Pacific Journal of Mathematics

HEEGAARD FLOER HOMOLOGY OF L-SPACE LINKS WITH TWO COMPONENTS

BEIBEI LIU

Volume 298 No. 1 January 2019

PACIFIC JOURNAL OF MATHEMATICS
Vol. 298, No. 1, 2019
dx.doi.org/10.2140/pjm.2019.298.83

HEEGAARD FLOER HOMOLOGY OF L-SPACE LINKS WITH TWO COMPONENTS

BEIBEI LIU

We compute different versions of link Floer homology HFL^- and $\widehat{\mathrm{HFL}}$ for any L-space link with two components. The main approach is to compute the h-function of the filtered chain complex which is determined by Alexander polynomials of all sublinks of the L-space link. As an application, the Thurston norm of its complement is explicitly determined by Alexander polynomials of the link and its components.

1. Introduction

Heegaard Floer homology is an invariant for closed, oriented 3-manifolds, defined using Heegaard diagrams [Ozsváth and Szabó 2004b]. This construction can be extended to give an invariant, *Heegaard Floer link homology* (also called *link Floer homology*), for oriented links in S^3 [Ozsváth and Szabó 2008a]. In general, it is very hard to compute the Heegaard Floer link homology HFL^- and $\widehat{\mathrm{HFL}}$. For any L-space link with two components (see Definition 2.2), Yajing Liu [2017] computed the link Floer homology HFL^-. Based on his work, we come up with a method to compute the link Floer homology $\widehat{\mathrm{HFL}}$ of any L-space link with two components. By the work of Ozsváth and Szabó [2008b], we compute the Thurston polytope and the Thurston norm of its complement. For an r-component L-space link with a given generic admissible multipointed Heegaard diagram, one can associate it with *generalized Floer complexes* $A^-(\boldsymbol{s})$ filtered by Alexander gradings [Manolescu and Ozsváth 2010]. In this article, we work over $\mathbb{F} = \mathbb{F}_2$ and $\boldsymbol{s} \in \mathbb{H}$, where $\mathbb{H}$ is some r-dimensional lattice; see Definition 2.3 and [Manolescu and Ozsváth 2010]. If the link L is an L-space link, we have the following result for $A^-(\boldsymbol{s})$:

Proposition 1.1 [Liu 2017, Proposition 1.11]. *For any L-space link*,

$$H_*(A^-(\boldsymbol{s})) = \mathbb{F}[\![U]\!] \quad \text{with } \boldsymbol{s} \in \mathbb{H}.$$

Here U has homological grading -2. Define $-2h(\boldsymbol{s})$ as the homological grading of the generator in $H_*(A^-(\boldsymbol{s}))$. By the work of Gorsky, Némethi and Yajing Liu,

MSC2010: 57M99.
Keywords: L-space links, link Floer homology, Thurston polytopes.

$h(\boldsymbol{s})$ is determined by Alexander polynomials $\Delta_L(t_1, t_2)$, $\Delta_{L_1}(t)$ and $\Delta_{L_2}(t)$ for any 2-component L-space link and $\boldsymbol{s} \in \mathbb{H}$. There is a spectral sequence which converges to $\mathrm{HFL}^-(L, \boldsymbol{s})$ [Gorsky and Némethi 2015]. It collapses at the E^2-page, and $h(\boldsymbol{s})$ determines its E^1-page; see [Gorsky and Némethi 2015, Theorem 2.2.10; Liu 2017].

The computation of $\widehat{\mathrm{HFL}}(L, \boldsymbol{s})$ is more complicated. We introduce a bigraded "iterated cone" complex $(\mathfrak{C}(s_1, s_2), d + d_1)$ in Section 3. There exists a spectral sequence associated with this bigraded complex where the E^1-page is defined by HFL^- and $E^3 = \widehat{\mathrm{HFL}}(L, s_1, s_2)$. Theorem 3.2 shows that the E^1-page of this spectral sequence is

$$\mathrm{HFL}^-(s_1 + 1, s_2 + 1) \oplus \mathrm{HFL}^-(s_1, s_2 + 1) \oplus \mathrm{HFL}^-(s_1 + 1, s_2) \oplus \mathrm{HFL}^-(s_1, s_2),$$

and the differential d_1 is induced by actions of U_1 and U_2. Lemma 3.3 indicates how U_i acts on the Heegaard Floer link homology $\mathrm{HFL}^-(L, \boldsymbol{s})$ for any $\boldsymbol{s} \in \mathbb{H}$ and $i = 1, 2$. So we can compute the E^2-page of the spectral sequence. If $d_2 = 0$, the spectral sequence collapses at the E^2-page. If d_2 is nonzero, we need to use another strategy to compute $\widehat{\mathrm{HFL}}(L, \boldsymbol{s})$. We first find all possible cases where d_2 may be nontrivial. In order to compute $\widehat{\mathrm{HFL}}(L, \boldsymbol{s})$, we use the symmetric property of Heegaard Floer link homology: $\widehat{\mathrm{HFL}}(L, \boldsymbol{s}) \cong \widehat{\mathrm{HFL}}(L, -\boldsymbol{s})$, up to some grading shift [Ozsváth and Szabó 2006, Equation 5]. In Section 3, we find that in all cases where d_2 may be nontrivial, the spectral sequence corresponding to $\widehat{\mathrm{HFL}}(L, -\boldsymbol{s})$ collapses at its E^2-page. Then we can compute $\widehat{\mathrm{HFL}}(L, -\boldsymbol{s})$, and hence $\widehat{\mathrm{HFL}}(L, \boldsymbol{s})$. Therefore, we compute $\widehat{\mathrm{HFL}}$ for all L-space links with two components and obtain the main theorem of this paper.

Theorem 1.2. *For any L-space link $L = L_1 \cup L_2$ with two components*, $\widehat{\mathrm{HFL}}(s_1, s_2)$ *is determined by the h-function and hence determined by symmetrized Alexander polynomials $\Delta_L(t_1, t_2)$, $\Delta_{L_1}(t)$, $\Delta_{L_2}(t)$, and the linking number* lk *of L_1 and L_2.*

Remark 1.3. The Heegaard Floer link homology depends on the orientation of the link. For any L-space link $L = L_1 \cup L_2$, we need to give it an orientation, which determines the linking number of L_1 and L_2.

Yajing Liu [2017] showed that $\mathrm{rank}_{\mathbb{F}}(\mathrm{HFL}^-(L, \boldsymbol{s})) \leq 2$. We show that 4 is a bound for the rank of link Floer homology $\widehat{\mathrm{HFL}}$ for any L-space link with two components. Then we give examples for all possible ranks from 0 to 4 in Section 3.

Corollary 1.4. *For* 2*-component L-space links $L = L_1 \cup L_2$ and $\boldsymbol{s} \in \mathbb{H}$,*

$$\mathrm{rank}_{\mathbb{F}}(\widehat{\mathrm{HFL}}(L, \boldsymbol{s})) \leq 4.$$

In particular, $|\chi(\widehat{\mathrm{HFL}}(L, \boldsymbol{s}))| \leq 4$.

In Section 4, we present an application of Theorem 1.2. It is known from [Ozsváth and Szabó 2008b] that $\widehat{\mathrm{HFL}}(L)$ detects the Thurston norm of the link complement. For any compact, oriented surface with boundary $F = \bigcup_{i=1}^{n} F_i$ (maybe disconnected), define its *complexity* as

$$\chi_-(F) = \sum_{\{F_i \mid \chi(F_i) \le 0\}} -\chi(F_i).$$

For any link $L \subseteq S^3$, and any homology class $h \in H_2(S^3, L)$, there exists a compact oriented surface F with boundary embedded in $S^3 \setminus \mathrm{nd}(L)$ which represents this homology class (i.e., $[F] = h$). So for any homology class $h \in H_2(S^3, L; \mathbb{Z})$, we can assign a function

$$x(h) = \min_{\{F \hookrightarrow S^3 \setminus \mathrm{nd}(L),\, [F]=h\}} \chi_-(F).$$

This function can be naturally extended to a seminorm, the *Thurston seminorm*, denoted by $x : H_2(S^3, L; \mathbb{R}) \to \mathbb{R}$ [Ozsváth and Szabó 2008b]. The unit ball for the norm x is called the *Thurston polytope*. Consider the convex hull of lattice points $\boldsymbol{s} \in \mathbb{H}$, where $\widehat{\mathrm{HFL}}(L, \boldsymbol{s}) \neq 0$, which is also called the *link Floer homology polytope*. We can compute the dual Thurston polytope, and thus the Thurston norm by [Ozsváth and Szabó 2008b]. So for any 2-component L-space link $L = L_1 \cup L_2$, the Thurston polytope and the Thurston norm are determined by Alexander polynomials of all sublinks, but in a very nontrivial way.

Theorem 1.5. *If $L = L_1 \cup L_2$ is an L-space link with two components in S^3, then the Thurston norm of its complement is determined by Alexander polynomials $\Delta_L(t_1, t_2)$, $\Delta_{L_1}(t)$, $\Delta_{L_2}(t)$ and the linking number of L_1 and L_2.*

Ozsváth and Szabó pointed out that for any alternating link, up to a scalar, the Thurston polytope is dual to the Newton polytope of its multivariable Alexander polynomial [Ozsváth and Szabó 2008b], which is contained in the dual Thurston polytope by [McMullen 2002]. We compute dual Thurston polytopes of two nonalternating L-space links with two components in Examples 4.4 and 4.5. They both agree with Newton polytopes of their Alexander polynomials. A natural question arises:

Question 1.6. For any 2-component L-space link which is not a split union of two L-space knots, is the Thurston polytope dual to the Newton polytope of its multivariable Alexander polynomial?

Remark 1.7. In Example 4.4, we present a 2-component L-space link where the set $\mathrm{supp}(\widehat{\mathrm{HFL}}) = \{(s_1, s_2) \in \mathbb{H} \mid \widehat{\mathrm{HFL}}(s_1, s_2) \neq 0\}$ is larger than $\mathrm{supp}(\chi(\widehat{\mathrm{HFL}})) = \{(s_1, s_2) \in \mathbb{H} \mid \chi(\widehat{\mathrm{HFL}}(s_1, s_2)) \neq 0\}$. But the convex hull of $\mathrm{supp}(\widehat{\mathrm{HFL}})$ is the same as that of $\mathrm{supp}(\chi(\widehat{\mathrm{HFL}}))$, since lattice points (s_1, s_2) for which $\chi(\widehat{\mathrm{HFL}}(s_1, s_2)) = 0$ and $\widehat{\mathrm{HFL}}(s_1, s_2) \neq 0$ are inside the convex hull of $\mathrm{supp}(\chi(\widehat{\mathrm{HFL}}))$.

For any split L-space link, the answer to Question 1.6 is negative since its Alexander polynomial vanishes, but the dual Thurston polytope is nonempty. Example 5.5 gives the link Floer homology polytope of the split union of two right-handed trefoils. The split union of two L-space knots is an L-space link [Liu 2017], and the h-function of the link satisfies $h(s_1, s_2) = h_1(s_1) + h_2(s_2)$, where h_1 and h_2 are h-functions of L_1 and L_2, respectively. We compute $\widehat{\mathrm{HFL}}$ for any split union of two L-space knots. In general, we compute $\widehat{\mathrm{HFL}}$ for all 2-component L-space links with Alexander polynomials $\Delta(t_1, t_2) = 0$.

Theorem 1.8. *For any* 2*-component L-space link $L = L_1 \cup L_2$ and $(s_1, s_2) \in \mathbb{H}$, if $\Delta_L(t_1, t_2) = 0$, then*

$$\widehat{\mathrm{HFL}}(L, s_1, s_2) \cong \widehat{\mathrm{HFL}}(L_1 \sqcup L_2, s_1, s_2) \cong \widehat{\mathrm{HFL}}(L_1, s_1) \otimes \widehat{\mathrm{HFL}}(L_2, s_2) \otimes (\mathbb{F} \oplus \mathbb{F}_{-1}),$$

where $L_1 \sqcup L_2$ denotes the split union of L_1 and L_2 .

In this paper, we use $L = L_1 \cup L_2$ to denote L-space links with two components L_1, L_2, unless otherwise stated.

2. Heegaard Floer link homology

2A. ***L-space links.*** The concept of L-spaces was introduced in [Ozsváth and Szabó 2005].

Definition 2.1. A 3-manifold Y is an L-space if it is a rational homology sphere and its Heegaard Floer homology has minimal possible rank: for any Spin^c-structure s, $\widehat{\mathrm{HF}}(Y, s) = \mathbb{F}$ has rank 1, and $\mathrm{HF}^-(Y, s)$ is a free $\mathbb{F}[U]$-module of rank 1.

Gorsky and Némethi [2016] defined L-space links in terms of large surgeries.

Definition 2.2. An l-component link $L \subseteq S^3$ is an L-space link if there exist integers $p_1, p_2, \dots, p_l$ such that for all integers $n_i \geq p_i$, $1 \leq i \leq l$, the $(n_1, n_2, \dots, n_l)$-surgery $S^3_{n_1,n_2,\dots,n_l}$ is an L-space.

The computation of Heegaard Floer link homology is not easy. However, L-space links have some nice properties which make the computation of Heegaard Floer link homology easier. In particular, we only consider L-space links $L = L_1 \cup L_2$ with two components in this article.

For a 2-component L-space link $L = L_1 \cup L_2$ in S^3, consider a generic admissible multipointed Heegaard diagram with each component L_i having only two basepoints w_i, z_i. One can associate a generalized Floer complex $A^-(s_1, s_2)$ with $(s_1, s_2) \in \mathbb{H}$, which is introduced in [Manolescu and Ozsváth 2010, Section 4]. It is a free $\mathbb{F}[U_1, U_2]$-module. The operations U_1 and U_2 are homotopic to each other on each $A^-(s_1, s_2)$ (see [Ozsváth and Szabó 2008a]), and both have homological degree -2.

Definition 2.3. For an oriented link $L = L_1 \cup L_2$ with two components, define $\mathbb{H}$ to be an affine lattice over $\mathbb{Z}^2$,

$$\mathbb{H} = \mathbb{H}_1 \oplus \mathbb{H}_2, \quad \mathbb{H}_i = \mathbb{Z} + \frac{\operatorname{lk}(L_1, L_2)}{2} \quad (i = 1, 2),$$

where $\operatorname{lk}(L_1, L_2)$ denotes the linking number of L_1 and L_2.

By Proposition 1.1, for any L-space link L with two components, we have $H_*(A^-(s_1, s_2)) = \mathbb{F}[\![U]\!]$, where $(s_1, s_2) \in \mathbb{H}$. Let $-2h(s_1, s_2)$ denote the homological grading of the generator in $H_*(A^-(s_1, s_2))$. The function $h(s_1, s_2)$ is the HFL-weight function of an L-space link defined in [Gorsky and Némethi 2015]. In this article, we call it the h-function. On each $A^-(s_1, s_2)$, the operations U_1 and U_2 are homotopic, and we denote them by U.

Lemma 2.4 [Gorsky and Némethi 2015, Lemma 2.2.3]. *Let $\boldsymbol{e}_1 = (1, 0)$ and $\boldsymbol{e}_2 = (0, 1)$. For any $\boldsymbol{s} = (s_1, s_2) \in \mathbb{H}$, there are inclusions $j : A^-(s_1, s_2) \hookrightarrow A^-(\boldsymbol{s} + \boldsymbol{e_i})$ for $i = 1, 2$ which induce injections on homology as follows*:

$$j_* : H_*(A^-(s_1, s_2)) \to H_*(A^-(\boldsymbol{s} + \boldsymbol{e_i})),$$

where $j_* = U_i^{\delta(i)}$ and $\delta(i) = 0$ or 1.

Remark 2.5. The actions U_i induce maps $U_i : A^-(\boldsymbol{s} + \boldsymbol{e_i}) \to A^-(\boldsymbol{s})$ for $i = 1, 2$, and induce maps on homology. By Proposition 1.1, $H_*(A^-(\boldsymbol{s})) \cong \mathbb{F}[\![U]\!]$ for any $\boldsymbol{s} \in \mathbb{H}$. Assume that a, b are the generators of $H_*(A^-(\boldsymbol{s}))$ and $H_*(A^-(\boldsymbol{s} + \boldsymbol{e_i}))$. Then $j_*(a) = U^{\delta(i)} b$ and $U_i(b) = U^{1-\delta(i)} a$.

Corollary 2.6. *For any L-space link with two components and $\boldsymbol{s} \in \mathbb{H}$, either $h(\boldsymbol{s}) = h(\boldsymbol{s} + \boldsymbol{e_i})$ or $h(\boldsymbol{s}) = h(\boldsymbol{s} + \boldsymbol{e_i}) + 1$, where $i = 1, 2$, $\boldsymbol{e}_1 = (1, 0)$, and $\boldsymbol{e}_2 = (0, 1)$.*

Proof. By Lemma 2.4, we have $-2h(\boldsymbol{s}) = -2h(\boldsymbol{s} + \boldsymbol{e_i}) - 2\delta(i)$, where $\delta(i) = 0$ or 1. So $h(\boldsymbol{s}) = h(\boldsymbol{s} + \boldsymbol{e_i})$ or $h(\boldsymbol{s}) = h(\boldsymbol{s} + \boldsymbol{e_i}) + 1$. □

Next, we revisit Yajing Liu's work [2017] about how to use the h-function to compute $\mathrm{HFL}^-(L)$ for any 2-component L-space link $L = L_1 \cup L_2$.

Lemma 2.7 [Gorsky and Némethi 2015, Lemma 2.2.9]. *For any $(s_1, s_2) \in \mathbb{H}$, the chain complex $\mathrm{CFL}^-(s_1, s_2)$ of the L-space link $L = L_1 \cup L_2$ is quasi-isomorphic to the "iterated cone" complex*

$$\left[\begin{array}{ccc} A^-(s_1 - 1, s_2) & \xrightarrow{i_1} & A^-(s_1, s_2) \\ \uparrow{\scriptstyle i_2} & & \uparrow{\scriptstyle i_2} \\ A^-(s_1 - 1, s_2 - 1) & \xrightarrow{i_1} & A^-(s_1, s_2 - 1) \end{array}\right]$$

where i_1 and i_2 are inclusion maps in Lemma 2.4.

Let d denote the differential in the generalized Floer complex $A^-(s_1, s_2)$ and $i = i_2 - i_1$. The above "iterated cone" complex has two differentials d and i. The differential d acts in Floer complexes on vertices of the cube, and i acts between Floer complexes. Let the cube grading $|K|$ of the upper-right corner of the cube be 0. The differential d decreases the homological grading by 1, and preserves the cube grading. The differential i preserves the homological grading, and decreases $|K|$ by 1. The total grading is defined as the sum of the homological grading and the cube grading. Let $D = d + i$ and $\mathfrak{K}(s_1, s_2)$ denote the "iterated cone" complex. There exists a spectral sequence whose E^∞-page is the homology of $\mathfrak{K}(s_1, s_2)$ under D.

Theorem 2.8 [Gorsky and Némethi 2015, Theorem 2.2.10]. *Let $L = L_1 \cup L_2$ be an L-space link with two components. For any $(s_1, s_2) \in \mathbb{H}$, there exists a spectral sequence which converges to* $\mathrm{HFL}^-(s_1, s_2)$ *and collapses at its E^2-page. Its E^2-page is isomorphic to $H_*(H_*(A^-(s_1, s_2), d), i)$.*

So $\mathrm{HFL}^-(s_1, s_2)$ is isomorphic to $H_*(H_*(A^-(s_1, s_2), d), i)$. By Proposition 1.1, for any $(s_1, s_2) \in \mathbb{H}$, $H_*(A^-(s_1, s_2), d) \cong \mathbb{F}[[U]][-2h(s_1, s_2)]$, where $-2h(s_1, s_2)$ is the homological grading of the generator in $H_*(A^-(s_1, s_2), d)$, and U_1, U_2 act as U, homotopic to each other on $A^-(s_1, s_2)$ [Ozsváth and Szabó 2008a]. To compute $\mathrm{HFL}^-(s_1, s_2)$, we just need to compute the homology of the mapping cone of i:

$$\begin{tikzcd}
\mathbb{F}[[U]][-2h(s_1-1, s_2)][b] \arrow[r, "i_1"] & \mathbb{F}[[U]][-2h(s_1, s_2)][a] \\
\mathbb{F}[[U]][-2h(s_1-1, s_2-1)][c] \arrow[r, "i_1"] \arrow[u, "i_2"] & \mathbb{F}[[U]][-2h(s_1, s_2-1)][d] \arrow[u, "i_2"]
\end{tikzcd}$$

where a, b, c, d denote the generators in $\mathbb{F}[[U]][-2h(s_1, s_2)]$, $\mathbb{F}[[U]][-2h(s_1-1, s_2)]$, $\mathbb{F}[[U]][-2h(s_1-1, s_2-1)]$, and $\mathbb{F}[[U]][-2h(s_1, s_2-1)]$, respectively. Let $h = h(s_1, s_2)$. By Corollary 2.6, there are 6 cases for the h-function corresponding to the mapping cone.

$$\boxed{\begin{array}{cc} h & h \\ h & h \end{array}} \quad \boxed{\begin{array}{cc} h & h \\ h+1 & h+1 \end{array}} \quad \boxed{\begin{array}{cc} h+1 & h \\ h+1 & h \end{array}}$$

Case (1) Case (2) Case (3)

$$\boxed{\begin{array}{cc} h & h \\ h+1 & h \end{array}} \quad \boxed{\begin{array}{cc} h+1 & h \\ h+1 & h+1 \end{array}} \quad \boxed{\begin{array}{cc} h+1 & h \\ h+2 & h+1 \end{array}}$$

Case (4) Case (5) Case (6)

Figure 1. Possible local behaviors of the h-function.

According to the h-function in Figure 1, we can compute the corresponding $\mathrm{HFL}^-(s_1, s_2)$ in each case.

Case (1): $i(b) = a$, $i(c) = b - d$, $i(d) = a$ and $i(a) = 0$, so $\mathrm{HFL}^-(s_1, s_2) = 0$.

Case (2): $i(b) = a$, $i(c) = Ub - d$, $i(d) = Ua$ and $i(a) = 0$, so $\mathrm{HFL}^-(s_1, s_2) = 0$.

Case (3): $i(b) = Ua$, $i(c) = b - Ud$, $i(d) = a$ and $i(a) = 0$, so $\mathrm{HFL}^-(s_1, s_2) = 0$.

Case (4): $i(b)=a$, $i(c)=Ub-Ud$, $i(d)=a$ and $i(a)=0$, so $\mathrm{HFL}^-(s_1, s_2)=\langle b-d\rangle$. Both b and d have homological grading $-2h$ and cube grading 1. The total grading of $b-d$ is $-2h+1$. Thus $\mathrm{HFL}^-(s_1, s_2) = \mathbb{F}[-2h+1]$.

Case (5): $i(b) = Ua$, $i(c) = b - d$, $i(d) = Ua$ and $i(a) = 0$, so $\mathrm{HFL}^- = \langle a\rangle$ with total grading $-2h$. Thus $\mathrm{HFL}^-(s_1, s_2) = \mathbb{F}[-2h]$.

Case (6): $i(b) = Ua$, $i(c) = Ub - Ud$, $i(d) = Ua$, and $i(a) = 0$, so in this case $\mathrm{HFL}^-(s_1, s_2) = \langle a, b-d\rangle$. Here a has total grading $-2h$ and $b-d$ has total grading $-2(h+1)+1 = -2h-1$. Thus $\mathrm{HFL}^-(s_1, s_2) = \mathbb{F}[-2h] \oplus \mathbb{F}[-2h-1]$.

Moreover, we also determine the Euler characteristics $\chi(\mathrm{HFL}^-(s_1, s_2))$ in these six cases. In Case (1), Case (2), Case (3) and Case (6), $\chi(\mathrm{HFL}^-(s_1, s_2)) = 0$. In Case (4), $\chi(\mathrm{HFL}^-(s_1, s_2)) = -1$, and in Case (5), $\chi(\mathrm{HFL}^-(s_1, s_2)) = 1$. Thus for any L-space link with two components, once the h-function is determined, we can compute $\mathrm{HFL}^-(s_1, s_2)$ for any $(s_1, s_2) \in \mathbb{H}$.

Corollary 2.9. *For any* 2*-component L-space link and* $(s_1, s_2) \in \mathbb{H}$, $\mathrm{HFL}^-(s_1, s_2)$ *is spanned by a or $b-d$ or both, where a has even grading and $b-d$ has odd grading.*

2B. ***Alexander polynomials of L-space links.*** In this section, we mainly introduce Yajing Liu's work [2017] about how to determine the h-function of any 2-component L-space link $L = L_1 \cup L_2$ by Alexander polynomials $\Delta_L(t_1, t_2)$, $\Delta_{L_1}(t)$, and $\Delta_{L_2}(t)$. Recall that for any L-space link $L = L_1 \cup L_2$, we have

$$\Delta_L(t_1, t_2) \doteq \sum_{(s_1,s_2)\in\mathbb{H}} \chi(\mathrm{HFL}^-(s_1, s_2)) t_1^{s_1} t_2^{s_2},$$

$$\Delta_L(t, 1) \doteq \frac{1-t^{\mathrm{lk}}}{1-t} \Delta_{L_1}(t), \tag{2-1}$$

where $f \doteq g$ means that f and g differ by multiplication by units. Yajing Liu [2017] defined normalization of Alexander polynomials.

Definition 2.10 [Liu 2017, Definition 5.12]. Let the symmetrized Alexander polynomial of L be $\Delta_L(x_1, x_2)$ in the form of

$$\Delta_L(t_1, t_2) = \sum_{i,j} a^L_{i,j} \cdot t_1^i \cdot t_2^j,$$

where t_i corresponds to the link component L_i for $i = 1, 2$. Let the symmetrized Alexander polynomials of L_1 and L_2 be $\Delta_{L_1}(t)$, $\Delta_{L_2}(t)$ in the form of

$$\frac{t}{t-1}\Delta_{L_1}(t) = \sum_{k\in\mathbb{Z}} a_k^{L_1}\cdot t^k, \qquad \frac{t}{t-1}\Delta_{L_2}(t) = \sum_{k\in\mathbb{Z}} a_k^{L_2}\cdot t^k.$$

Let (i_0, j_0) be such that

$$j_0 = \max\Big\{ j \in \mathbb{Z} + \frac{\mathrm{lk}-1}{2} \mid a_{i,j}^L \neq 0\Big\}, \qquad i_0 = \max\Big\{ i \in \mathbb{Z} + \frac{\mathrm{lk}-1}{2} \mid a_{i,j_0}^L \neq 0\Big\}.$$

Then these Alexander polynomials are called *normalized* if

(1) the leading coefficient of $\Delta_{L_i}(t)$ is 1 for both $i = 1, 2$,

(2) if $a^{L_2}_{j_0-\mathrm{lk}/2+1/2} = 1$, then $a^L_{i_0,j_0} = 1$, while if $a^{L_2}_{j_0-\mathrm{lk}/2+1/2} = 0$, then $a^L_{i_0,j_0} = -1$, where lk is the linking number of L_1 and L_2.

For the normalized Alexander polynomials of the 2-component L-space link $L = L_1 \cup L_2$, $\chi(\mathrm{HFL}^-)(s_1, s_2) = a^L_{s_1-1/2,s_2-1/2}$ and $\chi(\mathrm{HFK}^-(L_i, s)) = a_s^{L_i}$ for $i = 1, 2$ [Liu 2017]. Moreover, Yajing Liu gave the following formulas to determine the h-function in [Liu 2017, (5.8)]:

$$h(s_1, s_2-1) - h(s_1, s_2) = a^{L_2}_{s_2-\mathrm{lk}/2} - \sum_{i=1}^{\infty} a^L_{s_1+i-1/2,s_2-1/2} = 0 \quad \text{or} \quad 1. \tag{2-2}$$

Similarly,

$$h(s_1-1, s_2) - h(s_1, s_2) = a^{L_1}_{s_1-\mathrm{lk}/2} - \sum_{i=1}^{\infty} a^L_{s_1-1/2,s_2+i-1/2} = 0 \quad \text{or} \quad 1. \tag{2-3}$$

When $s_1 \to +\infty$ or $s_2 \to +\infty$,

$$h(+\infty, s_2) = h_2(s_2 - \mathrm{lk}/2), \quad h(s_1, +\infty) = h_1(s_1 - \mathrm{lk}/2), \tag{2-4}$$

$$h_1(s-1) - h_1(s) = a_s^{L_1}, \quad h_2(s-1) - h_2(s) = a_s^{L_2}, \tag{2-5}$$

where $h_1(s_1 - \mathrm{lk}/2)$ and $h_2(s_2 - \mathrm{lk}/2)$ are h-functions for link components L_1 and L_2, and $s \in \mathbb{Z}$. For sufficiently large s, $h_1(s) = h_2(s) = 0$. By using the formulas above, we can compute the h-function, and hence $\mathrm{HFL}^-(s_1, s_2)$ for any 2-component L-space link $L = L_1 \cup L_2$.

Remark 2.11. The link components L_1 and L_2 of 2-component L-space links are both L-space knots [Liu 2017, Lemma 1.10].

Corollary 2.12 [Dawra 2015; Gorsky and Némethi 2015; Liu 2017]. *For any L-space link $L = L_1 \cup L_2$ with two components*, $\mathrm{HFL}^-(L)$ *is determined by Alexander polynomials $\Delta_L(t_1, t_2)$, $\Delta_{L_1}(t)$ and $\Delta_{L_2}(t)$.*

3. Computation of $\widehat{\mathrm{HFL}}$ for 2-component L-space links

3A. ***The spectral sequence corresponding to*** $\widehat{\mathrm{HFL}}$**.** In Section 2, we proved that for any L-space link $L = L_1 \cup L_2$ with $(s_1, s_2) \in \mathbb{H}$, $\mathrm{HFL}^-(s_1, s_2)$ is determined by the h-function. Now we are going to prove Theorem 1.2 that the Heegaard Floer link homology $\widehat{\mathrm{HFL}}(s_1, s_2)$ is also determined by the h-function.

Let $\mathfrak{C}(s_1, s_2)$ denote

$$\mathrm{CFL}^-(s_1+1, s_2+1) \oplus \mathrm{CFL}^-(s_1+1, s_2) \oplus \mathrm{CFL}^-(s_1, s_2+1) \oplus \mathrm{CFL}^-(s_1, s_2).$$

For any $(s_1, s_2) \in \mathbb{H}$, we have maps $U_1 : \mathrm{CFL}^-(s_1, s_2) \to \mathrm{CFL}^-(s_1 - 1, s_2)$ and $U_2 : \mathrm{CFL}^-(s_1, s_2) \to \mathrm{CFL}^-(s_1, s_2 - 1)$. The action of U_1 (or U_2) is defined by the h-function (see Lemma 3.3). Let $D = d + d_1$, where d is the differential in chain complex $\mathrm{CFL}^-(s_1, s_2)$ and $d_1 = U_1 - U_2$. Then we have the "iterated cone" complex $(\mathfrak{C}(s_1, s_2), d + d_1)$ in the following form:

$$\begin{tikzcd}
\mathrm{CFL}^-(s_1, s_2+1) \arrow[d, "U_2"'] & \mathrm{CFL}^-(s_1+1, s_2+1) \arrow[l, "U_1"] \arrow[d, "U_2"'] \\
\mathrm{CFL}^-(s_1, s_2) & \mathrm{CFL}^{-1}(s_1+1, s_2) \arrow[l, "U_1"]
\end{tikzcd}$$

Lemma 3.1. *Suppose that $L = L_1 \cup L_2$ is an L-space link. Let $\widehat{\mathrm{CFL}}(s_1, s_2)$ denote the chain complex of the hat-version of Heegaard Floer link homology of L with $(s_1, s_2) \in \mathbb{H}$. Then $\widehat{\mathrm{CFL}}(s_1, s_2)$ is quasi-isomorphic to the "iterated cone" complex $(\mathfrak{C}(s_1, s_2), d + d_1)$.*

Proof. We can write $\widehat{\mathrm{CFL}}(s_1, s_2)$ as

$$\frac{\mathrm{CFL}^-(s_1, s_2)/U_1(\mathrm{CFL}^-(s_1+1, s_2))}{U_2(\mathrm{CFL}^-(s_1, s_2+1)/U_1(\mathrm{CFL}^-(s_1+1, s_2+1)))}.$$

The quotient $\mathrm{CFL}^-(s_1, s_2)/U_1(\mathrm{CFL}^-(s_1+1, s_2))$ can be realized as the mapping cone of $U_1 : \mathrm{CFL}^-(s_1+1, s_2) \to \mathrm{CFL}^-(s_1, s_2)$, and similarly the quotient $\mathrm{CFL}^-(s_1, s_2+1)/U_1(\mathrm{CFL}^-(s_1+1, s_2+1))$ can be realized as the mapping cone of $U_1 : \mathrm{CFL}^-(s_1+1, s_2+1) \to \mathrm{CFL}^-(s_1, s_2+1)$. Thus $\widehat{\mathrm{CFL}}(s_1, s_2)$ can be realized as the cone of the natural map induced by U_2 between these two cones. $\square$

Theorem 3.2. *Let $L = L_1 \cup L_2$ be an L-space link with two components. For any $(s_1, s_2) \in \mathbb{H}$, there exists a spectral sequence with the following properties*:

(a) *Its E^2-page is isomorphic* (*as a graded $\mathbb{F}$-module*) *to $H_*(H_*(\mathfrak{C}(s_1, s_2), d), d_1)$.*

(b) *Its E^∞-page is isomorphic* (*as a graded $\mathbb{F}$-module*) *to $\widehat{\mathrm{HFL}}(s_1, s_2)$.*

(c) *The spectral sequence collapses at the E^3-page.*

Proof. For the "iterated cone" complex $\mathfrak{C}(s_1, s_2)$, it is doubly graded. One is the homological grading ν in the chain complex $\mathrm{CFL}^-(s_1, s_2)$ with $(s_1, s_2) \in \mathbb{H}$. We define *cube grading* $|C|$ in the cube of the "iterated cone" complex $\mathfrak{C}(s_1, s_2)$. Fix $(s_1, s_2) \in \mathbb{H}$. The cube grading is defined as $(s_1+s_2)-(v_1+v_2)$, where $(v_1, v_2) \in \mathbb{H}$. It is equivalent to saying that the cube grading of the lower left corner is 0, and U_1 (or U_2) increases the cube grading by 1.

The spectral sequence corresponding to the doubly-graded complex $\mathfrak{C}(s_1, s_2)$ with two (anti)commuting differentials d and d_1 converges to $H_*(\mathfrak{C}(s_1, s_2), d+d_1)$. By Lemma 3.1, its E^∞-page is isomorphic to $\widehat{\mathrm{HFL}}(s_1, s_2)$. Its E^1-page is written as $\mathrm{HFL}^-(s_1+1, s_2+1) \oplus \mathrm{HFL}^-(s_1+1, s_2) \oplus \mathrm{HFL}^-(s_1, s_2+1) \oplus \mathrm{HFL}^-(s_1, s_2)$. Its E^2-page is $H_*(H_*(\mathfrak{C}(s_1, s_2), d), d_1)$. The differential $d_0 = d$ preserves the cube grading $|C|$ and decreases the homological degree ν by 1. The differential d_1 in the E^1-page increases $|C|$ by 1 and decreases ν by 2. For any nonnegative integer k, the differential d_k increases $|C|$ by k and decreases ν by $k+1$. The total homological grading is $\nu + |C|$. By grading reasons, the cube grading is less than or equal to 2. Thus, for the integer $k > 2$, $d_k = 0$ and this spectral sequence collapses at the E^3-page. □

By Theorem 3.2, $\widehat{\mathrm{HFL}}(s_1, s_2) \cong E^3$. Then we can compute $\widehat{\mathrm{HFL}}(s_1, s_2)$ by computing the E^3-page of the spectral sequence. The following lemma describes the action of U_1 (or U_2) on the E^1-page.

Lemma 3.3. *Consider the map* $U_1 : \mathrm{HFL}^-(s_1+1, s_2+1) \to \mathrm{HFL}^-(s_1, s_2+1)$. *Let* α *be a generator of* $\mathrm{HFL}^-(s_1+1, s_2+1)$ *with total homological grading* x. *If there exists a generator* β *in* $\mathrm{HFL}^-(s_1, s_2+1)$ *with total homological grading* $x-2$, *then* $U_1(\alpha) = \beta$.

Proof. As shown in Figure 2, let a_1, b_1, c_1 and d_1 denote the generators of $H_*(A^-(s_1, s_2+1))$, $H_*(A^-(s_1-1, s_2+1))$, $H_*(A^-(s_1-1, s_2))$ and $H_*(A^-(s_1, s_2))$, respectively, and likewise a, b, c and d the generators of $H_*(A^-(s_1+1, s_2+1))$, $H_*(A^-(s_1, s_2+1))$, $H_*(A^-(s_1, s_2))$ and $H_*(A^-(s_1+1, s_2))$. Here a_1 and b have different cube gradings as generators of $H_*(A^-(s_1, s_2+1))$ and d_1 and c have different cube gradings as generators of $H_*(A^-(s_1, s_2))$. By the computation of HFL^- in Section 2A, $h(s_1, s_2+1) = h(s_1+1, s_2)$ if $\mathrm{HFL}^-(s_1+1, s_2+1)$ is nonempty. Similarly, $h(s_1-1, s_2+1) = h(s_1, s_2)$ since $\mathrm{HFL}^-(s_1, s_2)$ is also nonempty. Assume that $\alpha = b-d$. Then it has total homological grading $-2h(s_1, s_2+1)+1$. The generator a_1 has total homological grading $-2h(s_1, s_2+1)$, and $b_1 - d_1$ has total homological grading $-2h(s_1-1, s_2+1)+1$. By the assumption of this lemma, the total homological grading of β is $-2h(s_1, s_2+1)-1$. So β can only be $b_1 - d_1$, and $h(s_1-1, s_2+1) = h(s_1, s_2+1)+1$.

Now consider the map $U_1 : H_*(A^-(s_1, s_2+1)) \to H_*(A^-(s_1-1, s_2+1))$, where $H_*(A^-(s_1, s_2+1)) = \langle b \rangle$ and $H_*(A^-(s_1-1, s_2+1)) = \langle b_1 \rangle$. Since U_1 has

$A^-(s_1-1, s_2+1)[b_1]$	$A^-(s_1, s_2+1)[a_1]$
$A^-(s_1-1, s_2)[c_1]$	$A^-(s_1, s_2)[d_1]$

$\mathrm{HFL}^-(s_1, s_2+1)$

$A^-(s_1, s_2+1)[b]$	$A^-(s_1+1, s_2+1)[a]$
$A^-(s_1, s_2)[c]$	$A^-(s_1+1, s_2)[d]$

$\mathrm{HFL}^-(s_1+1, s_2+1)$

Figure 2. Generators for Lemma 3.3.

homological degree -2, $U_1(d) = d_1$ by Lemma 2.4 and Remark 2.5. Similarly, $U_1(c) = c_1$. Then $U_1(\alpha) = U_1(b-d) = b_1 - d_1 = \beta$. If $\alpha = a$, then $\beta = a_1$, and we can use a similar argument to prove $U_1(\alpha) = \beta$ in this case. ☐

Remark 3.4. The map $U_2 : \mathrm{HFL}^-(s_1+1, s_2+1) \to \mathrm{HFL}^-(s_1+1, s_2)$ can be described similarly to Lemma 3.3.

3B. ***Proof of the main theorem.*** In this subsection, we prove Theorem 1.2, and show that 4 is an upper bound for the rank of link Floer homology $\widehat{\mathrm{HFL}}(s_1, s_2)$ for any 2-component L-space link and $(s_1, s_2) \in \mathbb{H}$. Example 3.8 gives a 2-component L-space link where the rank of $\widehat{\mathrm{HFL}}(s_1, s_2)$ ranges from 0 to 4.

In order to prove Theorem 1.2, we need the symmetric property of Heegaard Floer link homology.

Lemma 3.5 [Ozsváth and Szabó 2006, Equation 5]. *For an oriented L-space link $L = L_1 \cup L_2$ with two components and $\boldsymbol{s} = (s_1, s_2) \in \mathbb{H}$, there exists a relatively graded isomorphism*

$$\widehat{\mathrm{HFL}}(L, \boldsymbol{s}) \cong \widehat{\mathrm{HFL}}(L, -\boldsymbol{s}).$$

Remark 3.6. In particular, the h-functions satisfy $h(-\boldsymbol{s}) = h(\boldsymbol{s}) + |\boldsymbol{s}|$, [Liu 2017, Lemma 5.5], where $|\boldsymbol{s}| = s_1 + s_2$.

Proof of Theorem 1.2. Let $h = h(s_1+1, s_2+1)$. If $d_2 = 0$, then the spectral sequence in Theorem 3.2 collapses at its E^2-page. We can use the computation of HFL^- in Section 2A and Lemma 3.3 to compute $\widehat{\mathrm{HFL}}(s_1, s_2)$. For example, suppose that the h-function corresponding to $\widehat{\mathrm{HFL}}(s_1, s_2)$ is the following:

$h+1$	h	h
$h+1$	$h+1$	h
$h+2$	$h+1$	$h+1$

Then the E^2-page of the spectral sequence is:

$$\begin{tikzcd} \mathbb{F}[-2h] \arrow[d, "U_2"'] & \mathbb{F}[-2h+1] \arrow[l, "U_1"'] \arrow[d, "U_2"'] \\ \mathbb{F}[-2h-1] & \mathbb{F}[-2h] \arrow[l, "U_1"'] \end{tikzcd}$$

Since U_1 and U_2 both have homological grading -2, $U_1 = U_2 = 0$. By Theorem 3.2, $d_2 = 0$ since it increases the cube grading by 2, and decreases the homological grading ν by 3. Thus $\widehat{\mathrm{HFL}}(s_1, s_2) \cong \mathbb{F}[-2h-1] \oplus \mathbb{F}[-2h-1] \oplus \mathbb{F}[-2h-1] \oplus \mathbb{F}[-2h-1]$. Here the cube grading for the generator in $\mathbb{F}[-2h-1]$ is 0. We can use this method to compute $\widehat{\mathrm{HFL}}$ in all cases where $d_2 = 0$. Now it suffices to consider cases where d_2 may be nontrivial.

If $d_2 \neq 0$, then $\mathrm{HFL}^-(s_1+1, s_2+1)$ and $\mathrm{HFL}^-(s_1, s_2)$ are both nonzero and contain generators such that their homological grading difference is 3. For nonzero $\mathrm{HFL}^-(s_1+1, s_2+1)$, we have the following three possibilities for the corresponding h-function:

h	h
$h+1$	h

Case (1)

$h+1$	h
$h+1$	$h+1$

Case (2)

$h+1$	h
$h+2$	$h+1$

Case (3)

In Case (1), $\mathrm{HFL}^-(s_1+1, s_2+1) = \mathbb{F}[-2h+1]$. In order to have nontrivial d_2, $\mathrm{HFL}^-(s_1, s_2)$ must contain a generator with homological grading $-2h-2$. So the h-function corresponding to $\mathrm{HFL}^-(s_1, s_2)$ can only have the pattern as in Case (2) or Case (3). Once the h-function in $\mathrm{HFL}^-(s_1, s_2)$ is determined, its values in $\mathrm{HFL}^-(s_1, s_2+1)$ and $\mathrm{HFL}^-(s_1+1, s_2)$ are also determined by Corollary 2.6. Thus there are two possibilities for the h-function corresponding to $\widehat{\mathrm{HFL}}(s_1, s_2)$:

$h+1$	h	h
$h+2$	$h+1$	h
$h+2$	$h+2$	$h+1$

Case (1a)

$h+1$	h	h
$h+2$	$h+1$	h
$h+3$	$h+2$	$h+1$

Case (1b)

In both cases, we have $\mathrm{HFL}^-(s_1+1, s_2+1) = \mathbb{F}[-2h+1]$, $\mathrm{HFL}^-(s_1, s_2+1) = \mathbb{F}[-2h] \oplus \mathbb{F}[-2h-1]$ and $\mathrm{HFL}^-(s_1+1, s_2) = \mathbb{F}[-2h] \oplus \mathbb{F}[-2h-1]$. By Lemma 3.3, $U_1 a = b$ and $U_2 a = c$, where a is the generator in $\mathrm{HFL}^-(s_1+1, s_2+1)$, and b and c are generators with homological grading $-2h-1$ in $\mathrm{HFL}^-(s_1, s_2+1)$ and $\mathrm{HFL}^-(s_1+1, s_2)$, respectively. So the image of a under the differential d_1 is nonzero, and a does not survive in the E^2-page. Thus d_2 is trivial in both Case (1a) and Case (1b).

In Case (2), $\mathrm{HFL}^-(s_1+1, s_2+1) = \mathbb{F}[-2h]$. In order to have nontrivial d_2, $\mathrm{HFL}^-(s_1, s_2)$ must contain a generator with homological grading $[-2h-3]$. So the h-function in $\mathrm{HFL}^-(s_1, s_2)$ must have the pattern in Case (3). Then $\mathrm{HFL}^-(s_1, s_2) \cong \mathbb{F}[-2h-2] \oplus \mathbb{F}[-2h-3]$. Corresponding to this case, there are four possibilities for the h-function in $\widehat{\mathrm{HFL}}(s_1, s_2)$:

$h+1$	$h+1$	h
$h+2$	$h+1$	$h+1$
$h+3$	$h+2$	$h+1$

Case (2a)

$h+2$	$h+1$	h
$h+2$	$h+1$	$h+1$
$h+3$	$h+2$	$h+2$

Case (2b)

$h+2$	$h+1$	h
$h+2$	$h+1$	$h+1$
$h+3$	$h+2$	$h+1$

Case (2c)

$h+1$	$h+1$	h
$h+2$	$h+1$	$h+1$
$h+3$	$h+2$	$h+2$

Case (2d)

We use the symmetric property of Heegaard Floer link homology to compute $\widehat{\mathrm{HFL}}(s_1, s_2)$. Let $h^* = h(-s_1, -s_2)$. By Remark 3.6,

$$h(-s_1, -s_2-1) - h(-s_1, -s_2) = 1 - (h(s_1, s_2) - h(s_1, s_2+1))$$

and

$$h(-s_1-1, -s_2) - h(-s_1, -s_2) = 1 - (h(s_1, s_2) - h(s_1+1, s_2)).$$

So the h-function in $\widehat{\mathrm{HFL}}(-s_1, -s_2)$ corresponding to these four subcases are

h^*	h^*	h^*
h^*+1	h^*	h^*
h^*+1	h^*+1	h^*

dual-h (2a)

h^*+1	h^*	h^*
h^*+1	h^*	h^*
h^*+1	h^*+1	h^*+1

dual-h (2b)

h^*	h^*	h^*
h^*+1	h^*	h^*
h^*+1	h^*+1	h^*+1

dual-h (2c)

h^*+1	h^*	h^*
h^*+1	h^*	h^*
h^*+1	h^*+1	h^*

dual-h (2d)

Note that in all these four cases for $\widehat{\mathrm{HFL}}(-s_1, -s_2)$, $\mathrm{HFL}^-(-s_1+1, -s_2+1) = 0$. So $d_2 = 0$ in the spectral sequence corresponding to $\widehat{\mathrm{HFL}}(-s_1, -s_2)$. Now the computation of $\widehat{\mathrm{HFL}}(-s_1, -s_2)$ is quite straightforward.

In dual-h (2a),

$$\widehat{\mathrm{HFL}}(-s_1,-s_2)\cong\left[\begin{tikzcd} \mathbb{F}[-2h^*+1] \arrow[d, "U_2"'] & 0 \arrow[l, "U_1"] \arrow[d, "U_2"'] \\ \mathbb{F}[-2h^*] & \mathbb{F}[-2h^*+1] \arrow[l, "U_1"] \end{tikzcd}\right]$$

By grading reasons, $d_2=U_1=U_2=0$. Then it is easy to obtain $\widehat{\mathrm{HFL}}(-s_1,-s_2)\cong\mathbb{F}[-2h^*]\oplus\mathbb{F}[-2h^*]\oplus\mathbb{F}[-2h^*]$, and the Euler characteristic $\chi=3$. By Lemma 3.5, $\widehat{\mathrm{HFL}}(s_1,s_2)$ contains 3 generators with the same total grading. Observe that $\mathrm{HFL}^-(s_1,s_2)=\mathbb{F}[-2h-2]\oplus\mathbb{F}[-2h-3]$. Then the generator with total grading $-2h-2$ survives in $\widehat{\mathrm{HFL}}(s_1,s_2)$. Thus

$$\widehat{\mathrm{HFL}}(s_1,s_2)\cong\mathbb{F}[-2h-2]\oplus\mathbb{F}[-2h-2]\oplus\mathbb{F}[-2h-2],$$

and the Euler characteristic χ is 3.

In dual-h (2b),

$$\widehat{\mathrm{HFL}}(-s_1,-s_2)\cong\left[\begin{tikzcd} 0 \arrow[d, "U_2"'] & 0 \arrow[l, "U_1"] \arrow[d, "U_2"'] \\ \mathbb{F}[-2h^*] & 0 \arrow[l, "U_1"] \end{tikzcd}\right]$$

In this case, $\widehat{\mathrm{HFL}}(-s_1,-s_2)\cong\mathbb{F}[-2h^*]$. By an argument similar to the one in dual-h (2a), we obtain that $\widehat{\mathrm{HFL}}(L)(s_1,s_2)\cong\mathbb{F}[-2h-2]$, and the Euler characteristic χ is 1.

In dual-h (2c),

$$\widehat{\mathrm{HFL}}(-s_1,-s_2)\cong\left[\begin{tikzcd} \mathbb{F}[-2h^*+1] \arrow[d, "U_2"'] & 0 \arrow[l, "U_1"] \arrow[d, "U_2"'] \\ \mathbb{F}[-2h^*] & 0 \arrow[l, "U_1"] \end{tikzcd}\right]$$

By grading reasons, $d_2=U_1=U_2=0$. Then $\widehat{\mathrm{HFL}}(-s_1,-s_2)\cong\mathbb{F}[-2h^*]\oplus\mathbb{F}[-2h^*]$. So $\widehat{\mathrm{HFL}}(s_1,s_2)\cong\mathbb{F}[-2h-2]\oplus\mathbb{F}[-2h-2]$, and the Euler characteristic is $\chi=2$.

In dual-h (2d),

$$\widehat{\mathrm{HFL}}(-s_1,-s_2)\cong\left[\begin{tikzcd} 0 \arrow[d, "U_2"'] & 0 \arrow[l, "U_1"] \arrow[d, "U_2"'] \\ \mathbb{F}[-2h^*] & \mathbb{F}[-2h^*+1] \arrow[l, "U_1"] \end{tikzcd}\right]$$

Hence, $\widehat{\mathrm{HFL}}(L)(s_1,s_2)\cong\mathbb{F}[-2h-2]\oplus\mathbb{F}[-2h-2]$, and the Euler characteristic is $\chi=2$.

Now we consider Case (3). In this case, we have $\mathrm{HFL}^-(s_1+1, s_2+1) \cong \mathbb{F}[-2h] \oplus \mathbb{F}[-2h-1]$. Then there are three possibilities for $\mathrm{HFL}^-(s_1, s_2)$ if d_2 is nontrivial: $\mathrm{HFL}^-(s_1, s_2)$ is either $\mathbb{F}[-2h-4]$ or $\mathbb{F}[-2h-4] \oplus \mathbb{F}[-2h-5]$ or $\mathbb{F}[-2h-3]$. If $\mathrm{HFL}^-(s_1, s_2) = \mathbb{F}[-2h-4]$, its h-function is shown in Case (3a), and if $\mathrm{HFL}^-(s_1, s_2) \cong \mathbb{F}[-2h-4] \oplus \mathbb{F}[-2h-5]$, its h-function is shown in Case (3b):

$h+2$	$h+1$	h
$h+3$	$h+2$	$h+1$
$h+3$	$h+3$	$h+2$

Case (3a)

$h+2$	$h+1$	h
$h+3$	$h+2$	$h+1$
$h+4$	$h+3$	$h+2$

Case (3b)

In Case (3a) and Case (3b), we observe that both generators in $\mathrm{HFL}^-(s_1+1, s_2+1)$ have nontrivial images in $\mathrm{HFL}^-(s_1, s_2+1)$ and $\mathrm{HFL}^-(s_1+1, s_2)$ by Lemma 3.3. So these two generators have nontrivial images under the differential d_1, and cannot survive in the E^2-page. Thus d_2 is trivial in both cases.

If $\mathrm{HFL}^-(s_1, s_2) \cong \mathbb{F}[-2h-3]$, there are four possibilities for the h-function corresponding to $\widehat{\mathrm{HFL}}(s_1, s_2)$:

$h+1$	$h+1$	h
$h+2$	$h+2$	$h+1$
$h+3$	$h+2$	$h+1$

Case (3c)

$h+2$	$h+1$	h
$h+2$	$h+2$	$h+1$
$h+3$	$h+2$	$h+2$

Case (3d)

$h+1$	$h+1$	h
$h+2$	$h+2$	$h+1$
$h+3$	$h+2$	$h+2$

Case (3e)

$h+2$	$h+1$	h
$h+2$	$h+2$	$h+1$
$h+3$	$h+2$	$h+1$

Case (3f)

Let $h^* = h(-s_1, -s_2) = h(s_1, s_2) + s_1 + s_2$. By Remark 3.6, we find the h-function in $\widehat{\mathrm{HFL}}(-s_1, -s_2)$ corresponding to each case:

h^*-1	h^*-1	h^*-1
h^*	h^*	h^*-1
h^*	h^*	h^*-1

dual-h (3c)

h^*	h^*-1	h^*-1
h^*	h^*	h^*-1
h^*	h^*	h^*

dual-h (3d)

h^*	h^*-1	h^*-1
h^*	h^*	h^*-1
h^*	h^*	h^*-2

dual-h (3e)

h^*-1	h^*-1	h^*-1
h^*	h^*	h^*-1
h^*	h^*	h^*

dual-h (3f)

Observe that in these four cases, $\mathrm{HFL}^-(-s_1, -s_2) = 0$. So d_2 is trivial in the spectral sequence corresponding to $\widehat{\mathrm{HFL}}(-s_1, -s_2)$. We compute $\widehat{\mathrm{HFL}}(-s_1, -s_2)$, and hence $\widehat{\mathrm{HFL}}(s_1, s_2)$.

In dual-h (3c), $\widehat{\mathrm{HFL}}(-s_1, -s_2) \cong \mathbb{F}[-2h^* + 1]$. By Lemma 3.5, $\widehat{\mathrm{HFL}}(s_1, s_2) \cong \mathbb{F}[-2h - 3]$ with the Euler characteristic $\chi = -1$.

In Case (3d), $\widehat{\mathrm{HFL}}(L)(s_1, s_2) \cong \mathbb{F}[-2h-3] \oplus \mathbb{F}[-2h-3] \oplus \mathbb{F}[-2h-3]$, and the Euler characteristic is $\chi = -3$ by a similar computation.

In Case (3e), $\widehat{\mathrm{HFL}}(L)(s_1, s_2) \cong \mathbb{F}[-2h-3] \oplus \mathbb{F}[-2h-3]$, and the Euler characteristic is $\chi = -2$.

In Case (3f), $\widehat{\mathrm{HFL}}(L)(s_1, s_2) \cong \mathbb{F}[-2h-3] \oplus \mathbb{F}[-2h-3]$, and the Euler characteristic is $\chi = -2$.

Thus we conclude that for any L-space link $L = L_1 \cup L_2$ with two components, if the h-function is determined, we can compute $\widehat{\mathrm{HFL}}(s_1, s_2)$ with any $(s_1, s_2) \in \mathbb{H}$. By equations in Section 2B, the h-function is determined by Alexander polynomials $\Delta_L(x_1, x_2)$, $\Delta_{L_1}(t)$, $\Delta_{L_2}(t)$ and the linking number $\mathrm{lk}(L_1, L_2)$. □

Furthermore, we also get a bound for $\mathrm{rank}_{\mathbb{F}}(\widehat{\mathrm{HFL}}(s_1, s_2))$ and the Euler characteristic $\chi(\widehat{\mathrm{HFL}}(s_1, s_2))$ with any $(s_1, s_2) \in \mathbb{H}$.

Proof of Corollary 1.4. Consider the short exact sequence

$$0 \to \mathrm{CFL}^-(s_1+1, s_2+1) \xrightarrow{U_1} \mathrm{CFL}^-(s_1, s_2+1) \to C_1(s_1, s_2+1) \to 0, \tag{3-1}$$

where $C_1(s_1, s_2+1)$ is the quotient complex with $(s_1, s_2+1) \in \mathbb{H}$. By Lemma 3.1,

$$\widehat{\mathrm{CFL}}(s_1, s_2) \cong C_1(s_1, s_2)/U_2(C_1(s_1, s_2+1)). \tag{3-2}$$

Now we claim that $\mathrm{rank}_{\mathbb{F}}(H_*(C_1(s_1, s_2+1))) \leq 2$ for any $(s_1, s_2) \in \mathbb{H}$. From the short exact sequence (3-1), we have

$$\begin{aligned} &\mathrm{rank}_{\mathbb{F}}(H_*(C_1(s_1, s_2+1))) \\ &\qquad \leq \mathrm{rank}_{\mathbb{F}}(\mathrm{HFL}^-(s_1+1, s_2+1)) + \mathrm{rank}_{\mathbb{F}}(\mathrm{HFL}^-(s_1, s_2+1)). \end{aligned} \tag{3-3}$$

If $\mathrm{rank}_{\mathbb{F}}(H_*(C_1(s_1, s_2+1))) \geq 3$, then at least one of $\mathrm{HFL}^-(s_1+1, s_2+1)$ and $\mathrm{HFL}^-(s_1, s_2+1)$ should have rank at least 2, and the other one should have rank at least 1. By the computation in Section 2A, the h-functions corresponding to $\mathrm{HFL}^-(s_1+1, s_2+1)$ and $\mathrm{HFL}^-(s_1, s_2+1)$ have the following possibilities:

$h+1$	h	$h-1$
$h+1$	$h+1$	h

Case (1)

$h+1$	h	h
$h+2$	$h+1$	h

Case (2)

$h+1$	h	$h-1$
$h+2$	$h+1$	h

Case (3)

Here we assume that the generator of $H_*(A^-(s_1, s_2+1))$ has homological grading $-2h$. In Case (1), we have $U_1 : \mathbb{F}[-2h+2] \oplus \mathbb{F}[-2h+1] \to \mathbb{F}[-2h]$. Let α denote the generator of $\mathbb{F}[-2h+2] \subseteq \mathrm{HFL}^-(s_1+1, s_2+1)$, and β the generator of $\mathbb{F}[-2h] \cong \mathrm{HFL}^-(s_1, s_2+1)$. By Lemma 3.3, $U(\alpha) = \beta$. Then $H_*(C_1(s_1)) \cong \mathbb{F}[-2h+1]$, and the rank in this case is 1.

In Case (2), we have $U_1 : \mathbb{F}[-2h+1] \to \mathbb{F}[-2h] \oplus \mathbb{F}[-2h-1]$. Similarly $H_*(C_1(s_1, s_2+1)) \cong \mathbb{F}[-2h]$, and it has rank 1.

In Case (3), we have $U_1 : \mathbb{F}[-2h+2] \oplus \mathbb{F}[-2h+1] \to \mathbb{F}[-2h] \oplus \mathbb{F}[-2h-1]$. By Lemma 3.3, $H_*(C_1(s_1, s_2+1)) = 0$.

Thus for any $(s_1, s_2) \in \mathbb{H}$, $\mathrm{rank}_{\mathbb{F}}(H_*(C_1(s_1, s_2+1))) \le 2$. By (3-2),

$$\mathrm{rank}_{\mathbb{F}}(\widehat{\mathrm{HFL}}(s_1, s_2)) \le \mathrm{rank}_{\mathbb{F}}(H_*(C_1(s_1, s_2+1))) + \mathrm{rank}_{\mathbb{F}}(H_*(C_1(s_1, s_2))) \le 2+2 = 4$$

for any $(s_1, s_2) \in \mathbb{H}$. Therefore, $-4 \le \chi(\widehat{\mathrm{HFL}}(L, s_1, s_2)) \le 4$. □

In fact, we construct an example with $\chi(\widehat{\mathrm{HFL}}(L, s_1, s_2)) = -4$, given in the proof of Theorem 1.2, where $d_2 = 0$. Similarly, we construct an example with $\chi(\widehat{\mathrm{HFL}}(L, s_1, s_2)) = 4$.

Example 3.7. Assume that the h-function corresponding to $\widehat{\mathrm{HFL}}(s_1, s_2)$ is the following:

$h+1$	$h+1$	h
$h+2$	$h+1$	$h+1$
$h+2$	$h+2$	$h+1$

In this case, $\widehat{\mathrm{HFL}}(s_1, s_2) \cong \mathbb{F}[-2h-2] \oplus \mathbb{F}[-2h-2] \oplus \mathbb{F}[-2h-2] \oplus \mathbb{F}[-2h-2]$, and hence $\chi(\widehat{\mathrm{HFL}}(s_1, s_2)) = 4$.

Example 3.8. Figure 3 depicts the two-bridge link $b(20, -3)$.

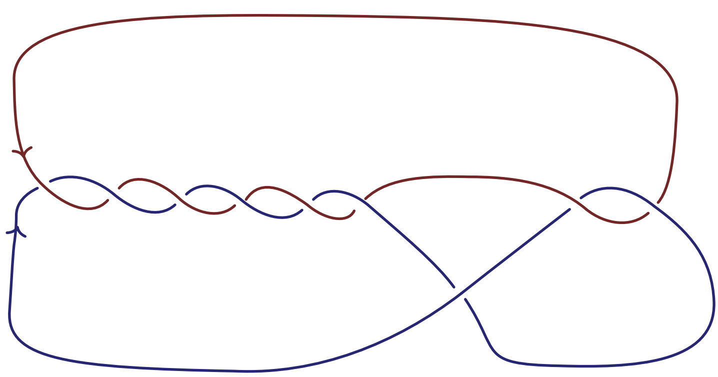

Figure 3. $b(20, -3)$.

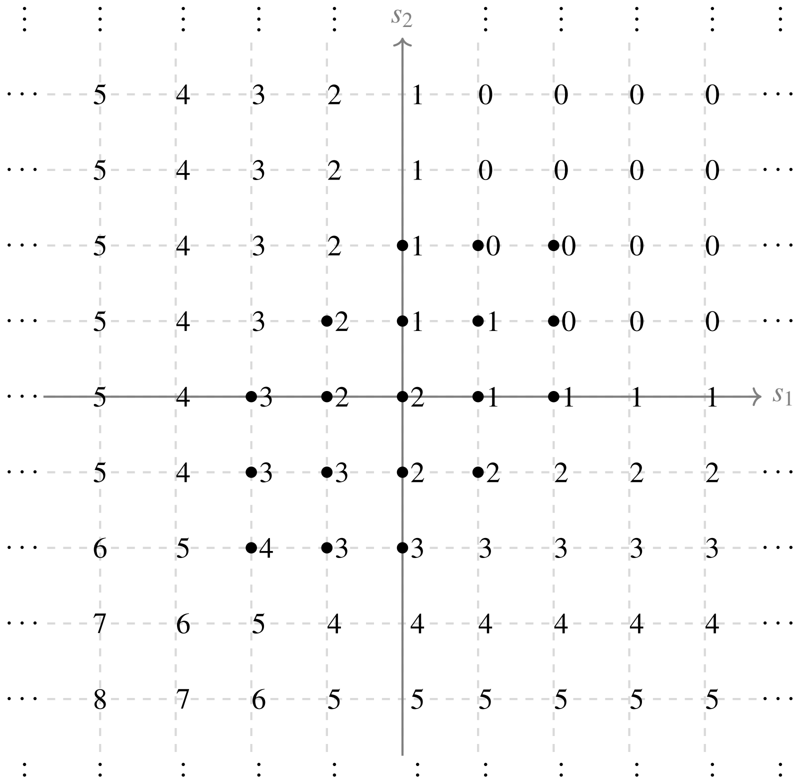

Figure 4. The h-function for $b(20, -3)$.

Yajing Liu proved that $b(20, -3)$ is an L-space link [2017, Theorem 3.8]. Its two components are both unknots with linking number 2. By [Dawra 2015], its normalized multivariable Alexander polynomial is

$$\text{(3-4)}\quad \Delta_L(t_1, t_2) = t_1^{1/2}t_2^{3/2} + t_1^{3/2}t_2^{1/2} + t_1^{1/2}t_2^{-1/2} + t_1^{-1/2}t_2^{1/2} + t_1^{-3/2}t_2^{-1/2} + t_1^{-1/2}t_2^{-3/2} - t_1^{3/2}t_2^{3/2} - t_1^{1/2}t_2^{1/2} - t_1^{-1/2}t_2^{-1/2} - t_1^{-3/2}t_2^{-3/2}.$$

Let L_1 and L_2 denote the unknot components. We obtain normalized Alexander polynomials of L_1 and L_2:

$$\frac{t}{t-1}\Delta_{L_1}(t) = \frac{t}{t-1}\Delta_{L_2}(t) = 1 + t^{-1} + t^{-2} + t^{-3} + t^{-4} + \cdots .$$

Using results of Section 2B, we compute the h-function for $\widehat{\mathrm{HFL}}(s_1, s_2)$ with any $(s_1, s_2) \in \mathbb{H}$ by Alexander polynomials. The h-function is shown in Figure 4, where numbers denote $h(s_1, s_2)$ for any $(s_1, s_2) \in \mathbb{H}$. For example, $h(0, 0) = h(-1, 0) = 2$. The black dots $\bullet$ denote the lattice points $(s_1, s_2) \in \mathbb{H}$ where $\widehat{\mathrm{HFL}}(s_1, s_2)$ is nonzero.

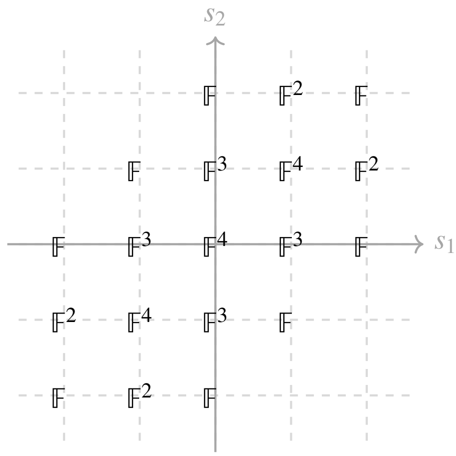

Figure 5. $\widehat{\mathrm{HFL}}(b(20,-3))$.

By an explicit computation, the link Floer homology $\widehat{\mathrm{HFL}}(s_1, s_2)$ is shown in Figure 5. We observe that $|\chi(s_1, s_2)| = \mathrm{rank}_{\mathbb{F}}(\widehat{\mathrm{HFL}}(s_1, s_2))$, and the rank of $\widehat{\mathrm{HFL}}(s_1, s_2)$ ranges from 0 to 4. This indicates that the bound for the rank in Corollary 1.4 can be realized by some L-space link with some $(s_1, s_2) \in \mathbb{H}$. More precisely, $\mathrm{rank}_{\mathbb{F}}(\widehat{\mathrm{HFL}}(2,2)) = 1$, $\mathrm{rank}_{\mathbb{F}}(\widehat{\mathrm{HFL}}(2,1)) = 2$, $\mathrm{rank}_{\mathbb{F}}(\widehat{\mathrm{HFL}}(1,0)) = 3$, $\mathrm{rank}_{\mathbb{F}}(\widehat{\mathrm{HFL}}(0,0)) = 4$ and $\mathrm{rank}_{\mathbb{F}}(\widehat{\mathrm{HFL}}(3,0)) = 0$.

4. An application of $\widehat{\mathrm{HFL}}$ to the Thurston norm

The Thurston norm was studied by many people, and some lower bounds were obtained in [McMullen 2002; Friedl and Kim 2008; Friedl and Vidussi 2015; Agol and Dunfield 2015]. Ozsváth and Szabó [2008b] showed that the link Floer homology detects the Thurston norm of the link complement. In Section 3, for any 2-component L-space link $L = L_1 \cup L_2$ and $\boldsymbol{s} \in \mathbb{H}$, we computed $\widehat{\mathrm{HFL}}(L, \boldsymbol{s})$ by using Alexander polynomials $\Delta_L(t_1, t_2)$, $\Delta_{L_1}(t)$, $\Delta_{L_2}(t)$ and the linking number $\mathrm{lk}(L_1, L_2)$. Thus we can compute the link Floer homology polytope for L, and also compute the dual Thurston polytope and the Thurston (semi)norm [Ozsváth and Szabó 2008b, Theorem 1.1].

In Section 1, we introduced complexity $\chi_-(F)$ for any compact oriented surface F with boundary. To any link $L \subseteq S^3$, and any homology class $h \in H_2(S^3, L)$, we can assign a function

$$x(h) = \min_{\{F \hookrightarrow S^3 \backslash \mathrm{nd}(L),\, [F]=h\}} \chi_-(F).$$

This function can be naturally extended to a seminorm, the *Thurston seminorm*, denoted by $x : H_2(S^3, L; \mathbb{R}) \to \mathbb{R}$.

Theorem 4.1 [Thurston 1986, Theorem 1]. *The function* $x : H_2(S^3, L; \mathbb{R}) \to \mathbb{R}$ *is a seminorm that vanishes exactly on the subspace spanned by embedded surfaces of nonnegative Euler characteristic.*

Assume that $L \subseteq S^3$ is a link with l components in S^3. Let u_i denote the meridian of the i-th component L_i of L. Recall that every lattice point $\boldsymbol{s} \in \mathbb{H}$ can be written as

$$\sum_{i=1}^{l} s_i \cdot [u_i],$$

where $s_i \in \mathbb{Q}$ satisfies the property that

$$2s_i + \mathrm{lk}(L_i, L - L_i)$$

is an even integer for $i = 1, \ldots, l$.

In [Ozsváth and Szabó 2008b], the Heegaard Floer link homology provides a function $y : H^1(S^3 - L; \mathbb{R}) \to \mathbb{R}$ defined by the formula

$$y(h) = \max_{\{\boldsymbol{s} \in \mathbb{H} \subseteq H_1(S^3 - L; \mathbb{R}) \,|\, \widehat{\mathrm{HFL}}(L, \boldsymbol{s}) \neq 0\}} |\langle \boldsymbol{s}, h\rangle|.$$

Ozsváth and Szabó proved the following formula for the link Floer homology and the Thurston norm.

Theorem 4.2 [Ozsváth and Szabó 2008b, Theorem 1.1]. *For an oriented link* $L \subseteq S^3$ *with no trivial components*, *the Heegaard Floer link homology detects the Thurston* (*semi*)*norm of its complement. For each* $h \in H^1(S^3 - L; \mathbb{R})$, *we have*

$$x(\mathrm{PD}[h]) + \sum_{i=1}^{l} |\langle h, u_i\rangle| = 2y(h),$$

where u_i *is the meridian of the* i*-th component of* L *and* $|\langle h, u_i\rangle|$ *denotes the absolute value of the Kronecker pairing of* $h \in H^1(S^3 - L; \mathbb{R})$ *and* $u_i \in H_1(S^3 - L; \mathbb{R})$.

Remark 4.3. A trivial component of a link L is an unknot component which is also unlinked from the rest of the link.

The unit ball for the norm x is called the *Thurston polytope*, and the unit ball for the norm y is called the *link Floer homology polytope*, which is also the convex hull of those $\boldsymbol{s} \in \mathbb{H}$ for which $\widehat{\mathrm{HFL}}(L, \boldsymbol{s}) \neq 0$. The unit ball for the dual norm x^* of x in $H_1(S^3 - L; \mathbb{R})$ is called the *dual Thurston polytope*. By Theorem 4.2, twice the link Floer homology polytope can be written as the sum of the dual Thurston polytope and an element of the symmetric hypercube in $H^1(S^3 - L)$ with edge-length two [Ozsváth and Szabó 2008b]. We give some examples of L-space links with two components, and compute their link Floer homology polytopes by using Alexander polynomials and linking numbers in detail. Moreover, we compute the

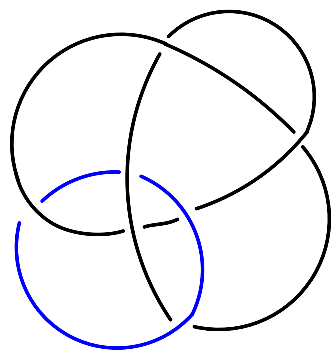

Figure 6. $L7n1$.

dual Thurston polytopes and Thurston norms of their complements by Theorem 4.2. We also compare the link Floer homology polytope and the convex hull of those $\boldsymbol{s} \in \mathbb{H}$ for which $\chi(\widehat{\mathrm{HFL}}(L, \boldsymbol{s})) \neq 0$.

Example 4.4 (the dual Thurston polytope for the L-space link $L7n1$). The link $L7n1$ in Figure 6 is an L-space link [Liu 2017, Example 3.17]. The link component L_1 is an unknot and the other link component L_2 is a right-handed trefoil. The linking number is 2 and its multivariable Alexander polynomial is

$$\Delta_L(t_1, t_2) = t_1^{1/2} t_2^{3/2} + t_1^{-1/2} t_2^{-3/2}.$$

Normalized Alexander polynomials of L_1 and L_2 are

$$\frac{t}{t-1}\Delta_{L_1}(t) = 1 + t^{-1} + t^{-2} + t^{-3} + t^{-4} + \cdots,$$
$$\frac{t}{t-1}\Delta_{L_2}(t) = t + t^{-1} + t^{-2} + t^{-3} + t^{-4} + \cdots.$$

The h-function in $\widehat{\mathrm{HFL}}(s_1, s_2)$ is shown in Figure 7. In this figure, the numbers denote the h-function, and $\bullet$ denotes the lattice points $(s_1, s_2) \in \mathbb{H}$ where $\widehat{\mathrm{HFL}}(s_1, s_2)$ is nonzero. By an explicit computation, the link Floer homology $\widehat{\mathrm{HFL}}(s_1, s_2)$ is shown in Figure 8. Moreover, $\widehat{\mathrm{HFL}}(0, 0) \cong \mathbb{F}[-2] \oplus \mathbb{F}[-3]$, so $\chi(\widehat{\mathrm{HFL}}(0, 0))$ is zero. For any other lattice point (s_1, s_2) labeled by $\bullet$ except $(0, 0)$, $\widehat{\mathrm{HFL}}(s_1, s_2)$ has rank one and $\chi(\widehat{\mathrm{HFL}}(s_1, s_2))$ is also nonzero. Thus in this example, the link Floer homology polytope is the same as the convex hull of those $(s_1, s_2) \in \mathbb{H}$ for which $\chi(\widehat{\mathrm{HFL}}(s_1, s_2))$ are nonzero. By Theorem 4.2, the dual Thurston polytope in $H_1(S^3 - L; \mathbb{R})$ is shown in Figure 9.

In Figure 9, the thick red line is the dual Thurston polytope for $L7n1$. It is the same as the Newton polytope of the Alexander polynomial $\Delta_L(t_1, t_2)$. The

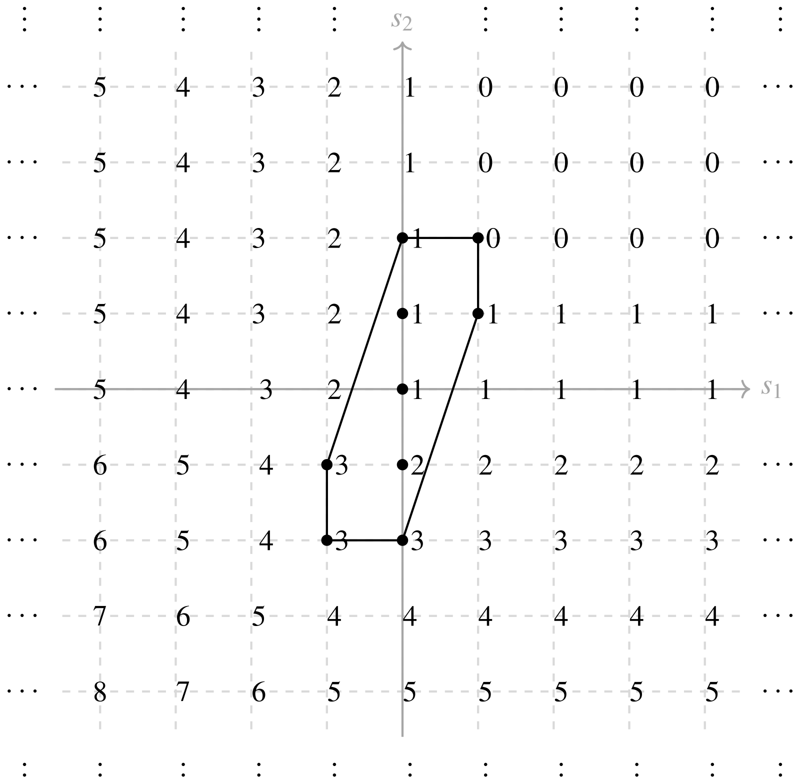

Figure 7. The h-function for $L7n1$.

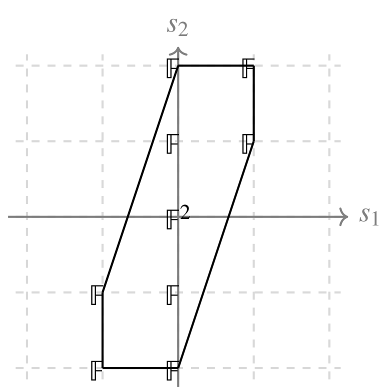

Figure 8. The link Floer homology polytope for $L7n1$.

unknot component of $L7n1$ bounds a surface F_{L_1} with Euler characteristic -1, and the right-handed trefoil link component L_2 bounds a surface F_{L_2} with Euler

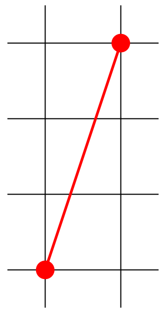

Figure 9. The dual Thurston polytope for $L7n1$.

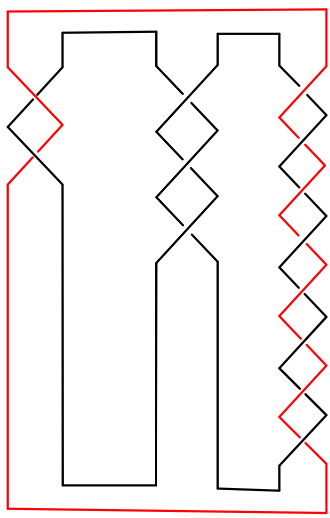

Figure 10. $b(-2, 3, 8)$.

characteristic -3. The surfaces F_{L_1} and F_{L_2} have maximal Euler characteristic in their respective homology classes.

Example 4.5 (the dual Thurston polytope for the pretzel link $L = b(-2, 3, 8)$). We claim that the pretzel link $b(-2, 3, 8)$ is an L-space link with two components. The link component L_1 is an unknot and the other link component L_2 is a right-handed trefoil as shown in Figure 10. The linking number of L_1 and L_2 is 5. Let P_1 be the knot obtained from $b(-2, 3, 8)$ by 1-Dehn surgery on L_1. It is the twisted torus knot $K(5, 6; 2, 1)$ [Remigio-Juárez and Rieck 2012, Proposition 3.1], and it is an L-space knot as proved by F. Vafaee [2015, Theorem 1]. Then for sufficiently large d, $S^3_{1,d}(L) = S^3_{d-25}(P_1)$ is an L-space. The link components L_1 and L_2 are

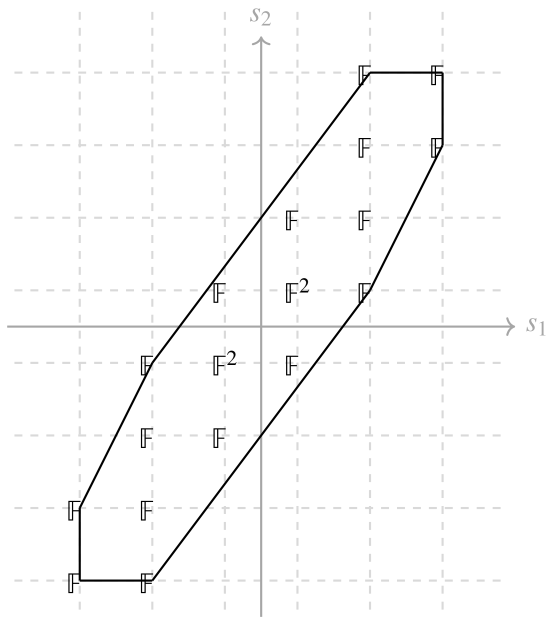

Figure 11. The link Floer homology polytope for $b(-2, 3, 8)$.

L-space knots, so $S^3_1(L_1)$ and $S^3_d(L_2)$ are both L-spaces. Observe that $d - 25 > 0$, so the pretzel link $b(-2, 3, 8)$ is an L-space link by L-space surgery criterion [Liu 2017, Lemma 2.6]. The symmetrized Alexander polynomial of $b(-2, 3, 8)$ is

$$\Delta_L(t_1, t_2) = t_1^{-2}t_2^{-3} + t_1^{-1}t_2^{-2} + 1 + t_1t_2 + t_1^2t_2^3.$$

The h-function corresponding to $\widehat{\mathrm{HFL}}(s_1, s_2)$ with $(s_1, s_2) \in \mathbb{H}$ is shown in Figure 13. By an explicit computation, the link Floer homology $\widehat{\mathrm{HFL}}(s_1, s_2)$ is as shown in Figure 11. We have $\mathrm{rank}_{\mathbb{F}}(\widehat{\mathrm{HFL}}(1/2, 1/2)) = \chi(\widehat{\mathrm{HFL}}(1/2, 1/2)) = 2$, and $\mathrm{rank}_{\mathbb{F}}(\widehat{\mathrm{HFL}}(-1/2, -1/2)) = \chi(\widehat{\mathrm{HFL}}(-1/2, -1/2)) = 2$. Observe that the link Floer homology polytope is the same as the convex hull of those $(s_1, s_2) \in \mathbb{H}$ for which $\chi(\widehat{\mathrm{HFL}}(s_1, s_2))$ are nonzero. By Theorem 4.2, the dual Thurston polytope is the shaded area in Figure 12.

Remark 4.6. For L-space links $L7n1$ and $b(-2, 3, 8)$, the Thurston polytopes are both dual to Newton polytopes of their symmetrized Alexander polynomials $\Delta_L(t_1, t_2)$. Ozsváth and Szabó [2008b] pointed out that the Thurston polytope of an alternating link is dual to the Newton polytope of its multivariable Alexander polynomial. This is also true for L-space knots. A natural question is whether the Thurston polytope of an L-space link with two components (which is not a split union of two L-space knots) is dual to the Newton polytope of its symmetrized Alexander polynomial.

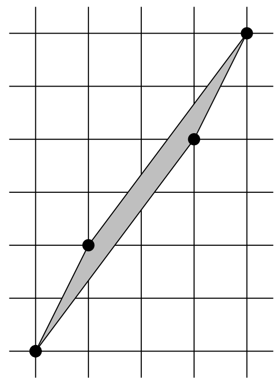

Figure 12. The dual Thurston polytope for $b(-2, 3, 8)$.

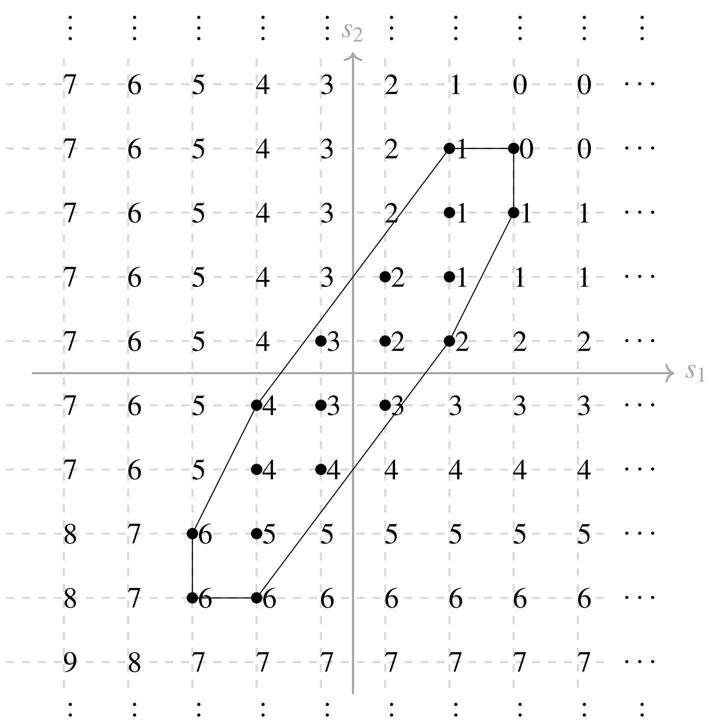

Figure 13. The h-function for $b(-2, 3, 8)$.

5. Two-component L-space links with vanishing Alexander polynomials

In Section 4, we have given examples of L-space links where Thurston polytopes are dual to Newton polytopes of their symmetrized Alexander polynomials. In this section, we mainly discuss 2-component L-space links with vanishing Alexander polynomials, especially split L-space links. Recall that multivariable Alexander polynomials for split links are 0. So Newton polytopes for split L-space links are empty, but link Floer homology polytopes may be nontrivial. To see this in detail, we need some lemmas first.

Lemma 5.1 [Liu 2017, Example 1.13(A)]. *Split disjoint unions of L-space knots are L-space links.*

Lemma 5.2 [Borodzik and Gorsky 2016, Proposition 3.11]. *For a split L-space link $L = L_1 \sqcup L_2$ with two components which are both L-space knots and $(s_1, s_2) \in \mathbb{H}$, the h-function $h(s_1, s_2)$ satisfies*

$$h(s_1, s_2) = h_1(s_1) + h_2(s_2),$$

where $h_1(s_1)$ and $h_2(s_2)$ are h-functions of L_1 and L_2, respectively.

Remark 5.3. L-space knots can be regarded as special L-space links with just one component. For any L-space knot $K \subseteq S^3$, we can associate it with a chain complex $A^-(s_1)$ filtered by the Alexander grading, and $H_*(A^-(s_1))$ has a unique generator for any s_1. Let $-2h(s_1)$ be the homological grading of the generator.

Proposition 5.4. *Let $L = L_1 \sqcup L_2$ be a split union of two L-space knots L_1 and L_2. Then $\widehat{\mathrm{HFL}}(L, s_1, s_2) \cong \widehat{\mathrm{HFL}}(L_1, s_1) \otimes \widehat{\mathrm{HFL}}(L_2, s_2) \otimes (\mathbb{F} \oplus \mathbb{F}_{(-1)})$ for any $(s_1, s_2) \in \mathbb{H}$.*

Proof. The proof is quite straightforward using our computation of $\widehat{\mathrm{HFL}}(s_1, s_2)$ in Section 3. For any $(s_1, s_2) \in \mathbb{H}$, the h-function corresponding to $\widehat{\mathrm{HFK}}(L_1, s_1)$ has the following possibilities:

$$\begin{array}{ccc} x & x & x \\ \bullet & \bullet & \bullet \\ s_1 - 1 & s_1 & s_1 + 1 \end{array} \qquad \begin{array}{ccc} x+1 & x & x \\ \bullet & \bullet & \bullet \\ s_1 - 1 & s_1 & s_1 + 1 \end{array}$$

Case (1) Case (2)

$$\begin{array}{ccc} x & x & x-1 \\ \bullet & \bullet & \bullet \\ s_1 - 1 & s_1 & s_1 + 1 \end{array} \qquad \begin{array}{ccc} x+1 & x & x-1 \\ \bullet & \bullet & \bullet \\ s_1 - 1 & s_1 & s_1 + 1 \end{array}$$

Case (3) Case (4)

Here $h_1(s_1) = x$ and x is any positive integer. Observe that

$$H_*(A^-(s_1)/A^-(s_1 - 1)) \cong \mathrm{HFK}^-(L_1, s_1),$$

$$\cdots \to \mathrm{HFK}^-_{i+2}(s_1 + 1) \xrightarrow{U} \mathrm{HFK}^-_i(s_1) \to \widehat{\mathrm{HFK}}_i(s_1) \\ \to \mathrm{HFK}^-_{i+1}(s_1 + 1) \xrightarrow{U} \mathrm{HFK}^-_{i-1}(s_1) \cdots .$$

The long exact sequence is induced by the short exact sequence

$$0 \to \mathrm{CFK}^-(s_1+1) \xrightarrow{U} \mathrm{CFK}^-(s_1) \to \widehat{\mathrm{CFK}}(s_1) \to 0.$$

By the long exact sequence, we compute $\widehat{\mathrm{HFL}}(L_1, s_1)$ as follows:

Case (1) $\widehat{\mathrm{HFK}}(L_1, s_1) \cong 0$.

Case (2) $\widehat{\mathrm{HFK}}(L_1, s_1) \cong \mathbb{F}[-2x]$.

Case (3) $\widehat{\mathrm{HFK}}(L_1, s_1) \cong \mathbb{F}[-2x+1]$.

Case (4) $\widehat{\mathrm{HFK}}(L_1, s_1) \cong 0$.

Similarly, for the link component L_2, we assume that $h_2(s_2) = y$. There are also four possibilities for the h-function corresponding to $\widehat{\mathrm{HFK}}(L_2, s_2)$. By Lemma 5.2, $h(s_1, s_2) = h_1(s_1) + h_2(s_2)$. We find that there are only four possibilities for the h-function such that $\widehat{\mathrm{HFL}}(L, s_1, s_2) \neq 0$:

$h+1$	h	h
$h+1$	h	h
$h+2$	$h+1$	$h+1$

Case (a)

$h+1$	h	h
$h+2$	$h+1$	$h+1$
$h+2$	$h+1$	$h+1$

Case (b)

$h+1$	$h+1$	h
$h+1$	$h+1$	h
$h+2$	$h+2$	$h+1$

Case (c)

$h+1$	$h+1$	h
$h+2$	$h+2$	$h+1$
$h+2$	$h+2$	$h+1$

Case (d)

In Case (a), h-functions for L_1 and L_2 are both like Case (2): $(x+1) \quad x \quad x$ and $(y+1) \quad y \quad y$. Then $\widehat{\mathrm{HFL}}(s_1, s_2) \cong \mathbb{F}[-2(x+y)] \oplus \mathbb{F}[-2(x+y)-1]$, $\widehat{\mathrm{HFK}}(L_1, s_1) \cong \mathbb{F}[-2x]$ and $\widehat{\mathrm{HFK}}(L_2, s_2) \cong \mathbb{F}[-2y]$. So

$$\widehat{\mathrm{HFL}}(s_1, s_2) \cong \widehat{\mathrm{HFK}}(L_1, s_1) \otimes \widehat{\mathrm{HFK}}(L_2, s_2) \otimes (\mathbb{F} \oplus \mathbb{F}_{(-1)}). \tag{5-1}$$

In Case (b), the h-function for L_1 is like Case (2): $(x+1) \quad x \quad x$, and the h-function for L_2 is like Case (3): $y \quad y \quad y-1$. In Case (c), the h-function for L_1 is like Case (3), and for L_2, the h-function is like Case (2). In Case (d), h-functions for L_1 and L_2 are like Case (3). Thus we can compute (5-1) in these cases as well.

If the h-function corresponding to $\widehat{\mathrm{HFL}}(s_1, s_2)$ is not in these four cases, then $\widehat{\mathrm{HFL}}(s_1, s_2) = 0$, and at least one of $\widehat{\mathrm{HFK}}(L_1, s_1)$ and $\widehat{\mathrm{HFK}}(L_2, s_2)$ is zero. Thus the conclusion also holds. □

Proof of Theorem 1.8. Let $L = L_1 \cup L_2$ be an L-space link with vanishing Alexander polynomial. The linking number of L_1 and L_2 is 0 by (2-1). By Theorem 1.2, the Heegaard Floer link homology $\widehat{\mathrm{HFL}}(s_1, s_2)$ is determined by $\Delta_L(t_1, t_2)$, $\Delta_{L_1}(t)$ and $\Delta_{L_2}(t)$. So

$$\widehat{\mathrm{HFL}}(L, s_1, s_2) \cong \widehat{\mathrm{HFL}}(L_1 \sqcup L_2, s_1, s_2) \cong \widehat{\mathrm{HFK}}(L_1, s_1) \otimes \widehat{\mathrm{HFK}}(L_2, s_2) \otimes (\mathbb{F} \oplus \mathbb{F}_{(-1)})$$

for any $(s_1, s_2) \in \mathbb{H}$. $\square$

Example 5.5 (the link Floer homology polytope for the split disjoint union of two right-handed trefoils). Let $L = L_1 \sqcup L_2$ be the split disjoint union of two right-handed trefoils. Recall that the right-handed trefoil is an L-space knot with Alexander polynomial $\Delta_{L_1}(t) = t - 1 + t^{-1}$, and

$$\sum_{s_1 \in \mathbb{Z}} \chi(\mathrm{HFK}^-(L_1, s_1)) t^{s_1} = \frac{\Delta_{L_1}}{1 - t^{-1}} = t + t^{-1} + t^{-2} + t^{-3} + t^{-4} + \cdots .$$

Observe the short exact sequence $0 \to A^-(s_1 - 1) \to A^-(s_1) \to \mathrm{CFK}^-(s_1) \to 0$. We have

$$\mathrm{HFK}^-(L_1, s_1) = H_*(A^-(s_1)/A^-(s_1 - 1)),$$
$$\chi(\mathrm{HFK}^-(L_1, s_1)) = h_1(s_1 - 1) - h_1(s_1),$$

which is also the coefficient of t^{s_1} in $\Delta_{L_1}(t)/(1 - t^{-1})$. Since L_1 is an L-space knot, $h_1(s_1) = 0$ for sufficiently large $s_1 \gg 0$. So the h-function $h_1(s_1)$ can be determined as follows:

$$\ldots, 7, 6, 5, 4, 3, 2, 1, 1, 0, 0, 0, 0, 0, \ldots,$$

where $h_1(0) = h_1(-1) = 1$, $h_1(s) = 0$ if $s \geq 1$, and $h_1(s) = -s$ if $s \leq -1$. Similarly, for another right-handed trefoil L_2, the h-function $h_2(s_2)$ is the same as $h_1(s_1)$. By Proposition 5.4, we can find all $(s_1, s_2) \in \mathbb{H}$ where $\widehat{\mathrm{HFL}}(L, s_1, s_2)$ are nonzero. So

$$\begin{aligned}
\widehat{\mathrm{HFL}}(L, 1, 1) &= \mathbb{F}[0] \oplus \mathbb{F}[-1], \\
\widehat{\mathrm{HFL}}(L, 0, 1) = \widehat{\mathrm{HFL}}(L, 1, 0) &= \mathbb{F}[-1] \oplus \mathbb{F}[-2], \\
\widehat{\mathrm{HFL}}(L, -1, 1) = \widehat{\mathrm{HFL}}(L, 0, 0) = \widehat{\mathrm{HFL}}(L, 1, -1) &= \mathbb{F}[-2] \oplus \mathbb{F}[-3], \\
\widehat{\mathrm{HFL}}(L, -1, 0) = \widehat{\mathrm{HFL}}(L, 0, -1) &= \mathbb{F}[-3] \oplus \mathbb{F}[-4], \\
\widehat{\mathrm{HFL}}(L, -1, -1) &= \mathbb{F}[-4] \oplus \mathbb{F}[-5].
\end{aligned}$$

For other lattice points $(s_1, s_2) \in \mathbb{H}$, $\widehat{\mathrm{HFL}}(L, s_1, s_2) = 0$. Thus the link Floer homology polytope is the shaded square in Figure 14.

Remark 5.6. In general, let $L = L_1 \sqcup L_2$ be the split union of any two L-space knots. The genus of a knot K is defined as

$$g(K) = \min\{\mathrm{genus}(F) \mid F \subseteq S^3 \text{ is an oriented, embedded surface with } \partial F = K\}.$$

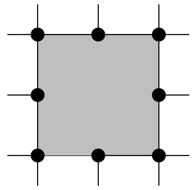

Figure 14. The link Floer homology polytope for L.

Observe that $g(L_i) = \max\{s \geq 0 \mid \widehat{\mathrm{HFK}}_*(L_i, s) \neq 0\}$ for $i = 1, 2$ [Ozsváth and Szabó 2004a, Theorem 1.2], and $\widehat{\mathrm{HFK}}(L_1, g(L_1)) \cong \mathbb{Z}, \widehat{\mathrm{HFK}}(L_2, g(L_2)) \cong \mathbb{Z}$, [Ozsváth and Szabó 2005, Theorem 1.2]. The link Floer homology polytope of L_i is the interval $[-g(L_i), g(L_i)]$, where $i = 1, 2$. By Proposition 5.4, the link Floer homology polytope for L is a rectangle with vertices $(g(L_1), g(L_2))$, $(g(L_1), -g(L_2))$, $(-g(L_1), g(L_2))$ and $(-g(L_1), -g(L_2))$ (see Figure 14).

Acknowledgements

I deeply appreciate Eugene Gorsky for introducing this interesting topic to me and his patient teaching on Heegaard Floer homology, and also for his constant guidance and discussions during the project. I am also grateful to Allison Moore, Yi Ni and Jacob Rasmussen for useful discussions on L-space links. I especially wish to thank the referee for helpful suggestions and corrections. The paper is inspired by the work of Yajing Liu, and the project is partially supported by NSF-1559338.

References

[Agol and Dunfield 2015] I. Agol and N. M. Dunfield, "Certifying the Thurston norm via SL(2, C)-twisted homology", 2015. To appear in Thurston memorial conference proceedings, Princeton Univ. Press. arXiv

[Borodzik and Gorsky 2016] M. Borodzik and E. Gorsky, "Immersed concordances of links and Heegaard Floer homology", preprint, 2016. arXiv

[Dawra 2015] N. Dawra, "On the link Floer homology of L-space links", preprint, 2015. arXiv

[Friedl and Kim 2008] S. Friedl and T. Kim, "Twisted Alexander norms give lower bounds on the Thurston norm", *Trans. Amer. Math. Soc.* **360**:9 (2008), 4597–4618. MR Zbl

[Friedl and Vidussi 2015] S. Friedl and S. Vidussi, "The Thurston norm and twisted Alexander polynomials", *J. Reine Angew. Math.* **707** (2015), 87–102. MR Zbl

[Gorsky and Némethi 2015] E. Gorsky and A. Némethi, "Lattice and Heegaard Floer homologies of algebraic links", *Int. Math. Res. Not.* **2015**:23 (2015), 12737–12780. MR Zbl

[Gorsky and Némethi 2016] E. Gorsky and A. Némethi, "Links of plane curve singularities are L-space links", *Algebr. Geom. Topol.* **16**:4 (2016), 1905–1912. MR Zbl

[Liu 2017] Y. Liu, "L-space surgeries on links", *Quantum Topol.* **8**:3 (2017), 505–570. MR Zbl

[Manolescu and Ozsváth 2010] C. Manolescu and P. Ozsváth, "Heegaard Floer homology and integer surgeries on links", preprint, 2010. arXiv

[McMullen 2002] C. T. McMullen, “The Alexander polynomial of a 3-manifold and the Thurston norm on cohomology”, *Ann. Sci. École Norm. Sup.* (4) **35**:2 (2002), 153–171. MR Zbl

[Ozsváth and Szabó 2004a] P. Ozsváth and Z. Szabó, “Holomorphic disks and genus bounds”, *Geom. Topol.* **8** (2004), 311–334. MR Zbl

[Ozsváth and Szabó 2004b] P. Ozsváth and Z. Szabó, “Holomorphic disks and topological invariants for closed three-manifolds”, *Ann. of Math.* (2) **159**:3 (2004), 1027–1158. MR Zbl

[Ozsváth and Szabó 2005] P. Ozsváth and Z. Szabó, “On knot Floer homology and lens space surgeries”, *Topology* **44**:6 (2005), 1281–1300. MR Zbl

[Ozsváth and Szabó 2006] P. Ozsváth and Z. Szabó, “Heegaard diagrams and Floer homology”, pp. 1083–1099 in *International Congress of Mathematicians, II*, edited by M. Sanz-Solé et al., Eur. Math. Soc., Zürich, 2006. MR Zbl arXiv

[Ozsváth and Szabó 2008a] P. Ozsváth and Z. Szabó, “Holomorphic disks, link invariants and the multi-variable Alexander polynomial”, *Algebr. Geom. Topol.* **8**:2 (2008), 615–692. MR Zbl

[Ozsváth and Szabó 2008b] P. Ozsváth and Z. Szabó, “Link Floer homology and the Thurston norm”, *J. Amer. Math. Soc.* **21**:3 (2008), 671–709. MR Zbl

[Remigio-Juárez and Rieck 2012] J. Remigio-Juárez and Y. Rieck, “The link volumes of some prism manifolds”, *Algebr. Geom. Topol.* **12**:3 (2012), 1649–1665. MR Zbl

[Thurston 1986] W. P. Thurston, “A norm for the homology of 3-manifolds”, pp. 99–130 in Mem. Amer. Math. Soc. **339**, Amer. Math. Soc., Providence, RI, 1986. MR Zbl

[Vafaee 2015] F. Vafaee, “On the knot Floer homology of twisted torus knots”, *Int. Math. Res. Not.* **2015**:15 (2015), 6516–6537. MR Zbl

Received May 9, 2017. Revised February 9, 2018.

BEIBEI LIU
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA
DAVIS, CA
UNITED STATES
bxliu@math.ucdavis.edu

See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2019 is US $490/year for the electronic version, and $665/year for print and electronic.
Subscriptions, requests for back issues and changes of subscriber address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

PACIFIC JOURNAL OF MATHEMATICS

Volume 298 No. 1 January 2019

0030-8730(201901)298:1;1-S