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THE ASYMPTOTIC BOUNDS OF VISCOSITY SOLUTIONS OF THE CAUCHY PROBLEM FOR HAMILTON–JACOBI EQUATIONS

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We study the Cauchy problem for time-periodic Hamilton–Jacobi equations with Tonelli Hamiltonians. It is well known that the Cauchy problem admits a unique bounded viscosity solution. We provide a more precise description of the boundedness of the viscosity solution. We introduce the notion of asymptotic bounds of the viscosity solution of the Cauchy problem. An asymptotic bound is a 1-periodic viscosity solution of the Hamilton–Jacobi equation. We show how to obtain the optimal asymptotic bounds, i.e., minimal asymptotic upper bound and maximal asymptotic lower bound. Our method relies upon Mather theory and weak KAM theory on Lagrangian dynamics.

1. Introduction and main result

Consider the Hamilton–Jacobi equation

$$u_t + H(x, u_x, t) = c(H), \quad x \in M,\ t \in [0, +\infty), \tag{1-1}$$

where H is a Tonelli Hamiltonian, the constant $c(H)$ is the Mañé critical value of H [Mañé 1997], and M is a compact and connected smooth manifold without boundary. We choose, once and for all, a C^∞ Riemannian metric g on M. A C^2 function $H : T^*M \times \mathbb{R}^1 \to \mathbb{R}^1$ is called a Tonelli Hamiltonian if:

- (periodicity) H is 1-periodic in t.
- (strict convexity) For each $(x, p, t) \in T^*M \times \mathbb{R}^1$, the second partial derivative $\partial^2 H/\partial p^2(x, p, t)$ is positive definite.
- (superlinear growth) $\lim_{\|p\|_x \to +\infty} H(x, p, t)/\|p\|_x = +\infty$ uniformly on $x \in M$, $t \in \mathbb{R}^1$, where $\|\cdot\|_x$ denotes the norm on $T^*_x M$ induced by g.
- (completeness of the Hamiltonian vector field) Each integral curve of the Hamiltonian vector field is defined on all of $\mathbb{R}^1$.

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For the Hamiltonian H, we can define the associated Lagrangian as its Fenchel–Legendre transform:

$$L : TM \times \mathbb{R}^1 \to \mathbb{R}^1, \quad (x, v, t) \mapsto \sup_{p \in T_x^* M} \{\langle p, v\rangle_x - H(x, p, t)\},$$

where $\langle \cdot , \cdot \rangle_x$ represents the canonical pairing between the tangent and cotangent space. Since H is a Tonelli Hamiltonian, one can easily prove that L is finite everywhere, a C^2 function, 1-periodic in t, superlinear and strictly convex in v, and that the Euler–Lagrange flow is complete. Such a Lagrangian will be called a Tonelli Lagrangian.

The Cauchy problem for (1-1) with Tonelli Hamiltonian H is well posed in the viscosity sense: given a continuous function $\varphi : M \to \mathbb{R}^1$, (1-1) admits a unique viscosity solution $u : M \times [0, +\infty) \to \mathbb{R}^1$ with $u|_{t=0} = \varphi$; see, e.g., [Lions 1982]. The notion of viscosity solutions was introduced by Crandall and Lions [1983] in the study of Hamilton–Jacobi equations.

Let $\tilde{H} = H - c(H)$. Then (1-1) can be rewritten as

$$u_t + \tilde{H}(x, u_x, t) = 0.$$

Since $\tilde{H}$ is still a Tonelli Hamiltonian and the Mañé critical value of $\tilde{H}$ is 0 (see Section 2A), then in the following we always assume that $c(H) = 0$ and consider the Hamilton–Jacobi equation

$$u_t + H(x, u_x, t) = 0, \quad x \in M, \ t \in [0, +\infty). \tag{1-2}$$

There exist viscosity solutions of (1-2) which are 1-periodic in time; see, e.g., [Wang and Yan 2012]. More precisely, Wang and Yan [2012] introduced a new kind of Lax–Oleinik type operators in the context of weak KAM theory [Fathi 2005]. The family of the new operators with an arbitrary continuous function φ on M as initial condition converges to a 1-periodic viscosity solution of (1-2). Moreover, using this method one can obtain all the 1-periodic viscosity solutions of (1-2). There is a nice representation formula for 1-periodic viscosity solutions: $u(x, t) = \inf_{y \in M}(\varphi(y) + h_{0,[t]}(y, x))$, where h is the Peierls barrier and $[t] = t \bmod 1$; see Section 2 for details. In addition, Wang and Yan also showed [2012] that weak KAM solutions and 1-periodic viscosity solutions of (1-2) are the same.

All the viscosity solutions of (1-2) are bounded; see Proposition 2.1. In general, it is not true that the viscosity solution converges, as $t \to +\infty$, to a 1-periodic viscosity solution; see [Barles and Souganidis 2000; Fathi and Mather 2000]. Roquejoffre [2001] and Bernard and Roquejoffre [2004] proved that the viscosity solution converges to a T-periodic viscosity solution in several nontrivial special cases, where T may be greater than 1.

In the present paper we aim to give a more precise description of the boundedness of the viscosity solution of (1-2). Denote by $u_\varphi(x, t)$ the unique viscosity solution of the Cauchy problem

$$\begin{cases} u_t + H(x, u_x, t) = 0 & \text{in } M \times (0, +\infty), \\ u(x, 0) = \varphi(x) & \text{on } M, \end{cases} \tag{1-3}$$

where $\varphi : M \to \mathbb{R}^1$ is a continuous function.

Definition 1.1 (asymptotic bounds). (i) We say that a 1-periodic viscosity solution v of (1-2) is an asymptotic upper bound (AUB) of u_φ, if for each $\epsilon > 0$ there exists $T > 0$ such that

$$u_\varphi(x, t) \le v(x, t) + \epsilon, \quad \forall (x, t) \in M \times [T, +\infty). \tag{1-4}$$

Furthermore, a function $\bar{v} : M \times [0, +\infty) \to \mathbb{R}^1$ is called the minimal asymptotic upper bound (min AUB) of u_φ, if it is an AUB and for each AUB v, we have

$$\bar{v}(x, t) \le v(x, t), \quad \forall (x, t) \in M \times [0, +\infty).$$

(ii) We say that a 1-periodic viscosity solution w of (1-2) is an asymptotic lower bound (ALB) of u_φ, if for each $\epsilon > 0$ there exists $T > 0$ such that

$$u_\varphi(x, t) \ge w(x, t) - \epsilon, \quad \forall (x, t) \in M \times [T, +\infty). \tag{1-5}$$

Furthermore, a function $\underline{w} : M \times [0, +\infty) \to \mathbb{R}^1$ is called the maximal asymptotic lower bound (max ALB) of u_φ, if it is an ALB and for each ALB w, we have

$$\underline{w}(x, t) \ge w(x, t), \quad \forall (x, t) \in M \times [0, +\infty).$$

Remark 1.1. (i) The existence of the AUB and the ALB of u_φ follows immediately from the existence of 1-periodic viscosity solutions of (1-2) and the boundedness of u_φ.

(ii) We assert that if the min AUB $\bar{v}$ and the max ALB $\underline{w}$ of u_φ exist, then for each $\epsilon > 0$ and each $T > 0$, there exist $(\bar{x}, \bar{t})$, $(\underline{x}, \underline{t}) \in M \times [T, +\infty)$ such that

$$u_\varphi(\bar{x}, \bar{t}) > \bar{v}(\bar{x}, \bar{t}) - \epsilon, \quad u_\varphi(\underline{x}, \underline{t}) < \underline{\mathrm{w}}(\underline{x}, \underline{t}) + \epsilon.$$

To show the first assertion, we argue by contradiction. For, otherwise, there would be $\epsilon_0 > 0$ and $T_0 > 0$ such that

$$u_\varphi(x, t) \le \bar{v}(x, t) - \epsilon_0 =: v'(x, t), \quad \forall (x, t) \in M \times [T_0, +\infty).$$

Note that v' is a 1-periodic viscosity solution of (1-2) and that for each $\epsilon > 0$,

$$u_\varphi(x, t) \le v'(x, t) \le v'(x, t) + \epsilon, \quad \forall (x, t) \in M \times [T_0, +\infty).$$

By definition, v' is an AUB of u_φ. Recall that $v'(x, t) := \bar{v}(x, t) - \epsilon_0$. Thus, we obtain a contradiction to the assumption that $\bar{v}$ is the min AUB of u_φ. By a similar argument, one can prove the second assertion.

The major result of the paper is as follows:

Theorem 1.1. *For every continuous function $\varphi : M \to \mathbb{R}^1$, both the min AUB and the max ALB of u_φ exist. Let $\bar{\varphi}$ and $\underline{\varphi}$ denote the min AUB and the max ALB respectively. Then there is a constant $C > 0$ such that*

$$|\bar{\varphi}(x,t) - \underline{\varphi}(x,t)| \le C, \quad \forall M \times [0,+\infty), \tag{1-6}$$

where C is independent of φ.

For a given $\varphi \in C(M, \mathbb{R}^1)$, we will show how to obtain $\bar{\varphi}$ and $\underline{\varphi}$ in Section 3. An outline of this paper is as follows. Section 2 includes some basic definitions and preliminary results. Section 3 is devoted to the proof of Theorem 1.1.

In recent years, many convergence results on the asymptotic behavior of viscosity solutions of Hamilton–Jacobi equations with the Hamiltonian independent of t have been obtained by various authors since the pioneering work of Lions [1982] and Barles [1985]. Among them, it is worth mentioning in particular that dynamical techniques were used first by Fathi [1998] and Roquejoffre [2001] to attack such problems. See [Ishii 2006] for more details.

2. Preliminaries

In this section we introduce the notation used in the sequel and review some definitions and results of Mather theory and weak KAM theory [Fathi 2005; Mañé 1997; Mather 1991; 1993]. We view $\mathbb{S}^1$ as a fundamental domain in $\mathbb{R}^1 : \bar{I} = [0,1]$ with the two endpoints identified. The standard universal covering projection $\pi : \mathbb{R}^1 \to \mathbb{S}^1$ takes the form $\pi(t) = [t]$, where $[t] = t \mod 1$ denotes the fractional part of t.

The L-action of a continuous and piecewise C^1 curve $\gamma : [a,b] \to M$ is defined by

$$A_L(\gamma) = \int_a^b L(d\gamma(\sigma), \sigma)\, d\sigma,$$

where $d\gamma : [a,b] \to TM$ denotes the differential of γ.

2A. ***Mañé critical value.*** The notion of the critical value of autonomous Tonelli Hamiltonians (or Lagrangians) was introduced by Mañé [1997]; see also [Contreras et al. 1997]. We can define the critical value of time-periodic Tonelli Hamiltonians (or Lagrangians) in a similar way. Contreras et al. [2013] gave the following property of the critical value for time-periodic case, which can also be regarded as equivalent definitions of the critical value.

$$\begin{aligned} c(H) &:= \inf\{k \in \mathbb{R}^1 : A_{L+k}(\gamma) \ge 0 \text{ for all absolutely continuous closed curves } \gamma\} \\ &= \sup\{k \in \mathbb{R}^1 : A_{L+k}(\gamma) < 0 \text{ for some absolutely continuous closed curve } \gamma\}, \end{aligned}$$

where $L(x,v,t) = \sup_{p \in T_x^*M}\{\langle p, v\rangle_x - H(x,p,t)\}$ and a curve $\gamma : [a,b] \to M$

will be called closed if $\gamma(a)=\gamma(b)$ and $b-a$ is an integer. It is straightforward to verify that the critical value of $H-c(H)$ is 0.

2B. ***Lax–Oleinik semigroup and viscosity solutions.*** For each $t \geq 0$ and each $\varphi \in C(M, \mathbb{R}^1)$, let

$$T_t\varphi(x) = \inf_{y \in M}\big\{\varphi(y) + \inf_{\gamma} A_L(\gamma)\big\}$$

for all $x \in M$, where the second infimum is taken among the continuous and piecewise C^1 paths $\gamma : [0, t] \to M$ with $\gamma(0) = y$ and $\gamma(t) = x$. For each $t \geq 0$, T_t is an operator from $C(M, \mathbb{R}^1)$ to itself. Since L is time-periodic, $\{T_n\}_{n\in\mathbb{N}}$ is a one-parameter semigroup of operators, called the Lax–Oleinik semigroup associated with L, where $\mathbb{N} = \{0, 1, 2, \ldots\}$. By the definition of T_t, one can easily verify the following properties:

(i) $T_{n+t}\varphi(x) = T_t \circ T_n\varphi(x), \quad \forall n \in \mathbb{N},\ \forall t \geq 0.$

(ii) The function $(x, t) \mapsto T_t\varphi(x)$ is continuous on $M \times [0, +\infty)$, for each $\varphi \in C(M, \mathbb{R}^1)$.

(iii) For each $\varphi_1, \varphi_2 \in C(M, \mathbb{R}^1)$ and each $t \geq 0$, we have

$$\varphi_1 \leq \varphi_2 \Rightarrow T_t\varphi_1 \leq T_t\varphi_2. \qquad \text{(monotonicity)}$$

(iv) For each $\varphi_1, \varphi_2 \in C(M, \mathbb{R}^1)$ and each $t \geq 0$, we have

$$\|T_t\varphi_1 - T_t\varphi_2\|_\infty \leq \|\varphi_1 - \varphi_2\|_\infty, \qquad \text{(nonexpansiveness)}$$

where $\|\cdot\|_\infty$ denotes the supremum norm in the space $C(M, \mathbb{R}^1)$.

As mentioned in Section 1, the Cauchy problem (1-3) is well posed in the viscosity sense. Furthermore, $u_\varphi(x, t) = T_t\varphi(x)$, for all $(x, t) \in M \times [0, +\infty)$, which means that $T_{\cdot}\varphi(\,\cdot\,)$ is the unique viscosity solution of (1-3); see, e.g., [Fathi and Mather 2000].

2C. ***Peierls barrier.*** As in [Mather 1993], it is convenient to introduce, for all $t < t' \in \mathbb{R}^1$ and $x, x' \in M$, the quantity

$$F_{t,t'}(x, x') = \inf_{\gamma} A_L(\gamma),$$

where the infimum is taken over the continuous and piecewise C^1 paths $\gamma : [t, t'] \to M$ such that $\gamma(t) = x$ and $\gamma(t') = x'$. For all $t < t' \in \mathbb{R}$ and all $x, x' \in M$, A_L takes a finite minimum value over the set of continuous and piecewise C^1 paths $\gamma : [t, t'] \to M$ such that $\gamma(t) = x$ and $\gamma(t') = x'$ [Mather 1991]. For each $t \geq 0$, each $\varphi \in C(M, \mathbb{R}^1)$ and each $x \in M$, it is easy to see that

$$T_t\varphi(x) = \inf_{y \in M}\big\{\varphi(y) + F_{0,t}(y, x)\big\}. \tag{2-1}$$

The following lemma [Bernard 2002] will be useful later. Recall that the critical value of the Lagrangian is 0. Such a Lagrangian is called critical in [Bernard 2002].

Lemma 2.1. *The function*

$$F : \mathbb{R}^1 \times \mathbb{R}^1 \times M \times M \to \mathbb{R}^1, \quad (t, t', x, x') \mapsto F_{t,t'}(x, x')$$

is Lipschitz and bounded on $\{t' \geq t + 1\}$.

In light of (ii) in Section 2B, (2-1) and Lemma 2.1, we have the following proposition.

Proposition 2.1. *All the viscosity solutions of* (1-2) *are bounded on* $M \times [0, +\infty)$.

Recall the notion of Peierls barrier introduced in [Mather 1993], which is the main ingredient in Mather's approach. Define the Peierls barrier as

$$h_{s,s'}(x, x') = \liminf_{t'-t\to+\infty} F_{t,t'}(x, x'), \quad s, s' \in \mathbb{S}^1, \ x, x' \in M,$$

where the lim inf is restricted to the set of $(t, t') \in \mathbb{R}^2$ such that $s = [t]$, $s' = [t']$. In view of Lemma 2.1, the lim inf in the definition exists. It is clear that

$$h_{s,s'}(x, x') = \liminf_{n\to+\infty} F_{s,s'+n}(x, x') = \lim_{n\to+\infty} \inf_{k\geq n} F_{s,s'+k}(x, x').$$

Again by Lemma 2.1, the family of functions $\{\inf_{k\geq n} F_{s,s'+k}(x, x')\}_n$ is equi-Lipschitz and thus

$$h_{s,s'}(x, x') = \lim_{n\to+\infty} \inf_{k\geq n} F_{s,s'+k}(x, x') \tag{2-2}$$

uniformly on $\mathbb{S}^1 \times \mathbb{S}^1 \times M \times M$. An important property of the Peierls barrier is that it is Lipschitz; see [Contreras et al. 2013].

Lemma 2.2. *Given any* $s \in \mathbb{S}^1$, $t \in \mathbb{R}^1$ *with* $[t] = s$ *and any* $x, y \in M$,

(i) $h_{0,s}(x, y) = \inf_{z\in M}(h_{0,0}(x, z) + F_{0,t}(z, y))$,

(ii) $h_{0,s}(x, y) = \inf_{z\in M}(F_{0,t}(x, z) + h_{0,0}(z, y))$.

Proof. (i) The inequality

$$\Big|\inf_{z\in M}\Big(\inf_{k\geq n} F_{0,k}(x, z) + F_{0,t}(z, y)\Big) - \inf_{z\in M}(h_{0,0}(x, z) + F_{0,t}(z, y))\Big| \leq \sup_{z\in M}\Big|\inf_{k\geq n} F_{0,k}(x, z) - h_{0,0}(x, z)\Big|,$$

together with (2-2) implies that

$$\lim_{n\to+\infty} \inf_{z\in M}\Big(\inf_{k\geq n} F_{0,k}(x, z) + F_{0,t}(z, y)\Big) = \inf_{z\in M}(h_{0,0}(x, z) + F_{0,t}(z, y)).$$

This equality can be rewritten as

$$\begin{aligned}\inf_{z\in M}(h_{0,0}(x,z)+F_{0,t}(z,y)) &= \lim_{n\to+\infty}\inf_{z\in M}\Big(\inf_{k\geq n}F_{0,k}(x,z)+F_{0,t}(z,y)\Big)\\ &= \lim_{n\to+\infty}\inf_{z\in M}\Big(\inf_{k\geq n}(F_{0,k}(x,z)+F_{k,k+t}(z,y))\Big)\\ &= \lim_{n\to+\infty}\inf_{k\geq n}\Big(\inf_{z\in M}(F_{0,k}(x,z)+F_{k,k+t}(z,y))\Big)\\ &= \lim_{n\to+\infty}\inf_{k\geq n}F_{0,k+t}(x,y)\\ &= h_{0,s}(x,y),\end{aligned}$$

and therefore (i) holds.

The proof of (ii) follows in a similar manner. □

2D. *Mañé potential and 1-periodic viscosity subsolutions.* For each $(s,s')\in\mathbb{S}^1\times\mathbb{S}^1$, let

$$\Phi_{s,s'}(x,x')=\inf F_{t,t'}(x,x')$$

for all $(x,x')\in M\times M$, where the infimum is taken on the set of $(t,t')\in\mathbb{R}^2$ such that $s=[t]$, $s'=[t']$ and $t'\geq t+1$. This quantity is commonly called Mañé potential [Mañé 1997].

Lemma 2.3. *A continuous function $u: M\times\mathbb{S}^1\to\mathbb{R}^1$ is a viscosity subsolution of (1-2) only if*

$$u(x',s')-u(x,s)\leq\Phi_{s,s'}(x,x'),\quad \forall(x,s),\ (x',s')\in M\times\mathbb{S}^1.$$

See, e.g., [Fathi 2005] for a proof.

2E. *Weak KAM solutions and 1-periodic viscosity solutions.* A function $u: M\times\mathbb{S}^1\to\mathbb{R}^1$ is called a weak KAM solution of (1-2) if u is a viscosity subsolution of (1-2) and if, for every $(x,s)\in M\times\mathbb{S}^1$ there exists a curve $\gamma:(-\infty,s]\to M$ with $\gamma(s)=x$ such that

$$w(x,s)-w(\gamma(t),[t])=\int_t^s L(d\gamma(\sigma),\sigma)\,d\sigma,\quad \forall t\in(-\infty,s].$$

Denote by $\mathcal{S}$ the set of weak KAM solutions.

Let us recall two elementary results [Contreras et al. 2013] about weak KAM solutions.

(i) Given $(x_0,s_0)\in M\times\mathbb{S}^1$, define $u^*(x,s):=h_{s_0,s}(x_0,x)$. Then $u^*\in\mathcal{S}$.

(ii) If $\mathcal{U}\subset\mathcal{S}$, let $u_*(x,s):=\inf_{u\in\mathcal{U}}u(x,s)$, then either $u_*\equiv-\infty$ or $u_*\in\mathcal{S}$.

In view of (i) and (ii), it is clear that for each $\varphi\in C(M,\mathbb{R}^1)$,

$$\underline{\varphi}(x,s):=\inf_{y\in M}(\varphi(y)+h_{0,s}(y,x))\in\mathcal{S}. \tag{2-3}$$

In Section 3, we will show that $\underline{\varphi}$ is the max ALB of u_φ.

The following result was proved in [Wang and Yan 2012].

Proposition 2.2. *Weak KAM solutions and* 1*-periodic viscosity solutions of* (1-2) *are the same.*

2F. *Projected Aubry sets and weak KAM solutions.* Recall the definition of the projected Aubry set $\mathcal{A}_0$:

$$\mathcal{A}_0 := \{(x, s) \in M \times \mathbb{S}^1 \mid h_{s,s}(x, x) = 0\}.$$

Define an equivalence relation on $\mathcal{A}_0$ by saying that (x, s) and (x', s') are equivalent if and only if

$$h_{s,s'}(x, x') + h_{s',s}(x', x) = 0.$$

The equivalent classes of this relation are called static classes. Let A be the set of static classes. For each static class $\Gamma \in$ A choose a point $(x, 0) \in \Gamma$ and let $\mathbb{A}_0$ be the set of such points.

The following result in [Contreras et al. 2013] characterizes weak KAM solutions of (1-2) in terms of their values at each static class and the Peierls barrier.

Proposition 2.3. *Let u be a weak KAM solution of* (1-2)*. Then we have*

$$u(x, [t]) = \min_{(p,0)\in\mathbb{A}_0} (u(p, 0) + h_{0,[t]}(p, x)), \quad \forall (x, t) \in M \times [0, +\infty). \tag{2-4}$$

3. Proof of Theorem 1.1

To prove the main result, we need some more auxiliary results.

Proposition 3.1. *Given a continuous function* $\varphi : M \to \mathbb{R}^1$, *we have*

(i) $\liminf_{n\to+\infty} T_{n+t}\varphi(x) = \inf_{y\in M}(\varphi(y)+h_{0,[t]}(y, x)) = \underline{\varphi}(x, [t])$, *for all* $(x, t) \in M \times [0, +\infty)$, *where* $\underline{\varphi}$ *denotes the function we have defined in* (2-3).

(ii) $\liminf_{n\to+\infty} T_{n+t}\varphi(x) = T_t(\liminf_{n\to+\infty} T_n\varphi)(x)$, *for all* $(x, t) \in M\times[0, +\infty)$.

Proof. (i) Note that

$$\begin{aligned}
\Big|\inf_{k\geq n} T_{k+t}\varphi(x) - \inf_{y\in M}(\varphi(y) + h_{0,[t]}(y, x))\Big| \\
&= \Big|\inf_{k\geq n}\inf_{y\in M}(\varphi(y) + F_{0,k+t}(y, x)) - \inf_{y\in M}(\varphi(y) + h_{0,[t]}(y, x))\Big| \\
&= \Big|\inf_{y\in M}(\varphi(y) + \inf_{k\geq n} F_{0,k+t}(y, x)) - \inf_{y\in M}(\varphi(y) + h_{0,[t]}(y, x))\Big| \\
&\leq \sup_{y\in M}\Big|\inf_{k\geq n} F_{0,k+t}(y, x) - h_{0,[t]}(y, x)\Big|.
\end{aligned}$$

Taking (2-2) into consideration, we have

$$\liminf_{n\to+\infty} T_{n+t}\varphi(x) = \lim_{n\to+\infty}\inf_{k\geq n} T_{k+t}\varphi(x) = \inf_{y\in M}(\varphi(y) + h_{0,[t]}(y, x)).$$

(ii) Let

$$\psi(x) = \underline{\varphi}(x, 0), \quad \forall x \in M.$$

Then by (i), we have

$$\psi(x) = \inf_{y\in M}(\varphi(y) + h_{0,0}(y, x)) = \liminf_{n\to+\infty} T_n\varphi(x),$$

and thus it suffices to show that $T_t\psi(x) = \inf_{y\in M}(\varphi(y)+h_{0,[t]}(y, x))$, for all $x \in M$, for all $t \geq 0$. It is clear that

(3-1)
$$\begin{aligned} T_t\psi(x) &= \inf_{y\in M}(\psi(y) + F_{0,t}(y, x)) \\ &= \inf_{y\in M}\big(\inf_{z\in M}(\varphi(z) + h_{0,0}(z, y)) + F_{0,t}(y, x)\big) \\ &= \inf_{z\in M}\big(\varphi(z) + \inf_{y\in M}(h_{0,0}(z, y) + F_{0,t}(y, x))\big). \end{aligned}$$

Combining (3-1) and (i) of Lemma 2.2, we get

$$T_t\psi(x) = \inf_{z\in M}(\varphi(z) + h_{0,[t]}(z, x)).$$ □

Proposition 3.2. *Given a continuous function* $\varphi : M \to \mathbb{R}^1$, $\limsup_{n\to+\infty} T_{n+t}\varphi(x)$ *exists for all* $(x, t) \in M \times [0, +\infty)$. *Let*

$$\tilde{\varphi}(x, t) = \limsup_{n\to+\infty} T_{n+t}\varphi(x), \quad \forall (x, t) \in M \times [0, +\infty).$$

Then $\tilde{\varphi}$ *is a* 1*-periodic viscosity subsolution of* (1-2).

Proof. It is apparent from Lemma 2.1 that $\limsup_{n\to+\infty} T_{n+t}\varphi(x)$ exists for all $(x, t) \in M \times [0, +\infty)$. Since

$$\begin{aligned} \tilde{\varphi}(x, t+1) &= \lim_{n\to+\infty} \sup_{k\geq n} T_{k+t+1}\varphi(x) = \lim_{n\to+\infty} \sup_{k\geq n+1} T_{k+t}\varphi(x) \\ &= \lim_{n\to+\infty} \sup_{k\geq n} T_{k+t}\varphi(x) = \tilde{\varphi}(x, t) \end{aligned}$$

for all $(x, t) \in M \times [0, +\infty)$, then $\tilde{\varphi}$ is 1-periodic in t. Therefore, in order to complete the proof, by Lemma 2.3 we only need to show that

$$\tilde{\varphi}(x', s') - \tilde{\varphi}(x, s) \leq \Phi_{s,s'}(x, x'), \quad \forall (x, s),\ (x', s') \in M \times \mathbb{S}^1.$$

For any positive integer m, we have

$$\begin{aligned} &\sup_{k\geq n}\inf_{y\in M}(\varphi(y) + F_{0,k+m+s'}(y, x')) - \sup_{k\geq n}\inf_{y\in M}(\varphi(y) + F_{0,k+s}(y, x)) \\ &\quad\leq \sup_{k\geq n}\big(\inf_{y\in M}(\varphi(y) + F_{0,k+m+s'}(y, x')) - \inf_{y\in M}(\varphi(y) + F_{0,k+s}(y, x))\big) \\ &\quad\leq \sup_{k\geq n}\sup_{y\in M}(F_{0,k+m+s'}(y, x') - F_{0,k+s}(y, x)) \\ &\quad\leq \sup_{k\geq n} F_{k+s,k+m+s'}(x, x') = F_{s,m+s'}(x, x'). \end{aligned}$$

By taking the limit for $n \to +\infty$, we find

$$\tilde{\varphi}(x', m+s') - \tilde{\varphi}(x, s) \leq F_{s,m+s'}(x, x').$$

Since m is an arbitrary positive integer and $\tilde{\varphi}$ is 1-periodic in t, by the definition of Mañé potential, we have

$$\tilde{\varphi}(x', s') - \tilde{\varphi}(x, s) \leq \Phi_{s,s'}(x, x'). \qquad \square$$

Proposition 3.3. *Let $\varphi \in C(M, \mathbb{R}^1)$ and $\hat{\varphi}(x) = \limsup_{n\to+\infty} T_n\varphi(x)$, for all $x \in M$. Then*

$$\hat{\varphi}(x) \leq T_1\hat{\varphi}(x) \leq \cdots \leq T_n\hat{\varphi}(x) \leq \cdots, \quad \forall x \in M, \tag{3-2}$$

and the uniform limit $\lim_{n\to+\infty} T_n\hat{\varphi}(x)$ exists. Let $\varphi_\infty(x) = \lim_{n\to+\infty} T_n\hat{\varphi}(x)$ and $\bar{\varphi}(x, t) = T_t\varphi_\infty(x)$, for all $x \in M$, for all $t \geq 0$. Then φ_∞ is a fixed point of T_1 and $\bar{\varphi}$ is a 1-periodic viscosity solution of (1-2).

Remark 3.1. It is easy to check that

$$\bar{\varphi}(x, t) = \lim_{n\to+\infty} T_{n+t}\hat{\varphi}(x), \quad \forall x \in M, \ \forall t \geq 0.$$

Proof of Proposition 3.3. By Proposition 3.2, $\hat{\varphi}$ is well defined. In view of the definition of $\hat{\varphi}$, for any $\epsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that

$$T_n\varphi(x) \leq \hat{\varphi}(x) + \epsilon,$$

for all $x \in M$ and all $n \geq N_0$. Using the monotonicity of the Lax–Oleinik operator, we see that

$$T_1 \circ T_n(x) \leq T_1\hat{\varphi}(x) + \epsilon,$$

for all $x \in M$ and all $n \geq N_0$. Combining the above inequality, (i) in Section 2B and the definition of $\hat{\varphi}$, we have

$$\hat{\varphi}(x) = \lim_{n\to+\infty} \sup_{k\geq n+1} T_k\varphi(x) = \limsup_{n\to+\infty} T_{n+1}\varphi(x) \leq T_1\hat{\varphi}(x) + \epsilon, \quad \forall x \in M.$$

It follows that

$$\hat{\varphi}(x) \leq T_1\hat{\varphi}(x), \quad \forall x \in M,$$

since ϵ may be taken arbitrarily small. Again by the monotonicity of the Lax–Oleinik operator, we get that

$$\hat{\varphi}(x) \leq T_1\hat{\varphi}(x) \leq \cdots \leq T_n\hat{\varphi}(x) \leq \cdots, \quad \forall x \in M.$$

From Lemma 2.1, it is easy to see that $\{T_n\hat{\varphi}\}_n$ is uniformly bounded and equi-Lipschitz. Therefore, the uniform limit $\lim_{n\to+\infty} T_n\hat{\varphi}(x)$ exists, i.e.,

$$\varphi_\infty(x) := \lim_{n\to+\infty} T_n\hat{\varphi}(x), \tag{3-3}$$

uniformly on $x \in M$.

The following inequality which comes from the nonexpansiveness of the Lax–Oleinik semigroup;

$$\|T_n\hat{\varphi} - T_1\varphi_\infty\|_\infty \le \|T_{n-1}\hat{\varphi} - \varphi_\infty\|_\infty,$$

together with (3-3), implies that

$$\varphi_\infty(x) = \lim_{n\to+\infty} T_n\hat{\varphi}(x) = T_1\varphi_\infty(x), \quad \forall x \in M, \tag{3-4}$$

namely φ_∞ is a fixed point of T_1.

Finally we prove that $\bar{\varphi}(x,t) := T_t\varphi_\infty(x)$ is a 1-periodic viscosity solution of (1-2). As mentioned in Section 2B, $T_t\varphi_\infty(x)$ is a viscosity solution of (1-2). Thus, it suffices to show that $\bar{\varphi}(x,t)$ is 1-periodic in t. By (i) in Section 2B and (3-4),

$$\bar{\varphi}(x,t+1) = T_{t+1}\varphi_\infty(x) = T_t \circ T_1\varphi_\infty(x) = T_t\varphi_\infty(x) = \bar{\varphi}(x,t), \ \forall x \in M, \ \forall t \ge 0.$$

The proof of the proposition is now complete. □

Lemma 3.1. *For $\bar{\varphi}$ and $\hat{\varphi}$ defined in Proposition 3.3, we have $\bar{\varphi}(x,0) = \hat{\varphi}(x)$, for all $(x,0) \in \mathbb{A}_0$.*

Proof. By (3-2), we have $\hat{\varphi}(x) \le T_n\hat{\varphi}(x)$ for all $x \in M$ and all $n \in \mathbb{N}$, which implies

$$\hat{\varphi}(x) \le \lim_{n\to+\infty} T_n\hat{\varphi}(x) = \bar{\varphi}(x,0), \quad \forall x \in M.$$

On the other hand, for each $(x,0) \in \mathbb{A}_0$, we have

$$\begin{aligned}\bar{\varphi}(x,0) &= \lim_{n\to+\infty} T_n\hat{\varphi}(x)\\ &= \lim_{n\to+\infty} \inf_{y\in M}(\hat{\varphi}(y) + F_{0,n}(y,x)) \le \hat{\varphi}(x) + \liminf_{n\to+\infty} F_{0,n}(x,x)\\ &= \hat{\varphi}(x) + h_{0,0}(x,x) = \hat{\varphi}(x).\end{aligned}$$

□

Proof of Theorem 1.1. We divide our proof in three steps. First, we show that $\bar{\varphi}$ in Proposition 3.3 is an AUB of u_φ and that $\underline{\varphi}$ in (2-3) is an ALB of u_φ. Since we have shown that $\bar{\varphi}$ and $\underline{\varphi}$ are 1-periodic viscosity solutions of (1-2), it suffices to prove that $\bar{\varphi}$ and $\underline{\varphi}$ satisfy (1-4) and (1-5), respectively. Next, we prove that $\bar{\varphi}$ is the min AUB of u_φ and that $\underline{\varphi}$ is the max ALB of u_φ. Finally, we need to show that (1-6) holds for some constant $C > 0$ which depends only on L.

Step 1. Our task now is to verify that $\bar{\varphi}$ and $\underline{\varphi}$ are AUB and ALB of u_φ, respectively.

First, we show that $\bar{\varphi}$ satisfies (1-4), which implies that $\bar{\varphi}$ is an AUB of u_φ. From the definition of $\hat{\varphi}$, for every $\epsilon > 0$, there exists $N_1 \in \mathbb{N}$ such that

$$T_n\varphi(x) - \hat{\varphi}(x) \le \sup_{k\ge n} T_k\varphi(x) - \hat{\varphi}(x) \le \epsilon, \quad \forall x \in M, \ \forall n \ge N_1. \tag{3-5}$$

Combining (3-2) and (3-5), we have

$$T_n\varphi(x) - \varphi_\infty(x) \le \epsilon, \quad \forall x \in M, \ \forall n \ge N_1.$$

Again by the monotonicity of the Lax–Oleinik operator, we get

$$T_\tau \circ T_n\varphi(x) \le T_\tau\varphi_\infty(x) + \epsilon, \quad \forall x \in M,\ \forall \tau \in [0, 1],\ \forall n \ge N_1.$$

In view of (i) in Section 2B and the definition of $\bar{\varphi}$, the above inequality implies

$$T_{n+\tau}\varphi(x) \le \bar{\varphi}(x, \tau) + \epsilon, \quad \forall x \in M,\ \forall \tau \in [0, 1],\ \forall n \ge N_1.$$

Since $u_\varphi(x, t) = T_t\varphi(x)$ and $\bar{\varphi}(x, t)$ is 1-periodic in t, we have

$$u_\varphi(x, n+\tau) \le \bar{\varphi}(x, n+\tau) + \epsilon, \quad \forall x \in M,\ \forall \tau \in [0, 1],\ \forall n \ge N_1,$$

i.e.,

$$u_\varphi(x, t) \le \bar{\varphi}(x, t) + \epsilon, \quad \forall (x, t) \in M \times [N_1, +\infty).$$

Hence, $\bar{\varphi}$ is an AUB of u_φ.

Then, we show that $\underline{\varphi}$ satisfies (1-5), which implies that $\underline{\varphi}$ is an ALB of u_φ. For each $\epsilon > 0$, by (2-2) there exists $N_2 \in \mathbb{N}$ such that

$$\inf_{k\ge n} F_{0,k+\tau}(y, x) - h_{0,[\tau]}(y, x) \ge -\epsilon, \quad \forall x,\ y \in M,\ \forall \tau \in [0, 1], \tag{3-6}$$

if $n \ge N_2$. Since

$$\inf_{y\in M}(\varphi(y) + h_{0,[\tau]}(y, x)) - \inf_{y\in M}(\varphi(y) + F_{0,n+\tau}(y, x)) \le \sup_{y\in M}(h_{0,[\tau]}(y, x) - F_{0,n+\tau}(y, x)),$$

then by (3-6), we have

$$\inf_{y\in M}(\varphi(y) + h_{0,[\tau]}(y, x)) - \inf_{y\in M}(\varphi(y) + F_{0,n+\tau}(y, x)) \le \epsilon, \quad \forall x \in M,\ \forall \tau \in [0, 1],\ \forall n \ge N_2. \tag{3-7}$$

From the definition of $\underline{\varphi}$, (2-1) and $u_\varphi(x, t) = T_t\varphi(x)$, (3-7) becomes

$$\underline{\varphi}(x, \tau) - u_\varphi(x, n+\tau) \le \epsilon, \quad \forall x \in M,\ \forall \tau \in [0, 1],\ \forall n \ge N_2.$$

Since $\underline{\varphi}$ is 1-periodic in t, we get

$$\underline{\varphi}(x, n+\tau) - u_\varphi(x, n+\tau) \le \epsilon, \quad \forall x \in M,\ \forall \tau \in [0, 1],\ \forall n \ge N_2,$$

i.e.,

$$u_\varphi(x, t) \ge \underline{\varphi}(x, t) - \epsilon, \quad \forall (x, t) \in M \times [N_2, +\infty).$$

Hence, $\underline{\varphi}$ is an ALB of u_φ.

Step 2. We are now in a position to show that $\bar{\varphi}$ is the min AUB of u_φ and that $\underline{\varphi}$ is the max ALB of u_φ.

First, we prove that $\bar{\varphi}$ is the min AUB of u_φ, by contradiction. Otherwise, there would be an AUB v and a point $(x_0, t_0) \in M \times [0, +\infty)$ such that

$$v(x_0, t_0) < \bar{\varphi}(x_0, t_0). \tag{3-8}$$

Note that v and $\bar{\varphi}$ are both 1-periodic viscosity solutions of (1-2). In view of Propositions 2.2 and 2.3, we have

$$
\begin{aligned}
v(x,t) &= \min_{(p,0)\in\mathbb{A}_0} (v(p,0) + h_{0,[t]}(p,x)),\\
\bar{\varphi}(x,t) &= \min_{(p,0)\in\mathbb{A}_0} (\bar{\varphi}(p,0) + h_{0,[t]}(p,x)),
\end{aligned}
\tag{3-9}
$$

for all $(x,t) \in M \times [0,+\infty)$. We assert that there exists a point $(p_0,0) \in \mathbb{A}_0$ such that

$$
v(p_0,0) < \bar{\varphi}(p_0,0). \tag{3-10}
$$

Suppose otherwise. Then

$$
v(p,0) \geq \bar{\varphi}(p,0), \quad \forall (p,0) \in \mathbb{A}_0.
$$

Therefore, we have

$$
\min_{(p,0)\in\mathbb{A}_0} (v(p,0) + h_{0,[t]}(p,x)) \geq \min_{(p,0)\in\mathbb{A}_0} (\bar{\varphi}(p,0) + h_{0,[t]}(p,x))
$$

for all $(x,t) \in M \times [0,+\infty)$. The above inequality and (3-9) imply that

$$
v(x,t) \geq \bar{\varphi}(x,t), \quad \forall (x,t) \in M \times [0,+\infty),
$$

which contradicts (3-8). Hence (3-10) holds. Let $\delta_0 = \bar{\varphi}(p_0,0) - v(p_0,0)$. Then $\delta_0 > 0$ and by Lemma 3.1, we have

$$
v(p_0,0) = \hat{\varphi}(p_0) - \delta_0. \tag{3-11}
$$

Since $\hat{\varphi}(p_0) = \limsup_{n\to+\infty} T_n\varphi(p_0)$, then for the above δ_0, there exists $N_3 \in \mathbb{N}$ such that $\sup_{k\geq n} T_k\varphi(p_0) > \hat{\varphi}(p_0) - \frac{\delta_0}{2}$, if $n \geq N_3$, which implies that there exists $k_n \geq n$ such that

$$
T_{k_n}\varphi(p_0) > \hat{\varphi}(p_0) - \frac{\delta_0}{2}.
$$

From the above inequality and (3-11), we deduce that

$$
T_{k_n}\varphi(p_0) > \hat{\varphi}(p_0) - \frac{\delta_0}{2} = v(p_0,0) + \frac{\delta_0}{2}.
$$

It follows that there exist $\{k_n\}_n \subset \mathbb{N}$ with $k_n \to +\infty$ as $n \to +\infty$ such that

$$
u_\varphi(p_0,k_n) > v(p_0,k_n) + \frac{\delta_0}{2},
$$

which contradicts the assumption that v is an AUB of u_φ.

Next, we show that $\underline{\varphi}$ is the max ALB of u_φ. Suppose not. There exist an ALB w of u_φ and a point $(x_1,t_1) \in M \times [0,+\infty)$ such that

$$
w(x_1,t_1) > \underline{\varphi}(x_1,t_1).
$$

Let $\delta_1 = w(x_1, t_1) - \underline{\varphi}(x_1, t_1)$. From Proposition 3.1, we have

$$\underline{\varphi}(x_1, t_1) = \liminf_{n\to+\infty} T_{n+t_1}\varphi(x_1).$$

Thus, there exists $N_4 \in \mathbb{N}$ such that $\inf_{j\geq n} T_{j+t_1}\varphi(x_1) < \underline{\varphi}(x_1, t_1) + \frac{\delta_1}{2}$ if $n \geq N_4$. It follows that there exists $j_n \geq n$ such that

$$T_{j_n+t_1}\varphi(x_1) < \underline{\varphi}(x_1, t_1) + \frac{\delta_1}{2}.$$

This inequality and the definition of δ_1 imply that

$$T_{j_n+t_1}\varphi(x_1) < \underline{\varphi}(x_1, t_1) + \frac{\delta_1}{2} = w(x_1, t_1) - \frac{\delta_1}{2},$$

which means that there exist $\{j_n\}_n \subset \mathbb{N}$ with $j_n \to +\infty$ as $n \to +\infty$ such that

$$u_\varphi(x_1, j_n + t_1) < w(x_1, j_n + t_1) - \frac{\delta_1}{2},$$

and we have a contradiction to the assumption that w is an ALB of u_φ.

Step 3. It remains to show that (1-6) holds. Note that

$$\begin{aligned}
\text{(3-12)}\quad & \underline{\varphi}(x, 0) - \bar{\varphi}(x, 0) \\
&= \inf_{y\in M}(\varphi(y) + h_{0,0}(y, x)) - \lim_{n\to+\infty} T_n\big(\limsup_{n\to+\infty} T_n\varphi\big)(x) \\
&\leq \inf_{y\in M}(\varphi(y) + h_{0,0}(y, x)) - \limsup_{n\to+\infty} T_n\varphi(x) \\
&= \inf_{y\in M}(\varphi(y) + h_{0,0}(y, x)) - \lim_{n\to+\infty} \sup_{k\geq n} \inf_{y\in M}(\varphi(y) + F_{0,k}(y, x)) \\
&= \lim_{n\to+\infty} \sup_{k\geq n}\big(\inf_{y\in M}(\varphi(y) + h_{0,0}(y, x)) - \inf_{y\in M}(\varphi(y) + F_{0,k}(y, x))\big) \\
&\leq \lim_{n\to+\infty} \sup_{k\geq n} \sup_{y\in M}(h_{0,0}(y, x) - F_{0,k}(y, x))
\end{aligned}$$

for all $x \in M$, where for the first inequality we have used (3-2). In view of (2-2),

$$\lim_{n\to+\infty} \inf_{k\geq n} F_{0,k}(y, x) = h_{0,0}(y, x), \quad \text{uniformly on } (y, x) \in M \times M.$$

Therefore, for any $\epsilon > 0$, there exists $N_5 \in \mathbb{N}$ such that whenever $n \geq N_5$ it follows that

$$\inf_{k\geq n} F_{0,k}(y, x) \geq h_{0,0}(y, x) - \epsilon,$$

for all $x, y \in M$, which implies that

$$\text{(3-13)}\qquad \epsilon \geq h_{0,0}(y, x) - F_{0,k}(y, x),$$

for all $k \geq N_5$ and all $x, y \in M$. Combining (3-12) and (3-13), we have

$$\text{(3-14)}\qquad \underline{\varphi}(x, 0) - \bar{\varphi}(x, 0) \leq 0, \quad \forall x \in M,$$

since ϵ may be taken arbitrarily small.

On the other hand, we have

$$\begin{aligned}
\bar{\varphi}(x,0)-\underline{\varphi}(x,0) &= \lim_{n\to+\infty} T_n\big(\limsup_{n\to+\infty} T_n\varphi\big)(x) - \inf_{y\in M}(\varphi(y)+h_{0,0}(y,x)) \\
&= \lim_{n\to+\infty} T_n\hat{\varphi}(x) - \inf_{y\in M}(\varphi(y)+h_{0,0}(y,x)) \\
&= \lim_{n\to+\infty}\big(\inf_{y\in M}(\hat{\varphi}(y)+F_{0,n}(y,x)) - \inf_{y\in M}(\varphi(y)+h_{0,0}(y,x))\big) \\
&\le \limsup_{n\to+\infty}\sup_{y\in M}(\hat{\varphi}(y)-\varphi(y)+F_{0,n}(y,x)-h_{0,0}(y,x)) \\
&= \limsup_{n\to+\infty}\sup_{y\in M}\big(\limsup_{m\to+\infty} T_m\varphi(y)-\varphi(y)+F_{0,n}(y,x)-h_{0,0}(y,x)\big) \\
&= \limsup_{n\to+\infty}\sup_{y\in M}\big(\lim_{m\to+\infty}\sup_{k\ge m} T_k\varphi(y)-\varphi(y)+F_{0,n}(y,x)-h_{0,0}(y,x)\big) \\
&= \limsup_{n\to+\infty}\sup_{y\in M}\big(\lim_{m\to+\infty}\sup_{k\ge m}\inf_{z\in M}(\varphi(z)+F_{0,k}(z,y)) \\
&\qquad\qquad -\varphi(y)+F_{0,n}(y,x)-h_{0,0}(y,x)\big) \\
&\le \limsup_{n\to+\infty}\sup_{y\in M}\big(\lim_{m\to+\infty}\sup_{k\ge m} F_{0,k}(y,y)+F_{0,n}(y,x)-h_{0,0}(y,x)\big)
\end{aligned}$$

for all $x\in M$. From Lemma 2.1, we get

$$\bar{\varphi}(x,0)-\underline{\varphi}(x,0)\le C,\quad \forall x\in M, \tag{3-15}$$

where $C>0$ depends only on L.

Combining (3-14) and (3-15), we obtain

$$\underline{\varphi}(x,0)\le\bar{\varphi}(x,0)\le\underline{\varphi}(x,0)+C,\quad \forall x\in M.$$

By the monotonicity of the Lax–Oleinik operator, we have

$$T_t\underline{\varphi}(x,0)\le T_t\bar{\varphi}(x,0)\le T_t\underline{\varphi}(x,0)+C,\quad \forall x\in M,\ \forall t\ge 0.$$

In view of Propositions 3.1 and 3.3, we have

$$|\bar{\varphi}(x,t)-\underline{\varphi}(x,t)|\le C,\quad \forall (x,t)\in M\times[0,+\infty),$$

which means that (1-6) holds true, and the proof is complete. □

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