a null-homotopic morphism of complexes of left S^e-modules.

Lemma 4.2.1. *There exists a mapping, which need not be linear,*

$$\begin{aligned} \operatorname{Der}_R(S) &\to \operatorname{Hom}_R(P^\bullet, P^\bullet), \\ \partial &\mapsto \partial^\bullet \end{aligned} \tag{4-7}$$

such that:

(1) *For all $\partial \in \operatorname{Der}_R(S)$, the mapping $\partial^\bullet$ is a derivation relative to ∂.*

(2) *For all $\partial_1, \partial_2 \in \operatorname{Der}_R(S)$ and $r \in R$, there exist morphisms of graded left S^e-modules,*

$$\ell, \ell' : P^\bullet \to P^\bullet[-1],$$

such that

$$\begin{cases} [\partial_1, \partial_2]^\bullet - [\partial_1^\bullet, \partial_2^\bullet] &= \ell \circ d + d \circ \ell, \\ (\partial_1 + r\partial_2)^\bullet - (\partial_1^\bullet + r\partial_2^\bullet) &= \ell' \circ d + d \circ \ell'. \end{cases} \tag{4-8}$$

(3) *For all $s \in S$ and $\partial \in \operatorname{Der}_R(S)$, there exists a morphism of graded R-modules*

$$h : P^\bullet \to P^\bullet[-1],$$

such that

$$(s\partial)^\bullet - \lambda_s \circ \partial^\bullet = h \circ d + d \circ h \tag{4-9}$$

and, for all $p \in P^\bullet$ and $t_1, t_2 \in S$,

$$h((t_1 \otimes t_2) \cdot p) = (t_1 \otimes t_2) \cdot h(p) - (t_1 \otimes \partial(t_2)) \cdot k_s(p). \tag{4-10}$$

Recall that $k_s : P^\bullet \to P^\bullet[-1]$ is a morphism of graded left S^e-modules such that $\lambda_s - \rho_s = k_s \circ d + d \circ k_s$ (see (4-1) and (4-2)).

Proof. (1) Let $\partial \in \operatorname{Der}_R(S)$. For convenience, denote ∂ by $\partial^1 : S \to S$. The proof is an induction on $n \leqslant 1$. Let $n \leqslant 0$, and assume that a commutative diagram is given

$$\begin{array}{ccccccccccc} P^n & \xrightarrow{d^n} & P^{n+1} & \longrightarrow & \cdots & \longrightarrow & P^0 & \xrightarrow{d^0} & P^1 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \partial^{n+1}} & & & & \downarrow{\scriptstyle \partial^0} & & \downarrow{\scriptstyle \partial^1} & & \\ P^n & \xrightarrow{d^n} & P^{n+1} & \longrightarrow & \cdots & \longrightarrow & P^0 & \xrightarrow{d^1} & P^1 & \longrightarrow & 0 \end{array}$$

where $\partial^i : P^i \to P^i$ is a derivation relative to ∂ for all $i \in \{n+1, n+2, \cdots, 0\}$. Let

$$((p_i, \varphi^i))_{i \in I}$$

be a coordinate system of the projective left S^e-module P^n (see the proof in 4.1). Then, for all $i \in I$, there exists $p'_i \in P^n$ such that

$$\partial^{n+1} \circ d^n(p_i) = d^n(p'_i).$$

Denote by ∂^n the R-linear mapping from P^n to P^n such that, for all $p \in P^n$,

$$\partial^n(p) = \sum_{i \in I} \partial(\varphi^i(p)) \cdot p_i + \varphi^i(p) \cdot p'_i.$$

Then, for all $p \in P^n$,

$$\begin{aligned} d^n \circ \partial^n(p) &= \sum_{i \in I} \partial(\varphi^i(p)) \cdot d^n(p_i) + \varphi^i(p) \cdot d^n(p'_i) \\ &= \sum_{i \in I} \partial(\varphi^i(p)) \cdot d^n(p_i) + \varphi^i(p) \cdot \partial^{n+1} \circ d^n(p_i) \\ &= \partial^{n+1} \circ d^n\left(\sum_{i \in I} \varphi^i(p) \cdot p_i\right) \\ &= \partial^{n+1} \circ d^n(p). \end{aligned}$$

Thus,

$$d^n \circ \partial^n = \partial^{n+1} \circ d^n.$$

Moreover, ∂^n is a derivation of P^n relative to ∂ because ∂ is a derivation of S^e and $\varphi^i \in \mathrm{Hom}_{S^e}(P^n, S^e)$ for all $i \in I$.

(2) Note that $[\partial_1, \partial_2]^\bullet$ and $[\partial_1^\bullet, \partial_2^\bullet]$ (or, $(\partial_1 + r\partial_2)^\bullet$ and $\partial_1^\bullet + r\partial_2^\bullet$) are derivations of $P^\bullet$ relative to $[\partial_1, \partial_2]$ (or, to $\partial_1 + r\partial_2$, respectively). The conclusion therefore follows from the remark preceding Lemma 4.2.1.

(3) Denote by ψ the mapping $(s\partial)^\bullet - \lambda_s \circ \partial^\bullet$ given by

$$\begin{aligned} &P^\bullet \to P^\bullet, \\ &p \mapsto (s\partial)^\bullet(p) - (s \otimes 1) \cdot \partial^\bullet(p). \end{aligned}$$

Then, for all $p \in P^\bullet$ and $t \in S$,

$$\begin{aligned} &\psi((t\otimes 1)\cdot p) \\ &\quad = (s\partial)^\bullet((t\otimes 1)\cdot p) - (s\otimes 1)\cdot\partial^\bullet((t\otimes 1)\cdot p) \\ &\quad = (s\partial(t)\otimes 1)\cdot p + (t\otimes 1)\cdot(s\partial)^\bullet(p) - (s\otimes 1)\cdot(\partial(t)\otimes 1)\cdot p - (s\otimes 1)\cdot(t\otimes 1)\cdot\partial^\bullet(p) \\ &\quad = (t\otimes 1)\cdot\psi(p) \end{aligned}$$

and

$$
\begin{aligned}
&\psi((1\otimes t)\cdot p)\\
&\quad = (s\partial)^\bullet((1\otimes t)\cdot p) - (s\otimes 1)\cdot\partial^\bullet((1\otimes t)\cdot p)\\
&\quad = (1\otimes s\partial(t))\cdot p + (1\otimes t)\cdot(s\partial)^\bullet(p) - (s\otimes 1)\cdot(1\otimes\partial(t))\cdot p - (s\otimes 1)\cdot(1\otimes t)\cdot\partial^\bullet(p)\\
&\quad = (1\otimes t)\cdot\psi(p) + (1\otimes\partial(t))\cdot(\rho_s - \lambda_s)(p)\\
&\quad \underset{(4\text{-}2)}{=} (1\otimes t)\cdot\psi(p) - (1\otimes\partial(t))\cdot(k_s\circ d + d\circ k_s)(p).
\end{aligned}
$$

Hence, Lemma 4.1.1 may be applied, which yields (3). □

Remark. Using the remark preceding Lemma 4.2.1, it may be checked that, although the mapping $\mathrm{Der}_R(S) \to \mathrm{Hom}_R(P^\bullet, P^\bullet)$ of the lemma is not unique, two such mappings induce the same mapping from $\mathrm{Der}_R(S)$ to $H^0\mathrm{Hom}_R(P^\bullet, P^\bullet)$, which is R-linear.

When S is projective in $\mathrm{Mod}(R)$, it is possible to be more explicit on a possible mapping, $\partial \mapsto \partial^\bullet$. Indeed, the Hochschild complex $B(S) = S^{\otimes\bullet+2}$ is a projective resolution of S. For all $\partial \in \mathrm{Der}_R(S)$, define the following mapping:

$$
\begin{aligned}
L_\partial : B(S) &\to B(S),\\
(s_0|\cdots|s_{n+1}) &\mapsto \sum_{i=0}^{n+1}(s_0|\cdots|s_{i-1}|\partial(s_i)|\cdots|s_{i+1}|\cdots|s_n).
\end{aligned}
$$

This is a derivation of $B(S)$ relative to ∂. It is direct to check that the mapping

$$
\begin{aligned}
\mathrm{Der}_R(S) &\to \mathrm{Hom}_R(B(S), B(S)),\\
\partial &\mapsto L_\partial
\end{aligned}
$$

is a morphism of Lie algebras over R. Now, consider homotopy equivalences of complexes of S^e-modules,

$$
P^\bullet \underset{g}{\overset{f}{\rightleftarrows}} B(S)\,,
$$

and, for all $\partial \in \mathrm{Der}_R(S)$, define $\partial^\bullet$ as

$$
\partial^\bullet = g\circ L_\partial \circ f\,;
$$

this is a derivation relative to ∂ because so is L_∂ and because f and g are morphisms of resolutions of S in $\mathrm{Mod}(S^e)$. The following mapping satisfies the conclusion of the preceding lemma, it is moreover R-linear:

$$
\begin{aligned}
\mathrm{Der}_R(S) &\to \mathrm{Hom}_R(P^\bullet, P^\bullet),\\
\partial &\mapsto \partial^\bullet.
\end{aligned}
$$

4.3. ***Lie derivatives.*** Consider a mapping $\partial \mapsto \partial^\bullet$ such as in Lemma 4.2.1. Let M be an S-bimodule and $\partial : S \to S$ be an R-linear derivation. Let $\partial_M : M \to M$ be a derivation relative to ∂. Given $n \in \mathbb{N}$ and $\psi \in \mathrm{Hom}_{S^e}(P^{-n}, M)$, denote by $\mathcal{L}_\partial(\psi)$ the mapping

$$\mathcal{L}_\partial(\psi) = \partial_M \circ \psi - \psi \circ \partial^{-n}. \tag{4-11}$$

This is a morphism in $\mathrm{Mod}(S^e)$ because so is ψ and because ∂_M and ∂^{-n} are derivations relative to ∂; moreover, it is a cocycle (or a coboundary) as soon as ψ is, because $\partial^\bullet : P^\bullet \to P^\bullet$ is a morphism of complexes. Denote by $\mathcal{L}_\partial$ the resulting mapping in cohomology

$$\mathcal{L}_\partial : \mathrm{Ext}^\bullet_{S^e}(S, M) \to \mathrm{Ext}^\bullet_{S^e}(S, M)$$

such that for all $c \in \mathrm{Ext}^\bullet_{S^e}(S, M)$, say represented by a cocycle ψ, then $\mathcal{L}_\partial(c)$ is represented by the cocycle $\mathcal{L}_\partial(\psi)$. In the situations considered later in the article, there is no ambiguity on ∂_M, whence its omission in the notation.

Following similar considerations denote also by $\mathcal{L}_\partial$ the mapping

$$\mathcal{L}_\partial : \mathrm{Tor}^{S^e}_\bullet(S, M) \to \mathrm{Tor}^{S^e}_\bullet(S, M)$$

such that for all $\omega \in \mathrm{Tor}^{S^e}_\bullet(S, M)$, say represented by a cocycle $m \otimes p \in M \otimes_{S^e} P^\bullet$ with sum sign omitted, $\mathcal{L}_\partial(\omega)$ is represented by the cocycle

$$\mathcal{L}_\partial(m \otimes p) := m \otimes \partial^\bullet(p) + \partial_M(m) \otimes p.$$

When S is projective in $\mathrm{Mod}(R)$, these operations may be written explicitly in terms of the Hochschild resolution. When ψ is a Hochschild cocycle lying in $\mathrm{Hom}_R(S^{\otimes n}, M)$, the mapping $\mathcal{L}_\partial(\psi)$ is given by

$$(s_1|\cdots|s_n) \mapsto \partial_M(f(s_1|\cdots|s_n)) - \sum_{i=1}^{n} f(s_1|\cdots|\partial(s_i)|\cdots|s_n). \tag{4-12}$$

Likewise, the operation in Hochschild homology is induced by the following mapping at the level of Hochschild chains,

$$M \otimes S^{\otimes n} \to M \otimes S^{\otimes n}$$

$$(m|s_1|\cdots|s_n) \mapsto (\partial_M(m)|s_1|\cdots|s_n) + \sum_{i=1}^{n}(m|s_1|\cdots|\partial(s_i)|\cdots|s_n).$$

The operator $\mathcal{L}_\partial$ is of course called the *Lie derivative* of ∂. When $M = S$ and S is projective in $\mathrm{Mod}(R)$, this is nothing else but the classical Lie derivative defined in [Rinehart 1963, Section 9]. In view of the remark following Lemma 4.2.1, these constructions depend only on ∂ and ∂_M and not on the choices of $P^\bullet$ and the mapping $\partial \mapsto \partial^\bullet$.

In the sequel these constructions are considered mainly in the following cases:

- $M = S$ and $\partial_M = \partial$.
- $M = S^e$ and $\partial_M = \partial^e$.
- $M = \operatorname{Ext}^n_{S^e}(S, S^e)$ ($n \in \mathbb{N}$) and $\partial_M = \mathcal{L}_\partial$, which makes sense according to the result below.

In the sequel the following construction is also used. Consider S-bimodules M, N. Let $m, n \in \mathbb{N}$. Let $\partial \in \operatorname{Der}_R(S)$ and let $\partial_M : M \to M$ and $\partial_N : N \to N$ be R-linear derivations relative to ∂. Then, for all $f \in \operatorname{Hom}_R(\operatorname{Ext}^m_{S^e}(S, M), \operatorname{Tor}^{S^e}_n(S, N))$, define $\mathcal{L}_\partial(f)$ as

$$\mathcal{L}_\partial \circ f - f \circ \mathcal{L}_\partial.$$

Recall that for all $M \in \operatorname{Mod}(S^e)$, the spaces $\operatorname{Ext}^\bullet_{S^e}(S, M)$ and $\operatorname{Tor}^{S^e}_\bullet(S, M)$ are left S-modules by means of $\lambda_s : M \to M$ for all $s \in S$; the corresponding multiplication by s on these (co)homology spaces is denoted by λ_s.

Lemma 4.3.1. *Let $M \in \operatorname{Mod}(S^e)$, $n \in \mathbb{N}$ and $s \in S$. Let $\partial, \partial' : S \to S$ be R-linear derivations. Let $\partial_M, \partial'_M : M \to M$ be R-linear derivations relative to ∂ and ∂', respectively. Then, the following hold in* $\operatorname{Ext}^\bullet_{S^e}(S, M)$:

(1) $\mathcal{L}_\partial \circ \lambda_s = \lambda_{\partial(s)} + \lambda_s \circ \mathcal{L}_\partial$.

(2) $\mathcal{L}_{[\partial,\partial']} = [\mathcal{L}_\partial, \mathcal{L}_{\partial'}]$.

(3) *Let $m \in \mathbb{N}$, let N be another S-bimodule and let $\partial_N : N \to N$ be a derivation relative to ∂. Consider the contraction mapping*

$$\begin{aligned}\operatorname{Tor}^{S^e}_m(S, M) &\to \operatorname{Hom}_R(\operatorname{Ext}^n_{S^e}(S, N), \operatorname{Tor}^{S^e}_{m-n}(S, M \otimes_S N)),\\ \omega &\mapsto (c \mapsto \iota_c(\omega)).\end{aligned}$$

If $m = n$, then it is $\mathcal{L}_\partial$-equivariant. When S is projective in $\operatorname{Mod}(R)$, *it is $\mathcal{L}_\partial$ equivariant for all $m, n \in \mathbb{N}$.*

(4) *If M is symmetric as an S-bimodule, $\mathcal{L}_{s\partial} = \lambda_s \circ \mathcal{L}_\partial$.*

(5) *When $M = S^e$ (and $\partial_M = \partial^e$), the following equality holds in* $\operatorname{Ext}^\bullet_{S^e}(S, M)$:

$$\mathcal{L}_{s\partial} = \lambda_s \circ \mathcal{L}_\partial - \lambda_{\partial(s)}.$$

Proof. (1) The equality is checked on cochains. Let $\psi \in \operatorname{Hom}_{S^e}(P^{-n}, M)$. Then,

$$\begin{aligned}\mathcal{L}_\partial \circ \lambda_s(\psi) &= \partial_M \circ \lambda_s \circ \psi - \lambda_s \circ \psi \circ \partial^\bullet\\ &= (\lambda_{\partial(s)} + \lambda_s \circ \partial_M) \circ \psi - \lambda_s \circ \psi \circ \partial^\bullet\\ &= (\lambda_{\partial(s)} + \lambda_s \circ \mathcal{L}_\partial)(\psi).\end{aligned}$$

(2) Note that $\mathcal{L}_{[\partial,\partial']}$ is defined with respect to $[\partial_M, \partial'_M]$, which is a derivation of M relative to $[\partial, \partial']$. Following Lemma 4.2.1, there exists a morphism of graded S^e-modules,

$$\ell : P^\bullet \to P^\bullet[-1],$$

such that

$$[\partial, \partial']^\bullet - [\partial^\bullet, \partial'^\bullet] = \ell \circ d + d \circ \ell.$$

Let $\psi \in \mathrm{Hom}_{S^e}(P^{-n}, M)$. If this is a cocycle, then

$$\begin{aligned}\mathcal{L}_{[\partial,\partial']}(\psi) &= [\partial_M, \partial'_M] \circ \psi - \psi \circ ([\partial^\bullet, \partial'^\bullet] + \ell \circ d + d \circ \ell)\\ &= [\mathcal{L}_\partial, \mathcal{L}_{\partial'}](\psi) - \psi \circ \ell \circ d - \underbrace{\psi \circ d}_{=0} \circ \ell,\end{aligned}$$

which is cohomologous to $[\mathcal{L}_\partial \mathcal{L}_{\partial'}](\psi)$. This proves (2).

(3) Note that the mapping

$$\begin{aligned}\partial_{M\otimes_S N} : M \otimes_S N &\to M \otimes_S N,\\ x \otimes y &\mapsto \partial_M(x) \otimes y + x \otimes \partial_N(y),\end{aligned}$$

is a well-defined derivation relative to ∂, which defines $\mathcal{L}_\partial$ on

$$\mathrm{Tor}^{S^e}_{m-n}(S, M \otimes_S N).$$

Assume first that $m = n$. Let p^0 be any element of the preimage of 1_S under the augmentation mapping $P^0 \to S$. Let $x \otimes p \in M \otimes_{S^e} P^{-m}$ and $\psi \in \mathrm{Hom}_{S^e}(P^{-m}, N)$, and use the notation

$$\iota_\psi(x \otimes p) := (x \otimes \psi(p)) \otimes p^0.$$

Recall that the contraction mapping is induced by the mapping

$$\begin{aligned}M \otimes_{S^e} P^{-m} &\to \mathrm{Hom}_R(\mathrm{Hom}_{S^e}(P^{-m}, N), (M \otimes_S N) \otimes_{S^e} P^0),\\ x \otimes p &\mapsto \iota_?(x \otimes p).\end{aligned}$$

Denote $\mathcal{L}_\partial(\iota_\psi(x \otimes p)) - \iota_{\mathcal{L}_\partial(\psi)}(x \otimes p)$ by δ. Then,

$$\begin{aligned}\delta &= \mathcal{L}_\partial((x \otimes \psi(p)) \otimes p^0) - (x \otimes \mathcal{L}_\partial(\psi)(p)) \otimes p^0\\ &= \partial_M(x) \otimes \psi(p) \otimes p^0 + x \otimes \partial_N(\psi(p)) \otimes p^0 + x \otimes \psi(p) \otimes \partial^0(p^0)\\ &\qquad - x \otimes \partial_N(\psi(p)) \otimes p^0 + x \otimes \psi(\partial^{-m}(p)) \otimes p^0\\ &= \iota_\psi(\mathcal{L}_\partial(x \otimes p)) + x \otimes \psi(p) \otimes \partial^0(p^0).\end{aligned}$$

Note that $\partial^0(p_0)$ lies in the image of $d : P^{-1} \to P^0$ because the image of p^0 under $P^0 \to S$ is 1 and $H^0(\partial^\bullet) = \partial$. These considerations therefore prove (3) when $m = n$.

Now assume that S is projective in Mod(R). Then, the equivariance may be checked at the level of Hochschild (co)chains. Let $o = (x|s_1|\cdots|s_m) \in S^{\otimes m}$ and $\psi \in \mathrm{Hom}_R(S^{\otimes n}, N)$. Then,

$$\begin{aligned}
&\mathcal{L}_\partial(\iota_\psi(o)) - \iota_{\mathcal{L}_\partial(\psi)}(o) \\
&\quad= \mathcal{L}_\partial(x \otimes \psi(s_1|\cdots|s_n)|s_{n+1}|\cdots|s_m) - (x \otimes \mathcal{L}_\partial(\psi)(s_1|\cdots|s_n)|s_{n+1}|\cdots|s_m) \\
&\quad= (\partial_M(x) \otimes \psi(s_1|\cdots|s_n)|s_{n+1}|\cdots|s_m) + (x \otimes \partial_N(\psi(s_1|\cdots|s_n))|s_{n+1}|\cdots|s_m) \\
&\qquad\qquad + \sum_{j=n+1}^{m} (x \otimes \psi(s_1|\cdots|s_n)|s_{n+1}|\cdots|\partial(s_j)|\cdots|s_m) \\
&\qquad\qquad\qquad - (x \otimes \partial_N(\psi(s_1|\cdots|s_n))|s_{n+1}|\cdots|s_m) \\
&\qquad\qquad\qquad\qquad + \sum_{j=1}^{n} (x \otimes \psi(s_1|\cdots|\partial(s_j)|\cdots|s_n)|s_{n+1}|\cdots|s_m) \\
&\quad= \iota_\psi(\mathcal{L}_\partial(o)),
\end{aligned}$$

which proves (3) for all $m, n \in \mathbb{N}$ when S is projective in Mod(R).

(4) Note that $\mathcal{L}_{s\partial}$ is defined with respect to the derivation $s\partial_M$ $(= \lambda_s \circ \partial_M)$. Assume that M is symmetric as an S-bimodule. Therefore, the mapping

$$\lambda_s \circ \partial^\bullet : P^\bullet \to P^\bullet$$

is a derivation relative to $s\partial$. Let $\psi \in \mathrm{Hom}_{S^e}(P^\bullet, M)$ be a cocycle with cohomology class denoted by c. Since $\psi \circ \lambda_s = \lambda_s \circ \psi$,

$$\mathcal{L}_{s\partial}(\psi) = (\lambda_s \circ \partial_M) \circ \psi - \psi \circ (\lambda_s \circ \partial^\bullet) = \lambda_s \circ \mathcal{L}_\partial(\psi).$$

Taking cohomology classes yields that $\mathcal{L}_{s\partial}(c) = \lambda_s \circ \mathcal{L}_\partial(c)$.

(5) Recall that, here, ∂_M is taken equal to

$$\begin{aligned}
(s\partial)^e : S^e &\to S^e, \\
s_1 \otimes s_2 &\mapsto s\partial(s_1) \otimes s_2 + s_1 \otimes s\partial(s_2).
\end{aligned}$$

Let $\psi \in \mathrm{Hom}_{S^e}(P^{-n}, M)$ be a cocycle with cohomology class denoted by c. Let h be as in part (3) of Lemma 4.2.1. Then,

$$\begin{aligned}
\mathcal{L}_{s\partial}(\psi) &= (s\partial)^e \circ \psi - \psi \circ (s\partial)^\bullet \\
&= (s\partial \otimes 1 + 1 \otimes s\partial) \circ \psi - \psi \circ (s\partial)^\bullet \\
&= \lambda_s \circ (\partial \otimes 1) \circ \psi + \rho_s \circ (1 \otimes \partial) \circ \psi - \psi \circ (s\partial)^\bullet.
\end{aligned}$$

Using (4-9), the equality becomes

$$\mathcal{L}_{s\partial}(\psi) = \lambda_s \circ (\partial \otimes 1) \circ \psi + \rho_s \circ (1 \otimes \partial) \circ \psi - \lambda_s \circ \psi \circ \partial^\bullet - \psi \circ h \circ d - \underbrace{\psi \circ d}_{=0} \circ h.$$

Using $[\partial, \rho_s] = \rho_{\partial(s)}$, it then becomes

$$\begin{aligned}\mathcal{L}_{s\partial}(\psi) &= \lambda_s \circ (\partial \otimes 1) \circ \psi + (1 \otimes \partial) \circ \rho_s \circ \psi - \rho_{\partial(s)} \circ \psi - \lambda_s \circ \psi \circ \partial^\bullet - \psi \circ h \circ d \\ &= \lambda_s \circ (\partial \otimes 1) \circ \psi - \rho_{\partial(s)} \circ \psi + (1 \otimes \partial) \circ \psi \circ (\rho_s - \lambda_s) \\ &\qquad + (1 \otimes \partial) \circ \psi \circ \lambda_s - \lambda_s \circ \psi \circ \partial^\bullet - \psi \circ h \circ d.\end{aligned}$$

Using (4-2), this becomes

$$\begin{aligned}\mathcal{L}_{s\partial}(\psi) &= \lambda_s \circ (\partial \otimes 1) \circ \psi - \rho_{\partial(s)} \circ \psi - (1 \otimes \partial) \circ \underbrace{\psi \circ d}_{=0} \circ k_s \\ &\qquad - (1 \otimes \partial) \circ \psi \circ k_s \circ d + \underbrace{(1 \otimes \partial) \circ \psi \circ \lambda_s}_{=(1\otimes\partial)\circ\lambda_s\circ\psi=\lambda_s\circ(1\otimes\partial)\circ\psi} - \lambda_s \circ \psi \circ \partial^\bullet - \psi \circ h \circ d \\ &= \lambda_s \circ (\partial \otimes 1 + 1 \otimes \partial) \circ \psi - \rho_{\partial(s)} \circ \psi - \lambda_s \circ \psi \circ \partial^\bullet - (\psi \circ h + (1 \otimes \partial) \circ \psi \circ k_s) \circ d \\ &= \lambda_s \circ (\mathcal{L}_\partial(\psi)) - \rho_{\partial(s)} \circ \psi - (\psi \circ h + (1 \otimes \partial) \circ \psi \circ k_s) \circ d.\end{aligned}$$

Now, consider the following R-linear mapping denoted by f:

$$\psi \circ h + (1 \otimes \partial) \circ \psi \circ k_s : P^{-n+1} \to S^e.$$

This is a morphism of S-bimodules. Indeed,

- it is a morphism of left S-modules because so are ψ, $1 \otimes \partial$, k_s and h (see (4-10));
- since ψ and k_s are morphisms of S-bimodules, then, for all $t \in S$,

$$\begin{aligned}f \circ \rho_t &= \psi \circ h \circ \rho_t + (1 \otimes \partial) \circ \rho_t \circ \psi \circ k_s \\ &\underset{(4\text{-}10)}{=} \psi \circ (\rho_t \circ h - \rho_{\partial(t)} \circ k_s) + (1 \otimes \partial) \circ \rho_t \circ \psi \circ k_s \\ &= \rho_t \circ \psi \circ h - \rho_{\partial(t)} \circ \psi \circ k_s + (1 \otimes \partial) \circ \rho_t \circ \psi \circ k_s \\ &= \rho_t \circ \psi \circ h + \rho_t \circ (1 \otimes \partial) \circ \psi \circ k_s \\ &= \rho_t \circ f.\end{aligned}$$

Therefore, $\mathcal{L}_{s\partial}(\psi)$ and $\lambda_s \circ \mathcal{L}_\partial(\psi) - \rho_{\partial(s)} \circ \psi$ are cohomologous. Since so are $\lambda_{\partial(s)} \circ \psi$ and $\rho_{\partial(s)} \circ \psi$ it follows that

$$\mathcal{L}_{s\partial}(c) = \lambda_s \circ \mathcal{L}_\partial(c) - \lambda_{\partial(s)}(c).$$ □

4.4. ***The action of L on*** $\mathbf{Ext}^\bullet_{S^e}(S, S^e)$**.** According to Lemma 4.3.1, the mapping

$$L \times \mathrm{Ext}^n_{S^e}(S, S^e) \to \mathrm{Ext}^n_{S^e}(S, S^e), \quad (\alpha, e) \mapsto \alpha \cdot e := \mathcal{L}_{\partial_\alpha}(e) \tag{4-13}$$

endows $\operatorname{Ext}^{\bullet}_{S^e}(S, S^e)$ with a compatible left $S \rtimes L$-module structure in the sense of (3-13), that is, a left $S \rtimes L$-module structure such that, for all $e \in \operatorname{Ext}^{\bullet}_{S^e}(S, S^e)$, $\alpha \in L$ and $s \in S$,

$$(s\alpha) \cdot e = s \cdot (\alpha \cdot e) - \alpha(s) \cdot e. \tag{4-14}$$

This left $S \rtimes L$-module structure on $\operatorname{Ext}^{\bullet}_{S^e}(S, S^e)$ does not define a left U-module structure in general. However, Lemma 3.5.1 yields that $\operatorname{Ext}^{\bullet}_{S^e}(S, S^e)^{\vee}$ is a right U-module by defining $\theta \cdot \alpha$, for all $\theta \in \operatorname{Ext}^{\bullet}_{S^e}(S, S^e)^{\vee}$ and $\alpha \in L$, as

$$\begin{aligned} \theta \cdot \alpha : \operatorname{Ext}^{n}_{S^e}(S, S^e) &\to S, \\ e &\mapsto -\alpha(\theta(e)) + \theta(\alpha \cdot e). \end{aligned}$$

4.5. ***Particular case of Van den Bergh and Calabi–Yau duality.*** Recall that, whenever $\operatorname{Tor}^{S^e}_n(S, S) \simeq S$ as S-(bi)modules, a *volume form* is a free generator ω_S of $\operatorname{Tor}^{S^e}_n(S, S)$, and the associated *divergence*

$$\operatorname{div} : \operatorname{Der}_R(S) \to S$$

is defined such that, for all $\partial \in \operatorname{Der}_R(S)$,

$$\mathcal{L}_{\partial}(\omega_S) = \operatorname{div}(\partial)\omega_S. \tag{4-15}$$

When S is Calabi–Yau in dimension n, any free generator e_S of the left S-module $\operatorname{Ext}^{n}_{S^e}(S, S^e)$ defines an isomorphism of S-bimodules

$$\begin{aligned} \theta : S &\to \operatorname{Ext}^{n}_{S^e}(S, S^e), \\ s &\mapsto s e_S. \end{aligned}$$

In such a situation, the fundamental class $c_S \in \operatorname{Tor}^{S^e}_n(S, \operatorname{Ext}^{n}_{S^e}(S, S^e))$ (see 2.1) is a free generator of the left S-module $\operatorname{Tor}^{S^e}_n(S, \operatorname{Ext}^{n}_{S^e}(S, S^e))$, and hence the preimage ω_S of c_S under the bijective mapping

$$\theta_* : \operatorname{Tor}^{S^e}_n(S, S) \to \operatorname{Tor}^{S^e}_n(S, \operatorname{Ext}^{n}_{S^e}(S, S^e))$$

is a volume form for S, thus defining a divergence operator.

Proposition 4.5.1. (1) *Assume the following*:

- *R is Noetherian and S is finitely generated as an R-algebra.*
- *S is projective in* $\operatorname{Mod}(R)$.
- *S has Van den Bergh duality with dimension n.*

Then there is an isomorphism of S-modules compatible with Lie derivatives

$$\operatorname{Ext}^{n}_{S^e}(S, S^e) \simeq \Lambda^n_S \operatorname{Der}_R(S).$$

(2) *Assume that S is Calabi–Yau in dimension n. Let e_S be a free generator of the left S-module $\operatorname{Ext}^n_{S^e}(S, S^e)$. Let div be the resulting divergence operator. Then, for all $\partial \in \operatorname{Der}_R(S)$,*

$$\mathcal{L}_\partial(e_S) = -\operatorname{div}(\partial) e_S. \tag{4-16}$$

Proof. In both cases, S lies in $\operatorname{per}(S^e)$. Denote the fundamental class of S by c_S. In view of part (3) of Lemma 4.3.1, the definition of c_S gives that

$$\mathcal{L}_\partial(c_S) = 0. \tag{4-17}$$

(1) Denote $\operatorname{Ext}^n_{S^e}(S, S^e)$ by D. In view of Proposition 2.2.1, [Hochschild et al. 1962, Theorem 3.1] applies and yields an isomorphism of S-modules,

$$\operatorname{Tor}^{S^e}_n(S, S) \simeq \Lambda^n_S \Omega_{S/R}. \tag{4-18}$$

Following [Rinehart 1963, Section 9], this isomorphism is compatible with Lie derivatives. Identify D^{-1} with $\operatorname{Hom}_S(D, S)$ and define $\partial_{D^{-1}}$ as follows, for all $\partial \in \operatorname{Der}_R(S)$:

$$\begin{aligned} \partial_{D^{-1}} : \operatorname{Hom}_S(D, S) &\to \operatorname{Hom}_S(D, S), \\ f &\mapsto \partial \circ f - f \circ \mathcal{L}_\partial. \end{aligned}$$

The evaluation isomorphism

$$\operatorname{ev} : D \otimes_S D^{-1} \xrightarrow{\sim} S \tag{4-19}$$

is compatible with Lie derivatives in the following sense, where $\partial \in \operatorname{Der}_R(S)$:

$$\partial \circ \operatorname{ev} = \operatorname{ev} \circ (\mathcal{L}_\partial \otimes \operatorname{Id} + \operatorname{Id} \otimes \partial_{D^{-1}}). \tag{4-20}$$

Besides, the duality isomorphism

$$\iota_?(c_S) : \operatorname{Ext}^0_{S^e}(S, D^{-1}) \to \operatorname{Tor}^{S^e}_n(S, D \otimes_S D^{-1}) \tag{4-21}$$

is compatible with the action of Lie derivatives because of (4-17) (see part (3) of Lemma 4.3.1). Combining (4-18), (4-19), (4-20) and (4-21) yields an isomorphism that is compatible with Lie derivatives

$$D^{-1} \simeq \Lambda^n_S \Omega_{S/R}.$$

This proves (1).

(2) Keep the notation c_S, ω_S, θ, θ_* for the objects defined from e_S before the statement of the proposition. Let $\partial \in \operatorname{Der}_R(S)$. There exists $\lambda \in S$ such that

$$\mathcal{L}_\partial(e_S) = \lambda e_S.$$

Now, for all $s\otimes p\in S\otimes_{S^e}P^{-n}$,

$$\begin{aligned}\mathcal{L}_\partial(\theta_*(s\otimes p)) &= \mathcal{L}_\partial(se_S\otimes p)\\ &= \partial(s)e_S\otimes p+s\mathcal{L}_\partial(e_S)\otimes p+se_S\otimes\partial^\bullet(p)\\ &= \theta_*(\mathcal{L}_\partial(s\otimes p))+\lambda\theta_*(s\otimes p).\end{aligned}$$

Therefore,

$$0=\mathcal{L}_\partial(c_S)=\mathcal{L}_\partial(\theta_*(\omega_S))=\theta_*(\underbrace{\mathcal{L}_\partial(\omega_S)}_{=\operatorname{div}(\partial)\omega_S})+\lambda\theta_*(\omega_S)=(\lambda+\operatorname{div}(\partial))c_S.$$

Since c_S is regular, $\lambda=-\operatorname{div}(\partial)$. □

5. Proof of the main theorems

The main results of this article are proved in this section. For this purpose, a description of $\operatorname{Ext}^\bullet_{U^e}(U,U^e)$ is made in Section 5.1, the underlying S-module is expressed in terms of $\operatorname{Ext}^\bullet_{S^e}(S,S^e)$ and $\operatorname{Ext}^\bullet_U(S,U)$, and the U-bimodule structure is described using the functor $F:\operatorname{Mod}(U)\to\operatorname{Mod}(U^e)$ and the action of L on $\operatorname{Ext}^\bullet_{S^e}(S,S^e)$ introduced in Section 4. This description is applied in Section 5.2 in order to prove Theorem 1. And Theorem 2 and Corollary 1 are proved in Sections 5.3 and 5.4 by specialising to the situations where $\operatorname{Ext}^{\mathrm{top}}_{S^e}(S,S^e)$ and $\operatorname{Ext}^{\mathrm{top}}_U(S,U)$ are free, and where (S,L) arises from a Poisson bracket on S, respectively.

Throughout the section, $\operatorname{Ext}^\bullet_{S^e}(S,S^e)$ is endowed with its compatible left $S\rtimes L$-module structure introduced in Section 4.4.

5.1. ***The inverse dualising bimodule of U.*** This subsection proves the following result.

Proposition 5.1.1. *Let R be a commutative ring and $d\in\mathbb{N}$. Let (S,L) be a Lie–Rinehart algebra over R. Assume the following*:

(a) *S is flat as an R-module.*

(b) *For all $n\in\mathbb{N}$, the S-module* $\operatorname{Ext}^n_{S^e}(S,S^e)$ *is projective.*

(c) $S\in\operatorname{per}(S^e)$.

(d) *L is finitely generated and projective with constant rank equal to d in* $\operatorname{Mod}(S)$.

Then, $\Lambda^d_S L^\vee\otimes_S\operatorname{Ext}^\bullet_{S^e}(S,S^e)$ is a graded left U-module such that, for all $\alpha\in L$, $c\in\operatorname{Ext}^\bullet_{S^e}(S,S^e)$ and $\varphi\in\Lambda^d_S L^\vee$,

$$\alpha\cdot(\varphi\otimes c)=-\varphi\cdot\alpha\otimes c+\varphi\otimes\alpha\cdot c.$$

Moreover, U is homologically smooth. Finally, there is an isomorphism of graded right U^e-modules,

$$\operatorname{Ext}^\bullet_{U^e}(U,U^e)\simeq F(\Lambda^d_S L^\vee\otimes_S\operatorname{Ext}^{\bullet-d}_{S^e}(S,S^e)).$$

For this subsection, assume (a), (b), (c) and (d) are true, and consider

- a bounded resolution $Q^\bullet \to S$ in $\mathrm{Mod}(U)$ by finitely generated and projective modules (see [Rinehart 1963, Lemma 4.1]),
- a bounded resolution $\pi : P^\bullet \to S$ in $\mathrm{Mod}(S^e)$ by finitely generated and projective modules,
- an injective resolution $j : U^e \to I^\bullet$ in $\mathrm{Mod}(U^e \otimes (U^e)^{\mathrm{op}})$.

Since S is flat over R and L is projective in $\mathrm{Mod}(S)$, part (2) of Lemma 3.0.1 gives that U^e is flat over R. Therefore, the extension-of-scalars functor

$$- \otimes U^e : \mathrm{Mod}(U^e) \to \mathrm{Mod}(U^e \otimes (U^e)^{\mathrm{op}})$$

is exact. Hence, the restriction-of-scalars-functor transforms injective U^e-bimodules into injective left U^e-modules. Thus, $I^\bullet$ is an injective resolution of U^e in $\mathrm{Mod}(U^e)$. Therefore, there is an isomorphism of graded right U^e-modules,

$$\mathrm{Ext}^\bullet_{U^e}(U, U^e) \simeq H^\bullet \mathrm{Hom}_{U^e}(U, I^*). \tag{5-1}$$

The right-hand side is a right U^e-module by means of I^*.

The proof of the above proposition is divided into separate lemmas.

Lemma 5.1.2. *U is homologically smooth.*

Proof. Since U is projective in $\mathrm{Mod}(S)$ (see part (2) of Lemma 3.0.1), the functor

$$F : \mathrm{Mod}(U) \to \mathrm{Mod}(U^e)$$

is exact. Moreover, $F(S) \simeq U$ and $S \in \mathrm{per}(U)$. Therefore, in order to prove that U is homologically smooth, it suffices to prove that $F(U) \in \mathrm{per}(U^e)$, which is equivalent to $F(U)$ being compact in the derived category $\mathcal{D}(U^e)$ of complexes of U-bimodules. Here is a proof of this fact. Let $(M_k)_{k\in K}$ be a family in $\mathcal{D}(U^e)$, denote $\oplus_{k\in K} M_k$ by M, and consider fibrant resolutions of complexes of U-bimodules $M_k \to i(M_k)$, for all $k \in K$, and $M \to i(M)$. Since S is homologically smooth, S is compact in $\mathcal{D}(S^e)$, and hence the following natural mapping is a quasi-isomorphism:

$$\bigoplus_{k\in K} \mathrm{Hom}_{S^e}(P^\bullet, M_k) \to \mathrm{Hom}_{S^e}(P^\bullet, M).$$

Since $P^\bullet$ is a right bounded complex of projective S-bimodules, the functor $\mathrm{Hom}_{S^e}(P^\bullet, -)$ preserves quasi-isomorphisms, and hence the following natural mapping is a quasi-isomorphism:

$$\bigoplus_{k\in K} \mathrm{Hom}_{S^e}(P^\bullet, i(M_k)) \to \mathrm{Hom}_{S^e}(P^\bullet, i(M)).$$

Since U is projective over S on both sides, U^e is projective in $\mathrm{Mod}(S^e)$. Therefore, for all fibrant complexes I of U-bimodules, the functor $\mathrm{Hom}_{S^e}(-, I)$ preserves quasi-isomorphisms. Accordingly, the following natural mapping is a quasi-isomorphism:

$$\bigoplus_{k\in K} \mathrm{Hom}_{S^e}(S, i(M_k)) \to \mathrm{Hom}_{S^e}(S, i(M))\,.$$

Since the pair (F, G) is adjoint and G is induced by the functor $\mathrm{Hom}_{S^e}(S, -)$, the following natural mapping is a quasi-isomorphism:

$$\bigoplus_{k\in K} \mathrm{Hom}_{U^e}(F(U), i(M_k)) \to \mathrm{Hom}_{U^e}(F(U), i(M)).$$

Taking cohomology in degree 0 yields that the following natural mapping is bijective:

$$\bigoplus_{k\in K} \mathcal{D}(U^e)(F(U), i(M_k)) \to \mathcal{D}(U^e)(F(U), i(M))\,.$$

This proves that $F(U)$ is compact in $\mathcal{D}(U^e)$. Thus, U is homologically smooth. □

The authors thank Bernhard Keller for having pointed out an incorrect argument in a previous version of this proof.

Lemma 5.1.3. *There is an isomorphism of graded right U^e-modules,*

$$\mathrm{Ext}^\bullet_{U^e}(U, U^e) \simeq H^\bullet(\mathrm{Hom}_U(Q^*, U) \otimes_U G(I^*)). \tag{5-2}$$

Proof. Because of the isomorphism $F(S) \simeq U$ in $\mathrm{Mod}(U^e)$ and the adjunction (F, G), there is a functorial isomorphism of complexes of right U^e-modules,

$$\mathrm{Hom}_{U^e}(U, I^\bullet) \simeq \mathrm{Hom}_U(S, G(I^\bullet)). \tag{5-3}$$

Since F is exact and the pair (F, G) is adjoint, $G(I^\bullet)$ is a left bounded complex of injective left U-modules. Hence, $\mathrm{Hom}_U(-, G(I^\bullet))$ preserves quasi-isomorphisms. Thus, the quasi-isomorphism $Q^\bullet \to S$ induces a quasi-isomorphism of complexes of right U^e-modules,

$$\mathrm{Hom}_U(S, G(I^\bullet)) \to \mathrm{Hom}_U(Q^\bullet, G(I^\bullet)). \tag{5-4}$$

Since $Q^\bullet$ is bounded and consists of finitely generated projective left U-modules, the following canonical mapping is a functorial isomorphism:

$$\mathrm{Hom}_U(Q^\bullet, U) \otimes_U G(I^\bullet) \to \mathrm{Hom}_U(Q^\bullet, G(I^\bullet)). \tag{5-5}$$

Note that, whether in (5-3), (5-4), or (5-5), the involved right U^e-module structures are inherited from $I^\bullet$. Thus, the announced isomorphism is proved. □

In order to examine the right-hand side of (5-2) by means of a spectral sequence, the following lemma describes $H^\bullet(G(I^*))$ as a graded $U - U^e$-bimodule.

Lemma 5.1.4. *Consider* $\mathrm{Ext}^{\bullet}_{S^e}(S, S^e)$ *as a left* $S \rtimes L$*-module as in Section 4.4. Then, there is a* $U - U^e$*-bimodule structure on* $\mathrm{Ext}^{\bullet}_{S^e}(S, S^e) \otimes_{S^e} U^e$ *such that the right* U^e*-module structure is inherited from* U^e *and for all* $\alpha \in L$, $c \in \mathrm{Ext}^{\bullet}_{S^e}(S, S^e)$ *and* $u, v \in U$,

$$\alpha \cdot (c \otimes (u \otimes v)) = \alpha \cdot c \otimes (u \otimes v) + c \otimes ((\alpha \otimes 1 - 1 \otimes \alpha) \cdot (u \otimes v)).$$

For this structure, there is an isomorphism of graded $U - U^e$*-bimodules,*

$$H^{\bullet}(G(I^{*})) \simeq \mathrm{Ext}^{\bullet}_{S^e}(S, S^e) \otimes_{S^e} U^e.$$

Proof. The object $G(I^{\bullet})$ is $\mathrm{Hom}_{S^e}(S, I^{\bullet})$ as a complex of S-modules, its right U^e-module structure is inherited from $I^{\bullet}$, and the one of left U-module is given in Section 3.2.

First, since U^e is projective in $\mathrm{Mod}(S^e)$ and $I^{\bullet}$ consists of injective left U^e-modules, $I^{\bullet}$ is a left bounded complex of injective left S^e-modules. Hence, $\mathrm{Hom}_{S^e}(-, I^{\bullet})$ preserves quasi-isomorphisms. Thus, $\pi : P^{\bullet} \to S$ induces a quasi-isomorphism of complexes of right S^e-modules,

$$\pi' : \mathrm{Hom}_{S^e}(S, I^{\bullet}) \to \mathrm{Hom}_{S^e}(P^{\bullet}, I^{\bullet}). \tag{5-6}$$

For all $\alpha \in L$, let $\partial^{\bullet}_{\alpha} : P^{\bullet} \to P^{\bullet}$ be a derivation relative to $\partial_{\alpha} : S \to S$ (see Section 4.2), and denote by $\delta^{\bullet}_{\alpha}$ the mapping from $I^{\bullet}$ to $I^{\bullet}$ given by

$$i \mapsto (\alpha \otimes 1 - 1 \otimes \alpha) \cdot i.$$

Then, define $\alpha \cdot f$ and $\alpha \cdot g$, for all $f \in \mathrm{Hom}_{S^e}(S, I^{\bullet})$ and $g \in \mathrm{Hom}_{S^e}(P^{\bullet}, I^{\bullet})$, by

$$\alpha \cdot f = \delta^{\bullet}_{\alpha} \circ f - f \circ \partial_{\alpha}$$
$$\alpha \cdot g = \delta^{\bullet}_{\alpha} \circ g - g \circ \partial^{\bullet}_{\alpha};$$

since $\pi \circ \partial^{\bullet}_{\alpha} = \partial_{\alpha} \circ \pi$,

$$\pi'(\alpha \cdot f) = \alpha \cdot \pi'(f).$$

The hypotheses on $P^{\bullet}$ yield an isomorphism of complexes of right U^e-modules,

$$\mathrm{ev} : \mathrm{Hom}_{S^e}(P^{\bullet}, S^e) \otimes_{S^e} I^{\bullet} \to \mathrm{Hom}_{S^e}(P^{\bullet}, I^{\bullet}). \tag{5-7}$$

Endow the left-hand side term with the following action of L. For all $\alpha \in L$ and $\varphi \otimes i \in \mathrm{Hom}_{S^e}(P^{\bullet}, S^e) \otimes_{S^e} I^{\bullet}$, denote by $\alpha \cdot (\varphi \otimes i)$ the (well-defined) element of $\mathrm{Hom}_{S^e}(P^{\bullet}, S^e) \otimes_{S^e} I^{\bullet}$,

$$\alpha \cdot \varphi \otimes i + \varphi \otimes (\delta^{\bullet}_{\alpha} i).$$

The assignment $\varphi \otimes i \mapsto \alpha \cdot (\varphi \otimes i)$ is a morphism of complexes of R-modules from $\mathrm{Hom}_{S^e}(P^{\bullet}, S^e) \otimes_{S^e} I^{\bullet}$ to itself. In view of (4-8) and of the identity

$$(\alpha \otimes 1 - 1 \otimes \alpha) \cdot ((s \otimes t) \cdot j) = \partial_{\alpha}(s \otimes t) \cdot j + (s \otimes t) \cdot (\alpha \otimes 1 - 1 \otimes \alpha) \cdot j$$

in $I^\bullet$, for all $s, t \in S$ and $j \in I^\bullet$, the following holds:

$$\mathrm{ev}(\alpha \cdot (\varphi \otimes i)) = \alpha \cdot \mathrm{ev}(\varphi \otimes i). \tag{5-8}$$

$\mathrm{Hom}_{S^e}(P^\bullet, S^e)$ is also a bounded complex of projective right S^e-modules. Hence, the functor $\mathrm{Hom}_{S^e}(P^\bullet, S^e) \otimes_{S^e} -$ preserves quasi-isomorphisms. Thus, $j : U^e \to I^\bullet$ induces a quasi-isomorphism of right U^e-modules,

$$\mathrm{Id} \otimes j : \mathrm{Hom}_{S^e}(P^\bullet, S^e) \otimes_{S^e} U^e \to \mathrm{Hom}_{S^e}(P^\bullet, S^e) \otimes_{S^e} I^\bullet. \tag{5-9}$$

Endow the left-hand side term with the following action of L. For all $\alpha \in L$, $\varphi \in \mathrm{Hom}_{S^e}(P^\bullet, S^e)$ and $u, v \in U$, denote by $\alpha \cdot (\varphi \otimes (u \otimes v))$ the following (well-defined) element of $\mathrm{Hom}_{S^e}(P^\bullet, S^e) \otimes_{S^e} U^e$:

$$\alpha \cdot \varphi \otimes (u \otimes v) + \varphi \otimes ((\alpha \otimes 1 - 1 \otimes \alpha) \cdot (u \otimes v)).$$

The assignment $\varphi \otimes (u \otimes v) \mapsto \alpha \cdot (\varphi \otimes (u \otimes v))$ is a morphism of complexes of R-modules from $\mathrm{Hom}_{S^e}(P^\bullet, S^e) \otimes_{S^e} U^e$ to itself, and

$$(\mathrm{Id} \otimes j)(\alpha \cdot (\varphi \otimes (u \otimes v)) = \alpha \cdot ((\mathrm{Id} \otimes j)(\varphi \otimes (u \otimes v)))$$

because $j : U^e \to I^\bullet$ is a morphism of complexes of $U^e - U^e$-bimodules.

Since U^e is projective in $\mathrm{Mod}(S^e)$, there is an isomorphism of right U^e-modules,

$$H^\bullet(\mathrm{Hom}_{S^e}(P^*, S^e) \otimes_{S^e} U^e) \simeq \mathrm{Ext}^\bullet_{S^e}(S, S^e) \otimes_{S^e} U^e. \tag{5-10}$$

For all cocycles $\varphi \in \mathrm{Hom}_{S^e}(P^\bullet, S^e)$, with cohomology class denoted by c, and for all $\alpha \in L$ and $u, v \in U$, the image under (5-10) of the cohomology class of

$$\alpha \cdot (\varphi \otimes (u \otimes v))$$

is

$$\alpha \cdot c \otimes (u \otimes v) + c \otimes ((\alpha \otimes 1 - 1 \otimes \alpha) \cdot (u \otimes v)), \tag{5-11}$$

where $\alpha \cdot c$ is defined in Section 4.4 (see (4-13)).

Combining (5-6), (5-7), (5-9), (5-10) yields an isomorphism of right U^e-modules,

$$\mathrm{Ext}^\bullet_{S^e}(S, S^e) \otimes_{S^e} U^e \xrightarrow{\sim} H^\bullet(G(I^*)), \tag{5-12}$$

such that, for all $\alpha \in L$, $c \in \mathrm{Ext}^\bullet_{S^e}(S, S^e)$ and $u, v \in U$, if γ denotes the image of $c \otimes (u \otimes v)$ under (5-12), then $\alpha \cdot \gamma$ is the image of (5-11).

Thus, applying part (1) of Lemma 3.5.2 to $N = \mathrm{Ext}^\bullet_{S^e}(S, S^e)$ yields the announced conclusion. □

Proof of Proposition 5.1.1. The statement relative to the left U-module structure on $\Lambda^d_S L^\vee \otimes \mathrm{Ext}^\bullet_{S^e}(S, S^e)$ follows from Lemma 3.5.1, and Lemma 5.1.2 shows that U

is homologically smooth. The (first quadrant, cohomological) spectral sequence of the bicomplex

$$(\mathrm{Hom}_U(Q^p, U) \otimes_U G(I^q))_{p,q} \tag{5-13}$$

converges to $H^\bullet(\mathrm{Hom}_U(Q^*, U) \otimes_U G(I^*))$ and its $E_2^{p,q}$-term is, for all $p, q \in \mathbb{Z}$,

$$H_h^p(H_v^q(\mathrm{Hom}_U(Q^\bullet, U) \otimes_U G(I^\bullet)).$$

Since $\mathrm{Hom}_U(Q^\bullet, U)$ consists of projective right U-modules, there is an isomorphism of right U^e-modules, for all $p, q \in \mathbb{Z}$,

$$H^q(\mathrm{Hom}_U(Q^p, U) \otimes_U G(I^\bullet)) \simeq \mathrm{Hom}_U(Q^p, U) \otimes_U H^q(G(I^\bullet)). \tag{5-14}$$

The description of $H^\bullet(G(I^*))$ made in Lemma 5.1.4 combines with (5-14) into the following isomorphism of right U^e-modules, for all $p, q \in \mathbb{Z}$:

$$H^q(\mathrm{Hom}_U(Q^p, U) \otimes_U G(I^\bullet)) \simeq \mathrm{Hom}_U(Q^p, U) \otimes_U (\mathrm{Ext}^q_{S^e}(S, S^e) \otimes_{S^e} U^e). \tag{5-15}$$

Using Lemma 3.5.2 (part (2)), this isormorphism yields an isomorphism of right U^e-modules, for all $p, q \in \mathbb{Z}$:

$$H^q(\mathrm{Hom}_U(Q^p, U) \otimes_U G(I^\bullet)) \simeq F(\mathrm{Hom}_U(Q^p, U) \otimes_S \mathrm{Ext}^q_{S^e}(S, S^e)). \tag{5-16}$$

Given that F is an exact functor, that $\mathrm{Ext}^q_{S^e}(S, S^e)$ is projective in $\mathrm{Mod}(S)$ for all q and that (S, L) has duality in dimension d, it follows from (5-16) that there is an isomorphism of right U^e-modules, for all $p, q \in \mathbb{Z}$,

$$H_h^p(H_v^q(\mathrm{Hom}_U(Q^\bullet, U) \otimes_U G(I^\bullet))) \simeq \begin{cases} F(\mathrm{Ext}^d_U(S, U) \otimes_S \mathrm{Ext}^q_{S^e}(S, S^e)) & \text{if } p = d, \\ 0 & \text{if } p \neq d. \end{cases}$$

Therefore, the spectral sequence of the bicomplex (5-13) degenerates at E_2. Thus,

$$H^\bullet(\mathrm{Hom}_U(Q^*, U) \otimes_U G(I^*)) \simeq F(\mathrm{Ext}^d_U(S, U) \otimes_S \mathrm{Ext}^{\bullet-d}_{S^e}(S, S^e)) \text{ in } \mathrm{Mod}(S^e).$$

The conclusion follows from (5-2) and from the isomorphism $\mathrm{Ext}^d_U(S, U) \simeq \Lambda^d_S L^\vee$ in $\mathrm{Mod}(U)$ established in [Huebschmann 1999, Theorem 2.10] □

5.2. *Proof of the main theorem.*

Proof of Theorem 1. Following Proposition 5.1.1, U is homologically smooth and there is an isomorphism of graded right U^e-modules,

$$\mathrm{Ext}^\bullet_{U^e}(U, U^e) \simeq F(\Lambda^d_S L^\vee \otimes_S \mathrm{Ext}^{\bullet-d}_{S^e}(S, S^e)).$$

According to Proposition 3.6.1, the functor F transforms left U-modules that are invertible as S-modules into invertible U-bimodules. Note that

- $\Lambda_S^d L^\vee$ is invertible as an S-module because L is projective with constant rank, and
- $\mathrm{Ext}^\bullet_{S^e}(S, S^e)$ is concentrated in degree n and $\mathrm{Ext}^n_{S^e}(S, S^e)$ is invertible as an S-module because S has Van den Bergh duality.

Thus, $\mathrm{Ext}^\bullet_{U^e}(U, U^e)$ is concentrated in degree $n+d$ and $\mathrm{Ext}^{n+d}_{U^e}(U, U^e)$ is invertible as a U-bimodule. Hence, U has Van den Bergh duality in dimension $n+d$. □

5.3. ***Proof of Theorem 2.*** The hypotheses of Theorem 2 are assumed throughout this subsection. Let φ_L be a free generator of the S-module $\Lambda_S^d L^\vee$. Let e_S be a free generator of the S-module $\mathrm{Ext}^n_{S^e}(S, S^e)$. Therefore, there exist mappings

$$\lambda_L, \lambda_S : L \to S$$

such that, for all $\alpha \in L$,

$$\begin{cases} \alpha \cdot e_S = \lambda_S(\alpha) \cdot e_S, \\ \varphi_L \cdot \alpha = \lambda_L(\alpha) \cdot \varphi_S. \end{cases}$$

Some basic properties of these are summarised below.

Lemma 5.3.1. *Let λ be either one of λ_S or λ_L. Then, for all $\alpha, \beta \in L$ and $s \in S$,*

(1) $\lambda(s\alpha) = s\lambda(\alpha) - \alpha(s)$,

(2) $\lambda([\alpha, \beta]) = \alpha(\lambda(\beta)) - \beta(\lambda(\alpha))$.

Proof. Assume that $\lambda = \lambda_S$. Let $s \in S$ and $\alpha \in L$. Then, using Section 4.4,

$$\begin{aligned} (s\alpha) \cdot e_S &= s \cdot (\alpha \cdot e_S) - \alpha(s) \cdot e_S \\ &= (s\lambda(\alpha) - \alpha(s)) \cdot e_S, \end{aligned}$$

which proves (1), and

$$\begin{aligned} \alpha \cdot (\beta \cdot e_S) &= \alpha \cdot (\lambda(\beta) \cdot e_S) \\ &= \alpha(\lambda(\beta)) \cdot e_S + \lambda(\beta) \cdot (\alpha \cdot e_S) \\ &= (\alpha(\lambda(\beta)) + \lambda(\alpha)\lambda(\beta)) \cdot e_S, \end{aligned}$$

from which (2) may be proved directly. The proof when $\lambda = \lambda_L$ is analogous, using the right U-module structure of $\Lambda_S^d L^\vee$ instead of Section 4.4. □

As proved later, the following automorphism is a Nakayama automorphism for U.

Lemma 5.3.2. *There exists a unique R-algebra homomorphism,*

$$\nu : U \to U,$$

such that, for all $s \in S$ and $\alpha \in L$,

$$\begin{cases} \nu(s) = s, \\ \nu(\alpha) = \alpha + \lambda_L(\alpha) - \lambda_S(\alpha). \end{cases}$$

This is an automorphism of R-algebra.

Proof. The uniqueness is immediate. For all $\alpha \in L$, denote $\alpha + \lambda_L(\alpha) - \lambda_S(\alpha)$ by ν_α. Then, for all $s \in S$ and $\alpha, \beta \in L$,

$$\begin{aligned} [\nu_\alpha, \nu_\beta] &= [\alpha + \lambda_L(\alpha) - \lambda_S(\alpha), \beta + \lambda_L(\beta) - \lambda_S(\beta)] \\ &\underset{\text{Lemma 5.3.1}}{=} [\alpha, \beta] + \lambda_L([\alpha, \beta]) - \lambda_S([\alpha, \beta]) = \nu_{[\alpha,\beta]}, \\ \nu_{s\alpha} &= s\alpha + \lambda_L(s\alpha) - \lambda_S(s\alpha) \\ &\underset{\text{Lemma 5.3.1}}{=} s\alpha + s\lambda_L(\alpha) - s\lambda_S(\alpha) = s\nu_\alpha, \\ [\nu_\alpha, s] &= [\alpha + \lambda_L(\alpha) - \lambda_L(\alpha), s] = \alpha(s). \end{aligned}$$

This proves the existence of ν. Note that ν preserves the filtration of U by the powers of L and that $\mathrm{gr}(\nu)$ is the identity mapping of U. Accordingly, ν is bijective. □

Now it is possible to prove Theorem 2.

Proof of Theorem 2. From Theorem 1, U has Van den Bergh duality in dimension $n+d$ and there is an isomorphism of U-bimodules,

$$\mathrm{Ext}^{n+d}_{U^e}(U, U^e) \simeq F(\Lambda^d_S \Lambda^\vee \otimes_S \mathrm{Ext}^n_{S^e}(S, S^e)), \tag{5-17}$$

where the tensor product inside $F(\bullet)$ is a left U-module by (3-8).

Recall that $\Lambda^d_S L^\vee$ and $\mathrm{Ext}^n_{S^e}(S, S^e)$ are freely generated by φ_L and e_S, respectively. Therefore, the following mapping is an isomorphism of left U-modules (see Section 3.3):

$$\begin{aligned} \Phi : U &\to F(\Lambda^d_S L^\vee \otimes_S \mathrm{Ext}^n_{S^e}(S, S^e)), \\ u &\mapsto u \otimes (\varphi_L \otimes e_S). \end{aligned} \tag{5-18}$$

For all $s \in S$, $\alpha \in L$ and $u \in U$,

$$\Phi(u)s = (u \otimes (\varphi_L \otimes e_S)) \cdot s = us \otimes (\varphi_L \otimes e_S) = \Phi(us),$$

$$\begin{aligned} \Phi(u)\alpha &= (u \otimes (\varphi_L \otimes e_S)) \cdot \alpha \\ &= u\alpha \otimes (\varphi_L \otimes e_S) - u \otimes \alpha \cdot (\varphi_L \otimes e_S) \\ &= u\alpha \otimes (\varphi_L \otimes e_S) - (-u \otimes (\varphi_L \cdot \alpha \otimes e_S) + u \otimes (\varphi_L \otimes \alpha \cdot e_S)) \\ &= (u(\alpha + \lambda_L(\alpha) - \lambda_S(\alpha))) \otimes (\varphi_L \otimes e_S) \\ &= \Phi(u(\alpha + \lambda_L(\alpha) - \lambda_S(\alpha))). \end{aligned}$$

Thus, denoting by ν the automorphism of U considered in Lemma 5.3.2, then, for all $u, v \in U$,

$$\Phi(u) \cdot v = \Phi(u\nu(v)). \tag{5-19}$$

Combining (5-17), (5-18) and (5-19) yields that there is an isomorphism of bimodules,

$$\mathrm{Ext}^{n+d}_{U^e}(U, U^e) \simeq U^{\nu}.$$

Since $\lambda_S = -\mathrm{div}$ (see Proposition 4.5.1), this proves Theorem 2. □

5.4. ***Case of Poisson algebras.***

Proof of Corollary 1. From Proposition 2.2.1, S has Van den Bergh duality in dimension n. Moreover, Proposition 4.5.1 yields an isomorphism of S-modules $\Lambda^n_S \mathrm{Der}_R(S) \simeq \mathrm{Ext}^n_{S^e}(S, S^e)$ which is compatible with the action of Lie derivatives. Finally, according to (1-3), the dualising module of $(S, \Omega_{S/R})$ is $\Lambda^n_S \mathrm{Der}_R(S)$ with right U-module structure such that, for all $s \in S$ and $\varphi \in \Lambda^n_S \mathrm{Der}_R(S)$,

$$\varphi \cdot ds = -\mathcal{L}_{\{s,-\}}(\varphi).$$

Using these considerations, the corollary follows from Theorems 1 and 2. □

6. Examples

6.1. ***The case where L is free as an S-module.*** In this subsection, it is assumed that L is free as an S-module. Consider a basis $(\alpha_1, \dots, \alpha_d)$ of L over S. Denote the dual basis of $L^\vee$ by $(\alpha_1^*, \dots, \alpha_d^*)$. In particular, $\Lambda^d_S L^\vee$ is free of rank one in $\mathrm{Mod}(S)$, with a generator denoted by φ_L,

$$\varphi_L = \alpha_1^* \wedge \cdots \wedge \alpha_d^*.$$

For all $i \in \{1, \dots, d\}$, consider the matrix of ad_{α_i}, denoted by $(s^i_{j,k})_{j,k} \in M_d(S)$. Hence, for all $i, k \in \{1, \dots, d\}$,

$$[\alpha_i, \alpha_k] = \sum_{j=1}^{d} s^i_{j,k} \alpha_j.$$

In this situation, the action of L on $\Lambda^\bullet_S L$ by Lie derivatives specialises as follows. For all $i, j, k \in \{1, \dots, d\}$,

$$(\lambda_{\alpha_i}(\alpha_j^*))(\alpha_k) = \alpha_i(\alpha_j^*(\alpha_k)) - \alpha_j^*([\alpha_i, \alpha_k]) = -s^i_{j,k}.$$

Hence, for all $i, j \in \{1, \dots, d\}$,

$$\lambda_{\alpha_i}(\alpha_j^*) = -\sum_{k=1}^{d} s^i_{j,k} \alpha_k^*.$$

Thus, the right U-module structure of $\Lambda^d_S L^\vee$ is such that, for all $\alpha \in L$,

$$\varphi_L \cdot \alpha = \mathrm{Tr}(\mathrm{ad}_\alpha)\varphi_L. \tag{6-1}$$

Using this simplified description of $\Lambda^d_S L^\vee$ yields the following corollary of the main theorems of this article.

Corollary 6.1.1. *Let R be a commutative ring. Let (S, L) be a Lie–Rinehart algebra of R. Denote by U its enveloping algebra. Assume that*

- *S is flat as an R-module,*
- *S has Van den Bergh duality in dimension n,*
- *L is free of rank d as an S-module.*

Let $(\alpha_1, \dots, \alpha_d)$ be a basis of L over S as considered previously. Then, U has Van den Bergh duality in dimension $n+d$ and there is an isomorphism of U-bimodules,

$$\mathrm{Ext}^{n+d}_{U^e}(U, U^e) \simeq U \otimes_S \mathrm{Ext}^n_{S^e}(S, S^e),$$

where the left U-module structure on $U \otimes_S \mathrm{Ext}^n_{S^e}(S, S^e)$ is the natural one and the right U-module structure is such that, for all $u \in U$, $e \in \mathrm{Ext}^n_{S^e}(S, S^e)$ and $\alpha \in L$,

$$(u \otimes e) \cdot \alpha = u\alpha \otimes e + u \otimes \mathrm{Tr}(\mathrm{ad}_\alpha)e - u \otimes \mathcal{L}_{\partial_\alpha}(e).$$

If, moreover, S is Calabi–Yau, then U is skew Calabi–Yau and each volume form on S determines a Nakayama automorphism $\nu \in \mathrm{Aut}_R(U)$ such that, for all $s \in S$ and $\alpha \in L$,

$$\begin{cases} \nu(s) = s, \\ \nu(\alpha) = \alpha + \mathrm{Tr}(\mathrm{ad}_\alpha) + \mathrm{div}(\partial_\alpha), \end{cases}$$

where div *denotes the divergence of the chosen volume form.*

Proof. In view of (6-1), there is an isomorphism of right U-modules,

$$\Lambda^d_S L^\vee \simeq S,$$

where the right U-module structure on the right-hand side term is such that, for all $\alpha \in L$,

$$1 \cdot \alpha = \mathrm{Tr}(\mathrm{ad}_\alpha).$$

The corollary therefore follows directly from Theorems 1 and 2. □

The previous corollary applies to any Lie–Rinehart algebra arising from a Poisson structure on $R[x_1, \dots, x_n]$, $n \in \mathbb{N}\backslash\{0, 1\}$.

Example 6.1.2. Let $S = R[x, y]$. Let $P \in S$. This defines a Poisson structure on S such that

$$\{x, y\} = P.$$

Let $L := \Omega_{S/R}$ and consider (S, L) as a Lie–Rinehart algebra over R such that, for all $s, t \in S$,

- $[ds, dt] = d\{s, t\}$;
- $\partial_{ds} = \{s, -\}$.

Then (dx, dy) is a basis of $\Omega_{S/R}$ over S. Note that

$$\begin{cases} \mathrm{Tr}(\mathrm{ad}_{dx}) = \mathrm{div}(\partial_{dx}) = \dfrac{\partial P}{\partial y}, \\ \mathrm{Tr}(\mathrm{ad}_{dy}) = \mathrm{div}(\partial_{dy}) = -\dfrac{\partial P}{\partial x}. \end{cases}$$

From Corollary 6.1.1, U is skew Calabi–Yau in dimension 4 and has a Nakayama automorphism $\nu \in \mathrm{Aut}_R(S)$ such that

$$\begin{cases} \nu(x) = x, \quad \nu(dx) = dx + 2\dfrac{\partial P}{\partial y}, \\ \nu(y) = y, \quad \nu(dy) = dy - 2\dfrac{\partial P}{\partial x}. \end{cases}$$

By considering the filtration of U by the powers of the image of L in U, with associated graded algebra the symmetric algebra of L over S (see [Rinehart 1963, Theorem 3.1]), it appears that $U^\times = S^\times = R^\times$. Accordingly, U has no nontrivial inner automorphism. Consequently, U is Calabi–Yau if and only if $\nu = \mathrm{Id}_U$, that is, if and only if $\mathrm{char}(R) = 2$, or else $P \in R$.

Example 6.1.3. Let $S = R[x, y, z]$. Let $P_x, P_y, P_z \in S$ be such that

$$\overrightarrow{P} \wedge \mathrm{curl}(\overrightarrow{P}) = 0,$$

where $\overrightarrow{P}$ denotes

$$\begin{pmatrix} P_x \\ P_y \\ P_z \end{pmatrix}.$$

Hence, the following defines a Poisson bracket on S,

$$\{x, y\} = P_z, \qquad \{y, z\} = P_x, \qquad \{z, x\} = P_y.$$

As in the previous example, let $(S, L := \Omega_{S/R})$ be the associated Lie–Rinehart algebra over R. As is well-known,

$$\{x, -\} = P_z\frac{\partial}{\partial y} - P_y\frac{\partial}{\partial z}, \qquad \{y, -\} = P_x\frac{\partial}{\partial z} - P_z\frac{\partial}{\partial x}, \qquad \{z, -\} = P_y\frac{\partial}{\partial x} - P_x\frac{\partial}{\partial y}.$$

Therefore, using the basis (dx, dy, dz) of $\Omega_{S/R}$ over S,

$$\begin{pmatrix}\operatorname{div}(\partial_{dx})\\ \operatorname{div}(\partial_{dy})\\ \operatorname{div}(\partial_{dz})\end{pmatrix} = \begin{pmatrix}\operatorname{Tr}(\operatorname{ad}_{dx})\\ \operatorname{Tr}(\operatorname{ad}_{dy})\\ \operatorname{Tr}(\operatorname{ad}_{dz})\end{pmatrix} = \operatorname{curl}(\overrightarrow{P}).$$

Using Corollary 6.1.1, it follows that U is skew Calabi–Yau in dimension 6 and has a Nakayama automorphism $\nu \in \operatorname{Aut}_R(S)$ such that

$$\begin{pmatrix}\nu(x)\\ \nu(y)\\ \nu(z)\end{pmatrix} = \begin{pmatrix}x\\ y\\ z\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}\nu(dx)\\ \nu(dy)\\ \nu(dz)\end{pmatrix} = \begin{pmatrix}dx\\ dy\\ dz\end{pmatrix} + 2\operatorname{curl}(\overrightarrow{P}).$$

As in the previous example, there are no nontrivial inner automorphisms for U. Hence, U is Calabi–Yau if and only if $\operatorname{char}(R) = 2$, or else $\operatorname{curl}(\overrightarrow{P}) = 0$. In particular, when R contains $\mathbb{Q}$ as a subring, then U is Calabi–Yau if and only if the Poisson bracket is Jacobian, that is, there exists $Q \in S$ such that $P = \overrightarrow{\operatorname{grad}}(Q)$.

By the Quillen–Suslin Theorem, when R is a field and $n \in \mathbb{N}$, any $R[x_1, \ldots, x_n]$-module that is finitely generated and projective is free. Hence, Corollary 6.1.1 also applies to all Lie–Rinehart algebras of the shape $(R[x_1, \ldots, x_n], L)$, where R is a field.

6.2. ***On two-dimensional Nambu–Poisson structures.*** Following Corollary 1, U is skew Calabi–Yau when S is flat over R and Calabi–Yau and (S, L) is given by a Poisson bracket on S. Assuming these properties, this section computes a Nakayama automorphism of U for a class of examples of two-dimensional Nambu–Poisson structures (see [Laurent-Gengoux et al. 2013, Section 8.3]).

Let $S = R[x, y, z]/(P)$ where $P = 1 + T$ for some $T \in R[x, y, z]$ which is (p, q, r)-homogeneous in the sense that $p, q, r \in R$ and $t := p\alpha + q\beta + r\gamma$ is a unit in R which does not depend on the monomial $x^\alpha y^\beta z^\gamma$ appearing in T. The hypotheses imply the following equality in S:

$$px\frac{\partial P}{\partial x} + qy\frac{\partial P}{\partial y} + rz\frac{\partial P}{\partial z} = -t \ (\in R^\times). \tag{6-2}$$

Let $Q \in R[x, y, z]$ and endow S with the Poisson structure such that

$$\{x, y\} = Q\frac{\partial P}{\partial z}, \qquad \{y, z\} = Q\frac{\partial P}{\partial x}, \qquad \{z, x\} = Q\frac{\partial P}{\partial y}. \tag{6-3}$$

Consider $(S, L := \Omega_{S/R})$ as a Lie–Rinehart algebra such that, for all $s, t, s' \in S$,

- $[ds, dt] = d\{s, t\}$,
- $(sdt)(s') = s\{t, s'\}$.

Consider the following 2-form on S:

$$\omega_S = px dy \wedge dz + qy dz \wedge dx + rz dx \wedge dy.$$

According to (6-2), $\Omega_{S/R}$ is a projective S-module of rank 2. And the relation

$$\frac{\partial P}{\partial x} dx + \frac{\partial P}{\partial y} dy + \frac{\partial P}{\partial z} dz = 0$$

in $\Omega_{S/R}$ yields the following relations in $\Lambda^2_S \Omega_{S/R}$:

$$\frac{\partial P}{\partial x}\, dx \wedge dy = \frac{\partial P}{\partial z}\, dy \wedge dz,$$
$$\frac{\partial P}{\partial y}\, dy \wedge dz = \frac{\partial P}{\partial x}\, dz \wedge dx,$$
$$\frac{\partial P}{\partial z}\, dz \wedge dx = \frac{\partial P}{\partial y}\, dx \wedge dy.$$

Combining with (6-2) yields

$$dx \wedge dy = -t^{-1} \frac{\partial P}{\partial z} \omega_S,$$
$$dy \wedge dz = -t^{-1} \frac{\partial P}{\partial x} \omega_S,$$
$$dz \wedge dx = -t^{-1} \frac{\partial P}{\partial y} \omega_S.$$

Thus, ω_S is a volume form of S.

In order to determine the divergence of ω_S, consider the derivations $\delta_x, \delta_y, \delta_z \in \mathrm{Der}_R(S)$ given by

$$\begin{aligned} \delta_x : x &\mapsto 0 & \delta_y : x &\mapsto -\frac{\partial P}{\partial z} & \delta_z : x &\mapsto \frac{\partial P}{\partial y} \\ y &\mapsto \frac{\partial P}{\partial z} & y &\mapsto 0 & y &\mapsto -\frac{\partial P}{\partial x} \\ z &\mapsto -\frac{\partial P}{\partial y} & z &\mapsto \frac{\partial P}{\partial x} & z &\mapsto 0. \end{aligned}$$

Note that

$$\{x, -\} = Q\delta_x, \qquad \{y, -\} = Q\delta_y \quad \text{and} \quad \{z, -\} = Q\delta_z.$$

Then,

$$\begin{aligned} \iota_{\delta_x}(\omega_S) &= \iota_{\delta_x}(px dy \wedge dz + qy dz \wedge dx + rz dx \wedge dy) \\ &= px\Big(\frac{\partial P}{\partial z} dz + \frac{\partial P}{\partial y} dy\Big) - qy \frac{\partial P}{\partial y} dx - rz \frac{\partial P}{\partial z} dx \\ &= t dx \quad \text{(see (6-2))}. \end{aligned}$$

Therefore, using the symmetry between x, y and z,

$$\operatorname{div}(\delta_x) = \operatorname{div}(\delta_y) = \operatorname{div}(\delta_z) = 0.$$

Apply Lemma 5.3.1, taking into account that $\lambda_S = -\operatorname{div}$ (see (4-16); then,

$$\operatorname{div}(\{x, -\}) = \operatorname{div}(Q\delta_x) = Q\operatorname{div}(\delta_x) + \delta_x(Q).$$

Therefore,

$$\operatorname{div}(\{x, -\}) = \frac{\partial Q}{\partial y}\frac{\partial P}{\partial z} - \frac{\partial Q}{\partial z}\frac{\partial P}{\partial y}. \tag{6-4}$$

Applying Corollary 1 gives that the enveloping algebra U of $(S, \Omega_{S/R})$ is skew Calabi–Yau. It has a Nakayama automorphism $\nu : U \to U$ such that, for all $s \in S$,

$$\begin{cases} \nu(s) = s, \\ \begin{pmatrix} \nu(dx) \\ \nu(dy) \\ \nu(dz) \end{pmatrix} = \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix} + 2\,\overrightarrow{\operatorname{grad}}(Q) \wedge \overrightarrow{\operatorname{grad}}(P). \end{cases}$$

References

[Artin and Schelter 1987] M. Artin and W. F. Schelter, "Graded algebras of global dimension 3", *Adv. in Math.* **66**:2 (1987), 171–216. MR Zbl

[Cartan and Eilenberg 1956] H. Cartan and S. Eilenberg, *Homological algebra*, Princeton Univ. Press, 1956. MR Zbl

[Chemla 1999] S. Chemla, "A duality property for complex Lie algebroids", *Math. Z.* **232**:2 (1999), 367–388. MR Zbl

[Ginzburg 2006] V. Ginzburg, "Calabi–Yau algebras", preprint, 2006. arXiv math/0612139

[Hochschild et al. 1962] G. Hochschild, B. Kostant, and A. Rosenberg, "Differential forms on regular affine algebras", *Trans. Amer. Math. Soc.* **102** (1962), 383–408. MR Zbl

[Huebschmann 1999] J. Huebschmann, "Duality for Lie–Rinehart algebras and the modular class", *J. Reine Angew. Math.* **510** (1999), 103–159. MR Zbl

[Krähmer 2007] U. Krähmer, "Poincaré duality in Hochschild (co)homology", pp. 117–125 in *New techniques in Hopf algebras and graded ring theory* (Brussels, 2006), edited by S. Caenepeel and F. Van Oystaeyen, K. Vlaam. Acad. Belgie Wet. Kunsten, Brussels, 2007. MR Zbl

[Lambre 2010] T. Lambre, "Dualité de Van den Bergh et structure de Batalin–Vilkoviskiĭ sur les algèbres de Calabi–Yau", *J. Noncommut. Geom.* **4**:3 (2010), 441–457. MR Zbl

[Lambre et al. 2017] T. Lambre, C. Ospel, and P. Vanhaecke, "Poisson enveloping algebras and the Poincaré–Birkhoff–Witt theorem", *J. Algebra* **485** (2017), 166–198. MR Zbl

[Laurent-Gengoux et al. 2013] C. Laurent-Gengoux, A. Pichereau, and P. Vanhaecke, *Poisson structures*, Grundlehren der Math. Wissenschaften **347**, Springer, 2013. MR Zbl

[Lü et al. 2017] J. Lü, X. Wang, and G. Zhuang, "Homological unimodularity and Calabi–Yau condition for Poisson algebras", *Lett. Math. Phys.* **107**:9 (2017), 1715–1740. MR Zbl

[Reyes et al. 2014] M. Reyes, D. Rogalski, and J. J. Zhang, "Skew Calabi–Yau algebras and homological identities", *Adv. Math.* **264** (2014), 308–354. MR Zbl

[Rinehart 1963] G. S. Rinehart, "Differential forms on general commutative algebras", *Trans. Amer. Math. Soc.* **108** (1963), 195–222. MR Zbl

[Van den Bergh 1998] M. Van den Bergh, "A relation between Hochschild homology and cohomology for Gorenstein rings", *Proc. Amer. Math. Soc.* **126**:5 (1998), 1345–1348. MR Zbl

Received January 9, 2018. Revised June 6, 2018.

THIERRY LAMBRE
LABORATOIRE DE MATHÉMATIQUES BLAISE PASCAL
UMR6620 CNRS
UNIVERSITÉ CLERMONT AUVERGNE
CAMPUS DES CÉZEAUX
3 PLACE VASARELY
63178 AUBIÈRE CEDEX
FRANCE

thierry.lambre@uca.fr

PATRICK LE MEUR
LABORATOIRE DE MATHÉMATIQUES BLAISE PASCAL
UMR 6620 CNRS
UNIVERSITÉ CLERMONT AUVERGNE
CAMPUS DES CÉZEAUX
3 PLACE VASARELY
63178 AUBIÈRE CEDEX
FRANCE
Current address:
UNIVERSITÉ PARIS DIDEROT
SORBONNE UNIVERSITÉ, CNRS
INSTITUT DE MATHÉMATIQUES DE JUSSIEU-PARIS RIVE GAUCHE
IMJ-PRG, F-75013
PARIS
FRANCE

patrick.le-meur@imj-prg.fr

PACIFIC JOURNAL OF MATHEMATICS

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

msp.org/pjm

See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2018 is US $475/year for the electronic version, and $640/year for print and electronic.
Subscriptions, requests for back issues and changes of subscriber address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFlow® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

PACIFIC JOURNAL OF MATHEMATICS

Volume 297 No. 2 December 2018