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ENHANCED ADJOINT ACTIONS AND THEIR ORBITS FOR THE GENERAL LINEAR GROUP

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We study an enhanced adjoint action of the general linear group on a product of its Lie algebra and a vector space consisting of several copies of defining representations and its duals. We determine regular semisimple orbits (i.e., closed orbits of maximal dimension) and the structure of enhanced null cone, including its irreducible components and their dimensions.

Introduction

Let G be a reductive algebraic group over the complex number field $\mathbb{C}$, and $\mathfrak{g}$ its Lie algebra. The adjoint action of G on $\mathfrak{g}$ is a basic tool for many aspects of representation theory, and is also useful for invariant theory, the theory of singularities, and so on.

Achar, Henderson and Johnson [Achar and Henderson 2008; Achar et al. 2011; Johnson 2010] considered an enhanced version of nilpotent varieties and classified the nilpotent orbits (there are only finitely many of them). Kato [2009] considered an "exotic" nilpotent cone and derived the Deligne–Langlands theory for those exotic nilpotent orbits. There are many related works based on algebraic geometry, combinatorial theory, and the theory of character sheaves [Travkin 2009; Finkelberg et al. 2009; Henderson and Trapa 2012; Fresse and Nishiyama 2016; Rosso 2012].

In these papers, enhancement of the nilpotent cone is only "one-sided" to get a criterion of finiteness of orbits. However, from the viewpoint of symmetric spaces and invariant theory, it seems better to enhance all the adjoint orbits in two-sided directions. In this respect, we already had two results that relate the orbit structure of two enhanced actions [Ohta 2008; Nishiyama 2014], but we did not know the explicit orbit structures of individual enhanced adjoint actions.

In this paper, we begin to study (two-sided) "enhanced adjoint action" of G for $G = \mathrm{GL}_n(\mathbb{C})$ (type A). The big difference from those one-sided enhanced (or exotic) ones is that there exist infinitely many nilpotent orbits. So the analysis becomes more difficult, but involves less combinatorics. In the easiest cases, we can

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describe enhanced adjoint orbits fairly explicitly, but in general, we have obtained coarser structures, like regular orbits of maximal possible dimensions, the structure of invariants, irreducible components of nilpotent variety.

To state the main results more explicitly, let us introduce some notation. Let $V = \mathbb{C}^n$ be a vector space of dimension n. We consider a natural action of $G = \mathrm{GL}(V) = \mathrm{GL}_n(\mathbb{C})$ on

$$W = (\mathbb{C}^n)^{\oplus p} \oplus (\mathbb{C}^{*n})^{\oplus q} \oplus \mathrm{M}_n = \mathrm{M}_{n,p} \oplus \mathrm{M}_{q,n} \oplus \mathrm{M}_n,$$

with the action of $g \in G$ given by

$$g \cdot (B, C, A) = (gB, Cg^{-1}, \mathrm{Ad}(g)A) \quad \text{for } (B, C, A) \in \mathrm{M}_{n,p} \oplus \mathrm{M}_{q,n} \oplus \mathrm{M}_n.$$

Thus, the part M_n is considered to be $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$ and the action is the adjoint action. For the other parts, $\mathrm{M}_{n,p}$ is a p-copy of natural representations and $\mathrm{M}_{q,n}$ is a q-copy of its dual, i.e., as a representation space we will study

$$W \simeq V^{\oplus p} \oplus (V \otimes V^*) \oplus (V^*)^{\oplus q}.$$

The space W is the *fully* enhanced adjoint representation as we explained. Here we note that, from the opposite view point, the space W is also considered as an extension of $V^{\oplus p} \oplus (V^*)^{\oplus q}$ by adding $V \otimes V^*$. Hence it is a generalization of what H. Weyl considered in the course of his study of classical invariant theory [1939].

There are obvious invariants for the action of $G = \mathrm{GL}_n(\mathbb{C})$ on W. We put

$$\tau_k := \operatorname{trace} A^k \qquad (1 \le k \le n-1),$$
$$\gamma_{i,j}^{\ell} := (CA^{\ell}B)_{i,j} \quad (0 \le \ell \le n-1,\ 1 \le i \le q,\ 1 \le j \le p).$$

These invariants are generators of the whole invariant ring $\mathbb{C}[W]^G$, and they seem to be known to experts in various forms, including in quiver theory (see Theorem 1.1). Thus, we can define a quotient map $\pi_W : W \to \mathbb{C}^n \times (\mathrm{M}_{q,p})^n$ using these invariants (see (2-3)).

If $p = 1$ or $q = 1$, the quotient map has a very good property. Namely, we get:

Theorem 0.1 (Theorem 2.1(2)). *If $p = 1$ or $q = 1$, the map $\pi_W : W \to \mathbb{C}^n \times (\mathrm{M}_{q,p})^n$ is an affine categorical quotient map* (*note that* $\mathrm{M}_{q,p} = \mathbb{C}^p$ *or* $\mathbb{C}^q$). *In particular, the quotient map π_W is coregular, and $\mathbb{C}[W]^G$ is a polynomial ring generated by the fundamental invariants listed above.*

For general $p \ge 1$ and $q \ge 1$, the following theorem gives a generic structure of enhanced adjoint orbits.

Theorem 0.2 (Theorem 2.1, Corollary 2.2). *The dimension of the image* $\dim \operatorname{Im} \pi_W$ *is equal to $n(p+q)$, and a general fiber of π_W is a single G-orbit of dimension n^2. This implies that general orbits for the enhanced adjoint action are closed of dimension n^2.*

These orbits are called *regular semisimple orbits*. Another extreme case are nilpotent orbits. We investigate the null cone $\mathfrak{N}(W) \subset W$ in Section 3, and get the following results.

Theorem 0.3 (Theorem 3.3). *The null cone $\mathfrak{N}(W)$ is reducible and it has $n+1$ irreducible components $C_k \subset \mathfrak{N}(W)$ $(0 \le k \le n)$ given in Lemma 3.2. The dimension of the null cone is $n^2 - n + n \cdot \max\{p, q\}$ and $\mathfrak{N}(W)$ is equidimensional if and only if $p = q$.*

Finally, we get the structure of general (enhanced) nilpotent orbits contained in each component C_k in Theorem 3.4.

1. Setting

Let $V = \mathbb{C}^n$ be a vector space of dimension n. We consider a natural action of $G = \mathrm{GL}(V)$ on

$$W = W(p, q; r) := V^{\oplus p} \oplus (V^*)^{\oplus q} \oplus (V \otimes V^*)^{\oplus r}$$

in the obvious manner. In explicit matrix form, we can identify

$$W = (\mathbb{C}^n)^{\oplus p} \oplus (\mathbb{C}^{*n})^{\oplus q} \oplus (\mathrm{M}_n)^{\oplus r} = \mathrm{M}_{n,p} \oplus \mathrm{M}_{q,n} \oplus \mathrm{M}_n^r,$$

with the action of $g \in G$ on

$$(B, C, (A_1, \dots, A_r)) \in \mathrm{M}_{n,p} \oplus \mathrm{M}_{q,n} \oplus \mathrm{M}_n^r$$

given by

$$g \cdot (B, C, (A_1, \dots, A_r)) = (gB, Cg^{-1}, (\mathrm{Ad}(g)A_i)_{i=1}^r).$$

There are obvious invariants, which we list below. For a multi-index

$$I = (i_1, i_2, \dots, i_\ell) \quad (1 \le i_k \le r),$$

let us write $A_I = A_{i_1} A_{i_2} \cdots A_{i_\ell}$. We denote $[n] = \{1, 2, \dots, n\}$ as usual; then the multi-index I above is an element in $[r]^\ell$. We put

$$\begin{aligned} \tau_I &:= \mathrm{trace}(A_I) \qquad (I \in [r]^\ell), \\ \gamma_{i,j}^K &:= (C A_K B)_{i,j} \quad (K \in [r]^\ell, 1 \le i \le q, 1 \le j \le p), \end{aligned}$$

where we allow $\ell = 0$ for K, which means $A_K = 1_n$ (the identity matrix). These invariants are generators of the whole invariant ring, which is essentially due to a more general result of Le Bruyn and Procesi [1990, § 3, Theorem 1] (see also [Le Bruyn and Procesi 1987; Itoh 2013]).

Theorem 1.1. *The invariant ring* $\mathbb{C}[W]^G$ *is generated by the elements* τ_I *with* $I \in [r]^\ell$ $(\ell \geq 0)$ *and the elements* $\gamma_{i,j}^K$ *with* $K \in [r]^\ell$ $(\ell \geq 0)$, $i \in [q]$, $j \in [p]$; *that is,*

$$\mathbb{C}[W]^G = \mathbb{C}\big[\tau_I, \gamma_{i,j}^K \mid I, K \in [r]^\ell \ (\ell \geq 0), i \in [q], j \in [p]\big].$$

Proof. We largely follow the notation of [Le Bruyn and Procesi 1990]. We denote a connected quiver by Q and by α its dimension vector. For a representation space $R(Q,\alpha)$ of Q, Theorem 1 in [loc. cit.] states that the invariant ring $\mathbb{C}[Q,\alpha]^{\mathrm{GL}(\alpha)}$ is generated by traces of oriented cycles. So we will consider a quiver Q of two vertices $Q_0 = \{1, 2\}$ with arrows

$$Q_1 = \{a_i \mid 1 \leq i \leq r\} \cup \{b_i \mid 1 \leq i \leq p\} \cup \{c_i \mid 1 \leq i \leq q\},$$

where the a_i are loops connecting 1 and itself (i.e., $h(a_i) = t(a_i) = 1$), the b_i are arrows from 2 to 1 ($h(b_i) = 2,\ t(b_i) = 1$), and the c_i are arrows from 1 to 2 ($h(c_i) = 1,\ t(c_i) = 2$). Take a dimension vector $\alpha = (\alpha(1), \alpha(2)) = (n, 1)$, so that $V(1) = \mathbb{C}^n$ and $V(2) = \mathbb{C}$. Then our $W = W(p,q,r)$ coincides with the representation space $R(Q,\alpha)$.

The invariants are considered with respect to the action of $G(\alpha) = \mathrm{GL}_n \times \mathrm{GL}_1$. However, the representation image of $G(\alpha)$ on $W = R(Q,\alpha)$ and that of GL_n are the same because the action of the torus GL_1 on $V(2) = \mathbb{C}$ can be recaptured by the center of GL_n. So both invariant rings for $G(\alpha)$ and GL_n are the same.

Let us consider any closed cycles. Since we take traces, we can start from any vertices contained in the cycle. If it only contains the vertex 1, the traces are τ_I's. If it contains the vertex 2, we will start from 2 which necessarily ends in 2. Decompose the cycle into several cycles which start from 2 and end in 2. Since $V(2) = \mathbb{C}$ is 1-dimensional, a decomposed cycle starting from 2 represents a scalar being equal to its trace. Thus the trace of the cycle which we are considering is a product of various $\gamma_{i,j}^K$'s. □

Let us denote $\pi = \pi_W : W \to W /\!/ G$, an affine quotient map by the action above. As a set, the quotient $W/\!/G$ corresponds to the set of closed G-orbits in W. It is known that these closed orbits are precisely the set of equivalence classes of completely reducible representations of a quiver corresponding to W.

Let $\mathfrak{N}(W) = \pi_W^{-1}(\pi_W(0))$ be the nilpotent variety, which consists of the nilpotent elements x with the property $\overline{G \cdot x} \ni 0$. The nilpotent variety $\mathfrak{N}(W)$ is the "worst" fiber. So we are strongly interested in its structure. In particular, we are interested in $\dim \mathfrak{N}(W)$, its irreducible components, its orbit structure, and whether it is reduced or not. For the dimensions and irreducible components, we have a complete result, which is stated in Section 3 in detail. The problems of orbit structure and reducibility of $\mathfrak{N}(W)$ also seem very interesting but these are our future subjects.

On the other hand, general fibers are supposed to have "best" properties we can expect. This will be helpful for studying the quotient space (at least its smooth part), which we shall do in Section 2.

It would be too ambitious to expect to get a very explicit orbit structure of the whole space W. Also it seems to be a difficult problem to clarify the structure of the singularities of the quotient space.

2. Enhanced adjoint action

In the following, we restrict ourselves to the case $r = 1$, so $W = \mathrm{M}_{n,p} \oplus \mathrm{M}_{q,n} \oplus \mathrm{M}_n$, on which $G = \mathrm{GL}_n$ acts. In matrix form, $g \in \mathrm{GL}_n$ acts on

$$(B, C, A) \in \mathrm{M}_{n,p} \oplus \mathrm{M}_{q,n} \oplus \mathrm{M}_n$$

via $g \cdot (B, C, A) = (gB, Cg^{-1}, \mathrm{Ad}(g)A)$. We call this action the *enhanced adjoint action*.

Now Theorem 1.1 gives a set of generators of G-invariants:

$$\tau_k := \mathrm{trace}(A^k) \quad (1 \le k \le n), \tag{2-1}$$

$$\gamma^k_{i,j} := (CA^kB)_{i,j} \quad (0 \le k \le n-1,\ 1 \le i \le q,\ 1 \le j \le p). \tag{2-2}$$

Note that A^n is a linear combination of A^k's $(0 \le k \le n-1)$ thanks to the Cayley–Hamilton formula, so we don't need higher powers of A in τ_k or $\gamma^k_{i,j}$. Let us denote the affine quotient map by

$$\begin{aligned} \pi_W &: W \to \mathbb{C}^n \oplus (\mathrm{M}_{q,p})^n, \\ (A, B, C) &\mapsto \big((\tau_k)_{k=1}^n; ((\gamma^k_{i,j})_{i,j})_{k=0}^{n-1}\big) = \big((\tau_k)_{k=1}^n; (CA^kB)_{k=0}^{n-1}\big). \end{aligned} \tag{2-3}$$

By the general theory of quotients, we know the image $\mathrm{Im}\,\pi_W$ is a closed subvariety of $\mathbb{C}^n \oplus (\mathrm{M}_{q,p})^n$. Let us denote by $\mathrm{Det}_r(\mathrm{M}_{q,p})$ the determinantal variety consisting of matrices in $\mathrm{M}_{q,p}$ of rank less than or equal to r. Clearly, if we put $m = \min\{p, q, n\}$, $\mathrm{Im}\,\pi_W$ is contained in $\mathbb{C}^n \times \mathrm{Det}_m(\mathrm{M}_{q,p})^n$. However, it is much smaller, as you can see from the theorem below.

Theorem 2.1. *Under the setting above, the image* $\mathrm{Im}\,\pi_W$ *is isomorphic to the affine quotient* $W/\!/G = \mathrm{Spec}\,(\mathbb{C}[W]^G)$. *Moreover*:

(1) *There is a dominant map*

$$\Psi : \mathbb{C}^n \times (\mathrm{Det}_1(\mathrm{M}_{q,p}))^n \to \mathrm{Im}\,\pi_W,$$

whose restriction to a dense open subset of $\mathbb{C}^n \times (\mathrm{Det}_1(\mathrm{M}_{q,p}))^n$ *gives an affine quotient map under the diagonal action of* S_n (*permuting both coordinates*) *to a dense open subset of* $\mathrm{Im}\,\pi_W$. *Consequently*, *we get* $\dim W/\!/G = \dim \mathrm{Im}\,\pi_W = n(p+q)$, *and a general fiber of* π_W *is of dimension* n^2.

(2) *If* $p = 1$ *or* $q = 1$, *the quotient map* π_W *is surjective*, *and*

$$\operatorname{Im} \pi_W = \mathbb{C}^n \oplus (\mathrm{M}_{q,p})^{\oplus n}$$

is an affine space. In particular, *the quotient map* π_W *is coregular*, *and* $\mathbb{C}[W]^G$ *is a polynomial ring of the fundamental invariants listed in* (2-1) *and* (2-2).

Proof. Let us fix a generic diagonal matrix $A = t = \operatorname{diag}(t_1, \dots, t_n)$, where $t_i \neq t_j$ $(i \neq j)$. For $1 \le r \le n$, put

$$X^{(r)} = \begin{pmatrix} c_{1,r} \\ c_{2,r} \\ \vdots \\ c_{p,r} \end{pmatrix} (b_{r,1}, b_{r,2}, \dots, b_{r,q}) \in \operatorname{Det}_1(\mathrm{M}_{q,p}),$$

where $c_{i,j}$ denotes the (i, j)-element of the matrix $C \in \mathrm{M}_{q,n}$ and similarly $b_{i,j}$ for $B \in \mathrm{M}_{n,p}$. We get

$$CA^kB = (\gamma^k_{i,j})_{i,j} = \Big(\sum_{r=1}^{n} c_{i,r} t_r^k b_{r,j}\Big)_{i,j} = \sum_{r=1}^{n} t_r^k X^{(r)} =: \Gamma^{(k)}. \tag{2-4}$$

Thus, in the matrix expression,

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ t_1 & t_2 & \cdots & t_n \\ \vdots & \vdots & \ddots & \vdots \\ t_1^{n-1} & t_2^{n-1} & \cdots & t_n^{n-1} \end{pmatrix} \begin{pmatrix} X^{(1)} \\ X^{(2)} \\ \vdots \\ X^{(n)} \end{pmatrix} = \begin{pmatrix} \Gamma^{(0)} \\ \Gamma^{(1)} \\ \vdots \\ \Gamma^{(n-1)} \end{pmatrix}, \tag{2-5}$$

hence

$$\begin{pmatrix} X^{(1)} \\ X^{(2)} \\ \vdots \\ X^{(n)} \end{pmatrix} = D(t)^{-1} \begin{pmatrix} \Gamma^{(0)} \\ \Gamma^{(1)} \\ \vdots \\ \Gamma^{(n-1)} \end{pmatrix}, \tag{2-6}$$

where $D(t) = (t_j^{i-1})_{i,j}$ denotes the Vandermonde matrix in (2-5). Bearing this calculation in mind, we define a map $\Psi : \mathbb{C}^n \times (\operatorname{Det}_1(\mathrm{M}_{q,p}))^n \to \mathbb{C}^n \oplus (\mathrm{M}_{q,p})^n$ by

$$\Psi(t; (X^{(k)})_{k=1}^n) = \left(\left(\sum_{i=1}^{n} t_i^k \right)_{k=1}^{n} ; (\Gamma^{(k)})_{k=1}^n = D(t)(X^{(k)})_{k=1}^n \right). \tag{2-7}$$

We will show that $\operatorname{Im} \Psi \subset \operatorname{Im} \pi_W$ so that we get a map from $U := \mathbb{C}^n \times (\operatorname{Det}_1(\mathrm{M}_{q,p}))^n$ to $\operatorname{Im} \pi_W$, denoted by the same letter Ψ.

To see $\operatorname{Im} \Psi \subset \operatorname{Im} \pi_W$, take $(\tau; (\Gamma^{(k)})_k) \in \operatorname{Im} \pi_W$ for which τ is in an image of regular semisimple A. For this A, we can pick a diagonal matrix $t = \operatorname{diag}(t_1, \dots, t_n)$ in the same adjoint orbit of A, which implies $\tau_k(A) = \sum_{i=1}^n t_i^k$. Using this t, we can recover $X^{(k)}$'s via (2-6), since $t_i \neq t_j \ (i \neq j)$. As we saw above, if $(\tau;\, (\Gamma^{(k)})_k) \in \operatorname{Im}\pi_W$ then $\operatorname{rank} X^{(k)} \leq 1$. Here we require that those $X^{(k)}$'s are all exactly rank one matrices. This is an open condition (in $\operatorname{Im}\pi_W$) and it does not depend on the choice of the diagonal representatives of A. We define an open dense set $(\operatorname{Im}\pi_W)' \subset \operatorname{Im}\pi_W$ consisting of $(\tau;\, (\Gamma^{(k)})_k) \in \operatorname{Im}\pi_W$ for which (i) τ is in an image of regular semisimple A; and (ii) $\operatorname{rank} X^{(k)} = 1$ for $1 \leq k \leq n$. Thus we conclude that Ψ is a surjective map from an open dense subset of U to $(\operatorname{Im}\pi_W)'$. Consequently, the image $\operatorname{Im}\Psi$ is contained in the closed subvariety $\operatorname{Im}\pi_W$, and we get a well defined map from $U = \mathbb{C}^n \times (\operatorname{Det}_1(\mathrm{M}_{q,p}))^n$ to $\operatorname{Im}\pi_W$ by

$$\begin{aligned} &\Psi : U \to \operatorname{Im}\pi_W, \\ &(t;\, (X^{(k)})_{k=1}^n) \mapsto \left(\left(\sum_{i=1}^n t_i^k \right)_{k=1}^n ;\, (\Gamma^{(k)})_{k=1}^n = D(t)(X^{(k)})_{k=1}^n \right). \end{aligned} \tag{2-8}$$

The map Ψ is generically an $n!$-fold covering map, and it is invariant under S_n which acts on U by the diagonal coordinate permutation on both factors.[1]

Since Ψ is a dominant map with generically finite fibers, we conclude that

$$\dim \operatorname{Im}\pi_W = \dim U = n + n(p+q-1) = n(p+q),$$

where we used $\dim \operatorname{Det}_1(\mathrm{M}_{q,p}) = p+q-1$. Comparing the dimension, we know the dimension of a generic fiber of π_W is $n^2 = \dim W - \dim \operatorname{Im}\pi_W$.

Now let us assume $p = 1$ or $q = 1$. Then $\mathrm{M}_{q,p} = \mathbb{C}^q$ or $\mathbb{C}^p$, and

$$\dim(\mathbb{C}^n \oplus (\mathrm{M}_{q,p})^{\oplus n}) = n(p+q) = \dim \operatorname{Im}\pi_W$$

(the last equality follows from Theorem 2.1(1)). Since the image $\operatorname{Im}\pi_W$ is closed in $\mathbb{C}^n \oplus (\mathrm{M}_{q,p})^{\oplus n}$, we have a surjective quotient map $\pi_W : W \to \mathbb{C}^n \oplus (\mathrm{M}_{q,p})^{\oplus n}$ so $W/\!/G \simeq \mathbb{C}^n \oplus (\mathrm{M}_{q,p})^{\oplus n}$, an affine space. This means the invariants are algebraically independent and $\mathbb{C}[W]^G$ is a polynomial ring. □

Corollary 2.2. *Let us denote the quotient map by* $\pi_W : W \to \mathbb{C}^n \oplus (\mathrm{M}_{q,p})^n$ *as in* (2-3). *Assume that* $(\tau;\, \Gamma) = (\tau;\, (\Gamma^{(k)})_{k=1}^n) \in \mathbb{C}^n \oplus (\mathrm{M}_{q,p})^n$ *satisfies the following conditions*:

(i) *There exists a regular diagonal matrix t with $\tau = (\tau_k(t))_{k=1}^n$, i.e., $\tau \in \mathbb{C}^n$ with the k-th coordinate being $\tau_k = \sum_{i=1}^n t_i^k$, where $t_i \neq t_j$ if $i \neq j$.*

(ii) $\Gamma^{(k)}$ $(0 \leq k \leq n-1)$ *corresponds to* $X^{(k)}$ *via* (2-6), *which are of rank* 1.

Then $(\tau;\, \Gamma)$ is in the image $\operatorname{Im}\pi_W$ and $\dim \pi_W^{-1}(\tau;\, \Gamma) = n^2$, i.e., the fiber of $(\tau;\, \Gamma)$ is generic and of dimension n^2. Moreover, it is a single closed G-orbit.

[1] Unfortunately, Ψ may not be a quotient map. See Remark 2.4

Proof. By condition (i), we can choose a regular diagonal matrix t with $\tau = (\tau_k(t))_{k=1}^n$. Thus we can define $(X^{(k)}) = D(t)^{-1}\Gamma$ via (2-6). If $X^{(k)}$ is of rank 1, then we can write $X^{(k)} = c_k\, {}^t b_k$ for certain $c_k \in \mathbb{C}^q$ and $b_k \in \mathbb{C}^p$. From these vectors, we can restore ${}^tB = (b_1, \dots, b_n)$ and $C = (c_1, \dots, c_n)$. Thus

$$(\tau; \Gamma) = \pi_W(t, B, C) \in \operatorname{Im} \pi_W.$$

There is not so much choice for the fiber. We know the fiber over τ of the adjoint quotient is just the conjugation of t, which is of dimension $n^2 - n$. For B and C, since any column of B and C is nonzero, we can only multiply scalars column by column, which is of dimension n.

It is now clear that any element in the fiber can be obtained from (t, B, C) through the action of G. Since the stabilizer of the fiber (t, B, C) is trivial, we again get the right dimension n^2. □

Remark 2.3. Let us assume $p = 1$ or $q = 1$. In this case, the action of $G = \mathrm{GL}_n(\mathbb{C})$ on W is coregular, i.e., the quotient space is an affine space and the generators listed in (2-1) and (2-2) are algebraically independent.

However, if we consider an action of the simple group $\mathrm{SL}_n(\mathbb{C})$ instead of $\mathrm{GL}_n(\mathbb{C})$, this action is not coregular (coregular actions are classified for simple groups; see [Schwarz 1978; Adamovich and Golovina 1979]).

To see this, let us assume $p = q = 1$ for simplicity. Consider two invariants D_1 and D_2, with respect to the action of SL_n defined as follows. For

$$(u, v, A) \in V \oplus V^* \oplus \mathrm{M}_n$$

(we consider $V = \mathbb{C}^n$ as a column vector), we put

$$D_1(u, v, A) = \det \begin{pmatrix} v \\ vA \\ vA^2 \\ \vdots \\ vA^{n-1} \end{pmatrix}, \qquad D_2(u, v, A) = \det(u, Au, A^2u, \dots, A^{n-1}u).$$

Both D_1 and D_2 are clearly SL_n-invariants, and they are not GL_n-invariants so they cannot be expressible by using τ_k and γ^k above.[2] However, it is easy to see

$$D_1 \cdot D_2 = \det\bigl(vA^{i+j}u\bigr)_{i,j} = \det(\gamma^{i+j})_{i,j},$$

which gives a relation. This shows that the action of SL_n is not coregular.

When $p > 1$ or $q > 1$, similar arguments lead to the same conclusion.

However, even if it is not coregular, it seems the SL_n-orbit structure has good properties. We will discuss it in the future.

[2]Note that, since $p = q = 1$, we do not need subscripts i and j for $\gamma^k_{i,j}$

Remark 2.4. Let us consider a toy model for the map (2-8), as illustrated below. Assume that V is a vector space and S_n acts on $\mathbb{C}^n \times V^n$ as the diagonal coordinate permutation.

$$
\begin{array}{ccc}
\mathbb{C}^n \times V^n \ni (a_1, \dots, a_n; v_1, \dots, v_n) & & \\
\big\downarrow \psi & \overset{\pi}{\searrow} & \\
\mathbb{C}^n \times V^n \ni \Big((\sum_{i=1}^n a_i^k)_{k=1}^n ; (\sum_{i=1}^n a_i^k v_i)_{k=1}^n \Big) & \overset{\varphi}{\longleftarrow} & (\mathbb{C}^n \times V^n)/S_n
\end{array}
$$

Consider a closed set $Z = \{(a; v) \mid a_i v_i = u \ (1 \leq i \leq n)\}$ for a fixed nonzero vector u, which is stable under the S_n-action. The image $\psi(Z)$ does not contain an element of the form $(0; w)$, however its closure contains $(0; (n\,u, 0, \dots, 0))$. Thus the image $\psi(Z)$ is not closed, hence ψ is not a quotient map.

Remark 2.5. Let us consider a semidirect sum $L = \mathfrak{gl}(V) \ltimes (V \oplus V^*)$ and the corresponding Lie group S. Then L admits a deformed universal enveloping algebra called "infinitesimal Cherednik algebra". The infinitesimal invariant ring $\mathbb{C}[L^*]^S$ is isomorphic to the center of the infinitesimal Cherednik algebra, which is a polynomial ring of n-variables ($n = \dim V$). Our invariant ring naturally contains it as a subalgebra if $p = q = 1$. For details, see [Tikaradze 2010; Panyushev 2007; Raïs 2009].

3. Structure of the null cone

We will study the structure of the null cone $\mathfrak{N}(W) = \pi_W^{-1}(\pi_W(0))$ in this section. For this, we follow the strategy of [Popov 2003] and [Kraft and Wallach 2006]. We briefly recall their theory.

3A. In this subsection, we consider a general situation so that the notation is independent of those in the former (sub)sections.

Let G be a connected reductive algebraic group G over $\mathbb{C}$, which acts on a vector space V linearly. Let $\pi : V \to V/\!/G$ be the quotient map, and

$$\mathcal{N}_V := \pi^{-1}(\pi(0)) = \{v \in V \mid \overline{Gv} \ni 0\}$$

be the null cone. For any one parameter subgroup (abbreviated as "1-PSG") $\lambda : \mathbb{C}^\times \to G$, we define $V(\lambda) := \{v \in V \mid \lim_{t \to 0} \lambda(t) v = 0\}$. Then $v \in V$ is in the null cone $\mathcal{N}_V$ if and only if $v \in V(\lambda)$ for a suitable 1-PSG λ (the Hilbert–Mumford criterion).

Let $T \subset G$ be a maximal torus. We fix T once and for all, and denote by $X^*(T)$ the character group of T. Then V has the weight space decomposition

$$V = \bigoplus_{\gamma \in X^*(T)} V_\gamma, \qquad V_\gamma := \{v \in V \mid tv = \gamma(t) v, \ t \in T\}.$$

We denote the set of 1-PSGs $\lambda : \mathbb{C}^\times \to T$ by $X_*(T)$. Then there is a natural pairing $\langle -, - \rangle : X_*(T) \times X^*(T) \to \mathbb{Z}$ determined as follows. For $(\lambda, \gamma) \in X_*(T) \times X^*(T)$, $m = \langle \lambda, \gamma \rangle$ if $\gamma(\lambda(t)) = t^m$ $(t \in \mathbb{C}^\times)$.

With these notations, for a 1-PSG $\lambda : \mathbb{C}^\times \to T \subset G$, we have

$$V(\lambda) = \bigoplus_{\langle \lambda, \gamma \rangle > 0} V_\gamma.$$

Since every 1-PSG of G is conjugate to a certain $\lambda \in X_*(T)$, we get

$$\mathcal{N}_V = \bigcup_{\lambda \in X_*(T)} G \cdot V(\lambda).$$

In this decomposition, there appear only finitely many different $V(\lambda) \neq 0$. Thus, a maximal $V(\lambda)$ may contribute to an irreducible component of $\mathcal{N}_V$ (but not always). We call such $U = V(\lambda)$ a maximal unstable subspace, and put

$$\mathcal{X}_U := \{\gamma \in X^*(T) \mid V_\gamma \subset U\} = \{\gamma \mid \langle \lambda, \gamma \rangle > 0\},$$

a maximal unstable subset of weights. Let $\mathcal{X}_1, \ldots, \mathcal{X}_s$ be a complete set of representatives of maximal unstable subsets of weights up to the conjugation of the Weyl group $W_G(T)$, and $U_i = \bigoplus_{\gamma \in \mathcal{X}_i} V_\gamma$ $(1 \le i \le s)$ be the corresponding maximal unstable subspace.

For a 1-PSG λ, put

$$P(\lambda) := \{g \in G \mid \text{the limit } \lim_{t \to 0} \mathrm{Ad}(\lambda(t))\, g \text{ exists}\}.$$

Then $P(\lambda)$ is a parabolic subgroup which leaves $V(\lambda)$ stable; see Kempf [1978]. If $U = V(\lambda)$ is a maximal unstable subspace, then the stabilizer $\mathrm{Stab}_G(U)$ contains $P(\lambda)$ and hence it is a parabolic subgroup.

Define $P_i := \mathrm{Stab}_G(U_i)$ for each $1 \le i \le s$. Thus, we get a natural multiplication map $G \times_{P_i} U_i \to C_i \subset \mathcal{N}_V$, where $C_i = G \cdot U_i$. Since G/P_i is projective, the image C_i is closed and irreducible. Thus we can choose $C_1, \ldots, C_r$ which give irreducible components of $\mathcal{N}_V$, after renumbering if necessary. In this way, we can determine the irreducible decomposition of $\mathcal{N}_V$:

$$\mathcal{N}_V = \bigcup_{k=1}^{r} C_k. \tag{3-1}$$

Let us apply this theory to our situation of the enhanced adjoint representation.

3B. Now let us return to our original notation, so $G = \mathrm{GL}_n(\mathbb{C})$ which acts on $W = \mathrm{M}_{n,p} \oplus \mathrm{M}_{q,n} \oplus \mathrm{M}_n$ as before. It is easy to see that the set of weights of W is given by

$$\Lambda = \Lambda(W) := \{0\} \cup \Delta_n \cup \{\pm\varepsilon_i \mid 1 \le i \le n\},$$
$$\Delta_n = \{\varepsilon_i - \varepsilon_j \mid 1 \le i \neq j \le n\}.$$

Here, Δ_n denotes the set of roots of type A_{n-1} and ε_i denotes the standard basis in $\mathfrak{t}^*$, where $\mathfrak{t}$ is the Lie algebra of the diagonal torus $T \subset G$. The multiplicity of $\alpha \in \Delta_n$ is 1, while the multiplicity of $\alpha = 0$ is n; that of ε_i is p and that of $-\varepsilon_i$ is q. We describe a family of maximal unstable subsets of weights up to the Weyl group conjugation. Take a standard positive system $\Delta_n^+ = \{\varepsilon_i - \varepsilon_j \mid 1 \le i < j \le n\}$ of Δ_n.

Lemma 3.1. *For $0 \le k \le n$, put $X_k := \Delta_n^+ \cup \{\varepsilon_i \mid 1 \le i \le k\} \cup \{-\varepsilon_j \mid k < j \le n\}$. Then $X_0, X_1, \dots, X_n$ gives a complete system of representatives of maximal unstable subset of weights up to the conjugation of the Weyl group $W_G(T) = S_n$.*

Proof. Let X be a maximal unstable subset corresponding to a 1-PSG λ. Taking conjugation of λ by S_n, we can assume $\lambda = (\lambda_1, \dots, \lambda_n)$, with $\lambda_1 > \lambda_2 > \cdots > \lambda_n$. Note that, if an equality appears among λ_i's or one of λ_i's is equal to zero, the corresponding unstable subset is not maximal. If $\lambda_k > 0 > \lambda_{k+1}$, X is given by X_k. □

Let $U_k \subset W$ be the maximal unstable subspace corresponding to X_k, so that

$$U_k = \bigoplus_{\alpha \in X_k} W_\alpha = \big\{(\xi, \eta, v) \in \mathrm{M}_{n,p} \oplus \mathrm{M}_{q,n} \oplus \mathrm{M}_n \mid \xi_{i,j} = 0 \ (i > k), \eta_{i',j'} = 0 \ (j' \le k), v \in \mathfrak{n}^+\big\}, \tag{3-2}$$

where $\mathfrak{n}^+$ denotes a maximal nilpotent subalgebra consisting of upper triangular matrices with 0's on the diagonal. It is the Lie algebra of the unipotent radical of a Borel subgroup B of upper triangular matrices in $G = \mathrm{GL}_n$. Note that

$$\xi = \begin{pmatrix} \xi_1 \\ 0 \end{pmatrix} \quad (\xi_1 \in \mathrm{M}_{k,p}), \qquad \text{while} \qquad \eta = (0, \eta_2) \quad (\eta_2 \in \mathrm{M}_{q,n-k}).$$

Lemma 3.2. *Let U_k $(0 \le k \le n)$ be a maximal unstable subspace as above. Then the stabilizer $P_k = \mathrm{Stab}_G(U_k)$ of U_k is the Borel subgroup B for any k and $\psi_k : G \times_B U_k \to C_k \subset \mathfrak{N}(W)$ is a resolution of singularity. In particular, C_k is an irreducible closed subvariety in $\mathfrak{N}(W)$ of dimension $(n^2 - n) + pk + q(n-k)$.*

Proof. Since P_k stabilizes $\mathfrak{n}^+$, it is contained in B. On the other hand, clearly B stabilizes U_k, hence $P_k = B$.

Let us show that a generic fiber of the map ψ_k is a one-point set. Since $C_k \supset U_k$, we will examine the fiber of $(\xi, \eta, v) \in U_k$, where $v \in \mathfrak{n}^+$ is a principal nilpotent element. Take an element $[g, (\xi', \eta', u)] \in \psi_k^{-1}((\xi, \eta, v))$. Then $(\xi, \eta, v) = \psi_k([g, (\xi', \eta', u)]) = (g\xi', \eta' g^{-1}, \mathrm{Ad}(g)u)$. In particular, $v = \mathrm{Ad}(g)u \in \mathrm{Ad}(g)\mathfrak{b} =: \mathfrak{b}^g$. It is well known that a principal element belongs to a unique Borel subalgebra. Since $v \in \mathfrak{b}$, we conclude $\mathfrak{b} = \mathfrak{b}^g$, hence $g \in B$. Now we know $[g, (\xi', \eta', u)] \sim [1_n, (\xi, \eta, v)]$, which means the element in the fiber is uniquely determined.

The set of elements $\{(\xi, \eta, v) \in C_k \mid v \text{ is principal nilpotent}\}$ is open dense in C_k, so the map ψ_k is generically one-to-one, hence it is birational. Since $G \times_B U_k$ is a vector bundle over a projective variety, the map ψ_k is proper and it is a resolution. □

Theorem 3.3. *Let $\mathfrak{N}(W)$ be the null cone, and let $C_k \subset \mathfrak{N}(W)$ $(0 \le k \le n)$ be as in Lemma 3.2.*

(1) $\mathfrak{N}(W) = \bigcup_{k=0}^{n} C_k$ *gives the irreducible decomposition. So the null cone has* $(n+1)$ *components, the number of which is independent of* $p \ge 1$ *and* $q \ge 1$. *The dimension of* $\mathfrak{N}(W)$ *is* $n^2 - n + n \cdot \max\{p, q\}$.

(2) *The null cone* $\mathfrak{N}(W)$ *is equidimensional if and only if* $p = q$. *In this case, the dimension of* $\mathfrak{N}(W)$ *is* $n^2 - n + pn$.

(3) *The dimension of* $\mathfrak{N}(W)$ *is* n^2 *if and only if* $p = q = 1$. *If this is the case, any fiber* $\pi_W^{-1}((\tau; \Gamma))$ *of* $(\tau; \Gamma) \in \operatorname{Im} \pi_W$ *is of dimension* n^2.

Proof. From Lemma 3.2, the subvariety C_k is closed and irreducible. The general theory described in Section 3A gives the irreducible decomposition of $\mathfrak{N}(W)$ (cf. (3-1)). Since $\dim C_k = (n^2 - n) + pk + q(n-k)$,

$$\dim \mathfrak{N}(W) = \max_{0 \le k \le n} \{(n^2 - n) + pk + q(n-k)\} = n^2 - n + n \cdot \max\{p, q\}.$$

This proves (1). The claim (2) follows immediately from (1).

Let us prove (3). For any $(\tau; \Gamma) \in \operatorname{Im} \pi_W$, the dimension of the fiber $\pi_W^{-1}((\tau; \Gamma))$ is greater than or equal to that of a general fiber, which is n^2 by Theorem 2.1. On the other hand, the dimension of the null cone is the greatest among those of the fibers (see [Popov and Vinberg 1994]). □

3C. *Orbits in the null cone.* Let us investigate orbits in an irreducible component $C_k = G \cdot U_k \subset \mathfrak{N}(W)$ (cf. (3-2)). So pick $w = (\xi, \eta, v) \in U_k$, where $v \in \mathfrak{n}^+$ is a principal nilpotent element. We denote the G orbit through w by $\mathbb{O}(w)$.

We compute the stabilizer $Z_G(w)$ of w. Up to G conjugacy, we can assume

$$v = e := \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix}.$$

By direct calculation, we get

$$Z_G(e) = \exp\left(\left\{\sum_{i=0}^{n-1} \sigma_i e^i \mid \sigma_i \in \mathbb{R}\right\}\right) \ni \sum_{j=1}^{n} x_j e^{j-1} =: g. \tag{3-3}$$

Assume that $k \ge n-k$, and denote $\xi \in \mathrm{M}_{n,p}$ and $\eta \in \mathrm{M}_{q,n}$ as

$$\xi = \begin{pmatrix} \xi_1 \\ 0 \end{pmatrix} \quad (\xi_1 \in \mathrm{M}_{k,p}) \quad \text{and} \quad \eta = (\, 0 \mid \eta_1 \,) \quad (\eta_1 \in \mathrm{M}_{q,n-k}). \tag{3-4}$$

Here we take

$$\xi_1 = (\boldsymbol{e}_k, \xi_1') \qquad (\xi_1' \in \mathrm{M}_{k,p-1}), \tag{3-5}$$

where $\boldsymbol{e}_k \in \mathbb{C}^k$ is the k-th elementary vector whose k-th coordinate is 1 and whose other coordinates are 0. Then, the element g in (3-3) stabilizes ξ and η if and only if $x_1 = 1,\ x_2 = \cdots = x_k = 0$. Thus we get

$$Z_G(w) = \Big\{ 1_n + \sum_{j=k+1}^{n} x_j e^{j-1} \Big\}.$$

In particular, we know $\operatorname{codim} \mathbb{O}(w) = n - k$. For the orbit $\mathbb{O}(w)$, we can take ξ_1' in (3-5) and η_1 in (3-4) freely, and they are uniquely determined by the orbit. So there is a fibration of orbits $\mathbb{O}(w)$ with the base space $\mathrm{M}_{k,p-1} \times \mathrm{M}_{q,n-k}$ of dimension

$$\begin{aligned} \dim \mathbb{O}(w) + \dim \mathrm{M}_{k,p-1} \times \mathrm{M}_{q,n-k} &= n^2 - (n-k) + k(p-1) + q(n-k) \\ &= n^2 - n + kp + (n-k)q = \dim C_k. \end{aligned}$$

This means the family of orbits $\{\mathbb{O}(w)\}$ makes up an open dense subset of the irreducible component C_k. Since the orbits of the largest possible dimension constitute an open set, $\dim \mathbb{O}(w) = n^2 - n + k$ is the largest among the orbits in C_k. For the family parametrized by $\mathrm{M}_{k,p-1} \times \mathrm{M}_{q,n-k}$, there is no reason to specialize the first column of ξ. So, if the k-th row of ξ does not vanish, we can follow the same arguments.

This construction also applies to the case of $k \le n - k$, if we take η instead of ξ.

Let us summarize what we have proven here.

Theorem 3.4. *Let $C_k \subset \mathfrak{N}(W)$ $(0 \le k \le n)$ be an irreducible component of the null cone* $\mathfrak{N}(W)$ (see Lemma 3.2). *The largest dimension of the nilpotent orbits in C_k is $n^2 - \min\{k, n-k\}$. Moreover, there exists an open dense subset of C_k which is fibered over an affine space of dimension $kp + q(n-k) - \max\{k, n-k\}$ with the fiber of isomorphic nilpotent orbits $\mathbb{O}$ of the largest dimension.*

In particular, an irreducible component C_k contains a nilpotent orbit of dimension n^2 if and only if $k = 0$ or n.

Remark 3.5. Let us consider $w = (\xi, \eta, v) \in U_k$ as above. Even if v is not principal, a G-orbit $\mathbb{O}(w)$ through w can attain the largest possible dimension in the irreducible component C_k. This seems difficult to describe when an orbit $\mathbb{O}(w)$ has the largest dimension.

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