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NONDEGENERACY OF THE GAUSS CURVATURE EQUATION WITH NEGATIVE CONIC SINGULARITY

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We study the Gauss curvature equation with negative singularities. For a local mean field type equation with only one negative index we prove a uniqueness property. For a global equation with one or two negative indexes we prove the nondegeneracy of the linearized equations.

1. Introduction

In this article we study two closely related equations, defined locally and globally in $\mathbb{R}^2$, respectively. The first equation is defined in $\Omega \subset \mathbb{R}^2$, which is simply connected, open and bounded. Throughout the whole article we shall always assume that the boundary of Ω, denoted as $\partial\Omega$, is a rectifiable Jordan curve, and we say Ω is regular. Let $p_0, p_1, \dots, p_m \in \Omega$ be a finite set in Ω. Then we consider v as a solution of

$$\begin{cases} \Delta v + \lambda \frac{e^v}{\int_\Omega e^v} = -4\pi\alpha_0\delta_{p_0} + \sum_{i=1}^m 4\pi\alpha_i\delta_{p_i} & \text{in } \Omega, \\ v = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1-1}$$

where $\alpha_0 \in (0, 1)$, $\alpha_1, \dots, \alpha_m > 0$ and $\lambda \in \mathbb{R}$.

The second equation is concerned with the stability of the following global equation, which we suppose has u as a solution:

$$\Delta u + e^u = \sum_{i=1}^{N} 4\pi\beta_i\delta_{p_i} \quad \text{in } \mathbb{R}^2, \tag{1-2}$$

where $\beta_1, \dots, \beta_n$ are constants greater than -1 and $p_1, \dots, p_n$ are the locations of singular sources in $\mathbb{R}^2$. For this equation we shall prove that under some restrictions of β_i, any bounded solution of the linearized equation has to be the trivial solution.

The background of both equations is incredibly rich not only in mathematics but also in physics. In particular, the study of (1-1) reveals core information on the configuration of vortices in the electroweak theory of Glashow–Salam–Weinberg [Lai 1981] and self-dual Chern–Simons theories [Dunne 1995; Hong et al. 1990;

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Jackiw and Weinberg 1990]. Also in statistical mechanics the behavior of solutions in (1-1) is closely related to Onsager's model of two-dimensional turbulence with vortex sources [Caglioti et al. 1995; Chanillo and Kiessling 1994]. Most of the motivation and applications of both equations come from their connection with conformal geometry. The singular sources represent conic singularities on a surface with constant curvature. There is a large number of interesting works that discuss the qualitative properties of solutions of such equations. We mention [Chang et al. 2003; Bartolucci and Lin 2009; 2014; Bartolucci and Malchiodi 2013; Bartolucci and Tarantello 2002; Chanillo and Kiessling 1994; Chen et al. 2004; Chen and Lin 2010; 2015; Chen and Li 1993; 1995; Li 1999; Lin et al. 2012; Luo and Tian 1992; Malchiodi and Ruiz 2011; 2013; Nolasco and Tarantello 2000; Ohtsuka and Suzuki 2007; Spruck and Yang 1992; Struwe and Tarantello 1998; Tarantello 2010; 2017; Troyanov 1989; 1991; Zhang 2006; 2009]. It is important to observe that it seems there are very few works which discuss singularities with negative strength and even fewer about the comparison between the negative indexes and positive ones. In this article, using an improved version of the Alexandrov–Bol inequality, we discuss the uniqueness property and the nondegeneracy for a local equation and a global equation. Our proof is based on techniques developed in a number of works of Bartolucci, Lin, Chang, Chen and Lin, etc.

To state the main result on the local equation, we first rewrite (1-1) using the following Green's function.

For $p \in \Omega$, let $G_\alpha(x, p)$ satisfy

$$\begin{cases} -\Delta G_\alpha(x, p) = 4\pi\alpha\delta_p & \text{in } \Omega, \\ G_\alpha(x, p) = 0, & x \in \partial\Omega, \end{cases}$$

and

$$u = v - G_{\alpha_0} + \sum_{j=1}^{m} G_{\alpha_j}(x, p_j).$$

Then u satisfies

$$\begin{cases} \Delta u + \lambda(He^u)/\left(\int_\Omega He^u\right) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1-3}$$

where

$$H(x) = e^{G_0(x,p_0)} e^{\sum_{j=1}^m G_{\alpha_j}(x,p)} = e^{h(x)}|x - p_0|^{-2\alpha_0} \prod_{i=1}^{m} |x - p_i|^{2\alpha_i}, \tag{1-4}$$

where h is harmonic in Ω and is continuous up to the boundary.

The first main result is the following theorem:

Theorem 1.1. *Let u be a solution of* (1-3) *and H be defined by* (1-4). *Assume that Ω is regular, then for any $\lambda \le 8\pi(1 - \alpha_0)$ there exists at most one solution to* (1-1).

Here we note that for $\lambda < 8\pi(1-\alpha_0)$, the existence result has been established by Bartolucci and Malchiodi [2013]. The existence result for $\lambda = 8\pi(1-\alpha_0)$ will be discussed in a separate work.

The second main goal of this article is to consider the nondegeneracy of (1-2) when there are exactly two negative indexes:

$$\begin{cases} \Delta u + e^u = -4\pi\alpha_1\delta_{p_1} - 4\pi\alpha_2\delta_{p_2} + \sum_{i=3}^n 4\pi\beta_i\delta_{p_i} \text{ in } \mathbb{R}^2, \\ u(x) = -4\log|x| + \text{ a bounded function near } \infty, \end{cases} \tag{1-5}$$

where $\alpha_1, \alpha_2 \in (0, 1)$ and $\beta_i > 0$ for $i = 3, \ldots, n$ and we assume that $n \geq 3$. The assumption of u at infinity says that ∞ is not a singularity of u when $\mathbb{R}^2$ is identified with $\mathbb{S}^2$.

Let

$$u_1(x) = u(x) + \sum_{i=1}^{2} 2\alpha_i \log|x-p_i| - 2\sum_{i=3}^{n} \beta_i \log|x-p_i|; \tag{1-6}$$

then clearly u_1 satisfies

$$\begin{cases} \Delta u_1 + H_1 e^{u_1} = 0 \quad \text{in } \mathbb{R}^2, \\ u_1(x) = (-4 - 2\alpha_1 - 2\alpha_2 + 2\sum_{i=3}^n \beta_i)\log|x| + O(1), \quad \text{for } |x| > 1, \end{cases}$$

where

$$H_1(x) = \prod_{i=1}^{2} |x-p_i|^{-2\alpha_i} \prod_{i=3}^{n} |x-p_i|^{2\beta_i}, \quad \text{for } x \in \mathbb{R}^2. \tag{1-7}$$

Our second main result is the following theorem:

Theorem 1.2. *Let u, u_1 and H_1 be defined as in* (1-5), (1-6) *and* (1-7), *respectively. Suppose ϕ is a classical solution of*

$$\Delta\phi + H_1(x)e^{u_1}\phi = 0 \quad \textit{in } \mathbb{R}^2. \tag{1-8}$$

If $\lim_{x\to\infty}|\phi(x)|/\log|x| = 0$ *and $\alpha_1, \alpha_2, \beta_i$ satisfy the condition*

$$-\max\{\alpha_1, \alpha_2\} + \min\{\alpha_1, \alpha_2\} + \sum_{i=3}^{n} \beta_i \leq 0, \tag{1-9}$$

then $\phi \equiv 0$.

Here we recall that the total angles at singularities are $2\pi(1-\alpha_1)$, $2\pi(1-\alpha_2)$, $2\pi(1+\beta_i)$ $(i = 3, \ldots, n)$. For a surface S with conic singularities, let

$$\chi(S, \theta) = \chi(S) + \sum_i \left(\frac{\theta_i}{2\pi} - 1\right),$$

where θ_i is the total angle at a conic singularity, and $\chi(S)$ is the Euler characteristic of S. The purpose of introducing $\chi(S,\theta)$ is to put all surfaces with conic singularities into three cases:

(i) The subcritical case if $\chi(S,\theta) < \min_i\{2, \theta_i/\pi\}$,

(ii) The critical case if $\chi(S,\theta) = \min_i\{2, \theta_i/\pi\}$,

(iii) The supercritical case if $\chi(S,\theta) > \min_i\{2, \theta_i/\pi\}$.

In our case $\chi(S) = 2$ because S is the standard sphere. It is easy to see that (1-9) refers to the supercritical case. For the subcritical case, Troyanov's well-known result [1991] states that every conic singular metric is pointwise conformal to a metric with constant curvature.

Finally, if there is only one negative singular source, a similar result still holds: Let u satisfy

$$\begin{cases} \Delta u + e^u = -4\pi\alpha\delta_{p_1} + \sum_{i=2}^n 4\pi\beta_i\delta_{p_i} \quad \text{in } \mathbb{R}^2, \\ u(x) = -4\log|x| + \text{a bounded function near } \infty, \end{cases} \tag{1-10}$$

where $\alpha \in (0, 1)$ and $\beta_i > 0$ for $i = 2, \dots, n$ and we assume that $n \geq 3$.

Let

$$u_1(x) = u(x) + 2\alpha\log|x - p_1| - 2\sum_{i=2}^n \beta_i \log|x - p_i|;$$

then clearly u_1 satisfies

$$\begin{cases} \Delta u_1 + H_2 e^{u_1} = 0 \quad \text{in } \mathbb{R}^2, \\ u_1(x) = (-4 - 2\alpha + 2\sum_{i=2}^n \beta_i)\log|x| + O(1), \qquad \text{for } |x| > 1, \end{cases} \tag{1-11}$$

where

$$H_2(x) = |x - p_1|^{-2\alpha}\prod_{i=2}^n |x - p_i|^{2\beta_i}, \qquad \text{for } x \in \mathbb{R}^2. \tag{1-12}$$

Our third main result is:

Theorem 1.3. *Let u_1 be a solution of* (1-11) *with H_2 defined in* (1-12)*. Let ϕ be a classical solution of*

$$\Delta\phi + H_2(x)e^{u_1}\phi = 0 \quad \text{in } \mathbb{R}^2. \tag{1-13}$$

If $\lim_{x\to\infty}|\phi(x)|/\log|x| = 0$ *and α, β_i satisfy*

$$-\alpha + \sum_{i=2}^n \beta_i \leq 0, \tag{1-14}$$

then $\phi \equiv 0$.

The organization of this article is as follows. In Section 2 we derive a Bol's inequality with one negative singular source. Then in Section 3 the first two eigenvalues of the linearized local equation are discussed. The proofs of the major theorems are arranged in Sections 4 and 5. The main approach of this article follows closely from previous works of Bartolucci, Chang, Chen and Lin, etc.

2. On Bol's inequality and the first eigenvalues of the local equation

One of the major tools we shall use is Bol's inequality:

Proposition 2.1. *Let $\Omega \Subset \mathbb{R}^2$ be a simply connected, open and bounded domain in $\mathbb{R}^2$. Let u be a solution of*

$$\Delta u + Ve^u = 0 \quad \text{in } \Omega$$

for

$$V = |x-p_1|^{-2\alpha_0} \prod_{i=2}^{n} |x-p_i|^{2\beta_i} e^g \tag{2-1}$$

and $\Delta g \geq 0$ in Ω. Here $p_1, \dots, p_n$ $(n \geq 2)$ are distinct points in Ω. Let $\omega \subset \Omega$ be an open subset of Ω such that $\partial\omega$ is a finite union of rectifiable Jordan curves. Let

$$L_{\alpha_0}(\partial\omega) = \int_{\partial\omega} (Ve^u)^{\frac{1}{2}}\, ds, \qquad M_{\alpha_0}(\omega) = \int_{\omega} Ve^u\, dx.$$

Then

$$2L_{\alpha_0}^2(\partial\omega) \geq (8\pi(1-\alpha_0) - M_{\alpha_0}(\omega))M_{\alpha_0}(\omega). \tag{2-2}$$

The strict inequality holds if ω contains more than one singular source or is multiply connected.

Our proof of Proposition 2.1 is motivated by the argument in [Bartolucci and Castorina 2016] and [Bartolucci and Lin 2009; 2014]. For the case where $\alpha_0 = 0$, the proposition was established in [Bartolucci and Lin 2014], and for the case where V has only a singular source at 0, it was established by Bartolucci and Castorina. It all starts from an inequality of Huber:

Theorem A [Huber 1954]. *Let ω be an open, bounded, simply connected domain with $\partial\omega$ being a rectifiable Jordan curve, $\tilde{V} = |x|^{-2\alpha_0}e^g$, for some $\Delta g \geq 0$ in ω. Then*

$$\left(\int_{\partial\omega} \tilde{V}^{\frac{1}{2}}\, ds\right)^2 \geq 4\pi(1-\alpha_0)\int_{\omega} \tilde{V}\, dx \qquad \text{if } 0 \in \omega,$$

$$\left(\int_{\partial\omega} \tilde{V}^{\frac{1}{2}}\, ds\right)^2 \geq 4\pi \int_{\omega} \tilde{V} dx \qquad \text{if } 0 \notin \omega.$$

Huber's theorem can be adjusted to the following version:

Theorem B (Bartolucci–Castorina). *Let $\omega \subset \mathbb{R}^2$ be an open bounded domain such that $\partial\omega$ is a rectifiable Jordan curve. Suppose $\bar{\omega}_B$ is the closure of a possibly disconnected bounded component of $\mathbb{R}^2 \setminus \omega$ and ω_B is the interior of $\bar{\omega}_B$. Let $\tilde{V} = |x|^{-2\alpha_0} e^g$ for some g satisfying $\Delta g \geq 0$ in the interior of $\bar{\omega} \cup \bar{\omega}_B$. Then*

$$\left(\int_{\partial\omega} \tilde{V}^{\frac{1}{2}}\, ds \right)^2 \geq 4\pi(1-\alpha_0) \int_\omega \tilde{V}\, dx,$$

if 0 is in the interior of $\bar{\omega} \cup \bar{\omega}_B$, and

$$\left(\int_{\partial\omega} \tilde{V}^{\frac{1}{2}}\, ds \right)^2 \geq 4\pi \int_\omega \tilde{V}\, dx,$$

if 0 is not in the interior of $\bar{\omega} \cup \bar{\omega}_B$.

Proof of Proposition 2.1. We shall only consider the first case mentioned in Theorem B because the other case corresponds to $\alpha_0 = 0$. Let

$$\begin{cases} \Delta q = 0 & \text{in } \omega, \\ q = u & \text{on } \partial\omega. \end{cases}$$

and let $\eta = u - q$. Then the equation for η is

$$\begin{cases} \Delta \eta + V e^q e^\eta = 0 & \text{in } \omega, \\ \eta = 0 & \text{on } \partial\omega, \end{cases} \tag{2-3}$$

and we use

$$t_m = \max_{\bar{\omega}} \eta.$$

Then we set

$$\Omega(t) = \{x \in \omega;\ \eta(x) > t\}, \quad \Gamma(t) = \partial\Omega(t), \quad \mu(t) = \int_{\Omega(t)} V e^q\, dx.$$

Clearly $\Omega(0) = \omega$, $\mu(0) = \int_\omega V e^q\, dx$, and $\mu(t_m) = \lim_{t \to t_m -} \mu(t) = 0$. Since μ is continuous and strictly decreasing, it is easy to see that

$$\frac{d\mu(t)}{dt} = -\int_{\Gamma(t)} \frac{V e^q}{|\nabla \eta|}\, ds, \quad \text{for almost every } t \in [0, t_m]. \tag{2-4}$$

For all $s \in [0, \mu(0)]$, set

$$\eta^*(s) = |\{t \in [0, t_m], \quad \mu(t) > s\}|,$$

where $|E|$ is the Lebesgue measure of the measurable set $E \in \mathbb{R}$. It is easy to see that η^* is the inverse of μ on $[0, t_m]$ and is continuous, strictly monotone and differentiable almost everywhere. By (2-4) we have, for almost all $s \in [0, \mu(0)]$,

$$\frac{d\eta^*}{ds} = -\left(\int_{\Gamma(\eta^*(s))} \frac{V e^q}{|\nabla \eta|}\, dt \right)^{-1}. \tag{2-5}$$

Let

$$F(s)=\int_{\Omega(\eta^*(s))} e^{\eta} V e^{q}\,dx, \quad \text{for almost every } s\in[0,\mu(0)].$$

Then by the definition of $\Omega(t)$ we see that

$$F(s)=\int_{\eta^*(s)}^{t_m} e^{t}\left(\int_{\Gamma_t}\frac{Ve^{q}}{|\nabla\eta|}\,ds\right)dt.$$

Using $\beta=\mu(t)$, we further have

$$F(s)=\int_0^s e^{\eta^*(\beta)}\,d\beta, \tag{2-6}$$

where $\eta^*=\mu^{-1}$ and (2-4) are used. The definition of F also gives

$$F(0)=\int_{\Omega(\eta^*(0))} e^{\eta}Ve^{q}=\int_{\Omega(t_m)} e^{\eta}Ve^{q}=0$$

and $F(\mu(0))=\int_\omega e^{\eta}d\tau=M(\omega)$. Consequently, from (2-6) we obtain

$$\frac{dF}{ds}=e^{\eta^*(s)},\qquad \frac{d^2F}{ds^2}=\frac{d\eta^*}{ds}e^{\eta^*(s)}=\frac{d\eta^*}{ds}\frac{dF}{ds},\quad \text{for almost every } s. \tag{2-7}$$

Here we use the argument from [Bartolucci and Castorina 2016] to show that η^* is locally Lipschitz in $(0,\mu(0))$:

Lemma 2.1. *For any $0<\bar a\le a<b\le\bar b<\bar u(0)$, there exists $C(\bar a,\bar b,\beta_1,\dots,\beta_k)>0$ such that*

$$\eta^*(a)-\eta^*(b)\le C(b-a).$$

Proof of Lemma 2.1. First we find $\Omega_{a,b}$ that satisfies

$$\{x\in\omega;\ \ \eta^*(b)\le\eta(x)\le\eta^*(a)\}\Subset\Omega_{a,b}\Subset\omega.$$

Using Green's representation formula we have

$$|\nabla\eta(x)|\le C+C\int_{\Omega_{a,b}}\frac{1}{|x-y|}|y-p_1|^{-2\alpha_0}\,dy.$$

A standard estimate gives

$$|\nabla\eta(x)|\le C+C|x-p_0|^{1-2\alpha_0}. \tag{2-8}$$

Recall that $d\eta = Ve^q dx$. Thus

$$\begin{aligned} b-a &= \mu(\eta^*(b)) - \mu(\eta^*(a)) \\ &= \int_{\eta>\eta^*(b)} d\tau - \int_{\eta>\eta^*(a)} d\tau \geq \int_{\eta^*(b)<\eta<\eta^*(a)} d\tau \\ &= \int_{\eta^*(b)}^{\eta^*(a)} \left(\int_{\Gamma(t)} \frac{Ve^q}{|\nabla\eta|}\, ds \right) dt. \end{aligned}$$

Using the expression of V in (2-1) and (2-8) we further have

$$\begin{aligned} b-a &\geq \frac{1}{C} \int_{\eta^*(b)}^{\eta^*(a)} \left(\int_{\Gamma(t)} \frac{1}{|x-p_0|^{2\alpha_0} + |x-p_0|} \right) dt \\ &\geq \frac{1}{C} \int_{\eta^*(b)}^{\eta^*(a)} L_1(\Gamma(t))\, dt \\ &\geq \min_{\eta^*(b)\leq t\leq \eta^*(a)} L_1(\Gamma(t)) \int_{\eta^*(b)}^{\eta^*(a)} dt \\ &\geq C(\eta^*(a) - \eta^*(b)), \end{aligned}$$

where the estimate of $\nabla\eta$ was used, $L_1(\Gamma(t))$ stands for the Lebesgue measure of Γ and in the last inequality, the standard isoperimetric inequality

$$L_1(\Gamma(t)) \geq 4\pi|\Omega(t)| \geq 4\pi|\Omega(\eta^*(\bar{a})| > 0$$

is used. Lemma 2.1 is established. □

Now we go back to the proof of Proposition 2.1. By Cauchy's inequality

$$\begin{aligned} &\left(\int_{\Gamma(\eta^*(s))} (Ve^q)^{\frac{1}{2}}\, ds \right)^2 \\ &\leq \left(\int_{\Gamma(\eta^*(s))} \frac{Ve^q}{|\nabla\eta|}\, ds \right) \left(\int_{\Gamma(\eta^*(s))} |\nabla\eta|\, ds \right) \\ &= \left(-\frac{d\eta^*}{ds} \right)^{-1} \left(\int_{\Gamma(\eta^*(s))} \left(-\frac{\partial\eta}{\partial\nu} \right) ds \right), \qquad \text{for almost every } s \in [0, \mu(0)], \end{aligned} \tag{2-9}$$

where $\nu = \nabla\eta/|\nabla\eta|$. Moreover from (2-3)

$$\int_{\Gamma(\eta^*(s))} \left(-\frac{\partial\eta}{\partial\nu} \right) ds = \int_{\Omega(\eta^*(s))} Ve^q e^\eta\, dx = F(s), \qquad \text{for almost every } s \in [0, \mu(0)]. \tag{2-10}$$

By Theorem A, the following inequality holds for almost all $s \in [0, \mu(0)]$:

$$\left(\int_{\Gamma(\eta^*(s))} (Ve^q)^{\frac{1}{2}} \right)^2 \geq 4\pi(1-\alpha_0)\mu(\eta^*(s)) = 4\pi(1-\alpha_0)s. \tag{2-11}$$

Putting (2-10) into (2-9) yields

$$\left(\int_{\Gamma(\eta^*(s))}(Ve^q)^{\frac{1}{2}}\,ds\right)^2 \le \left(-\frac{d\eta^*}{ds}\right)^{-1}F(s). \tag{2-12}$$

Using (2-11) in (2-12), we have

$$4\pi(1-\alpha_0)s \le \left(-\frac{d\eta^*}{ds}\right)^{-1}F(s), \quad \text{for almost every } s\in[0,\mu(0)],$$

which is equivalent to

$$4\pi(1-\alpha_0)s\frac{d\eta^*}{ds}+F(s)\ge 0, \quad \text{for almost every } s\in[0,\mu(0)]. \tag{2-13}$$

By (2-7) and (2-13), we obtain

$$\frac{d}{ds}\left[4\pi(1-\alpha_0)\left(s\frac{dF}{ds}-F(s)\right)+\tfrac{1}{2}F^2(s)\right]\ge 0, \quad \text{for almost every } s\in[0,\mu(0)].$$

Let $P(s)$ denote the function in the brackets, then P is well defined, continuous, nondecreasing on $[0,\mu(0)]$. By the Lipschitz property of η^*, P is absolutely continuous on $[0,\mu(0)]$;

$$P(\mu(0))-P(0)=\lim_{b\to\mu(0)^-}\lim_{a\to 0^+}\int_a^b\frac{dP}{ds}\,ds.$$

Using $F(0)=0$, $F(\mu(0))=M(\omega)$, and $\frac{dF}{ds}|_{s=\mu(0)}=e^0=1$, we have

$$8\pi(1-\alpha_0)(\mu(0)-M(\omega))+M(\omega)^2\ge 0.$$

Then Huber's inequality and $\Gamma(0)=\partial\omega$ further yield

$$\begin{aligned}2l^2(\partial\omega)=2\left(\int_{\partial\omega}(Ve^v)^{\frac{1}{2}}\,ds\right)^2&=2\left(\int_{\partial\omega}(Ve^q)^{\frac{1}{2}}\,ds\right)^2\\&\ge 8\pi(1-\alpha_0)\mu(0)\ge M(\omega)(8\pi(1-\alpha_0)-M(\omega)),\end{aligned}$$

where we have used the fact that $v=q$ on $\partial\omega$. The Bol's inequality is established. The equality holds if $Ve^q=|x-p_0|^{-2\alpha_0}|\Phi_t'|^2e^k$ on $\Omega(t)$ for almost all $t\in(0,t_m)$ where k is a constant. In particular for $t=0$, Φ_0 maps Ω to a ball. In this case g must be harmonic. On the other hand from the equality of Cauchy's inequality we have

$$Ve^q=c_t|\nabla\eta|^2 \quad \text{on } \Gamma(t), \quad \text{for almost every } t\in(0,t_m),$$

for some $c_t>0$. Putting $w=\Phi_0(z)$ and $\xi(w)=\eta(\Phi_0^{-1}(w))+k$, we see that ξ satisfies

$$\Delta\xi+|x|^{-2\alpha_0}e^{\xi}=0,$$

and ξ is radial. This ξ is a scaling of

$$\log \frac{8(1-\alpha_0)^2}{1+|x|^{2(1-\alpha_0))^2}}.$$

Thus we have strict inequality in Bol's inequality if at least one of the following situations occurs:

(1) $p_1 \notin \omega$,

(2) ω has at least two singular sources

(3) ω is not simply connected. □

3. The first eigenvalues of the linearized local equation

Proposition 3.1. *Let Ω be an open, bounded domain of $\mathbb{R}^2$ with rectifiable boundary $\partial\Omega$, $V=|x|^{-\alpha_0}\prod_{i=1}^k|x-p_i|^{2\beta_i}e^g$ for some subharmonic and smooth function g, $\alpha_0\in(0,1)$, $\beta_1,\dots,\beta_k>0$, and assume that all the singular points are in Ω: $0, p_1,\dots,p_k\in\Omega$. Let w be a classical solution of*

$$\Delta w+Ve^w=0 \quad \text{in } \Omega.$$

Suppose $\hat\nu_1$ is the first eigenvalue of

$$\begin{cases}-\Delta\phi-Ve^w\phi=\hat\nu_1 Ve^w\phi & \text{in } \Omega,\\ \phi=0 & \text{on } \partial\Omega.\end{cases} \tag{3-1}$$

Then if $\int_\Omega Ve^w\le 4\pi(1-\alpha_0)$ we have $\hat\nu_1>0$. Moreover if $\int_\Omega Ve^w\le 8\pi(1-\alpha_0)$ we have $\hat\nu_2>0$.

Proof. Let $\nu_1=\hat\nu_1+1$ and ϕ be the eigenfunction corresponding to $\hat\nu_1$, then we have $\phi>0$ and

$$\begin{cases}-\Delta\phi=\nu_1 Ve^w\phi & \text{in } \Omega,\\ \phi=0 & \text{on } \partial\Omega.\end{cases}$$

Let

$$U_0(x)=(-2)\log(1+|x|^{2(1-\alpha_0)})+\log(8(1-\alpha_0)^2).$$

Then clearly U_0 solves

$$\Delta U_0+|x|^{-2\alpha_0}e^{U_0}=0 \quad \text{in } \mathbb{R}^2.$$

For $t\in(0,t_+)$ where $t_+=\max_{\overline\Omega}\phi$, we set $\Omega(t)=\{x\in\Omega,\ \phi(x)>t\}$ and we set $R(t)$ to satisfy

$$\int_{\Omega(t)}Ve^w=\int_{B_{R(t)}}e^{U_0}|x|^{-2\alpha_0}.$$

Clearly $\Omega(0)=\Omega$, $R_0=\lim_{t\to 0+}R(t)$, $\lim_{t\to t_+-}R(t)=0$. Let ϕ^* be a radial function from $B_{R_0}\to\mathbb{R}$. For $y\in B_{R_0}$ and $|y|=r$, set

$$\phi^*(r)=\sup\{t\in(0,t_+)\mid R(t)>r\}.$$

Then $\phi^*(R_0)=\lim_{r\to R_0-}\phi^*(r)=0$, and the definition implies

$$B_{R(t)}=\{y\in\mathbb{R}^2,\ \phi^*(y)>t\}.$$

$$\int_{\phi^*>t}e^{U_0}|x|^{-2\alpha_0}=\int_{\Omega(t)}Ve^w,\quad t\in[0,t_+].$$

$$\int_{B_{R_0}}|x|^{-2\alpha_0}e^{U_0}|\phi^*|^2=\int_\Omega Ve^w\phi^2.$$

Then for almost all t

$$\begin{aligned}
-\frac{d}{dt}\int_{\Omega(t)}|\nabla\phi|^2&=\int_{\phi=t}|\nabla\phi| \\
&\ge\left(\int_{\phi=t}(Ve^w)^{\frac12}\,ds\right)^2\left(\int_{\phi=t}\frac{Ve^w}{|\nabla\phi|}\,ds\right)^{-1}, \\
&=\left(-\frac{d}{dt}\int_{\Omega(t)}Ve^w\right)^{-1}\left(\int_{\phi=t}(Ve^w)^{\frac12}\,ds\right)^2 \\
&\ge\frac12\left(8\pi(1-\alpha_0)-\int_{\Omega(t)}Ve^w\right)\left(\int_{\Omega_t}Ve^w\right)\left(-\frac{d}{dt}\int_{\Omega(t)}Ve^w\right)^{-1}, \\
&=\frac12\left(8\pi(1-\alpha_0)-\int_{\phi^*>t}e^{U_0}|x|^{-2\alpha_0}\right) \\
&\qquad\times\left(\int_{\phi^*>t}e^{U_0}|x|^{-2\alpha_0}\right)\left(-\frac{d}{dt}\int_{\phi^*>t}e^{U_0}|x|^{-2\alpha_0}\right)^{-1}.
\end{aligned}\tag{3-2}$$

Applying the same computation to ϕ^* we see that for almost all t, since ϕ^* is radial, we have

$$\begin{aligned}
-\frac{d}{dt}\int_{\Omega(t)}|\nabla\phi^*|^2&=\int_{\phi^*=t}|\nabla\phi^*| \\
&=\left(\int_{\phi^*=t}|x|^{-\alpha_0}e^{U_0/2}\,ds\right)^2\left(\int_{\phi^*=t}\frac{|x|^{-2\alpha_0}e^{U_0}}{|\nabla\phi^*|}\,ds\right)^{-1} \\
&=\left(-\frac{d}{dt}\int_{\Omega(t)}|x|^{-2\alpha_0}e^{U_0}\right)^{-1}\left(\int_{\phi^*=t}|x|^{-\alpha_0}e^{U_0/2}\,ds\right)^2.
\end{aligned}$$

Direct computation on U_0 gives

$$\left(-\frac{d}{dt}\int_{\Omega(t)}|x|^{-2\alpha_0}e^{U_0}\right)^{-1}=\frac{1}{2}\left(8\pi(1-\alpha_0)-\int_{\phi^*>t}e^{U_0}|x|^{-2\alpha_0}\right)\left(\int_{\phi^*>t}e^{U_0}|x|^{-2\alpha_0}\right).$$

Thus the combination of the two equations above gives

$$-\frac{d}{dt}\int_{\Omega(t)}|\nabla\phi^*|^2=\frac{1}{2}\left(8\pi(1-\alpha_0)-\int_{\phi^*>t}e^{U_0}|x|^{-2\alpha_0}\right)\times\left(\int_{\phi^*>t}e^{U_0}|x|^{-2\alpha_0}\right)\left(-\frac{d}{dt}\int_{\phi^*>t}e^{U_0}|x|^{-2\alpha_0}\right)^{-1}\tag{3-3}$$

for almost all $t\in(0,t_+)$.

Integrating (3-2) and (3-3) for $t\in(0,t_+)$ we have

$$\int_{B_{R_0}}|\nabla\phi^*|^2\le\int_\Omega|\nabla\phi|^2.$$

If $\nu_1\le 1$, we obtain from (3-1) that

$$\begin{aligned}0\ge(\nu_1-1)\int_\Omega Ve^w|\phi|^2&=\int_\Omega|\nabla\phi|^2-\int_\Omega Ve^w|\phi|^2\\&\ge\int_{B_{R_0}}|\nabla\phi^*|^2-\int_{B_{R_0}}e^{U_0}|x|^{-2\alpha_0}|\phi^*|^2.\end{aligned}$$

Thus the first eigenvalue of

$$-\Delta-|x|^{-2\alpha_0}e^{U_0}$$

on B_{R_0} with Dirichlet boundary condition is nonpositive. Since

$$\psi=2(1-\alpha_0)\frac{1-|x|^{2(1-\alpha_0)}}{1+|x|^{2(1-\alpha_0)}}$$

satisfies

$$-\Delta\psi-|x|^{-2\alpha_0}e^{U_0}\psi=0\quad\text{in }\mathbb{R}^2,$$

we see that $R_0\ge 1$. But

$$\int_{B_1}|x|^{-2\alpha_0}e^{U_0}=4\pi(1-\alpha_0),$$

so we clearly have $\hat{\nu}\ge 0$. From the proof of Bol's inequality we see that the strict inequality holds because Ω has more than one singular point in its interior.

The proof of $\hat{\nu}_2>0$ for a higher threshold of $\int_\Omega Ve^w$ is very similar. If we consider Ω_+ and Ω_-, which are the set of points where ϕ is positive or negative, respectively, the integral of Ve^w on at least one of them is less than or equal to $4\pi(1-\alpha_0)$. The argument of redistribution of mass can be applied to at least one of them. Then we see that either one of them has the integral of Ve^w strictly less than $4\pi(1-\alpha_0)$, which leads to a contradiction, or both regions have their integral equal to $4\pi(1-\alpha_0)$. In the latter case, the equality cannot hold because 0 can only be in the interior of at most one region. Then at least one region either does not contain 0 in its interior, or is not simply connected. The strict inequality holds in at least one region. Thus $\hat{\nu}_2 > 0$ if $\int_\Omega Ve^w \le 8\pi(1-\alpha_0)$. □

4. The proof of Theorem 1.2

First we claim that ϕ in the linearized equation is actually bounded. Recall that u_1 satisfies

$$\begin{aligned}&\Delta u_1 + H_1 e^{u_1} = 0 \quad \text{in } \mathbb{R}^2,\\ &u_1(x) = \Bigl(-4 + 2\alpha_1 + 2\alpha_2 - 2\sum_{i=3}^{n} \beta_i\Bigr) \log|x| + O(1) \quad \text{at } \infty.\end{aligned}$$

By the equation for ϕ and the mild growth rate of ϕ at infinity, we have

$$\phi(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \log|x-y| H_1(y) e^{u_1(y)} \phi(y)\, dy + c, \quad x \in \mathbb{R}^2,$$

for some $c \in \mathbb{R}$.

Differentiating the equation above, we have

$$\partial_i \phi(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{x_i - y_i}{|x-y|^2} H_1 e^{u_1} \phi(y)\, dy, \quad i = 1, 2, \;\; x \in \mathbb{R}^2.$$

By standard estimates in different regions of $\mathbb{R}^2$, it is easy to see that

$$\partial_i \phi(x) = A \frac{x_i}{|x|^2} + O(|x|^{-1-\delta}), \quad |x| > 1, \;\; i = 1, 2,$$

for $A = \frac{1}{2\pi}\int_{\mathbb{R}^2} H_1 e^{u_1}\phi$ and some $\delta > 0$. Thus the assumption $\phi(x) = o(\log|x|)$ actually implies

$$\int_{\mathbb{R}^2} H_1 e^{u_1} \phi = 0. \tag{4-1}$$

and

$$\phi(x) = C + O(|x|^{-\delta}), \quad |x| > 1, \tag{4-2}$$

for some $\delta > 0$.

Next we make a transformation on the equation for u_1. Without loss of generality we assume $p_1 = 0$ and we write H_1 as

$$H_1(x) = |x|^{-2\alpha_1} V_1.$$

Let

$$u_2(x) = u_1\left(\frac{x}{|x|^2}\right) - (4-2\alpha_1)\log|x|,$$

then direct computation shows that

$$\Delta u_2 + V_2 e^{u_2} = 0 \quad \text{in } \mathbb{R}^2$$

and

$$u_2(x) = (-4+2\alpha_1)\log|x| + O(1) \quad \text{at } \infty,$$

where $V_2(x) = V_1(x/|x|^2)$. It is also easy to verify that

$$\int_{\mathbb{R}^2} H_1 e^{u_1} = \int_{\mathbb{R}^2} V_2 e^{u_2}. \tag{4-3}$$

Setting $\phi_1(x) = \phi(x/|x|^2)$, we see that

$$\Delta\phi_1 + V_2 e^{u_2}\phi_1 = 0 \quad \text{in } \mathbb{R}^2.$$

Here we note that by the bound of ϕ_1 near the origin, the equation above holds in the whole $\mathbb{R}^2$.

First, by the asymptotic behavior of u_1 at infinity, integration of the equation for u_1 gives

$$\frac{1}{2\pi}\int_{\mathbb{R}^2} H_1 e^{u_1} = 4 - 2(\alpha_1+\alpha_2) + 2\sum_{i=3}^{n}\beta_i \le 4(1-\alpha_2). \tag{4-4}$$

From the definition of ϕ we have $\phi_1(x) \to c_0$ as $x \to \infty$ for some $c_0 \in \mathbb{R}$. Without loss of generality we assume $c_0 \le 0$. By the same estimate for ϕ we have

$$\int_{\mathbb{R}^2} V_2 e^{u_2}\phi_1 = 0. \tag{4-5}$$

By (4-3) and (4-4) we have

$$\int_{\mathbb{R}^2} V_2 e^{u_2} \le 8\pi(1-\alpha_2).$$

Let ϕ_2 be an eigenfunction corresponding to eigenvalue $\hat{\nu}$:

$$\begin{cases} -\Delta\phi_2 - V_2 e^{u_2}\phi_2 = \hat{\nu} V_2 e^{u_2}\phi_2 & \text{in } \mathbb{R}^2, \\ \lim_{x\to\infty}\phi_2(x) = c_0 \le 0, \\ \int_{\mathbb{R}^2} V_2 e^{u_2}\phi_2 = 0. \end{cases}$$

We claim that $\hat{\nu} > 0$.

By way of contradiction we assume that $\hat{\nu} \le 0$. By setting $\nu = 1 + \hat{\nu}$ we clearly have $\nu \le 1$ and

$$\Delta\phi_2 + \nu V_2 e^{u_2}\phi_2 = 0 \quad \text{in } \mathbb{R}^2.$$

Let $\Omega^+ = \{x;\ \phi_2(x) > c_0\}$, then by the same argument as in the proof of the previous proposition we must have

$$\int_{\Omega^+} V_2 e^{u_2} = c_2(c_0) \geq 4\pi(1-\alpha_2)$$

and if the equality holds, we have $c_0 = 0$. Then there is one singular source with negative index $-4\pi\alpha_2$ in the interior of Ω_+, which has to be simply connected at the same time. All other singular sources (which have positive indexes) are not in the interior of Ω_+.

Let ϕ^* be the rearrangement of ϕ_2 in Ω_+. By the previous argument we have

$$\int_{\Omega_+} |\nabla\phi_2|^2 \leq \int_{B_{R_1}} |\nabla\phi^*|^2$$

and $c_2(c_0) = \int_{B_{R_1}} |x|^{-2\alpha_2} e^{U_0}$. Let

$$c_1 = \min_{\mathbb{R}^2} \phi_2$$

and we set R_2 to make

$$\int_{B_{R_2}\setminus B_{R_1}} |x|^{-2\alpha_2} e^{U_0} = \int_{\mathbb{R}^2\setminus\Omega_+} V_1 e^{u_2}.$$

Note that R_2 could be ∞. Then we define a radial function ϕ^{**} from $B_{R_2}\setminus B_{R_1} \to \mathbb{R}$: for any $y \in B_{R_2}\setminus B_{R_1},\ |y| = r$,

$$\phi^{**}(r) = \inf\{t \in (c_1, c_0) \mid R^{(-)}(t) < r\},$$

where $R^{(-)}(t)$ is defined by

$$\int_{B_{R_2}\setminus B_{R^{(-)}(t)}} |x|^{-2\alpha_2} e^{U_0} = \int_{\phi_2 < t} V_2 e^{u_2}, \quad \text{for all } t \in (c_1, c_0).$$

The definition of ϕ^{**} implies

$$\int_{B_{R_2}\setminus B_{R_1}} |x|^{-2\alpha_2} e^{U_0} |\phi^{(**)}|^2 = \int_{\Omega^-} V_2 e^{u_2} |\phi_2|^2, \quad \Omega^- = \mathbb{R}^2\setminus\Omega_+,$$

and

$$\int_{B_{R_2}\setminus B_{R_1}} |x|^{-2\alpha_2} e^{U_0} \phi^{(**)} = \int_{\Omega^-} V_2 e^{u_2} \phi_2, \quad \Omega^- = \mathbb{R}^2\setminus\Omega_+.$$

The symmetrization also gives

$$\int_{B_{R_2}\setminus B_{R_1}} |\nabla\phi^{**}|^2 \leq \int_{\Omega^-} |\nabla\phi_2|^2.$$

Now we set

$$\phi_* : B_{R_2} \to \mathbb{R}, \quad \phi_* \text{ radial } \phi_*(r) = \begin{cases} \phi^*(r) & \text{for } r \in [0, R_1], \\ \phi^{**}(r) & \text{for } r \in [R_1, R_2). \end{cases}$$

Since ϕ_* is continuous, monotone, we have

$$\int_{B_{R_2}} |\nabla \phi_*|^2 \le \int_{\mathbb{R}^2} |\nabla \phi_2|^2 = \int_{\mathbb{R}^2} V_2 e^{u_2} |\phi_2|^2 = \int_{B_{R_2}} |x|^{-2\alpha_2} e^{U_0} |\phi_*|^2.$$

From the definition of ϕ_* we also have

$$\int_{B_{R_2}} |x|^{-2\alpha_2} e^{U_0} \phi_* = 0.$$

Let

$K^* = \inf\bigl\{\int_{\mathbb{R}^2} |\nabla \psi|^2 \, dx,\ \ \psi \text{ is radial},\ \ \int_{\mathbb{R}^2} |x|^{-2\alpha_2} e^{U_0} \psi = 0,\ \ \int_{\mathbb{R}^2} |x|^{-2\alpha_2} e^{U_0} \psi^2 = 1\bigr\}.$

By Hölder's inequality we have

$$\left| \int_{\mathbb{R}^2} |x|^{-2\alpha_2} e^{U_0} \psi \, dx \right| \le \left(\int_{\mathbb{R}^2} |x|^{-2\alpha_2} e^{U_0} \psi^2 \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^2} |x|^{-2\alpha_2} e^{U_0} \right)^{\frac{1}{2}},$$

which implies that the minimizer (say ψ^*) also satisfies

$$\int_{\mathbb{R}^2} |x|^{-2\alpha_2} e^{U_0} \psi^* = 0.$$

Clearly the minimizer ψ^* satisfies

$$\Delta \psi^* + K^* |x|^{-2\alpha_2} e^{U_0} \psi^* = 0 \quad \text{in } \mathbb{R}^2.$$

From ϕ_* and the definition of K^* we already know $K^* \in (0, 1)$. Our goal is to show that $K^* = 1$ by an argument of Chang, Chen and Lin [Chang et al. 2003]. The minimizer ψ^* should only change sign once. Let ξ_0 be the zero of ψ^*.

Integrating the equation for ψ^*, we have

$$r \frac{d}{dr} \psi^*(r) = -K^* \int_0^r |s|^{1-2\alpha_2} e^{U_0(s)} \psi^*(s) \, ds = K^* \int_r^\infty s^{1-2\alpha_2} e^{U_0} \psi^*(s) \, ds < 0,$$

for $r > \xi_0$. Thus ψ^* is decreasing for $r \ge \xi_0$ and $r \frac{d}{dr} \psi^*(r) \to 0$ as $r \to \infty$. The equation for ψ^* also gives

$$\left| r \frac{d}{dr} \psi^*(r) \right| \le K^* \left(\int_r^\infty |s|^{1-2\alpha_2} e^{U_0} (\psi^*(s))^2 \, ds \right)^{\frac{1}{2}} \left(\int_r^\infty s^{1-2\alpha_2} e^{U_0(s)} \, ds \right)^{\frac{1}{2}} \le C r^{-1},$$

for large r. Therefore $\lim_{r \to \infty} \psi^*(r)$ exists and is a negative constant.

Let

$$\psi(r) = 2(1-\alpha_2) \frac{1 - r^{2(1-\alpha_2)}}{1 + r^{2(1-\alpha_2)}}.$$

Then ψ satisfies

$$\Delta\psi + r^{-2\alpha_2} e^{U_0}\psi = 0 \quad \text{in } \mathbb{R}^2.$$

It is easy to obtain the following from the equations for ψ and for ψ^*:

$$r\left(\frac{\psi^*}{\psi(r)}\right)' = \frac{1-K^*}{\psi^2(r)} \int_0^r s^{1-2\alpha_2} e^{U_0(s)} \psi^*(s)\psi(s)\, ds.$$

If $\xi_0 < 1$, $\frac{\psi^*(r)}{\psi(r)}$ is increasing from $r \in (0, \xi_0]$. Clearly this is not possible because otherwise this could happen:

$$0 < \frac{\psi^*(0)}{\psi(0)} < \frac{\psi^*(\xi_0)}{\psi(\xi_0)} = 0.$$

On other hand, we observe that it is also absurd to have $\xi_0 > 1$, indeed, had this happened, we would start from

$$\lim_{R\to\infty} R\left(\frac{\psi^*}{\psi}\right)'(R)\psi^2(R) - r\left(\frac{\psi^*}{\psi}\right)'(r)\psi^2(r) = (1-K^*)\int_r^\infty s^{1-2\alpha_2} e^{U_0}\psi^*(s)\psi(s)\, ds.$$

Since

$$\lim_{R\to\infty} R\Big(\frac{d}{dr}\psi^*(R)\psi(R) - \psi'(R)\psi^*(R)\Big) = 0,$$

we have

$$-r\left(\frac{\psi^*}{\psi}\right)'\psi^2(r) = (1-K^*)\int_r^\infty s^{1-2\alpha_2} e^{U_0(s)}\psi^*(s)\psi(s)\, ds.$$

If $\xi_0 > 1$, $(\psi^*(r))/(\psi(r))$ is decreasing for $r > 1$, which yields

$$0 = \frac{\psi^*(\xi_0)}{\psi(\xi_0)} > \lim_{r\to\infty} \frac{\psi^*(r)}{\psi(r)} = -\frac{1}{2(1-\alpha_2)} \lim_{r\to\infty} \psi^*(r) > 0.$$

This contradiction proves that $\xi_0 = 1$ and $\psi^*(r)\psi(r) > 0$ for all $r \neq 1$. Furthermore

$$\begin{aligned} 0 &= \lim_{r\to\infty}\left(\frac{d}{dr}\psi^*(r)\psi(r) - \frac{d}{dr}\psi(r)\psi^*(r)\right)r \\ &= (1-K^*)\int_0^\infty s^{1-2\alpha_2} e^{U_0}\psi^*(s)\psi(s)\, ds. \end{aligned}$$

Thus we have proved that $K^* = 1$ and the desired contradiction. Theorem 1.2 is established. □

The proof of Theorem 1.3 is very similar, we just use Kelvin transformation to move the negative singularity to infinity, then use the same argument with the standard Bol's inequality for nonnegative indexes.

5. The proof of Theorem 1.1.

Our argument follows from a previous result of Bartolucci and Lin [2009] for nonnegative indexed singularities. We prove by way of contradiction. Suppose u is a solution of (1-3) and a nonzero function $\tilde{\phi} \in H_0^1(\Omega)$ is a solution of

$$\begin{cases} -\Delta\tilde{\phi} - \lambda(He^u)/\big(\int_\Omega He^u\,dx\big)\tilde{\phi} + \lambda\big(\int_\Omega He^u\tilde{\phi}\big)(He^u)/\big(\int_\Omega He^u\big)^2 = 0 & \text{in } \Omega, \\ \tilde{\phi} = 0 \quad \text{on } \partial\Omega. \end{cases}$$

Let $w = u + \log\lambda - \log\big(\int_\Omega He^u\,dx\big)$ and

$$\phi = \tilde{\phi} - \frac{\int_\Omega He^u\tilde{\phi}}{\int_\Omega He^u};$$

we have

$$\begin{cases} \Delta\phi + He^w\phi = 0 & \text{in } \Omega, \\ \phi = c_0 & \text{on } \partial\Omega, \\ \int_\Omega He^w\phi = 0, \\ \lambda = \int_\Omega He^w \le 8\pi(1-\alpha_0). \end{cases} \tag{5-1}$$

Without loss of generality we assume $c_0 \le 0$. Our goal is to show that $\phi \equiv c_0$, which further leads to $c_0 = 0$, obviously. If $c_0 = 0$, ϕ must change sign if not identically equal to 0. But this situation is ruled out by Proposition 3.1 that $\nu_2 > 0$. So we only consider $c_0 < 0$. Let

$$\Omega_+ = \{x \in \Omega,\ \phi(x) > 0\}, \qquad \Omega_- = \{x \in \Omega,\ \phi(x) < 0\}.$$

Clearly $\mathrm{dist}(\Omega_+, \partial\Omega) > 0$. Then if $\int_{\Omega_+} He^w \le 4\pi(1-\alpha_0)$ there is no way for ϕ to satisfy (5-1) on Ω_+ without being identically zero. Then using the same rearrangement argument as in the proof of Theorem 1.2 we can also reach the following conclusion: if ϕ_2 is a solution of

$$\begin{cases} -\Delta\phi_2 - \lambda e^u w\phi_2 = \nu e^u w\phi_2 & \text{in } \Omega, \\ \phi_2 = c_0 & \text{on } \partial\Omega, \end{cases}$$

then $\nu > 0$. The remaining part of the proof of Theorem 1.1 follows by standard argument in [Chang et al. 2003] and [Bartolucci and Lin 2009]. We include it with necessary modification.

If we use L_λ to denote the linearized operator of (1-3), we know that all eigenvalues of L_λ are strictly positive for $\lambda \in [0, 8\pi(1-\alpha_0)]$. By using the improved Moser–Trudinger inequality [Malchiodi and Ruiz 2011], one can easily find a solution of (1-3) by the direct minimization method. By the uniform estimate of the linearized equation and standard elliptic estimate we have: for any $\epsilon \in (0, 8\pi(1-\alpha_0))$,

$$\|u_\lambda\|_\infty \le \lambda C_\epsilon, \tag{5-2}$$

for some $C_\epsilon > 0$, $\lambda \in [0, 8\pi(1-\alpha_0)]$ and u_λ as a solution of (1-3). Let S_λ be the solution's branch for (1-3) bifurcating from $(u, \lambda) = (0, 0)$. The standard bifurcation theory of Crandall and Rabinowitz [1975] gives that S_λ is a simple branch near $\lambda = 0$. This means that for $\lambda > 0$ small there exists one and only solution for (1-3) and S_λ is smooth in $C^2(\Omega) \times \mathbb{R}$. By the implicit function theorem (because L_λ has positive first eigenvalue) S_λ can be extended uniquely for $\lambda \in (0, 8\pi(1-\alpha_0))$. If for any given $\lambda \in (0, 8\pi(1-\alpha_0)$ there is another solution, it implies the other solution's branch does not bend in $[0, 8\pi(1-\alpha_0))$. By the uniform estimate (5-2), this second branch intersects S_λ at $(u, \lambda) = (0, 0)$. This contradiction proves the uniqueness for $\lambda \in [0, 8\pi(1-\alpha_0))$. If a solution exists for $\lambda = 8\pi(1-\alpha_0)$, the implicit function theorem and the uniqueness result can be combined to prove the uniqueness in this case as well. Theorem 1.1 is established. □

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