Pacific Journal of Mathematics

Volume 298 No. 2 February 2019

PACIFIC JOURNAL OF MATHEMATICS

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

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The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFlow® from Mathematical Sciences Publishers.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 298, No. 2, 2019
dx.doi.org/10.2140/pjm.2019.298.257

UNIQUENESS QUESTIONS FOR C^*-NORMS ON GROUP RINGS

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We provide a large class of discrete amenable groups for which the complex group ring has several C*-completions, thus providing partial evidence towards a positive answer to a question raised by Rostislav Grigorchuk, Magdalena Musat and Mikael Rørdam.

1. Introduction

The interplay between group theory and operator algebras dates back to the seminal papers by Murray and von Neumann [1936] and by choosing different completions of a discrete countable group Γ one obtains interesting analytic objects; for instance the Banach algebra $\ell^1(\Gamma)$, the full and reduced C^*-algebras $\mathrm{C}^*(\Gamma)$ and $\mathrm{C}^*_{\mathrm{r}}(\Gamma)$, and the group von Neumann algebra $L\Gamma$. In general there are many norms on, say, $\ell^1(\Gamma)$ such that the completion with respect to this norm gives a C^*-algebra, and the question of when the C^*-completion is unique (in which case Γ is said to be C^*-*unique*) has been studied by various authors [Leung and Ng 2004; Boidol 1984; Barnes 1983]. A C^*-unique discrete group is evidently amenable and it is, to the best of the authors' knowledge, an open question whether the converse is true, although it is known to be false in the more general context of locally compact groups [Leung and Ng 2004]. More recently, the paper [Grigorchuk et al. 2018] put emphasis on the question of when the complex group algebra $\mathbb{C}\Gamma$ has a unique C^*-completion. As is easily seen [Grigorchuk et al. 2018, Proposition 6.7], if Γ is locally finite (i.e., if every finitely generated subgroup is finite) then $\mathbb{C}\Gamma$ has a unique C^*-completion, and [Grigorchuk et al. 2018, Question 6.8] asks if the converse is true. The present paper provides partial evidence towards a positive answer to this, in that we prove that the following classes of nonlocally finite groups have several C^*-completions.

Theorem A (see Proposition 2.4 and Corollary 3.7). *The class of countable groups* Γ *for which* $\mathbb{C}\Gamma$ *does not have a unique* C^**-norm includes the following*:

(i) *Infinite groups of polynomial growth.*

MSC2010: 16S34, 46L05, 46L10.
Keywords: group rings, C*-norms, the Atiyah conjecture.

(ii) *Torsion free, elementary amenable groups with a nontrivial, finite conjugacy class.*

(iii) *Groups with a central element of infinite order.*

The key to the proof of (i) and (ii) is the so-called strong Atiyah conjecture (see Section 3A), which predicts a concrete restriction on the von Neumann dimension of kernels of elements in the complex group algebra under the left regular representation — notably these are predicted to be either zero or one if the group in question is torsion free.

2. Basic results on C*-uniqueness

In what follows, all discrete groups are implicitly assumed to be at most countable. We will use several operator algebras associated to a discrete group Γ: the maximal C*-algebra $C^*(\Gamma)$, the reduced C*-algebra $C^*_r(\Gamma)$ and the von Neumann algebra $L\Gamma$. For more information on these, we refer to [Brown and Ozawa 2008, §2.5]. We recall that $L\Gamma = (\lambda(\mathbb{C}\Gamma))'' \subset \mathbb{B}(\ell^2\Gamma)$ is generated by the left regular representation $\lambda : \mathbb{C}\Gamma \to \mathbb{B}(\ell^2\Gamma)$ and carries a canonical, faithful, normal trace given by $\tau(x) = \langle x\delta_e, \delta_e\rangle$. In what follows, tr will denote the normalized trace on $\mathbb{M}_n(\mathbb{C})$ while Tr will denote the nonnormalized trace.

We begin by formally introducing the notion of C*-uniqueness. In order to avoid a notational conflict with the already existing notions studied in [Leung and Ng 2004; Boidol 1984], we emphasize that we are investigating the uniqueness of C*-norms on the complex group algebra in contrast to the ℓ^1-algebra.

Definition 2.1. Let Γ be a discrete group. $\mathbb{C}\Gamma$ is said to be

(i) C*-unique if it carries a unique C*-norm;

(ii) C^*_r-unique if no C*-norm on $\mathbb{C}\Gamma$ is properly majorised by the reduced C*-norm.

Γ is said to be algebraically C*-unique if $\mathbb{C}\Gamma$ is C*-unique, and it is said to be C^*_r-unique if $\mathbb{C}\Gamma$ is C^*_r-unique.

Amenable groups are characterized by the property that the maximal and reduced C*-algebras coincide, and thus a nonamenable group is never algebraically C*-unique; on the other hand, for amenable groups the above notions coincide. Note also that the class of C*-simple groups, which has recently received a lot of attention [Breuillard et al. 2017; Le Boudec 2017], falls within the class of algebraically C^*_r-unique groups. As already mentioned in the introduction, algebraic C*-uniqueness appeared in the recent paper [Grigorchuk et al. 2018] in which the authors observed that locally finite groups have this property and asked if this characterizes the class of locally finite groups. Below we prove a few basic permanence results regarding algebraic C*-uniqueness, but before doing so we give an alternative characterization, which is a straightforward algebraic adaptation of the similar result for ℓ^1-algebras [Barnes 1983, Proposition 2.4].

Lemma 2.2. *Let* Γ *be a discrete group. Then* $\mathbb{C}\Gamma$ *is* C^**-unique* (*respectively*, C^*_r*-unique*) *if and only if every nontrivial closed*, *two-sided ideal in* $\mathrm{C}^*(\Gamma)$ (*respectively*, $\mathrm{C}^*_\mathrm{r}(\Gamma)$) *intersects* $\mathbb{C}\Gamma$ *nontrivially.*

Proof. We give the proof for the statement about algebraic C^*-uniqueness; the other case is obtained by replacing $\mathrm{C}^*(\Gamma)$ by $\mathrm{C}^*_\mathrm{r}(\Gamma)$ throughout the proof. Assume that there is a nontrivial ideal $J \trianglelefteq \mathrm{C}^*(\Gamma)$ intersecting $\mathbb{C}\Gamma$ trivially and denote by $q : \mathrm{C}^*(\Gamma) \to \mathrm{C}^*(\Gamma)/J$ the quotient map. Composing q with the inclusion $\mathbb{C}\Gamma \hookrightarrow \mathrm{C}^*(\Gamma)$ yields a faithful representation of $\mathbb{C}\Gamma$ and it defines a C^*-norm on it that is properly majorised by the maximal norm by nontriviality of J. Conversely, if there is a C^*-norm on $\mathbb{C}\Gamma$ which is properly majorised by the norm coming from $\mathrm{C}^*(\Gamma)$, then $\mathrm{C}^*(\Gamma)$ surjects onto the corresponding quotient, and the kernel of this surjection is a nontrivial ideal intersecting $\mathbb{C}\Gamma$ trivially. □

Corollary 2.3. *Let* Γ *and* Λ *be discrete groups. If* $\mathbb{C}(\Gamma \times \Lambda)$ *is* C^**-unique* (*respectively*, C^*_r*-unique*), *then so are* $\mathbb{C}\Gamma$ *and* $\mathbb{C}\Lambda$.

Proof. Let $J \trianglelefteq \mathrm{C}^*_\mathrm{r}(\Gamma)$ be a nontrivial ideal intersecting $\mathbb{C}\Gamma$ trivially. Then

$$J \otimes_{\max} \mathrm{C}^*_\mathrm{r}(\Lambda) \trianglelefteq \mathrm{C}^*(\Gamma) \otimes_{\max} \mathrm{C}^*(\Lambda) = \mathrm{C}^*(\Gamma \times \Lambda)$$

is a nontrivial ideal intersecting $\mathbb{C}(\Gamma \times \Lambda) = \mathbb{C}\Gamma \otimes_{\mathrm{alg}} \mathbb{C}\Lambda$ trivially. The same proof with C^* replaced by C^*_r and $\otimes_{\max}$ replaced by $\otimes_{\min}$ works for the reduced case. □

Proposition 2.4. *If* Γ *is a discrete group with a central element of infinite order then* $\mathbb{C}\Gamma$ *is not* C^*_r*-unique.*

Proof. Denote by Z the subgroup in Γ generated by a central element of infinite order. Then $\mathrm{C}^*_\mathrm{r}(Z) \cong C(S^1)$ and $LZ \cong L^\infty(S^1)$ via the Fourier transform and we denote by $p \in LZ$ the projection corresponding to the characteristic function of the upper half-circle $\{e^{i\theta} : \theta \in [0, \pi]\}$. Define $\pi := \lambda_\Gamma p$; i.e., the left regular representation of Γ restricted to the invariant subspace $p\ell^2(\Gamma)$. Choosing a nonzero function $f \in C(S^1)$ supported in the *lower* half-circle we obtain a nonzero element $x \in \mathrm{C}^*_\mathrm{r}(Z) \subset \mathrm{C}^*_\mathrm{r}(\Gamma)$ with $xp = 0$ and hence the norm on $\mathrm{C}^*_\mathrm{r}(\Gamma)$ induced by π is not the one induced by λ_Γ. We now only need to show that π is faithful on $\mathbb{C}\Gamma$. To this end, consider the trace-preserving conditional expectation $E : L\Gamma \to LZ$ [Brown and Ozawa 2008, Lemma 1.5.11] and assume that $a \in \mathbb{C}\Gamma$ is in the kernel of π. Then a^*a is also in the kernel of π and since E is an LZ-bimodule map [Brown and Ozawa 2008, Proposition 1.5.7], we get

$$0 = E(\lambda_\Gamma(a^*a)p) = E(\lambda_\Gamma(a^*a))p.$$

However, $E(\mathbb{C}\Gamma) \subset \mathbb{C}Z \simeq \mathrm{Pol}(z, \bar{z}) \subset C(S^1)$ and therefore $E(\lambda_\Gamma(a^*a)) = 0$ and since E is trace-preserving and the trace on $L\Gamma$ is faithful we conclude that a^*a, and hence a, is zero. □

Corollary 2.5. *An abelian group is algebraically* C**-unique if and only if it is locally finite* (*i.e.*, *pure torsion*).

Remark 2.6. The result in Corollary 2.5 was also observed, independently and with different proofs, by Rostislav Grigorchuk, Magdalena Musat and Mikael Rørdam (unpublished).

Remark 2.7. The class of locally finite groups has many stability properties — for instance it is closed under subgroups, quotients and extensions and, moreover, being virtually locally finite is the same as being locally finite. However, verifying these properties for the class of C*-unique groups seems to be a much bigger challenge.

3. The strong Atiyah conjecture and $\mathrm{C}^*_{\mathrm{r}}$-uniqueness

3A. ***The strong Atiyah conjecture.*** The key to our main result is the so-called strong Atiyah conjecture which is briefly described in the following. A good general reference is [Lück 2002, Chapter 10] where all of the results below can be found, and to which we also refer for the original references. Let Γ be a discrete group and denote by $1/|\operatorname{FIN}(\Gamma)|\,\mathbb{Z}$ the additive subgroup in $\mathbb{Q}$ generated by the set

$$\left\{\frac{1}{|\Lambda|} : \Lambda \leqslant \Gamma \text{ a finite subgroup}\right\}.$$

Given a matrix $A \in \mathbb{M}_n(\mathbb{C}\Gamma)$ we denote by $L_A \in L\Gamma \otimes \mathbb{M}_n(\mathbb{C}) \subset \mathbb{B}(\ell^2(\Gamma)^n)$ the bounded operator given by left multiplication with A (via the left regular representation of Γ). The strong Atiyah conjecture then predicts that

$$\dim_{L\Gamma} \ker(L_A) := (\tau \otimes \operatorname{Tr})(P_{\ker L_A}) \in \frac{1}{|\operatorname{FIN}(\Gamma)|}\mathbb{Z}.$$

Here $\dim_{L\Gamma}(-)$ denotes the von Neumann dimension of the (right) Hilbert $L\Gamma$-module $\ker(L_A)$ defined as the nonnormalized trace of the kernel projection $P_{\ker L_A}$; see [Lück 2002] for details on this. It should be noted that the strong Atiyah conjecture is false in general [Lück 2002, Theorem 10.23], but is known to hold for all groups which have a bound on the order of finite subgroups and belong to Linnell's class $\mathscr{C}$ [Lück 2002, Theorem 10.19], the latter being the smallest class of groups which contain all free groups, and is closed under directed unions and extensions by elementary amenable groups (i.e., if $\Lambda \trianglelefteq \Gamma$, $\Lambda \in \mathscr{C}$ and Γ/Λ is elementary amenable, then $\Gamma \in \mathscr{C}$). The above discussion motivates the following notion.

Definition 3.1. Let Γ be a countable group. The *torsion multiplier* of Γ is defined as

$$\theta(\Gamma) = \frac{1}{\operatorname{lcm}\{|H| : H \leqslant \Gamma \text{ finite}\}} \in [0, 1].$$

In this definition, and in what follows, we use the convention that the least common multiple (lcm) of an infinite set of natural numbers is infinity and that $\frac{1}{\infty} = 0$. Note that if Γ has an upper bound on the set of finite subgroups, then

$$\frac{1}{|\mathrm{FIN(G)}|}\mathbb{Z} = \{n\theta(\Gamma) : n \in \mathbb{Z}\},$$

and $1/|\mathrm{FIN(G)}|\,\mathbb{Z}$ has 0 as an accumulation point otherwise. In view of this, the strong Atiyah conjecture for a group Γ with $\theta(\Gamma) > 0$ implies that the possible kernel dimensions are properly quantized in the sense that they can only take values in the discrete set $\{n\theta(\Gamma) : n \in \mathbb{N}\} \subset \mathbb{R}$. Theorem A (i) and (ii) will follow directly from our main technical result, Theorem 3.6. The key idea in the proof is to play the aforementioned "quantization" of the kernel dimensions against an abundance of central projections in $L\Gamma$ with small traces which provide representations of $\mathrm{C}^*_\mathrm{r}(\Gamma)$ with nontrivial kernels. To quantify this, we need the following definition.

Definition 3.2. The *central granularity* of Γ is defined as

$$\sigma(\Gamma) = \inf\{\tau(p) : p \in \mathrm{Proj}(Z(L\Gamma)),\ p \neq 0\} \in [0, 1].$$

We note that $\sigma(\Gamma) < 1$ if and only if $Z(L\Gamma)$ is nontrivial which is equivalent to Γ not being icc.[1] The next proposition computes the central granularity of Γ in group-theoretic terms. Recall that the FC-*centre* Γ_{fc} is the normal subgroup of Γ consisting of all elements with finite conjugacy classes.

Proposition 3.3. *Let* $\Gamma_{\mathrm{fc}} \trianglelefteq \Gamma$ *be the* FC-*centre of* Γ. *Then*

$$\sigma(\Gamma) = \frac{1}{|\Gamma_{\mathrm{fc}}|},$$

where the right-hand side is interpreted as 0 *if* $|\Gamma_{\mathrm{fc}}| = \infty$.

Proof. Γ_{fc} is an increasing union of a sequence of finitely generated normal subgroups $\Lambda_n \trianglelefteq \Gamma$; to see this, note that Γ_{fc} is clearly an increasing union of a sequence of finitely generated subgroups Λ'_n, and defining Λ_n to be generated by the Γ-conjugacy classes of a finite system of generators for Λ'_n yields the desired sequence of finitely generated subgroups which are normal in Γ. We now have two cases to consider:

(i) All Λ_n are finite (equivalently, Γ_{fc} is a torsion group),

(ii) Λ_n is infinite for some n.

In case (i), setting $p_n := 1/|\Lambda_n| \sum_{g\in\Lambda_n} g$, we get a projection $p_n \in L\Gamma_{\mathrm{fc}}$ with $\tau(p_n) = 1/|\Lambda_n|$; moreover, p_n is central in $L\Gamma$ since Λ_n is normal in Γ. This proves that $\sigma(\Gamma) = 0$ if Γ_{fc} is an infinite torsion group (in this case $|\Lambda_n| \to \infty$). If Γ_{fc} is

[1] Recall that a group is *icc* if all nontrivial conjugacy classes are infinite

finite, then the sequence stabilizes, and therefore we get a central projection p in $L\Gamma$ with trace $1/|\Gamma_{\rm fc}|$. The centre of $L\Gamma$ consists of elements whose associated Fourier series in $\ell^2(\Gamma) = L^2(L\Gamma, \tau)$ are supported only on $\Gamma_{\rm fc}$ and are constant along conjugacy classes, and is therefore contained in the centre of $L\Gamma_{\rm fc}$; hence we get $1/|\Gamma_{\rm fc}| \geqslant \sigma(\Gamma) \geqslant \sigma(\Gamma_{\rm fc})$. But we also have $L\Gamma_{\rm fc} = \mathbb{C}\Gamma_{\rm fc}$ which by representation theory of finite groups is isomorphic to a direct sum of matrix algebras $\bigoplus_\pi \mathbb{M}_{d_\pi}(\mathbb{C})$ with the trace given by $\bigoplus_\pi d_\pi^2/|\Gamma_{\rm fc}|\,\mathrm{tr}$; thus, the minimal central projection has trace $1/|\Gamma_{\rm fc}| = \sigma(\Gamma_{\rm fc})$; this proves the claim.

In case (ii) we fix an $n \in \mathbb{N}$ such that $\Lambda_n =: \Lambda$ is infinite and note that since Λ is generated by a finite number of elements with finite conjugacy classes, its centralizer $C_\Gamma(\Lambda)$ is of finite index in Γ. We now claim that $L\Lambda$ has a diffuse von Neumann subalgebra and thus projections of arbitrarily small trace. This can be seen as follows: if $L\Lambda$ has a direct summand of type II_1, it is clear because such von Neumann algebras are diffuse. Otherwise $L\Lambda$ is of type I, but then Λ is virtually abelian [Lück 1997, Lemma 3.3], and hence, being infinite by assumption and finitely generated by construction, contains a copy of $\mathbb{Z}$ which generates a diffuse von Neumann algebra $L\mathbb{Z} \cong L^\infty(S^1)$. In view of the above, for an arbitrary $\varepsilon > 0$, there is a projection $p \in L\Lambda \subset L\Gamma_{\rm fc}$ of trace $\tau(p) < \varepsilon/[\Gamma : C_\Gamma(\Lambda)]$. Now let

$$q := \bigvee_{g\in\Gamma} {}^g p,$$

where ${}^g p := gpg^{-1}$. Then q is a central projection in $L\Gamma$. Moreover, p is invariant under the centralizer $C_\Gamma(\Lambda)$ and upon choosing coset representatives $g_1, \dots, g_{[\Gamma:C_\Gamma(\Lambda)]}$ for $\Gamma/C_\Gamma(\Lambda)$ we obtain

$$q = \bigvee_{i=1}^{[\Gamma:C_\Gamma(\Lambda)]} {}^{g_i} p$$

and hence $\tau(q) \leqslant [\Gamma : C_\Gamma(\Lambda)] \cdot \tau(p) < \varepsilon$. Thus $\sigma(\Gamma) = 0$. □

Lemma 3.4. *Let Γ be a discrete non-icc group. For every $\varepsilon > 0$ there exists a nonzero projection $p \in Z(L\Gamma)$ with $\tau(p) < \sigma(\Gamma) + \varepsilon$ and a nonzero, central element $x \in \mathrm{C}^*_{\mathrm{r}}(\Gamma)$ with $xp^\perp = 0$.*

Proof. Since Γ is non-icc, $Z(L\Gamma) \neq \mathbb{C}1$ so $\sigma(\Gamma) < 1$. Let $\varepsilon > 0$ be given and assume, without loss of generality, that $\sigma(\Gamma) + \varepsilon < 1$. One has $Z(L\Gamma) = Z(\mathrm{C}^*_{\mathrm{r}}(\Gamma))'' = Z(\mathbb{C}\Gamma)''$, as can be seen for instance by using Kaplansky's density theorem together with the centre valued trace, and noting that $Z(\mathbb{C}\Gamma)$ consists of the elements whose coefficients are constant along conjugacy classes. By Gelfand duality, $Z(\mathrm{C}^*_{\mathrm{r}}(\Gamma))$ is isomorphic to the C^*-algebra $C(Z)$ of continuous functions on its Gelfand spectrum Z, which is a compact Hausdorff space; it is metrizable because $\mathrm{C}^*_{\mathrm{r}}(\Gamma)$ is separable. The canonical trace τ thus gives a regular Borel probability measure μ on Z [Rudin 1966, Theorem 2.14] and an isomorphism $Z(L\Gamma) = Z(\mathrm{C}^*_{\mathrm{r}}(\Gamma))'' \cong L^\infty(Z, \mu)$ compatible with the natural inclusions. Projections in $Z(L\Gamma)$ correspond via this isomorphism to measurable subsets of Z (up to null sets), and we therefore obtain a measurable subset $A \subset Z$ such that $0 < \mu(A) < \sigma(\Gamma) + \varepsilon/2$. By regularity of μ, there exists $U \supseteq A$ open such that

$$0 < \mu(A) \leqslant \mu(U) < \sigma(\Gamma) + \varepsilon < 1.$$

Now, there is a nonzero element $x \in C(Z)$ vanishing on the compact set $K := Z \setminus U$ (for instance, the distance function to K); letting p be the projection corresponding to U finishes the proof. □

The following lemma gives a concrete description of the decomposition of the left regular representation of a discrete group Γ over the cosets of a finite index normal subgroup Λ.

Lemma 3.5. *Let $\Lambda \trianglelefteq \Gamma$ be a normal subgroup of finite index. For every choice of coset representatives $g_1, \dots, g_{[\Gamma:\Lambda]} \in \Gamma$ there exists a trace-preserving inclusion of von Neumann algebras $\pi : (L\Gamma, \tau) \hookrightarrow (\mathbb{M}_{[\Gamma:\Lambda]}(L\Lambda), \tau \otimes \mathrm{tr})$ which restricts to corresponding inclusions at the level of reduced* C**-algebras and complex group rings, and which for $x \in L\Lambda$ is given by*

$$\pi(x) = \mathrm{diag}({}^{g_1}x, {}^{g_2}x, \dots, {}^{g_{[\Gamma:\Lambda]}}x), \tag{3-1}$$

where ${}^g x = gxg^{-1}$ is the conjugation action of $g \in \Gamma$ on $L\Lambda$.

Proof. Choose coset representatives $g_1, g_2, \dots, g_{[\Gamma:\Lambda]}$ of Γ/Λ and consider the isomorphisms of Hilbert spaces

$$\ell^2(\Gamma) \cong \bigoplus_{i=1}^{[\Gamma:\Lambda]} \ell^2(g_i^{-1}\Lambda) \cong \bigoplus_{i=1}^{[\Gamma:\Lambda]} \ell^2(\Lambda).$$

These induce a $*$-isomorphism $\pi : \mathbb{B}(\ell^2\Gamma) \xrightarrow{\cong} \mathbb{M}_n(\mathbb{B}(\ell^2(\Lambda)))$. It is routine to check that π restricts to a trace-preserving inclusion of $\mathbb{C}\Gamma$ into $\mathbb{M}_{[\Gamma:\Lambda]}(\mathbb{C}\Lambda)$ which automatically implies the corresponding results for the reduced C*-algebras and von Neumann algebras. Finally, for $h \in \Lambda$ we have

$$\pi(h) = \mathrm{diag}(\lambda(h), \dots, \lambda(h)) \in \bigoplus_{i=1}^{[\Gamma:\Lambda]} \mathbb{B}(\ell^2(g_i^{-1}\Lambda)),$$

and thus formula (3-1) follows in view of the identity

$$hg_i^{-1}h' = g_i^{-1}({}^{g_i}h)h', \quad h, h' \in \Lambda.$$

□

Theorem 3.6. *Let Λ be a discrete group satisfying the strong Atiyah conjecture and let $\Lambda \trianglelefteq \Gamma$ be a finite index inclusion of Λ into a group Γ as a normal subgroup. If $[\Gamma : \Lambda]^2 \cdot \sigma(\Lambda) < \theta(\Lambda)$ then $\mathbb{C}\Gamma$ is not C^*_r-unique.*

Proof. The assumption $[\Gamma : \Lambda]^2 \cdot \sigma(\Lambda) < \theta(\Lambda)$ forces Λ to be non-icc and applying Lemma 3.4 we get a projection $p \in Z(L\Lambda)$ with $\tau(p) < \theta(\Lambda)/[\Gamma : \Lambda]^2$ and a nonzero central element $x \in \mathrm{C}^*_\mathrm{r}(\Lambda)$ with $xp^\perp = 0$. We are going to construct a representation of $\mathrm{C}^*_\mathrm{r}(\Gamma)$ which is injective on $\mathbb{C}\Gamma$ but with x in the kernel. To this end, consider a set of coset representatives $g_1, \dots, g_{[\Gamma:\Lambda]}$ for Γ/Λ and the $*$-homomorphism $\pi : L\Gamma \to \mathbb{M}_{[\Gamma:\Lambda]}(L\Lambda)$ provided by Lemma 3.5. From this we obtain a central projection $q := \bigvee_{i=1}^{[\Gamma:\Lambda]} {}^{g_i}p \in Z(L\Lambda)$, and cutting π with the complement of $\tilde{q} := \mathrm{diag}(q, \dots, q) \in Z(\mathbb{M}_{[\Gamma:\Lambda]}(L\Lambda))$, we get a representation

$$\pi_q : \mathrm{C}^*_\mathrm{r}(\Gamma) \to \mathbb{B}\big(\ell^2(\Lambda)^{[\Gamma:\Lambda]}\tilde{q}^\perp\big), \qquad a \mapsto \pi(a)\tilde{q}^\perp.$$

As $q^\perp = \bigwedge_{i=1}^{[\Gamma:\Lambda]} {}^{g_i}(p^\perp)$ and $xp^\perp = 0$, it follows that $x \in \ker \pi_q$ in view of (3-1). Let $a \in \mathbb{C}\Gamma \cap \ker \pi_q$. This means that $\pi(a)\tilde{q}^\perp = 0$, and thus the kernel projection r of $\pi(a)$ satisfies $r \geqslant \tilde{q}^\perp$. Therefore

$$(\tau \otimes \mathrm{Tr})(r) \geqslant (\tau \otimes \mathrm{Tr})(\tilde{q}^\perp) \geqslant [\Gamma : \Lambda](1 - [\Gamma : \Lambda]\tau(p)) > [\Gamma : \Lambda] - \theta(\Lambda).$$

On the other hand, the assumption $[\Gamma : \Lambda]^2 \cdot \sigma(\Lambda) < \theta(\Lambda)$ forces an upper bound on the order of finite subgroups in Λ, i.e., $\theta(\Lambda) > 0$, and since Λ is furthermore assumed to satisfy the strong Atiyah conjecture we obtain (using the notation of Section 3A) that

$$\dim_{L\Lambda}(\ker(L_A)) = (\tau \otimes \mathrm{Tr})(P_{\ker L_A}) \in \{n\theta(\Lambda) : n \in \mathbb{Z}\}$$

for any matrix $A \in \mathbb{M}_{[\Gamma:\Lambda]}(\mathbb{C}\Lambda)$. Thus $(\tau \otimes \mathrm{Tr})(r) \leqslant [\Gamma : \Lambda] - \theta(\Lambda)$ unless $\pi(a) = 0$. This proves that π_q is injective on $\mathbb{C}\Gamma$ and hence completes the proof. □

As a corollary, we deduce that some important families of groups are not C^*_r-unique. In particular, this includes the groups mentioned in Theorem A (i) and (ii), and together with Proposition 2.4 this completes the proof of Theorem A.

Corollary 3.7. *All groups in following classes are not* C^*_r*-unique*:

(i) *Torsion free*, *non-icc groups satisfying the strong Atiyah conjecture*; *in particular all elementary amenable*, *non-icc*, *torsion free groups.*

(ii) *Virtually polycyclic groups with infinite* FC*-centre*; *in particular*, *all infinite groups of polynomial growth.*

Proof. To see (i), note that the existence of a nontrivial finite conjugacy class implies the existence of a nontrivial central element in $\mathbb{C}\Gamma$ (namely the sum of the elements in the finite conjugacy class) and hence a nontrivial projection in $Z(L\Gamma)$; thus $\sigma(\Gamma) < 1$. Moreover, since Γ is torsion free, $\theta(\Gamma) = 1$ and since Γ is assumed to satisfy the strong Atiyah conjecture it follows that it is C^*_r-unique by Theorem 3.6. The last statement in (i) follows directly from this since the elementary amenable groups are contained in Linnell's class $\mathscr{C}$ (see Section 3A) for which the strong Atiyah conjecture is known to hold in the presence of a bound on the order of finite subgroups [Lück 2002, Theorem 10.19].

To see (ii), let $\Lambda \trianglelefteq \Gamma$ be a normal finite index polycyclic subgroup of Γ. As Γ has infinite FC-centre, so does Λ and the FC-centre of Λ is moreover finitely generated by polycyclicity. A classical result by Hirsch [1946, Theorem 3.21] implies that the orders of finite subgroups of Λ are bounded; thus $\theta(\Lambda) > 0$. On the other hand, $\sigma(\Lambda) = 0$ by Proposition 3.3(ii). Moreover, polycyclic groups, being elementary amenable, satisfy the strong Atiyah conjecture. Thus, it follows that $\mathbb{C}\Gamma$ is non-C^*_r-unique by Theorem 3.6.

Finally, the claim about infinite groups of polynomial growth follows by first observing that by Gromov's theorem [1981], these are exactly finitely generated virtually nilpotent groups. As finitely generated nilpotent groups are polycyclic, the claim follows once we argue that virtually nilpotent groups automatically have infinite FC-centre. To see this, recall that a finitely generated virtually nilpotent group Γ contains a finite index torsion free nilpotent normal subgroup Λ (by polycyclicity and [Hirsch 1946, Theorem 3.21]). Now it follows that the centre of Λ is infinite, and therefore so is the FC-centre Λ_{fc}; but as $\Lambda \trianglelefteq \Gamma$ is a finite index inclusion, $\Lambda_{\mathrm{fc}} \subseteq \Gamma_{\mathrm{fc}}$. Thus, Γ_{fc} is infinite. □

Acknowledgements

The authors would like to thank Mikael Rørdam and Thomas Schick for helpful discussions revolving around the topics of the present paper, and the anonymous referee for valuable remarks improving Corollary 2.3 and Proposition 3.3. Kyed gratefully acknowledges the financial support from the Villum Foundation (grant no. 7423) and from the Independent Research Fund Denmark (grant no. 7014-00145B).

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Received August 17, 2018.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 298, No. 2, 2019
dx.doi.org/10.2140/pjm.2019.298.267

EXPECTED DEPTH OF RANDOM WALKS ON GROUPS

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For G a finitely generated group and $g \in G$, we say g is detected by a normal subgroup $N \lhd G$ if $g \notin N$. The depth $D_G(g)$ of g is the lowest index of a normal, finite index subgroup N that detects g. In this paper we study the expected depth, $\mathbb{E}[D_G(X_n)]$, where X_n is a random walk on G. We give several criteria that imply that

$$\mathbb{E}[D_G(X_n)] \xrightarrow[n\to\infty]{} 2 + \sum_{k \geq 2} \frac{1}{[G : \Lambda_k]},$$

where Λ_k is the intersection of all normal subgroups of index at most k. In particular, the equality holds in the class of all nilpotent groups and in the class of all linear groups satisfying Kazhdan's property (T). We explain how the right-hand side above appears as a natural limit and also give an example where the convergence does not hold.

1. Introduction

Let G be a finitely generated group. The *depth* of an element in G encodes how well approximated that element is by finite quotients of the group. The goal of this article is to find the average depth of an element of G. As such, this question is ill-posed, and a more precise one is: *what is the asymptotic expected depth of a random walk on the Cayley graph of G?* This question arises naturally when quantifying *residual finiteness*, or in other words, when studying statistics surrounding the *depth function*.

For $g \in G$ and N a normal subgroup of G, we say g is *detected by* N if $g \notin N$ (in other words, if g is mapped onto a nontrivial element of G/N). The *depth* of g is the lowest index of a normal, finite index subgroup N that detects g. Formally, for $g \in G$, $g \neq e$, set

$$D_G(g) := \min\{|G/N| : N \lhd_{\text{finite index}} G \text{ and } g \notin N\}.$$

For $g = e$, the above definition would produce a depth equal to $\min \varnothing = \infty$. In

Bou-Rabee was supported in part by NSF grant DMS-1405609. Myropolska was supported by Swiss NSF grant P2GEP2_162064. Manolescu is affiliated to NCCR SwissMAP.
MSC2010: 05C81, 20F65, 60B15.
Keywords: expectation, residual finiteness growth, intersection growth, residual averages, depth function.

the context of random walks, this singularity would produce trivial results. To circumvent this triviality, we instead define $D_G(e) := 0$. With this definition, G is *residually finite* if and only if $D_G(g) < \infty$ for all elements $g \in G$.

Let G be residually finite and S be a finite generating set, which will be always considered symmetric. Then the *residual finiteness growth function* is

$$F_G^S(n) = \max_{g \in B_G^S(n)} D_G(g),$$

where $B_G^S(n)$ is the ball of radius n in the Cayley graph $\mathrm{Cay}(G, S)$. This notion was introduced in [Bou-Rabee 2010] and has been studied for various classes of groups; the relevant results to this paper are listed in Section 2A.

While the residual finiteness growth function reflects the largest depth of an element in the ball of radius n, the question of interest in this paper is what we can say about the "average" depth of a "uniform" element of the group. If G is discrete but infinite, there is no natural definition of a uniform probability measure on G, hence no good notion of a uniform element. However, we may try to approach the desired "average" by averages of well defined measures. Two approaches come to mind:

- For $n \geq 0$, let Z_n be a uniform element in $B_G^S(n)$, and let

  $$a_n = \mathbb{E}[D_G(Z_n)] = \frac{1}{|B_G^S(n)|} \sum_{g \in B_G^S(n)} D_G(g).$$

  We could then say that the average depth of an element of G is $\lim_n a_n$, provided that this limit exists.

- Alternatively, one may define a random walk $(X_n)_{n \in \mathbb{N}}$ on a Cayley graph $\mathrm{Cay}(G, S)$ of G, starting from the neutral element e, and set

  $$b_n = \mathbb{E}[D_G(X_n)].$$

  Then define the average depth as $\lim_n b_n$, again under the condition that the limit exist.

One expects that for compliant groups, both limits exist and are equal. We will focus on the second situation, but will make reference to the first to stress similarities. The exact definition of $(X_n)_{n \geq 0}$ as well as a discussion on random walks on groups is deferred to Section 2B. We mention here only that $(X_n)_{n \geq 0}$ is a *lazy* random walk, that is a process that at every step remains unchanged with probability $1/2$ and takes a step otherwise.

It is a general fact that for an integer valued nonnegative random variable Y,

$$\mathbb{E}(Y) = \sum_{k \geq 0} \mathbb{P}(Y > k).$$

It may therefore be interesting to study $\mathbb{P}[D_G(X_n) > k]$ for $(X_n)_{n \geq 0}$ as above.

For $k \geq 2$, let Λ_k be the intersection of all normal subgroups of G of index at most k. (For $k = 0, 1$, set $\Lambda_k = G$). Then, for $g \in G \setminus \{e\}$, $D_G(g) > k$ if and only if $g \in \Lambda_k$. Thus,

$$\mathbb{E}[D_G(X_n)] = \sum_{k \geq 0} \mathbb{P}(X_n \in \Lambda_k \setminus \{e\}). \tag{1}$$

As we will see in Corollary 2.5,

$$\mathbb{P}(X_n \in \Lambda_k \setminus \{e\}) \xrightarrow[n\to\infty]{} \frac{1}{[G : \Lambda_k]}.$$

One may therefore expect that

$$\mathbb{E}[D_G(X_n)] \xrightarrow[n\to\infty]{} 2 + \sum_{k \geq 2} \frac{1}{[G : \Lambda_k]}, \tag{2}$$

where the factor 2 appears since $[G : \Lambda_1] = [G : \Lambda_0] = [G : G] = 1$. For this reason, we call the right-hand side of the above the *presumed limit*. However, the convergence above is far from obvious. The main goal of this paper is to provide criteria for G under which (2) holds. We will also provide an example where this is not valid.

The finiteness of $\sum_{k \geq 2} 1/|G : \Lambda_k|$ in (2) depends on the group G and is related to the *intersection growth* $i_G(k) = [G : \Lambda_k]$ of G. It follows from Equation (7) and Theorem 3.2 that $\sum_{k \geq 2} 1/|G : \Lambda_k| < \infty$ for finitely generated linear groups. Moreover, finitely generated nilpotent groups enjoy this property as it is a classical result that they are linear (see [Segal 1983, Chapter 5, §B, Theorem 2] or [Hall 1969, p. 56, Theorem 7.5]).

Our two results ensuring (2) are the following:

Theorem 1.1. *Let G be a linear group with Kazhdan's property (T). Then*

$$\lim_{n\to\infty} \mathbb{E}[D_G(X_n)] = 2 + \sum_{k \geq 2} \frac{1}{[G : \Lambda_k]} < \infty$$

for any finite generating set S of G. In particular, $\lim_{n\to\infty} \mathbb{E}[D_G(X_n)]$ *is finite for the special linear groups $SL_k(\mathbb{Z})$ with $k \geq 3$.*

Theorem 1.2. *Let G be a finitely generated nilpotent group. Then*

$$\lim_{n\to\infty} \mathbb{E}[D_G(X_n)] = 2 + \sum_{k \geq 2} \frac{1}{[G : \Lambda_k]} < \infty$$

for any finite generating set S of G.

In Section 4A, it will also be shown that the convergence holds whenever the presumed limit is infinite. Considering these examples, one may think that the convergence in (2) is always valid. However, in Proposition 4.8, we exhibit a 3-generated group for which the presumed limit is finite but $\lim_{n\to\infty} \mathbb{E}[D_G(X_n)] = \infty$.

Henceforth, when no ambiguity is possible, we drop the index G from the notation $D_G(\cdot)$.

2. Preliminaries

2A. ***Depth function and residual finiteness growth.*** This short subsection includes some results on the residual finiteness growth function that we will use in the sequel.

Theorem 2.1 [Bou-Rabee 2010]. *Let G be a finitely generated nilpotent group with a generating set S. Then*

$$F_G^S(n) \leq C \log(n)^{h(G)}, \quad \textit{for all } n \geq 2,$$

where $h(G)$ is the Hirsch length of G and $C = C(G, S)$ is a constant independent of n.

The prime number theorem and Hall's embedding theorem play key roles in the proof of Theorem 2.1. In [Bou-Rabee and McReynolds 2015], the following is proved using Gauss's counting lemma to help quantify Mal'cev's classical proof of residual finiteness of finitely generated linear groups.

Theorem 2.2 [Bou-Rabee and McReynolds 2015]. *Let K be a field. Let G be a finitely generated subgroup of* $\mathrm{GL}(m, K)$ *with a generating set S. Then there exists a positive integer b such that*

$$F_G^S(n) \leq C n^b, \quad \textit{for all } n \geq 1,$$

where $C = C(G, S)$ is a constant independent of n.

The above results bound from above the residual finiteness growth. Conversely, the following states that there exist groups with arbitrary large residual finiteness growth.

Theorem 2.3 [Bou-Rabee and Seward 2016]. *For any function $f : \mathbb{N} \to \mathbb{N}$, there exists a residually finite group G and a two element generating set S for G, such that $F_G^S(n) \geq f(n)$ for all $n \geq 8$.*

The proof of Theorem 2.3 in [Bou-Rabee and Seward 2016] involves an explicit construction of a finitely generated group embedded in an infinite product of finite simple groups.

2B. ***Random walks on groups.*** Let G be a finitely generated group with a finite symmetric generating set

$$S = \{s_1, \ldots, s_k\},$$

i.e., such that $S^{-1} = S$. A random walk $(X_n)_{n \geq 0}$ on G is a Markov chain with state space G and such that $X_0 = e_G$ and $X_{n+1} = X_n \cdot Y_n$ for $n \geq 0$ where $Y_0, Y_1, \ldots$ are independent and uniform in $\{s_1, \ldots, s_k\}$.

If G is finite with $|G| = m$, one may consider the transition matrix P of the random walk $(X_n)_{n\geq 0}$ on G defined by

$$P(x, y) = \frac{1}{|S|}\sum_{s\in S}\mathbf{1}_{\{y=xs\}},$$

where $\mathbf{1}_{\{y=xs\}} = 1$ if $y = xs$ and 0 otherwise. It is simply the adjacency matrix of the Cayley graph $\mathrm{Cay}(G, S)$, normalized by $1/|S|$. The generating set is considered symmetric so as to have an unoriented Cayley graph, or equivalently to have P symmetric.

Let $1 = \lambda_1 \geq \cdots \geq \lambda_m \geq -1$ be the eigenvalues of P and $x_1, \ldots, x_m$ be a basis of orthonormal eigenvectors of P (such a basis necessarily exists since P is real and symmetric). Let σ be an initial distribution on G seen as a probability vector of dimension m, and let $p_u = \left(\frac{1}{m}, \ldots, \frac{1}{m}\right)$ be the uniform distribution on G. It is well known that the distribution of such a random walk converges to the uniform distribution whenever the graph is assumed to not be bipartite. For a general convergence statement one considers a *lazy* random walk instead; that is a walk with transition matrix $L = \frac{1}{2}I + \frac{1}{2}P$. The lazy random walk at time n takes a step of the original random walk with probability $\frac{1}{2}$ and stays at the current vertex with probability $\frac{1}{2}$. Notice that the eigenvectors of L are $x_1, \ldots, x_m$ and the corresponding eigenvalues are all nonnegative:

$$\mu_1 = \tfrac{1}{2} + \tfrac{1}{2}\lambda_1 = 1 > \mu_2 = \tfrac{1}{2} + \tfrac{1}{2}\lambda_2 \geq \cdots \geq \mu_m = \tfrac{1}{2} + \tfrac{1}{2}\lambda_m \geq 0.$$

Lemma 2.4. *Let G be a finite group with a finite symmetric generating set S. With the above notation, $\|\sigma L^n - p_u\|_2 \leq \mu_2^n$. In particular, $\left|\sigma L^n(g) - \frac{1}{m}\right| \leq \mu_2^n$ for every $g \in G$.*

Proof. For $n \geq 1$, the matrix $\sigma \cdot L^n$ is a probability distribution and it represents the distribution of the n-th step of the lazy random walk on G that starts at a random vertex selected according to σ.

We write $\sigma = \alpha_1x_1+\alpha_2x_2+\cdots+\alpha_mx_m$, with $\alpha_1, \ldots, \alpha_m \in \mathbb{R}$. Since $x_1, \ldots, x_m$ are eigenvectors, we have $\sigma L^n = \alpha_1\mu_1^n x_1 + \alpha_2\mu_2^n x_2 + \cdots + \alpha_m\mu_m^n x_m$. Notice that $\mu_1 = 1$, $x_1 = 1/\sqrt{m}(1, \ldots, 1)$ and $\alpha_1 = \sigma \cdot x_1^T = 1/\sqrt{m}$ which implies that $\alpha_1x_1 = \left(\frac{1}{m}, \ldots, \frac{1}{m}\right) = p_u$. We deduce that

$$\|\sigma L^n - p_u\|_2 = \|\alpha_2\mu_2^n x_2 + \cdots + \alpha_m\mu_m^n x_m\|_2 \leq \max_{i=2,\ldots,m} |\mu_i|^n \cdot \sqrt{\alpha_2^2 + \cdots + \alpha_m^2} \leq \mu_2^n \cdot \|\sigma\|_2.$$

In the last line we used the orthonormality of the base $x_1, \ldots, x_m$. Finally, $\|\sigma\|_2 \leq \sum_{i=1}^m \sigma_i = 1$, which shows that the above is bounded by μ_2^n as required. □

Corollary 2.5. *Let G be a finitely generated infinite group with a finite symmetric generating set S and let N be a normal subgroup of G of finite index. Consider the* *lazy random walk* $(X_n)_{n\geq 0}$ *on* $\mathrm{Cay}(G, S)$. *Then*

$$\left|\mathbb{P}(X_n \in N) - \frac{1}{|G:N|}\right| \leq \mu^n, \tag{3}$$

where μ *is the second-largest eigenvalue of the transition matrix of* $\tilde{X}_n$, *the lazy random walk on* $\mathrm{Cay}(G/N, S)$ *induced by* $(X_n)_{n\geq 0}$. *Moreover*,

$$\mathbb{P}(X_n \in N \setminus \{e\}) \xrightarrow[n\to\infty]{} \frac{1}{|G:N|}. \tag{4}$$

Proof. To prove (3) it suffices to observe that $\mathbb{P}(X_n \in N) = \mathbb{P}(\tilde{X}_n = e_N)$ and conclude by Lemma 2.4. Let us now show (4).

It is a standard fact (see for instance [Varopoulos et al. 1992, Theorems VI.3.3 and VI.5.1]) that, since G is infinite,

$$\mathbb{P}(X_n = e) \to 0, \qquad \text{as } n \to \infty.$$

Moreover, as discussed above, the eigenvalue μ appearing in (3) is strictly smaller than 1. These two facts, together with (3), imply (4). □

In the rest of the paper, we will always consider lazy random walks as described above. Straightforward generalizations are possible, such as to random walks with nonuniform symmetric transition probabilities — that is, walks taking steps according to a finitely supported, symmetric probability on G, with e having a positive probability (which is to say that the walk has, at any given step, a positive probability of staying at the same place). Certain transition probabilities with infinite support (but finite first moment) may also be treated, but small complications arise in specific parts of the proof. For the sake of readability, we limit ourselves to the simple framework of uniform probabilities on symmetric generating sets.

2C. ***Asymptotic density.*** For a given infinite group G generated by a finite set S, consider the so-called *asymptotic density* (as defined in [Burillo and Ventura 2002]) of a subset X in G defined as

$$\rho_S(X) = \limsup_{n\to\infty} \frac{|X \cap B_G^S(n)|}{|B_G^S(n)|}. \tag{5}$$

If G satisfies

$$\lim_{n\to\infty} \frac{|B_G^S(n+1)|}{|B_G^S(n)|} = 1, \tag{6}$$

then by [Burillo and Ventura 2002], ρ_S is left- and right-invariant; and in particular, $\rho(H) = 1/|G:H|$ for a finite index subgroup H. Moreover, if (6) holds, the lim sup in (5) is actually a limit.

Condition (6) holds for groups of polynomial growth (as a consequence of [Pansu 1983]). Thus, for all $k \geq 2$, recalling the random variables X_n and Z_n defined in the introduction,

$$\lim_{n\to\infty} \mathbb{P}(X_n \in \Lambda_k \setminus \{e\}) = \lim_{n\to\infty} \mathbb{P}(Z_n \in \Lambda_k \setminus \{e\}) = \frac{1}{|G : \Lambda_k|}.$$

For groups with exponential growth, however, condition (6) fails, and the second limit in the above display does not necessarily exist. We give next an example for $G = \mathrm{F}_{\{a,b\}}$, the free group generated by two elements $\{a, b\}$ and for a normal subgroup $N \lhd \mathrm{F}_{\{a,b\}}$. Take

$$S = \{a, b, a^{-1}, b^{-1}\}.$$

For $g \in \mathrm{F}_{\{a,b\}}$, let $\|g\|$ be the word-length of g, that is, the graph distance from g to e in $\mathrm{Cay}(\mathrm{F}_{\{a,b\}}, S)$. Set

$$N = \big\{g \in \mathrm{F}_{\{a,b\}} : \|g\| \in 2\mathbb{N}\big\}.$$

It is straightforward to check that N is a normal subgroup of $\mathrm{F}_{\{a,b\}}$ of index 2. However,

$$\frac{|N \cap B_{\mathrm{F}_{\{a,b\}},S}(n)|}{|B_{\mathrm{F}_{\{a,b\}},S}(n)|} = \begin{cases} \dfrac{3^{n+1}-1}{4\cdot 3^n - 2} & \text{for } n \text{ even,} \\[2ex] \dfrac{3^n - 1}{4\cdot 3^n - 2} & \text{for } n \text{ odd.} \end{cases}$$

It is immediate from the above that $\mathbb{P}(Z_n \in N)$ does not converge when $n \to \infty$.

The above example, together with Corollary 2.5, explains the choice of the random walk $(X_n)_{n\geq 0}$ rather than of the uniform variables $(Z_n)_{n\geq 0}$ on $B^S_G(n)$.

Another reason for this choice relates to sampling. Suppose we have sampled an instance of the variable Z_n for some $n \geq 0$. In order to then obtain a sample of Z_{n+1}, one needs to restart the relatively costly process of sampling a uniform point in $B^S_G(n+1)$. For the random walk, however, if X_n is simulated for some $n \geq 0$, X_{n+1} is easily obtained by multiplying X_n with a random element in S. This makes the sampling of a sequence $(X_1, X_2, \ldots)$ much easier than that of a sequence $(Z_1, Z_2, \ldots)$.

3. Residual average

We mentioned in the introduction that our goal is to compute the "average" depth of an element in G. In addition to the two methods proposed above, that is, taking the limit of $\mathbb{E}[D_G(X_n)]$ or $\mathbb{E}[D_G(Z_n)]$, one may compactify G so that it has a Haar probability measure and take the average depth with respect to it. The natural way to render G compact is by considering its *profinite completion*, which we will denote by $\widehat{G}$. It is a compact group, with a unique uniform Haar measure which we denote by μ. The depth function D_G may be extended by continuity to $\widehat{G}$, and we call $D_{\widehat{G}}$ this extension. Then the *residual average* of G, denoted by $\mathrm{Ave}(G)$, is

$$\mathrm{Ave}(G) := \int_{\widehat{G}} D_{\widehat{G}}\, d\mu.$$

For details of the profinite completion construction, see [Wilson 1998]. For further details of the residual average construction, see [Bou-Rabee and McReynolds 2010].

Lemma 3.1. *For any linear group G,*

$$\mathrm{Ave}(G) = 2 + \sum_{k=2}^{\infty} \frac{1}{|G : \Lambda_k|}. \tag{7}$$

Note that $\mathrm{Ave}(G)$ in (7) is equal to the limit in (2).

Proof. Recall the fact that we have conveniently defined $\Lambda_0 = \Lambda_1 = G$ and therefore that $\mu(\Lambda_0) = \mu(\Lambda_1) = 1$. The residual average is then

$$\begin{aligned}\mathrm{Ave}(G) = \sum_{k=1}^{\infty} k \cdot [\mu(\Lambda_{k-1}) - \mu(\Lambda_k)] &= \sum_{k\geq 1} \sum_{\ell=0}^{k-1} [\mu(\Lambda_{k-1}) - \mu(\Lambda_k)] \\ &= \sum_{\ell\geq 0} \sum_{k>\ell} [\mu(\Lambda_{k-1}) - \mu(\Lambda_k)] \\ &= \sum_{\ell\geq 0} \mu(\Lambda_\ell)\end{aligned}$$

We are authorized to change the order of summation in the third equality, since all the terms in the sum are nonnegative. In the last equality, we have used the telescoping sum and the fact that $\mu(\Lambda_k) \to 0$ as $k \to \infty$. The latter convergence is due to G being residually finite.

The first two terms in the last sum above are equal to 1; for $\ell \geq 2$, $\mu(\Lambda_\ell) = 1/|G : \Lambda_\ell|$. The lemma follows immediately. □

The following theorem, taken from [Bou-Rabee and McReynolds 2010], will be necessary when proving Theorem 1.1.

Theorem 3.2 [Bou-Rabee and McReynolds 2010, Theorem 1.4]. *Let Γ be any finitely generated linear group. Then the residual average of Γ is finite.*

For completeness, we give a proof of the above. The present proof is based on the one in the original paper, with some adjustments meant to correct certain points. The main difference with the original proof is that here we focus on the connection to intersection growth. One has to be especially careful in proving this result, as the residual finiteness growth may vary when passing to subgroups of finite index (see [Bou-Rabee and Kaletha 2012, Example 2.5]).

Proof. We follow the proof [Bou-Rabee and McReynolds 2010, Theorem 1.4], with some changes and expansions. According to [Bou-Rabee and McReynolds 2010, Proposition 5.2], there exists an infinite representation

$$\rho : \Gamma \to \mathrm{GL}(n, K)$$

for some n and $K/\mathbb{Q}$ finite. By [Bou-Rabee and McReynolds 2010, Lemma 2.5], it suffices to show that the normal residual average of $\rho(\Gamma)$ is finite. Set $\Lambda = \rho(\Gamma)$ and set S to be the coefficient ring of Λ.

For each $\delta > 0$, from the proof of [Bou-Rabee and McReynolds 2010, Proposition 5.1], there exists a normal residual system $\mathfrak{F}_\delta$ on Λ given by $\Delta_j = \Lambda \cap \ker r_j$, where

$$r_j : \mathrm{GL}(n, S) \to \mathrm{GL}(n, S/\mathfrak{p}_{S,j}^{k_j}),$$

and $[\Lambda : \Delta_j] \le [\Lambda : \Delta_{j+1}] \le [\Lambda : \Delta_j]^{1+\delta}$. In addition, we have

$$|r_j(\Lambda)| = O_j p_j^{\ell_j},$$

where

$$1 \le O_j < p_j^{n^2}. \tag{8}$$

We also have for constants $N > (n^2)!$ and $C > 4$ that

$$\ell_j > N + Cjn^2 \tag{9}$$

and

$$\ell_j + Cn^2 < \ell_{j+1} \le \ell_j + (C+1)n^2.$$

For each $i < j$, we claim that the largest power of p_j that divides $[\Lambda : \Delta_i]$ is $p_j^{n^2}$. To see this claim, note that if p_j^m divides $O_i p_i^{\ell_i}$, since p_i, p_j are distinct primes, p_j^m must divide O_i, however $O_i < p_i^{n^2}$ and $p_i < p_j$. The claim follows. Since

$$[\Delta_i : \Delta_i \cap \Delta_j] = [\Lambda : \Delta_j][\Delta_j : \Delta_i \cap \Delta_j]/[\Lambda : \Delta_i] = O_j p_j^{\ell_j} [\Delta_j : \Delta_i \cap \Delta_j]/[\Lambda : \Delta_i],$$

the aforementioned claim implies that

$$[\Delta_i : \Delta_i \cap \Delta_j] \ge p_j^{\ell_j - n^2}.$$

Set $\Lambda_i := \cap_{n=1}^{i} \Delta_n$, then

$$[\Lambda_{j-1} : \Lambda_j] = O_j p_j^{\ell_j} [\Delta_j : \Lambda_j]/[\Lambda : \Lambda_{j-1}],$$

and $[\Lambda : \Lambda_{j-1}]$ divides $[\Lambda : \Delta_1] \cdots [\Lambda : \Delta_{j-1}]$ for which $p_j^{(j-1)n^2}$ is the largest power of p_j that appears as a factor. Hence, we obtain

$$[\Lambda_{j-1} : \Lambda_j] \ge p_j^{\ell_j - (j-1)n^2}.$$

From this, we obtain the important inequality

$$[\Lambda : \Lambda_k] \geq \prod_{j=1}^{k} p_j^{\ell_j-(j-1)n^2}. \tag{10}$$

To employ (10), we need a comparison function. This is where we deviate from the proof in [Bou-Rabee and McReynolds 2010]. Define, for $g \in \Lambda \setminus \{1\}$,

$$M(g) = \min\{[\Lambda : \Delta_i] : g \notin \Delta_i\}.$$

Let $\hat{M}$ be the unique continuous extension of M to $\hat{\Gamma}$. Then clearly, $D_\Gamma(g) \leq M(g)$, and so

$$\int \hat{D}_\Gamma(g)\, d\mu \leq \int \hat{M}(g)\, d\mu.$$

By studying the partial sums that define $\int \hat{M}(g)\, d\mu$, we obtain, for any n,

$$\begin{aligned}\sum_{k=1}^{n}[\Lambda : \Delta_k]\mu(\Lambda_{k-1}\setminus\Lambda_k) &= \sum_{k=1}^{n}[\Lambda : \Delta_k]\Big(\frac{1}{[\Lambda : \Lambda_{k-1}]} - \frac{1}{[\Lambda : \Lambda_k]}\Big)\\ &= \frac{[\Lambda : \Delta_1]}{[\Lambda : \Lambda_0]} - \frac{[\Lambda : \Delta_n]}{[\Lambda : \Lambda_n]} + \sum_{k=1}^{n-1}\frac{[\Lambda : \Delta_{k+1}] - [\Lambda : \Delta_k]}{[\Lambda : \Lambda_k]}\\ &< \frac{[\Lambda : \Delta_1]}{[\Lambda : \Lambda_0]} + \sum_{k=1}^{n}\frac{[\Lambda : \Delta_k]^{1+\delta}}{[\Lambda : \Lambda_k]}.\end{aligned}$$

The last inequality follows from the conclusion of [Bou-Rabee and McReynolds 2010, Proposition 5.1] that $[\Lambda : \Delta_{k+1}] \leq [\Lambda : \Delta_k]^{1+\delta}$. Applying (8) and (10) while plugging in the value for $[\Lambda : \Lambda_k]$ yields

$$\begin{aligned}\sum_{k=1}^{n}\frac{[\Lambda : \Delta_k]^{1+\delta}}{[\Lambda : \Lambda_k]} &\leq \sum_{k=1}^{n}\frac{O_k^{1+\delta}p_k^{(1+\delta)\ell_k}}{\prod_{j=1}^{k}p_j^{\ell_j-(j-1)n^2}}\\ &\leq \sum_{k=1}^{n}\frac{p_k^{(1+\delta)(n^2+\ell_k)}}{\prod_{j=1}^{k}p_j^{\ell_j-(j-1)n^2}} = \sum_{k=1}^{n}\frac{p_k^{(k+\delta)n^2+(\delta)\ell_k}}{\prod_{j=1}^{k-1}p_j^{\ell_j-(j-1)n^2}}.\end{aligned}$$

We compute the ratio (k-th term)/(($k+1$)-th term) of the series above:

$$\frac{\dfrac{p_k^{(k+\delta)n^2+(\delta)\ell_k}}{\prod_{j=1}^{k-1}p_j^{\ell_j-(j-1)n^2}}}{\dfrac{p_{k+1}^{(k+\delta+1)n^2+(\delta)\ell_{k+1}}}{\prod_{j=1}^{k}p_j^{\ell_j-(j-1)n^2}}} = \frac{p_k^{(1+\delta)n^2+(\delta+1)\ell_k}}{p_{k+1}^{(k+\delta+1)n^2+\delta\ell_{k+1}}} \leq \frac{p_k^{(1+\delta)n^2+(\delta+1)\ell_k}}{p_k^{(k+\delta+1)n^2+\delta\ell_{k+1}}} = p_k^{-kn^2+\delta(\ell_k-\ell_{k+1})+\ell_k}.$$

Thus, if δ is sufficiently small and k is sufficiently large, we have that the exponent above is greater than 1, by (9). Hence, as p_k is an increasing sequence of integers, the ratio test implies that the resulting series above converges, and $\int \hat{M}(g)\, d\mu$ is finite. We conclude that the normal residual average of Λ is finite, as desired. □

4. Expected depth of random walks on groups

Fix for the whole section a finitely generated residually finite group G and a finite symmetric generating set S. Consider the simple lazy random walk $(X_n)_{n\geq 0}$ on the Cayley graph $\mathrm{Cay}(G, S)$, as defined in Section 2B. Recall that we are interested in

$$\mathbb{E}[D_G(X_n)] = \sum_{k\geq 2} k\mathbb{P}[D(X_n) = k] = \sum_{k\geq 0} \mathbb{P}[D(X_n) > k] = \sum_{k\geq 0} \mathbb{P}(X_n \in \Lambda_k \setminus \{e\}).$$

The second equality is obtained through the same double-sum argument as in the proof of (7).

4A. *First estimates.*

Proposition 4.1. *We have*

$$\liminf_{n\to\infty} \mathbb{E}[D(X_n)] \geq 2 + \sum_{k\geq 2} \frac{1}{|G : \Lambda_k|}.$$

Proof. Recall the expression (1) for $E[D(X_n)]$:

$$\mathbb{E}[D(X_n)] = \sum_{k\geq 0} \mathbb{P}(X_n \in \Lambda_k \setminus \{e\}).$$

Also recall from Corollary 2.5 that

$$\mathbb{P}(X_n \in \Lambda_k \setminus \{e\}) \xrightarrow[n\to\infty]{} \begin{cases} \dfrac{1}{[G : \Lambda_k]} & \text{if } k \geq 2, \\ 1 & \text{if } k = 0, 1. \end{cases}$$

The result follows from Fatou's lemma. □

Corollary 4.2. *Suppose G is such that $\sum_{k\geq 2} 1/|G : \Lambda_k|$ diverges. Then*

$$\lim_{n\to\infty} \mathbb{E}(D(X_n)) = \infty.$$

Proof. This is a direct consequence of Proposition 4.1. □

Proposition 4.3. *Suppose there exists a sequence of positive numbers $\{p_k\}_{k\geq 2}$ with*

- $\sum_{k\geq 2} p_k < \infty$;
- $\mathbb{P}(X_n \in \Lambda_k \setminus \{e\}) \leq p_k$ *for all $n \geq 1$ and $k \geq 2$.*

Then

$$\lim_{n\to\infty} \mathbb{E}[D(X_n)] = 2 + \sum_{k\geq 2} \frac{1}{|G : \Lambda_k|} < \infty.$$

Proof. Fix a sequence $(p_k)_{k\geq 2}$ as above, and set $p_0 = p_1 = 1$. Then the convergence of $\mathbb{P}(X_n \in \Lambda_k \setminus \{e\})$ to $1/[G:\Lambda_k]$ is dominated by p_k. Since p_k is summable, the dominated convergence theorem implies the desired result. □

Below, when applying Proposition 4.3, we will do so using the sequence

$$p_k = \sup_{n\geq 0} \mathbb{P}(X_n \in \Lambda_k \setminus \{e\}), \quad \text{for } k \geq 2.$$

This sequence obviously satisfies the domination criterion; one needs to show it is summable in order to apply the proposition.

4B. ***Sufficient condition using spectral properties.***

Proof of Theorem 1.1. Fix a linear group G with property (T). We will apply Proposition 4.3 to show the desired convergence. Fix some $k \geq 2$ and let us bound $\mathbb{P}(X_n \in \Lambda_k \setminus \{e\})$ for arbitrary n.

First, notice that $X_n \in B_G^S(n)$ and therefore $D_G(X_n) \leq F_G^S(n)$. It follows that

$$\mathbb{P}(X_n \in \Lambda_k \setminus \{e\}) = 0 \quad \text{if } F_G^S(n) \leq k.$$

Suppose now that n is such that $F_G^S(n) > k$. Recall from Corollary 2.5 that

$$\left| \mathbb{P}(X_n \in \Lambda_k) - \frac{1}{|G:\Lambda_k|} \right| \leq \mu_k^n,$$

where μ_k is the second largest eigenvalue of the transition matrix of the induced lazy random walk on $\mathrm{Cay}(G/\Lambda_k, S)$. Now, since G has property (T), there exists a constant $0 < \theta < 1$ such that, for any normal finite index subgroup $N \lhd G$, the second largest eigenvalue of the Cayley graph of G/N is bounded above by $\theta < 1$ (see [Bekka et al. 2008]; the exact value of θ does depend on the generating set S of G). In particular,

$$\mathbb{P}(X_n \in \Lambda_k \setminus \{e\}) \leq \mathbb{P}(X_n \in \Lambda_k) \leq \frac{1}{|G:\Lambda_k|} + \mu_k^n \leq \frac{1}{|G:\Lambda_k|} + \theta^n.$$

Observe that the right-hand side above is decreasing in n, and therefore is maximal when n is minimal. Set $N_k = \inf\{n \geq 1 : F_G^S(n) > k\}$. Then, by the above two cases, we deduce that

$$\mathbb{P}(X_n \in \Lambda_k \setminus \{e\}) \leq \frac{1}{|G:\Lambda_k|} + \theta^{N_k} =: p_k \quad \text{for all } n \geq 1 \text{ and } k \geq 2.$$

The values $(p_k)_{k\geq 2}$ defined above satisfy the second property of Proposition 4.3; we will show now that they also satisfy the first.

By [Bou-Rabee and McReynolds 2015], there exist $b \in \mathbb{N}$ and $C > 0$ such that $F_G^S(n) \leq Cn^b$ for all $n \geq 1$. In particular, for any $k \geq 2$, $N_k \geq C'k^{1/b}$ for some constant $C' > 0$ that does not depend on k. Moreover, $\sum_{k\geq 2} 1/|G:\Lambda_k|$ is finite by Equation (7) and Theorem 3.2. Thus

$$\sum_{k\geq 2} p_k \leq \sum_{k\geq 2} \frac{1}{|G:\Lambda_k|} + \sum_{k\geq 2} \theta^{C'k^{1/b}} < \infty.$$

Applying Proposition 4.3 yields the desired result. □

4C. *Sufficient condition: abelian groups.*

Lemma 4.4. *Let $(X_n)_{n\geq 0}$ be a lazy random walk on $\mathbb{Z}$ (that is on a Cayley graph of $\mathbb{Z}$, as in Section 2B). Then there exists a constant $C > 0$ such that, for all $m \geq 1$,*

$$\sup_{n\geq 0} \mathbb{P}(X_n \in m\mathbb{Z} \setminus \{0\}) \leq \frac{C}{\sqrt{m}}. \tag{11}$$

In particular, there exists $c > 0$ such that, for $k \geq 2$,

$$\sup_{n\geq 0} \mathbb{P}(X_n \in \Lambda_k(\mathbb{Z}) \setminus \{0\}) \leq e^{-ck}. \tag{12}$$

Proof. We start with the proof of (11). Let $c_0 > 0$ be such that $|X_1| < 1/(2c_0)$ almost surely. Below, write $c_0 m$ for the integer part of $c_0 m$ so as not to overburden notation. Then, for $n \leq c_0 m$, $\mathbb{P}(X_n \in m\mathbb{Z} \setminus \{0\}) = 0$. For $n \geq c_0 m$, write

$$\mathbb{P}(X_n \in m\mathbb{Z} \setminus \{0\}) = \sum_{\ell\in\mathbb{Z}} \mathbb{P}(X_n \in m\mathbb{Z} \setminus \{0\} \mid X_{n-c_0 m} = \ell)\mathbb{P}(X_{n-c_0 m} = \ell). \tag{13}$$

Now notice that, due to the choice of c_0, $|X_n - X_{n-c_0 m}| < m/2$ almost surely. However, for any fixed $\ell \in \mathbb{Z}$, there exists at most one element $m(\ell) \in m\mathbb{Z}$ with $|\ell - m(\ell)| < m/2$. If no such element exists, choose $m(\ell) \in m\mathbb{Z}$ arbitrarily. Thus

$$\begin{aligned}\mathbb{P}(X_n \in m\mathbb{Z} \setminus \{0\} \mid |X_{n-c_0 m} = \ell) &\leq \mathbb{P}(X_n = m(\ell) \mid |X_{n-c_0 m} = \ell) \\ &= \mathbb{P}(X_{c_0 m} = m(\ell) - \ell) \leq \frac{C}{\sqrt{c_0 m}},\end{aligned}$$

where the last inequality is due to [Varopoulos et al. 1992, Theorem VI.5.1] and $C > 0$ is some fixed constant depending only on the transition probability of the random walk. When injecting the above in (13), we find

$$\mathbb{P}(X_n \in m\mathbb{Z} \setminus \{0\}) \leq \sum_{\ell\in\mathbb{Z}} \frac{C}{\sqrt{c_0 m}} \mathbb{P}(X_{n-c_0 m} = \ell) = \frac{C}{\sqrt{c_0 m}}.$$

Since the right-hand side does not depend on n, this implies (11) with an adjusted value of C.

We move on to proving (12). The (normal) subgroups of $\mathbb{Z}$ are of the form $k\mathbb{Z}$, with k being their index. Thus, for $k \geq 2$,

$$\Lambda_k(\mathbb{Z}) = m_k\mathbb{Z},$$

where m_k is the least common multiple of $1, \dots, k$ (see [Biringer et al. 2017]). It follows from the prime number theorem that there exists a constant $c > 0$ such that

$$m_k \geq \exp(ck), \quad \text{for all } k \geq 2.$$

The above bound, together with (11), implies (12) with an adjusted value of c. □

Corollary 4.5 (expected depth for $\mathbb{Z}$). *Let $(X_n)_{n\geq 0}$ be a lazy random walk on $\mathbb{Z}$ (that is, on a Cayley graph of $\mathbb{Z}$, as in Section 2B). Then*

$$\mathbb{E}[D_{\mathbb{Z}}(X_n)] \xrightarrow[n\to\infty]{} 2 + \sum_{k\geq 2} \frac{1}{|\mathbb{Z} : \Lambda_k|} < \infty. \tag{14}$$

Proof. For $k \geq 2$, set $p_k = \sup_{n\geq 0} \mathbb{P}(X_n \in \Lambda_k(\mathbb{Z}) \backslash \{0\})$. By Lemma 4.4, $\sum_{k\geq 2} p_k < \infty$.

Further, the sequence $(p_k)_{k\geq 2}$ dominates the convergence of $\mathbb{P}(X_n \in \Lambda_k(\mathbb{Z}) \setminus \{0\})$, as required in Proposition 4.3. The conclusion follows. □

Proposition 4.6. *Let G and H be two finitely generated residually finite groups. Let $(X_n)_{n\geq 0}$ be a random walk on a Cayley graph of $G \times H$, as in Section 2B. Then*

$$\mathbb{P}[D_{G\times H}(X_n) > k] \leq \mathbb{P}[D_G(Y_n) > k] + \mathbb{P}[D_H(Z_n) > k] \quad \textit{for all } k \geq 0,$$

where $(Y_n)_{n\geq 0}$ and $(Z_n)_{n\geq 0}$ are the random walks on G and H, respectively, induced by $(X_n)_{n\geq 0}$.

Proof. Notice that for $g = (g_1, g_2) \in G \times H$, we have estimates

$$D_{G\times H}(g) \leq D_G(g_1) \text{ if } g_1 \neq e,$$
$$D_{G\times H}(g) \leq D_G(g_2) \text{ if } g_2 \neq e.$$

Therefore $\mathbb{P}[D_{G\times H}(X_n) > k] \leq \mathbb{P}[D_G(Y_n) > k] + \mathbb{P}[D_H(Z_n) > k]$. □

Corollary 4.7. *Let G be a finitely generated abelian group and let $(X_n)_{n\geq 0}$ be a lazy random walk on its Cayley graph, as in Section 2B. Then*

$$\mathbb{E}[D_G(X_n)] \xrightarrow[n\to\infty]{} 2 + \sum_{k\geq 2} \frac{1}{|G : \Lambda_k|} < \infty,$$

for any finite generating set S of G.

Proof. Let G be a finitely generated abelian group. Then it may be written as $G = \mathbb{Z} \times \cdots \times \mathbb{Z} \times H$, where the product contains j copies of $\mathbb{Z}$ and H is a finite abelian group. The depth of elements of H is bounded by $|H|$. By Proposition 4.6 and Lemma 4.4, there exists $c > 0$ such that

$$\mathbb{P}(X_n \in \Lambda_k(G) \setminus \{0\}) \leq je^{-ck}, \quad \text{for all } k > |H|.$$

We conclude using the same domination argument as in the proof of (14). □

4D. ***Nilpotent groups: proof of Theorem 1.2.***

Torsion free case. Suppose first that G is a finitely generated and torsion free nilpotent group, different from $\mathbb{Z}$. Observe that G is a poly-C_∞ group, and in particular, $G = \mathbb{Z} \ltimes H$ where H is a nontrivial finitely generated nilpotent group. Let S be a system of generators of G and consider the lazy random walk on G with steps taken uniformly in S. We will write it in the product form $(X_n, Y_n)_{n\geq 0}$, where $X_n \in \mathbb{Z}$ and $Y_n \in H$, for all $n \geq 0$. Notice then that $(X_n)_{n\geq 0}$ is a lazy random walk on $\mathbb{Z}$, as treated in Lemma 4.4. This is not true on the second coordinate: $(Y_n)_{n\geq 0}$ is not a random walk on H, it is not even a Markov process.

For $k \geq 2$, let

$$p_k(G) = \sup_{n\geq 0} \mathbb{P}[D_G(X_n, Y_n) > k], \tag{15}$$

and recall from Proposition 4.3 (and the commentary below it) that our goal is to prove that $\sum_k p_k < \infty$.

One may easily check that, for any $m \in \mathbb{N}$, $m\mathbb{Z} \ltimes H$ is a normal subgroup of G. Thus, for all $x \in \mathbb{Z} \setminus \{0\}$ and $y \in H$,

$$D_G(x, y) \leq D_{\mathbb{Z}}(x).$$

We may therefore bound p_k by

$$\begin{aligned} p_k(G) &\leq \sup_{n\geq 0} \mathbb{P}[D_G(X_n) > k] + \mathbb{P}[X_n = 0 \text{ and } D_G(0, Y_n) > k] \\ &\leq p_k(\mathbb{Z}) + \sup_{n\geq 0} \mathbb{P}[X_n = 0 \text{ and } D_G(0, Y_n) > k]. \end{aligned}$$

In the above, $p_k(\mathbb{Z}) = \sup_{n\geq 0} \mathbb{P}[D_{\mathbb{Z}}(X_n) > k]$. We have shown in Lemma 4.4 that $\sum_k p_k(\mathbb{Z}) < \infty$, and we may focus on whether the second supremum is summable.

Fix $k \geq 2$. Since G is nilpotent, there exists $c > 0$ such that

$$F_G^S(n) \leq c(\log n)^{h(G)},$$

where $h(G)$ is the Hirsch length of G (see Theorem 2.1). This should be understood as follows. In order for an element $g \in G$ to have $D_G(g) \geq k$, it is necessary that $\|g\|_S \geq \exp(Ck^{1/h(G)})$, where $\|g\|_S$ denotes the length of g with respect to the generating set S and $C > 0$ is a constant independent of g.

In particular, we conclude that

$$\begin{aligned} \mathbb{P}[X_n = 0 \text{ and } D_G(0, Y_n) > k] &= 0 \quad \text{if } n < \exp(Ck^{1/h(G)}); \\ \mathbb{P}[X_n = 0 \text{ and } D_G(0, Y_n) > k] &\leq \mathbb{P}(X_n = 0) \quad \text{if } n \geq \exp(Ck^{1/h(G)}). \end{aligned}$$

In treating the second case, observe that, since $(X_n)_{n\geq 0}$ is a random walk on $\mathbb{Z}$,

$$\mathbb{P}(X_n = 0) \leq c_0 n^{-1/2},$$

for some constant $c_0 > 0$ (see [Varopoulos et al. 1992, Theorem VI.5.1]). Thus

$$\mathbb{P}[X_n = 0 \text{ and } D_G(0, Y_n) > k] \le c_0 \exp\left(-\frac{C}{2} k^{1/(h(G))}\right), \quad \text{for all } n \in \mathbb{N}.$$

We conclude that

$$\sum_{k\ge 2} \sup_{n\ge 0} \mathbb{P}[X_n = 0 \text{ and } D_G(0, Y_n) > k] < \infty,$$

and therefore that $\sum_{k\ge 0} p_k(G) < \infty$.

General nilpotent case. Let now G be a finitely generated nilpotent group. Consider the set $T(G)$ of all torsion elements of G. Since G is nilpotent, the set $T(G)$ is a finite normal subgroup in G. Consider an epimorphism $\pi : G \to G/T(G)$. Denote by H the quotient $G/T(G)$ and notice that H is a finitely generated torsion-free nilpotent group.

Observe that for any nontrivial element in H detected by a normal subgroup in H of index k there exists a normal subgroup in G of index at most k that detects its preimage. In other words, for all $g \in G \setminus T(G)$,

$$D_G(g) \le D_H(\pi(g)).$$

The random walk $(X_n)_{n\ge 0}$ on G induces a random walk $(\pi(X_n))_{n\ge 0}$ on H. Let

$$d = \max\{D_G(g), g \in T(G)\}.$$

Due to the observation above, for all $k > d$,

$$\mathbb{P}[D_G(X_n) \ge k] \le \mathbb{P}[D_H(\pi(X_n)) \ge k].$$

We deduce from the case of torsion free nilpotent groups that

$$\sum_{k>d} \sup_{n\ge 0} \mathbb{P}[D_G(X_n) \ge k] < \infty.$$

Then the second point of Proposition 4.3 applies and the proof is concluded. □

4E. *A counter example.*

Proposition 4.8 (groups with infinite expected depth). *There exists a finitely generated residually finite group G such that*

$$\lim_{n\to\infty} \mathbb{E}(D(X_n)) = \infty,$$

but for which the "presumed" limit $\sum_{k\ge 2} 1/|G : \Lambda_k|$ is finite.

Proof. The existence of finitely generated residually finite groups with arbitrary large residual finiteness growth was shown in [Bou-Rabee and Seward 2016].

Let H be a two-generated group (with generators a, b) such that, for any $n \geq 8$, there exists an element h_n in the ball of radius n of H with $D_H(h_n) \geq 24^n$. Let $(X_n)_{n\geq 0}$ be a lazy simple random walk on the Cayley graph of $G = H \times \mathbb{Z}$ with the natural choice of 3 generators (that is, $(a, 0)$, $(b, 0)$ and $(e_H, 1)$, where a and b are the two generators of H mentioned above) and their inverses.

Then, for any $n \geq 1$, $\mathbb{P}[X_n = (h_n, 0)] \geq 12^{-n}$. Therefore

$$\mathbb{E}[D_G(X_n)] \geq \mathbb{P}[X_n = (h_n, 0)] \cdot D_G[(h_n, 0)] = \mathbb{P}[X_n = (h_n, 0)] \cdot D_H(h_n) \geq 2^n.$$

Hence the expectation of the depth of X_n tends to infinity.

Furthermore, observe that $\Lambda_k(G)$ is a subgroup of $H \times \Lambda_k(\mathbb{Z})$ and hence

$$|G : \Lambda_k(G)| \geq |\mathbb{Z} : \Lambda_k(\mathbb{Z})|.$$

It follows that $\sum_{k\geq 2} 1/|G : \Lambda_k(G)| \leq \sum_{k\geq 2} 1/|\mathbb{Z} : \Lambda_k(\mathbb{Z})| < \infty$. □

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Received November 8, 2017.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 298, No. 2, 2019
dx.doi.org/10.2140/pjm.2019.298.285

SIGNATURE RANKS OF UNITS IN CYCLOTOMIC EXTENSIONS OF ABELIAN NUMBER FIELDS

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We prove the rank of the group of signatures of the circular units (hence also the full group of units) of $\mathbb{Q}(\zeta_m)^+$ tends to infinity with m. We also show the signature rank of the units differs from its maximum possible value by a bounded amount for all the real subfields of the composite of an abelian field with finitely many odd prime-power cyclotomic towers. In particular, for any prime p the signature rank of the units of $\mathbb{Q}(\zeta_{p^n})^+$ differs from $\varphi(p^n)/2$ by an amount that is bounded independent of n. Finally, we show conditionally that for general cyclotomic fields the unit signature rank can differ from its maximum possible value by an arbitrarily large amount.

1. Introduction

For any positive integer m with m odd or m divisible by 4, let ζ_m be a primitive m-th root of unity, $\mathbb{Q}(\zeta_m)$ the corresponding cyclotomic field and $\mathbb{Q}(\zeta_m)^+ = \mathbb{Q}(\zeta_m + \zeta_m^{-1})$ its maximal totally real subfield. The units in $\mathbb{Q}(\zeta_m)^+$ together with ζ_m are a subgroup of finite index in the group of units of $\mathbb{Q}(\zeta_m)$ (this index is 1 when m is a prime power and 2 otherwise; see [Washington 1982, Corollary 4.13]).

Under the various $\varphi(m)/2$ real embeddings of $\mathbb{Q}(\zeta_m)^+$, each unit ε of $\mathbb{Q}(\zeta_m)^+$ has a sign, and the collection of these signs is called the *signature* of ε. The collection of all such unit signatures is an elementary abelian 2-group, whose rank is called the *unit signature rank* of $\mathbb{Q}(\zeta_m)^+$ (or, by abuse, of $\mathbb{Q}(\zeta_m)$). The signature rank measures how many different possible signs arise from the units and determines the difference between the class number and the strict class number.

The purpose of this paper is to prove that the unit signature rank of $\mathbb{Q}(\zeta_m)^+$ tends to infinity with m. We do this by demonstrating an explicit lower bound for the signature rank of the subgroup of *cyclotomic units* of $\mathbb{Q}(\zeta_m)^+$. We then show that the difference between the signature rank of the units in the maximal real subfield of the cyclotomic field of $mp_1^{n_1} \cdots p_s^{n_s}$-th roots of unity (with all p_i odd) and its maximum possible value is bounded independent of $n_1, \ldots, n_s$ (and is constant if all the n_i are sufficiently large). This in particular provides infinitely many families

MSC2010: primary 11R18; secondary 11R27.
Keywords: signature rank of units, cyclotomic fields, abelian extensions, circular units.

of cyclotomic fields whose unit signature ranks are "nearly maximal". Finally, we show this difference can be arbitrarily large, conditional on the existence of infinitely many cyclic cubic fields with totally positive fundamental units.

2. Signatures

For any totally real field F of degree n over $\mathbb{Q}$ let $F_{\mathbb{R}} = F \otimes_{\mathbb{Q}} \mathbb{R} \simeq \prod_{v \text{ real}} \mathbb{R}$ where the product is taken over the real embeddings v of F. Define the *archimedean signature space* $V_{\infty,F}$ of F to be

$$V_{\infty,F} = F_{\mathbb{R}}^{*}/F_{\mathbb{R}}^{*2} \simeq \prod_{v \text{ real}} \{\pm 1\} \simeq \mathbb{F}_2^n,$$

where by identifying $\{\pm 1\}$ with the finite field $\mathbb{F}_2$ of two elements, we view the multiplicative group $V_{\infty,F}$ as a vector space over $\mathbb{F}_2$, written additively.

For any $\alpha \in F^*$ and $v : F \hookrightarrow \mathbb{R}$ a real place of F, let $\alpha_v = v(\alpha)$ and define the sign of α_v as usual by $\operatorname{sign}(\alpha_v) = \alpha_v/|\alpha_v| \in \{\pm 1\}$. Write $\operatorname{sgn}(\alpha_v) \in \mathbb{F}_2$ for $\operatorname{sign}(\alpha_v)$ when viewed in the additive group $\mathbb{F}_2$, i.e., $\operatorname{sgn}(\alpha_v) = 0$ if $\alpha_v > 0$ and $\operatorname{sgn}(\alpha_v) = 1$ if $\alpha_v < 0$. The (*archimedean*) *signature map* of F is the homomorphism

$$\operatorname{sgn}_{\infty,F} : F^* \to V_{\infty,F}, \qquad \alpha \mapsto (\operatorname{sgn}(\alpha_v))_v.$$

In the case when $F/\mathbb{Q}$ is Galois and one real embedding of F is fixed, we can index the real embeddings of F by the elements σ in $\operatorname{Gal}(F/\mathbb{Q})$, and

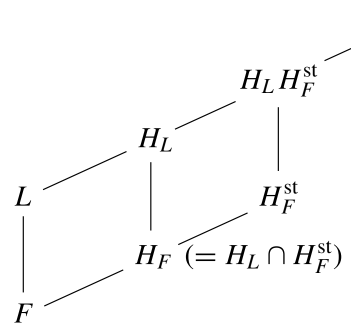
$$\operatorname{sgn}_{\infty,F}(\alpha) = (\operatorname{sgn}(\sigma(\alpha)))_{\sigma \in \operatorname{Gal}(F/\mathbb{Q})},$$

where sgn is the sign (viewed as an element of $\mathbb{F}_2$) in the fixed real embedding. The element $\operatorname{sgn}_{\infty,F}(\alpha)$ is called the *signature* of α.

The collection of all the signatures $\operatorname{sgn}_{\infty,F}(\varepsilon)$ where ε varies over the units of F is called the *unit signature group* of F; the rank of this subspace of $V_{\infty,F}$ is called the (*unit*) *signature rank* of F and, as previously mentioned, is a measure of how many different possible sign configurations arise from the units of F.

Define the (*unit signature rank*) *"deficiency" of* F, denoted $\delta(F)$, to be the corank of the unit signature group of F in $V_{\infty,F}$, i.e., $[F : \mathbb{Q}]$ minus the signature rank of the units of F. The deficiency of F is just the nonnegative difference between the unit signature rank of F and its maximum possible value — the deficiency is 0 if and only if there are units of every possible signature type. The deficiency is also the rank of the group of totally positive units of F modulo squares.

Remark 1. For any finite extension L/F of totally real fields, we have $\delta(F) \le \delta(L)$, a result due to Edgar, Mollin and Peterson [Edgar et al. 1986, Theorem 2.1]. We briefly recall the reason: the intersection of the Hilbert class field H_L of L with the strict (or narrow) Hilbert class field H_F^{st} of F is easily seen to be the Hilbert class



field H_F of F since L is totally real. The composite field $H_L H_F^{\mathrm{st}}$ is a subfield of the strict Hilbert class field, H_L^{st}, of L and has degree over H_L equal to $[H_F^{\mathrm{st}} : H_F]$ because $H_F = H_L \cap H_F^{\mathrm{st}}$. Since $[H_F^{\mathrm{st}} : H_F] = 2^{\delta(F)}$ and $[H_L^{\mathrm{st}} : H_L] = 2^{\delta(L)}$ (see [Dummit and Voight 2018, §2] for details), the result follows.

3. Circular units and signatures in cyclotomic fields

Suppose now that $F = \mathbb{Q}(\zeta_m)^+$ is the maximal (totally) real subfield of the cyclotomic field $\mathbb{Q}(\zeta_m)$ of m-th roots of unity (with m odd or divisible by 4), which is of degree $\varphi(m)/2$ over $\mathbb{Q}$.

We can fix an embedding of $\mathbb{Q}(\zeta_m)$ into $\mathbb{C}$ by mapping ζ_m to $e^{2\pi i/m}$, which also fixes an embedding of $\mathbb{Q}(\zeta_m)^+$ into $\mathbb{R}$.

The Galois group $\mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})$ consists of the automorphisms σ_a that map ζ_m to ζ_m^a for integers a, $1 \le a < m$, relatively prime to m. We identify $\mathrm{Gal}(\mathbb{Q}(\zeta_m)^+/\mathbb{Q})$ with $\mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})/\{1, \sigma_{-1}\}$ and take the elements $\bar{\sigma}_a \in \mathrm{Gal}(\mathbb{Q}(\zeta_m)^+/\mathbb{Q})$ with $1 \le a < m/2$ relatively prime to m as representatives.

To get information on the unit signature rank of $\mathbb{Q}(\zeta_m)^+$, we consider the signatures of the subgroup of *circular* (or *cyclotomic*) *units*, denoted $C_{\mathbb{Q}(\zeta_m)}$, which when m is a prime power has a set of generators given by -1 and the $\varphi(m)/2 - 1$ independent elements

$$\zeta_m^{(1-a)/2} \frac{1 - \zeta_m^a}{1 - \zeta_m} \tag{1}$$

for $1 < a < m/2$ and a relatively prime to m [Washington 1982, Lemma 8.1] (for m not a prime power the definition of $C_{\mathbb{Q}(\zeta_m)}$ is more complicated; see [Washington 1982, Chapter 8]).

The group $C_{\mathbb{Q}(\zeta_m)}$ is contained in $\mathbb{Q}(\zeta_m)^+$ and, when m is a prime power, is isomorphic (as an additive abelian group) to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}^{\varphi(m)/2-1}$, with -1 and the elements in (1) serving as independent generators over $\mathbb{Z}$.

If we choose an ordering of the elements of $\mathrm{Gal}(\mathbb{Q}(\zeta_m)^+/\mathbb{Q})$ and an ordering of a set of generators for the cyclotomic units, their corresponding signatures in

$\mathbb{F}_2^{\varphi(m)/2}$ define the rows of a $\varphi(m)/2 \times \varphi(m)/2$ matrix over $\mathbb{F}_2$, whose rank is the rank of the subgroup $\mathrm{sgn}_{\infty,\mathbb{Q}(\zeta_m)^+}(C_{\mathbb{Q}(\zeta_m)})$ of signatures of the cyclotomic units.

We first consider the case where m is an odd prime power.

The case m = p^n for an odd prime p. When $m = p^n$ for an odd prime p, every primitive m-th root of unity is the square of another primitive m-th root of unity, so the circular units in (1) can be written in the form

$$\frac{\zeta_m^a - \zeta_m^{-a}}{\zeta_m - \zeta_m^{-1}} \tag{2}$$

for $1 < a < m/2$ and a relatively prime to m. If we order the elements in $\mathrm{Gal}(\mathbb{Q}(\zeta_m)^+/\mathbb{Q})$ by $\bar{\sigma}_b$ with $1 \le b < m/2$ relatively prime to m, then the signature of the element in (2) is given by the signs of the elements

$$\bar{\sigma}_b\left(\frac{\zeta_m^a - \zeta_m^{-a}}{\zeta_m - \zeta_m^{-1}}\right) = \frac{\zeta_m^{ab} - \zeta_m^{-ab}}{\zeta_m^b - \zeta_m^{-b}}, \quad 1 \le b < m/2, \ \ (b, m) = 1.$$

Under the embedding $\zeta_m = e^{2\pi i/m}$ we have

$$\frac{\zeta_m^{ab} - \zeta_m^{-ab}}{\zeta_m^b - \zeta_m^{-b}} = \frac{\sin(2\pi ab/m)}{\sin(2\pi b/m)}. \tag{3}$$

Since $1 \le b < m/2$, the denominator $\sin(2\pi b/m)$ is positive. Hence,

$$\bar{\sigma}_b((\zeta_m^a - \zeta_m^{-a})/(\zeta_m - \zeta_m^{-1}))$$

is positive if and only if $\sin(2\pi ab/m)$ is positive, which happens precisely when the least positive residue of ab modulo m is in $(0, m/2)$.

It follows that the rank of the subspace $\mathrm{sgn}_{\infty,\mathbb{Q}(\zeta_m)^+}(C_{\mathbb{Q}(\zeta_m)})$ of $V_{\infty,\mathbb{Q}(\zeta_m)^+}$ is equal to the rank of a $\varphi(m)/2 \times \varphi(m)/2$ matrix $C = (c_{a,b})$ over $\mathbb{F}_2$ (referred to as the *circular unit signature matrix*) whose rows are indexed by the elements a relatively prime to m with $1 \le a < m/2$ and whose columns are indexed by the elements b relatively prime to m with $1 \le b < m/2$, as follows. The first row of C is $(1, 1, \dots, 1)$, corresponding to the signs $(-1, -1, \dots, -1)$ of the element -1, viewed in the additive group $\mathbb{F}_2$, so

$$c_{1,b} = 1,$$

for $1 \le b < m/2$, with b relatively prime to m. For $2 \le a < m/2$, $1 \le b < m/2$, with a and b relatively prime to m, we have

$$c_{a,b} = \begin{cases} 0 & \text{if } ab \pmod m \in (0, m/2), \\ 1 & \text{if } ab \pmod m \in (m/2, m), \end{cases}$$

where $ab \pmod m$ is taken to be the least positive residue of ab modulo m.

For computational purposes, it is useful to note the row indexed by a is the i-th row of the matrix where $i = a - \lfloor (a-1)/p \rfloor$ and the i-th row is indexed by $a = i + \lfloor (i-1)/(p-1) \rfloor$ (and similarly for the numbering of the columns).

If we add the first row of C to the remaining rows (which does not affect the rank of the matrix), we obtain the *modified circular unit signature matrix* $M = (c'_{a,b})$ with

$$c'_{a,b} = \begin{cases} 1 & \text{if } ab \pmod m \in (0, m/2), \\ 0 & \text{if } ab \pmod m \in (m/2, m), \end{cases}$$

for $1 \le a < m/2$ and $1 \le b < m/2$, with a and b relatively prime to m and as before $ab \pmod m$ is taken to be the least positive residue of ab modulo m (this matrix appears in [Davis 1969] in the case when m is an odd prime).

The entry $c'_{2^d,b}$ of the matrix M in the column indexed by b ($b = 1, \ldots, m/2$, with b prime to m) and the row indexed by 2^d ($1 \le 2^d < m/2$) is 1 if the least positive residue of $2^d b$ modulo m lies in $(0, m/2)$ and is 0 if the least positive residue lies in $(m/2, m)$. Writing $2^d b = Am + r$ with an integer A and least positive remainder r with $0 \le r < m$, i.e., $2^{d+1} b/m = 2A + (2r/m)$, it follows that $r \in (0, m/2)$ implies $\lfloor 2^{d+1} b/m \rfloor = 2A$ and $r \in (m/2, m)$ implies $\lfloor 2^{d+1} b/m \rfloor = 2A+1$. As a consequence, the entry of M in the row indexed by 2^d and column indexed by b is 0 if $\lfloor 2^{d+1} b/m \rfloor$ is odd and is 1 if $\lfloor 2^{d+1} b/m \rfloor$ is even.

Lemma. *Suppose $m = p^n$ where p is an odd prime and $n \ge 1$. Let $k \ge 1$ be any integer with $m > 2^{k+2}$. If*

$$b_0(k) = \left\lfloor \frac{(2^k - 2)m}{2^{k+1}} \right\rfloor + 1 \quad \textit{and} \quad b_1(k) = \left\lfloor \frac{(2^k - 2)m}{2^{k+1}} \right\rfloor + 2,$$

then $\lfloor 2^{d+1} b_0(k)/m \rfloor$ and $\lfloor 2^{d+1} b_1(k)/m \rfloor$ are both odd for $d = 1, 2, \ldots, k-1$ and both even for $d = k$.

Proof. Write

$$\frac{(2^k - 2)m}{2^{k+1}} = \left\lfloor \frac{(2^k - 2)m}{2^{k+1}} \right\rfloor + \theta$$

where $0 \le \theta < 1$. Then

$$b_0(k) = \frac{2^k - 2}{2^{k+1}} m + (1 - \theta) \quad \text{and} \quad b_1(k) = \frac{2^k - 2}{2^{k+1}} m + (2 - \theta),$$

so

$$\frac{2^{d+1} b_0(k)}{m} = 2^d - \frac{2}{2^{k-d}} + \frac{2^{d+1}}{m}(1-\theta) \ \text{ and } \ \frac{2^{d+1} b_1(k)}{m} = 2^d - \frac{2}{2^{k-d}} + \frac{2^{d+1}}{m}(2-\theta).$$

Since $0 < 1 - \theta \le 1$, $1 < 2 - \theta \le 2$, and $m > 2^{k+2}$, when $d = k$ this gives

$$\lfloor 2^{k+1} b_0(k)/m \rfloor = \lfloor 2^{k+1} b_1(k)/m \rfloor = 2^k - 2,$$

so $\lfloor 2^{k+1}b_0(k)/m \rfloor$ and $\lfloor 2^{k+1}b_1(k)/m \rfloor$ are both even. For $1 \le d < k$, we have

$$2^d - 1 < 2^d - \frac{2}{2^{k-d}} + \frac{2^{d+1}}{m}(1-\theta) < 2^d - \frac{2}{2^{k-d}} + \frac{2^{d+1}}{m}(2-\theta) < 2^d,$$

so that

$$\lfloor 2^{d+1}b_0(k)/m \rfloor = \lfloor 2^{d+1}b_1(k)/m \rfloor = 2^d - 1,$$

and $\lfloor 2^{d+1}b_0(k)/m \rfloor$ and $\lfloor 2^{d+1}b_1(k)/m \rfloor$ are both odd, completing the proof of the lemma. □

We can use the lemma to give the following lower bound for the number of independent signatures for the circular units in this case.

Proposition 2. *Suppose p is an odd prime and $n \ge 1$. Then the rank of the group of signatures of the circular units in $\mathbb{Q}(\zeta_{p^n})^+$ is at least $\lfloor \log_2(p^n) \rfloor - 2$.*

Proof. Since $b_0(k)$ and $b_1(k)$ in the lemma differ by 1 and $m = p^n$, at least one is relatively prime to m. It then follows from the lemma that for each $k \ge 1$ with $2^{k+2} < m$ there is a $B(k)$, relatively prime to m and satisfying $1 \le B(k) < m/2$, such that $\lfloor 2^{d+1}B(k)/m \rfloor$ is odd for $d = 1, 2, \dots, k-1$ and even for $d = k$.

By the remarks before the lemma, it follows that for each $k \ge 1$ with $2^{k+2} < m$, the entries of the matrix M in the column indexed by $B(k)$ and belonging to the rows indexed by $2, 4, 8, \dots, 2^k$ are $0, 0, \dots, 0, 1$, respectively. In particular, this shows that the row of M indexed by 2^k is not in the span of the rows indexed by $2, 4, \dots, 2^{k-1}$. Applying this successively for $k = 1, 2, \dots, \lfloor \log_2 m \rfloor - 2$ shows that all these rows are linearly independent, implying that the rank of M is at least $\lfloor \log_2 m \rfloor - 2$, which proves the proposition. □

The case $m = 2^n$, $n \ge 2$. In this case let $\zeta_{2^{n+1}}$ be a 2^{n+1}-st root of unity with $\zeta_{2^n} = \zeta_{2^{n+1}}^2$. Fix an embedding into $\mathbb{C}$ mapping $\zeta_{2^{n+1}}$ to $e^{2\pi i/2^{n+1}}$ and order the elements of $\mathrm{Gal}(\mathbb{Q}(\zeta_m)^+/\mathbb{Q})$ as $\bar{\sigma}_b$ with odd $b = 3, 5, \dots, 2^{n-1} - 1$. Then the conjugates of the elements in (1) are given by

$$\frac{\zeta_{2^{n+1}}^{ab} - \zeta_{2^{n+1}}^{-ab}}{\zeta_{2^{n+1}}^{b} - \zeta_{2^{n+1}}^{-b}} = \frac{\sin(\pi ab/2^n)}{\sin(\pi b/2^n)} \tag{4}$$

for $1 < a < 2^{n-1}$, $1 \le b < 2^{n-1}$, with a and b both odd. The sign of the unit in (4) is $+1$ if the least positive residue of ab modulo 2^{n+1} lies in $(0, 2^n)$ and is -1 if the least positive residue lies in $(2^n, 2^{n+1})$.

In this case the circular unit signature matrix has first row consisting of all 1's and the entry in the row indexed by a and column indexed by b is given by

$$c_{1,b} = 1 \tag{5}$$

for b odd, $1 < b < 2^{n-1}$, and

$$c_{a,b} = \begin{cases} 0 & \text{if } ab \pmod{2^{n+1}} \in (0, 2^n), \\ 1 & \text{if } ab \pmod{2^{n+1}} \in (2^n, 2^{n+1}), \end{cases} \tag{6}$$

for a and b odd, $1 < a < 2^{n-1}$ and $1 \le b < 2^{n-2}$, where $ab \pmod{2^{n+1}}$ is taken to be the least positive residue of ab modulo 2^{n+1}.

An argument similar to that for odd prime powers (here for the rows indexed by $2^d - 1$ and column indexed by $2^n - 2^{n-k+1} + 1$) shows the rank of the circular unit signature matrix is at least $n - 2 = \lfloor \log_2 m \rfloor - 2$, but for this case Weber proved the following definitive result.

Proposition 3 [Weber 1899, B, p. 821]. *Suppose $n \ge 2$. Then the rank of the group of signatures of the circular units in the maximal totally real subfield of the cyclotomic field of 2^n-th roots of unity is maximal, i.e., equal to 2^{n-2}.*

This result was generalized by Hasse [1952], whose simpler and more conceptual proof used the fact that the circular unit $(\zeta_{2^{n+1}}^5 - \zeta_{2^{n+1}}^{-5})/(\zeta_{2^{n+1}} - \zeta_{2^{n+1}}^{-1})$ in $\mathbb{Q}(\zeta_{2^n})^+$ has norm -1 and showed the signatures of its Galois conjugates generate a group of maximal signature rank (see [Hasse 1952, pp. 28–29]). A nice proof of this, using the fact that over $\mathbb{F}_2$ the only irreducible representation of a 2-group is the trivial representation, can be found in [Garbanati 1976]. Another nice proof of Weber's result (due to Iwasawa) can be found in [Dummit 2018].

Remark 4. Unlike the situation for the 2-power cyclotomic fields, not every possible signature type occurs for the circular units in general cyclotomic fields, even for $m = p$ an odd prime (for example, in the case $p = 29$ the circular unit signature group has rank 11, not the maximal possible rank of 14 [Davis 1969, Appendix I, p. 70]).

4. Signatures in composites of extensions

Propositions 2 and 3 already imply that the signature rank of the units of $\mathbb{Q}(\zeta_m)^+$ tends to infinity with m (since $\mathbb{Q}(\zeta_{p^n})^+ \subset \mathbb{Q}(\zeta_m)^+$ if p^n divides m and the largest prime power divisor of m tends to infinity as m tends to infinity), but this can be made more precise using the following result, which may be of independent interest.

Suppose $F/\mathbb{Q}$ and $F'/\mathbb{Q}$ are two totally real Galois extensions of $\mathbb{Q}$ with $F \cap F' = \mathbb{Q}$. Then the composite field FF' has Galois group

$$\mathrm{Gal}(FF'/\mathbb{Q}) = \mathrm{Gal}(F/\mathbb{Q}) \times \mathrm{Gal}(F'/\mathbb{Q}),$$

and independent signatures in F and F' produce essentially independent signatures in FF':

Proposition 5. *With F and F' as above, suppose $\alpha_1, \dots, \alpha_r$ are nonzero elements of F whose signatures $\mathrm{sgn}_{\infty,F}(\alpha_i)$, $i = 1, \dots, r$, are linearly independent in the*

$\mathbb{F}_2$-vector space $V_{\infty,F}$. Suppose similarly that $\beta_1, \ldots, \beta_s$ are nonzero elements of F' whose signatures $\mathrm{sgn}_{\infty,F'}(\beta_j)$, $j = 1, \ldots, s$, are linearly independent in the $\mathbb{F}_2$-vector space $V_{\infty,F'}$. Then the dimension of the space generated by the signatures of $\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_s$ in the $\mathbb{F}_2$-vector space $V_{\infty,FF'}$ is $r+s$ unless the subgroups generated by $\alpha_1, \ldots, \alpha_r$ and by $\beta_1, \ldots, \beta_s$ both contain totally negative elements, in which case the dimension is $r+s-1$.

Proof. If the signatures in $V_{\infty,FF'}$ of $\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_s$ are linearly dependent, then there is an element ${\alpha_1}^{a_1} \cdots {\alpha_r}^{a_r} {\beta_1}^{b_1} \cdots {\beta_s}^{b_s}$ in FF' with $a_1, \ldots, a_r, b_1, \ldots, b_s \in \{0, 1\}$, not all 0, that is totally positive. Assume without loss that at least one of $a_1, \ldots, a_r$ is not 0, and let $\alpha = {\alpha_1}^{a_1} \cdots {\alpha_r}^{a_r}$ and $\beta = {\beta_1}^{b_1} \cdots {\beta_s}^{b_s}$. Then $\sigma\tau(\alpha\beta) = \sigma(\alpha)\tau(\beta)$ is positive for every $\sigma \in \mathrm{Gal}(F/\mathbb{Q})$ and every $\tau \in \mathrm{Gal}(F'/\mathbb{Q})$. Since the signatures $\mathrm{sgn}_{\infty,F}(\alpha_i)$, $i = 1, \ldots, r$, are linearly independent and some a_i is nonzero, there exists a σ_0 such that $\sigma_0(\alpha)$ is negative. This implies $\tau(\beta)$ is negative for every τ, i.e., β is totally negative. Then, since there is a τ_0 with $\tau_0(\beta)$ negative, it follows that also $\sigma(\alpha)$ is negative for every σ, i.e., α is totally negative. Hence there is at most one nontrivial relation among the signatures of $\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_s$ in $V_{\infty,FF'}$ — this nontrivial relation occurs if and only if $(1, 1, \ldots, 1) \in V_{\infty,F}$ is in the $\mathbb{F}_2$-space spanned by $\mathrm{sgn}_{\infty,F}(\alpha_i)$, $i = 1, \ldots, r$, and $(1, 1, \ldots, 1) \in V_{\infty,F'}$ is in the $\mathbb{F}_2$-space spanned by $\mathrm{sgn}_{\infty,F'}(\beta_j)$, $j = 1, \ldots, s$, completing the proof. □

5. The signature rank of the units in $\mathbb{Q}(\zeta_m)$ for general m

If we combine Propositions 2 and 3 on the signature ranks in the prime power case with Proposition 5 we obtain the following result.

Theorem 6. *Suppose the positive integer m is odd or is divisible by 4. Then the rank of the group of signatures of the group of circular units in $\mathbb{Q}(\zeta_m)^+$ is at least $\log_2(m) - 4\omega(m) + 1$, where $\omega(m)$ is the number of distinct prime factors of m. In particular, the signature rank of the units in $\mathbb{Q}(\zeta_m)^+$ tends to infinity with m.*

Proof. Write $m = p_1^{a_1} \ldots p_k^{a_k}$. Then $\mathbb{Q}(\zeta_m)^+$ contains the composite of the totally real fields $\mathbb{Q}(\zeta_{{p_i}^{a_i}})^+$, $i = 1, \ldots, k$. By Propositions 2 and 3, the signature rank of the circular units in these latter fields is at least $\lfloor \log_2({p_i}^{a_i}) \rfloor - 2$ (and much better when $p = 2$). Applying the previous proposition repeatedly shows that the signature rank of the group generated by the circular units is at least

$$\sum_{i=1}^{k} (\lfloor \log_2({p_i}^{a_i}) \rfloor - 2) - (k-1).$$

Since $\lfloor \log_2({p_i}^{a_i}) \rfloor > \log_2({p_i}^{a_i}) - 1$, the lower bound in the theorem follows. The final statement in the theorem follows from standard bounds on the growth of $\omega(m)$ (see, for example, [Hardy and Wright 2008, Section 22.10]). □

Corollary 7. *With the exception of $m = 12$, no maximal real subfield of any cyclotomic field of m-th roots of unity has a fundamental system of units that are all totally positive.*

Proof. The signature rank of the circular units is at least 2 for $m = 2^3, 3^2, 5, 7, 11, 13$ by direct computation and for all $m = p$ for primes $p \geq 17$ by Proposition 2. It follows that the signature rank of the circular units is at least 2 for all m divisible by 2^3, 3^2 or any odd prime $p \geq 5$. The only remaining possible values for m are $m = 3, 4, 12$. The first two have no units of infinite order, and the third has maximal real subfield $\mathbb{Q}(\sqrt{3})$ with totally positive fundamental unit $2+\sqrt{3}$. □

6. Signatures in cyclotomic towers over cyclotomic fields

Computations suggest that the signature rank of the units in the real subfield of the cyclotomic field of m-th roots of unity is in fact always close to the maximal possible rank of $\varphi(m)/2$ (equivalently, the unit signature rank deficiency for these fields should be close to 0), i.e., nearly all possible signature types arise for units. This is in keeping with the heuristics in [Dummit and Voight 2018] suggesting that "most" totally real fields have nearly maximal unit signature rank (although these abelian extensions are hardly "typical").

In this section we prove that for infinitely many different families of cyclotomic fields the units do indeed have nearly maximal signature rank. We do this by showing the unit signature rank deficiency is bounded in (finitely many composites of) prime power cyclotomic towers over cyclotomic fields.

Theorem 8. *Suppose $p_1, \dots, p_s$ $(s \geq 1)$ are distinct odd primes and suppose m is any positive integer that is either odd or divisible by 4 and that is relatively prime to $p_1, \dots, p_s$.*

Let $\delta(m; n_1, \dots, n_s) = \delta(\mathbb{Q}(\zeta_{mp_1^{n_1}\cdots p_s^{n_s}})^+) \geq 0$ denote the unit signature rank deficiency of the maximal real subfield of the cyclotomic field of $mp_1^{n_1}\cdots p_s^{n_s}$-th roots of unity defined in Section 2, i.e., the nonnegative difference between the signature rank of the units of $\mathbb{Q}(\zeta_{mp_1^{n_1}\cdots p_s^{n_s}})^+$ and its maximum possible value $\varphi(mp_1^{n_1}\cdots p_s^{n_s})/2$. Then:

(a) *$\delta(m; n_1, \dots, n_s) \leq \delta(m; n'_1, \dots, n'_s)$ if $n_i \leq n'_i$ for all $1 \leq i \leq s$.*

(b) *$\delta(m; n_1, \dots, n_s)$ is bounded independent of $n_1, \dots, n_s$.*

(c) *$\delta(m; n_1, \dots, n_s)$ is constant (depending on m) if $n_1, \dots, n_s$ are all large enough.*

Proof. The maximal real subfield of the cyclotomic field of $mp_1^{n_1}\cdots p_s^{n_s}$-th roots of unity is a subfield of the maximal real subfield of the cyclotomic field of $mp_1^{n'_1}\cdots p_s^{n'_s}$-th roots of unity if $n_i \leq n'_i$, $i = 1, \dots, s$, so (a) follows immediately from Remark 1.

Suppose that $n_i \geq 1$ for $1 \leq i \leq s$ and let L denote the cyclotomic field

$\mathbb{Q}(\zeta_{mp_1^{n_1}\cdots p_s^{n_s}})$, with maximal real subfield $L^+ = \mathbb{Q}(\zeta_{mp_1^{n_1}\cdots p_s^{n_s}})^+$. Then the strict class number of L^+ divides twice the class number of L, as follows. The quadratic extension L/L^+ is ramified at a finite prime (if $m = s = 1$) or is ramified only at infinity (otherwise). Hence, if $H_{L^+}^{\mathrm{st}}$ is the strict Hilbert class field of L^+, then the degree over L of the composite $LH_{L^+}^{\mathrm{st}}$ is either the strict class number of L^+ (if $m = s = 1$) or half that (otherwise). Since $LH_{L^+}^{\mathrm{st}}$ is an abelian extension of L that is unramified at finite primes, it is contained in the Hilbert class field of the complex field L, so $[LH_{L^+}^{\mathrm{st}} : L]$ divides the class number of L, which gives the desired divisibility.

Next observe that the cyclotomic fields $\mathbb{Q}(\zeta_{mp_1^{n_1}\cdots p_s^{n_s}})$ with $n_i \geq 1$ for $1 \leq i \leq s$ are the subfields of the composite of the cyclotomic $\mathbb{Z}_{p_i}$-extensions of $\mathbb{Q}(\zeta_{mp_1\cdots p_s})$ for $1 \leq i \leq s$. Hence the 2-primary part of the class number of $\mathbb{Q}(\zeta_{mp_1^{n_1}\cdots p_s^{n_s}})$ is bounded for all s-tuples $(n_1, \dots, n_s)$ and is constant if $n_1, \dots, n_s$ are all sufficiently large by a theorem of Friedman [1981/82] extending a result of Washington. By the previous observation, the strict class number of $L^+ = \mathbb{Q}(\zeta_{mp_1^{n_1}\cdots p_s^{n_s}})^+$ is therefore bounded for all s-tuples $(n_1, \dots, n_s)$ (and is in fact constant if $n_1, \dots, n_s$ are all sufficiently large).

Finally, since the strict class number of L^+ is the product of the usual class number of L^+ with $2^{\delta(L^+)}$, we obtain (b). Then by (a), we obtain (c). □

Corollary 9. *With notation as in Theorem 8, the unit signature rank deficiencies for all totally real abelian fields F whose conductor is a product of a divisor of m with an integer whose prime divisors are among the set $\{p_1, \dots, p_s\}$, are uniformly bounded.*

Proof. Any totally real abelian field F having conductor $dp_1^{n_1} \cdots p_s^{n_s}$ where d is a divisor of m and with $n_i \geq 0$ for $1 \leq i \leq s$ is contained in $\mathbb{Q}(\zeta_{mp_1^{n_1}\cdots p_s^{n_s}})^+$. Hence, $\delta(F) \leq \delta(m; n_1, \dots, n_s)$ by Remark 1, and the result follows immediately from Theorem 8. □

Remark 10. Corollary 9 shows that among all the abelian fields whose conductors are supported in a fixed finite set of primes, almost all have nearly maximal unit signature rank (in the precise sense that the signature rank deficiencies are uniformly bounded by a constant depending only on the set of primes chosen).

We highlight some particular special cases:

Corollary 11. *Let k be a finite totally real abelian extension of $\mathbb{Q}$. For p an odd prime, let $k^{p,\infty}$ denote the cyclotomic $\mathbb{Z}_p$-extension of k. If k_n is the subfield of $k^{p,\infty}$ of degree p^n over k, then the signature rank of the units of k_n differs from $[k_n : \mathbb{Q}]$ by a constant amount for k_n sufficiently far up the tower.*

Applying Corollary 11 to $k = \mathbb{Q}(\zeta_p)^+$ for p odd, together with Weber's result in Proposition 3, gives the following.

Corollary 12. *For any prime p, the difference between the signature rank of the units of $\mathbb{Q}(\zeta_{p^n})^+$ and $\varphi(p^n)/2$ is constant for n sufficiently large* (*the constant depending on p*).

Corollary 12 gives another proof of the results in Section 3 that the signature ranks of the units in the fields $\mathbb{Q}(\zeta_{p^n})^+$ tend to infinity as n tends to infinity. Corollary 12 is far superior, asymptotically, to the results in Section 3 for the cyclotomic fields of odd prime power conductor since it shows the signature rank is "nearly" maximal. The result yields relatively little information for any specific $\mathbb{Q}(\zeta_{p^N})^+$, however, since the unit signature rank deficiency for this field could conceivably be close to $\varphi(p^N)/2$ (although, as mentioned, this is not expected to happen). The only explicit lower bounds for the unit signature rank for $\mathbb{Q}(\zeta_{p^n})$ for odd p and, more significantly, for general $\mathbb{Q}(\zeta_m)$ (for example if m is the product of distinct primes, for which the results in this section have little to say), are those in Section 3.

Remark 13. We have done some computations of the signature ranks for the subgroup of circular units in towers of prime power cyclotomic fields. While the computations are somewhat modest (since $\varphi(p^n)$ grows rapidly with n), these computations have exhibited the following behavior: if the signature rank of the circular units in $\mathbb{Q}(\zeta_p)^+$ is $\frac{1}{2}\varphi(p)-\delta$, $(\delta \geq 0)$, then the signature rank of the circular units in the fields $\mathbb{Q}(\zeta_{p^n})^+$ is $\frac{1}{2}\varphi(p^n)-\delta$, i.e., the circular unit signature rank deficiency is constant and equal to its value in the first layer. Whether this behavior persists in general is an extremely interesting question.

We also note that the deficiency of the circular units is at least the deficiency for the full group of units, but may be strictly larger: for the field $\mathbb{Q}(\zeta_{163})^+$ the circular unit deficiency is 2, while the deficiency for the full group of units is 0 (see [Dummit 2018]).

7. Unit signature rank deficiencies in cyclotomic fields

In this section we show that the signature rank deficiency in the maximal real subfields of cyclotomic fields can be arbitrarily large, under the assumption that there exist infinitely many cyclic cubic extensions having a system of totally positive fundamental units.

Suppose k is a cyclic cubic extension of $\mathbb{Q}$ with totally positive fundamental units $\varepsilon_1, \varepsilon_2$. If E_k is the unit group of k, then $E_k = \{\pm 1\} \times \langle \varepsilon_1, \varepsilon_2 \rangle$ and the subgroup $\langle \varepsilon_1, \varepsilon_2 \rangle$ consists of the totally positive units in k.

If the Galois group G of k is generated by σ, then $\langle \varepsilon_1, \varepsilon_2 \rangle$ is a module for the quotient $\mathbb{Z}[G]/(\sigma^2+\sigma+1)$ of the group ring $\mathbb{Z}[G]$ of G since ε_1 and ε_2 both have norm $+1$. This quotient of the group ring is isomorphic to the ring of integers in $\mathbb{Q}(\sqrt{-3})$, which is a principal ideal domain, and it follows that $\langle \varepsilon_1, \varepsilon_2 \rangle \simeq \mathbb{Z}[G]/(\sigma^2+\sigma+1)$ as G-modules (and not just as abelian groups).

Modulo squares, $\langle \varepsilon_1, \varepsilon_2 \rangle$ is therefore isomorphic to $\mathbb{F}_2[G]/(\sigma^2+\sigma+1)$ as a module over the group ring $\mathbb{F}_2[G]$, and hence affords the unique irreducible 2-dimensional representation of $\mathbb{F}_2[G]$. In particular, G acts irreducibly and with no nontrivial fixed points on $\langle \varepsilon_1, \varepsilon_2 \rangle$ modulo squares.

With these preliminaries, we consider the composite of cyclic cubic fields having a system of totally positive fundamental units:

Proposition 14. *Suppose $k_1, \dots, k_n$ are linearly disjoint cyclic cubic fields, each with a totally positive system of fundamental units, i.e., with unit signature rank deficiency $\delta(k_i)$ equal to 2, $i = 1, \dots, n$. Then the unit signature rank deficiency $\delta(k_1 \cdots k_n)$ for the composite field $k_1 \cdots k_n$ is at least $2n$.*

Proof. We need to prove there are at least $2n$ totally positive units in $k_1 \cdots k_n$ that are multiplicatively independent modulo squares (in $k_1 \cdots k_n$). We proceed by induction on n, the case $n = 1$ being trivial. Suppose by induction that the composite $k_1 \cdots k_s$ contains $2s$ totally positive units $\varepsilon'_1, \dots, \varepsilon'_{2s}$ that are multiplicatively independent modulo squares in $k_1 \cdots k_s$. By assumption, the field k_{s+1} contains two totally positive units $\varepsilon_1, \varepsilon_2$ that are multiplicatively independent modulo squares in k_{s+1}.

Suppose $\varepsilon \in \langle \varepsilon_1, \varepsilon_2 \rangle$ and $\varepsilon' \in \langle \varepsilon'_1, \dots, \varepsilon'_{2s} \rangle$ with $\varepsilon\varepsilon' = \alpha^2$ for some α in the composite field $F = k_1 \cdots k_s k_{s+1}$.

Let $\sigma \in \mathrm{Gal}(F/\mathbb{Q})$ be a lift of a generator for the cyclic group $\mathrm{Gal}(k_{s+1}/\mathbb{Q})$ that is the identity on $k_1 \cdots k_s$. Then $\sigma(\varepsilon)\varepsilon' = \sigma(\alpha)^2$, so $\sigma(\varepsilon)/\varepsilon = (\sigma(\alpha)/\alpha)^2$ is a square in F, hence $k_{s+1}(\sqrt{\sigma(\varepsilon)/\varepsilon})$ is a subfield of F. Since F has degree 3^s over k_{s+1}, $k_{s+1}(\sqrt{\sigma(\varepsilon)/\varepsilon})$ cannot be a quadratic extension, so $\sigma(\varepsilon)/\varepsilon$ is in fact a square in k_{s+1}. Since σ acting on $\langle \varepsilon_1, \varepsilon_2 \rangle$ modulo squares in k_{s+1} has no nontrivial fixed point, it follows that ε is the square of a unit in k_{s+1}. Then $\varepsilon' = \alpha^2/\varepsilon$ would be a square in F, a cubic extension of $k_1 \cdots k_s$, and, as before, this implies that ε' would be a square in $k_1 \cdots k_s$.

This shows that the totally positive units $\varepsilon_1, \varepsilon_2, \varepsilon'_1, \dots, \varepsilon'_{2s}$ are multiplicatively independent modulo squares in F, completing the proof by induction. □

For a cyclic cubic field, either there is a totally positive system of fundamental units or the units have all possible signatures. Heuristics, supported by computations, in [Breen et al. ≥ 2019] suggest that, when counted by discriminant, there is a positive proportion of cyclic cubic fields of either type. Roughly 3% of cyclic cubic fields (see [Breen et al. ≥ 2019] for the precise value) are predicted to have unit signature rank deficiency 2, so in particular there should exist infinitely many such cubic fields that are linearly disjoint.

Theorem 15. *Suppose, as expected, that there exist infinitely many cyclic cubic fields having a totally positive system of fundamental units. Then the difference between $\varphi(m)/2$ and the unit signature rank of $\mathbb{Q}(\zeta_m)^+$ can be arbitrarily large.*

Proof. By Proposition 14 and Remark 1, to obtain a unit signature rank deficiency at least $2n$ it suffices to take the cyclotomic field whose conductor is the product of the distinct primes (which are congruent to 1 mod 3) dividing the conductors of n linearly disjoint cyclic cubic fields having totally positive fundamental units. □

Remark 16. The same sort of arguments could be applied to composites of other abelian fields of, for example, odd prime degree. As in the case for cubic fields, it is expected that there are infinitely many such cyclic extensions of $\mathbb{Q}$ with nonzero deficiencies (see [Breen et al. $\geq$ 2019] for specific predictions). This suggests that the unit signature rank deficiency can increase without bound as one moves "horizontally" among cyclotomic fields, that is, over fields $\mathbb{Q}(\zeta_m)^+$ where m is the product of an increasing number of distinct primes, as opposed to the results of Section 6 which show the deficiency is bounded as one moves "vertically" among cyclotomic fields.

8. Remarks on 2-adic unit signature rank deficiencies

The results above have implications for analogous deficiencies for the "2-adic signatures" of units in the sense of [Dummit and Voight 2018, Section 4], which we now very briefly outline. We use the notation of [Dummit and Voight 2018].

If F is a totally real field with $[F : \mathbb{Q}] = n$ then there is a structure theorem for the image of the 2-Selmer group, $\mathrm{Sel}_2(F)$, under the 2-Selmer signature map φ [Dummit and Voight 2018, Theorem A.13]. The space $\varphi(\mathrm{Sel}_2(F))$ is n-dimensional over $\mathbb{F}_2$ and is an orthogonal direct sum $U \perp S \perp U'$, where U is the subspace of elements whose 2-adic signature is trivial, U' is the subspace of elements whose archimedean signature is trivial, and S is a diagonal subspace. Since F is totally real, the dimension of U' is the same as the dimension of U [Dummit and Voight 2018, Theorem A.13(a)] and so S has dimension $n - 2\dim(U)$.

Suppose now that the unit signature rank deficiency of F is $\delta(F)$. Then the set of signatures of units is a subspace of dimension $n - \delta(F)$. It follows that the dimension of U is at most $\delta(F)$ (so S has dimension at least $n - 2\delta(F)$) and that the dimension of the image $\varphi(E_F)$ of the units E_F of F is at least $n - \delta(F)$. Then $\dim(\varphi(E_F) + S)$ is at most $\dim \varphi(\mathrm{Sel}_2(F)) = n$ and

$$\begin{aligned}\dim(\varphi(E_F) \cap S) &= \dim \varphi(E_F) + \dim S - \dim(\varphi(E_F) + S)\\ &\geq (n - \delta(F)) + (n - 2\delta(F)) - n = n - 3\delta(F).\end{aligned}$$

Since S is a diagonal subspace, it follows that the dimension of the subspace of 2-adic signatures of the units of F is at least $n - 3\delta(F)$. Hence the 2-adic signature deficiency of the units of F is at most $3\delta(F)$.

As a consequence, Theorem 8(b) and Corollaries 9, 11, and 12 remain true if the (archimedean) signature rank of the units is replaced by the 2-adic signature rank of the units.

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Received May 31, 2018. Revised August 20, 2018.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 298, No. 2, 2019
dx.doi.org/10.2140/pjm.2019.298.299

SEMISTABLE DEFORMATION RINGS IN EVEN HODGE–TATE WEIGHTS

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Let p be a prime number and r a positive even integer less than $p-1$. In this paper, we find a Galois stable lattice in each two-dimensional semistable noncrystalline representation of $G_{\mathbb{Q}_p}$ with Hodge–Tate weights $(0, r)$ by constructing the corresponding strongly divisible module. We also compute the Breuil modules corresponding to the mod p reductions of these strongly divisible modules, and determine the semisimplification of the mod p reduction of the original representations. We use these results to construct the irreducible components of the semistable deformation rings in Hodge–Tate weights $(0, r)$ of the absolutely irreducible residual representations of $G_{\mathbb{Q}_p}$.

MSC2010: 11F80.
Keywords: semistable representations, strongly divisible modules, Breuil modules, semistable deformation rings.

1. Introduction

Let p be a prime number and r a positive even integer less than $p-1$. Our goal in this paper is to construct the irreducible components of deformation spaces whose characteristic 0 closed points give rise to the semistable lifts with Hodge–Tate weights $(0, r)$ of a fixed irreducible representation $\bar{\rho} : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$. The existence of these deformation rings was proved by Kisin [2008]. The geometric structure of these local deformation spaces is described by a conjecture of Breuil–Mézard [2002] as well as a refinement due to Emerton–Gee [2014]. Thanks to the work of Kisin [2009], this conjecture is known for GL_2 over $\mathbb{Q}_p$. Gee and Kisin [2014] have recently proved the Breuil–Mézard conjecture for 2-dimensional potentially Barsotti–Tate representations of G_K, where K is a finite extension of $\mathbb{Q}_p$.

The Breuil–Mézard conjecture, however, does not predict the whole structure of deformation rings. Moreover, for a given (potentially) semistable representation, the conjecture rarely gives information about what the mod p reduction of the original representation is. It measures how far the special fibers of deformation rings is to be smooth. Breuil and Mézard [2002] describe the irreducible components of the semistable deformation rings in Hodge–Tate weights $(0, r)$ for r a positive odd integer less than $p-1$. In this paper, the case that r is a positive even integer less than $p-1$ is treated, following the basic strategy in [Breuil and Mézard 2002; Savitt 2005]. We construct strongly divisible modules to find Galois stable lattices, and compute Breuil modules to determine mod p reductions.

Constructing strongly divisible modules is often very difficult. We find that a phenomenon, which didn't occur in the case r is odd, occurs in the case r is even. More precisely, if we write D for the admissible filtered (ϕ, N)-modules, that exhaust all the 2-dimensional semistable noncrystalline representations of $G_{\mathbb{Q}_p}$ with Hodge–Tate weights $(0, r)$ (see Example 3.1.1), then D is parameterized by λ and $\mathfrak{L}$, where λ is a Frobenius eigenvalue of D and $\mathfrak{L}$ is the $\mathfrak{L}$-invariant in the filtration of D. By (weakly) admissible condition, the valuation of $\lambda \in \overline{\mathbb{Q}}_p$ should be $\frac{1}{2}(r-1)$, while $\mathfrak{L}$ varies over the whole $\overline{\mathbb{Q}}_p$-line. The coefficients of the strongly divisible modules in [Breuil and Mézard 2002] are rational functions of λ and $\mathfrak{L}$, while some of those in this paper are limits of sequences of rational functions of λ and $\mathfrak{L}$. These limits satisfy polynomial equations of degree 2. A similar phenomenon was observed in an earlier work of the second author [Park 2017]. In fact, the situation in this paper is more complicated than the one in [Park 2017], since in [Park 2017] the author only considers Hodge–Tate weights (0, 1, 2) while in this paper we consider Hodge–Tate weights $(0, r)$ for positive even integers r less than $p-1$, so that we should construct those sequences for each such r (which is more or less equivalent to constructing those polynomial equations of degree 2; see Remark 4.2.4).

Once we finish constructing strongly divisible modules, we compute the Breuil modules corresponding to the mod p reductions of the strongly divisible modules. By using a result of Caruso [2006], we compute the semisimplification of the mod p reduction corresponding to the Breuil modules. In particular, we determine which semistable representations have absolutely irreducible mod p reductions.

The following is our main results on mod p reduction:

Theorem 5.0.5. *Let $r = 2m$ be a positive even integer less than $p-1$ and write $a(r)$ for a rational number depending on r, which is explicitly defined in* (4.2.1).

The mod p reduction of $\rho := \mathrm{V}^r_{\mathrm{st}}(D)$ is absolutely irreducible if and only if one of the following cases holds:

(1) $-\frac{1}{2}-\ell < v_p(\mathfrak{L}-a(r)) < \frac{1}{2}-\ell$ *for* $\ell \in \{0, 1, 2, \ldots, m-2\}$, *in which case*
$$\bar{\rho}|_{I_{\mathbb{Q}_p}} \cong \omega_2^{m-\ell-1+p(m+\ell+1)} \oplus \omega_2^{m+\ell+1+p(m-\ell-1)};$$

(2) $v_p(\mathfrak{L}-a(r)) < \frac{3}{2}-m$, *in which case*
$$\bar{\rho}|_{I_{\mathbb{Q}_p}} \cong \omega_2^{r} \oplus \omega_2^{pr}.$$

Moreover,
$$\bar{\rho}^{ss}|_{I_{\mathbb{Q}_p}} \cong \begin{cases} \omega^{m-\ell-1} \oplus \omega^{m+\ell+1} & \text{if } v_p(\mathfrak{L}-a(r)) = -\frac{1}{2}-\ell \text{ for } \ell \in \{0, 1, \ldots, m-2\}; \\ \omega^{m} \oplus \omega^{m} & \text{if } v_p(\mathfrak{L}-a(r)) \geq \frac{1}{2}. \end{cases}$$

We use our construction of Galois stable lattices and their mod p reductions to construct the semistable deformation spaces in Hodge–Tate weights $(0, r)$ for absolutely irreducible residual representations $\bar{\rho}$ by making use of the parameterization of our families of strongly divisible modules. From our computations of Breuil modules, we can readily figure out which semistable representations have the same absolutely irreducible mod p reductions. Once we fix an absolutely irreducible residual representation of $G_{\mathbb{Q}_p}$, we parameterize the lifts of the fixed residual representation to determine the irreducible components of the semistable deformation rings. We state our main results on semistable deformation rings.

Theorem 9.0.1. *Let $r = 2m$ be a positive even integer less than $p-1$ and $\mathfrak{R}^{(0,r)}_{\bar{\rho}_0}$ the semistable deformation ring in Hodge–Tate weights $(0, r)$ of the 2-dimensional absolutely irreducible residual representation $\bar{\rho}_0$.*

(1) *If $\bar{\rho}_0|_{I_{\mathbb{Q}_p}} \cong \omega_2^{m-\ell-1+p(m+\ell+1)} \oplus \omega_2^{m+\ell+1+p(m-\ell-1)}$ for $\ell \in \{0, 1, 2, \dots, m-2\}$, then*
$$\mathfrak{R}^{(0,r)}_{\bar{\rho}_0} \sim \frac{\mathbb{O}[\![D, X, Y]\!]}{(XY-p)} \times \frac{\mathbb{O}[\![D, X, Y]\!]}{(XY-p)};$$

(2) *If $\bar{\rho}_0|_{I_{\mathbb{Q}_p}} \cong \omega_2^{r} \oplus \omega_2^{pr}$, then*
$$\mathfrak{R}^{(0,r)}_{\bar{\rho}_0} \sim' \mathbb{O}[\![D_1, D_2]\!] \times \mathbb{O}[\![D, X]\!] \times \mathbb{O}[\![D, X]\!].$$

Let $\mathfrak{R}, \mathfrak{R}_1, \dots, \mathfrak{R}_n$ be complete Noetherian local rings. By $\mathfrak{R} \sim' \prod_{i=1}^{n} \mathfrak{R}_i$ we mean that $\mathfrak{R}_i$'s are the irreducible components of $\mathfrak{R}$, and for $\mathfrak{R} \sim \prod_{i=1}^{n} \mathfrak{R}_i$ we follow the notation in [Breuil and Mézard 2002] (see (9.0.2)). Note that the case (2) in the preceding theorem gives rise to examples of deformation rings whose generic fibers are not formally smooth, while the ones in case (1) have smooth generic fibers (see Remark 9.3.15). It is known that the generic fibers of potentially crystalline deformation rings are formally smooth. We also note that the deformation rings in case (1) parameterize only semistable noncrystalline representations, while the residual representations in case (2) have crystalline lifts with Hodge–Tate weights $(0, r)$.

This paper is organized as follows. In Section 2, we quickly review necessary integral p-adic Hodge theory, and introduce the notation that will be used throughout the paper. In Section 3, we recall the family of (weakly) admissible filtered (ϕ, N)-modules of rank 2 that parameterize all the 2-dimensional semistable noncrystalline representations of $G_{\mathbb{Q}_p}$, and we also recall the corresponding S-modules. Then we introduce examples of Breuil modules of rank 2, which will appear as mod p reductions of the semistable representations of $G_{\mathbb{Q}_p}$. In Section 4, we construct certain elements in $S_{\mathbb{O}}$, denoted by $\delta_\bullet$ for $\bullet \in \{\infty, 0, 1, \dots, m-2, -\infty\}$, which will appear in the coefficients of our strongly divisible modules. To construct these $\delta_\bullet$, we first study various properties of certain matrices, which are elementary but nontrivial. We give a special attention to δ_0: to define δ_0, we construct a sequence for each r and the limit of the sequence appears in the coefficients of δ_0. In Section 5, we state our main results, Theorem 5.0.1 on Galois stable lattices and Theorem 5.0.5

on their mod p reductions. To prove these theorems, we divide the proofs into three cases, according to the valuation $v_p(\mathfrak{L}-a(r))$, as follows:

- $\frac{1}{2} \le v_p(\mathfrak{L}-a(r))$;
- $-\frac{1}{2}-\ell \le v_p(\mathfrak{L}-a(r)) < \frac{1}{2}-\ell$ for $\ell \in \{0, 1, 2, \dots, m-2\}$;
- $v_p(\mathfrak{L}-a(r)) < \frac{3}{2}-m$.

The proof for each case are in Section 6, Section 7, and Section 8, respectively. In the last section, Section 9, we use our construction of strongly divisible modules and computation of Breuil modules to construct the irreducible components of semistable deformation rings in Hodge–Tate weights $(0, r)$ of the absolutely irreducible residual representations of $G_{\mathbb{Q}_p}$.

2. Review of integral p-adic Hodge theory

In this section, we quickly review filtered (ϕ, N)-modules, strongly divisible modules, and Breuil modules, which correspond to representations of $G_{\mathbb{Q}_p}$. We note that all of the results in this section are already known. We closely follow the exposition of [Park 2017], and we refer the reader to [Breuil and Mézard 2002; Emerton et al. 2013] for more detail.

Let E be a finite extension of $\mathbb{Q}_p$ with the ring of integers $\mathbb{O}$, maximal ideal $\mathfrak{m}$, a uniformizer π, and residue field $\mathbb{F}$.

2.1. ***Filtered (ϕ, N)-modules over E.*** We fix a prime number $p \in \mathbb{Q}_p$, thereby fixing an embedding $\boldsymbol{B}_{\mathrm{st}} \hookrightarrow \boldsymbol{B}_{\mathrm{dR}}$. (See [Fontaine 1994] for detail.) A *filtered (ϕ, N)-module* over E is a finite dimensional E-vector space D together with a triple $(\phi, N, \{\mathrm{Fil}^i D\}_{i\in\mathbb{Z}})$ where

- the *Frobenius map* $\phi : D \to D$ is an E-linear automorphism;
- the *monodromy operator* $N : D \to D$ is a (nilpotent) E-linear endomorphism such that $N\phi = p\phi N$;
- the *Hodge filtration* $\{\mathrm{Fil}^i D\}_{i\in\mathbb{Z}}$ is a decreasing filtration on D such that an E-subspace $\mathrm{Fil}^i D$ is D if $i \ll 0$ and 0 if $i \gg 0$.

A filtered (ϕ, N)-module D is said to be *admissible* if it is in the sense of [Breuil and Mézard 2002]. The *Hodge–Tate weights* of a filtered (ϕ, N)-module D are the integers r such that $\mathrm{Fil}^r D \ne \mathrm{Fil}^{r+1} D$, each counted with multiplicity $\dim_E(\mathrm{Fil}^r D/\mathrm{Fil}^{r+1} D)$.

Let V be a finite-dimensional continuous E-representation of $G_{\mathbb{Q}_p}$, and define

$$\mathrm{D}_{\mathrm{st}}(V) := (\boldsymbol{B}_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V)^{G_{\mathbb{Q}_p}}.$$

Then $\dim_E \mathrm{D}_{\mathrm{st}}(V) \le \dim_E V$ in general. If the equality holds, then we say that V is *semistable*, in which case $\mathrm{D}_{\mathrm{st}}(V)$ inherits from $\boldsymbol{B}_{\mathrm{st}}$ the structure of an admissible filtered (ϕ, N)-module. We say that V is *crystalline* if V is semistable and the monodromy operator N on $\mathrm{D}_{\mathrm{st}}(V)$ is 0. Thanks to Colmez and Fontaine [2000], the functor D_{st} provides an equivalence between the category of semistable E-representations of $G_{\mathbb{Q}_p}$ and the category of admissible filtered (ϕ, N)-modules over E.

If V is a finite dimensional continuous E-representations of $G_{\mathbb{Q}_p}$, we let $V^\vee$ be the dual representation of $G_{\mathbb{Q}_p}$. It is known that V is semistable (resp. crystalline) if and only if so is $V^\vee$. If we denote $\mathrm{D}^*_{\mathrm{st}}(V) := \mathrm{D}_{\mathrm{st}}(V^\vee)$, then the functor $\mathrm{D}^*_{\mathrm{st}}$ gives rise to an antiequivalence between the category of semistable E-representations of $G_{\mathbb{Q}_p}$ and the category of admissible filtered (ϕ, N)-modules over E, whose quasi-inverse is given by

$$\mathrm{V}^*_{\mathrm{st}}(D) := \mathrm{Hom}_{\phi, N, \mathrm{Fil}^*}(D, \boldsymbol{B}_{\mathrm{st}}).$$

If V is semistable, then when we refer to the Hodge–Tate weights of V, we mean those of $\mathrm{D}^*_{\mathrm{st}}(V)$. Our normalization implies that the cyclotomic character $\varepsilon : G_{\mathbb{Q}_p} \to E^\times$ has Hodge–Tate weight 1. Twisting V by a power ε^n of the cyclotomic character has the effect of shifting all the Hodge–Tate weights of V by n, so that we are therefore free to assume that the lowest Hodge–Tate weight is 0 after a suitable twist.

2.2. ***Strongly divisible modules.*** We fix the uniformizer p in $\mathbb{Q}_p$, and let $E(u) := u - p \in \mathbb{Z}_p[u]$. We also let S be the p-adic completion of $\mathbb{Z}_p[u^i/i!]_{i\in\mathbb{N}}$. We endow S with the following structure:

- a continuous Frobenius-semilinear map $\phi : S \to S$ with $\phi(u) = u^p$;
- a continuous $\mathbb{Z}_p$-linear derivation $N : S \to S$ with $N(u^i/i!) = -iu^i/i!$;
- a decreasing filtration $\{\mathrm{Fil}^i S\}_{i\in\mathbb{Z}_{\geq 0}}$ where $\mathrm{Fil}^i S$ is the p-adic completion of $\sum_{j\geq i} E(u)^j/j!\, S$.

Note that $N\phi = p\phi N$ and $\phi(\mathrm{Fil}^i S) \subset p^i S$ for $0 \leq i \leq p-1$.

We also let $S_{\mathbb{O}} := S \otimes_{\mathbb{Z}_p} \mathbb{O}$ and $S_E := S_{\mathbb{O}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, and extend the definitions of Fil, ϕ, and N to $S_{\mathbb{O}}$ and S_E $\mathbb{O}$-linearly and E-linearly, respectively. Let $\mathcal{MF}(\phi, N)$ be the category whose objects are finite free S_E-modules $\mathcal{D}$ with

- a ϕ-semilinear and E-linear morphism $\phi : \mathcal{D} \to \mathcal{D}$ such that the determinant of ϕ with respect to some choice of $S_{\mathbb{Q}_p}$-basis is invertible in $S_{\mathbb{Q}_p}$ (which does not depend on the choice of basis);
- a decreasing filtration of $\mathcal{D}$ by S_E-submodules $\mathrm{Fil}^i \mathcal{D}$, $i \in \mathbb{Z}$, with $\mathrm{Fil}^i \mathcal{D} = \mathcal{D}$ for $i \leq 0$ and $\mathrm{Fil}^i S_E \cdot \mathrm{Fil}^j \mathcal{D} \subset \mathrm{Fil}^{i+j} \mathcal{D}$ for all j and all $i \geq 0$;
- a E-linear map $N : \mathcal{D} \to \mathcal{D}$ such that
  - $N(sx) = N(s)x + sN(x)$ for all $s \in S_E$ and $x \in \mathcal{D}$,

- $N\phi = p\phi N$,
- $N(\mathrm{Fil}^i \mathscr{D}) \subset \mathrm{Fil}^{i-1} \mathscr{D}$ for all i.

For a filtered (ϕ, N)-module D with $\mathrm{Fil}^0 D = D$, one can associate an object $\mathscr{D} \in \mathcal{MF}(\phi, N)$ by the following:

- $\mathscr{D} := S \otimes_{W(k)} D$;
- $\phi := \phi \otimes \phi : \mathscr{D} \to \mathscr{D}$;
- $N := N \otimes \mathrm{Id} + \mathrm{Id} \otimes N : \mathscr{D} \to \mathscr{D}$;
- $\mathrm{Fil}^0 \mathscr{D} := \mathscr{D}$ and, by induction,
$$\mathrm{Fil}^{i+1} \mathscr{D} := \{x \in \mathscr{D} \mid N(x) \in \mathrm{Fil}^i \mathscr{D} \text{ and } f_p(x) \in \mathrm{Fil}^{i+1} D\},$$
where $f_p : \mathscr{D} \to D$ is defined by $s(u) \otimes x \mapsto s(p)x$.

By a result of Breuil [1997], the functor $\mathscr{D} : D \mapsto S \otimes_{W(k)} D$ gives rise to an equivalence between the category of filtered (ϕ, N)-modules with $\mathrm{Fil}^0 D = D$ and the category $\mathcal{MF}(\phi, N)$.

Fix a positive integer $r \le p-2$. The category $\mathfrak{M}\mathfrak{D}^r_{\mathcal{O}}$ of *strongly divisible modules of weight r* is defined to be the category of free $S_{\mathcal{O}}$-modules $\mathfrak{M}$ of finite rank with an $S_{\mathcal{O}}$-submodule $\mathrm{Fil}^r \mathfrak{M}$ and additive maps $\phi, N : \mathfrak{M} \to \mathfrak{M}$ such that the following properties hold:

- $\mathrm{Fil}^r S_{\mathcal{O}} \cdot \mathfrak{M} \subset \mathrm{Fil}^r \mathfrak{M}$;
- $\mathrm{Fil}\, r\mathfrak{M} \cap I\mathfrak{M} = I \,\mathrm{Fil}^r \mathfrak{M}$ for all ideals I in $\mathcal{O}$;
- $\phi(sx) = \phi(s)\phi(x)$ for all $s \in S_{\mathcal{O}}$ and for all $x \in \mathfrak{M}$;
- $\phi(\mathrm{Fil}^r \mathfrak{M})$ is contained in $p^r \mathfrak{M}$ and generates it over $S_{\mathcal{O}}$;
- $N(sx) = N(s)x + sN(x)$ for all $s \in S_{\mathcal{O}}$ and for all $x \in \mathfrak{M}$;
- $N\phi = p\phi N$;
- $E(u)N(\mathrm{Fil}^r \mathfrak{M}) \subset \mathrm{Fil}\, r\mathfrak{M}$.

For a strongly divisible module $\mathfrak{M}$ of weight r, there exists a unique admissible filtered (ϕ, N)-module D with Hodge–Tate weights lying in $[0, r]$ such that $\mathfrak{M}[1/p] \simeq S \otimes_{W(k)} D$ described as follows. We define a free S_E-module $\mathscr{D} := \mathfrak{M} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, and extend ϕ and N on $\mathscr{D}$. We also define a filtration on $\mathscr{D}$ as follows: $\mathrm{Fil}^r \mathscr{D} = \mathrm{Fil}^r \mathfrak{M}[1/p]$ and

$$\mathrm{Fil}^i \mathscr{D} := \begin{cases} \mathscr{D} & \text{if } i \le 0; \\ \{x \in \mathscr{D} \mid E(u)^{r-i} x \in \mathrm{Fil}^r \mathscr{D}\} & \text{if } 0 \le i \le r; \\ \sum_{j=0}^{i-1} (\mathrm{Fil}^{i-j} S_{\mathbb{Q}_p})(\mathrm{Fil}^j \mathscr{D}) & \text{if } i > r, \text{ inductively.} \end{cases} \tag{2.2.1}$$

Let $s_0 : S_{\mathbb{Q}_p} \to \mathbb{Q}_p$ and $s_p : S_{\mathbb{Q}_p} \to \mathbb{Q}_p$ be defined by $u \mapsto 0$ and $u \mapsto p$ respectively, and let $D := \mathscr{D} \otimes_{S_{\mathbb{Q}_p}, s_0} \mathbb{Q}_p$. The map s_0 induces ϕ and N on D, and the map s_p

induces the filtration on D by taking $s_p(\mathrm{Fil}^i \mathcal{D})$. Then it turns out that D is a weakly admissible filtered (ϕ, N)-module over E with $\mathrm{Fil}^{r+1} D = 0$.

Hence, one has the following equivalent definition: let D be an admissible filtered (ϕ, N)-module such that $\mathrm{Fil}^0 D = D$ and $\mathrm{Fil}^{r+1} D = 0$. *A strongly divisible module* in $\mathcal{D} := \mathcal{D}(D)$ is an $S_{\mathbb{O}}$-submodule $\mathfrak{M}$ of $\mathcal{D}$ such that

- $\mathfrak{M}$ is a free $S_{\mathbb{O}}$-module of finite rank such that $\mathfrak{M}[1/p] \simeq \mathcal{D}$;
- $\mathfrak{M}$ is stable under ϕ and N;
- $\phi(\mathrm{Fil}^r \mathfrak{M}) \subset p^r \mathfrak{M}$ where $\mathrm{Fil}^r \mathfrak{M} := \mathfrak{M} \cap \mathrm{Fil}^r \mathcal{D}$.

For a strongly divisible module $\mathfrak{M}$, we define an $\mathbb{O}[G_{\mathbb{Q}_p}]$-module $\mathrm{T}^*_{\mathrm{st}}(\mathfrak{M})$ as follows:

$$\mathrm{T}^*_{\mathrm{st}}(\mathfrak{M}) := \mathrm{Hom}_{S, \mathrm{Fil}^r, \phi, N}(\mathfrak{M}, \widehat{A}_{\mathrm{st}}).$$

(See [Breuil 1999] for detail.) The functor $\mathrm{T}^*_{\mathrm{st}}$ provides an antiequivalence of categories between the category $\mathfrak{MD}^r_{\mathbb{O}}$ of strongly divisible modules of weight r and the category of $G_{\mathbb{Q}_p}$-stable $\mathbb{O}$-lattices in semistable E-representations of $G_{\mathbb{Q}_p}$ with Hodge–Tate weights lying in $[0, r]$, provided that $0 \le r \le p-2$. Moreover, there is a compatibility: if $\mathfrak{M}$ is a strongly divisible module in $\mathcal{D} := \mathcal{D}(D)$ for an admissible filtered (ϕ, N)-module D, then $\mathrm{T}^*_{\mathrm{st}}(\mathfrak{M})$ is a Galois stable $\mathbb{O}$-lattice in $\mathrm{V}^*_{\mathrm{st}}(D)$. This was conjectured by Breuil and proved by Liu [2008] (for the group G_K for a finite extension K of $\mathbb{Q}_p$).

2.3. *Breuil modules.* We write $\bar{S}_{\mathbb{F}}$ for $S_{\mathbb{O}}/(\pi, \mathrm{Fil}^p S) = \mathbb{F}[u]/u^p$. The category $\mathrm{BrMod}^r_{\mathbb{F}}$ of *Breuil modules of weight r* consists of quadruples $(\mathcal{M}, \mathrm{Fil}^r \mathcal{M}, \phi_r, N)$ where

- $\mathcal{M}$ is a finitely generated $\bar{S}_{\mathbb{F}}$-module, free over $\bar{S}_{\mathbb{F}}$;
- $\mathrm{Fil}^r \mathcal{M}$ is a $\bar{S}_{\mathbb{F}}$-submodule of $\mathcal{M}$ containing $u^r \mathcal{M}$;
- $\phi_r : \mathrm{Fil}^r \mathcal{M} \to \mathcal{M}$ is $\mathbb{F}$-linear and ϕ-semilinear (where $\phi : \mathbb{F}_p[u]/u^p \to \mathbb{F}_p[u]/u^p$ is the p-th power map) with image generating $\mathcal{M}$ as a $\bar{S}_{\mathbb{F}}$-module;
- $N : \mathcal{M} \to \mathcal{M}$ is $\mathbb{F}$-linear and satisfies
  - $N(ux) = uN(x) - ux$ for all $x \in \mathcal{M}$,
  - $u^e N(\mathrm{Fil}^r \mathcal{M}) \subset \mathrm{Fil}^r \mathcal{M}$, and
  - $\phi_r(u^e N(x)) = N(\phi_r(x))$ for all $x \in \mathrm{Fil}^r \mathcal{M}$.

The morphisms are $\bar{S}_{\mathbb{F}}$-module homomorphisms that preserve $\mathrm{Fil}^r \mathcal{M}$ and commute with ϕ_r and N.

If $\mathfrak{M}$ is an object of $\mathfrak{MD}^r_{\mathbb{O}}$, then $\mathcal{M} := \mathfrak{M}/(\pi, \mathrm{Fil}^p S)\mathfrak{M}$ is naturally an object of $\mathrm{BrMod}^r_{\mathbb{F}}$ where

- $\mathrm{Fil}^r \mathcal{M}$ is the image of $\mathrm{Fil}^r \mathfrak{M}$ in $\mathcal{M}$;
- the map ϕ_r is induced by $1/p^r \, \phi|_{\mathrm{Fil}^r \mathfrak{M}}$;

- N is induced by the one on $\mathfrak{M}$.

Note that this association gives rise to a functor from the category $\mathfrak{MD}^r_{\mathbb{O}}$ to the category $\mathrm{BrMod}^r_{\mathbb{F}}$.

For a Breuil module $\mathcal{M}$, we define a $\mathbb{F}[G_{\mathbb{Q}_p}]$-module $\mathrm{T}^*_{\mathrm{st}}(\mathcal{M})$ as follows:

$$\mathrm{T}^*_{\mathrm{st}}(\mathcal{M}) := \mathrm{Hom}_{\mathbb{F}_p[u]/u^p,\mathrm{Fil}^r,\phi_r,N}(\mathcal{M}, \widehat{\boldsymbol{A}}).$$

(See [Emerton et al. 2013] for detail.) Here, $\mathrm{T}^*_{\mathrm{st}}$ gives rise to a fully faithful contravariant functor from the category $\mathrm{BrMod}^r_{\mathbb{F}}$ to the category of finite-dimensional $\mathbb{F}$-representations of $G_{\mathbb{Q}_p}$ with $\dim_{\mathbb{F}} \mathrm{T}^*_{\mathrm{st}}(\mathcal{M}) = \mathrm{rank}_{\bar{S}_{\mathbb{F}}} \mathcal{M}$. There is a compatibility, that is, if $\mathfrak{M}$ is a strongly divisible module of weight r and $\mathcal{M} := \mathfrak{M}/(\pi, \mathrm{Fil}^p\, S)\mathfrak{M}$ denotes the Breuil module corresponding to the mod p reduction of $\mathfrak{M}$, then $\mathrm{T}^*_{\mathrm{st}}(\mathfrak{M}) \otimes_{\mathbb{O}} \mathbb{F} \cong \mathrm{T}^*_{\mathrm{st}}(\mathcal{M})$.

2.4. ***Notation.*** We let

$$\gamma := \frac{(u-p)^p}{p} \in S \quad \text{and} \quad c := \frac{1}{p}\phi(E(u)) \in S^\times.$$

It is easy to check that $c \equiv \gamma - 1$ modulo pS and that $\phi(\gamma) \in p^{p-1}S$. We often write v for $E(u) = u - p$ to lighten the notation.

It will often be convenient to use covariant functors. We define covariant functors $\mathrm{T}^r_{\mathrm{st}}(\bullet)$ and $\mathrm{V}^r_{\mathrm{st}}(\bullet)$ as follows:

$$\mathrm{T}^r_{\mathrm{st}}(\bullet) := \mathrm{T}^*_{\mathrm{st}}(\bullet)^\vee \otimes \varepsilon^r \quad \text{and} \quad \mathrm{V}^r_{\mathrm{st}}(\bullet) := \mathrm{V}^*_{\mathrm{st}}(\bullet)^\vee \otimes \varepsilon^r,$$

where ε is the cyclotomic character.

We write $\mathrm{M}_{m\times n}(R)$ for the group of $m \times n$-matrices over a ring R. Let I_n be the $n\times n$ identity matrix, and J_n the antidiagonal $n\times n$ matrix with 1 in the antidiagonal entries. We also write $0_{m\times n}$ for the trivial matrix of size $m \times n$. By $\mathrm{col}_i(A)$, we mean the i-th column of a matrix A. Similarly, by $\mathrm{row}_i(A)$, we mean the i-th row of a matrix A.

If we let R be a commutative ring with unity, then by $M = R(e_1, \dots, e_n)$ we mean that M is a free module over R of rank n with a basis $\underline{e} := (e_1, \dots, e_n)$. For an R-module homomorphism $f : M \to M$ we define an $n \times n$-matrix $\mathrm{Mat}_{\underline{e}}(f)$ by the following equation:

$$(f(e_1), \dots, f(e_n)) = (e_1, \dots, e_n) \cdot \mathrm{Mat}_{\underline{e}}(f).$$

If $x \in M$ is written as $x = a_1e_1 + a_2e_2 + \cdots + a_ne_n$ then we let $[x]_{e_i} = a_i$ for all $1 \le i \le n$.

Let $\mathcal{M}$ be a Breuil module of weight r over $\bar{S}_{\mathbb{F}}$ with a basis $\underline{e} = (e_1, \dots, e_n)$ and let $\underline{f} := (f_1, \dots, f_n)$ be a system of generators for $\mathrm{Fil}^r \mathcal{M}$ modulo $\mathrm{Fil}^r S \cdot \mathcal{M}$. We define an $n \times n$-matrix $\mathrm{Mat}_{\underline{e},\underline{f}}(\mathrm{Fil}^r \mathcal{M})$ by the equation

$$(f_1, \dots, f_n) = (e_1, \dots, e_n) \cdot \mathrm{Mat}_{\underline{e},\underline{f}}(\mathrm{Fil}^r \mathcal{M}).$$

Similarly, for Frobenius morphism $\phi_r : \mathrm{Fil}^r \mathcal{M} \to \mathcal{M}$, we define an $n \times n$-matrix $\mathrm{Mat}_{\underline{e},\underline{f}}(\phi_r)$ by the equation

$$(\phi_r(f_1), \dots, \phi_r(f_n)) = (e_1, \dots, e_n) \cdot \mathrm{Mat}_{\underline{e},\underline{f}}(\phi_r).$$

By v_p we mean a valuation on $\overline{\mathbb{Q}}_p$ normalized as $v_p(p) = 1$. By ω_f we mean Serre's fundamental character of niveau f. If $f = 1$ then we write ω for ω_1. We also note that if $a > b$ then $\sum_{i=a}^{b} f(i) = 0$ by convention.

3. Examples

In this section, we introduce certain examples of various modules, such as filtered (ϕ, N)-modules, filtered (ϕ, N)-modules over S, and Breuil modules. The necessary studies of weakly admissible filtered (ϕ, N)-modules and their corresponding filtered (ϕ, N)-modules over S are already done in [Breuil and Mézard 2002], so we just import their results. The Breuil modules we introduce in this subsection will appear as mod p reductions of the semistable representations of $G_{\mathbb{Q}_p}$ with Hodge–Tate weights $(0, r)$ for r a positive integer less than $p - 1$, as we will see later.

3.1. *Examples of filtered* (ϕ, N)*-modules.* The following examples of filtered (ϕ, N)-modules exhaust all the 2-dimensional semistable noncrystalline representations of $G_{\mathbb{Q}_p}$ with Hodge–Tate weights $(0, r)$ for $r > 0$.

Example 3.1.1. For $\lambda \in E$ with $v_p(\lambda) = \frac{1}{2}(r-1)$ and $\mathfrak{L} \in E$, there exists a basis $\underline{\eta} := (\eta_1, \eta_2)$ satisfying

- $\mathrm{Fil}^i D = \begin{cases} D = E(\eta_1, \eta_2) & \text{if } i \le 0, \\ E(\eta_1 + \mathfrak{L}\eta_2) & \text{if } 0 < i \le r, \\ 0 & \text{if } i > r; \end{cases}$
- $\mathrm{Mat}_{\underline{\eta}}(\phi) = \begin{pmatrix} p\lambda & 0 \\ 0 & \lambda \end{pmatrix};$
- $\mathrm{Mat}_{\underline{\eta}}(N) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$

We write $D(\lambda, \mathfrak{L})$ for the filtered (ϕ, N)-modules in the preceding example.

Lemma 3.1.2. (1) *$D(\lambda, \mathfrak{L})$ are admissible.*

(2) *$D(\lambda, \mathfrak{L})$ is isomorphic to $D(\lambda', \mathfrak{L}')$ if and only if $\lambda = \lambda'$ and $\mathfrak{L} = \mathfrak{L}'$.*

(3) $D(\lambda, \mathfrak{L})$ *exhaust all the* 2*-dimensional semistable noncrystalline representations of* $G_{\mathbb{Q}_p}$ *with Hodge–Tate weights* $(0, r)$ *for* $r > 0$.

(4) $D(\lambda, \mathfrak{L})$ *is not simple if and only if* $r = 1$, *in which case it has a nontrivial proper submodule* $E\eta_2$.

Proof. This is already done in [Breuil and Mézard 2002]. □

3.2. *Examples of filtered* (ϕ, N)*-modules over S.* Let $S_E := S \otimes_{\mathbb{Q}_p} E$, and extend ϕ, N, and $\{\mathrm{Fil}^j S\}_{i\in\mathbb{N}}$ to S_E E-linearly. We also let $D := D(\lambda, \mathfrak{L})$ in Example 3.1.1 and let $\mathfrak{D} = \mathfrak{D}(\lambda, \mathfrak{L}) := S \otimes_{W(k)} D$. Then $\mathfrak{D}$ is a free S_E-module. The induced maps ϕ and N on $\mathfrak{D}$ are obvious by definition.

We compute $\mathrm{Fil}^r \mathfrak{D}$ as follows:

Lemma 3.2.1. (1) $\mathrm{Fil}^0 \mathfrak{D} = \mathfrak{D}$;

(2) *For* $1 \le j \le r$,

$$\mathrm{Fil}^j \mathfrak{D} = S_E\left(\eta_1 + \mathfrak{L}\eta_2 + \sum_{i=1}^{j-1} \frac{(-1)^{i-1}(u-p)^i}{ip^i}\eta_2\right) + \mathrm{Fil}^j S_E \cdot \mathfrak{D};$$

(3) *For* $r < j$,

$$\mathrm{Fil}^j \mathfrak{D} = \mathrm{Fil}^{j-r} S_E\left(\eta_1 + \mathfrak{L}\eta_2 + \sum_{i=1}^{r-1} \frac{(-1)^{i-1}(u-p)^i}{ip^i}\eta_2\right) + \mathrm{Fil}^j S_E \cdot \mathfrak{D}.$$

Proof. This is already done in [Breuil and Mézard 2002]. □

It is easy to see that every element in $\mathrm{Fil}^r \mathfrak{D}$ can be written as

$$\mathfrak{X}(C_0, C_1, \dots, C_{r-1}) := \sum_{i=0}^{r-1} C_i (u-p)^i \left(\eta_1 + \mathfrak{L}\eta_2 + \sum_{j=1}^{r-1-i} \frac{(-1)^{j+1}}{j \cdot p^j}(u-p)^j \eta_2\right)$$

modulo $\mathrm{Fil}^r S_E \cdot \mathfrak{D}$ for some $C_i \in E$, which is also easy to check that this can be rewritten as follows:

$$\mathfrak{X}(C_0, C_1, \dots, C_{r-1}) = \sum_{k=0}^{r-1} (u-p)^k \left(C_k(\eta_1 + \mathfrak{L}\eta_2) + \sum_{j=1}^{k} \frac{(-1)^{j+1} C_{k-j}}{j \cdot p^j}\eta_2\right). \tag{3.2.2}$$

We often write $\mathfrak{X}$ for $\mathfrak{X}(C_0, C_1, \dots, C_{r-1})$ for brevity.

3.3. *Examples of Breuil modules.* In this subsection, we provide some examples of Breuil modules of rank 2 which (as we will see later) occur as mod p reductions of semistable representations of $G_{\mathbb{Q}_p}$ with Hodge–Tate weights $(0, r)$, where r is an even positive integer less then $p - 1$.

The following example parameterizes simple Breuil modules of rank 2:

Example 3.3.1. Let a, b be positive integers with $0 \le b < a \le a+b \le r < p-1$, and $\alpha, \beta \in \mathbb{F}^\times$. The Breuil module $\mathcal{M}(a, b : \alpha, \beta) \in \mathbb{F}\text{-BrMod}^r$ is defined as follows: there exists a basis $\underline{e} := (e_1, e_2)$ for $\mathcal{M}(a, b : \alpha, \beta)$ and a system of generators $\underline{f} := (f_1, f_2)$ for $\mathrm{Fil}^r \mathcal{M}$ such that

- $\mathcal{M} := \bar{S}_{\mathbb{F}}(e_1, e_2)$;
- $\mathrm{Mat}_{\underline{e},\underline{f}}(\mathrm{Fil}^r \mathcal{M}) = \begin{pmatrix} u^a & 0 \\ 0 & u^b \end{pmatrix}$;
- $\mathrm{Mat}_{\underline{e},\underline{f}}(\phi_r) = \begin{pmatrix} 0 & \beta \\ \alpha & 0 \end{pmatrix}$;
- $\mathrm{Mat}_{\underline{e}}(N) = 0_{2\times 2}$.

Lemma 3.3.2. *$\mathcal{M}(a, b : \alpha, \beta)$ is isomorphic to $\mathcal{M}(a', b' : \alpha', \beta')$ if and only if $a = a'$, $b = b'$, and $\alpha\beta = \alpha'\beta'$.*

Proof. This is an easy linear algebra computation, chasing the definition. One can find a similar argument in [Park 2017, Lemma 2.2]. □

Lemma 3.3.3. *Assume that $a+b=r$ and let $\bar{\rho} := \mathrm{T}^r_{\mathrm{st}}(\mathcal{M}(a, b : \alpha, \beta)) \otimes_{\mathbb{F}} \bar{\mathbb{F}}$. Then*

$$\bar{\rho}|_{I_{\mathbb{Q}_p}} \cong \omega_2^{a+pb} \oplus \omega_2^{b+pa}.$$

In particular, $\bar{\rho}$ is absolutely irreducible.

Proof. The argument is similar to [Park 2017, Proposition 2.5], which is heavily relied on the result of [Caruso 2006, Theorem 5.2.2,]. □

The following example parameterizes Breuil modules of rank 2 whose corresponding representations are ordinary:

Example 3.3.4. Let a, b be positive integers with $0 \le b < a \le a+b \le r < p-1$, and $\alpha, \beta, \gamma \in \mathbb{F}^\times$. The Breuil module $\mathcal{M}(a, b : \alpha, \beta, \gamma) \in \mathbb{F}\text{-BrMod}^r$ is defined as follows: there exists a basis $\underline{e} := (e_1, e_2)$ for $\mathcal{M}(a, b : \alpha, \beta, \gamma)$ and a system of generators $\underline{f} := (f_1, f_2)$ for $\mathrm{Fil}^r \mathcal{M}$ such that

- $\mathcal{M} := \bar{S}_{\mathbb{F}}(e_1, e_2)$;
- $\mathrm{Mat}_{\underline{e},\underline{f}}(\mathrm{Fil}^r \mathcal{M}) = \begin{pmatrix} u^a & 0 \\ 0 & u^b \end{pmatrix}$;
- $\mathrm{Mat}_{\underline{e},\underline{f}}(\phi_r) = \begin{pmatrix} \gamma & \beta \\ \alpha & 0 \end{pmatrix}$;
- $\mathrm{Mat}_{\underline{e}}(N) = 0_{2\times 2}$.

Lemma 3.3.5. *Assume that $a+b=r$ and let $\bar{\rho} := \mathrm{T}^r_{\mathrm{st}}(\mathcal{M}(a, b : \alpha, \beta, \gamma)) \otimes_{\mathbb{F}} \bar{\mathbb{F}}$. Then*

$$\bar{\rho}^{ss}|_{I_{\mathbb{Q}_p}} \cong \omega^a \oplus \omega^b.$$

In particular, $\bar{\rho}$ is reducible.

Proof. We first define Breuil modules $\widetilde{\mathcal{M}}(a:\alpha) := \bar{S}_{\mathbb{F}}(\tilde{e})$ of rank 1 as follows:

- $\mathrm{Fil}^r\,\widetilde{\mathcal{M}}$ is generated by $u^a\tilde{e}$;
- $\phi_r : \mathrm{Fil}^r\,\widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}$ is induced by $u^a\tilde{e} \mapsto \alpha\tilde{e}$;
- $N : \widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}$ is induced by $N(\tilde{e}) = 0$.

The association $e_1 \mapsto -\alpha/\gamma\,\tilde{e}$ and $e_2 \mapsto \tilde{e}$ gives rise to a morphism from $\mathcal{M}(a, b : \alpha, \beta, \gamma)$ to $\widetilde{\mathcal{M}}(b : -\alpha\beta/\gamma)$. Moreover, the association $\tilde{e} \mapsto e_1 + \alpha/\gamma\, e_2$ gives rise to a morphism from $\widetilde{\mathcal{M}}(a : \gamma)$ to $\mathcal{M}(a, b : \alpha, \beta, \gamma)$. Namely, we have a short exact sequence

$$0 \to \widetilde{\mathcal{M}}(a:\gamma) \to \mathcal{M}(a, b : \alpha, \beta, \gamma) \to \widetilde{\mathcal{M}}\Big(b : -\frac{\alpha\beta}{\gamma}\Big) \to 0.$$

Now, applying [Caruso 2006, Theorem 5.2.2,], to the short exact sequence completes the proof. □

The following example parameterizes Breuil modules of rank 2 whose corresponding representations have scalar semisimplification:

Example 3.3.6. Let a be a positive integer with $a \le r < p-1$, and $A \in \mathrm{GL}_2(\mathbb{F})$. The Breuil module $\mathcal{M}(a : A) \in \mathbb{F}\text{-}\mathrm{BrMod}^r$ is defined as follows: there exists a basis $\underline{e} := (e_1, e_2)$ for $\mathcal{M}(a, A)$ and a system of generators $\underline{f} := (f_1, f_2)$ for $\mathrm{Fil}^r\,\mathcal{M}$ such that

- $\mathcal{M} := \bar{S}_{\mathbb{F}}(e_1, e_2)$;
- $\mathrm{Mat}_{\underline{e},\underline{f}}(\mathrm{Fil}^r\,\mathcal{M}) = \begin{pmatrix} u^a & 0 \\ 0 & u^a \end{pmatrix}$;
- $\mathrm{Mat}_{\underline{e},\underline{f}}(\phi_r) = A$;
- $\mathrm{Mat}_{\underline{e}}(N) = 0_{2\times 2}$.

Lemma 3.3.7. *Assume that $r = 2a$ and let $\bar{\rho} := \mathrm{T}^r_{\mathrm{st}}(\mathcal{M}(a : A)) \otimes_{\mathbb{F}} \bar{\mathbb{F}}$. Then*

$$\bar{\rho}^{ss}|_{I_{\mathbb{Q}_p}} \cong \omega^a \oplus \omega^a.$$

In particular, $\bar{\rho}$ is reducible.

Proof. First, notice that the argument in the proof of Lemma 3.3.5 does not work for these Breuil modules. But, due to the shape of filtration, it is enough to consider equivalent classes of similar matrices. Any 2×2 invertible matrix is similar to one of the following two matrices:

$$\begin{pmatrix} \alpha & 1 \\ 0 & \beta \end{pmatrix} \quad\text{and}\quad \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$$

for $\alpha \neq 0 \neq \beta$.

Let $\widetilde{\mathcal{M}}$ be a Breuil module of rank 1 defined in the proof of Lemma 3.3.5, and assume that A is one of the matrices above. Then the association $e_1 \mapsto 0$ and $e_2 \mapsto \tilde{e}$ gives rise to a nontrivial morphism from $\mathcal{M}(a : A)$ to $\widetilde{\mathcal{M}}(a : \beta)$. Moreover, the association $\tilde{e} \mapsto e_1$ gives rise to a nontrivial morphism from $\widetilde{\mathcal{M}}(a : \alpha)$ to $\mathcal{M}(a : A)$, i.e., we have a short exact sequence

$$0 \to \widetilde{\mathcal{M}}(a : \alpha) \to \mathcal{M}(a : A) \to \widetilde{\mathcal{M}}(a : \beta) \to 0.$$

Applying [Caruso 2006, Theorem 5.2.2], to these short exact sequences of Breuil modules completes the proof. □

4. Certain elements in $S_{\mathbb{O}}$

From now on, by r we always mean a positive even integer less than $p-1$, and we let $r = 2m$. In this section, we construct certain elements $\delta_\ell \in S_{\mathbb{O}}$ for integers $0 \le \ell \le m-2$, as well as elements that we denote δ_∞ and $\delta_{-\infty}$, that will be used to construct strongly divisible modules. In order to do this, we need several elementary but nontrivial lemmas, which will occupy the first subsection. In the second subsection, we define sequences depending on $\mathfrak{L}$ for each $0 < r = 2m < p-1$, and the limits of these sequences will appear in the coefficients of our strongly divisible modules in certain cases.

4.1. *Some matrices.* Let s and t be positive integers such that $t + 2s \le p$, and $M^{(s,t)} \in \mathrm{M}_{s\times(s+1)}(\mathbb{O})$ be the matrix defined as

$$M^{(s,t)} = \begin{pmatrix} \frac{(-1)^t}{t} & \frac{(-1)^{t+1}}{t+1} & \cdots & \frac{(-1)^{t+s-1}}{t+s-1} & \frac{(-1)^{t+s}}{t+s} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{(-1)^{t+s-1}}{t+s-1} & \frac{(-1)^{t+s}}{t+s} & \cdots & \frac{(-1)^{t+2s-2}}{t+2s-2} & \frac{(-1)^{t+2s-1}}{t+2s-1} \end{pmatrix}.$$

Lemma 4.1.1. *Suppose that $Y_0, \dots, Y_s \in \mathbb{O}$ satisfy*

$$M^{(s,t)} \cdot \begin{pmatrix} Y_s \\ \vdots \\ Y_0 \end{pmatrix} = 0.$$

Then for every $j \in \{0, 1, \dots, s\}$,

$$Y_j = x_j^{(s,t)} \cdot Y_0$$

where

$$x_j^{(s,t)} = \frac{(s+t-1)!}{(2s+t-1)!} \frac{(2s-j+t-1)!}{(s-j+t-1)!} \binom{s}{j}.$$

In particular, if we let $M'^{(s,t)}$ be the symmetric $s \times s$-matrix obtained from $M^{(s,t)}$ by deleting the last column, then $M'^{(s,t)}$ is invertible, that is, $M'^{(s,t)} \in \mathrm{GL}_s(\mathbb{O})$.

Proof. For $f, g \in \mathbb{O}[X]$ polynomials such that $\deg(f)+\deg(g)+t < p$, let

$$(f, g) = -\int_{-1}^{0} X^{t-1} f(X) g(X)\, dX.$$

Note that there is a unique monic $f \in \mathbb{O}[X]$ of degree s such that $(f, g) = 0$ for all $g \in \mathbb{O}[X]$ of degree less than s. Indeed, let

$$\tilde{f} = \frac{1}{X^{t-1}} \frac{d^s}{dX^s} (X+1)^s X^{s+t-1} = \sum_{j=0}^{s} \binom{s}{j} \frac{(j+s+t-1)!}{(j+t-1)!} X^j.$$

Then $\tilde{f}$ is orthogonal to all $g \in \mathbb{O}[X]$ of degree less than s. Dividing by its leading coefficient, we obtain

$$f = \frac{(s+t-1)!}{(2s+t-1)!} \sum_{j=0}^{s} \binom{s}{j} \frac{(j+s+t-1)!}{(j+t-1)!} X^j.$$

To see that f is unique, we can replace $\mathbb{O}$ by $\mathbb{C}$ by fixing an embedding $E \hookrightarrow \mathbb{C}$. Then we need to show that the rank of $M^{(s,t)}$ is $s-1$, but for this it's enough to show the uniqueness of f over $\mathbb{Q}$. If f' is another such monic polynomial, then $f - f'$ has degree less than s and is orthogonal to itself, from where the claim easily follows.

Let

$$g = \sum_{j=0}^{s} Y_{s-j} X^j.$$

If $0 \le i \le s-1$ and $0 \le j \le s$, then

$$\frac{(-1)^{t+j+i}}{t+j+i} = (X^j, X^i).$$

Thus the hypothesis says that $(g, X^i) = 0$ for $i = 0, \dots, s-1$, so that g is orthogonal to all the polynomials of degree less than s. Thus,

$$g = Y_0 f,$$

from where the lemma follows.

For the second part, we consider the square matrix

$$\mathrm{Diag}((-1)^t, (-1)^{t+1}, \dots, (-1)^{t+s-1}) \cdot M'^{(s,t)} \cdot \mathrm{Diag}(1, (-1), \dots, (-1)^{s-1})$$

which is a Cauchy matrix, i.e., it can be written as

$$\Big(\frac{1}{x_i - y_j}\Big)$$

where

$$x_i = t + s - 2 + i \quad \text{and} \quad y_j := s - j.$$

Hence, one can readily check from the well-known determinant formula of a Cauchy matrix that $\det M'^{(s,t)}$ is computed as

$$\det M'^{(s,t)} = \frac{(-1)^{s(t+s-1)} \prod_{i=2}^{s} \prod_{j=1}^{i-1} (i-j)^2}{\prod_{i=1}^{s} \prod_{j=1}^{s} (t - 2 + i + j)}.$$

Since $t + 2s \leq p$, we conclude that $\det M'^{(s,t)} \in \mathbb{O}^\times$, which completes the proof. □

Note that by Lemma 4.1.1 $x_0^{(s,t)} = 1$ for any positive integers s and t with $t + 2s \leq p$. But for any integers (not necessarily positive) s and t we also let $x_0^{(s,t)} = 1$ by convention.

Lemma 4.1.2. *Let* $0 \leq \ell \leq m-1$, $s = m - \ell - 1$ *and* $t = 2(\ell + 1)$. *Then*

$$\sum_{i=0}^{m-\ell-1} (-1)^i x_i^{(s,t)} = \frac{(m+\ell)!(m-\ell-1)!}{(r-1)!}.$$

Proof. Write $x_i = x_i^{(s,t)}$. By definition,

$$x_i = \frac{(m+\ell)!(r-1-i)!}{(r-1)!(m+\ell-i)!} \binom{m-\ell-1}{i},$$

and thus

$$\sum_{i=0}^{m-\ell-1} (-1)^i x_i = \frac{(m+\ell)!}{(r-1)!} \sum_{i=0}^{m-\ell-1} (-1)^i \binom{m-\ell-1}{i} P(i),$$

where $P = (r - 1 - X) \dots (m + \ell + 1 - X)$ is a polynomial of degree $m - \ell - 1$ whose leading coefficient is $(-1)^{m-\ell-1}$. We now apply the well-known identity

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} P(x + (n-i)d) = d^n n! a_n$$

with $n = m - \ell - 1$, $x = n$ and $d = -1$, where a_n is the leading coefficient of P, and we get the expression in the statement of the lemma. □

We now restrict our attention to the following $(m-1)\times(m+1)$-matrix:

$$M^{(0)} = \begin{pmatrix} \frac{(-1)^1}{1} & \frac{(-1)^2}{2} & \cdots & \frac{(-1)^m}{m} & \frac{(-1)^{m+1}}{m+1} \\ \frac{(-1)^2}{2} & \frac{(-1)^3}{3} & \cdots & \frac{(-1)^{m+1}}{m+1} & \frac{(-1)^{m+2}}{m+2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{(-1)^{m-1}}{m-1} & \frac{(-1)^m}{m} & \cdots & \frac{(-1)^{2m-2}}{2m-2} & \frac{(-1)^{2m-1}}{2m-1} \end{pmatrix}.$$

Note that the $(m-1)\times m$-matrix $M^{(s,t)}$ with $s=m-1$ and $t=2$ can be obtained from $M^{(0)}$ by deleting the left-most column.

Lemma 4.1.3. *Suppose that $Y_0,\dots,Y_m\in\mathbb{O}$ satisfy*

$$M^{(0)}\cdot\begin{pmatrix} Y_m \\ \vdots \\ Y_0\end{pmatrix}=0.$$

Then for every $j\in\{1,2,\dots,m-1\}$

$$Y_j = x_j^{(s,t)}\cdot Y_0 + y_j^{(0)}\cdot Y_m$$

where $x_j^{(s,t)}$ is defined as in Lemma 4.1.1 with $s=m-1$ and $t=2$ and $y_j^{(0)}$ is a uniquely determined element in $\mathbb{O}$.

Proof. By Lemma 4.1.1, we see that $[M'^{(s,t)}]^{-1}\in \mathrm{GL}_s(\mathbb{O})$. It is now immediate to get the results by looking at $[M'^{(s,t)}]^{-1}\cdot M^{(0)}$. □

4.2. ***Sequences.*** In this subsection, we define sequences, which are convergent in $\mathbb{O}$ and study their properties. Their limits will appear in the coefficients of our strongly divisible modules in certain cases. We keep the notation of Section 4.1; in particular, recall that we defined elements $x_i^{(s,t)}$ and $y_i^{(0)}$. By convention, we let $x_0^{(s,t)}=1$ for any integers s and t and $y_0^{(0)}=0$.

We start to define the following two quantities: for a positive integer k,

$$H_k := \sum_{i=1}^{k}\frac{1}{i} \quad\text{and}\quad a(r) := H_{m-1}+H_m. \tag{4.2.1}$$

Note that $a(2)=1$ by convention. We also note that $H_k\in\mathbb{O}$ for $k=1,2,\dots,r$ since $r<p-1$ and so $a(r)\in\mathbb{O}$.

Assume first that $r=2$. We start to define a sequence $\{G_{2,n}\}$ as follows: let $G_{2,0}=1$ and

$$G_{2,n+1} = \frac{(\mathfrak{L}-1)^2}{(\mathfrak{L}-1)^2 - pG_{2,n}}.$$

Lemma 4.2.2. *Assume that* $v_p(\mathfrak{L}-1) < \frac{1}{2}$:

(1) $\{G_{2,n}\}$ *converges to an element, denoted by* $\Delta_2 = \Delta_2(\mathfrak{L})$, *in* $1+\mathfrak{m}$.

(2) Δ_2 *satisfies the equation* $p(\Delta_2)^2 - (\mathfrak{L}-1)^2\Delta_2 + (\mathfrak{L}-1)^2 = 0$.

Proof. It is immediate, by induction on n, that for all $n \geq 0$ $G_{2,n} \in 1+\mathfrak{m}$ since $v_p(\mathfrak{L}-1) < \frac{1}{2}$. One can readily check that

$$G_{2,n+2} - G_{2,n+1} = \frac{p(\mathfrak{L}-1)^2}{[(\mathfrak{L}-1)^2 - pG_{2,n+1}][(\mathfrak{L}-1)^2 - pG_{2,n}]}(G_{2,n+1} - G_{2,n}),$$

so that we have

$$v_p(G_{2,n+1} - G_{2,n}) = (n+1)[1 - 2v_p(\mathfrak{L}-1)] + v_p(G_{2,1} - G_{2,0}).$$

This completes the proof of the part (1).

Part (2) is immediate by part (1) and by definition. □

For $r = 2m > 2$, we need a little more preparation to define the sequences we will use later on. For the rest of this subsection, we fix $s = m-1$ and $t = 2$. Let

$$X_0 := \sum_{j=1}^{m} \frac{(-1)^j}{j} x_{m-j}^{(s,t)} \quad \text{and} \quad Y_0 := \sum_{j=1}^{m-1} \frac{(-1)^j}{j} y_{m-j}^{(0)}.$$

We also let

$$S_0 := \frac{(-1)^m + \sum_{i=0}^{m-1}(-1)^i y_i^{(0)}}{\sum_{i=0}^{m-1}(-1)^i x_i^{(s,t)}} \quad \text{and} \quad T_0 := \frac{\sum_{k=1}^{m}(-1)^k \sum_{j=1}^{k}(-1)^j/j\, y_{k-j}^{(0)}}{\sum_{i=0}^{m-1}(-1)^i x_i^{(s,t)}}.$$

Note that $X_0,\ Y_0,\ S_0,\ T_0 \in \mathbb{O}$ by Lemmas 4.1.1, 4.1.2, and 4.1.3.

We are now ready to define the sequence: let $G_{r,0} = 1$ and

$$G_{r,n+1} = \frac{X_0(T_0 - a(r)S_0)(\mathfrak{L}-a(r))^2}{(\mathfrak{L}-a(r))^2 + pG_{r,n} + p(Y_0 - X_0S_0 - \mathfrak{L})(\mathfrak{L}-a(r))}.$$

Lemma 4.2.3. *Assume that* $r = 2m > 2$ *and that* $v_p(\mathfrak{L}-a(r)) < \frac{1}{2}$:

(1) $\{G_{r,n}\}$ *converges to an element, denoted by* $\Delta_r = \Delta_r(\mathfrak{L})$, *in* $\mathbb{O}$;

(2) Δ_r *satisfies the equation*

$$p(\Delta_r)^2 + (\mathfrak{L}-a(r))[(\mathfrak{L}-a(r)) + p(Y_0 - X_0S_0 - \mathfrak{L})]\Delta_r - X_0(T_0 - a(r)S_0)(\mathfrak{L}-a(r))^2 = 0.$$

Proof. The proof is very similar to Lemma 4.2.2. It is easy to check that

$$G_{r,n+2} - G_{r,n+1} = \frac{p \cdot X}{(\mathfrak{L}-a(r))^2}(G_{r,n+1} - G_{r,n})$$

for some $X \in \mathcal{O}$, which implies that the sequence $\{G_{r,n}\}$ is Cauchy, since $v_p(G_{r,n+1} - G_{r,n})$ approaches ∞ as n goes to ∞. The limit obviously sits in $\mathcal{O}$ since each G_n does. The second part is immediate from the first part. □

Remark 4.2.4. The limits Δ_r can be defined as solutions of the polynomial equations. One can check that the equation described as in Lemma 4.2.2 (2) for $r = 2$ (resp. in Lemma 4.2.3 (2) for $r = 2m > 2$) has a unique solution $\Delta_2 \in \mathcal{O}$ such that $\Delta_2 \equiv 1 \bmod \mathfrak{m}$ (resp. $\Delta_r \in \mathcal{O}$ such that $\Delta_r \equiv X_0(T_0 - a(r)S_0) \bmod \mathfrak{m}$) by Hensel's lemma.

4.3. *The elements $\delta_\bullet \in S_{\mathcal{O}}$.* In this subsection, we define certain elements $\delta_\bullet \in S_{\mathcal{O}}$ for $\bullet \in \{\infty, 0, 1, 2, \dots, m-2, -\infty\}$. Recall that the elements $x_j^{(s,t)}$ are defined in Lemma 4.1.1 for positive integers s and t with $t + 2s \leq p$, and that we let $x_0^{(s,t)} = 1$ for any s and t and $y_0^{(0)} = 0$ by convention.

For $\ell \in \{1, \dots, m-2\}$, we let $t = 2(\ell+1)$ and $s = m - \ell - 1$, and we define

$$\delta_\ell = \frac{\sum_{k=1}^{m+\ell} (\gamma-1)^k \sum_{j=\max\{1,\, k-m+\ell+1\}}^{k} (-1)^j/j \, x_{k-j}^{(s,t)}}{\sum_{i=0}^{m-\ell-1} x_i^{(s,t)} (\gamma-1)^i}. \tag{4.3.1}$$

Note that δ_ℓ is defined only when $r = 2m \geq 2(\ell+2) \geq 6$.

We define δ_∞ by letting $\ell = 0$ in the formula of δ_ℓ in (4.3.1). Namely, we fix $t = 2$ and $s = m - 1$, and let

$$\delta_\infty = \frac{\sum_{k=1}^{m} (\gamma-1)^k \sum_{j=1}^{k} (-1)^j/j \, x_{k-j}^{(s,t)}}{\sum_{i=0}^{m-1} x_i^{(s,t)} (\gamma-1)^i}. \tag{4.3.2}$$

For $\delta_{-\infty}$, we define $r = 2$ case separately. If $r = 2$ then we let

$$\delta_{-\infty} = -(\gamma-1) + \frac{p\Delta_2}{\mathfrak{L}-1}(\gamma-1)^2, \tag{4.3.3}$$

where Δ_2 is defined in Lemma 4.2.2. Note that we use $\delta_{-\infty}$ for $r = 2$ only when $v_p(\mathfrak{L}-1) < \frac{1}{2}$, so that $\delta_{-\infty} \in S_{\mathcal{O}}$. If $r = 2m > 2$ then we define $\delta_{-\infty}$ by letting $\ell = m - 1$ in the formula of δ_ℓ in (4.3.1). More precisely, for $r = 2m > 2$ we let

$$\delta_{-\infty} = \sum_{j=1}^{r-1} \frac{(-1)^j}{j} (\gamma-1)^j. \tag{4.3.4}$$

Remark 4.3.5. If $v_p(\mathfrak{L} - a(r)) < \frac{1}{2}$, $\delta_{-\infty}$ for $r = 2$ is in $S_{\mathcal{O}}$. Other $\delta_\bullet$ also belongs to $S_{\mathcal{O}}$. Indeed, the denominators in the expressions are all units in $S_{\mathcal{O}}$, since modulo γ they belong to $\mathcal{O}^\times$ by Lemma 4.1.2. Thus, $\delta_\bullet \in S_{\mathcal{O}}$ for $\bullet \in \{\infty, 1, 2, \dots, m-2, -\infty\}$, where they are defined.

We often write $\delta_\bullet^{(0)} \in \mathcal{O}$ for the constant determined by $\delta_\bullet^{(0)} \equiv \delta_\bullet$ modulo (γ).

Lemma 4.3.6. *$\delta_\infty \equiv a(r)$ modulo $\gamma S_{\mathbb{O}}$. Equivalently,*

$$a(r) = \frac{\sum_{k=1}^{m}(-1)^k \sum_{j=1}^{k}(-1)^j/j\, x_{k-j}^{(m-1,2)}}{\sum_{i=0}^{m-1}(-1)^i x_i^{(m-1,2)}}. \tag{4.3.7}$$

Proof. The case $r = 2$ is clear. We assume that $r \geq 4$ for the rest of the proof, so $m \geq 2$. It is obvious that it is enough to show the first congruence, since the quantity in (4.3.7) is the same as $\delta_\infty^{(0)}$.

Let $x_i = x_i^{(s,t)}$ with $s = m-1$ and $t = 2$. After changing the indices, we can write

$$\delta_\infty = \frac{\sum_{i=1}^{m} z_i(\gamma-1)^i}{\sum_{i=0}^{m-1} x_i(\gamma-1)^i} \tag{4.3.8}$$

with

$$z_i = \sum_{j=0}^{i-1} \frac{(-1)^{i-j}}{i-j} x_j.$$

Now, notice that the denominator of δ_∞ is congruent to $\sum_{i=0}^{m-1}(-1)^i x_i$ modulo $\gamma S_{\mathbb{O}}$, which equals

$$\frac{m!(m-1)!}{(r-1)!}$$

by Lemma 4.1.2. By Lemma 4.1.1 the numerator of δ_∞ is congruent to

$$\frac{m!}{(r-1)!}\sum_{i=1}^{m}\sum_{j=0}^{i-1}\frac{(-1)^j}{i-j}\frac{(r-1-j)!}{(m-j)!}\binom{m-1}{j},$$

which is equal to

$$\frac{m!}{(r-1)!}\sum_{j=0}^{m-1}(-1)^j\binom{m-1}{j}\frac{(r-1-j)!}{(m-j)!}H_{m-j}. \tag{4.3.9}$$

We define a power series $g(x)$ as follows:

$$g(x) := \sum_{i \geq m+1}(-1)^{i-m-1}\frac{m!(i-m-1)!}{i!}x^i.$$

It is easy to check that it satisfies the equation

$$(1+x)^m \ln(1+x) = \sum_{i=0}^{m}\binom{m}{i}(H_m - H_{m-i})x^i + g(x),$$

and so

$$\frac{\ln(1-x)}{1-x} - \frac{H_m}{1-x} - \frac{g(-x)}{(1-x)^{m+1}} = -\frac{1}{(1-x)^{m+1}}\sum_{i=0}^{m}(-1)^i\binom{m}{i}H_{m-i}x^i.$$

Rewriting the formula on the right-hand side, we get

$$\text{(4.3.10)}\quad \frac{\ln(1-x)}{1-x} - \frac{H_m}{1-x} - \frac{g(-x)}{(1-x)^{m+1}} \\ = \sum_{i\geq 0} x^i \left(\sum_{j=0}^{\min(m,i)} (-1)^{j-1} \binom{m}{j} \binom{m+i-j}{m} H_{m-j} \right).$$

Thus, (4.3.9) is equal to $-m!(m-1)!/(r-1)!$ times the coefficient of x^{m-1} in (4.3.10). Now, we have

$$\frac{\ln(1-x)}{1-x} = -\sum_{i\geq 1} H_i x^i, \quad -\frac{H_m}{1-x} = -H_m \sum_{i\geq 0} x^i,$$

and

$$-\frac{g(-x)}{(1-x)^{m+1}} = (-1)^m \sum_{i\geq m+1} x^i \sum_{j=0}^{i-m-1} \frac{m!j!}{(m+j+1)!} \binom{i-j-1}{m},$$

from where we get that (4.3.9) is equal to $m!(m-1)!/(r-1)!(H_m + H_{m-1})$. Putting everything together, we get that

$$\delta_\infty \equiv \frac{\dfrac{m!(m-1)!}{(r-1)!}(H_m + H_{m-1})}{\dfrac{m!(m-1)!}{(r-1)!}} = a(r)$$

modulo $\gamma S_{\mathbb{O}}$. □

We now define δ_0. In this case, we always fix $s = m-1$ and $t = 2$, and keep the notation as in Section 4.2. Note that δ_0 is defined only when $r = 2m > 2$, and that we are going to use it only when $v_p(\mathfrak{L} - a(r)) < \frac{1}{2}$. To lighten the notation we first let

$$Z_0 := \frac{X_0(\mathfrak{L}-a(r))^2}{(\mathfrak{L}-a(r))^2 + p\Delta_r + p(Y_0 - \mathfrak{L})(\mathfrak{L}-a(r))},$$

where Δ_r is defined in Lemma 4.2.3. Note that $Z_0 \in \mathbb{O}$ if $v_p(\mathfrak{L} - a(r)) < \frac{1}{2}$. We are now ready to define δ_0:

$$\text{(4.3.11)}\quad \delta_0 = \frac{\sum_{k=1}^m (\gamma-1)^k \sum_{j=1}^k \frac{(-1)^j}{j} \left(x_{k-j}^{(s,t)} - \frac{pZ_0}{\mathfrak{L}-a(r)} y_{k-j}^{(0)} \right)}{\sum_{k=0}^{m-1} \left(x_k^{(s,t)} - \frac{pZ_0}{\mathfrak{L}-a(r)} y_k^{(0)} \right) (\gamma-1)^k - \frac{pZ_0}{\mathfrak{L}-a(r)} (\gamma-1)^m}.$$

Note that $\delta_0 \in S_{\mathbb{O}}$ since $\delta_0 \equiv \delta_\infty$ modulo $p/(\mathfrak{L} - a(r))\, S_{\mathbb{O}}$ if $v_p(\mathfrak{L} - a(r)) < \frac{1}{2}$.

Lemma 4.3.12. *Assume that* $r = 2m > 2$ *and* $v_p(\mathfrak{L} - a(r)) < \frac{1}{2}$, *and let* $\delta_0^{(0)} \in \mathbb{O}$ *be the constant determined by* $\delta_0^{(0)} \equiv \delta_0$ *modulo* (γ). *Then*

$$\delta_0^{(0)} = a(r) - \frac{p}{\mathfrak{L} - a(r)} \Delta_r. \tag{4.3.13}$$

In particular, $v_p(\mathfrak{L} - a(r)) = v_p(\mathfrak{L} - \delta_0^{(0)})$.

Proof. It is easy to check that the equation in Lemma 4.2.3 (2) implies

$$\Delta_r = \frac{Z_0(T_0 - a(r)S_0)(\mathfrak{L} - a(r))}{(\mathfrak{L} - a(r)) - pZ_0S_0}.$$

On the other hand, we also readily get

$$\delta_0^{(0)} = \frac{a(r) - p/(\mathfrak{L} - a(r))\, Z_0T_0}{1 - p/(\mathfrak{L} - a(r))\, Z_0S_0}$$

from the definition of δ_0, by using the identity (4.3.7) in Lemma 4.3.6. Note that the denominator in the above two expressions never vanish due to our assumption $v_p(\mathfrak{L} - a(r)) < \frac{1}{2}$. Now it is easy to induce the identity (4.3.13) from the two identities above.

The second part is immediate from the identity (4.3.13) since

$$v_p\left(\mathfrak{L} - a(r) + \frac{p\Delta_r}{\mathfrak{L} - a(r)}\right) = v_p(\mathfrak{L} - a(r))$$

if $v_p(\mathfrak{L} - a(r)) < \frac{1}{2}$. □

Lemma 4.3.14. $N(\delta_0) \equiv 0$ *modulo* (p) *if* $v_p(\mathfrak{L} - a(r)) < \frac{1}{2}$ *as well as* $N(\delta_\bullet) \equiv 0$ *modulo* (p) *for* $\bullet \in \{\infty, 1, 2, \dots, m-2, -\infty\}$.

Proof. It is routine to check that $N(\gamma) = -p[\gamma + (u-p)^{p-1}]$. Now it is immediate to show that p divides $N(\delta_\bullet)$. □

5. Statements of main results on mod p reduction

In this section, we state our main results on Galois stable lattices in the semistable representations of $G_{\mathbb{Q}_p}$ with Hodge–Tate weights $(0, r)$, where r is a positive even integer less than $p-1$. Recall that we define an integer m by $r = 2m$. We also state our main results on mod p reduction of those semistable representations.

Recall that by $D = D(\lambda, \mathfrak{L})$ we mean the admissible filtered (ϕ, N)-modules in Example 3.1.1. These modules parameterize all the 2-dimensional semistable noncrystalline representations of $G_{\mathbb{Q}_p}$ with Hodge–Tate weights $(0, r)$. We also recall that we write $\mathfrak{D} = \mathfrak{D}(\lambda, \mathfrak{L})$ for $S \otimes_{\mathbb{Z}_p} D$, and let $h := v_p(\mathfrak{L} - a(r))$ for brevity.

Define $\delta \in S_{\mathbb{O}}$ and $\Theta \in E$ as follows:

$$\delta := \begin{cases} \delta_\infty & \text{if } \frac{1}{2} \le h; \\ \delta_\ell & \text{if } -\frac{1}{2} - \ell \le h < \frac{1}{2} - \ell \text{ for } \ell \in \{0, 1, \dots, m-2\}; \\ \delta_{-\infty} & \text{if } h < \frac{3}{2} - m; \end{cases}$$

and

$$\Theta := \begin{cases} \lambda / p^{m-1} & \text{if } \frac{1}{2} \le h; \\ \lambda(\mathfrak{L} - a(r))/p^{m-\ell-1} & \text{if } -\frac{1}{2} - \ell \le h < \frac{1}{2} - \ell \text{ for } \ell \in \{0, 1, \dots, m-2\}; \\ \lambda(\mathfrak{L} - a(r)) & \text{if } h < \frac{3}{2} - m. \end{cases}$$

Notice that the three cases according to the values of h cover all the h-line, and recall that the elements $\delta_\bullet \in S_{\mathbb{O}}$ for $\bullet \in \{\infty, 0, 1, \dots, m-2, -\infty\}$ are defined in Section 4.3. Finally, we let

$$E_1 = p\eta_1 + (\mathfrak{L} - \delta)\eta_2, \quad E_2 = \Theta\, \eta_2,$$

which are generators of our strongly divisible modules, as we will see below.

The following theorem is our main results on Galois stable lattices.

Theorem 5.0.1. *Let $r = 2m > 0$ be an even integer less than $p-1$. Then $\mathfrak{M} := S_{\mathbb{O}}(E_1, E_2)$ is a strongly divisible module in $\mathfrak{D}$.*

The following computations, which we will use later in the proof of Theorem 5.0.1, are very elementary. We let $\delta^{(0)} \in \mathbb{O}$ be the constant determined by $\delta^{(0)} \equiv \delta \pmod{\gamma S_{\mathbb{O}}}$. For instance, $\delta^{(0)}_\infty = a(r)$ by Lemma 4.3.6. One can readily check that

$$\phi(E_1) = p\lambda E_1 + \frac{\lambda[\mathfrak{L} - \phi(\delta) - p\mathfrak{L} + p\delta]}{\Theta} E_2, \quad \phi(E_2) = \lambda E_2, \tag{5.0.2}$$

and

$$N(E_1) = \frac{p - N(\delta)}{\Theta} E_2, \quad N(E_2) = 0. \tag{5.0.3}$$

Moreover, it follows from (3.2.2) that any element of $\mathrm{Fil}^r \mathfrak{D}$ can be written, modulo $\mathrm{Fil}^r S_E \mathfrak{D}$, as

$$\begin{aligned} \mathfrak{Y} &= \mathfrak{Y}(\vec{C}) \\ &= \sum_{k=0}^{r-1} v^k \Bigg[C_k \bigg(\frac{E_1}{p} + \frac{p\mathfrak{L} - (\mathfrak{L} - \delta^{(0)})}{p\Theta} E_2 \bigg) + \sum_{j=1}^{k} \frac{(-1)^{j+1}}{j} \frac{C_{k-j}}{p^j \Theta} E_2 \Bigg] \end{aligned} \tag{5.0.4}$$

for elements $C_0, \dots, C_{r-1} \in E$, where we denote $\vec{C} = (C_0, \dots, C_{r-1})$. Recall that we write v for $E(u) = u - p$ for brevity.

We now state our main results on mod p reduction.

Theorem 5.0.5. *Let $r = 2m > 0$ be an even integer less than $p-1$. The mod p reduction of $\rho := \mathrm{V}^r_{\mathrm{st}}(D)$ is absolutely irreducible if and only if one of the following cases holds*:

(1) $-\frac{1}{2} - \ell < h < \frac{1}{2} - \ell$ *for* $\ell \in \{0, 1, 2, \dots, m-2\}$, *in which case*

$$\bar{\rho}|_{I_{\mathbb{Q}_p}} \cong \omega_2^{m-\ell-1+p(m+\ell+1)} \oplus \omega_2^{m+\ell+1+p(m-\ell-1)};$$

(2) $h < \frac{3}{2} - m$, *in which case*

$$\bar{\rho}|_{I_{\mathbb{Q}_p}} \cong \omega_2^r \oplus \omega_2^{pr}.$$

Moreover,

$$\bar{\rho}^{ss}|_{I_{\mathbb{Q}_p}} \cong \begin{cases} \omega^{m-\ell-1} \oplus \omega^{m+\ell+1} & \text{if } h = -\frac{1}{2} - \ell \text{ for } \ell \in \{0, 1, \dots, m-2\}; \\ \omega^m \oplus \omega^m & \text{if } h \geq \frac{1}{2}. \end{cases}$$

The proof for the theorems above will occupy the next three sections. We divide their proofs into three cases according to the different values of h. Namely,

$$\begin{aligned} \tfrac{1}{2} \leq h & \qquad \text{in Section 6;} \\ -\tfrac{1}{2} - \ell \leq h < \tfrac{1}{2} - \ell \text{ for } \ell \in \{0, 1, 2, \dots, m-2\} & \qquad \text{in Section 7;} \\ h < \tfrac{3}{2} - m & \qquad \text{in Section 8.} \end{aligned}$$

We found that constructing strongly divisible modules when $0 \leq h < \frac{1}{2}$ is more difficult than other cases. For instance, the case $-\frac{1}{2} \leq h < \frac{1}{2}$ in Section 7, as well as the case $h < \frac{3}{2} - m$ in Section 8 when $r = 2$, is more subtle. The strongly divisible modules in these cases have coefficients defined by limits of sequences which satisfy equations given by polynomials of degree 2 (see Section 4.2).

6. The first case: $\bullet = \infty$

In this section, we prove Theorems 5.0.1 and 5.0.5 under the condition $h \geq \frac{1}{2}$. We keep the assumption and the notation as in Section 5. In particular, we let $\delta = \delta_\infty$ and $\Theta = \lambda / p^{m-1}$.

6.1. ***Matrices.*** In this subsection, we study some properties of certain matrices. These matrices will be used later to describe generators of the filtration of our strongly divisible modules. Recall that we let $x_0^{(s,t)} = 1$ for any integers s and t by convention.

We let $\kappa \in E^\times$ and construct a matrix $T_\infty \in \mathrm{M}_{m\times r}(E)$ as follows:

$$T_\infty = \begin{pmatrix} \kappa & 1 & -\frac{1}{2} & \cdots & \frac{(-1)^{m-1}}{m-2} & \frac{(-1)^m}{m-1} & \cdots & \frac{(-1)^r}{r-1} \\ 0 & \kappa & 1 & \cdots & \frac{(-1)^{m-2}}{m-3} & \frac{(-1)^{m-1}}{m-2} & \cdots & \frac{(-1)^{r-1}}{r-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -\frac{1}{2} & \cdots & \frac{(-1)^{m+3}}{m+2} \\ 0 & 0 & 0 & \cdots & \kappa & 1 & \cdots & \frac{(-1)^{m+2}}{m+1} \\ 0 & 0 & 0 & \cdots & 0 & \kappa & \cdots & \frac{(-1)^{m+1}}{m} \end{pmatrix}.$$

We let $P_\infty \in \mathrm{M}_{m\times m}(E)$ be the matrix obtained from the first m columns of T_∞, and let Q_∞ be the matrix obtained from the remaining columns of T_∞. It's easy to see that $Q_\infty = -J_m M'^{(m,1)}$, so that $Q_\infty \in \mathrm{GL}_m(\mathbb{O})$ by Lemma 4.1.1. We let $R_\infty = Q_\infty^{-1} \in \mathrm{GL}_m(\mathbb{O})$.

Lemma 6.1.1. *Keep the notation and the assumptions as above, and let $R = R_\infty$. Then*

$$v_p(R_{i,j}) \geq 0 \quad (1 \leq i, j \leq m).$$

Proof. This is immediate from the fact $Q_\infty = -J_m M'^{(m,1)} \in \mathrm{GL}_m(\mathbb{O})$. □

Lemma 6.1.2. *Keep the notation and the assumptions as above, and let $R = R_\infty$. Then for every $k = 0, \dots, m-1$, we have $R_{m-k,m} = x_k^{(m-1,2)} R_{m,m}$. Moreover, $R_{m,m} z_m = -1$ where*

$$z_m = \sum_{k=0}^{m-1} \frac{(-1)^{m-k}}{m-k} x_k^{(m-1,2)}.$$

Proof. Notice that $M^{(m-1,2)}$ is obtained from $M'^{(m,1)}$ by deleting the first row. Since $M'^{(m,1)} R = -J_m$, it follows that $M^{(m-1,2)}$ multiplied by the last column of R is 0. That is,

$$M^{(m-1,2)} \cdot \begin{pmatrix} R_{1,m} \\ \vdots \\ R_{m,m} \end{pmatrix} = 0.$$

The first part of the statement follows from Lemma 4.1.1.

Now, by definition, we have

$$R_{m,m} z_m = R_{m,m} \sum_{k=0}^{m-1} \frac{(-1)^{m-k}}{m-k} x_k^{(m-1,2)}.$$

By the first part, this is equal to

$$\sum_{k=0}^{m-1} \frac{(-1)^{m-k}}{m-k} R_{m-k,m} = \sum_{j=1}^{m} \frac{(-1)^j}{j} R_{j,m}.$$

Recall that $M'^{(m,1)} R = -J_m$. In particular, the first row of $M'^{(m,1)}$ multiplied by the last column of R is equal to -1. This completes the proof. □

Lemma 6.1.3. *Keep the notation and assumptions as above. Let $R = R_\infty$, $P = P_\infty$, and consider the matrix $RP \in \mathrm{M}_{m\times m}(E)$. If $1 \le i, j \le m$, then*

$$v_p((RP)_{i,j}) \ge \min\{0, v_p(\kappa)\}.$$

Proof. This follows immediately from the shape of P and from Lemma 6.1.1. □

6.2. ***Galois stable lattices.*** In this subsection, we prove Theorem 5.0.1 for the case $h \ge \frac{1}{2}$ when $r = 2m \ge 2$. We keep the notation as in Section 6.1. From the computations in (5.0.2) and in (5.0.3), it is easy to check that

$$\phi(E_1) \equiv \phi(E_2) \equiv N(E_1) \equiv N(E_2) \equiv 0$$

modulo $\mathfrak{m}\mathfrak{M}$. In particular, $\mathfrak{M}$ is stable under ϕ and N.

Let $\kappa = (a(r) - \mathfrak{L})/p$ in T_∞, and

$$A = \begin{pmatrix} pI_m & 0_{m\times m} \\ -pRP & p^m\Theta R \end{pmatrix}.$$

Define vectors $\vec{C}^{(i)} = (C_0^{(i)}, \dots, C_{r-1}^{(i)}) \in E^r$, for $i = 0, \dots, r-1$, such that

$$\begin{pmatrix} p^{r-1}C_{r-1}^{(i)} \\ p^{r-2}C_{r-2}^{(i)} \\ \vdots \\ p^0 C_0^{(i)} \end{pmatrix} = p^i \operatorname{col}_{r-i}(A).$$

Concretely, $C_k^{(i)} = p^{i-k} A_{r-k,r-i}$, so that when $0 \le i \le m-1$ we have

$$C_k^{(i)} = \begin{cases} 0 & \text{if } m \le k \le r-1; \\ p^{i-k+m}\Theta R_{m-k,m-i} & \text{if } 0 \le k \le m-1, \end{cases}$$

and when $m \le i \le r-1$ we have

$$C_k^{(i)} = \begin{cases} p^{i-k+1}(I_m)_{r-k,r-i} & \text{if } m \le k \le r-1; \\ -p^{i-k+1}(RP)_{m-k,r-i} & \text{if } 0 \le k \le m-1. \end{cases}$$

Using these vectors, we define $\hat{F}_i \in \mathrm{Fil}^r \mathfrak{D}$ by the formula $\hat{F}_i = \mathfrak{Y}(\vec{C}^{(i)})$, where $\mathfrak{Y}$ is defined in (5.0.4).

Lemma 6.2.1. *If* $0 \le i \le m-1$ *then*

$$\hat{F}_i = v^{m+i} E_2 + \sum_{k=0}^{m-1} v^k R_{m-k,m-i}\, p^{i+m-k} \Theta\left(\frac{E_1}{p} + \frac{p\mathfrak{L} - (\mathfrak{L} - a(r))}{p\Theta} E_2\right) + \sum_{k=0}^{m-1}\sum_{j=1}^{k} v^k \frac{(-1)^{j+1}}{j} R_{m-k+j,m-i}\, p^{i+m-k} E_2,$$

and if $m \le i \le r-1$ *then*

$$\hat{F}_i = v^i\left(E_1 + \frac{p\mathfrak{L}}{\Theta}E_2\right) - \sum_{k=0}^{m-1} v^k (RP)_{m-k,r-i}\, p^{i+1-k}\left(\frac{E_1}{p} + \frac{p\mathfrak{L}-(\mathfrak{L}-a(r))}{p\Theta}E_2\right) - \sum_{k=0}^{m-1}\sum_{j=1}^{k} v^k \frac{(-1)^{j+1}}{j\Theta}(RP)_{m-k+j,r-i}\, p^{i+1-k} E_2.$$

Proof. By definition, we have

$$\hat{F}_i = \sum_{k=0}^{r-1} v^k\left[C_k^{(i)}\left(\frac{E_1}{p} + \frac{\kappa+\mathfrak{L}}{\Theta}E_2\right) + \sum_{j=1}^{k} \frac{(-1)^{j+1}}{j}\frac{C_{k-j}^{(i)}}{p^j\Theta}E_2\right].$$

It is easy to see that

$$[\hat{F}_i]_{E_1} = \sum_{k=0}^{r-1} v^k \frac{C_k^{(i)}}{p} \tag{6.2.2}$$

and

$$[\hat{F}_i]_{E_2} = \sum_{k=0}^{r-1} v^k\left(C_k^{(i)}\frac{\kappa+\mathfrak{L}}{\Theta} + \sum_{j=1}^{k}\frac{(-1)^{j+1}}{j}\frac{C_{k-j}^{(i)}}{p^j\Theta}\right). \tag{6.2.3}$$

Suppose first that $0 \le i \le m-1$. Applying the definition of $C_k^{(i)}$ to (6.2.2), we get that

$$[\hat{F}_i]_{E_1} = \sum_{k=0}^{m-1} v^k p^{i-k+m-1}\Theta R_{m-k,m-i},$$

which proves that the coefficient of E_1 in $\hat{F}_i$ is as stated in the lemma. On the other hand, Applying the definition of $C_k^{(i)}$ to (6.2.3), we get

$$\sum_{k=0}^{m-1} v^k p^{i-k+m}(\kappa+\mathfrak{L})R_{m-k,m-i} + \sum_{k=0}^{r-1} v^k p^{i-k+m} \sum_{\substack{j=1\\ k-j\le m-1}}^{k} \frac{(-1)^{j+1}}{j} R_{m-k+j,m-i}.$$

Thus, to get the formula in the statement of the lemma, it's enough to show that

$$\sum_{k=0}^{r-1} v^k p^{i-k+m} \sum_{\substack{j=1\\ k-j\leq m-1}}^{k} \frac{(-1)^{j+1}}{j} R_{m-k+j,m-i} = v^{i+m} + \sum_{k=0}^{m-1} v^k p^{i-k+m} \sum_{j=1}^{k} \frac{(-1)^{j+1}}{j} R_{m-k+j,m-i}.$$

We write this as the sum of two terms in the following way:

$$\sum_{k=0}^{m-1} v^k p^{i-k+m} \sum_{j=1}^{k} \frac{(-1)^{j+1}}{j} R_{m-k+j,m-i} + \sum_{k=m}^{r-1} v^k p^{i-k+m} \sum_{j=k-m+1}^{k} \frac{(-1)^{j+1}}{j} R_{m-k+j,m-i}.$$

Thus, it's enough to show that

$$\sum_{k=m}^{r-1} v^k p^{i-k+m} \sum_{j=k-m+1}^{k} \frac{(-1)^{j+1}}{j} R_{m-k+j,m-i} = v^{m+i}. \tag{6.2.4}$$

Now, note that for any $j \geq 1$ and any $k \geq m$, we can write

$$\frac{(-1)^{j+1}}{j} = Q_{r-k,m-k+j}. \tag{6.2.5}$$

Hence, we see that

$$\sum_{j=k-m+1}^{k} \frac{(-1)^{j+1}}{j} R_{m-k+j,m-i} = \sum_{j=1}^{m} Q_{r-k,j} R_{j,m-i} = (I_m)_{r-k,m-i},$$

from where (6.2.4) follows. This finishes the proof of the formula for $\hat{F}_i$ in the case $0 \leq i \leq m-1$.

Suppose from now on that $m \leq i \leq r-1$. Applying the definition of $C_k^{(i)}$ to (6.2.2), we get that

$$[\hat{F}_i]_{E_1} = -\sum_{k=0}^{m-1} v^k p^{i-k} (RP)_{m-k,r-i} + \sum_{k=m}^{r-1} v^k p^{i-k} (I_m)_{r-k,r-i},$$

which proves that the coefficient of E_1 in $\hat{F}_i$ is as stated in the lemma. On the other hand, we apply the definition of $C_k^{(i)}$ to (6.2.3), and then we split the formula of

$[\hat{F}_i]_{E_2}$ into four terms as follows:

$$(6.2.6)\quad -\sum_{k=0}^{m-1} v^k \frac{p^{i-k+1}}{\Theta}(\kappa+\mathfrak{L})(RP)_{m-k,r-i} + v^i \frac{p(\kappa+\mathfrak{L})}{\Theta} - \sum_{k=0}^{r-1} v^k \frac{p^{i-k+1}}{\Theta} \sum_{\substack{j=1\\ k-j\le m-1}}^{k} \frac{(-1)^{j+1}}{j}(RP)_{m-k+j,r-i} + \sum_{k=0}^{r-1} v^k \frac{p^{i-k+1}}{\Theta} \sum_{\substack{j=1\\ k-j\ge m}}^{k} \frac{(-1)^{j+1}}{j}(I_m)_{r-k+j,r-i}.$$

Using the identity (6.2.5), as well as using the facts that $QRP = P$ and that $P_{r-k,r-i} = 0$ if $k < i$ and $P_{r-i,r-i} = \kappa$, we get

$$-\sum_{k=m}^{r-1} v^k \frac{p^{i-k+1}}{\Theta} \sum_{\substack{j=1\\ k-j\le m-1}}^{k} \frac{(-1)^{j+1}}{j}(RP)_{m-k+j,r-i} = -v^i \frac{p\kappa}{\Theta} - \sum_{k=i+1}^{r-1} v^k \frac{p^{i-k+1}}{\Theta} P_{r-k,r-i}.$$

It then follows that (6.2.6) is equal to

$$-\sum_{k=0}^{m-1} v^k \frac{p^{i-k+1}}{\Theta}\left((\kappa+\mathfrak{L})(RP)_{m-k+j,r-i} + \sum_{j=1}^{k} \frac{(-1)^{j+1}}{j}(RP)_{m-k+j,r-i}\right) + \sum_{k=0}^{r-1} v^k \frac{p^{i-k+1}}{\Theta} \sum_{\substack{j=1\\ k-j\ge m}}^{k} \frac{(-1)^{j+1}}{j}(I_m)_{r-k+j,r-i} - \sum_{k=i+1}^{r-1} v^k \frac{p^{i-k+1}}{\Theta} P_{r-k,r-i}.$$

It's easy to see that the last two terms cancel each other, using that

$$P_{r-k,r-i} = \frac{(-1)^{k-i+1}}{k-i}$$

for $i < k$. This ends the proof of the lemma. ☐

Lemma 6.2.7. *Every element $x \in \mathrm{Fil}^r \mathfrak{D}$ can be written as*

$$x = \sum_{i=0}^{r-1} D_i \hat{F}_i + x',$$

where $D_i \in E$ and $x' \in \mathrm{Fil}^r S_E \mathfrak{D}$.

Proof. As we've seen before, any element of $\mathrm{Fil}^r \mathfrak{D}$ can be written, modulo $\mathrm{Fil}^r S_E\mathfrak{D}$ as $\mathfrak{Y} = \mathfrak{Y}(\vec{C})$, with $\vec{C} = (C_0, \ldots, C_{r-1}) \in E^r$, as in (5.0.4). For $i = 0, \ldots, m-1$, let

$$D_i = -\frac{\mathfrak{L} - a(r)}{p\Theta} C_{m+i} + \sum_{j=1}^{m+i} \frac{(-1)^{j+1}}{j} \frac{C_{m+i-j}}{p^j \Theta},$$

and for $i = m, \ldots, r-1$, let

$$D_i = \frac{1}{p} C_i.$$

We can express the C_i in terms of the D_i as follows. It's easy to see that

$$\begin{pmatrix} p^{r-1} D_{r-1} \\ p^{r-2} D_{r-2} \\ \vdots \\ p^0 D_0 \end{pmatrix} = \frac{1}{p^m \Theta} \begin{pmatrix} p^{m-1}\Theta I_m & 0_{m\times m} \\ P & Q \end{pmatrix} \begin{pmatrix} p^{r-1} C_{r-1} \\ p^{r-2} C_{r-2} \\ \vdots \\ p^0 C_0 \end{pmatrix}. \tag{6.2.8}$$

We can then invert the matrix in (6.2.8) and obtain

$$\begin{pmatrix} p^{r-1} C_{r-1} \\ p^{r-2} C_{r-2} \\ \vdots \\ p^0 C_0 \end{pmatrix} = A \begin{pmatrix} p^{r-1} D_{r-1} \\ p^{r-2} D_{r-2} \\ \vdots \\ p^0 D_0 \end{pmatrix}.$$

Now, it follows from the definition of the vectors $\vec{C}^{(i)}$ that

$$\vec{C} = \sum_{i=0}^{r-1} D_i \vec{C}^{(i)}$$

and hence

$$\mathfrak{Y} = \mathfrak{Y}(\vec{C}) = \sum_{i=0}^{r-1} D_i \hat{F}_i. \qquad \square$$

Lemma 6.2.9. *$\hat{F}_i \equiv u^{m+i} E_2$ modulo $\mathfrak{m}\mathfrak{M}$ for $0 \le i \le m-1$ and $\hat{F}_i \equiv u^i E_1$ modulo $\mathfrak{m}\mathfrak{M}$ for $m \le i \le r-1$. In particular, the elements $\hat{F}_i$ belong to* $\mathrm{Fil}^r \mathfrak{M}$.

Proof. Since $\hat{F}_i \in \mathrm{Fil}^r \mathfrak{D}$ by construction, it remains to show that $\hat{F}_i \in \mathfrak{M}$. This is easy to see using that $R \in \mathrm{GL}_m(\mathbb{O})$ and that $v_p((RP)_{k,j}) \ge -\frac{1}{2}$ for any k, j by Lemma 6.1.3 and the fact that $v_p(\kappa) = h - 1 \ge -\frac{1}{2}$. $\square$

Lemma 6.2.10. *Let $D_0, \ldots, D_{r-1} \in E$. Then $\sum_{i=0}^{r-1} D_i \hat{F}_i \in \mathfrak{M}$ if and only if $D_i \in \mathbb{O}$ for every $i = 0, \ldots, r-1$.*

Proof. One direction is obvious from the previous lemma. To prove the other direction, suppose that $y = \sum_{i=0}^{r-1} D_i \hat{F}_i \in \mathfrak{M}$. Then $[y]_{E_1}$ is

$$\sum_{i=0}^{m-1} D_i\biggl(\sum_{k=0}^{m-1} v^k R_{m-k,m-i}\, p^{i+m-k-1}\Theta\biggr) + \sum_{i=m}^{r-1} D_i\biggl(v^i - \sum_{k=0}^{m-1} v^k (RP)_{m-k,r-i}\, p^{i-k}\biggr).$$

By grouping this expression according to the power of v, we see that v^i appears multiplied by D_i when $i = m, \dots, r-1$. Since the expression belongs to $S_{\mathbb{O}}$, we must have $D_i \in \mathbb{O}$ for $i = m, \dots, r-1$. Now, we do the same with $[y]_{E_2}$. The power v^i for $i = m, \dots, r-1$ appears multiplied by $D_{i-m} + D_i p\mathfrak{L}/\Theta$. Since this expression belongs to $S_{\mathbb{O}}$ and

$$v_p\Bigl(D_i \frac{p\mathfrak{L}}{\Theta}\Bigr) \geq 1 + v_p(\mathfrak{L}) - \tfrac{1}{2} \geq 0,$$

we must also have $D_{i-m} \in \mathbb{O}$. This proves that $D_i \in \mathbb{O}$ for any $i = 0, \dots, r-1$. □

From the previous lemmas, it follows that any element $x \in \mathrm{Fil}^r\,\mathfrak{M}$ can be written as $x = \sum_{i=0}^{r-1} D_i \hat{F}_i + x'$ for some $D_i \in \mathbb{O}$, with $x' \in \mathrm{Fil}^r\, S_{\mathbb{O}}\mathfrak{M}$. Moreover, to show that $\phi(\mathrm{Fil}^r\,\mathfrak{M}) \subset p^r\mathfrak{M}$, it is enough to show that $\phi(\hat{F}_i) \in p^r\mathfrak{M}$ for any $i = 0, \dots, r-1$. We state the result as a proposition, and we include some additional information for mod p reduction.

Proposition 6.2.11. *Assume that $h \geq \frac{1}{2}$. The module $\mathfrak{M} = S_{\mathbb{O}}(E_1, E_2)$ is a strongly divisible module with $\hat{F}_i$ generators of* $\mathrm{Fil}^r\,\mathfrak{M}$ *modulo* $\mathrm{Fil}^r\, S_{\mathbb{O}}\mathfrak{M}$. *Moreover, $\phi(\hat{F}_i) \in p^r\mathfrak{m}\mathfrak{M}$ for every $i \neq 0, m$,*

$$\phi(\hat{F}_0) \equiv p\lambda^2 R_{m,m} \sum_{k=0}^{m-1} (\gamma-1)^k x_k^{(m-1,2)} E_1$$

and

$$\begin{aligned}\phi(\hat{F}_m) \equiv{}& p^m\lambda(\mathfrak{L} - a(r)) R_{m,m} \sum_{k=0}^{m-1} (\gamma-1)^k x_k^{(m-1,2)} E_1 \\ &+ p^r\biggl[(\gamma-1)^m\delta - \sum_{k=0}^{m-1}(\gamma-1)^k\biggl(\sum_{i=1}^{m-1} R_{m-k,i} P_{i,m}\delta \\ &\qquad + \sum_{j=1}^{k} \frac{(-1)^{j+1}}{j} \sum_{i=1}^{m-1} R_{m-k+j,i} P_{i,m}\biggr)\biggr] E_2\end{aligned}$$

modulo $p^r\mathfrak{m}\mathfrak{M}$.

The only thing that we need to do to prove that $\mathfrak{M}$ is a strongly divisible module is to show that $\phi(\hat{F}_i) \in p^r\mathfrak{M}$ for every $i = 0, \dots, r-1$. For later use for mod p reduction, we compute $\phi(\hat{F}_i)$ modulo $p^r\mathfrak{m}\mathfrak{M}$, using (5.0.2).

Proof. First suppose that $0 \leq i \leq m-1$. Then

$$[\phi(\hat{F}_i)]_{E_1} = \Theta\lambda p^{i+m} \sum_{k=0}^{m-1} \Big(\frac{\phi(v)}{p}\Big)^k R_{m-k,m-i}.$$

Since $(\phi(v)/p)^k \in S_{\mathbb{O}}$ and $R_{m-k,m-i} \in \mathbb{O}$, we have $[\phi(\hat{F}_i)]_{E_1} \in p^r S_{\mathbb{O}}$, because $v_p(\Theta\lambda p^{i+m}) = r+i$. Moreover, $[\phi(\hat{F}_i)]_{E_1} \in p^r \mathfrak{m} S_{\mathbb{O}}$ if $i \geq 1$, and using that $(\phi(v)/p) \equiv (\gamma - 1) \pmod{pS_{\mathbb{O}}}$, we can write, combining with Lemma 6.1.2,

$$[\phi(\hat{F}_0)]_{E_1} \equiv R_{m,m} p^m \Theta\lambda \sum_{k=0}^{m-1} (\gamma-1)^k x_k^{(m-1,2)} \pmod{p^r \mathfrak{m} S_{\mathbb{O}}}. \tag{6.2.12}$$

Since $\phi(\gamma) \in p^{p-1} S_{\mathbb{O}} \subset p^r S_{\mathbb{O}}$, we get that $a(r) - \phi(\delta) \in p^r S_{\mathbb{O}}$ by Lemma 4.3.6. This implies that

$$[\phi(\hat{F}_i)]_{E_2} \equiv p^{m+i}\lambda\Bigg(\Big(\frac{\phi(v)}{p}\Big)^{m+i} + \delta\sum_{k=0}^{m-1} R_{m-k,m-i}\Big(\frac{\phi(v)}{p}\Big)^k + \sum_{k=0}^{m-1}\sum_{j=1}^{k} \frac{(-1)^{j+1}}{j} R_{m-k+j,m-i}\Big(\frac{\phi(v)}{p}\Big)^k\Bigg)$$

modulo $p^r \mathfrak{m} S_{\mathbb{O}}$. The element between the inner parentheses belongs to $S_{\mathbb{O}}$, so if $i \geq 1$, the whole expression belongs to $p^r \mathfrak{m} S_{\mathbb{O}}$. If $i = 0$, we use the fact that $(\phi(v)/p) \equiv (\gamma - 1) \pmod{pS_{\mathbb{O}}}$, together with Lemma 6.1.2, to get

$$[\phi(\hat{F}_0)]_{E_2} \equiv p^m\lambda\Bigg[(\gamma-1)^m + R_{m,m}\Bigg(\delta\sum_{k=0}^{m-1} x_k(\gamma-1)^k + \sum_{k=0}^{m-1}\sum_{j=1}^{k}\frac{(-1)^{j+1}}{j} x_{k-j}(\gamma-1)^k\Bigg)\Bigg]$$

modulo $p^r \mathfrak{m} S_{\mathbb{O}}$, where $x_i = x_i^{(s,t)}$ with $s = m-1$ and $t = 2$. The expression is 0 by formula (4.3.8) and the second statement of Lemma 6.1.2. This ends the proof that $\phi(\hat{F}_i) \in p^r \mathfrak{M}$ if $i = 0, \dots, m-1$, and combined with (6.2.12), proves the formula in the proposition for $\phi(\hat{F}_0)$ modulo $p^r \mathfrak{m}\mathfrak{M}$.

Suppose now that $m \leq i \leq r-1$. Then

$$[\phi(\hat{F}_i)]_{E_1} = \Big(\frac{\phi(v)}{p}\Big)^i p^{i+1}\lambda - \sum_{k=0}^{m-1}\Big(\frac{\phi(v)}{p}\Big)^k p^{i+1}\lambda (RP)_{m-k,r-i}. \tag{6.2.13}$$

Since $\phi(v)/p \in S_{\mathbb{O}}$, $v_p((RP)_{m-k,r-i}) \geq -\frac{1}{2}$ and $i \geq m$, Equation (6.2.13) belongs to $p^r S_{\mathbb{O}}$. Moreover, it belongs to $p^r \mathfrak{m} S_{\mathbb{O}}$ when $i > m$. When $i = m$, we use Lemma 6.1.2, the fact that $v_p((RP)_{m-k,m} - \kappa R_{m-k,m}) \geq 0$ and $v_p(p^{m+1}\lambda) = r + \frac{1}{2}$

to get

$$[\phi(\hat{F}_m)]_{E_1} \equiv -R_{m,m}p^{m+1}\lambda\kappa\sum_{k=0}^{m-1}(\gamma-1)^k x_k^{(m-1,2)} \tag{6.2.14}$$

modulo $p^r\mathfrak{m}S_{\mathcal{O}}$.

We now use a similar argument as in the previous case to get

$$[\phi(\hat{F}_i)]_{E_2} \equiv \frac{p^{i+1}\lambda}{\Theta}\Bigg[\Big(\frac{\phi(v)}{p}\Big)^i\Big(\frac{\mathfrak{L}-a(r)}{p}+\delta\Big)-\sum_{k=0}^{m-1}(RP)_{m-k,r-i}\Big(\frac{\phi(v)}{p}\Big)^k\delta \\ -\sum_{k=0}^{m-1}\sum_{j=1}^{k}\frac{(-1)^{j+1}}{j}(RP)_{m-k+j,r-i}\Big(\frac{\phi(v)}{p}\Big)^k\Bigg]$$

modulo $p^r\mathfrak{m}S_{\mathcal{O}}$. Note that $v_p(p^{i+1}\lambda/\Theta)=i+m\geq r$ (and strictly larger when $i\geq m+1$). Thus, if $i\geq m+1$, since the valuations of the entries of RP and of $(\mathfrak{L}-a(r))/p$ are at least $-\frac{1}{2}$, we have $[\phi(\hat{F}_i)]_{E_2}\in p^r\mathfrak{m}S_{\mathcal{O}}$.

Suppose now that $i=m$. Since $v_p(p^{m+1}\lambda/\Theta)=r$, we get that

$$[\phi(\hat{F}_m)]_{E_2} \equiv \frac{p^{m+1}\lambda}{\Theta}\Bigg[(\gamma-1)^m(\delta-\kappa)-\delta\sum_{k=0}^{m-1}(RP)_{m-k,m}(\gamma-1)^k \\ -\sum_{k=0}^{m-1}\sum_{j=1}^{k}\frac{(-1)^{j+1}}{j}(RP)_{m-k+j,m}(\gamma-1)^k\Bigg]$$

modulo $p^r\mathfrak{m}S_{\mathcal{O}}$. Note that for all $k\in\{0,1,\ldots,m-1\}$,

$$(RP)_{m-k,m}-\kappa R_{m-k,m}=\sum_{i=1}^{m-1}R_{m-k,i}P_{i,m} \quad\text{and}\quad v_p\Big(\sum_{i=1}^{m-1}R_{m-k,i}P_{im}\Big)\geq 0.$$

We now use that $\phi(v)/p\equiv\gamma-1 \pmod{pS_{\mathcal{O}}}$, $v_p(\kappa)\geq-\frac{1}{2}$ and $v_p(R_{i,j})\geq 0$, together with Lemma 6.1.2, and we get that $[\phi(\hat{F}_m)]_{E_2}$ is congruent to

$$p^r\Bigg[(\gamma-1)^m\delta-\sum_{k=0}^{m-1}(\gamma-1)^k\Big(\sum_{i=1}^{m-1}R_{m-k,i}P_{i,m}\delta+\sum_{j=1}^{k}\frac{(-1)^{j+1}}{j}\sum_{i=1}^{m-1}R_{m-k+j,i}P_{i,m}\Big)\Bigg] \\ -p^r\kappa\Bigg((\gamma-1)^m+R_{m,m}\delta\sum_{k=0}^{m-1}x_k(\gamma-1)^k+R_{m,m}\sum_{k=0}^{m-1}\sum_{j=1}^{k}\frac{(-1)^{j+1}}{j}x_{k-j}(\gamma-1)^k\Bigg)$$

modulo $p^r\mathfrak{m}S_{\mathcal{O}}$. The term multiplied by κ is equal to 0 by definition of $\delta=\delta_\infty$ and by Lemma 6.1.2, which completes the proof. □

6.3. ***Mod p reduction.*** In this subsection, we prove Theorem 5.0.5 in the case $h \geq \frac{1}{2}$. Throughout this subsection, we keep the notation and assumption as in Section 6.2. We first compute the Breuil module corresponding to the mod p reduction of the strongly divisible module $\mathfrak{M}$ in Theorem 5.0.1 when $h \geq \frac{1}{2}$.

Lemma 6.3.1. *The Breuil module* $\mathcal{M} := \mathfrak{M}/(\pi, \mathrm{Fil}^p S)$ *is described as follows: there exists a basis* $\underline{e} := (e_1, e_2)$ *for* $\mathcal{M}$ *and a system of generators* $\underline{f} := (f_1, f_2)$ *for* $\mathrm{Fil}^r \mathcal{M}$ *such that*

- $\mathcal{M} := \bar{S}_{\mathbb{F}}(e_1, e_2)$;
- $\mathrm{Mat}_{\underline{e},\underline{f}}(\mathrm{Fil}^r \mathcal{M}) = \begin{pmatrix} u^m & 0 \\ 0 & u^m \end{pmatrix}$;
- $\mathrm{Mat}_{\underline{e},\underline{f}}(\phi_r) = \begin{pmatrix} \gamma & \beta \\ \alpha & 0 \end{pmatrix}$ *where*

$$\alpha := (-1)^m a(r) - \sum_{k=0}^{m-1} (-1)^k \Bigg(\sum_{i=1}^{m-1} R_{m-k,i} P_{i,m} a(r) + \sum_{j=1}^{k} \frac{(-1)^{j+1}}{j} \sum_{i=1}^{m-1} R_{m-k+j,i} P_{i,m} \Bigg)$$

$$\beta := \frac{\lambda^2 \sum_{k=0}^{m-1} (-1)^{k+1} x_k^{(m-1,2)}}{p^{r-1} \sum_{k=0}^{m-1} (-1)^{m-k}/(m-k)\, x_k^{(m-1,2)}}$$

$$\gamma := \frac{\lambda(\mathfrak{L} - a(r)) \sum_{k=0}^{m-1} (-1)^{k+1} x_k^{(m-1,2)}}{p^m \sum_{k=0}^{m-1} (-1)^{m-k}/(m-k)\, x_k^{(m-1,2)}};$$

- $\mathrm{Mat}_{\underline{e}}(N) = 0_{2\times 2}$.

Proof. We keep the notation as in Section 6.2. We let $e_1 = E_1$ and $e_2 = E_2$ modulo $(\pi, \mathrm{Fil}^p S_{\mathbb{O}})$. We also let $f_1 := \hat{F}_m$ modulo $(\pi, \mathrm{Fil}^p S_{\mathbb{O}})$ and $f_2 := \hat{F}_0$ modulo $(\pi, \mathrm{Fil}^p S_{\mathbb{O}})$. By Lemma 6.2.9, $\hat{F}_i \equiv u^{m+i} E_2$ for $i \in \{0, 1, \ldots, m-1\}$ modulo $\mathfrak{m}\mathfrak{M}$ and $\hat{F}_i \equiv u^i E_1$ for $i \in \{m, m+1, \ldots, r-1\}$ modulo $\mathfrak{m}\mathfrak{M}$. Hence, $\mathrm{Fil}^r \mathcal{M}$ is generated by $f_2 = u^m e_2$ and $f_1 = u^m e_1$. By Proposition 6.2.11 and by Lemma 6.1.2, we get the description for ϕ_r as in the statement. It is obvious that $N(e_i) = 0$ from (5.0.3) since $v_p(\Theta) < 1$ and p divides $N(\delta_\infty)$. □

Note that the proof in Lemma 6.3.1 implies that $\mathrm{Fil}^r \mathfrak{M}$ is generated by $\hat{F}_0$ and $\hat{F}_m$ modulo $\mathrm{Fil}^r S_{\mathbb{O}}\mathfrak{M}$ by Nakayama's lemma.

Proposition 6.3.2. *Let* $h \geq \frac{1}{2}$ *and* $\bar{\rho} := \mathrm{V}_{\mathrm{st}}^r(D)$. *Then*

$$\bar{\rho}^{ss}|_{I_{\mathbb{Q}_p}} \cong \omega^m \oplus \omega^m.$$

In particular, $\bar{\rho}$ *is reducible.*

Proof. This is obvious by Lemmas 6.3.1 and 3.3.7. □

7. The second case: $\bullet \in \{0, 1, \dots, m-2\}$

In this section, we prove Theorems 5.0.1 and 5.0.5 under the condition $-\frac{1}{2} - \ell \le h < \frac{1}{2} - \ell$ for $\ell \in \{0, 1, 2, \dots, m-2\}$. We keep the assumption and the notation as in Section 5. In particular, we let $\delta = \delta_\ell$ and $\Theta = \lambda(\mathfrak{L} - a(r))/p^{m-\ell-1}$. Note that this case occurs only when $r = 2m \ge 2(\ell+2) \ge 4$. We also note that the case $\ell = 0$ is more difficult than others, in which case we need to define limits of sequences to construct δ_0 (see (4.3.11)).

7.1. *Matrices.* In this subsection, we study some properties of certain matrices. These matrices will be used later to describe generators of filtration of our strongly divisible modules.

Fix $\kappa \in E$ with $v_p(\kappa) < 0$ and let x, y be positive integers. We let U be a upper-triangular matrix of size x such that $U - \kappa I_x \in \mathrm{M}_{x\times x}(\mathbb{O})$ is nilpotent, $Q' \in \mathrm{M}_{y\times x}(\mathbb{O})$, $Q''' \in \mathrm{M}_{x\times y}(\mathbb{O})$, and $Q'' \in \mathrm{GL}_y(\mathbb{O})$. We define a square matrix of size $x + y$ as

$$Q := \begin{pmatrix} Q' & Q'' \\ U & Q''' \end{pmatrix}. \tag{7.1.1}$$

Lemma 7.1.2. *Keep the notation and assumptions as above, and let $R = R_\ell$. Then*

$$\det Q \equiv (-1)^{(y+2)x} \kappa^x \det Q'' \pmod{\kappa^{x-1}}.$$

In particular, Q is invertible.

Proof. We induct on x. For a given matrix A, we write $M_{i,j}(A)$ for the (i, j)-minor of A. If $x = 1$ then

$$\det Q = \sum_{i=1}^{1+y} Q_{i,1} M_{i,1}(Q) \equiv \kappa M_{1+y,1}(Q) = (-1)^{y+2} \kappa \det Q''$$

modulo $\mathbb{O}$, since $v_p(\kappa) < 0$. For general x,

$$\det Q = \sum_{i=1}^{x+y} Q_{i,1} M_{i,1}(Q) \equiv \kappa M_{y+1,1}(Q) \pmod{\kappa^{x-1}}$$

by induction hypothesis and by using the assumption $v_p(\kappa) < 0$. By induction hypothesis again,

$$M_{y+1,1}(Q) \equiv (-1)^{y+2} (-1)^{(y+2)(x-1)} \kappa^{x-1} \det Q'' \pmod{\kappa^{x-2}},$$

which completes the proof. $\square$

From now on, fix $r = 2m \geq 4$ an even integer. We let ℓ be an integer such that $0 \leq \ell \leq m-2$. We construct a matrix $T_\ell \in \mathrm{M}_{(m+\ell+1)\times r}(E)$ as follows:

$$T_\ell = \begin{pmatrix} \kappa & 1 & -\frac{1}{2} & \cdots & \dfrac{(-1)^{m+\ell}}{m+\ell-1} & \dfrac{(-1)^{m+\ell+1}}{m+\ell} & \cdots & \dfrac{(-1)^r}{r-1} \\ 0 & \kappa & 1 & \cdots & \dfrac{(-1)^{m+\ell-1}}{m+\ell-2} & \dfrac{(-1)^{m+\ell}}{m+\ell-1} & \cdots & \dfrac{(-1)^{r-1}}{r-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -\dfrac{1}{2} & \cdots & \dfrac{(-1)^{m-\ell+2}}{m-\ell+1} \\ 0 & 0 & 0 & \cdots & \kappa & 1 & \cdots & \dfrac{(-1)^{m-\ell+1}}{m-\ell} \\ 0 & 0 & 0 & \cdots & 0 & \kappa & \cdots & \dfrac{(-1)^{m-\ell}}{m-\ell-1} \end{pmatrix}.$$

Let $P_\ell \in \mathrm{M}_{(m+\ell+1)\times(m-\ell-1)}(E)$ be the matrix obtained from the first $m-\ell-1$ columns of T_ℓ, and Q_ℓ be the matrix obtained from the remaining columns of T_ℓ. Note that Q_ℓ is an example of Q in Lemma 7.1.2. More precisely, we let U_ℓ be the square matrix of size $2\ell+2$ in the lower left corner of Q_ℓ. Note that U_ℓ is upper-triangular. We also let $Q'_\ell \in \mathrm{M}_{(m-\ell-1)\times(2\ell+2)}(\mathbb{O})$ (resp. $Q''_\ell \in \mathrm{M}_{(m-\ell-1)\times(m-\ell-1)}(\mathbb{O})$) be the matrix obtained by using the first $m-\ell-1$ rows and the first $2\ell+2$ (resp. last $m-\ell-1$) columns of Q_ℓ. We let $Q'''_\ell \in \mathrm{M}_{(2\ell+2)\times(m-\ell-1)}(\mathbb{O})$ be the remaining part of Q_ℓ, so we have the same picture for Q_ℓ as in (7.1.1). Note that $Q''_\ell = -J_{m-\ell-1}M'^{(m-\ell-1,2\ell+3)}$. In particular, $Q''_\ell \in \mathrm{GL}_{m-\ell-1}(\mathbb{O})$ by Lemma 4.1.1. We let $R_\ell = Q_\ell^{-1} \in \mathrm{GL}_{m+\ell+1}(E)$.

Lemma 7.1.3. *Keep the notation and assumptions as above, and let $R = R_\ell$. Then*

(1) *provided that* $1 \leq i \leq 2\ell+2$ *and* $1 \leq j \leq m-\ell-1$, $v_p(R_{i,j}) = -v_p(\kappa)$;

(2) *provided that* $1 \leq i \leq 2\ell+2$ *and* $m-\ell \leq j \leq m+\ell+1$,

$$v_p(R_{i,j}) = \begin{cases} -v_p(\kappa) & \text{if } i = j-m+\ell+1, \\ -2v_p(\kappa) & \text{if } i \neq j-m+\ell+1; \end{cases}$$

(3) *provided that* $2\ell+3 \leq i \leq m+\ell+1$ *and* $1 \leq j \leq m-\ell-1$, $v_p(R_{i,j}) = 0$;

(4) *provided that* $2\ell+3 \leq i \leq m+\ell+1$ *and* $m-\ell \leq j \leq m+\ell+1$, $v_p(R_{i,j}) = -v_p(\kappa)$.

Moreover, $R_{i,j} \equiv r_{i,j} \cdot \kappa^{-a_{i,j}}$ *modulo* $(\kappa^{-a_{i,j}-1})$ *for some* $r_{i,j} \in \mathbb{O}^\times$, *where* $a_{i,j} \in \mathbb{Z}$ *are defined by the equation* $v_p(R_{i,j}) = -a_{i,j}v_p(\kappa)$.

Proof. Recall the following well-known identity:

$$R_\ell = \frac{1}{\det Q_\ell}\,\mathrm{Adj}(Q_\ell).$$

By Lemma 7.1.2, $v_p(\det Q_\ell) = 2(\ell+1)v_p(\kappa)$. Hence, for $v_p(R_{i,j})$ it is enough to compute $v_p(M_{j,i}(Q_\ell))$. We recall the definition of $M_{j,i}(Q_\ell)$. It is $(-1)^{i+j}$ times the determinant of the submatrix obtained from Q_ℓ by deleting the j-row and i-th column. But this submatrix can be translated to the shape of Q in Lemma 7.1.2 by interchanging rows and columns. Note that this operations change only the sign of determinant. Hence, by Lemma 7.1.2 $v_p(M_{j,i}(Q_\ell))$ is really $v_p(\kappa)$ times the number of κ that appears in the corresponding submatrix. This completes the proof. The last part is obvious by Lemma 7.1.2. □

Lemma 7.1.4. *Keep the notation and assumptions as above, and let $R = R_\ell$. Then*

$$v_p(R_{m+\ell+1-k,m+\ell+1} - x_k^{(m-\ell-1,2\ell+2)} R_{m+\ell+1,m+\ell+1}) \geq -2v_p(\kappa)$$

for every $k = 0, \ldots, m-\ell-1$.

Moreover, if $\ell = 0$, then we can say a bit more:

$$R_{1,m+1} = \frac{X_0}{\kappa - Y_0} R_{m+1,m+1}$$

and for every $k = 0, \ldots, m-1$

$$R_{m+1-k,m+1} = \left(x_k^{(m-1,2)} + y_k^{(0)} \frac{X_0}{\kappa - Y_0} \right) R_{m+1,m+1},$$

where X_0 and Y_0 are defined in Section 4.2.

Proof. By Lemma 7.1.3, the valuation of the first $2\ell+1$ entries of the last column of R are all greater than or equal to $-2v_p(\kappa)$. Let $\widetilde{Q}''$ be the matrix obtained from the last $m-\ell$ columns and first $m-\ell-1$ rows of Q. Thus, $\widetilde{Q}''$ consist of adjoining the last column of Q' to the left of Q''. We then have

$$\widetilde{Q}'' \begin{pmatrix} R_{2\ell+2,m+\ell+1} \\ R_{2\ell+3,m+\ell+1} \\ \vdots \\ R_{m+\ell+1,m+\ell+1} \end{pmatrix} \equiv 0 \pmod{\kappa^{-2}}.$$

Now, notice that $\widetilde{Q}'' = -J_{m-\ell-1} M^{(m-\ell-1,2\ell+2)}$. Then the first part of the lemma follows from Lemma 4.1.1.

Assume now that $\ell = 0$. Let $x_i = x_i^{(m-1,2)}$, $y_i = y_i^{(0)}$, and $\widetilde{Q}$ be the matrix obtained from Q_0 by deleting the lowest two rows. Then $\widetilde{Q} = -J_{m+1} M^{(0)}$. Thus, by Lemma 4.1.3 we have

$$R_{j,m+1} = x_{m+1-j} R_{m+1,m+1} + y_{m+1-j} R_{1,m+1}$$

for $j = 2, 3, \ldots, m+1$. From $(QR)_{m,m+1} = 0$, we have another identity

$$\kappa R_{1,m+1} + \sum_{j=1}^{m} \frac{(-1)^{j+1}}{j} R_{j+1,m+1} = 0.$$

Combining these identities, we have

$$\left(\kappa - \sum_{j=1}^{m} \frac{(-1)^j}{j} y_{m-j}\right) R_{1,m+1} = \left(\sum_{j=1}^{m} \frac{(-1)^j}{j} x_{m-j}\right) R_{m+1,m+1},$$

which completes the proof. □

Lemma 7.1.5. *Keep the assumptions and notation as above. Let $R = R_\ell$, $P = P_\ell$, and consider the matrix $RP \in \mathrm{M}_{(m+\ell+1)\times(m-\ell-1)}(E)$. We also let $1 \le j \le m-\ell-1$ be arbitrary.*

If $1 \le i \le 2\ell+2$, then

$$v_p((RP)_{i,j}) \ge 0,$$

and if $2\ell+3 \le i \le m+\ell+1$, then

$$v_p((RP)_{i,j}) \ge v_p(\kappa).$$

Proof. This follows immediately from the shape of P and from Lemma 7.1.3. □

Lemma 7.1.6. *Keep the notation and assumptions as above, and let $R = R_\ell$. Then*

$$\sum_{k=0}^{m-\ell-2} (-1)^k R_{m+\ell+1-k,m-\ell-1} \in \mathbb{O}^\times. \tag{7.1.7}$$

Proof. Let $Q'' = Q''_\ell$ and R'' be the submatrix of R determined by

$$R'' = (R_{i,j})_{2\ell+3\le i\le m+\ell+1,\, 1\le j\le m-\ell-1}.$$

Note that $R''_{i,j} = R_{i+2\ell+2,j}$ for all $1 \le i, j \le m-\ell-1$, so that

$$\sum_{k=0}^{m-\ell-2} (-1)^k R_{m+\ell+1-k,m-\ell-1} = \sum_{k=0}^{m-\ell-2} (-1)^k R''_{m-\ell-1-k,m-\ell-1}.$$

Then $Q'', R'' \in \mathrm{M}_{(m-\ell-1)\times(m-\ell-1)}(\mathbb{O})$. It is easy to see that $Q'' = -J_{m-\ell-1} M'^{(s,t)}$ with $s = m-\ell-1$ and $t = 2\ell+3$, so that $Q'' \in \mathrm{GL}_{m-\ell-1}(\mathbb{O})$ by Lemma 4.1.1. Since $QR = I_{m+\ell+1}$, we have $Q''R'' \equiv I_{m-\ell-1}$ modulo $(\mathfrak{m})$ by Lemma 7.1.3. Thus, we may regard Q'' and R'' as matrices in $\mathrm{GL}_{m-\ell-1}(\mathbb{F})$.

We define a linear automorphism T on $\mathbb{F}^{m-\ell-1}$ as follows: if $v \in \mathbb{F}^{m-\ell-1}$, regarding v as a row vector, then $T(v) = v \cdot R''$. We also define a quotient map

$\pi_{m-\ell-1} : \mathbb{F}^{m-\ell-1} \to \mathbb{F}$ sending $(x_1, \ldots, x_{m-\ell-1})$ to $x_{m-\ell-1}$. We let $F = \pi_{m-\ell-1} \circ T$. Then it is immediate to see that (7.1.7) holds if and only if

$$v_0 := ((-1)^{s-1}, (-1)^{s-2}, \ldots, (-1)^{s-(s-1)}, (-1)^{s-s}) \notin \mathrm{Ker}(F).$$

If $\ell = m-2$ then $v_0 = (1)$ and F is an identity map, so that $v_0 \notin \mathrm{Ker}(F)$. For the rest of the proof, we assume that $0 \leq \ell < m-2$. Note that $\mathrm{Ker}(F)$ is generated by $\mathrm{row}_1(Q'')$, $\mathrm{row}_2(Q'')$, $\ldots$, $\mathrm{row}_{m-\ell-2}(Q'')$, since $Q''R'' = I_{m-\ell-1}$ in $\mathrm{GL}_{m-\ell-1}(\mathbb{F})$ as well as $\dim_{\mathbb{F}} \mathrm{Ker}(F) = m-\ell-2$. Note also that if we let S'' be the matrix obtained from Q'' by deleting the last row, then $S'' = -J_{m-\ell-2} M^{(s',t')}$ with $s' = m-\ell-2$ and $t' = 2\ell+4$. By Lemma 4.1.1, S'' is row-equivalent to

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & -1 & x_1 \\ 0 & 0 & \cdots & -1 & 0 & x_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & -1 & \cdots & 0 & 0 & x_{m-\ell-3} \\ -1 & 0 & \cdots & 0 & 0 & x_{m-\ell-2} \end{pmatrix},$$

where $x_i = x_i^{(s',t')}$. Now it is easy to see that $v_0 \in \mathrm{Ker}(F)$ if and only if

$$1 + \sum_{i=1}^{m-\ell-2} (-1)^i x_i = \sum_{i=0}^{m-\ell-2} (-1)^i x_i = 0 \in \mathbb{F}.$$

But this sum is already computed in Lemma 4.1.2. Replacing ℓ with $\ell+1$ in the identity of Lemma 4.1.2, we get

$$\sum_{i=0}^{m-\ell-2} (-1)^i x_i = \frac{(m+\ell+1)!(m-\ell-2)!}{(r-1)!},$$

which is obviously in $\mathbb{F}^\times$ since $r = 2m < p-1$. This completes the proof. □

7.2. ***Galois stable lattices.*** In this subsection, we prove Theorem 5.0.1 for the case $-\frac{1}{2} - \ell \leq h < \frac{1}{2} - \ell$ for $\ell \in \{0, 1, 2, \ldots, m-2\}$. Recall that this case occurs only when $r = 2m \geq 4$. Note that $v_p(\mathfrak{L} - \delta_\ell^{(0)} - p\mathfrak{L}) = h$ for all $\ell \in \{0, 1, 2, \ldots, m-2\}$.

From the computations in (5.0.2) and in (5.0.3), it is easy to check that

$$\phi(E_1) \equiv \phi(E_2) \equiv N(E_1) \equiv N(E_2) \equiv 0$$

modulo $\mathfrak{m}\mathfrak{M}$. In particular, $\mathfrak{M}$ is stable under ϕ and N.

Let $\kappa = (p\mathfrak{L} - (\mathfrak{L} - \delta^{(0)}))/p$ in T_ℓ. We also let $P = P_\ell \in \mathrm{M}_{(m+\ell+1)\times(m-\ell-1)}(E)$, $Q = Q_\ell \in \mathrm{GL}_{m+\ell+1}(E)$, and $R = R_\ell = Q^{-1}$. Note that $v_p(\kappa) = h-1 < 0$. Let

$$A = \begin{pmatrix} pI_{m-\ell-1} & 0_{m-\ell-1\times m+\ell+1} \\ -pRP & p^{m-\ell-1}\Theta R \end{pmatrix}.$$

Define vectors $\vec{C}^{(i)} = (C_0^{(i)}, \dots, C_{r-1}^{(i)}) \in E^r$, for $i = 0, \dots, r-1$, such that

$$\begin{pmatrix} p^{r-1} C_{r-1}^{(i)} \\ p^{r-2} C_{r-2}^{(i)} \\ \vdots \\ p^0 C_0^{(i)} \end{pmatrix} = p^i \operatorname{col}_{r-i}(A).$$

Concretely, $C_k^{(i)} = p^{i-k} A_{r-k,r-i}$, so that when $0 \le i \le m+\ell$ we have

$$C_k^{(i)} = \begin{cases} p^{i-k+m-\ell-1} \Theta R_{m+\ell+1-k,m+\ell+1-i} & \text{if } 0 \le k \le m+\ell; \\ 0 & \text{if } m+\ell+1 \le k \le r-1, \end{cases}$$

and when $m+\ell+1 \le i \le r-1$ we have

$$C_k^{(i)} = \begin{cases} -p^{i-k+1} (RP)_{m+\ell+1-k,r-i} & \text{if } 0 \le k \le m+\ell; \\ p^{i-k+1} (I_{m-\ell-1})_{r-k,r-i} & \text{if } m+\ell+1 \le k \le r-1. \end{cases}$$

Using these vectors, we define $\hat{F}_i \in \operatorname{Fil}^r \mathfrak{D}$ by the formula $\hat{F}_i = \mathfrak{Y}(\vec{C}^{(i)})$, where $\mathfrak{Y}$ is defined in (5.0.4). For simplicity of notation, we will let R' (resp. $(RP)'$) be the matrix with entries $R'_{i,j} = R_{m+\ell+1-i,m+\ell+1-j}$ for $0 \le i, j \le m+\ell$ (resp. $(RP)'_{i,j} = (RP)_{m+\ell+1-i,r-j}$ for $0 \le i \le m+\ell$, $m+\ell+1 \le j \le r-1$).

Lemma 7.2.1. *If* $0 \le i \le m+\ell$ *then*

$$\hat{F}_i = \sum_{k=0}^{m+\ell} v^k p^{i-k+m-\ell-2} \Theta R'_{k,i} E_1 + \left[v^{i+m-\ell-1} + \sum_{k=0}^{m-\ell-2} v^k p^{i-k+m-\ell-1} \left(\kappa R'_{k,i} + \sum_{j=1}^{k} \frac{(-1)^{j+1}}{j} R'_{k-j,i} \right) \right] E_2,$$

and if $m+\ell+1 \le i \le r-1$ *then*

$$\hat{F}_i = \left(v^i - \sum_{k=0}^{m+\ell} v^k p^{i-k} (RP)'_{k,i} \right) E_1 - \sum_{k=0}^{m-\ell-2} v^k \frac{p^{i-k+1}}{\Theta} \left(\kappa (RP)'_{k,i} + \sum_{j=1}^{k} \frac{(-1)^{j+1}}{j} (RP)'_{k-j,i} \right) E_2.$$

Proof. By definition, we have

$$\hat{F}_i = \sum_{k=0}^{r-1} v^k \left[C_k^{(i)} \left(\frac{E_1}{p} + \frac{\kappa}{\Theta} E_2 \right) + \sum_{j=1}^{k} \frac{(-1)^{j+1}}{j} \frac{C_{k-j}^{(i)}}{p^j \Theta} E_2 \right].$$

It is easy to see that

$$[\hat{F}_i]_{E_1} = \sum_{k=0}^{r-1} v^k \frac{C_k^{(i)}}{p} \tag{7.2.2}$$

and

$$[\hat{F}_i]_{E_2} = \sum_{k=0}^{r-1} v^k \left(C_k^{(i)} \frac{\kappa}{\Theta} + \sum_{j=1}^{k} \frac{(-1)^{j+1}}{j} \frac{C_{k-j}^{(i)}}{p^j \Theta} \right). \tag{7.2.3}$$

Suppose first that $0 \le i \le m+\ell$. Applying the definition of $C_k^{(i)}$ to (7.2.2), we get that

$$[\hat{F}_i]_{E_1} = \sum_{k=0}^{m+\ell} v^k p^{i-k+m-\ell-2} \Theta R'_{k,i},$$

which proves that the coefficient of E_1 in $\hat{F}_i$ is as stated in the lemma. Similarly, applying the definition of $C_k^{(i)}$ to (7.2.3), we can write this as

$$\sum_{k=0}^{m+\ell} v^k p^{i-k+m-\ell-1} \kappa R'_{k,i} + \sum_{k=0}^{r-1} v^k p^{i-k+m-\ell-1} \sum_{\substack{j=1 \\ k-j \le m+\ell}}^{k} \frac{(-1)^{j+1}}{j} R'_{k-j,i}.$$

Thus, to get the formula in the statement of the lemma, it's enough to show that

$$\sum_{k=m-\ell-1}^{m+\ell} v^k p^{i-k+m-\ell-1} \kappa R'_{k,i} + \sum_{k=m-\ell-1}^{r-1} v^k p^{i-k+m-\ell-1} \sum_{\substack{j=1 \\ k-j \le m+\ell}}^{k} \frac{(-1)^{j+1}}{j} R'_{k-j,i}$$

is equal to $v^{i+m-\ell-1}$. We write this as the sum of two terms in the following way:

$$\begin{aligned} &\sum_{k=m-\ell-1}^{m+\ell} v^k p^{i-k+m-\ell-1} \left(\kappa R'_{k,i} + \sum_{j=1}^{k} \frac{(-1)^{j+1}}{j} R'_{k-j,i} \right) \\ &\qquad + \sum_{k=m+\ell+1}^{r-1} v^k p^{i-k+m-\ell-1} \sum_{j=k-m-\ell}^{k} \frac{(-1)^{j+1}}{j} R'_{k-j,i}. \end{aligned} \tag{7.2.4}$$

Now, note that for any $j \ge 1$ and any $k \ge m-\ell-1$, we can write

$$\frac{(-1)^{j+1}}{j} = Q_{r-k,m+\ell+1-k+j}. \tag{7.2.5}$$

Hence, we see that

$$\kappa R'_{k,i} + \sum_{j=1}^{k} \frac{(-1)^{j+1}}{j} R'_{k-j,i} = \kappa R_{m+\ell+1-k,m+\ell+1-i} + \sum_{j=m+\ell+2-k}^{m+\ell+1} Q_{r-k,j} R_{j,m+\ell+1-i}.$$

Also notice that $Q_{r-k,m+\ell+1-k}=\kappa$ and $Q_{r-k,j}=0$ if $j<m+\ell+1-k$, so that

$$\kappa R'_{k,i}+\sum_{j=1}^{k}\frac{(-1)^{j+1}}{j}R'_{k-j,i}=\sum_{j=1}^{m+\ell+1}Q_{r-k,j}R_{j,m+\ell+1-i}=(I_{m+\ell+1})_{r-k,m+\ell+1-i}.$$

Similarly,

$$\sum_{j=k-m-\ell}^{k}\frac{(-1)^{j+1}}{j}R'_{k-j,i}=\sum_{j=1}^{m+\ell+1}Q_{r-k,j}R_{j,m+\ell+1-i}=(I_{m+\ell+1})_{r-k,m+\ell+1-i}.$$

It then follows that (7.2.4) is equal to

$$\sum_{k=m-\ell-1}^{r-1}v^k p^{i-k+m-\ell-1}(I_{m+\ell+1})_{r-k,m+\ell+1-i}=v^{i+m-\ell-1},$$

which is what we wanted to show. This finishes the proof of the formula for $\hat{F}_i$ in the case $0\le i\le m+\ell$.

Suppose from now on that $m+\ell+1\le i\le r-1$. Applying the definition of $C_k^{(i)}$ to (7.2.2), we get that

$$[\hat{F}_i]_{E_1}=-\sum_{k=0}^{m+\ell}v^k p^{i-k}(RP)'_{k,i}+\sum_{k=m+\ell+1}^{r-1}v^k p^{i-k}(I_{m-\ell-1})_{r-k,r-i},$$

which proves that the coefficient of E_1 in $\hat{F}_i$ is as stated in the lemma. On the other hand, we apply the definition of $C_k^{(i)}$ to (7.2.3), and then we split the formula of $[\hat{F}_i]_{E_2}$ into four terms as follows:

$$\begin{aligned}(7.2.6)\quad & -\sum_{k=0}^{m+\ell}v^k\frac{p^{i-k+1}}{\Theta}\kappa(RP)'_{k,i}+v^i\frac{p\kappa}{\Theta}\\ &-\sum_{k=0}^{r-1}v^k\frac{p^{i-k+1}}{\Theta}\sum_{\substack{j=1\\ k-j\le m+\ell}}^{k}\frac{(-1)^{j+1}}{j}(RP)'_{k-j,i}\\ &+\sum_{k=0}^{r-1}v^k\frac{p^{i-k+1}}{\Theta}\sum_{\substack{j=1\\ k-j\ge m+\ell+1}}^{k}\frac{(-1)^{j+1}}{j}(I_{m-\ell-1})_{r-k+j,r-i}.\end{aligned}$$

Using the identity (7.2.5) as well as using the facts that $QRP=P$ and that $P_{r-k,r-i}=0$ if $k<i$ and $P_{r-i,r-i}=\kappa$, we get

$$-\sum_{k=m-\ell-1}^{m+\ell}v^k\frac{p^{i-k+1}}{\Theta}\sum_{\substack{j=1\\ k-j\le m+\ell}}^{k}\frac{(-1)^{j+1}}{j}(RP)'_{k-j,i}=\sum_{k=m-\ell-1}^{m+\ell}v^k\frac{p^{i-k+1}}{\Theta}\kappa(RP)'_{k,i}$$

and

$$-\sum_{k=m+\ell+1}^{r-1} v^k \frac{p^{i-k+1}}{\Theta} \sum_{\substack{j=1\\ k-j\leq m+\ell}}^{k} \frac{(-1)^{j+1}}{j}(RP)'_{k-j,i} = -v^i \frac{p\kappa}{\Theta} - \sum_{k=i+1}^{r-1} v^k \frac{p^{i-k+1}}{\Theta} P_{r-k,r-i}.$$

It then follows that (7.2.6) is equal to

$$-\sum_{k=0}^{m-\ell-2} v^k \frac{p^{i-k+1}}{\Theta}\left(\kappa(RP)'_{k,i} + \sum_{j=1}^{k} \frac{(-1)^{j+1}}{j}(RP)'_{k-j,i}\right) + \sum_{k=0}^{r-1} v^k \frac{p^{i-k+1}}{\Theta} \sum_{\substack{j=1\\ k-j\geq m+\ell+1}}^{k} \frac{(-1)^{j+1}}{j}(I_{m-\ell-1})_{r-k+j,r-i} - \sum_{k=i+1}^{r-1} v^k \frac{p^{i-k+1}}{\Theta} P_{r-k,r-i}.$$

It's easy to see that the last two terms cancel each other, using that

$$P_{r-k,r-i} = \frac{(-1)^{k-i+1}}{k-i}$$

for $i < k$. This ends the proof of the lemma. ☐

Lemma 7.2.7. *Every element $x \in \mathrm{Fil}^r \mathfrak{D}$ can be written as*

$$x = \sum_{i=0}^{r-1} D_i \hat{F}_i + x',$$

where $D_i \in E$ and $x' \in \mathrm{Fil}^r S_E\mathfrak{D}$.

Proof. As we've seen before, any element of $\mathrm{Fil}^r \mathfrak{D}$ can be written, modulo $\mathrm{Fil}^r S_E\mathfrak{D}$ as $\mathfrak{Y} = \mathfrak{Y}(\vec{C})$, with $\vec{C} = (C_0, \dots, C_{r-1}) \in E^r$, as in (5.0.4). For $i = 0, \dots, m+\ell$, let

$$D_i = \frac{p\mathfrak{L} - (\mathfrak{L} - \delta^{(0)})}{p\Theta} C_{i+m-\ell-1} + \sum_{j=1}^{i+m-\ell-1} \frac{(-1)^{j+1}}{j} \frac{C_{i+m-\ell-1-j}}{p^j\Theta}.$$

For $i = m+\ell+1, \dots, r-1$, let

$$D_i = \frac{C_i}{p}.$$

We can express the C_i in terms of the D_i as follows. Since

$$\begin{pmatrix} p^{r-1}D_{r-1} \\ p^{r-2}D_{r-2} \\ \vdots \\ p^0 D_0 \end{pmatrix} = \begin{pmatrix} \frac{1}{p} I_{m-\ell-1} & 0_{m-\ell-1\times m+\ell+1} \\ \frac{1}{p^{m-\ell-1}\Theta} P & \frac{1}{p^{m-\ell-1}\Theta} Q \end{pmatrix} \begin{pmatrix} p^{r-1}C_{r-1} \\ p^{r-2}C_{r-2} \\ \vdots \\ p^0 C_0 \end{pmatrix},$$

we have

$$\begin{pmatrix} p^{r-1}C_{r-1} \\ p^{r-2}C_{r-2} \\ \vdots \\ p^0 C_0 \end{pmatrix} = A \begin{pmatrix} p^{r-1}D_{r-1} \\ p^{r-2}D_{r-2} \\ \vdots \\ p^0 D_0 \end{pmatrix}.$$

Now, it follows from the definition of the vectors $\vec{C}^{(i)}$ that

$$\vec{C} = \sum_{i=0}^{r-1} D_i \vec{C}^{(i)}$$

and hence

$$\mathfrak{Y} = \mathfrak{Y}(\vec{C}) = \sum_{i=0}^{r-1} D_i \hat{F}_i. \qquad \square$$

Lemma 7.2.8. *$\hat{F}_i \equiv u^{i+m-\ell-1} E_2$ modulo $\mathfrak{m}\mathfrak{M}$ for $0 \le i \le m+\ell$ and $\hat{F}_i \equiv u^i E_1$ for $m+\ell+1 \le i \le r-1$ modulo $\mathfrak{m}\mathfrak{M}$. In particular, the elements $\hat{F}_i$ belong to* $\mathrm{Fil}^r \mathfrak{M}$.

Proof. Since $\hat{F}_i \in \mathrm{Fil}^r \mathfrak{D}$ by construction, it remains to show that $\hat{F}_i \in \mathfrak{M}$. Suppose first that $0 \le i \le m+\ell$. We have

$$(7.2.9) \qquad [\hat{F}_i]_{E_1} = \sum_{k=0}^{m+\ell} v^k p^{i-k+m-\ell-2} \Theta R_{m+\ell+1-k, m+\ell+1-i}.$$

We divide into four cases for k and i according to Lemma 7.1.3. In case (1), we get

$$v_p(p^{i-k+m-\ell-2}\Theta R'_{k,i}) \ge i-k+m-\tfrac{1}{2} \ge \ell+\tfrac{3}{2} > 0.$$

In case (2) with $i \ne k-m+\ell+1$, we get

$$v_p(p^{i-k+m-\ell-2}\Theta R'_{k,i}) \ge i-k+m-h+\tfrac{1}{2} \ge -\ell-h+\tfrac{1}{2} > 0.$$

In case (2) with $i = k-m+\ell+1$, we get

$$v_p(p^{i-k+m-\ell-2}\Theta R'_{k,i}) \ge \ell+\tfrac{1}{2} > 0.$$

In case (3), we get

$$v_p(p^{i-k+m-\ell-2}\Theta R'_{k,i}) \ge 3\ell+\tfrac{5}{2}+h \ge 2\ell+2 > 0.$$

Finally, in case (4), we get

$$v_p(p^{i-k+m-\ell-2}\Theta R'_{k,i}) \geq i-k+m-\tfrac{1}{2} \geq \ell+\tfrac{3}{2} > 0.$$

Thus we conclude that (7.2.9) belongs to $\mathfrak{m}S_{\mathbb{O}}$. On the other hand,

$$[\hat{F}_i]_{E_2} = v^{i+m-\ell-1} + \sum_{k=0}^{m-\ell-2} v^k p^{i-k+m-\ell-1}\Big(\kappa R'_{k,i} + \sum_{j=1}^{k} \frac{(-1)^{j+1}}{j} R'_{k-j,i}\Big). \tag{7.2.10}$$

Thus, in order to show $[\hat{F}_i]_{E_2} \equiv u^{i+m-\ell-1}$ modulo $\mathfrak{m}S_{\mathbb{O}}$, it's enough to see that

$$v_p(p^{i-k+m-\ell-1}\kappa R_{m+\ell+1-k,m+\ell+1-i}) > 0$$

and

$$v_p(p^{i-k+m-\ell-1}R_{m+\ell+1-k+j,m+\ell+1-i}) > 0$$

for every $k = 0, \dots, m-\ell-2$ and $j = 1, \dots, k$. Both follow by considering cases (3) and (4) of Lemma 7.1.3 for $R_{m+\ell+1-k,m+\ell+1-i}$ and $R_{m+\ell+1-k+j,m+\ell+1-i}$. For instance, consider the element $R_{m+\ell+1-k,m+\ell+1-i}$. Lemma 7.1.3, case (4), says that $v_p(R_{m+\ell+1-k,m+\ell+1-i}) \geq -v_p(\kappa)$ if $i \leq 2\ell+1$, in which case

$$v_p(p^{i-k+m-\ell-1}R_{m+\ell+1-k,m+\ell+1-i}\kappa) \geq i-k+m-\ell-1 \geq 1.$$

On the other hand, if $i \geq 2\ell+2$, then $R_{m+\ell+1-k,m+\ell+1-i}$ falls into case (3), and we have $v_p(R_{m+\ell+1-k,m+\ell+1-i}) \geq 0$. Then

$$v_p(p^{i-k+m-\ell-1}R_{m+\ell+1-k,m+\ell+1-i}\kappa) \geq i-k+m-\ell-1+h-1 \geq \ell+\tfrac{3}{2}.$$

The case of $R_{m+\ell+1-k+j,m+\ell+1-i}$ can be analyzed in a similar fashion.

Now suppose that $m+\ell+1 \leq i \leq r-1$. Then

$$[\hat{F}_i]_{E_1} = v^i - \sum_{k=0}^{m+\ell} v^k p^{i-k}(RP)_{m+\ell+1-k,r-i}. \tag{7.2.11}$$

We now use Lemma 7.1.5 to get that $v_p(p^{i-k}(RP)_{m+\ell+1-k,r-i}) > 0$ for any $k = 0, \dots, m+\ell$, so $[\hat{F}_i]_{E_1} \equiv u^i$ modulo $\mathfrak{m}S_{\mathbb{O}}$. On the other hand,

$$[\hat{F}_i]_{E_2} = \sum_{k=0}^{m-\ell-2} v^k \frac{p^{i-k+1}}{\Theta}\Big(\kappa(RP)'_{k,i} + \sum_{j=1}^{k} \frac{(-1)^{j+1}}{j}(RP)'_{k-j,i}\Big). \tag{7.2.12}$$

We see that in order to show that (7.2.12) is in $\mathfrak{m}S_{\mathbb{O}}$, it's enough to see that

$$v_p\Big(\frac{p^{i-k}}{\Theta}(p\mathfrak{L} - (\mathfrak{L}-\delta^{(0)}))(RP)_{m+\ell+1-k,r-i}\Big) > 0$$

and

$$v_p\Big(\frac{p^{i-k+1}}{\Theta}(RP)_{m+\ell+1-k+j,r-i}\Big) > 0$$

for every $k = 0, \dots, m-\ell-2$ and $j = 1, \dots, k$. Both follow from Lemma 7.1.5. This finishes the proof of the lemma. □

Lemma 7.2.13. *Let $D_0, \dots, D_{r-1} \in E$. Then $\sum_{i=0}^{r-1} D_i \hat{F}_i \in \mathfrak{M}$ if and only if $D_i \in \mathbb{O}$ for every $i = 0, \dots, r-1$.*

Proof. One direction is obvious from the previous lemma. To prove the other direction, suppose that $y = \sum_{i=0}^{r-1} D_i \hat{F}_i \in \mathfrak{M}$. If $0 \le i \le m+\ell$, then the coefficient of $v^{i+m-\ell-1}$ in $[y]_{E_2}$ is D_i. Since $[y]_{E_2} \in S_{\mathbb{O}}$, we have that $D_i \in \mathbb{O}$ for every $i = 0, \dots, m+\ell$. Similarly, if $m+\ell+1 \le i \le r-1$, then the coefficient of v^i in $[y]_{E_1}$ is D_i. Since $[y]_{E_1} \in S_{\mathbb{O}}$, we have that $D_i \in \mathbb{O}$ for every $i = m+\ell+1, \dots, r-1$. □

From the previous lemmas, it follows that any element $x \in \mathrm{Fil}^r \mathfrak{M}$ can be written as $x = \sum_{i=0}^{r-1} D_i \hat{F}_i + x'$ for some $D_i \in \mathbb{O}$, with $x' \in \mathrm{Fil}^r S_{\mathbb{O}}\mathfrak{M}$. Moreover, to show that $\phi(\mathrm{Fil}^r \mathfrak{M}) \subset p^r \mathfrak{M}$, it is enough to show that $\phi(\hat{F}_i) \in p^r \mathfrak{M}$ for any $i = 0, \dots, r-1$. We state the result as a proposition, and we include some additional information for mod p reduction.

Proposition 7.2.14. *Assume that $-\frac{1}{2}-\ell \le h < \frac{1}{2}-\ell$ for $\ell \in \{0, 1, \dots, m-2\}$. Then $\mathfrak{M}$ is a strongly divisible module with $\hat{F}_i$ generators for* $\mathrm{Fil}^r \mathfrak{M}$ *modulo* $\mathrm{Fil}^r S_{\mathbb{O}}\mathfrak{M}$. *Moreover, $\phi(\hat{F}_i) \in p^r \mathfrak{m}\mathfrak{M}$ for every $i \ne 0, m+\ell+1$,*

$$\phi(\hat{F}_0) \equiv \lambda^2(\mathfrak{L} - a(r)) R_{m+\ell+1,m+\ell+1} \sum_{k=0}^{m-\ell-1} (\gamma-1)^k x_k^{(m-\ell-1,2\ell+2)} E_1$$

and

$$\begin{aligned}\phi(\hat{F}_{m+\ell+1}) \equiv\ & p^{m+\ell+1}\lambda(\mathfrak{L} - a(r)) \sum_{k=0}^{m-\ell-2} (\gamma-1)^k R_{m+\ell+1-k,m-\ell-1} E_1 \\ & + p^r \Bigg[(\gamma-1)^{m+\ell+1} + \frac{\mathfrak{L} - a(r)}{p} \sum_{k=m-\ell-1}^{m+\ell} (\gamma-1)^k R_{m+\ell+1-k,m-\ell-1} \\ & + \sum_{k=0}^{m-\ell-2} (\gamma-1)^k \Bigg(\delta_\ell R_{m+\ell+1-k,m-\ell-1} \\ & + \sum_{j=1}^{k} \frac{(-1)^{j+1}}{j} R_{m+\ell+1-k+j,m-\ell-1} \Bigg) \Bigg] E_2\end{aligned}$$

modulo $p^r \mathfrak{m}\mathfrak{M}$.

The case $\ell = 0$ requires an extra work, which is the main reason why we first prove the following lemma for Proposition 7.2.14. This lemma will explain why we need to construct δ_0 using limits of sequences (see (4.3.11)).

Lemma 7.2.15. *Assume that* $\ell = 0$, *i.e.*, $-\frac{1}{2} \le h < \frac{1}{2}$. *Then*

$$[\phi(\hat{F}_0)]_{E_2} \in p^r \mathfrak{m} S_{\mathbb{O}}. \tag{7.2.16}$$

Proof. It is routine to check that

$$\begin{aligned}[\phi(\hat{F}_0)]_{E_2} &= p^{m-1}\lambda\Bigg[\Big(\frac{\phi(v)}{p}\Big)^{m-1} + \sum_{k=0}^{m}\Big(\frac{\phi(v)}{p}\Big)^k R'_{k,0}\frac{\mathfrak{L}-\phi(\delta)-p\mathfrak{L}+p\delta}{p} \\ &\quad + \sum_{k=0}^{m-2}\Big(\frac{\phi(v)}{p}\Big)^k\Big(\kappa R'_{k,0} + \sum_{j=1}^{k}\frac{(-1)^{j+1}}{j}R'_{k-j,0}\Big)\Bigg].\end{aligned} \tag{7.2.17}$$

Since $\delta \equiv \delta^{(0)} \pmod{\gamma S_{\mathbb{O}}}$ and $\phi(\gamma) \in p^{p-1}S_{\mathbb{O}}$, we get $\phi(\delta) \equiv \delta^{(0)} \pmod{p^{p-1}S_{\mathbb{O}}}$. Note that, by Lemma 7.1.3, $v_p(R_{m+1-k,m+1}) \ge 0$. Since $r < p-1$, we get that

$$v_p(R_{m+1-k,m+1}p^{m-2+p-1}\lambda) > r,$$

and thus from (7.2.17) we get

$$\begin{aligned}[\phi(\hat{F}_0)]_{E_2} \equiv p^{m-1}\lambda\Bigg[\Big(\frac{\phi(v)}{p}\Big)^{m-1} &- \sum_{k=0}^{m}\Big(\frac{\phi(v)}{p}\Big)^k\kappa R'_{k,0} + \delta\sum_{k=0}^{m}\Big(\frac{\phi(v)}{p}\Big)^k R'_{k,0} \\ &+ \sum_{k=0}^{m-2}\Big(\frac{\phi(v)}{p}\Big)^k\Big(\kappa R'_{k,0} + \sum_{j=1}^{k}\frac{(-1)^{j+1}}{j}R'_{k-j,0}\Big)\Bigg]\end{aligned}$$

modulo $p^r\mathfrak{m}S_{\mathbb{O}}$. Rearranging the terms, we can write this as

$$\begin{aligned}[\phi(\hat{F}_0)]_{E_2} &\equiv p^{m-1}\lambda\Bigg[\Big(\frac{\phi(v)}{p}\Big)^{m-1} - \sum_{k=m-1}^{m}\Big(\frac{\phi(v)}{p}\Big)^k\kappa R'_{k,0} \\ &\quad + \delta\sum_{k=0}^{m}\Big(\frac{\phi(v)}{p}\Big)^k R'_{k,0} + \sum_{k=0}^{m-2}\Big(\frac{\phi(v)}{p}\Big)^k\sum_{j=1}^{k}\frac{(-1)^{j+1}}{j}R'_{k-j,0}\Bigg]\end{aligned} \tag{7.2.18}$$

modulo $p^r\mathfrak{m}S_{\mathbb{O}}$.

Notice that if $m-1 \le k \le m$, then

$$\kappa R'_{k,0} = Q_{r-k,m+1-k}R_{m+1-k,m+1}.$$

Using this, and the fact that $QR = I_{m+1}$, we get

$$\kappa R'_{k,0} = (I_{m+1})_{r-k,m+1} - \sum_{j=1}^{k}\frac{(-1)^{j+1}}{j}R'_{k-j,0}.$$

Thus, (7.2.18) is equal to

$$[\phi(\hat{F}_0)]_{E_2} \equiv p^{m-1}\lambda\biggl(\delta\sum_{k=0}^{m}\Bigl(\frac{\phi(v)}{p}\Bigr)^k R'_{k,0}+\sum_{k=0}^{m}\Bigl(\frac{\phi(v)}{p}\Bigr)^k\sum_{j=1}^{k}\frac{(-1)^{j+1}}{j}R'_{k-j,0}\biggr)$$

modulo $p^r\mathfrak{m}S_{\mathbb{O}}$. Since $\phi(v)/p\equiv(\gamma-1) \pmod{pS_{\mathbb{O}}}$, we also get

$$(7.2.19)\quad [\phi(\hat{F}_0)]_{E_2} \equiv p^{m-1}\lambda\biggl(\delta\sum_{k=0}^{m}(\gamma-1)^k R'_{k,0}+\sum_{k=0}^{m}(\gamma-1)^k\sum_{j=1}^{k}\frac{(-1)^{j+1}}{j}R'_{k-j,0}\biggr)$$

modulo $p^r\mathfrak{m}S_{\mathbb{O}}$.

Applying Lemma 7.1.4 to (7.2.19), we get

$$(7.2.20)\quad [\phi(\hat{F}_0)]_{E_2} \equiv p^{m-1}\lambda R'_{0,0}\Biggl[\delta\Bigl(\frac{X_0}{\kappa-Y_0}(\gamma-1)^m+\sum_{k=0}^{m-1}\Bigl(x_k+y_k\frac{X_0}{\kappa-Y_0}\Bigr)(\gamma-1)^k\Bigr) + \sum_{k=0}^{m}(\gamma-1)^k\sum_{j=1}^{k}\frac{(-1)^{j+1}}{j}\Bigl(x_{k-j}+y_{k-j}\frac{X_0}{\kappa-Y_0}\Bigr)\Biggr]$$

modulo $p^r\mathfrak{m}S_{\mathbb{O}}$. Here, we write x_i and y_i for $x_i^{(m-1,2)}$ and $y_i^{(0)}$ respectively, to lighten the notation. Recall that Z_0 is defined in Section 4.3. It is easy to check that

$$\frac{X_0}{\kappa-Y_0}=-\frac{p}{\mathfrak{L}-a(r)}Z_0$$

using the identity (4.3.13) in Lemma 4.3.12. Thus, the quantity in (7.2.20) vanishes by definition of $\delta=\delta_0$. That is, $[\phi(\hat{F}_0)]_{E_2}\in p^r\mathfrak{m}S_{\mathbb{O}}$ when $\ell=0$. □

We now prove Proposition 7.2.14. The only thing that we need to do to prove that $\mathfrak{M}$ is a strongly divisible module is to show that $\phi(\hat{F}_i)\in p^r\mathfrak{M}$ for every $i=0,\dots,r-1$. For later use for mod p reduction, we compute $\phi(\hat{F}_i)$ modulo $p^r\mathfrak{m}\mathfrak{M}$.

Proof of Proposition 7.2.14. Suppose first that $0\le i\le m+\ell$. Then

$$(7.2.21)\qquad [\phi(\hat{F}_i)]_{E_1}=p^{i+m-\ell-1}\Theta\lambda\sum_{k=0}^{m+\ell}\Bigl(\frac{\phi(v)}{p}\Bigr)^k R_{m+\ell+1-k,m+\ell+1-i}.$$

Note that

$$v_p(p^{i+m-\ell-1}\Theta\lambda R_{m+\ell+1-k,m+\ell+1-i})=i+r+h-1+v_p(R_{m+\ell+1-k,m+\ell+1-i}).$$

Using Lemma 7.1.3, it's then easy to see that $[\phi(\hat{F}_i)]_{E_1} \in p^r S_{\mathbb{O}}$. Moreover, it's also easy to see that $[\phi(\hat{F}_i)]_{E_1} \in p^r \mathfrak{m} S_{\mathbb{O}}$ when $i \geq 1$, whereas

$$[\phi(\hat{F}_0)]_{E_1} \equiv p^{m-\ell-1}\Theta\lambda \sum_{k=0}^{m-\ell-1} \left(\frac{\phi(v)}{p}\right)^k R_{m+\ell+1-k,m+\ell+1} \pmod{p^r \mathfrak{m} S_{\mathbb{O}}}.$$

Moreover, since $\phi(v)/p \equiv (\gamma - 1) \pmod{p S_{\mathbb{O}}}$, we have

$$[\phi(\hat{F}_0)]_{E_1} \equiv p^{m-\ell-1}\Theta\lambda \sum_{k=0}^{m-\ell-1} (\gamma - 1)^k R_{m+\ell+1-k,m+\ell+1} \pmod{p^r \mathfrak{m} S_{\mathbb{O}}}.$$

Now, Lemma 7.1.4 implies that

$$v_p(R_{m+\ell+1-k,m+\ell+1} - x_k^{(m-\ell-1,2\ell+2)} R_{m+\ell+1,m+\ell+1}) \geq 2 - 2h > 1 - h.$$

Also, $v_p(p^{m-\ell-1}\Theta\lambda) = r + h - 1$, so we get

$$[\phi(\hat{F}_0)]_{E_1} \equiv R_{m+\ell+1,m+\ell+1} p^{m-\ell-1}\Theta\lambda \sum_{k=0}^{m-\ell-1} (\gamma - 1)^k x_k^{(m-\ell-1,2\ell+2)} \tag{7.2.22}$$

modulo $p^r \mathfrak{m} S_{\mathbb{O}}$.

On the other hand,

$$\begin{aligned}
[\phi(\hat{F}_i)]_{E_2} &\\
= p^{i+m-\ell-1}\lambda \Bigg[&\left(\frac{\phi(v)}{p}\right)^{i+m-\ell-1} + \sum_{k=0}^{m+\ell} \left(\frac{\phi(v)}{p}\right)^k R'_{k,i} \frac{\mathfrak{L} - \phi(\delta) - p\mathfrak{L} + p\delta}{p} \\
&+ \sum_{k=0}^{m-\ell-2} \left(\frac{\phi(v)}{p}\right)^k \left(\kappa R'_{k,i} + \sum_{j=1}^{k} \frac{(-1)^{j+1}}{j} R'_{k-j,i}\right)\Bigg].
\end{aligned} \tag{7.2.23}$$

Since $\phi(\gamma) \in p^{p-1} S_{\mathbb{O}}$, $\phi(\delta) \equiv \delta^{(0)} \pmod{p^{p-1} S_{\mathbb{O}}}$. Note that

$$v_p(R_{m+\ell+1-k,m+\ell+1-i}) \geq 0,$$

by Lemma 7.1.3. As $r < p-1$, we get $v_p(R_{m+\ell+1-k,m+\ell+1-i}\, p^{i+m-\ell-2+p-1}\lambda) > r$, and thus from (7.2.23) we get

$$\begin{aligned}
[\phi(\hat{F}_i)]_{E_2} \equiv p^{i+m-\ell-1}\lambda \Bigg[&\left(\frac{\phi(v)}{p}\right)^{i+m-\ell-1} - \sum_{k=0}^{m+\ell} \left(\frac{\phi(v)}{p}\right)^k \kappa R'_{k,i} + \delta \sum_{k=0}^{m+\ell} \left(\frac{\phi(v)}{p}\right)^k R'_{k,i} \\
&+ \sum_{k=0}^{m-\ell-2} \left(\frac{\phi(v)}{p}\right)^k \left(\kappa R'_{k,i} + \sum_{j=1}^{k} \frac{(-1)^{j+1}}{j} R'_{k-j,i}\right)\Bigg]
\end{aligned}$$

modulo $p^r\mathfrak{m}S_{\mathbb{O}}$. Rearranging the terms, we can write this as

$$[\phi(\hat{F}_i)]_{E_2} \equiv p^{i+m-\ell-1}\lambda\Bigg[\Big(\frac{\phi(v)}{p}\Big)^{i+m-\ell-1} - \sum_{k=m-\ell-1}^{m+\ell}\Big(\frac{\phi(v)}{p}\Big)^k \kappa R'_{k,i} + \delta\sum_{k=0}^{m+\ell}\Big(\frac{\phi(v)}{p}\Big)^k R'_{k,i} + \sum_{k=0}^{m-\ell-2}\Big(\frac{\phi(v)}{p}\Big)^k \sum_{j=1}^{k}\frac{(-1)^{j+1}}{j}R'_{k-j,i}\Bigg]$$

modulo $p^r\mathfrak{m}S_{\mathbb{O}}$. Note that the case $i=0$ and $\ell=0$ is already treated in Lemma 7.2.15. So we assume that $1\le \ell \le m-2$ for $[\phi(\hat{F}_i)]_{E_2}$ if $i=0$. Using Lemma 7.1.3, we see that $v_p(p^{i+m-\ell-1}\lambda R'_{k,i}) > r$ when $m-\ell\le k\le m+\ell$, and thus we can remove some of the summands multiplying δ. We get

$$[\phi(\hat{F}_i)]_{E_2} \equiv p^{i+m-\ell-1}\lambda\Bigg[\Big(\frac{\phi(v)}{p}\Big)^{i+m-\ell-1} - \sum_{k=m-\ell-1}^{m+\ell}\Big(\frac{\phi(v)}{p}\Big)^k \kappa R'_{k,i} + \delta\sum_{k=0}^{m-\ell-1}\Big(\frac{\phi(v)}{p}\Big)^k R'_{k,i} + \sum_{k=0}^{m-\ell-2}\Big(\frac{\phi(v)}{p}\Big)^k \sum_{j=1}^{k}\frac{(-1)^{j+1}}{j}R'_{k-j,i}\Bigg] \tag{7.2.24}$$

modulo $p^r\mathfrak{m}S_{\mathbb{O}}$. Suppose that $2\ell+2\le i\le m+\ell$. It's easy to see, again using Lemma 7.1.3, that $[\phi(\hat{F}_i)]_{E_2}\in p^r\mathfrak{m}S_{\mathbb{O}}$. Suppose now that $1\le i\le 2\ell+1$. After multiplying by $p^{i+m-\ell-1}\lambda$, the terms in the second line of (7.2.24) are in $p^r\mathfrak{m}S_{\mathbb{O}}$. Similarly, in the sum at the end of the first line, we can ignore the terms with $k\ne i+m-\ell-1$. Thus

$$[\phi(\hat{F}_i)]_{E_2} \equiv p^{i+m-\ell-1}\lambda\Big(\frac{\phi(v)}{p}\Big)^{i+m-\ell-1}(1-\kappa R_{2\ell+2-i,m+\ell+1-i}) \pmod{p^r\mathfrak{m}S_{\mathbb{O}}}. \tag{7.2.25}$$

Using that $QR = I_{m+\ell+1}$, we know that 1 is equal to the $m+\ell+1-i$-th row of Q, multiplied by the $m+\ell+1-i$-th column of R, so

$$1 = \kappa R_{2\ell+2-i,m+\ell+1-i} + \sum_{j=1}^{i+m-\ell-1}\frac{(-1)^{j+1}}{j}R_{2\ell+2-i+j,m+\ell+1-i}.$$

Using Lemma 7.1.3, this implies that the expression in (7.2.25) is in $p^r\mathfrak{m}S_{\mathbb{O}}$.

To sum up, we have thus far proved that $\phi(\hat{F}_i)\in p^r\mathfrak{m}\mathfrak{M}$ for $1\le i\le m+\ell$. Suppose from now on that $i=0$ (and so we also assume $\ell>0$ by Lemma 7.2.15).

The coefficient of E_1 in $\phi(\hat{F}_0)$ has already been dealt above. From (7.2.24), we get

$$[\phi(\hat{F}_0)]_{E_2} \equiv p^{m-\ell-1}\lambda\Bigg[\Big(\frac{\phi(v)}{p}\Big)^{m-\ell-1} - \sum_{k=m-\ell-1}^{m+\ell}\Big(\frac{\phi(v)}{p}\Big)^{k}\kappa R'_{k,0} + \delta\sum_{k=0}^{m-\ell-1}\Big(\frac{\phi(v)}{p}\Big)^{k}R'_{k,0} + \sum_{k=0}^{m-\ell-2}\Big(\frac{\phi(v)}{p}\Big)^{k}\sum_{j=1}^{k}\frac{(-1)^{j+1}}{j}R'_{k-j,0}\Bigg]$$

modulo $p^r\mathfrak{m}S_{\mathcal{O}}$. Now, notice that if $m-\ell-1 \le k \le m+\ell$, then

$$\kappa R'_{k,0} = Q_{r-k,m+\ell+1-k}R_{m+\ell+1-k,m+\ell+1}.$$

Using this, and the fact that $QR = I_{m+\ell+1}$, we get

$$\kappa R'_{k,0} = (I_{m+\ell+1})_{r-k,m+\ell+1} - \sum_{j=1}^{k}\frac{(-1)^{j+1}}{j}R'_{k-j,0}.$$

Thus,

$$[\phi(\hat{F}_0)]_{E_2} \equiv p^{m-\ell-1}\lambda\Bigg(\delta\sum_{k=0}^{m-\ell-1}\Big(\frac{\phi(v)}{p}\Big)^{k}R'_{k,0} + \sum_{k=0}^{m+\ell}\Big(\frac{\phi(v)}{p}\Big)^{k}\sum_{j=1}^{k}\frac{(-1)^{j+1}}{j}R'_{k-j,0}\Bigg) \tag{7.2.26}$$

modulo $p^r\mathfrak{m}S_{\mathcal{O}}$. If $k-j \ge m-\ell$, then $v_p(R'_{k-j,0}) \ge 2-2h$ by Lemma 7.1.3, which implies that these terms can be ignored in the last sum. When $k-j \le m-\ell-1$, we can use Lemma 7.1.4 to get $v_p(R'_{k-j,0} - R'_{0,0}x_{k-j}) \ge 2-2h$, where $x_i = x_i^{(s,t)}$, with $s = m-\ell-1$ and $t = 2\ell+2$. We do the same with the first sum. Finally, $\phi(v)/p \equiv (\gamma-1) \pmod{pS_{\mathcal{O}}}$, and this easily implies that

$$[\phi(\hat{F}_0)]_{E_2} \equiv p^{m-\ell-1}\lambda R'_{0,0}\Bigg(\delta\sum_{k=0}^{m-\ell-1}(\gamma-1)^k x_k + \sum_{k=0}^{m+\ell}(\gamma-1)^k\sum_{\substack{j=1\\ k-j\le m-\ell-1}}^{k}\frac{(-1)^{j+1}}{j}x_{k-j}\Bigg) \tag{7.2.27}$$

modulo $p^r\mathfrak{m}S_{\mathcal{O}}$. Since $\delta = \delta_\ell$ it follows that (7.2.27) is equal to 0. This finishes the proof that $\phi(\hat{F}_0) \in p^r\mathfrak{M}$, and the formula for this element modulo $p^r\mathfrak{m}\mathfrak{M}$ follows from (7.2.22) and the above computation.

Suppose from now on that $m+\ell+1 \le i \le r-1$. Then

$$[\phi(\hat{F}_i)]_{E_1} = p^{i+1}\lambda\Bigg(\Big(\frac{\phi(v)}{p}\Big)^{i} - \sum_{k=0}^{m+\ell}\Big(\frac{\phi(v)}{p}\Big)^{k}(RP)'_{k,i}\Bigg).$$

Using Lemma 7.1.5, it's easy to see that $[\phi(\hat{F}_i)]_{E_1} \in p^r S_{\mathbb{O}}$. Moreover, if $i > m+\ell+1$, then $[\phi(\hat{F}_i)]_{E_1} \in p^r \mathfrak{m} S_{\mathbb{O}}$. We also get that

$$[\phi(\hat{F}_{m+\ell+1})]_{E_1} \equiv -p^{m+\ell+2}\lambda \sum_{k=0}^{m-\ell-2} (\gamma-1)^k (RP)_{m+\ell+1-k,m-\ell-1}$$

modulo $p^r \mathfrak{m} S_{\mathbb{O}}$. We now write

$$(RP)_{m+\ell+1-k,m-\ell-1} = \sum_{j=1}^{m-\ell-1} R_{m+\ell+1-k,j} P_{j,m-\ell-1}.$$

When $j = m-\ell-1$, $P_{j,m-\ell-1} = \kappa$, and $v_p(R_{m+\ell+1-k,j}) \geq 0$ by Lemma 7.1.3. When $j < m-\ell-1$, $P_{j,m-\ell-1} \in \mathbb{O}$, and $v_p(R_{m+\ell+1-k,j}) \geq 0$ by the same lemma. It follows that

$$v_p((RP)_{m+\ell+1-k,m-\ell-1} - R_{m+\ell+1-k,m-\ell-1}\kappa) \geq 0.$$

Since $v_p(p^{m+\ell+2}\lambda) = r + \ell + \frac{3}{2} > r$, we get that

$$[\phi(\hat{F}_{m+\ell+1})]_{E_1} \equiv -p^{m+\ell+2}\lambda\kappa \sum_{k=0}^{m-\ell-2} (\gamma-1)^k R_{m+\ell+1-k,m-\ell-1} \tag{7.2.28}$$

modulo $p^r \mathfrak{m}\mathfrak{M}$.

On the other hand, for any $m+\ell+1 \leq i \leq r-1$, we have

$$[\phi(\hat{F}_i)]_{E_2} = \frac{p^i\lambda[\mathfrak{L} - \phi(\delta) - p(\mathfrak{L}-\delta)]}{\Theta}\left(\left(\frac{\phi(v)}{p}\right)^i - \sum_{k=0}^{m+\ell}\left(\frac{\phi(v)}{p}\right)^k (RP)'_{k,i}\right)$$
$$- \frac{p^{i+1}\lambda}{\Theta}\sum_{k=0}^{m-\ell-2}\left(\frac{\phi(v)}{p}\right)^k\left((RP)'_{k,i}\kappa + \sum_{j=1}^{k}\frac{(-1)^{j+1}}{j}(RP)'_{k-j,i}\right).$$

Now, we know that $\phi(\delta) \equiv \delta^{(0)} \pmod{p^{p-1}S_{\mathbb{O}}}$. Note that, by Lemma 7.1.5, $v_p((RP)'_{k,i}) \geq h-1$. Since $r < p-1$, this implies that

$$[\phi(\hat{F}_i)]_{E_2} \equiv p^{i+m-\ell-1}\left(\frac{\phi(v)}{p}\right)^i + \frac{p^{i+1}\lambda}{\Theta}\sum_{k=m-\ell-1}^{m+\ell}\left(\frac{\phi(v)}{p}\right)^k \kappa (RP)'_{k,i}$$
$$- \frac{p^{i+1}\lambda}{\Theta}\sum_{k=0}^{m-\ell-2}\left(\frac{\phi(v)}{p}\right)^k\left(\delta(RP)'_{k,i} + \sum_{j=1}^{k}\frac{(-1)^{j+1}}{j}(RP)'_{k-j,i}\right)$$

modulo $p^r \mathfrak{m} S_{\mathbb{O}}$. Using Lemma 7.1.5, it's easy to see that $[\phi(\hat{F}_i)]_{E_2} \in p^r S_{\mathbb{O}}$, and moreover if $i \geq m+\ell+2$, then $[\phi(\hat{F}_i)]_{E_2} \in p^r \mathfrak{m} S_{\mathbb{O}}$. Furthermore, since $\phi(v)/p \equiv$

$(\gamma-1) \pmod{pS_{\mathbb{O}}}$, if $i=m+\ell+1$ then the same argument implies that

$$[\phi(\hat{F}_{m+\ell+1})]_{E_2} \equiv p^r(\gamma-1)^{m+\ell+1} + \frac{p^{m+\ell+2}\lambda}{\Theta}\sum_{k=m-\ell-1}^{m+\ell}(\gamma-1)^k\kappa(RP)'_{k,m+\ell+1}$$
$$-\frac{p^{m+\ell+2}\lambda}{\Theta}\sum_{k=0}^{m-\ell-2}(\gamma-1)^k\left(\delta(RP)'_{k,m+\ell+1}+\sum_{j=1}^{k}\frac{(-1)^{j+1}}{j}(RP)'_{k-j,m+\ell+1}\right)$$

modulo $p^r\mathfrak{m}S_{\mathbb{O}}$. We then write

$$(RP)'_{k,m+\ell+1} = \kappa R_{m+\ell+1-k,m-\ell-1} + \sum_{j=1}^{m-\ell-2} R_{m+\ell+1-k,j}P_{j,m-\ell-1}.$$

Note that $P_{j,m-\ell-1}\in\mathbb{O}$ for $j=1,\dots,m-\ell-2$. In the end, by Lemma 7.1.3, we get

$$[\phi(\hat{F}_{m+\ell+1})]_{E_2} \equiv p^r(\gamma-1)^{m+\ell+1} + \frac{p^{m+\ell+2}\lambda}{\Theta}\sum_{k=m-\ell-1}^{m+\ell}(\gamma-1)^k\kappa^2R_{m+\ell+1-k,m-\ell-1}$$
$$-\frac{p^{m+\ell+2}\lambda\kappa}{\Theta}\sum_{k=0}^{m-\ell-2}(\gamma-1)^k\left(\delta R_{m+\ell+1-k,m-\ell-1}+\sum_{j=1}^{k}\frac{(-1)^{j+1}}{j}R_{m+\ell+1-k+j,m-\ell-1}\right)$$

modulo $p^r\mathfrak{m}S_{\mathbb{O}}$. When considering only modulo $p^rS_{\mathbb{O}}$, we can see that this is equal to 0 by Lemma 7.1.4, as in the case $i=0$. This finishes the proof of the proposition. The formula for $\phi(\hat{F}_{m+\ell+1})$ modulo $p^r\mathfrak{m}\mathfrak{M}$ follows from the above computation and (7.2.28). □

7.3. ***Mod p reduction.*** In this subsection, we prove Theorem 5.0.5 in the case $-\frac{1}{2}-\ell\le h<\frac{1}{2}-\ell$ for $\ell\in\{0,1,2,\dots,m-2\}$. Throughout this subsection, we keep the notation and assumption as in Section 7.2. We first compute the Breuil modules corresponding to the mod p reduction of the strongly divisible module $\mathfrak{M}$ in Theorem 5.0.1 when $-\frac{1}{2}-\ell\le h<\frac{1}{2}-\ell$ for $\ell\in\{0,1,\dots,m-2\}$.

Lemma 7.3.1. *The Breuil module* $\mathcal{M}:=\mathfrak{M}/(\pi,\mathrm{Fil}^pS)$ *is described as follows: there exists a basis* $\underline{e}:=(e_1,e_2)$ *for* $\mathcal{M}$ *and a system of generators* $\underline{f}:=(f_1,f_2)$ *for* $\mathrm{Fil}^r\mathcal{M}$ *such that*

- $\mathcal{M}:=\bar{S}_{\mathbb{F}}(e_1,e_2)$;
- $\mathrm{Mat}_{\underline{e},\underline{f}}(\mathrm{Fil}^r\mathcal{M})=\begin{pmatrix}u^{m+\ell+1}&0\\0&u^{m-\ell-1}\end{pmatrix}$;

- $\mathrm{Mat}_{\underline{e},\underline{f}}(\phi_r) = \begin{pmatrix} \gamma & \beta \\ \alpha & 0 \end{pmatrix}$, *where*

$$\begin{aligned}\alpha :=& (-1)^{m+\ell+1} + \frac{\mathfrak{L}-a(r)}{p} \sum_{k=m-\ell-1}^{m+\ell} (-1)^k R_{m+\ell+1-k,m-\ell-1} \\ &+ \sum_{k=0}^{m-\ell-2} (-1)^k \left(R_{m+\ell+1-k,m-\ell-1}\delta_\ell^{(0)} + \sum_{j=1}^{k} \frac{(-1)^{j+1}}{j} R_{m+\ell+1-k+j,m-\ell-1} \right), \\ \beta :=& \frac{\lambda^2(\mathfrak{L}-a(r))}{p^r} R_{m+\ell+1,m+\ell+1} \sum_{k=0}^{m-\ell-1} (-1)^k x_k^{(m-\ell-1,2\ell+2)}, \\ \gamma :=& \frac{\lambda(\mathfrak{L}-a(r))}{p^{m-\ell-1}} \sum_{k=0}^{m-\ell-2} (-1)^k R_{m+\ell+1-k,m-\ell-1};\end{aligned}$$

- $\mathrm{Mat}_{\underline{e}}(N) = 0_{2\times 2}$.

Proof. We keep the notation as in Section 7.2. We let $e_1 = E_1$ and $e_2 = E_2$ modulo $(\pi, \mathrm{Fil}^p S_{\mathbb{O}})$. We also let $f_1 := \hat{F}_{m+\ell+1}$ modulo $(\pi, \mathrm{Fil}^p S_{\mathbb{O}})$ and $f_2 := \hat{F}_0$ modulo $(\pi, \mathrm{Fil}^p S_{\mathbb{O}})$. By Lemma 7.2.8, $\hat{F}_i \equiv u^{m-\ell-1+i} E_2$ for $i \in \{0, 1, \dots, m+\ell\}$ modulo $\mathfrak{m}\mathfrak{M}$ and $\hat{F}_i \equiv u^i E_1$ for $i \in \{m+\ell+1, \dots, r-1\}$ modulo $\mathfrak{m}\mathfrak{M}$. Hence, $\mathrm{Fil}^r \mathcal{M}$ is generated by $f_1 = u^{m+\ell+1} e_1$ and $f_2 = u^{m-\ell-1} e_2$. By Proposition 7.2.14 and by Lemma 7.1.4, we get the description for ϕ_r as in the statement. It is obvious that $N(e_i) = 0$ from (5.0.3) since $v_p(\Theta) < 1$ and p divides $N(\delta_0)$. □

Note that the proof in Lemma 7.3.1 implies that $\mathrm{Fil}^r \mathfrak{M}$ is generated by $\hat{F}_0$ and $\hat{F}_{m+\ell+1}$ modulo $\mathrm{Fil}^r S_{\mathbb{O}}\mathfrak{M}$ by Nakayama's lemma.

Proposition 7.3.2. *Fix* $\ell \in \{0, 1, 2, \dots, m-2\}$, *and let* $-\frac{1}{2} - \ell \le h < \frac{1}{2} - \ell$ *and* $\rho := \mathrm{V}_{\mathrm{st}}^r(D)$. *Then* $\bar{\rho}$ *is absolutely irreducible if and only if* $-\frac{1}{2} - \ell < h < \frac{1}{2} - \ell$, *in which case*

$$\bar{\rho}|_{I_{\mathbb{Q}_p}} \cong \omega_2^{m-\ell-1+p(m+\ell+1)} \oplus \omega_2^{m+\ell+1+p(m-\ell-1)}.$$

Moreover, if $v_p(\mathfrak{L} - a(r)) = -\frac{1}{2} - \ell$ *then*

$$\bar{\rho}^{ss}|_{I_{\mathbb{Q}_p}} \cong \omega^{m-\ell-1} \oplus \omega^{m+\ell+1}.$$

Proof. Recall that γ is the (1, 1)-entry of the matrix $\mathrm{Mat}_{\underline{e},\underline{f}}(\phi_r)$ in Lemma 7.3.1. By Lemma 7.1.6, $\gamma = 0$ if and only if $-\frac{1}{2} - \ell < v_p(\mathfrak{L} - a(r)) < \frac{1}{2} - \ell$. Now it is obvious by Lemmas 7.3.1, 3.3.3, and 3.3.5. □

8. The third case: $\bullet = -\infty$

In this section, we prove Theorems 5.0.1 and 5.0.5 under the condition $h < \frac{3}{2} - m$. We keep the assumption and the notation as in Section 5. In particular, we let

$\delta = \delta_{-\infty}$ and $\Theta = \lambda(\mathfrak{L} - a(r))$. Notice that the definition of $\delta_{-\infty}$ for $r = 2$ is quite different to the one for $r = 2m > 2$, so that we prove the case $r = 2$ separately, and then prove for general $r = 2m > 2$.

8.1. ***The case r = 2.*** In this subsection, we prove Theorem 5.0.1 for the case $h < \frac{3}{2} - m = \frac{1}{2}$ when $r = 2m = 2$. Recall that $\mathfrak{X}$ in (3.2.2) and $\mathfrak{Y}$ in (5.0.4), and that $a(2) = 1$. In this subsection, we write Δ for Δ_2 to lighten the notation.

From the computations (5.0.2) and (5.0.3), it is easy to check that

$$\phi(E_1) \equiv E_2 \quad \text{and} \quad \phi(E_2) \equiv N(E_1) \equiv N(E_2) \equiv 0 \tag{8.1.1}$$

modulo $\mathfrak{m}\mathfrak{M}$. In particular, $\mathfrak{M}$ is stable under ϕ and N.

Rewriting $\mathfrak{X}(C_0, C_1)$ in terms of E_1, E_2,

$$\mathfrak{X}(C_0, C_1) \in \mathfrak{Y}(C_0, C_1) + \mathrm{Fil}^p\, S_E\mathfrak{D},$$

where

$$\mathfrak{Y}(C_0, C_1) = C_0\biggl(\frac{1}{p}E_1 - \frac{1}{p\lambda}E_2 + \frac{\Delta}{\lambda(\mathfrak{L}-1)^2}E_2 + \frac{\mathfrak{L}}{\lambda(\mathfrak{L}-1)}E_2\biggr) + (u-p)\biggl(\frac{C_1}{p}E_1 + \frac{(\mathfrak{L}-1)(C_0 + p\mathfrak{L}C_1) - [(\mathfrak{L}-1)^2 - p\Delta]C_1}{p\lambda(\mathfrak{L}-1)^2}E_2\biggr).$$

One can readily induce the identity

$$\frac{\Delta C_0 + (\mathfrak{L}-1)C_1}{p\lambda\Delta(\mathfrak{L}-1)} = \frac{(\mathfrak{L}-1)C_0 + [(\mathfrak{L}-1)^2 - p\Delta]C_1}{p\lambda(\mathfrak{L}-1)^2}$$

from the equation in Lemma 4.2.2 (2). Using this identity, and rewriting C_1 as a linear combination of $C_0/(p\lambda)$ and $\Delta C_0 - (\mathfrak{L}-1)C_1/(p\lambda\Delta(\mathfrak{L}-1))$, we have

$$\mathfrak{Y}(C_0, C_1) = \frac{C_0}{p\lambda}\hat{F}_1 - \frac{\Delta C_0 - (\mathfrak{L}-1)C_1}{p\lambda\Delta(\mathfrak{L}-1)}\hat{F}_2$$

where

$$\hat{F}_1 = \lambda E_1 - E_2 + \frac{p\Delta}{(\mathfrak{L}-1)^2}E_2 + \frac{p\mathfrak{L}}{\mathfrak{L}-1}E_2 + (u-p)\biggl(\frac{\lambda\Delta}{\mathfrak{L}-1}E_1 + \frac{p\mathfrak{L}\Delta}{(\mathfrak{L}-1)^2}E_2\biggr),$$
$$\hat{F}_2 = (u-p)\biggl(\lambda\Delta E_1 - E_2 + \frac{p\Delta\mathfrak{L}}{\mathfrak{L}-1}E_2\biggr).$$

It is easy to check that

$$\hat{F}_1 \equiv -E_2 \quad \text{and} \quad \hat{F}_2 \equiv -uE_2$$

modulo $\mathfrak{m}\mathfrak{M}$. Hence, $\mathfrak{Y}(C_0, C_1) \in \mathrm{Fil}^2\,\mathfrak{M}$ if and only if

$$\begin{aligned} v_p(C_0) &\geq 1 + v_p(\lambda) = \tfrac{3}{2} \\ v_p(\Delta C_0 - (\mathfrak{L}-1)C_1) &\geq \tfrac{3}{2} + (\mathfrak{L}-1). \end{aligned} \tag{8.1.2}$$

We often write $\mathfrak{Y}$ for $\mathfrak{Y}(C_0, C_1)$ for brevity. For any $d \in \mathrm{Fil}^2 \mathscr{D}$, we have $d \in \mathrm{Fil}^2 \mathfrak{M}$ if and only if $d \in \mathfrak{Y}(C_0, C_1) + \mathrm{Fil}^2 S\mathfrak{M}$ for some $C_0, C_1 \in E$ with $\mathfrak{Y}(C_0, C_1) \in \mathfrak{M}$. So it is enough to check that $\phi(\mathfrak{Y}) \in p^2\mathfrak{M}$ whenever $\mathfrak{Y}(C_0, C_1) \in \mathrm{Fil}^2 \mathfrak{M}$.

It is also routine to check, by our computation of $\phi(E_i)$, that

$$\begin{aligned}
\phi(\hat{F}_1) = p\lambda^2 E_1 &+ \frac{\lambda[\phi(\gamma) - p(\gamma-1)]}{\mathfrak{L}-1} E_2 \\
&- \frac{p\lambda\Delta[\phi(\gamma)(\phi(\gamma)-2) - p(\gamma-1)^2]}{(\mathfrak{L}-1)^2} E_2 \\
&+ \phi(v)\biggl(\frac{p\lambda^2\Delta}{\mathfrak{L}-1} E_1 + \frac{\lambda\Delta[\phi(\gamma) + \mathfrak{L} - 1 - p(\gamma-1)]}{(\mathfrak{L}-1)^2} E_2 \\
&\qquad - \frac{p\lambda\Delta^2[(\phi(\gamma)-1)^2 - p(\gamma-1)^2]}{(\mathfrak{L}-1)^3} E_2\biggr), \\
\phi(\hat{F}_2) = \phi(v)\biggl(p\lambda^2\Delta E_1 &+ \frac{\lambda\Delta[\phi(\gamma) + \mathfrak{L} - 1 - p(\gamma-1)]}{\mathfrak{L}-1} E_2 \\
&- \frac{p\lambda\Delta^2[(\phi(\gamma)-1)^2 - p(\gamma-1)^2]}{(\mathfrak{L}-1)^2} E_2 - \lambda E_2\biggr).
\end{aligned}$$

We claim that

$$\phi(\hat{F}_1) \equiv p\lambda^2 E_1 \quad \text{and} \quad \phi(\hat{F}_2) \equiv 0$$

modulo $p^2\mathfrak{m}\mathfrak{M}$.

Indeed, it is easy to check that

$$\phi(\hat{F}_2) \equiv -c\frac{p\lambda[p\Delta^2 - (\mathfrak{L}-1)^2\Delta + (\mathfrak{L}-1)^2]}{(\mathfrak{L}-1)^2} E_2$$

modulo $p^2\mathfrak{m}\mathfrak{M}$, where $c := 1/p\,\phi(v)$. By the equation in Lemma 4.2.2 (2), we conclude that $\phi(\hat{F}_2) \equiv 0$ modulo $p^2\mathfrak{m}\mathfrak{M}$. It is also easy to check that

$$\begin{aligned}
\phi(\hat{F}_1) \equiv p\lambda^2 E_1 - \frac{p\lambda(\gamma-1)}{\mathfrak{L}-1} E_2 &+ \frac{p^2\lambda\Delta(\gamma-1)^2}{(\mathfrak{L}-1)^2} E_2 \\
&+ \phi(v)\biggl(\frac{\lambda\Delta[(\mathfrak{L}-1)^2 - p\Delta]}{(\mathfrak{L}-1)^3} - \frac{p\lambda\Delta(\gamma-1)}{(\mathfrak{L}-1)^2}\biggr) E_2.
\end{aligned}$$

By the equation in Lemma 4.2.2 (2) again,

$$\begin{aligned}
\phi(\hat{F}_1) \equiv p\lambda^2 E_1 &- \frac{\lambda}{(\mathfrak{L}-1)}[p(\gamma-1) - (u^p - p)] E_2 \\
&+ \frac{p\lambda\Delta(\gamma-1)}{(\mathfrak{L}-1)^2}[p(\gamma-1) - (u^p - p)] E_2.
\end{aligned}$$

Since $p(\gamma-1) \equiv u^p - p$ modulo $p^2 S$, we conclude $\phi(\hat{F}_1) \equiv p\lambda^2 E_1 \bmod p^2\mathfrak{m}\mathfrak{M}$.

Therefore, we conclude that

$$\frac{1}{p^2}\phi(\mathfrak{Y}) \equiv \frac{\lambda C_0}{p^2} E_1 \tag{8.1.3}$$

modulo $\mathfrak{m}\mathfrak{M}$ if the inequalities in (8.1.2) hold, and so $\phi(\mathrm{Fil}^2\,\mathfrak{M}) \subset p^2\mathfrak{M}$.

8.2. ***Matrices.*** In this subsection, we study some properties of certain matrices. These matrices will be used later to describe generators of the filtration of our strongly divisible modules.

From now on, fix $r = 2m \geq 4$ an even integer. We let $\kappa \in E^\times$ be an element such that $v_p(\kappa) < 0$. We construct a matrix $T_{-\infty} \in \mathrm{GL}_r(E)$ be the upper-triangular matrix defined as

$$T_{-\infty} = \begin{pmatrix} \kappa & 1 & -\frac{1}{2} & \dots & \dfrac{(-1)^{r-1}}{r-2} & \dfrac{(-1)^r}{r-1} \\ 0 & \kappa & 1 & \dots & \dfrac{(-1)^{r-2}}{r-3} & \dfrac{(-1)^{r-1}}{r-2} \\ 0 & 0 & \kappa & \dots & \dfrac{(-1)^{r-3}}{r-4} & \dfrac{(-1)^{r-2}}{r-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \kappa & 1 \\ 0 & 0 & 0 & \dots & 0 & \kappa \end{pmatrix}.$$

There is no $P_{-\infty}$ in this case, and we let $Q_{-\infty} := T_{-\infty}$. We also let $R_{-\infty} := Q_{-\infty}^{-1}$. We often write Q and R for $Q_{-\infty}$ and $R_{-\infty}$ respectively, to lighten the notation.

Lemma 8.2.1. *Keep the notation and the assumptions as above, and let* $R = R_{-\infty}$.

(1) *For any* $i = 1, \dots, r$, $R_{i,i} = 1/\kappa$.

(2) *For any* $1 \leq i < j \leq r$,

$$R_{i,j} \equiv (-1)^{j-i}/j - i\frac{1}{\kappa^2} \left(\mathrm{mod}\ \frac{1}{\kappa^3}\right).$$

Proof. Since $Q_{-\infty}$ is upper-triangular, the case (1) is immediate.

For the case (2), we induct on j for each i. Fix i and consider $\mathrm{row}_i(R)$. Note that the first $i-1$ entries of $\mathrm{row}_i(R)$ are 0, as R is upper-triangular, and the i-th entry of $\mathrm{row}_i(R)$ is $R_{i,i} = 1/\kappa$. When $j = i+1$, $(RQ)_{i,i+1} = \mathrm{row}_i(R) \cdot \mathrm{col}_{i+1}(Q) = 1/\kappa + \kappa R_{i,i+1} = 0$ so that we have $R_{i,i+1} = -1/\kappa^2$. Assume that the assertion holds for $i+1, i+2, \dots, j-1$. From the equation

$$(RQ)_{i,j} = \mathrm{row}_i(R) \cdot \mathrm{col}_j(Q) = \frac{(-1)^{j-1+1}}{j-i}\frac{1}{\kappa} + \sum_{k=i+1}^{j-1} \frac{(-1)^{j-k+1}}{j-k} R_{i,k} + \kappa R_{i,j} = 0,$$

we conclude that $R_{i,j} \equiv \dfrac{(-1)^{j-i}}{j-i}\dfrac{1}{\kappa^2} \left(\mathrm{mod}\ \dfrac{1}{\kappa^3}\right)$ by induction hypothesis. □

8.3. ***Galois stable lattices.*** In this subsection, we prove Theorem 5.0.1 for the case $h < \frac{3}{2} - m$ when $r = 2m > 2$. We let $\delta = \delta_{-\infty}$ and $\Theta = \lambda(\mathfrak{L} - a(r))$. We also let $\kappa = (p\mathfrak{L} - (\mathfrak{L} - \delta^{(0)}))/p$ in $T_{-\infty} = Q_{-\infty}$. Note that $v_p(\mathfrak{L} - \delta^{(0)} - p\mathfrak{L}) = h$ since $h < 0$, and so $v_p(\kappa) = h - 1$.

By Lemma 8.2.1, we have that

$$v_p(R_{i,i}) = 1 - h \quad (i = 1, \dots, r) \tag{8.3.1}$$

and

$$v_p(R_{i,j}) = 2 - 2h \quad (1 \le i < j \le r). \tag{8.3.2}$$

Note that

$$v_p(R_{i,j}) > 0 \quad (1 \le i, j \le r). \tag{8.3.3}$$

From the computations (5.0.2) and (5.0.3), it is easy to check that

$$\phi(E_1) \equiv E_2 \text{ and } \phi(E_2) \equiv N(E_1) \equiv N(E_2) \equiv 0 \tag{8.3.4}$$

modulo $\mathfrak{m}\mathfrak{M}$. In particular, $\mathfrak{M}$ is stable under ϕ and N.

Let $A = \Theta R$. Define vectors $\vec{C}^{(i)} = (C_0^{(i)}, \dots, C_{r-1}^{(i)}) \in E^r$, for $i = 0, \dots, r-1$, such that

$$\begin{pmatrix} p^{r-1}C_{r-1}^{(i)} \\ p^{r-2}C_{r-2}^{(i)} \\ \vdots \\ p^0 C_0^{(i)} \end{pmatrix} = p^i \operatorname{col}_{r-i}(A).$$

Concretely, $C_k^{(i)} = p^{i-k} A_{r-k,r-i} = p^{i-k}\Theta R_{r-k,r-i}$. Using these vectors, we define $\hat{F}_i \in \operatorname{Fil}^r \mathfrak{D}$ by the formula $\hat{F}_i = \mathfrak{Y}(\vec{C}^{(i)})$, where $\mathfrak{Y}$ is defined in (5.0.4).

Lemma 8.3.5. *If* $0 \le i \le r-1$ *then*

$$\hat{F}_i = v^i E_2 + \sum_{k=i}^{r-1} v^k p^{i-k-1} \Theta R_{r-k,r-i} E_1.$$

Proof. By definition, we have

$$\hat{F}_i = \sum_{k=0}^{r-1} v^k \left[p^{i-k}\Theta R_{r-k,r-i}\left(\frac{E_1}{p} + \frac{\kappa}{\Theta}E_2\right) + \sum_{j=1}^{k} \frac{(-1)^{j+1}}{j} p^{i-k} R_{r-k+j,r-i} E_2 \right].$$

It is immediate to get $[\hat{F}_i]_{E_1}$ as in the statement. But we need a little more computation for $[\hat{F}_i]_{E_2}$. We readily get

$$[\hat{F}_i]_{E_2} = \sum_{k=0}^{r-1} p^{i-k} v^k \left(\kappa R_{r-k,r-i} + \sum_{j=1}^{k} \frac{(-1)^{j+1}}{j} R_{r-k+j,r-i} \right).$$

But one can readily check that

$$\kappa R_{r-k,r-i} + \sum_{j=1}^{k} \frac{(-1)^{j+1}}{j} R_{r-k+j,r-i} = (QR)_{r-k,r-i},$$

so that we have

$$[\hat{F}_i]_{E_2} = \sum_{k=0}^{r-1} p^{i-k} v^k (QR)_{r-k,r-i}.$$

Since $QR = I_r$, we complete the proof. □

Lemma 8.3.6. *Every element $x \in \mathrm{Fil}^r \mathfrak{D}$ can be written as*

$$x = \sum_{i=0}^{r-1} D_i \hat{F}_i + x',$$

where $D_i \in E$ and $x' \in \mathrm{Fil}^r S_E \mathfrak{D}$.

Proof. As we've seen before, any element of $\mathrm{Fil}^r \mathfrak{D}$ can be written, modulo $\mathrm{Fil}^r S_E \mathfrak{D}$ as $\mathfrak{Y} = \mathfrak{Y}(\vec{C})$, with $\vec{C} = (C_0, \dots, C_{r-1}) \in E^r$, as in (5.0.4). For $i = 0, \dots, r-1$, let

$$D_i = \frac{p\mathfrak{L} - (\mathfrak{L} - \delta^{(0)})}{p\Theta} C_i + \sum_{j=1}^{i} \frac{(-1)^{j+1}}{j} \frac{C_{i-j}}{p^j \Theta}.$$

We can express the C_i in terms of the D_i as follows. Since

$$\begin{pmatrix} p^{r-1} D_{r-1} \\ p^{r-2} D_{r-2} \\ \vdots \\ p^0 D_0 \end{pmatrix} = \frac{1}{\Theta} Q \begin{pmatrix} p^{r-1} C_{r-1} \\ p^{r-2} C_{r-2} \\ \vdots \\ p^0 C_0 \end{pmatrix},$$

we have

$$\begin{pmatrix} p^{r-1} C_{r-1} \\ p^{r-2} C_{r-2} \\ \vdots \\ p^0 C_0 \end{pmatrix} = A \begin{pmatrix} p^{r-1} D_{r-1} \\ p^{r-2} D_{r-2} \\ \vdots \\ p^0 D_0 \end{pmatrix}.$$

Now, it follows from the definition of the vectors $\vec{C}^{(i)}$ that

$$\vec{C} = \sum_{i=0}^{r-1} D_i \vec{C}^{(i)}$$

and hence

$$\mathfrak{Y} = \mathfrak{Y}(\vec{C}) = \sum_{i=0}^{r-1} D_i \hat{F}_i. \qquad \square$$

Lemma 8.3.7. *$\hat{F}_i \equiv u^i E_2$ modulo $\mathfrak{m}\mathfrak{M}$ for $0 \le i \le r-1$. In particular, the elements $\hat{F}_i$ belong to* $\mathrm{Fil}^r \mathfrak{M}$.

Proof. Since $\hat{F}_i \in \mathrm{Fil}^r \mathscr{D}$ by construction, it remains to show that $\hat{F}_i \in \mathfrak{M}$. We have

$$[\hat{F}_i]_{E_1} = \sum_{k=i}^{r-1} v^k R_{r-k,r-i} p^{i-k-1} \Theta. \tag{8.3.8}$$

If $k = i$,

$$v_p(R_{r-k,r-i} p^{i-k-1} \Theta) = m - \tfrac{1}{2} > 0$$

by (8.3.1). *If $k > i$, $v_p(R_{r-k,r-i} p^{i-k-1} \Theta) = m + \frac{1}{2} - h + i - k$ by* (8.3.2), *and $m + \frac{1}{2} - h + i - k > r - 1 + i - k \ge 0$ because $k \le r - 1$ and $h < \frac{3}{2} - m$. Thus,* (8.3.8) *belongs to $\mathfrak{m}S_{\mathbb{O}}$. On the other hand, $[\hat{F}_i]_{E_2} = v^i \in S_{\mathbb{O}}$. This finishes the proof.* $\square$

Lemma 8.3.9. *Let $D_0, \ldots, D_{r-1} \in E$. Then $\sum_{i=0}^{r-1} D_i \hat{F}_i \in \mathfrak{M}$ if and only if $D_i \in \mathbb{O}$ for every $i = 0, \ldots, r-1$.*

Proof. One direction is obvious from the previous lemma. To prove the other direction, suppose that $y = \sum_{i=0}^{r-1} D_i \hat{F}_i \in \mathfrak{M}$. We see that

$$[y]_{E_2} = \sum_{i=0}^{r-1} v^i D_i. \tag{8.3.10}$$

Since (8.3.10) belongs to $S_{\mathbb{O}}$, we have that $D_i \in \mathbb{O}$ for every $i = 0, \ldots, r-1$. $\square$

From the previous lemmas, it follows that any element $x \in \mathrm{Fil}^r \mathfrak{M}$ can be written as $x = \sum_{i=0}^{r-1} D_i \hat{F}_i + x'$ for some $D_i \in \mathbb{O}$, with $x' \in \mathrm{Fil}^r S_{\mathbb{O}}\mathfrak{M}$. Moreover, to show that $\phi(\mathrm{Fil}^r \mathfrak{M}) \subset p^r \mathfrak{M}$, it is enough to show that $\phi(\hat{F}_i) \in p^r \mathfrak{M}$ for any $i = 0, \ldots, r-1$. We state the result as a proposition, and we include some additional information for mod p reduction.

Proposition 8.3.11. *Assume that $h < \frac{3}{2} - m$ and $r = 2m > 2$. Then $\mathfrak{M}$ is a strongly divisible module with $\hat{F}_i$ generators for* $\mathrm{Fil}^r \mathfrak{M}$ *modulo* $\mathrm{Fil}^r S_{\mathbb{O}}\mathfrak{M}$. *Moreover, $\phi(\hat{F}_i) \in p^r \mathfrak{m}\mathfrak{M}$ for every $i \neq 0$, and*

$$\phi(\hat{F}_0) \equiv -p\lambda^2 E_1$$

modulo $p^r\mathfrak{m}\mathfrak{M}$.

The only thing that we need to do to prove that $\mathfrak{M}$ is a strongly divisible module is to show that $\phi(\hat{F}_i) \in p^r\mathfrak{M}$ for every $i = 0, \dots, r-1$. For later use for mod p reduction, we compute $\phi(\hat{F}_i)$ modulo $p^r\mathfrak{m}\mathfrak{M}$, using (5.0.2).

Proof. We have

$$[\phi(\hat{F}_i)]_{E_1} = \sum_{k=i}^{r-1}\Big(\frac{\phi(v)}{p}\Big)^k R_{r-k,r-i}\, p^i\,\Theta\lambda.$$

Since $(\phi(v)/p)^k \in S_{\mathcal{O}}$, we conclude that $[\phi(\hat{F}_i)]_{E_1} \in p^r S_{\mathcal{O}}$ using (8.3.1) and (8.3.2). Combining with (8.3.2) and the fact that $R_{r,r} = 1/\kappa$, we get that $[\phi(\hat{F}_i)] \in p^r\mathfrak{m}S_{\mathcal{O}}$ for $i \geq 1$, and

$$(8.3.12)\qquad [\phi(\hat{F}_0)]_{E_1} \equiv \frac{\Theta\lambda}{\kappa} \equiv -p\lambda^2 \pmod{p^r\mathfrak{m}S_{\mathcal{O}}}.$$

On the other hand,

$$(8.3.13)\quad [\phi(\hat{F}_i)]_{E_2} = p^i\lambda\bigg(\Big(\frac{\phi(v)}{p}\Big)^i + \sum_{k=i}^{r-1}\Big(\frac{\phi(v)}{p}\Big)^k R_{r-k,r-i}\,\frac{\mathfrak{L}-\phi(\delta)-p\mathfrak{L}+p\delta}{p}\bigg).$$

Since $\phi(\gamma) \in p^{p-1}S_{\mathcal{O}}$, we get $\phi(\delta) \equiv \delta^{(0)} \pmod{p^{p-1}S_{\mathcal{O}}}$. Since $r < p-1$, we get that

$$(8.3.14)\qquad [\phi(\hat{F}_i)]_{E_2} \equiv p^i\lambda\bigg(\Big(\frac{\phi(v)}{p}\Big)^i + \sum_{k=i}^{r-1}\Big(\frac{\phi(v)}{p}\Big)^k R_{r-k,r-i}(\delta-\kappa)\bigg)$$

modulo $p^r\mathfrak{m}S_{\mathcal{O}}$. Using (8.3.2) again, we see that $v_p(p^i\lambda R_{r-k,r-i}) > r$ when $i < k$, and thus we can remove some of the summands multiplying δ. Also, $R_{r-i,r-i} = \kappa^{-1}$, so we get

$$(8.3.15)\quad [\phi(\hat{F}_i)]_{E_2} \equiv -p^i\lambda\bigg(\sum_{k=i+1}^{r-1}\Big(\frac{\phi(v)}{p}\Big)^k R_{r-k,r-i}\kappa - \Big(\frac{\phi(v)}{p}\Big)^i R_{r-i,r-i}\delta\bigg)$$

modulo $p^r\mathfrak{m}S_{\mathcal{O}}$. Suppose first that $i \geq 1$. Then

$$v_p(p^i\lambda R_{r-k,r-i}\kappa) > r$$

by (8.3.2). Similarly,

$$v_p(p^i\lambda R_{r-i,r-i}) > r.$$

Thus, we see that $[\phi(\hat{F}_i)]_{E_2} \in p^r\mathfrak{m}\mathfrak{M}$ for $i \geq 1$. In particular, $\phi(\hat{F}_i) \in p^r\mathfrak{M}$ for $i \geq 1$. Suppose that $i = 0$. With a similar argument as in the case $i \geq 1$, together with the fact that $(\phi(v)/p)^k \equiv (\gamma-1)^k \pmod{pS_{\mathbb{O}}}$, we get that

$$[\phi(\hat{F}_0)]_{E_2} \equiv -\lambda\left(\sum_{k=1}^{r-1}(\gamma-1)^k R_{r-k,r}\kappa - R_{r,r}\delta\right) \tag{8.3.16}$$

modulo $p^r\mathfrak{m}S_{\mathbb{O}}$. Now, notice that if $0 \le k \le r-1$, then $\kappa R_{r-k,r} = Q_{r-k,r-k}R_{r-k,r}$. Using this, and the fact that $QR = I_r$, we get

$$\kappa R_{r-k,r} = (I_r)_{r-k,r} - \sum_{j=1}^{k}\frac{(-1)^{j+1}}{j}R_{r-k+j,r}.$$

Thus,

$$[\phi(\hat{F}_0)]_{E_2} \equiv \lambda\left(\sum_{k=1}^{r-1}(\gamma-1)^k\sum_{j=1}^{k}\frac{(-1)^{j+1}}{j}R_{r-k+j,r} + R_{r,r}\delta\right) \tag{8.3.17}$$

modulo $p^r\mathfrak{m}S_{\mathbb{O}}$. If $j < k$, then $v_p(R_{r-k+j,r}) \ge 2-2h$ by (8.3.2), which implies that these terms can be ignored. Thus,

$$[\phi(\hat{F}_0)]_{E_2} \equiv \lambda R_{r,r}\left(\sum_{k=1}^{r-1}(\gamma-1)^k\frac{(-1)^{k+1}}{k} + \delta\right) \tag{8.3.18}$$

modulo $p^r\mathfrak{m}S_{\mathbb{O}}$. This is equal to 0 by definition of $\delta = \delta_{-\infty}$. This finishes the proof that $\phi(\hat{F}_i) \in p^r\mathfrak{M}$ for every $i = 0, \dots, r-1$, and hence $\phi(\mathrm{Fil}^r\,\mathfrak{M}) \subset p^r\mathfrak{M}$. This finishes the proof of the proposition. The formula for $\phi(\hat{F}_0)$ modulo $p^r\mathfrak{m}\mathfrak{M}$ follows from the above computation and (8.3.12). ☐

8.4. ***Mod p reduction.*** In this subsection, we prove Theorem 5.0.5 in the case $h < \frac{3}{2} - m$. We keep the notation and assumption as in Section 8.3 if $r = 2m > 2$ and in Section 8.1 if $r = 2$. We first compute the Breuil module corresponding to the mod p reduction of the strongly divisible module $\mathfrak{M}$ in Theorem 5.0.1 when $h < \frac{3}{2} - m$.

Lemma 8.4.1. *The Breuil module $\mathcal{M} := \mathfrak{M}/(\pi, \mathrm{Fil}^p\, S)$ is described as follows: there exists a basis $\underline{e} := (e_1, e_2)$ for $\mathcal{M}$ and a system of generators $\underline{f} := (f_1, f_2)$ for* $\mathrm{Fil}^r\,\mathcal{M}$ *such that*

- $\mathcal{M} := \bar{S}_{\mathbb{F}}(e_1, e_2)$;
- $\mathrm{Mat}_{\underline{e},\underline{f}}(\mathrm{Fil}^r\,\mathcal{M}) = \begin{pmatrix} u^r & 0 \\ 0 & 1 \end{pmatrix}$;
- $\mathrm{Mat}_{\underline{e},\underline{f}}(\phi_r) = \begin{pmatrix} 0 & -\lambda^2/p^{r-1} \\ 1 & 0 \end{pmatrix}$;
- $\mathrm{Mat}_{\underline{e}}(N) = 0_{2\times 2}$.

Proof. We first consider the case $r > 2$, keeping the notation as in Section 8.3. We let $e_1 = E_1$ and $e_2 = E_2$ modulo $(\pi, \mathrm{Fil}^p S_{\mathbb{O}})$. We also let $f_1 := v^r E_1$ modulo $(\pi, \mathrm{Fil}^p S_{\mathbb{O}})$ and $f_2 := \hat{F}_0$ modulo $(\pi, \mathrm{Fil}^p S_{\mathbb{O}})$. By Lemma 8.3.7, $\hat{F}_i \equiv u^i E_2$ modulo $\mathfrak{m}\mathfrak{M}$. Hence, $\mathrm{Fil}^r \mathcal{M}$ is generated by $f_2 = e_2$ and $f_1 = u^r e_1$. By Proposition 8.3.11,

$$\phi_r(f_2) = -\frac{\lambda^2}{p^{r-1}} e_1$$

and, by (5.0.2),

$$\phi_r(f_1) \equiv \frac{1}{p^r}\phi(v^r E_1) \equiv (\gamma - 1)^r E_2 \equiv (-1)^r e_2 \equiv e_2$$

modulo $(\pi, \mathrm{Fil}^p S_{\mathbb{O}})$. It is obvious that $N(e_i) = 0$ from (5.0.3) since $v_p(\Theta) < 1$ and p divides $N(\delta_{m-1})$.

The case $r = 2$ is similar and easier. We keep the notation as in Section 8.1. In this case we let $f_1 := v^2 E_1$ modulo $(\pi, \mathrm{Fil}^p S_{\mathbb{O}})$ and $f_2 := -\hat{F}_1$ modulo $(\pi, \mathrm{Fil}^p S_{\mathbb{O}})$ as well as $e_1 = E_1$ and $e_2 = E_2$ modulo $(\pi, \mathrm{Fil}^p S_{\mathbb{O}})$. Then all the others follow immediately from Section 8.1. □

Note that if $r = 2m > 2$ then the proof in Lemma 8.4.1 implies that $\mathrm{Fil}^r \mathfrak{M}$ is generated by $\hat{F}_0$ modulo $\mathrm{Fil}^r S_{\mathbb{O}}\mathfrak{M}$ by Nakayama's lemma.

Proposition 8.4.2. *Let $h < \frac{3}{2} - m$ and $\rho := \mathrm{V}^r_{\mathrm{st}}(D)$. Then*

$$\bar{\rho}|_{I_{\mathbb{Q}_p}} \cong \omega_2^r \oplus \omega_2^{pr}.$$

In particular, $\bar{\rho}$ is absolutely irreducible.

Proof. This is obvious by Lemmas 8.4.1 and 3.3.3. □

9. Semistable deformation rings

In this section, we construct the irreducible components of the semistable deformation rings in Hodge–Tate weights $(0, r)$, where r is a positive even integer less than $p - 1$, of absolutely irreducible residual representations of $G_{\mathbb{Q}_p}$. The following is the main results in this section.

Theorem 9.0.1. *Let $r = 2m > 0$ be an even integer less than $p - 1$, and let $\mathfrak{R}^{(0,r)}_{\bar{\rho}_0}$ be the semistable deformation rings in Hodge–Tate weights $(0, r)$.*

(1) *If $\bar{\rho}_0|_{I_{\mathbb{Q}_p}} \cong \omega_2^r \oplus \omega_2^{pr}$, then*

$$\mathfrak{R}^{(0,r)}_{\bar{\rho}_0} \sim' \mathbb{O}[\![D_1, D_2]\!] \times \mathbb{O}[\![D, X]\!] \times \mathbb{O}[\![D, X]\!];$$

(2) *If $\bar{\rho}_0|_{I_{\mathbb{Q}_p}} \cong \omega_2^{m-\ell-1+p(m+\ell+1)} \oplus \omega_2^{m+\ell+1+p(m-\ell-1)}$ for $\ell \in \{0, 1, 2, \dots, m-2\}$, then*

$$\mathfrak{R}^{(0,r)}_{\bar{\rho}_0} \sim \frac{\mathbb{O}[\![D, X, Y]\!]}{(XY - p)} \times \frac{\mathbb{O}[\![D, X, Y]\!]}{(XY - p)}.$$

By $\mathfrak{R} \sim' \prod_{i=1}^{n} \mathfrak{R}_i$ we mean that $\mathfrak{R}_i$'s are just the irreducible components of $\mathfrak{R}$, and by $\mathfrak{R} \sim \prod_{i=1}^{n} \mathfrak{R}_i$ we mean in the sense of Breuil–Mézard [2002]. We recall their definition: for complete Noetherian local rings $\mathfrak{R}$ and $\mathfrak{R}_i$'s,

$$\mathfrak{R} \sim \prod_{i=1}^{n} \mathfrak{R}_i \tag{9.0.2}$$

if

(1) $\pi_i : \mathfrak{R} \to \mathfrak{R}_i$ is surjective for each i;

(2) the map $\prod_{i=1}^{n} \pi_i : \mathfrak{R} \to \prod_{i=1}^{n} \mathfrak{R}_i$ induces an isomorphism

$$\mathfrak{R}\Big[\frac{1}{p}\Big] \cong \prod_{i=1}^{n} \mathfrak{R}_i\Big[\frac{1}{p}\Big].$$

We quickly review some known results on Galois deformation theory and on integral p-adic Hodge theory over complete Noetherian local rings in the next two subsections, and then prove Theorem 9.0.1 in the third subsection.

9.1. ***Galois deformation theory.*** In this subsection, we quickly review Galois deformation theory and recall some known results. Galois deformation theory is initiated by B. Mazur [1989] and developed for n-dimensional representations of G for profinite groups G satisfying the p-finiteness condition. But in this subsection, we restrict to $n = 2$ and $G = G_{\mathbb{Q}_p}$ since this is our context. It is easy to check that $G_{\mathbb{Q}_p}$ satisfies the p-finiteness condition by local class field theory. In fact, $G_{\mathbb{Q}_p}$ is topologically finitely generated, which implies the p-finiteness condition.

Let $\hat{\mathscr{C}}$ be the category of complete Noetherian local $\mathbb{O}$-algebras with residue field $\mathbb{F}$, and fix a residual representation $\bar{\rho}_0 : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(\mathbb{F})$ with $\mathrm{End}_{\mathbb{F}[G_{\mathbb{Q}_p}]}(\bar{\rho}_0) = \mathbb{F}$. There is a functor $D_{\bar{\rho}_0}$ from the category $\hat{\mathscr{C}}$ to the category of sets, defined by

$$D_{\bar{\rho}_0}(A) := \{\rho : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(A) \mid \bar{\rho}_0 = \rho \pmod{\mathfrak{m}_A}\}/\sim,$$

where $\mathfrak{m}_A$ is the maximal ideal of A, for all objects A in $\hat{\mathscr{C}}$. By $\rho_1 \sim \rho_2$ for $\rho_1, \rho_2 \in D_{\bar{\rho}_0}(A)$, we mean that ρ_1 is (strictly) equivalent to ρ_2 over A, i.e., there exists $\gamma \in \mathrm{Ker}(\pi_A)$ such that $\gamma \cdot \rho_1 \cdot \gamma^{-1} \cong \rho_2$, where $\pi_A : \mathrm{GL}_2(A) \to \mathrm{GL}_2(\mathbb{F})$ is the map induced from the surjection $A \to \mathbb{F}$. The upshot is that this functor is representable, i.e., there exists $\mathfrak{R}_{\bar{\rho}_0} \in \hat{\mathscr{C}}$ such that $D_{\bar{\rho}_0}(\bullet) \cong \mathrm{Hom}_{\hat{\mathscr{C}}}(\mathfrak{R}_{\bar{\rho}_0}, \bullet)$. This ring $\mathfrak{R}_{\bar{\rho}_0}$ is called a universal deformation ring of $\bar{\rho}_0$. The object in $D_{\bar{\rho}_0}(\mathfrak{R}_{\bar{\rho}_0})$ corresponding to the identity $1_{\mathfrak{R}_{\bar{\rho}_0}}$ under the isomorphism $D_{\bar{\rho}_0}(\mathfrak{R}_{\bar{\rho}_0}) \cong \mathrm{Hom}_{\hat{\mathscr{C}}}(\mathfrak{R}_{\bar{\rho}_0}, \mathfrak{R}_{\bar{\rho}_0})$ is called a universal deformation of $\bar{\rho}_0$, which is denoted by $\rho^u : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(\mathfrak{R}_{\bar{\rho}_0})$.

One can ask if there exists a closed subspace of $\mathfrak{R}_{\bar{\rho}_0}$ whose geometric points parameterize potentially semistable representations of $G_{\mathbb{Q}_p}$ with fixed Hodge–Tate weights $\boldsymbol{v}$ and Galois type τ. This was a very difficult question, but thanks to Kisin it is now known.

Theorem 9.1.1. *Let $\boldsymbol{v} = (s, t)$ be a pair of integers and τ be a Galois type. Then there exists a quotient $\pi : \mathfrak{R}_{\bar{\rho}_0} \to \mathfrak{R}^{\boldsymbol{v},\tau}_{\bar{\rho}_0}$ such that a geometric point $x : \mathfrak{R}_{\bar{\rho}_0} \to \overline{\mathbb{Q}}_p$ induces a potentially semistable lift with Hodge–Tate weights $\boldsymbol{v}$ and with Galois type τ if and only if x factors through π.*

Moreover, $\mathfrak{R}^{\boldsymbol{v},\tau}_{\bar{\rho}_0}$ is p-torsion free and has a reduced generic fiber, and the relative dimension of $\mathfrak{R}^{\boldsymbol{v},\tau}_{\bar{\rho}_0}$ over $\mathbb{O}$ is 2.

Proof. This is a result of Kisin [2008]. □

Note that Kisin proves a much more general statement than the one above. We also note that a potentially semistable representation with trivial Galois type is semistable. In this case, we write $\mathfrak{R}^{\boldsymbol{v}}_{\bar{\rho}_0}$ for $\mathfrak{R}^{\boldsymbol{v},\tau}_{\bar{\rho}_0}$.

Fix a lift $\psi : G_{\mathbb{Q}_p} \to \mathbb{O}^\times$ of $\det \bar{\rho}_0$, and write $\mathfrak{R}^{\boldsymbol{v},\tau,\psi}_{\bar{\rho}_0}$ for the deformation ring whose characteristic 0 closed points parameterizes potentially semistable lifts of $\bar{\rho}_0$ with Hodge–Tate weights $\boldsymbol{v}$, Galois type τ, and a fixed determinant ψ. The ring $\mathfrak{R}^{\boldsymbol{v},\tau,\psi}_{\bar{\rho}_0}$ forms a closed subscheme of $\mathfrak{R}^{\boldsymbol{v},\tau}_{\bar{\rho}_0}$. Moreover, it is known from [Emerton and Gee 2014, Lemma 4.3.1] that if $p > 2$ then we have

$$\mathfrak{R}^{\boldsymbol{v},\tau}_{\bar{\rho}_0} \cong \mathfrak{R}^{\boldsymbol{v},\tau,\psi}_{\bar{\rho}_0}[\![D]\!]. \tag{9.1.2}$$

We also write $\mathfrak{R}^{\boldsymbol{v},\psi}_{\bar{\rho}_0}$ for $\mathfrak{R}^{\boldsymbol{v},\tau,\psi}_{\bar{\rho}_0}$ if τ is trivial.

9.2. *Strongly divisible modules with coefficients.* In this subsection, we review some integral p-adic Hodge theory over complete Noetherian local rings.

Let $\mathfrak{R}$ be an object in $\hat{\mathscr{C}}$. We define $S_{\mathfrak{R}}$ by $\mathfrak{m}_{\mathfrak{R}}$-completion of $S \otimes_{\mathbb{Z}_p} \mathfrak{R}$, and extend the definitions of Fil, ϕ, and N to $S_{\mathfrak{R}}$ $\mathfrak{R}$-linearly. Fix a positive integer $r < p - 1$. We now define the category $\mathfrak{MD}^r_{\mathfrak{R}}$ of strongly divisible $S_{\mathfrak{R}}$-modules as follows: an object in $\mathfrak{MD}^r_{\mathfrak{R}}$ consists of quadruples $(\mathfrak{M}, \operatorname{Fil}^r \mathfrak{M}, \phi_r, N)$ where

- $\mathfrak{M}$ is a finitely generated free $S_{\mathfrak{R}}$-module;
- $\operatorname{Fil}^r \mathfrak{M}$ is a submodule of $\mathfrak{M}$ over $S_{\mathfrak{R}}$;
- $\phi_r : \operatorname{Fil}^r \mathfrak{M} \to \mathfrak{M}$ and $N : \mathfrak{M} \to \mathfrak{M}$ are additive maps

satisfying the following conditions:

(1) $\operatorname{Fil}^r \mathfrak{M}$ contains $\operatorname{Fil}^r S_{\mathfrak{R}}\mathfrak{M}$;

(2) $\operatorname{Fil}^r \mathfrak{M} \cap I\mathfrak{M} = I \operatorname{Fil}^r \mathfrak{M}$ for all ideals I of $\mathfrak{R}$;

(3) $\phi_r(sx) = \phi(s)\phi_r(x)$ for all $s \in S_{\mathfrak{R}}$ and for all $x \in \operatorname{Fil}^r \mathfrak{M}$;

(4) $\phi_r(\operatorname{Fil}^r \mathfrak{M})$ is contained in $\mathfrak{M}$ and generates $\mathfrak{M}$ over $S_{\mathfrak{R}}$;

(5) $N(sx) = sN(x) + N(s)x$ for all $s \in S_{\mathfrak{R}}$ and for all $x \in \mathfrak{M}$;

(6) $E(u)N(\operatorname{Fil}^r \mathfrak{M}) \subset \operatorname{Fil}^r \mathfrak{M}$;

(7) $cN\phi_r(x) = \phi_r(E(u)N(x))$ for all $x \in \operatorname{Fil}^r \mathfrak{M}$, where $c := 1/p\,\phi(E(u))$.

The morphisms are $S_{\mathfrak{R}}$-linear maps that preserve $\mathrm{Fil}^r\,\mathfrak{M}$ and that commute with ϕ_r and N. If a strongly divisible module is defined over $S_{\mathfrak{R}}$ then we often say that it is defined over $\mathfrak{R}$.

Note that if $\mathfrak{R}$ is p-torsion free then it is easy to check that there is an additive map $\phi : \mathfrak{M} \to \mathfrak{M}$ such that $\phi(sx) = \phi(s)\phi(x)$ for all $s \in S_{\mathfrak{R}}$ and for all $x \in \mathfrak{M}$ and that $\phi_r = 1/p^r\,\phi$. Moreover, the condition (7) is equivalent to $N\phi = p\phi N$ on $\mathfrak{M}$. It is also easy to check that $\mathfrak{M}/(\mathfrak{m}_{\mathfrak{R}}, \mathrm{Fil}^p\, S_{\mathfrak{R}})$ naturally has a structure of Breuil module over $\bar{S}_{\mathbb{F}}$.

There is an exact faithful covariant functor $\mathrm{T}^r_{\mathrm{st}}$ from the category $\mathfrak{MD}^r_{\mathfrak{R}}$ to the category of $\mathfrak{R}$-representations of $G_{\mathbb{Q}_p}$, defined in [Breuil and Mézard 2002]. It satisfies various compatibilities:

Lemma 9.2.1. *Fix a positive integer $r < p-1$. Let $\mathfrak{R}$ be an object in $\hat{\mathscr{C}}$, I an ideal of $\mathfrak{R}$ containing $\mathfrak{m}^n_{\mathfrak{R}}$ for some $n > 0$, and $\mathfrak{M}$ a strongly divisible module of weight r over $S_{\mathfrak{R}}$.*

(1) *If $\mathfrak{R}'$ is a complete Noetherian local $\mathbb{O}$-algebra whose residue field is a finite extension of $\mathbb{F}$ with a morphism $\mathfrak{R}/I \to \mathfrak{R}'$ then*

$$\mathrm{T}^r_{\mathrm{st}}(\mathfrak{M} \otimes_{\mathfrak{R}} \mathfrak{R}') \cong \mathrm{T}^r_{\mathrm{st}}(\mathfrak{M}) \otimes_{\mathfrak{R}} \mathfrak{R}'.$$

(2) *The induced map $\mathrm{T}^r_{\mathrm{st}}(\mathfrak{M}) \to \mathrm{T}^r_{\mathrm{st}}(\mathfrak{M} \otimes_{\mathfrak{R}} \mathfrak{R}/I)$ is surjective.*

(3) *If $\mathcal{M}$ is the Breuil module $\mathfrak{M}/(\mathfrak{m}_{\mathfrak{R}}, \mathrm{Fil}^p\, S_{\mathfrak{R}})$ then*

$$\mathrm{T}^r_{\mathrm{st}}(\mathcal{M}) \cong \mathrm{T}^r_{\mathrm{st}}(\mathfrak{M}) \otimes_{\mathfrak{R}} \mathbb{F}.$$

Proof. These are proved in [Breuil and Mézard 2002, §3.2]. □

9.3. *Proof of Theorem 9.0.1.* In this subsection, we prove Theorem 9.0.1. Note that we are interested only in absolutely irreducible mod p reductions in this section. Hence, by Theorem 5.0.5 we divide its proof into three cases as follows:

$$\begin{aligned} h < \tfrac{3}{2} - m \quad &\text{and} \quad r = 2m = 2 && \text{in Section 9.3.1;}\\ h < \tfrac{3}{2} - m \quad &\text{and} \quad r = 2m > 2 && \text{in Section 9.3.2;}\\ -\tfrac{1}{2} - \ell < h < \tfrac{1}{2} - \ell \quad &\text{for} \quad \ell \in \{0, 1, 2, \dots, m-2\} && \text{in Section 9.3.3.} \end{aligned}$$

We start this subsection by proving the following lemma.

Lemma 9.3.1. *Let $a, b \in \mathbb{Z}_{\geq 0}$ with $0 < a + b < p - 1$, and let $\bar{\rho}_0 : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(\mathbb{F})$ be an irreducible representation such that $\bar{\rho}_0|_{I_{\mathbb{Q}_p}} \cong \omega_2^{a+pb} \oplus \omega_2^{b+pa}$ with $a \neq b$. Then*

(1) *$\bar{\rho}_0$ has a crystalline lift with Hodge–Tate weights (a, b);*

(2) *$\bar{\rho}_0$ does not have a crystalline lift with Hodge–Tate weights $(0, a+b)$ unless $ab = 0$.*

Proof. (2) is obvious by Fontaine–Laffaille theory. We now prove (1). By [Gee and Savitt 2011, Lemma 6.2], there exists a crystalline character $\varepsilon_{(a,b)} : G_{\mathbb{Q}_{p^2}} \to \mathbb{O}^\times$ such that $\mathrm{HT}_{\sigma^2}(\varepsilon_{(a,b)}) = \{a\}$ and $\mathrm{HT}_{\sigma}(\varepsilon_{(a,b)}) = \{b\}$, where σ is the generator of the cyclic group $\mathrm{Gal}(\mathbb{Q}_{p^2}/\mathbb{Q}_p)$ of order 2. Note that $\bar{\varepsilon}_{(a,b)} \cong \omega_2^{a+pb}$. $V_{(a,b)} := \mathrm{Ind}_{G_{\mathbb{Q}_{p^2}}}^{G_{\mathbb{Q}_p}} \varepsilon_{(a,b)}$ is a 2-dimensional crystalline lift of $\bar{\rho}_0$ with Hodge–Tate weights (a, b) up to twisting by an unramified character. This completes (1). □

9.3.1. *The case $h < \frac{3}{2} - m$ and $r = 2m = 2$.* Assume that $r = 2m = 2$ and that $h < \frac{3}{2} - m = \frac{1}{2}$. Recall that $a(2) = 1$ and that $v_p(\lambda) = \frac{1}{2}$. We keep the notation as in Section 8.1.

Fix a residual representation $\bar{\rho}_0$ such that $\bar{\rho}_0|_{I_{\mathbb{Q}_p}} \cong \omega_2^2 \oplus \omega_2^{2p}$. Since T_{st} is fully faithful, it is equivalent to fixing a Breuil module $\mathcal{M}(2, 0, \alpha, \beta)$ in Example 3.3.1. By Lemma 3.3.2, we need to fix $\alpha \cdot \beta$. Say, $-\mu = \alpha \cdot \beta$ for a fixed $\mu \in \mathbb{F}^\times$. By Lemma 8.4.1 for $r = 2$, we have the identity $\mu = \lambda^2/p \pmod{\mathfrak{m}}$. We also fix a lift $\psi : G_{\mathbb{Q}_p} \to \mathbb{O}^\times$ of $\det \bar{\rho}_0$, which is equivalent to fixing $\eta := \lambda^2 \in \mathbb{O}$.

Fix $\lambda \in \mathbb{O}$ satisfying the equation $\eta = \lambda^2$, extending E if necessary. We note that there are two λ satisfying the equation $\eta = \lambda^2$, say, $\lambda = \sqrt{\eta}$ and $\lambda = -\sqrt{\eta}$. For each such λ, the admissible filtered (ϕ, N)-modules $D(\lambda, \mathfrak{L})$ for $\mathfrak{L} \in E$ satisfying $v_p(\mathfrak{L} - 1) < \frac{1}{2}$ corresponds to the characteristic 0 closed points of $\mathfrak{R}_{\bar{\rho}_0}^{(0,2),\psi}$. Thus for each such λ, we may let $X = p/(\lambda(\mathfrak{L} - 1))$, keeping in mind that X varies over the maximal ideal $\mathfrak{m}$ in $\mathbb{O}$.

Lemma 9.3.2. *Keep the notation as above.*

(1) *For $\lambda = \sqrt{\eta}$, denoting $X = p/(\lambda(\mathfrak{L} - 1))$ in the strongly divisible modules in Section 8.1 determines a strongly divisible module over $S_{\mathbb{O}[\![X]\!]}$;*

(2) *For $\lambda = -\sqrt{\eta}$, denoting $X = p/(\lambda(\mathfrak{L} - 1))$ in the strongly divisible modules in Section 8.1 determines a strongly divisible module over $S_{\mathbb{O}[\![X]\!]}$.*

We denoted by $\mathfrak{M}_2^+(X)$ (resp. $\mathfrak{M}_2^-(X)$) for the strongly divisible module over $\mathbb{O}[\![X]\!]$ in (1) (resp. in (2)) in Lemma 9.3.2.

Proof. We only prove the case (1). The case (2) is identical. Assume $\lambda = \sqrt{\eta}$, and let $X = p/(\lambda(\mathfrak{L} - 1))$. Using these identities, if we replace $p/(\lambda(\mathfrak{L} - 1))$ with X, then

$$G_{2,n+1} = \frac{1}{1 - \eta/p\, X^2 G_{2,n}}.$$

Thus, we may regard $G_{2,n}$ as an element in $\mathbb{O}[\![X]\!]$, so that Δ_2 in $\mathbb{O}[\![X]\!]$ as well, which we denote by $\Delta_2(X)$. Note that $\Delta_2(X) \equiv 1$ modulo $((\pi) + (X)^2)$.

From Section 8.1 and from the proof of Lemma 8.4.1, we see that $\mathrm{Fil}^2 \mathfrak{M}$ is generated by $\hat{F}_1$ and $v^2 E_1$ modulo $\mathrm{Fil}^2 S_{\mathbb{O}}\mathfrak{M}$. If we replace $p/(\lambda(\mathfrak{L} - 1))$ with X in the coefficients of $\hat{F}_1$, $v^2 E_1$, $\phi_2(\hat{F}_1)$, $\phi_2(v^2 E_1)$, $N(E_1)$, and $N(E_2)$, then we claim that we get a strongly divisible module over $S_{\mathbb{O}[\![X]\!]}$. Indeed, the proof of this claim is equivalent to the proof that specializing X in $\mathfrak{m}$ gives rise to a strongly divisible module over $S_{\mathbb{O}}$. If $X \neq 0$, we do know that the corresponding object is a strongly divisible module over $S_{\mathbb{O}}$ by the results in Section 8.1. But if $X = 0$, we claim that $\mathfrak{M}_2^+(0)$ corresponds to a crystalline representation. We check this by showing that the filtered (ϕ, N)-module corresponding to $\mathfrak{M}_2^+(0)[1/p]$ has $N = 0$ and is admissible.

Using $\Delta_2(0) = 1$, $\mathfrak{M}_2^+(0)$ is computed as follows:

- $\mathfrak{M} := S_{\mathbb{O}}(E_1, E_2)$;
- $\phi(E_1) = p\lambda E_1 + (1-p)E_2$ and $\phi(E_2) = \lambda E_2$;
- $N(E_1) = 0 = N(E_2)$;
- $\mathrm{Fil}^2\,\mathfrak{M} = \langle \lambda E_1 - (1-p)E_2\rangle_{S_{\mathbb{O}}} + \mathrm{Fil}^2\, S\mathfrak{M}$.

Following the functor described in the paragraph of (2.2.1), one can readily compute a filtered (ϕ, N)-module D corresponding $\mathfrak{M}_2^+(0)$:

- $D := E(\eta_1, \eta_2)$ and $\underline{\eta} := (\eta_1, \eta_2)$;
- $\mathrm{Fil}^i\, D := \begin{cases} D & \text{if } i \leq 0; \\ E(\lambda\eta_1 - (1-p)\eta_2) & \text{if } i = 1, 2; \\ 0 & \text{if } 3 \leq i; \end{cases}$
- $\mathrm{Mat}_{\underline{\eta}}(\phi) = \begin{pmatrix} p\lambda & 0 \\ 1-p & \lambda \end{pmatrix}$ and $N = 0$.

The only ϕ-invariant subspaces of D of dimension 1 are $E(\lambda\eta_1 - \eta_2)$ and $E(\eta_2)$. Now it is easy to check that D is admissible. Since D is admissible and $N = 0$ on D, $\mathrm{T}^2_{\mathrm{st}}(\mathfrak{M}_2^+(0))$ is a Galois stable lattice in a crystalline lift of $\bar{\rho}_0$ with Hodge–Tate weights $(0, 2)$, which completes the proof. □

Considering $\mathrm{T}^2_{\mathrm{st}}(\mathfrak{M}_2^\bullet(X))$ for $\bullet \in \{\pm\}$, one sees that Lemma 9.3.2 gives rise to $\mathbb{O}$-algebra morphisms

$$\pi_2^\bullet : \mathfrak{R}_{\bar{\rho}_0} \to \mathbb{O}[\![X]\!].$$

Lemma 9.3.3. *For each $\bullet \in \{\pm\}$, the morphism $\pi_2^\bullet : \mathfrak{R}_{\bar{\rho}_0} \to \mathbb{O}[\![X]\!]$ is surjective.*

Proof. Let $\mathfrak{R}_0$ be the quotient ring $\mathbb{O}[\![X]\!]/((\pi)+(X)^2) \cong \mathbb{F}[X]/X^2$. By Nakayama's lemma and by Lemma 9.2.1 (2), it is enough to show that the induced morphism $\pi_2^\bullet : \mathfrak{R}_{\bar{\rho}_0} \to \mathfrak{R}_0$ is surjective for each $\bullet \in \{\pm\}$. We only consider the case $\bullet = +$. The case $\bullet = -$ is very similar. We compute the strongly divisible module $\mathfrak{M}_0 := \mathfrak{M}_2^+(X)/((\pi) + (X)^2)$ over $S_{\mathfrak{R}_0}$. We describe $\mathfrak{M}_0 = S_{\mathfrak{R}_0}(E_1, E_2)$ as follows:

- $\mathrm{Fil}^2\,\mathfrak{M}_0$ is generated by u^2E_1 and E_2 modulo $\mathrm{Fil}^p\, S_{\mathfrak{R}_0}\mathfrak{M}_0$;
- $\phi_2(u^2E_1) = E_2$ and $\phi_2(E_2) = -\mu E_1$;
- $N(E_1) = X(1 - [u^p + (u-p)^{p-1}])E_2$ and $N(E_2) = 0$.

Let $A \in \mathrm{GL}_2(S_{\mathfrak{R}_0})$ be a change of basis matrix. Then $A = A_0 + XA_1$ for some $A_0 \in \mathrm{GL}_n(S_{\mathbb{F}})$ and $A_1 \in \mathrm{M}_{2\times 2}(S_{\mathbb{F}})$. We also write $A^{-1} = B_0 + XB_1$ for some $B_0 \in \mathrm{GL}_n(S_{\mathbb{F}})$ and $B_1 \in \mathrm{M}_{2\times 2}(S_{\mathbb{F}})$. Let $C := \mathrm{Mat}_{\underline{e}}(N)$, where $\underline{e} := (E_1, E_2)$. Then

$$ACA^{-1} = (A_0 + XA_1)C(B_0 + XB_1) = XA_0 \begin{pmatrix} 0 & 0 \\ 1 - [u^p + u^{p-1}] & 0 \end{pmatrix} B_0.$$

Hence, one sees that $\mathfrak{M}_0$ is not defined over any proper subring of $\mathfrak{R}_0$, so that we conclude that π_2^+ is surjective. □

Lemma 9.3.4. *For each $\bullet \in \{\pm\}$, the morphism $\pi_2^\bullet : \mathfrak{R}_{\bar{\rho}_0} \to \mathbb{O}[\![X]\!]$ factors through the quotient $q : \mathfrak{R}_{\bar{\rho}_0} \to \mathfrak{R}_{\bar{\rho}_0}^{(0,2),\psi}$.*

Proof. Fix $\bullet \in \{\pm\}$, and let $f : \mathbb{O}[\![X]\!] \to \overline{\mathbb{Q}}_p$ be an $\mathbb{O}$-algebra morphism. Then by the universal property of $\mathfrak{M}_2^\bullet(X)$, there exists a semistable representation of $G_{\mathbb{Q}_p}$ with Hodge–Tate weights $(0, 2)$ corresponding to f. Hence, there exists an $\mathbb{O}$-algebra morphism $f' : \mathfrak{R}_{\bar{\rho}_0}^{(0,2),\psi} \to \overline{\mathbb{Q}}_p$ such that $f \circ \pi_2^\bullet = f' \circ q$. Since $\mathbb{O}[\![X]\!]$ is p-torsion free and has a reduced generic fiber, we have an $\mathbb{O}$-algebra morphism $q_\bullet : \mathfrak{R}_{\bar{\rho}_0}^{(0,2),\psi} \to \mathbb{O}[\![X]\!]$ such that $q_\bullet \circ q = \pi_2^\bullet$, which completes the proof. □

Proposition 9.3.5. *If $\bar{\rho}_0|_{I_{\mathbb{Q}_p}} \cong \omega_2^2 \oplus \omega_2^{2p}$, then*

$$\mathfrak{R}_{\bar{\rho}_0}^{(0,2)} \sim' \mathbb{O}[\![D_1, D_2]\!] \times \mathbb{O}[\![D, X]\!] \times \mathbb{O}[\![D, X]\!].$$

Proof. By a result of Kisin [2008], we know that the relative dimension of $\mathfrak{R}_{\bar{\rho}_0}^{(0,2)}$ over $\mathbb{O}$ is 2, so that we know that there are at least two irreducible components by Lemma 9.3.4, both of which are isomorphic to $\mathbb{O}[\![D, X]\!]$ by (9.1.2).

In fact, we have one more irreducible component, parameterizing the crystalline lifts with Hodge–Tate weights $(0, 2)$ of $\bar{\rho}_0$. By Lemma 9.3.1 (1), $\bar{\rho}_0$ has a crystalline lift with Hodge–Tate weights $(0, 2)$. By a result of [Clozel et al. 2008], we do know that it is formally smooth. Since it also has a relative dimension 2 over $\mathbb{O}$, we denote this crystalline deformation ring by $\mathbb{O}[\![D_1, D_2]\!]$. Note that the characteristic 0 closed points of these irreducible components exhaust all the 2-dimensional semistable lifts of $\bar{\rho}_0$ with Hodge–Tate weights $(0, 2)$, which completes the proof. □

9.3.2. *The case $h < \frac{3}{2} - m$ and $r = 2m > 2$.* Assume that $r = 2m > 2$ and that $h < \frac{3}{2} - m$. Recall that $v_p(\lambda) = \frac{1}{2}(r-1) = m - \frac{1}{2}$. We keep the notation as in Section 8.3.

Fix a residual representation $\bar{\rho}_0$ such that $\bar{\rho}_0|_{I_{\mathbb{Q}_p}} \cong \omega_2^r \oplus \omega_2^{pr}$. Since T_{st} is fully faithful, it is equivalent to fixing a Breuil module $\mathcal{M}(r, 0, \alpha, \beta)$ in Example 3.3.1. By Lemma 3.3.2, we need to fix $\alpha \cdot \beta$. Say, $-\mu = \alpha \cdot \beta$ for a fixed $\mu \in \mathbb{F}^\times$. By Lemma 8.4.1 for $r > 2$, we have the identity $\mu = \lambda^2/p^{r-1} \pmod{\mathfrak{m}}$. We also fix a lift $\psi : G_{\mathbb{Q}_p} \to \mathbb{O}^\times$ of $\det \bar{\rho}_0$, which is equivalent to fixing $\eta := \lambda^2 \in \mathbb{O}$.

Fix $\lambda \in \mathbb{O}$ satisfying the equation $\eta = \lambda^2$, extending E if necessary. We note that there are two λ satisfying the equation $\eta = \lambda^2$, say, $\lambda = \sqrt{\eta}$ and $\lambda = -\sqrt{\eta}$. For each such λ, the admissible filtered (ϕ, N)-modules $D(\lambda, \mathfrak{L})$ for $\mathfrak{L} \in E$ satisfying $v_p(\mathfrak{L} - a(r)) < \frac{3}{2} - m$ corresponds to the characteristic 0 closed points of $\mathfrak{R}^{(0,r),\psi}_{\bar{\rho}_0}$. Thus for each such λ, we may let $X = p/(\lambda(\mathfrak{L} - a(r)))$, keeping in mind that X varies over the maximal ideal $\mathfrak{m}$ in $\mathbb{O}$.

Lemma 9.3.6. *Keep the notation as above.*

(1) *For $\lambda = \sqrt{\eta}$, denoting $X = p/(\lambda(\mathfrak{L} - a(r)))$ in the strongly divisible modules in Section 8.3 determines a strongly divisible module over $S_{\mathbb{O}[\![X]\!]}$;*

(2) *For $\lambda = -\sqrt{\eta}$, denoting $X = p/(\lambda(\mathfrak{L} - a(r)))$ in the strongly divisible modules in Section 8.3 determines a strongly divisible module over $S_{\mathbb{O}[\![X]\!]}$.*

We denoted by $\mathfrak{M}_r^+(X)$ (resp. $\mathfrak{M}_r^-(X)$) for the strongly divisible module over $\mathbb{O}[\![X]\!]$ in (1) (resp. in (2)) in Lemma 9.3.6.

Proof. The proof is very similar to Lemma 9.3.2. We only prove the case (1). The case (2) is identical. Assume $\lambda = \sqrt{\eta}$, and let $X = p/(\lambda(\mathfrak{L} - a(r)))$.

From Section 8.3 and from the proof of Lemma 8.4.1, we see that $\mathrm{Fil}^r \mathfrak{M}$ is generated by $\hat{F}_0$ and $v^r E_1$ modulo $\mathrm{Fil}^r S_{\mathbb{O}}\mathfrak{M}$. If we replace $p/(\lambda(\mathfrak{L} - a(r)))$ with X in the coefficients of $\hat{F}_0$, $v^r E_1$, $\phi_r(\hat{F}_0)$, $\phi_r(v^r E_1)$, $N(E_1)$, and $N(E_2)$, then we claim that we get a strongly divisible module over $S_{\mathbb{O}[\![X]\!]}$. Indeed, the proof of this claim is equivalent to the proof that specializing X in $\mathfrak{m}$ gives rise to a strongly divisible module over $S_{\mathbb{O}}$. If $X \neq 0$, we do know that the corresponding object is a strongly divisible module over $S_{\mathbb{O}}$ by the results in Section 8.3. But if $X = 0$, we claim that $\mathfrak{M}_r^+(0)$ corresponds to a crystalline representation. We check this by showing that the filtered (ϕ, N)-module corresponding to $\mathfrak{M}_r^+(0)[1/p]$ has $N = 0$ and admissible.

$\mathfrak{M}_r^+(0)$ is computed as follows:

- $\mathfrak{M} := S_{\mathbb{O}}(E_1, E_2)$;
- $\phi(E_1) = p\lambda E_1 + (1-p)E_2$ and $\phi(E_2) = \lambda E_2$;
- $N(E_1) = 0 = N(E_2)$;
- $\mathrm{Fil}^r \mathfrak{M} = \langle \lambda E_1 - (1-p)E_2 \rangle_{S_{\mathbb{O}}} + \mathrm{Fil}^r S\mathfrak{M}$.

Following the functor described in the paragraph of (2.2.1), one can readily compute a filtered (ϕ, N)-module D corresponding $\mathfrak{M}_r^+(0)$:

- $D := E(\eta_1, \eta_2)$ and $\underline{\eta} := (\eta_1, \eta_2)$;
- $\mathrm{Fil}^i D := \begin{cases} D & \text{if } i \le 0; \\ E(\lambda\eta_1 - (1-p)\eta_2) & \text{if } 0 < i \le r; \\ 0 & \text{if } r < i; \end{cases}$

- $\mathrm{Mat}_{\underline{\eta}}(\phi) = \begin{pmatrix} p\lambda & 0 \\ 1-p & \lambda \end{pmatrix}$ and $N = 0$.

The only ϕ-invariant subspaces of D of dimension 1 are $E(\lambda\eta_1 - \eta_2)$ and $E(\eta_2)$. Now it is easy to check that D is admissible. Since D is admissible and $N = 0$ on D, $\mathrm{T}^r_{\mathrm{st}}(\mathfrak{M}^+_r(0))$ is a Galois stable lattice in a crystalline lift of $\bar{\rho}_0$ with Hodge–Tate weights $(0, r)$, which completes the proof. □

Considering $\mathrm{T}^r_{\mathrm{st}}(\mathfrak{M}^\bullet_r(X))$ for $\bullet \in \{\pm\}$, one sees that Lemma 9.3.6 gives rise to $\mathbb{O}$-algebra morphisms

$$\pi^\bullet_r : \mathfrak{R}_{\bar{\rho}_0} \to \mathbb{O}[\![X]\!].$$

Lemma 9.3.7. *For each $\bullet \in \{\pm\}$, the morphism $\pi^\bullet_r : \mathfrak{R}_{\bar{\rho}_0} \to \mathbb{O}[\![X]\!]$ is surjective.*

Proof. Let $\mathfrak{R}_0$ be the quotient ring $\mathbb{O}[\![X]\!]/((\pi) + (X)^2) \cong \mathbb{F}[X]/X^2$. By the same argument as in Lemma 9.3.3, it is enough to show that the induced morphism $\pi^\bullet_r : \mathfrak{R}_{\bar{\rho}_0} \to \mathfrak{R}_0$ is surjective for each $\bullet \in \{\pm\}$. We only consider the case $\bullet = +$. The case $\bullet = -$ is very similar. We compute the strongly divisible module $\mathfrak{M}_0 := \mathfrak{M}^+_r(X)/((\pi) + (X)^2)$ over $S_{\mathfrak{R}_0}$. We describe $\mathfrak{M}_0 = S_{\mathfrak{R}_0}(E_1, E_2)$ as follows:

- $\mathrm{Fil}^r\,\mathfrak{M}_0$ is generated by $u^r E_1$ and $E_2 - \frac{\mu}{r-1} X u^{r-1} E_1$ modulo $\mathrm{Fil}^p\, S_{\mathbb{O}}\mathfrak{M}_0$;
- $\phi_r(u^r E_1) = E_2$ and $\phi_r\Big(E_2 - \frac{\mu}{r-1} X u^{r-1} E_1\Big) = -(\mu + 2\sqrt{\mu} D) E_1$;
- $N(E_1) = X\Big(1 - \frac{N(\delta_{-\infty})}{p}\Big) E_2$ and $N(E_2) = 0$.

By the same argument as in Lemma 9.3.3, one sees that $\mathfrak{M}_0$ is not defined over any proper subring of $\mathfrak{R}_0$. Hence, we conclude that π^+_r is surjective. □

Lemma 9.3.8. *For each $\bullet \in \{\pm\}$, the morphism $\pi^\bullet_r : \mathfrak{R}_{\bar{\rho}_0} \to \mathbb{O}[\![X]\!]$ factors through the quotient $q : \mathfrak{R}_{\bar{\rho}_0} \to \mathfrak{R}^{(0,r),\psi}_{\bar{\rho}_0}$.*

Proof. The same argument as in Lemma 9.3.4 works. □

Proposition 9.3.9. *If $\bar{\rho}_0|_{I_{\mathbb{Q}_p}} \cong \omega_2^r \oplus \omega_2^{pr}$, then*

$$\mathfrak{R}^{(0,r)}_{\bar{\rho}_0} \sim' \mathbb{O}[\![D_1, D_2]\!] \times \mathbb{O}[\![D, X]\!] \times \mathbb{O}[\![D, X]\!].$$

Proof. The proof is very similar to Proposition 9.3.5. The only difference is that $\bar{\rho}_0$ has a crystalline lift with Hodge–Tate weights $(0, r)$, by Lemma 9.3.1 (1). Note that the characteristic 0 closed points of these irreducible components exhaust all the 2-dimensional semistable lifts of $\bar{\rho}_0$ with Hodge–Tate weights $(0, r)$. □

9.3.3. *The case* $-\frac{1}{2}-\ell < h < \frac{1}{2}-\ell$ *for* $\ell \in \{0, 1, 2, \dots, m-2\}$. Assume that $r = 2m \geq 4$ and that $-\frac{1}{2}-\ell < h < \frac{1}{2}-\ell$ for $\ell \in \{0, 1, 2, \dots, m-2\}$. Recall that $v_p(\lambda) = \frac{1}{2}(r-1) = m - \frac{1}{2}$. We keep the notation as in Section 7.2.

Fix a residual representation $\bar{\rho}_0$ such that

$$\bar{\rho}_0|_{I_{\mathbb{Q}_p}} \cong \omega_2^{m+\ell+1+p(m-\ell-1)} \oplus \omega_2^{m-\ell-1+p(m+\ell+1)}.$$

Since T_{st} is fully faithful, it is equivalent to fixing a Breuil module $\mathcal{M}(m+\ell+1, m-\ell-1, \alpha, \beta)$ in Example 3.3.1. By Lemma 3.3.2, we need to fix $\alpha \cdot \beta$. By Lemma 7.3.1, it is enough to fix λ^2/p^{r-1}, since all the other quantities do not depend on λ or on $\mathfrak{L}$. Say, $\mu := \lambda^2/p^{r-1} \pmod{\mathfrak{m}}$ for a fixed $\mu \in \mathbb{F}^\times$. We also fix a lift $\psi : G_{\mathbb{Q}_p} \to \mathbb{O}^\times$ of $\det \bar{\rho}_0$, which is equivalent to fixing $\eta := \lambda^2 \in \mathbb{O}$.

Fix $\lambda \in \mathbb{O}$ satisfying the equation $\eta = \lambda^2$, extending E if necessary. We note that there are two λ satisfying the equation $\eta = \lambda^2$, say $\lambda = \sqrt{\eta}$ and $\lambda = -\sqrt{\eta}$. For each such λ, the admissible filtered (ϕ, N)-modules $D(\lambda, \mathfrak{L})$ for $\mathfrak{L} \in E$ satisfying $-\frac{1}{2}-\ell < v_p(\mathfrak{L}-a(r)) < \frac{1}{2}-\ell$ corresponds to the characteristic 0 closed points of $\mathfrak{R}_{\bar{\rho}_0}^{(0,r),\psi}$. Thus for each such λ, we may let $X = p^{m-\ell}/(\lambda(\mathfrak{L}-a(r)))$, keeping in mind that X varies over the maximal ideal $\mathfrak{m}$ in $\mathbb{O}$ satisfying $0 < v_p(X) < 1$. We also let $Y = \lambda(\mathfrak{L}-a(r))/p^{m-\ell-1}$, so that X and Y satisfy the equation $XY = p$.

Lemma 9.3.10. *Keep the notation as above*:

(1) *For* $\lambda = \sqrt{\eta}$, *denoting* $X = p^{m-\ell}/(\lambda(\mathfrak{L}-a(r)))$ *and* $Y = \lambda(\mathfrak{L}-a(r))/p^{m-\ell-1}$ *in the strongly divisible modules in Section 7.2 determines a strongly divisible module over* $S_{\mathbb{O}[\![X,Y]\!]/(XY-p)}$;

(2) *For* $\lambda = -\sqrt{\eta}$, *denoting* $X = p^{m-\ell}/(\lambda(\mathfrak{L}-a(r)))$ *and* $Y = \lambda(\mathfrak{L}-a(r))/p^{m-\ell-1}$ *in the strongly divisible modules in Section 7.2 determines a strongly divisible module over* $S_{\mathbb{O}[\![X,Y]\!]/(XY-p)}$.

We denoted by $\mathfrak{M}_{r,\ell}^+(X)$ (resp. $\mathfrak{M}_{r,\ell}^-(X)$) for the strongly divisible module over $\mathbb{O}[\![X, Y]\!]/(XY-p)$ in (1) (resp. in (2)) in Lemma 9.3.10.

Proof. We only prove the case (1). The case (2) is identical. Assume $\lambda = \sqrt{\eta}$, and let $X = p^{m-\ell}/(\lambda(\mathfrak{L}-a(r)))$ and $Y = \lambda(\mathfrak{L}-a(r))/p^{m-\ell-1}$. The proof of this lemma is equivalent to the proof that specializing X and Y in $\mathfrak{m}$ satisfying $XY = p$ gives rise to a strongly divisible module over $S_{\mathbb{O}}$, as in Lemmas 9.3.2 and 9.3.6. But the proof of this lemma is, in fact, easier, since the results in Section 7.2 already tell us all the specializations are strongly divisible modules over $S_{\mathbb{O}}$. Hence, we only need to check that the resulting object replacing $p^{m-\ell}/(\lambda(\mathfrak{L}-a(r)))$ and $\lambda(\mathfrak{L}-a(r))/p^{m-\ell-1}$ with X and Y respectively are defined over $S_{\mathbb{O}[\![X,Y]\!]/(XY-p)}$.

From Section 7.2 and from the proof of Lemma 7.3.1, we see that the $\mathrm{Fil}^r\,\mathfrak{M}$ is generated by $\hat{F}_0$ and $\hat{F}_{m+\ell+1}$ modulo $\mathrm{Fil}^r\,S_{\mathbb{O}}\mathfrak{M}$. If we replace $p^{m-\ell}/(\lambda(\mathfrak{L}-a(r)))$ and $\lambda(\mathfrak{L}-a(r))/p^{m-\ell-1}$ with X and Y, respectively in the coefficients of $\hat{F}_0$,

$\hat{F}_{m+\ell+1}$, $\phi_r(\hat{F}_0)$, $\phi_r(\hat{F}_{m+\ell+1})$, $N(E_1)$, and $N(E_2)$, then it is easy and routine to check that we get a strongly divisible module, denoted by $\mathfrak{M}^{+}_{r,\ell}(X)$, defined over $S_{\mathbb{O}[\![X,Y]\!]/(XY-p)}$: to see this, one wants to check first that

$$R_{i,j},\ (RP)_{i,j} \in \frac{\mathbb{O}[\![X,Y]\!]}{(XY-p)}\left[\frac{1}{p}\right].$$

In the case $\ell = 0$, one also needs to show that $\Delta_r \in \mathbb{O}[\![X,Y]\!]/(XY-p)$ and $\delta_0 \in S_{\mathbb{O}[\![X,Y]\!]/(XY-p)}$. By a similar argument as in Section 9.3.3, one can readily check these properties as well as $\Delta_r \equiv X_0(T_0 - a(r)S_0)$ modulo $(\pi) + (X,Y)^2$. We leave the details for the reader. □

Considering $\mathrm{T}^r_{\mathrm{st}}(\mathfrak{M}^{\bullet}_{r,\ell}(X))$ for $\bullet \in \{\pm\}$, one sees that Lemma 9.3.10 gives rise to $\mathbb{O}$-algebra morphisms

$$\pi^{\bullet}_{r,\ell} : \mathfrak{R}_{\bar{\rho}_0} \to \frac{\mathbb{O}[\![X,Y]\!]}{(XY-p)}. \tag{9.3.11}$$

Lemma 9.3.12. *For each* $\bullet \in \{\pm\}$, *the morphism* (9.3.11) *is surjective.*

Proof. Let $\mathfrak{R}_0$ be the quotient ring of $\mathbb{O}[\![X,Y]\!]/(XY-p)$ by the ideal $((\pi) + (X,Y)^2)$, which is isomorphic to $\mathbb{F}[X,Y]/(X,Y)^2$. By the same argument as in Lemma 9.3.3, it is enough to show that the induced map $\pi^{\bullet}_{r,\ell} : \mathfrak{R}_{\bar{\rho}_0} \to \mathfrak{R}_0$ is surjective. To do that, we compute the strongly divisible module $\mathfrak{M}_0 := \mathfrak{M}^{\bullet}_{r,\ell}(X)/((\pi) + (X,Y)^2)$ over $S_{\mathfrak{R}_0}$. We only consider the case $\bullet = +$ since the case $\bullet = -$ is almost the same. Recall that $r_{i,j}$ is defined in Lemma 7.1.3. By $\bar{r}_{i,j}$ we mean the image of $r_{i,j}$ under the quotient map $\mathbb{O} \to \mathbb{F}$. We describe $\mathfrak{M}_0 = S_{\mathfrak{R}_0}(E_1, E_2)$ as follows:

- $\mathrm{Fil}^r\,\mathfrak{M}_0$ is generated by $u^{m+\ell+1}E_1$ and $\mu\bar{r}_{1,m+\ell+1}Xu^{m+\ell}E_1 + u^{m-\ell-1}E_2$ modulo $\mathrm{Fil}^p\, S_{\mathbb{O}}\mathfrak{M}_0$;
- ϕ_r is computed as follows:

$$\phi_r(\mu\bar{r}_{1,m+\ell+1}Xu^{m+\ell}E_1 + u^{m-\ell-1}E_2) = -\mu\bar{r}_{m+\ell+1,m+\ell+1}\sum_{k=0}^{m-\ell-1}(\gamma-1)^k x_k^{(m-\ell-1,2\ell+2)}E_1,$$

$$\begin{aligned}\phi_r(u^{m+\ell+1}E_1) &= Y\sum_{k=0}^{m-\ell-2}(\gamma-1)^k\bar{r}_{m+\ell+1-k,m-\ell-1}E_1\\ &+\Bigg[(\gamma-1)^{m+\ell+1} - \sum_{k=m-\ell-1}^{m+\ell}(\gamma-1)^k\bar{r}_{m+\ell+1-k,m-\ell-1}\\ &+\sum_{k=0}^{m-\ell-2}(\gamma-1)^k\Bigg(\delta_\ell\bar{r}_{m+\ell+1-k,m-\ell-1} + \sum_{j=1}^{k}\frac{(-1)^{j+1}}{j}\bar{r}_{m+\ell+1-k+j,m-\ell-1}\Bigg)\Bigg]E_2;\end{aligned}$$

- $N(E_1) = X(1 - N(\delta_\ell)/p)E_2$ and $N(E_2) = 0$.

Note that the parameters X and Y all survive in the coefficients of $\mathfrak{M}_0$. Moreover, we claim that $\mathfrak{M}_0$ is not defined over any proper subring of $\mathfrak{R}_0$. Assume that $\mathfrak{M}_0$ is defined over a subring $\mathfrak{R}'_0$ of $\mathfrak{R}_0$. By the same argument of Lemma 9.3.3, we see that $\mathfrak{R}'_0$ contains the element X. We now consider the strongly divisible module $\mathfrak{M}'_0 := \mathfrak{M}_0/(X)$ over $\mathfrak{R}_0/(X) \cong \mathbb{F}[Y]/(Y)^2$. We also let $\underline{e} := (E_1, E_2)$ and $\underline{f} := \underline{e}V$ where

$$V := \begin{pmatrix} u^{m+\ell+1} & 0 \\ 0 & u^{m-\ell-1} \end{pmatrix}.$$

Then $V = \mathrm{Mat}_{\underline{e},\underline{f}}(\mathrm{Fil}^r\, \mathfrak{M}'_0)$. We also let $A := \mathrm{Mat}_{\underline{e},\underline{f}}(\phi_r)$. Note that if we write $Y \cdot \gamma'$ for the (1, 1)-entry of A then $\gamma' \in S_{\mathbb{F}}^{\times}$ by Lemma 7.1.6. Let $R \in \mathrm{GL}_2(S_{\mathbb{F}[Y]/(Y)^2})$ be a matrix of change of basis. Letting $\underline{e}' := \underline{e}R^{-1}$ and $V' := RV$, we have

$$\phi_r(\underline{e}'V') = \phi_r(\underline{e}R^{-1}V') = \phi_r(\underline{e}V) = \underline{e}A = \underline{e}'RA,$$

so that $\mathrm{Mat}_{\underline{e}',\underline{f}'}(\mathrm{Fil}^r\, \mathfrak{M}'_0) = V'$ and $\mathrm{Mat}_{\underline{e}',\underline{f}'}(\phi_r) = RA$, where $\underline{f}' = \underline{e}V'$. We may write $R = R_0 + YR_1$ for some $R_0 \in \mathrm{GL}_2(S_{\mathbb{F}})$ and $R_1 \in \mathrm{M}_{2\times 2}(S_{\mathbb{F}})$, and write $A = A_0 + YA_1$ for some $A_0 \in \mathrm{GL}_2(S_{\mathbb{F}})$ and $A_1 \in \mathrm{M}_{2\times 2}(S_{\mathbb{F}})$. One easily sees that

$$RA = R_0A_0 + Y(R_1A_0 + R_0A_1).$$

If $V' \in \mathrm{M}_{2\times 2}(S_{\mathbb{F}})$, i.e., $R_1 = 0_{2\times 2}$, then $R_0A_1 \neq 0_{2\times 2}$ since R_0 is invertible and

$$A_1 = \begin{pmatrix} \gamma' & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence, $\mathfrak{M}'_0$ is not defined over any proper subring of $\mathbb{F}[Y]/(Y)^2$, which implies that $\mathfrak{R}'_0$ contains the elements X and Y. That is, $\mathfrak{R}'_0 = \mathfrak{R}_0$. Hence, $\pi^+_{r,\ell}$ is surjective. □

Lemma 9.3.13. *For each* $\bullet \in \{\pm\}$, *the morphism* $\pi^\bullet_{r,\ell} : \mathfrak{R}_{\bar\rho_0} \to \mathbb{O}[\![X, Y]\!]/(XY - p)$ *factors through the quotient* $q : \mathfrak{R}_{\bar\rho_0} \to \mathfrak{R}^{(0,r),\psi}_{\bar\rho_0}$.

Proof. The same argument as in Lemma 9.3.4 works. □

Proposition 9.3.14. *If* $\bar\rho_0|_{I_{\mathbb{Q}_p}} \cong \omega_2^{m-\ell-1+p(m+\ell+1)} \oplus \omega_2^{m+\ell+1+p(m-\ell-1)}$ *for* $0 \le \ell \le m-2$, *then*

$$\mathfrak{R}^{(0,r)}_{\bar\rho_0} \sim \frac{\mathbb{O}[\![D, X, Y]\!]}{(XY-p)} \times \frac{\mathbb{O}[\![D, X, Y]\!]}{(XY-p)}.$$

Proof. By Lemma 9.3.13 and by (9.1.2), we have a surjection $\pi^\bullet_{r,\ell} : \mathfrak{R}^{(0,r)}_{\bar\rho_0}[1/p] \to \mathbb{O}[\![D, X, Y]\!]/(XY - p)[1/p]$ for $\bullet \in \{\pm\}$. Since the two irreducible components are disjoint,

$$\pi^+_{r,\ell} \times \pi^-_{r,\ell} : \mathfrak{R}^{(0,r)}_{\bar\rho_0}\Big[\frac{1}{p}\Big] \to \Big(\frac{\mathbb{O}[\![D, X, Y]\!]}{(XY-p)} \times \frac{\mathbb{O}[\![D, X, Y]\!]}{(XY-p)}\Big)\Big[\frac{1}{p}\Big]$$

is also surjective. By Lemma 9.3.1 (2), $\bar{\rho}_0$ does not have a crystalline lift with Hodge–Tate weights $(0, r)$. In other words, $\mathfrak{R}^{(0,r)}_{\bar{\rho}_0}$ has only two irreducible components described as above. The map $\pi^+_{r,\ell} \times \pi^-_{r,\ell}$ induces one to one correspondence between the maximal ideals of $\mathfrak{R}^{(0,r)}_{\bar{\rho}_0}[1/p]$ and the ones of

$$\frac{\mathbb{O}[\![D, X, Y]\!]}{(XY-p)}\Big[\frac{1}{p}\Big] \times \frac{\mathbb{O}[\![D, X, Y]\!]}{(XY-p)}\Big[\frac{1}{p}\Big].$$

Since these rings are Jacobson, it also gives one to one correspondence between prime ideals. The fact that $\mathfrak{R}^{(0,r)}_{\bar{\rho}_0}[1/p]$ is reduced completes the proof. □

Remark 9.3.15. If $\bar{\rho}_0 : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(\mathbb{F})$ is given as in Proposition 9.3.14, then $\mathfrak{R}^{(0,r)}_{\bar{\rho}_0}$ has a smooth generic fiber, since the generic fibers of the irreducible components are disjoint.

On the other hand, let $\bar{\rho}_0 : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(\mathbb{F})$ be a residual representation such that $\bar{\rho}_0|_{I_{\mathbb{Q}_p}} \cong \omega_2^r \oplus \omega_2^{pr}$ (as either in Proposition 9.3.5 or in Proposition 9.3.9). It is obvious that the two irreducible components $\mathbb{O}[\![D, X]\!]$ are disjoint. More precisely, if we let $\mathfrak{R}^+ = \mathbb{O}[\![D, X]\!]$ when $\lambda = \sqrt{\eta}$ and $\mathfrak{R}^- = \mathbb{O}[\![D, X]\!]$ when $\lambda = -\sqrt{\eta}$, then the geometric points of $\mathfrak{R}^+$ give rise to semistable representations that are not isomorphic to the ones coming from $\mathfrak{R}^-$. However, these two irreducible components intersect the crystalline deformation ring $\mathbb{O}[\![D_1, D_2]\!]$, which happens exactly when $X = 0$ (see the proofs of Lemmas 9.3.2 and 9.3.6). These are examples of deformation rings whose generic fibers are not smooth.

Acknowledgements

The authors sincerely thank David Savitt for his constant interests and his helpful comments and suggestions. The authors also sincerely thank the anonymous referee for careful reading, many helpful comments and suggestions, and pointing out some mistakes.

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Received March 30, 2017. Revised January 30, 2018.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 298, No. 2, 2019
dx.doi.org/10.2140/pjm.2019.298.375

NONHOLOMORPHIC LEFSCHETZ FIBRATIONS WITH (−1)-SECTIONS

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We construct two types of nonholomorphic Lefschetz fibrations over S^2 with (−1)-sections — hence, they are fiber sum indecomposable — by giving the corresponding positive relators. One type of the two does not satisfy the slope inequality (a necessary condition for a fibration to be holomorphic) and has a simply connected total space, and the other has a total space that cannot admit any complex structure in the first place. These give an alternative existence proof for nonholomorphic Lefschetz pencils without Donaldson's theorem.

1. Introduction

The notion of Lefschetz fibrations in the smooth category was introduced from algebraic geometry by Moishezon [1977] to study complex surfaces from a topological viewpoint. It is therefore natural to ask how far smooth (symplectic) Lefschetz fibrations are from holomorphic ones. One approach to this question is to construct various nonholomorphic examples. Motivated by this, we give the following results.

Theorem 1.1. *For each $g \geq 3$, there is a genus-g nonholomorphic Lefschetz fibration $X \to S^2$ with a (-1)-section and $\pi_1(X) = 1$ such that it does not satisfy the "slope inequality".*

Theorem 1.2. *For each $g \geq 4$, there is a family of genus-g nonholomorphic Lefschetz fibrations $X_{\widehat{U}_n} \to S^2$ with two disjoint (-1)-sections (for each positive integer n) such that $X_{\widehat{U}_n}$ does not admit any complex structure with either orientation and is not homotopically equivalent to $X_{\widehat{U}_m}$ when $n \neq m$.*

Here, a nonholomorphic Lefschetz fibration means that it is not isomorphic to any holomorphic one. We would like to emphasize that we are able to give explicit monodromy factorizations of the above fibrations although we only give a procedure to get such factorizations without explicitly showing them. In the rest of this section, we give some background on Theorems 1.1 and 1.2.

MSC2010: 14D06, 55R55, 57R15, 57R20.
Keywords: Lefschetz fibrations, (−1)-sections, slope inequality, complex structure.

1A. ***Lefschetz fibrations with* (−1)*-sections.*** The reason that we focus on Lefschetz fibrations that have (-1)-sections is that they play an important role as follows. Blowing up at the base loci of a genus-g Lefschetz pencil yields a genus-g Lefschetz fibration with (-1)-sections, and conversely, blowing down of (-1)-sections of a genus-g Lefschetz fibration gives a genus-g Lefschetz pencil. Furthermore, a closed 4-manifold admits a symplectic structure if and only if it admits a Lefschetz pencil (Donaldson [1999] proved the "if" part, and the "only if" part was shown in [Gompf and Stipsicz 1999]). On the other hand, a Lefschetz fibration with a (-1)-section is fiber sum indecomposable (see [Stipsicz 2001; Smith 2001a]); hence, such a fibration can be considered "prime" with respect to the fiber sum operation. Therefore, as a corollary of Theorems 1.1 and 1.2, we obtain the following result.

Corollary 1.3. *For arbitrary $g \geq 3$, there exists a genus-g nonholomorphic Lefschetz pencil on a simply connected* 4*-manifold. For arbitrary $g \geq 4$, there exist infinitely many genus-g nonholomorphic Lefschetz pencils on* 4*-manifolds that cannot admit any complex structure with either orientation.*

Remark 1.4. Baykur [2015] constructed infinitely many nonholomorphic genus-3 Lefschetz pencils with explicit monodromies. The 4-manifolds obtained as the total spaces are not simply connected and do not admit any complex structure with either orientation.

Remark 1.5. Donaldson's construction of Lefschetz pencils on symplectic 4-manifolds immediately implies the existence of nonholomorphic Lefschetz pencils since there are symplectic 4-manifolds that cannot be complex. Yet this does not tell much about the genera of the resulting pencils. Our result shows the existence of nonholomorphic Lefschetz pencils for *arbitrary genus* $g \geq 3$.

1B. ***The slope inequality and simply connected examples.*** The "slope inequality" derives from the geography problem of relatively minimal holomorphic fibrations. Let us consider a relatively minimal genus-g holomorphic fibration $f : S \to C$ where S and C are a complex surface and a complex curve, respectively. Xiao [1987] defined a certain numerical invariant λ_f, called the "slope" of f, determined by the signature and Euler characteristic of S, the genera of C and a generic fiber. Then he showed that every relatively minimal genus-g holomorphic fibration f satisfies $4 - 4/g \leq \lambda_f$. We call this inequality the slope inequality.

The notion of the slope can be extended for (smooth) Lefschetz fibrations as λ_f is determined by topological invariants (see Section 3D); hence we can also consider the slope inequality in the smooth category. Note that the slope inequality can be rewritten as an inequality giving a lower bound on the signatures of Lefschetz fibrations in terms of the genus of a generic fiber and the number of singular fibers (see Remark 3.11). It is known that the slope inequality holds for any hyperelliptic Lefschetz fibration, especially any genus-2 Lefschetz fibration. Hain conjectured that every Lefschetz fibration over S^2 satisfies the slope inequality as well (see [Amorós et al. 2000; Endo and Nagami 2005]). This conjecture in fact fails; Monden [2014, Theorem 3.1] gave examples violating the slope inequality. In particular, those examples are nonholomorphic by Xiao's result. However, we do not know if they are fiber sum indecomposable. Hence, we ask the following question: *Is there a fiber sum indecomposable Lefschetz fibration violating the slope inequality?* Theorem 1.1 together with the above-mentioned work of Stipsicz [2001] and Smith [2001a] implies that the answer to this question is positive for any $g \geq 3$.

Let us consider a genus-g nonholomorphic Lefschetz fibration $X \to S^2$ with a (-1)-section such that $\pi_1(X) = 1$. To the best of our knowledge, all known such fibrations with explicit monodromy factorizations are Fuller's example ($g = 3$)[1] and those of Endo and Nagami [2005] ($g = 3, 4, 5$). Theorem 1.1 gives such examples with explicit monodromy factorizations for arbitrary $g \geq 3$.

Remark 1.6. We do not know whether the examples in [Smith 2001b; Endo and Nagami 2005] and Theorem 1.1 have noncomplex total spaces or not. On the other hand, Li [2008] constructed nonholomorphic Lefschetz pencils (fibrations with (-1)-sections) on complex surfaces. However, their genera are implicit.

1C. ***Lefschetz fibrations with noncomplex total spaces.*** Many Lefschetz fibrations with explicit monodromies and noncomplex total spaces have been constructed using the (twisted) fiber sum operation (see for instance [Smith 1998; Ozbagci and Stipsicz 2000; Fintushel and Stern 1998; Korkmaz 2001; Akhmedov and Ozbagci 2017[2] Akhmedov and Monden 2015; Baykur and Korkmaz 2017]). They are nonholomorphic, however, they do not have any (-1)-section since they are decomposable. On the other hand, Stipsicz [2001] and, independently, Smith [2001a] proved that there are infinitely many fiber sum indecomposable Lefschetz fibrations with noncomplex total spaces. Since the constructions of these fibrations are based on Donaldson's theorem [1999], their monodromy factorizations are not explicitly given. Theorem 1.2 gives infinitely many fiber sum indecomposable Lefschetz fibrations with explicit monodromy factorizations and noncomplex total spaces for any $g \geq 4$.

The fundamental group of the total space $X_{\widehat{U}_n}$ of a genus-g Lefschetz fibration in the family in Theorem 1.2 is $H_1(X_{\widehat{U}_n}) = \mathbb{Z} \oplus \mathbb{Z}_n$. By improving the work of [Ozbagci and Stipsicz 2000] (see also [Baykur 2012]) slightly, we see that the 4-manifold $X_{\widehat{U}_n}$ does not carry any complex structure with either orientation. For $g \geq 22$,

[1] It was shown by Smith [2001b] that Fuller's example is nonholomorphic.

[2] Baykur has informed us that the examples in [Akhmedov and Ozbagci 2017] should be fiber sum decomposable from Ozbagci's talk in Turkey a few years ago.

nonholomorphic Lefschetz fibrations with the same property of Theorem 1.2 were constructed in [Kobayashi and Monden 2016] based on the technique of this paper. Theorem 1.2 improves this result.

Remark 1.7. Nonholomorphic genus-2 Lefschetz fibrations with finite cyclic fundamental groups and without any (-1)-sections were constructed in [Akhmedov and Monden 2015] by rationally blowing down a twisted fiber sum of two copies of Matsumoto's fibration. However, we do not know whether these are decomposable.

2. Preliminaries

2A. ***Notation.*** From now on, we use the same letter for a loop and its homotopy class and homology class by abuse of notation. Similarly, we use the same letter for a diffeomorphism and its isotopy class, or for a simple closed curve and its isotopy class. A simple loop and a simple closed curve are even denoted by the same letter. It will cause no confusion as it will be clear from the context which one we mean.

For convenience's sake, we first fix the notation and the symbols for the curves which we use throughout the paper. Let Σ_g be the closed oriented surface of genus g standardly embedded in the 3-space and

$$a_1, b_1, a_2, b_2, \dots, a_g, b_g$$

be the standard generators of the fundamental group $\pi_1(\Sigma_g)$ of Σ_g as shown in Figure 1. We choose orientations of a_i, b_i so that $i(a_i, b_i) = 1$, where $i(a_i, b_i)$ is the algebraic intersection number of a_i and b_i. For loops a and b in $\pi_1(\Sigma_g)$, the product ab means that we traverse first a then b as usual. Let $c_1, c_2, \dots, c_g$ and a_{g+1} be the simple closed curves on Σ_g as shown in Figure 1. Note that in $\pi_1(\Sigma_g)$, $c_g = 1$ and $a_{g+1} = 1$. Then, the fundamental group $\pi_1(\Sigma_g)$ has the presentation

$$\pi_1(\Sigma_g) = \langle a_1, b_1, a_2, b_2, \dots, a_g, b_g \mid c_g \rangle.$$

Let $B^h_{0,1}, B^h_{0,2}, B^h_1, B^h_2, \dots, B^h_h$ $(h = 1, 2, \dots, g)$ and $a'_1, a'_2, \dots, a'_g$ be the simple closed curves on Σ_g as shown in Figures 2 and 3.

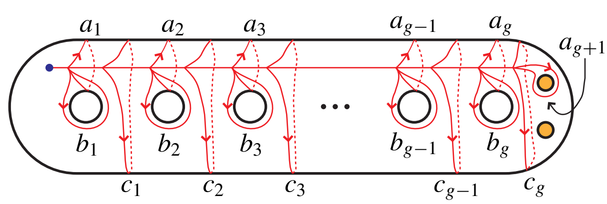

Figure 1. The standardly embedded Σ_g with two indicated disks on the rightmost position and the generators a_j, b_j of the fundamental group and loops c_j.

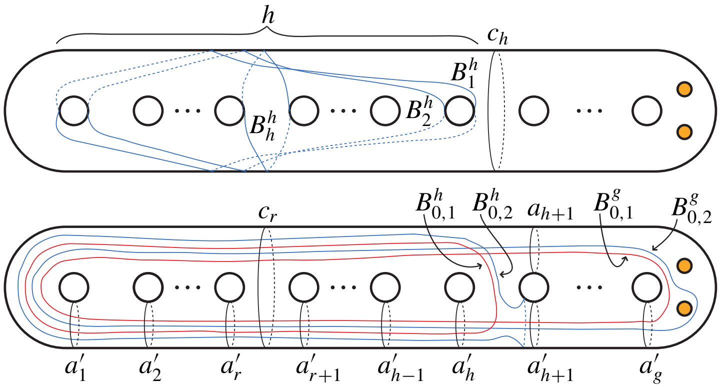

Figure 2. The curves $B^h_{0,1}, B^h_{0,2}, B^h_1, B^h_2, \ldots, B^h_h, a'_1, a'_2, \ldots, a'_g$ for $h = 2r$.

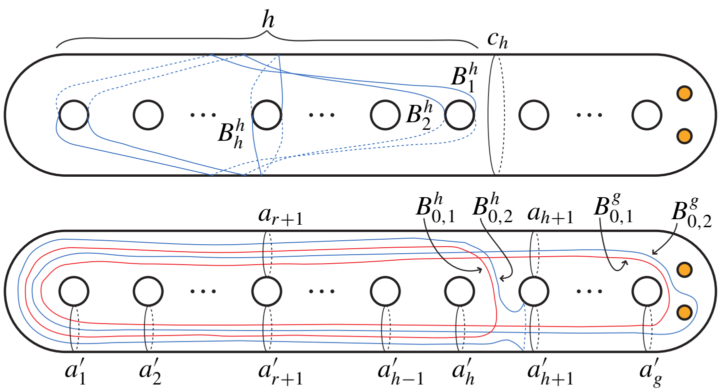

Figure 3. The curves $B^h_{0,1}, B^h_{0,2}, B^h_1, B^h_2, \ldots, B^h_h, a'_1, a'_2, \ldots, a'_g$ for $h = 2r+1$.

Suppose $h = 2r$. It is easy to check that the following equalities hold in $H_1(\Sigma_g)$:

(1) $B^h_{0,1} = b_1 + b_2 + \cdots + b_h, \quad B^h_{0,2} = b_1 + b_2 + \cdots + b_h + a_{h+1}, \quad 1 \le h \le g;$

(2) $B^h_{2k-1} = a_k + b_k + b_{k+1} + \cdots + b_{h+1-k} + a_{h+1-k}, \quad 1 \le k \le r, \ 1 \le h \le g;$

(3) $B^h_{2k} = a_k + b_{k+1} + b_{k+2} + \cdots + b_{h-k} + a_{h+1-k}, \quad 1 \le k \le r, \ 1 \le h \le g.$

In the case of $h = 2r+1$, the same equalities (1) and (3) hold without change, while the equality (2) holds for $1 \le k \le r+1, \ 1 \le h \le g$.

2B. ***Substitution technique.*** In this subsection, we introduce key techniques, called a *substitution* and a *partial conjugation*, for constructing a new word in mapping class groups from a word and a relator. We will utilize this technique to construct Lefschetz fibrations with (−1)-section in the later sections.

Let Σ^b_g be a compact oriented surface of genus g with b boundary components.

The *mapping class group* Γ_g^b of Σ_g^b is the group of isotopy classes of orientation-preserving self-diffeomorphisms of Σ_g^b, where all the maps involved are assumed to fix $\partial\Sigma_g^b$ pointwise. For simplicity, we write $\Sigma_g = \Sigma_g^0$ and $\Gamma_g = \Gamma_g^0$. For two elements ϕ_1 and ϕ_2 in Γ_g^b, the product $\phi_2\phi_1$ means that we first apply ϕ_1 then ϕ_2. We denote by t_c the right-handed *Dehn twist* along a simple closed curve c on Σ_g^b.

Definition 2.1. Let $v_1, v_2, \dots, v_n$ be simple closed curves on Σ_g^b. If $t_{v_n}^{\epsilon_n} \cdots t_{v_2}^{\epsilon_2} t_{v_1}^{\epsilon_1} = 1$ in Γ_g^b, where $\epsilon_i = \pm 1$, then this factorization is called a *relator*. In the special case where $\epsilon_i = 1$ for all i, namely, $t_{v_n} \cdots t_{v_2} t_{v_1} = 1$ holds in Γ_g, then this factorization is called a *positive relator*.

We introduce a key technique for constructing a new product of right-handed Dehn twists in Γ_g^b from old ones.

Definition 2.2. Let $v_1, v_2, \dots, v_k$ and $d_1, d_2, \dots, d_l$ be simple closed curves on Σ_g^b such that the following product, denoted by R, is a relator in Γ_g^b:

$$R := t_{v_1} t_{v_2} \cdots t_{v_k} t_{d_l}^{-1} \cdots t_{d_2}^{-1} t_{d_1}^{-1},$$

which equals the identity as a mapping class by definition. If a mapping class ϕ in Γ_g^b satisfies $\phi(d_i) = d_i$, then by the relation $t_{\phi(c)} = \phi t_c \phi^{-1}$, we obtain the following relator, denoted by R^ϕ, in Γ_g^b:

$$R^\phi = t_{\phi(v_1)} t_{\phi(v_2)} \cdots t_{\phi(v_k)} t_{d_l}^{-1} \cdots t_{d_2}^{-1} t_{d_1}^{-1}.$$

Let W be a product of right-handed Dehn twists including $t_{d_1} t_{d_2} \cdots t_{d_l}$ as a subword:

$$W = U \cdot t_{d_1} t_{d_2} \cdots t_{d_l} \cdot V,$$

where U and V are products of right-handed Dehn twists. Then, we get a new product of right-handed Dehn twists, denoted by W', as follows:

$$U \cdot R^\phi \cdot t_{d_1} t_{d_2} \cdots t_{d_l} \cdot V = U \cdot t_{\phi(v_1)} t_{\phi(v_2)} \cdots t_{\phi(v_k)} \cdot V =: W',$$

where the first equality means the equality as a mapping class. Then, W' is said to be obtained by applying a *R^ϕ-substitution* to W.

Remark 2.3. A R^ϕ-substitution is a combination of a substitution technique and a partial conjugation introduced by Fuller and Auroux [Auroux 2006b; Auroux 2006a], respectively.

2C. *Relators in mapping class groups.* In this subsection, we introduce some well-known relators in mapping class groups, called the braid relator B, the lantern relator L, the chain relators C_k, $\overline{C}_k$ and certain relators W_1^h, W_2^h.

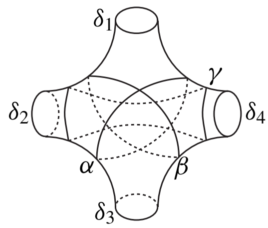

Figure 4. The curves δ_1, δ_2, δ_3, δ_4 and α, β, γ.

Definition 2.4 (braid relator). Let α and β be simple closed curves on Σ_g^b. If the geometric intersection number of α and β is equal to 0 (resp. 1), then we have the *braid relator B*:

$$B := t_\alpha t_\beta t_\alpha^{-1} t_\beta^{-1} \quad (\text{resp. } B := t_\alpha t_\beta t_\alpha t_\beta^{-1} t_\alpha^{-1} t_\beta^{-1}).$$

Definition 2.5 (lantern relator). Let δ_1, δ_2, δ_3 and δ_4 be the four boundary curves of Σ_0^4 and let α, β and γ be the interior curves as shown in Figure 4. Then, we have the *lantern relator L* in Γ_0^4:

$$L := t_\alpha t_\beta t_\gamma t_{\delta_4}^{-1} t_{\delta_3}^{-1} t_{\delta_2}^{-1} t_{\delta_1}^{-1}.$$

The lantern relator was discovered by Dehn [1938] and was rediscovered by Johnson [1979].

Definition 2.6 (chain relator). Suppose $h \geq 1$. Let $\alpha_1, \alpha_2, \ldots, \alpha_{2h+1}$ be simple closed curves on an oriented surface such that α_i and α_{i+1} intersect transversally at exactly one point for $1 \leq i \leq 2h$ and that α_i and α_j are disjoint if $|i - j| \geq 2$. Then, a regular neighborhood of $\alpha_1 \cup \alpha_2 \cup \cdots \cup \alpha_{2h}$ (resp. $\alpha_1 \cup \alpha_2 \cup \cdots \cup \alpha_{2h+1}$) is a subsurface of genus h with one boundary component (resp. two boundary components), say d (resp. d_1 and d_2). We then have the *even chain relator C_{2h}* and the *odd chain relator C_{2h+1}*:

$$C_{2h} := (t_{\alpha_1} t_{\alpha_2} \cdots t_{\alpha_{2h}})^{4h+2} t_d^{-1},$$
$$C_{2h+1} := (t_{\alpha_1} t_{\alpha_2} \cdots t_{\alpha_{2h+1}})^{2h+2} t_{d_2}^{-1} t_{d_1}^{-1}.$$

Definition 2.7. Suppose $g \geq 2$. Let Σ_g^2 be the surface of genus g with two boundary components obtained from Σ_g by removing two disjoint open disks (see Figures 1, 2 and 3). Let a_{g+1} be one of the boundary curves of Σ_g^2 as shown in Figure 1, and let a'_{g+1} be the other boundary curve of Σ_g^2 defined by $a'_{g+1} = c_g a_{g+1}$. We then have the following two relators $W_{1,h}$, $W_{2,h}$ in Γ_g^2 for each $h = 1, 2, \ldots, g$:

$$W_{1,h} := \begin{cases} (t_{B_{0,1}^h} t_{B_1^h} t_{B_2^h} \cdots t_{B_{h-1}^h} t_{B_h^h} t_{c_\ell})^2 t_{c_h}^{-1} & \text{if } h = 2\ell, \\ (t_{B_{0,1}^h} t_{B_1^h} t_{B_2^h} \cdots t_{B_{h-1}^h} t_{B_h^h} t_{a_{\ell+1}}^2 t_{a'_{\ell+1}}^2)^2 t_{c_h}^{-1} & \text{if } h = 2\ell + 1, \end{cases}$$

$$W_{2,h} := \begin{cases} (t_{B^h_{0,2}} t_{B^h_1} t_{B^h_2} \cdots t_{B^h_{h-1}} t_{B^h_h} t_{c_\ell})^2 t^{-1}_{a_{h+1}} t^{-1}_{a'_{h+1}} & \text{if } h = 2\ell, \\ (t_{B^h_{0,2}} t_{B^h_1} t_{B^h_2} \cdots t_{B^h_{h-1}} t_{B^h_h} t^2_{a_{\ell+1}} t^2_{a'_{\ell+1}})^2 t^{-1}_{a_{h+1}} t^{-1}_{a'_{h+1}} & \text{if } h = 2\ell+1. \end{cases}$$

Note that in Γ_g, the relator $W_{2,g}$ is a positive relator. Matsumoto [1996] discovered this positive relator for $g = 2$, and Cadavid [1998] and independently Korkmaz [2001] generalized Matsumoto's relator to $g \geq 3$. $W_{1,g}$ was shown to be a relator in Γ^1_g by Ozbagci and Stipsicz [2004]. In [Korkmaz 2009], it was claimed without proof that $W_{2,g}$ is a relator in Γ^2_g. Yet, we can show it to be true by applying the same argument in Section 2 of [Korkmaz 2001] (for example see Section 6 of [Kobayashi and Monden 2016]).

3. Lefschetz fibrations

3A. ***Basics on Lefschetz fibrations.*** We recall the definition and basic properties of Lefschetz fibrations. More details can be found in [Gompf and Stipsicz 1999].

Definition 3.1. Let X be a closed, oriented smooth 4-manifold. A smooth map $f : X \to S^2$ is a *Lefschetz fibration* if for each critical point p of f and $f(p)$, there are complex local coordinate charts agreeing with the orientations of X and S^2 on which f is of the form $f(z_1, z_2) = z_1 z_2$.

It follows that f has finitely many critical points $C = \{p_1, p_2, \ldots, p_n\}$. We can assume that f is injective on C and relatively minimal (i.e., no fiber contains a sphere with self-intersection number -1). Each fiber which contains a critical point, called a *singular fiber*, is obtained by "collapsing" a simple closed curve in the prescribed regular fiber to a point. We call the simple closed curve in the regular fiber the *vanishing cycle*. If the genus of the regular fiber of f is g, then we call f a *genus-g Lefschetz fibration*.

The monodromy of the fibration around a singular fiber $f^{-1}(f(p_i))$ is given by a right-handed Dehn twist along the corresponding vanishing cycle, denoted by v_i. Once we fix an identification of Σ_g with the fiber over a base point of S^2, we can characterize the Lefschetz fibration $f : X \to S^2$ by its *monodromy representation* $\pi_1(S^2 - f(C)) \to \Gamma_g$. Here, this map is indeed an antihomomorphism. Let $\gamma_1, \gamma_2, \ldots, \gamma_n$ be an ordered system of generating loops for $\pi_1(S^2 - f(C))$ such that each γ_i encircles only $f(p_i)$ and $\gamma_1\gamma_2\cdots\gamma_n = 1$ in $\pi_1(S^2 - f(C))$. Thus, the monodromy of f comprises a positive relator

$$t_{v_n} \cdots t_{v_2} t_{v_1} = 1 \text{ in } \Gamma_g.$$

Conversely, for any positive relator P in Γ_g, one can construct a genus-g Lefschetz fibration over S^2 whose monodromy is P. Therefore, we denote a genus-g Lefschetz fibration associated with a positive relator P in Γ_g by $f_P : X_P \to S^2$.

Two Lefschetz fibrations $f_{P_i} : X_{P_i} \to S^2$ $(i = 1, 2)$ are said to be *isomorphic* if there exist orientation-preserving diffeomorphisms $H : X_{P_1} \to X_{P_2}$ and $h : S^2 \to S^2$ such that $f_{P_2} \circ H = h \circ f_{P_1}$. According to theorems of Kas [1980] and Matsumoto [1996], if $g \geq 2$, then the isomorphism class of a Lefschetz fibration is determined by a positive relator modulo *simultaneous conjugations*

$$t_{v_n} \cdots t_{v_2} t_{v_1} \sim t_{\phi(v_n)} \cdots t_{\phi(v_2)} t_{\phi(v_1)} \quad \text{for any} \quad \phi \in \Gamma_g$$

and *elementary transformations*

$$t_{v_n} \cdots t_{v_{i+2}} t_{v_{i+1}} t_{v_i} t_{v_{i-1}} t_{v_{i-2}} \cdots t_{v_1} \sim t_{v_n} \cdots t_{v_{i+2}} t_{v_i} t_{t_{v_i}^{-1}(v_{i+1})} t_{v_{i-1}} t_{v_{i-2}} \cdots t_{v_1},$$

$$t_{v_n} \cdots t_{v_{i+2}} t_{v_{i+1}} t_{v_i} t_{v_{i-1}} t_{v_{i-2}} \cdots t_{v_1} \sim t_{v_n} \cdots t_{v_{i+2}} t_{v_{i+1}} t_{t_{v_i}(v_{i-1})} t_{v_i} t_{v_{i-2}} \cdots t_{v_1}.$$

Therefore, if P_2 is obtained by applying a series of elementary transformations and simultaneous conjugations to P_1, then

$$\sigma(X_{P_1}) = \sigma(X_{P_2}) \quad \text{and} \quad e(X_{P_1}) = e(X_{P_2}), \tag{4}$$

where $\sigma(X)$ and $e(X)$ stand for the signature and Euler characteristic of a 4-manifold X, respectively.

3B. *Sections of Lefschetz fibrations.*

Definition 3.2. Let $f : X \to S^2$ be a Lefschetz fibration. A map $\sigma : S^2 \to X$ is called a *k-section* of f if it satisfies $f \circ \sigma = \mathrm{id}_{S^2}$ and the self-intersection number $[\sigma(S^2)]^2 = k$, where $[\sigma(S^2)]$ is the homology class in $H_2(X; \mathbb{Z})$.

If the factorization $P = t_{v_n} \cdots t_{v_2} t_{v_1} (= 1)$ lifts from Γ_g to Γ_g^1 as

$$t_\delta^k = t_{\tilde{v}_n} \cdots t_{\tilde{v}_2} t_{\tilde{v}_1} \quad (\text{i.e., } 1 = t_{\tilde{v}_n} \cdots t_{\tilde{v}_2} t_{\tilde{v}_1} t_\delta^{-k}),$$

then the Lefschetz fibration f_P has a $(-k)$-section. Here, δ is the boundary curve of Σ_g^1 and $t_{\tilde{v}_i}$ is a Dehn twist mapped to t_{v_i} under $\Gamma_g^1 \to \Gamma_g$. Conversely, if a genus-g Lefschetz fibration admits a $(-k)$-section, we obtain a relator of the above type in Γ_g^1. A similar relator holds for b disjoint sections (in which case one has to work in the mapping class group Γ_g^b).

A necessary condition for a Lefschetz fibration to admit a (-1)-section was shown independently by Stipsicz [2001] and Smith [2001a]:

Theorem 3.3 [Stipsicz 2001; Smith 2001a]. *Let $g \geq 1$. If a genus-g Lefschetz fibration $f : X \to S^2$ admits a (-1)-section, then f is fiber sum indecomposable.*

Here, we recall the definition of fiber sum. Let $f_i : X_i \to S^2$ be a genus-g Lefschetz fibration for $i = 1, 2$, and let D_i be an open disk on S^2 which does not contain any critical values. Then, the *fiber sum* $f_1 \#_F f_2 : X_1 \#_F X_2 \to S^2$ is obtained by gluing $X_1 - f_1^{-1}(D_1)$ and $X_2 - f_2^{-1}(D_2)$ along their boundaries via a

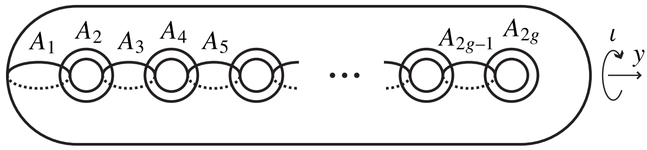

Figure 5. The involution ι of Σ and the curves $A_1, A_2, \ldots, A_{2g}$ on Σ_g.

fiber-preserving orientation-reversing diffeomorphism and extending f_1 and f_2 in a natural way. A Lefschetz fibration is said to be *fiber sum indecomposable* if it cannot be decomposed as a fiber sum of two Lefschetz fibrations each of which has at least one singular point.

For a Lefschetz fibration over S^2 with a positive relator and a section, we can determine the fundamental group of X as follows:

Lemma 3.4 (see [Gompf and Stipsicz 1999]). *Let P be a positive relator $P = t_{v_n} \cdots t_{v_2} t_{v_1}$ in Γ_g. Suppose that the corresponding genus-g Lefschetz fibration $f : X_P \to S^2$ admits a section σ. Then, the fundamental group $\pi_1(X)$ is isomorphic to the quotient of $\pi_1(\Sigma_g)$ by the normal subgroup generated by the vanishing cycles $v_1, v_2, \ldots, v_n$. The same holds for the first homology group $H_1(X)$.*

3C. ***Signatures of Lefschetz fibrations.*** This subsection gives two results about the signatures of Lefschetz fibrations.

Let Δ_g be the *hyperelliptic mapping class group* of genus g, i.e., the subgroup of Γ_g consisting of those mapping classes commuting with the isotopy class of the involution ι shown in Figure 5. Note that $\Delta_g = \Gamma_g$ for $g = 1, 2$ and that t_c is in Δ_g if and only if $\iota(c) = c$.

A genus-g Lefschetz fibration is said to be *hyperelliptic* if it is associated with a positive relator $P = t_{v_1} \cdots t_{v_n}$ such that each t_{v_i} is contained in Δ_g. To compute the signatures of Lefschetz fibrations, we present Matsumoto and Endo's signature formula for hyperelliptic Lefschetz fibrations.

Theorem 3.5 ([Matsumoto 1983; 1996] ($g = 1, 2$), [Endo 2000] ($g \geq 3$)). *Let us consider a genus-g hyperelliptic Lefschetz fibration $f_P : X_P \to S^2$ with n nonseparating and $s = \Sigma_{h=1}^{[g/2]} s_h$ separating vanishing cycles, where s_h is the number of separating vanishing cycles that separate Σ_g into two surfaces, one of which has genus h. Then, we have*

$$\sigma(X_P) = -\frac{g+1}{2g+1} n + \sum_{h=1}^{[g/2]} \left(\frac{4h(g-h)}{2g+1} - 1 \right) s_h.$$

By the work of Endo and Nagami [2005], we see the behavior of signatures of Lefschetz fibrations under a monodromy substitution as follows.

Proposition 3.6 [Endo and Nagami 2005, Theorem 4.3, Definition 3.3, Lemma 3.5 and Propositions 3.9, 3.10 and 3.12]. *Let B, L and C_{2h+1} be the braid relator, the* *lantern relator and the odd chain relator in Definitions 2.4, 2.5 and 2.6, respectively. We assume that those relators are in* Σ_g.

Let $f_{P_i} : X_{P_i} \to S^2$ *be a genus-g Lefschetz fibration with a positive relator* P_i $(i = 1, 2)$. *Suppose that* P_2 *is obtained by applying an* R^{ϕ}*-substitution to* P_1, *where* ϕ *is a mapping class and R is a relator in* Γ_g.

(1) *If* $R = B$, *then* $\sigma(X_{P_2}) = \sigma(X_{P_1})$.

(2) *If* $R = L$, *then* $\sigma(X_{P_2}) = \sigma(X_{P_1}) + 1$. *Hence, if* $R = L^{-1}$, *then* $\sigma(X_{P_2}) = \sigma(X_{P_1}) - 1$.

(3) *Assume that both* d_1 *and* d_2 *are not nullhomotopic in* Σ_g. *If* $R = C_{2h+1}$, *then* $\sigma(X_{P_2}) = \sigma(X_{P_1}) + 2h(h+2)$. *Hence, if* $R = C_{2h+1}^{-1}$, *then* $\sigma(X_{P_2}) = \sigma(X_{P_1}) - 2h(h+2)$.

3D. *Nonholomorphicity of Lefschetz fibrations.*

Definition 3.7. A Lefschetz fibration $f : X \to S^2$ is said to be *holomorphic* if there are complex structures on both X and S^2 with respect to which f is a holomorphic projection. We say f is *nonholomorphic* if it is not isomorphic to any holomorphic Lefschetz fibration.

Suppose that $g \geq 2$. In order to prove Theorems 1.1 and 1.2, we introduce two sufficient conditions for a Lefschetz fibration to be nonholomorphic.

One comes from the result of Xiao [1987]. For an almost complex 4-manifold X, we set $K^2(X) := 3\sigma(X) + 2e(X)$ and $\chi_h(X) := (\sigma(X) + e(X))/4$. Xiao proved the following theorem, called the *slope inequality*:

Theorem 3.8 [Xiao 1987]. *Every relatively minimal holomorphic genus-g fibration f on a complex surface X over a complex curve C of genus* $k \geq 0$ *satisfies the inequality*

$$4 - 4/g \leq \lambda_f,$$

where

$$\lambda_f := \frac{K^2(X) - 8(g-1)(k-1)}{\chi_h(X) - (g-1)(k-1)}.$$

As a consequence of Theorem 3.8, we have:

Proposition 3.9. *If a genus-g Lefschetz fibration* $f : X \to S^2$ *does not satisfy the slope inequality, namely,* $\lambda_f < 4 - 4/g$, *then f is nonholomorphic.*

The other comes from the result of Ozbagci and Stipsicz [2000]. We present a slightly improved version of their result where we replace π_1 by H_1, but this can be concluded from the proof of Theorem 1.3 in [Ozbagci and Stipsicz 2000]:

Theorem 3.10. *If a Lefschetz fibration* $f : X \to S^2$ *satisfies* $H_1(X) = \mathbb{Z} \oplus \mathbb{Z}_n$ *for some positive integer n, then X admits no complex structure with either orientation, so f is nonholomorphic.*

For the convenience of the readers, we give a proof of this theorem, which is merely a simplification of that in [Ozbagci and Stipsicz 2000].

Proof. Assume that X carries a complex structure and let X' be the minimal model of X. By the Enriques–Kodaira classification of complex surfaces, together with the fact that $b_1(X') = 1$ and $b_2^+(X') \geq 1$ (since X admits a symplectic structure and so does X'), we can observe that X' is an elliptic surface. If X' is an elliptic fibration over a Riemann surface Σ, we have $b_1(X') \geq b_1(\Sigma)$. Since $b_1(X') = 1$, Σ must be S^2. Since $b_1(X') = b_3(X') = 1$ and $b_2(X') \neq 0$, the Euler characteristic of X' cannot be 0. Now we suppose that X' is a minimal elliptic surface over S^2 with nonzero Euler characteristic. According to [Gompf 1991], a presentation for the fundamental group of such an elliptic surface is given as

$$\pi_1(X') = \langle x_1, \cdots, x_k \mid x_i^{p_i} = 1, i = 1, \ldots, k;\ x_1 \cdots x_k = 1 \rangle.$$

So it is clear that $H_1(X')$ has only torsion elements, which contradicts the assumption $H_1(X) = \mathbb{Z} \oplus \mathbb{Z}_n$. □

Remark 3.11. If X admits a genus-g Lefschetz fibration $f : X \to S^2$ with n singular fibers, then the Euler characteristic of X is $e(X) = -4(g-1) + n$. Using this fact, the slope λ_f of f can be written as

$$\lambda_f = 12 - \frac{4}{(\sigma(X)/n) + 1},$$

where $\sigma(X)$ is the signature of X. Therefore, we can regard the slope λ_f as the "average signature" $\sigma(X)/n$ per singular fiber. Moreover, the slope inequality $\lambda_f \geq 4 - 4/g$ can be rewritten as

$$\sigma(X) \geq -\frac{g+1}{2g+1} n,$$

that is, it gives a lower bound on σ in terms of g and n.

Remark 3.12. The work of Xiao [1987] was mainly motivated by the so-called Severi inequality, stating that every minimal surface of general type of maximal Albanese dimension satisfies $K^2 \geq 4\chi_h$. This is equivalent to stating that if a minimal complex surface S of general type satisfies $K^2 < 4\chi_h$, then S admits a relatively minimal holomorphic fibration over C of genus $b_1(S)/2$. The Severi inequality was stated in [di Severi 1932] (but the proof was not correct) and independently posed as a conjecture by Reid [1979] and by Catanese [1983]. Xiao proved it when S admits a relatively minimal holomorphic fibration over a curve of positive genus, that is, a complex surface S admitting a holomorphic genus-g fibration f over C of positive genus k with $K^2 < 4\chi_h + 4(g-1)(k-1)$ (i.e., $\lambda_f < 4$) satisfies $k = b_1(S)/2$. The Severi inequality has been studied by many authors (for example [di Severi 1932;

Reid 1979; Catanese 1983; Konno 1996; Manetti 2003; Pardini 2005]) and was proved by Pardini [2005].

Remark 3.13. We denote by $\overline{\mathcal{M}_g}$ the Deligne–Mumford compactified moduli space of stable curves of genus g. We can reformulate the slope inequality for Lefschetz fibrations in terms of $\overline{\mathcal{M}_g}$ as follows. For a genus-g Lefschetz fibration $f : X \to S^2$ with n singular fibers, there is a symplectic structure on X such that for all $x \in S^2$, $f^{-1}(x)$ is a pseudo-holomorphic curve. Since a 2-dimensional almost-complex structure is integrable, $f^{-1}(x)$ determines a point in $\overline{\mathcal{M}_g}$. By defining $\phi_f(x) = [f^{-1}(x)] \in \overline{\mathcal{M}_g}$ for $x \in S^2$, we obtain the moduli map $\phi_f : S^2 \to \overline{\mathcal{M}_g}$. Let $\mathcal{H}_g$ be the Hodge bundle on $\overline{\mathcal{M}_g}$ with fiber the determinant line $\wedge^g H^0(C; K_C)$, where C is the set of critical points of f. Then, by combining the signature formula $\sigma(X) = \langle c_1(\mathcal{H}_g), [\phi_f(S^2)]\rangle - n$ given by Smith [1999] and the slope inequality, we have

$$(2g+1)\langle c_1(\mathcal{H}_g), [\phi_f(S^2)]\rangle - g \cdot n \geq 0.$$

4. Nonholomorphic Lefschetz fibrations admitting (−1)-sections

In this section, we prove Theorem 1.1

Theorem 1.1. *For each $g \geq 3$, there is a genus-g nonholomorphic Lefschetz fibration $X \to S^2$ with a (-1)-section and $\pi_1(X) = 1$ such that it does not satisfy the slope inequality.*

To prove this, we need a lemma. Suppose $g \geq 3$. Let Σ_g^1 be the surface of genus g with one boundary component obtained from Σ_g by removing the open disk whose boundary curve is a_{g+1} (see Figure 1). Let us consider $A_1, A_2, \ldots, A_{2g}$ to be the simple closed curves on Σ_g^1 (see Figure 6) defined as follows: $A_1 = a_1$, $A_2 = b_1$, $A_{2h-1} = a_{h-1}a_h^{-1}$ and $A_{2h} = b_h$ for $h = 2, 3, \ldots, g$.

Lemma 4.1. $(t_{A_1}t_{A_2}\cdots t_{A_{2g}})^{2g+1} = (t_{A_1}t_{A_2}\cdots t_{A_{2g-1}})^{2g}t_{A_{2g}}\cdots t_{A_2}t_{A_1}t_{A_1}t_{A_2}\cdots t_{A_{2g}}$.

Proof. The proof follows from the braid relations $t_{A_i}t_{A_{i+1}}t_{A_i} = t_{A_{i+1}}t_{A_i}t_{A_{i+1}}$ and $t_{A_i}t_{A_j} = t_{A_j}t_{A_i}$ for $|i-j| > 1$ (i.e., by applying B-substitutions to the left side). □

We now prove Theorem 1.1.

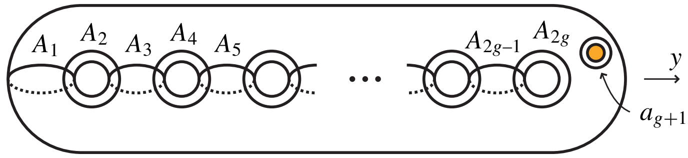

Figure 6. The curves $A_1, A_2, \ldots, A_{2g}$ on Σ_g^1.

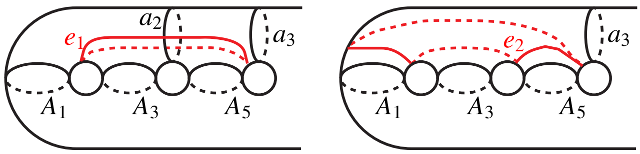

Figure 7. The curves that give a Lantern relator.

Proof of Theorem 1.1. Suppose $g \geq 3$. Let us consider the following chain relators, C_{2g} and C_{2g+1}:

$$C_{2g} = (t_{A_1} t_{A_2} \cdots t_{A_{2g}})^{4g+2} t_{a_{g+1}}^{-1}, \qquad C_{2g-1} = (t_{A_1} t_{A_2} \cdots t_{A_{2g-1}})^{2g} t_{a_g}^{-1} t_{a'_g}^{-1},$$

where a_g and a'_g are the curves as shown in Figures 2 and 3. By Lemma 4.1 and the even chain relator C_{2g}, we obtain the following relator C'_{2g}:

$$C'_{2g} = \{(t_{A_1} t_{A_2} \cdots t_{A_{2g-1}})^{2g} \cdot t_{A_{2g}} \cdots t_{A_2} t_{A_1} t_{A_1} t_{A_2} \cdots t_{A_{2g}}\}^2 t_{a_{g+1}}^{-1}.$$

By applying C_{2g-1}^{-1}-substitution to C'_{2g} twice, we get a new relator H in Γ_g^1:

$$H = (t_{a_g} t_{a'_g} \cdot t_{A_{2g}} \cdots t_{A_2} t_{A_1} t_{A_1} t_{A_2} \cdots t_{A_{2g}})^2 t_{a_{g+1}}^{-1}.$$

Consider the curves on Σ_g^1 in Figure 7. Since A_1, a_2, e_1, and e_2 are nonseparating curves on the subsurface of genus $g-1 (\geq 2)$ with two boundary components a_g and a'_g, there are diffeomorphisms ψ_1, ψ_2 and ψ_3 in Γ_g^1 such that $\psi_1(A_1) = a_2$, $\psi_2(A_1) = e_1$, $\psi_3(A_1) = e_2$, and each ψ_i is identical near a_g and a'_g. Then, we have the following relator H^{ψ_1}:

$$H^{\psi_1} = (t_{a_g} t_{a'_g} \cdot t_{\psi_1(A_{2g})} \cdots t_{\psi_1(A_2)} t_{a_2} t_{a_2} t_{\psi_1(A_2)} \cdots t_{\psi_1(A_{2g})})^2 \cdot t_{a_{g+1}}^{-1}.$$

Applying $C_{2g-1}^{\psi_2}$- and $C_{2g-1}^{\psi_3}$-substitutions to H^{ψ_1}, we get a relator H':

$$\begin{aligned} H' = {} & (t_{e_1} t_{\psi_2(A_2)} \cdots t_{\psi_2(A_{2g-1})})^{2g} t_{\psi_1(A_{2g})} \cdots t_{\psi_1(A_2)} t_{a_2} t_{a_2} t_{\psi_1(A_2)} \cdots t_{\psi_1(A_{2g})} \\ & \cdot (t_{e_2} t_{\psi_3(A_2)} \cdots t_{\psi_3(A_{2g-1})})^{2g} t_{\psi_1(A_{2g})} \cdots t_{\psi_1(A_2)} t_{a_2} t_{a_2} t_{\psi_1(A_2)} \cdots t_{\psi_1(A_{2g})} \cdot t_{a_{g+1}}^{-1}. \end{aligned}$$

Here, let us consider a word $t_c \cdot t_{v_1} t_{v_2} \cdots t_{v_k}$. By repeating elementary transformations on this word, we obtain the word $t_{t_c(v_1)} t_{t_c(v_2)} \cdots t_{t_c(v_k)} \cdot t_c$. Therefore, since H' is a positive relator including t_{e_1}, t_{a_2} and t_{e_2} in this order, we can put them together to the right side of the word to obtain a relator in the form

$$H'' = T \cdot t_{e_1} t_{a_2} t_{e_2} \cdot t_{a_{g+1}}^{-1},$$

where T is a product of $8g^2 + 4g - 3$ right-handed Dehn twists. Let L denote the lantern relator $L = t_{e_1} t_{a_2} t_{e_2} t_{A_1}^{-1} t_{a_3}^{-1} t_{A_5}^{-1} t_{A_3}^{-1}$. Finally, we do L^{-1}-substitution to H'', to obtain the following relator I in Γ_g^1:

$$I = T \cdot t_{A_3} t_{A_5} t_{a_3} t_{A_1} \cdot t_{a_{g+1}}^{-1}.$$

The relator I reduces to a positive relator $\widehat{I}$ in Γ_g. Thus, $\widehat{I}$ gives a genus-g Lefschetz fibration $f_{\widehat{I}} : X_{\widehat{I}} \to S^2$ which admits a (-1)-section.

We see that a genus-g Lefschetz fibration $f_{\widehat{I}} : X_{\widehat{I}} \to S^2$ has $2g(4g+2)+1$ singular fibers. Hence, we have

$$e(X_{\widehat{I}}) = 8g^2 + 5.$$

Here, note that C_{2g} is a positive relator in Γ_g. This gives a genus-g Lefschetz fibration $f_{C_{2g}} : X_{C_{2g}} \to S^2$ with $2g(4g+2)$ nonseparating singular fibers. In particular, this fibration is hyperelliptic since $\iota(A_i) = A_i$ for each $i = 1, 2, \dots, 2g$ (see Figure 5). Therefore, we have $\sigma(X_{C_{2g}}) = -4g(g+1)$ by Theorem 3.5. Since I is obtained from C_{2g} by some B-substitutions, two C_{2g-1}^{-1}-substitutions, $C_{2g-1}^{\psi_2}$- and $C_{2g-1}^{\psi_3}$-substitutions, several other B-substitutions, and one L^{-1}-substitution, by (4) and Proposition 3.6, we have

$$\sigma(X_{\widehat{I}}) = \sigma(X_{C_{2g}}) - 1 = -4g(g+1) - 1.$$

This gives $\lambda_{f_{\widehat{I}}} = 4 - 4/g - 1/g^2 < 4 - 4/g$. By Proposition 3.9, this fibration is nonholomorphic.

It is easy to check that $\widehat{I}$ includes the Dehn twist about the curve $t_{e_1}(\psi_1(A_i))$ for $1 \le i \le 2g$. Since $f_{\widehat{I}}$ admits a section, by Lemma 3.4 we have

$$\pi_1(X_{\widehat{I}}) \subset \pi_1(\Sigma_g)/\langle t_{e_1}(\psi_1(A_1)), \dots, t_{e_1}(\psi_1(A_{2g}))\rangle.$$

On the other hand, it is easy to check that

$$\pi_1(\Sigma_g)/\langle t_{e_1}(\psi_1(A_1)), \dots, t_{e_1}(\psi_1(A_{2g})) = \pi_1(\Sigma_g)/\langle A_1, \dots, A_{2g}\rangle = 1,$$

hence $\pi_1(X_{\widehat{I}}) = 1$. □

Remark 4.2. We do not provide a monodromy factorization of $f_{\widehat{I}}$ explicitly; however, we can obtain it by giving explicit $\psi_j(A_i)$ for $j = 1, 2, 3$ and $i = 1, 2, \dots, 2g$.

Remark 4.3. All vanishing cycles of the Lefschetz fibration $f_{\widehat{I}}$ are nonseparating since all curves of the lantern relator employed in the proof of Theorem 1.1 are nonseparating. For $g \ge 3$, we can consider a lantern relator such that six curves are nonseparating and one curve, denoted by s_h, is separating, which separates Σ_g^1 into two subsurfaces Σ_h^1 and Σ_{g-h}^2 for $h \ge 2$. Then, a similar argument to the proof of Theorem 1.1 gives a genus-g Lefschetz fibration with a (-1)-section on a simply connected total space having s_h as a vanishing cycle and violating the slope inequality, for each $h = 2, 3, \dots, g-1$. Therefore, we can construct at least $g-1$ different genus-g Lefschetz fibrations with the same conditions as in Theorem 1.1.

Remark 4.4. Miyachi and Shiga [2011] produced genus-g Lefschetz fibrations over Σ_{2m} $(m \geq 1)$ which do not satisfy the slope inequality.

5. Noncomplex Lefschetz fibrations admitting (−1)-sections

In this section, we prove Theorem 1.2.

Theorem 1.2. *For each $g \geq 4$ and each positive integer n, there is a genus-g nonholomorphic Lefschetz fibration $f_{\widehat{U}_n} : X_{\widehat{U}_n} \to S^2$ with two disjoint (-1)-sections such that $X_{\widehat{U}_n}$ does not admit any complex structure with either orientation.*

We assume $g \geq 4$ and $g = 4t, 4t+1, 4t+2, 4t+3$ throughout this section. To prove Theorem 1.2, we construct a relator U_n in Γ_g^2 by applying substitutions to the relator $W_{2,g}$ in Γ_g^2, which gives the Lefschetz fibration $f_{\widehat{U}_n} : X_{\widehat{U}_n} \to S^2$.

Let a_j, a'_j, b_j and c_j $(j = 1, 2, \dots, g)$ be the simple closed curves on Σ_g^2 in Figures 1, 2 and 3, and let a_{g+1} and a'_{g+1} be the boundary curves of Σ_g^2 as before. For a positive integer n, we define a map ϕ_n to be

$$\phi_n = t_{a_2}^n t_{a_3} t_{a_4} \cdots t_{a_t} t_{b_1} t_{b_2} \cdots t_{b_t}.$$

Note that $\phi_n(c_r) = c_r$ for $g = 2r$, that $\phi_n(a_{r+1}) = a_{r+1}$ and $\phi_n(a'_{r+1}) = a'_{r+1}$ for $g = 2r+1$, that $\phi_n(c_t) = c_t$ for $r = 2t$, and that $\phi_n(a_{t+1}) = a_{t+1}$ and $\phi_n(a'_{t+1}) = a'_{t+1}$ for $r = 2t+1$.

The relator $W_{2,g}$ in Γ_g^2 includes Dehn twist t_{c_r} twice if $g = 2r$ and the product $t_{a_{r+1}} t_{a'_{r+1}}$ of two Dehn twists four times if $g = 2r+1$. Therefore, we can apply $W_{1,r}$- and $W_{1,r}^{\phi_n}$-substitutions to W_2^g if $g = 2r$ and $W_{2,r}$- and $W_{2,r}^{\phi_n}$-substitutions to W_2^g if $g = 2r+1$. Then, for $g = 2r$ (resp. $g = 2r+1$), we denote by

$$U_n$$

a relator which is obtained by applying once trivial and once ϕ_n-twisted $W_{1,r}$- (resp. $W_{2,r}$-) substitutions to $W_{2,g}$. For the convenience of the reader we write the definition of the relator U_n in detail. Let us consider the following word in Γ_g^2 for $j = 1, 2$:

$$V_j := \begin{cases} (t_{B_{0,j}^r} t_{B_1^r} t_{B_2^r} \cdots t_{B_r^r} t_{c_t})^2 & \text{if } r = 2t, \\ (t_{B_{0,j}^r} t_{B_1^r} t_{B_2^r} \cdots t_{B_r^r} t_{a_{t+1}}^2 t_{a'_{t+1}}^2)^2 & \text{if } r = 2t+1. \end{cases}$$

Note that $V_1 = W_{1,r} t_{c_r}^{-1}$ and $V_2 = W_{2,r} t_{a'_{r+1}}^{-1} t_{a_{r+1}}^{-1}$. Then, we can write U_n as follows: If $g = 2r$, and therefore $g = 4t, 4t+2$, then

$$U_n := (t_{B_{0,2}^g} t_{B_1^g} t_{B_2^g} \cdots t_{B_g^g} V_1)(t_{B_{0,2}^g} t_{B_1^g} t_{B_2^g} \cdots t_{B_g^g} V_1^{\phi_n}) t_{a_{g+1}}^{-1} t_{a'_{g+1}}^{-1},$$

and if $g = 2r+1$, and therefore $g = 4t+1, 4t+3$, then

$$U_n := (t_{B_{0,2}^g} t_{B_1^g} t_{B_2^g} \cdots t_{B_g^g} V_2 t_{a_{r+1}} t_{a'_{r+1}})(t_{B_{0,2}^g} t_{B_1^g} t_{B_2^g} \cdots t_{B_g^g} V_2^{\phi_n} t_{a_{r+1}} t_{a'_{r+1}}) t_{a_{g+1}}^{-1} t_{a'_{g+1}}^{-1}.$$

Since the relator U_n in Γ_g^2 is a product of $t_{a_{g+1}}^{-1} t_{a'_{g+1}}^{-1}$ and positive Dehn twists, it reduces to a positive relator of Γ_g, which is denoted by $\widehat{U}_n$. This gives a genus-g Lefschetz fibration $f_{\widehat{U}_n} : X_{\widehat{U}_n} \to S^2$ with two disjoint (-1)-sections.

We prepare the following lemma.

Lemma 5.1. *For $g = 2r, 2r+1$ and $r = 2t, 2t+1$, the following holds in $H_1(\Sigma_g)$:*

$$\begin{aligned}
\phi_n(B_{0,j}^r) &= B_{0,j}^r + a_t \cdots + a_4 + a_3 + na_2.\\
\phi_n(B_1^r) &= B_1^r - b_1 + a_t + \cdots + a_4 + a_3 + na_2.\\
\phi_n(B_2^r) &= B_2^r - b_1 + a_t + \cdots + a_4 + a_3 + na_2.\\
\phi_n(B_3^r) &= B_3^r - b_2 + a_t + \cdots + a_4 + a_3.\\
\phi_n(B_4^r) &= B_4^r - b_2 + a_t + \cdots + a_4 + a_3 - na_2.\\
\phi_n(B_{2k-1}^r) &= B_{2k-1}^r - b_k + a_t + \cdots + a_{k+2} + a_{k+1}, \qquad 3 \le k \le t.\\
\phi_n(B_{2k}^r) &= B_{2k}^r - b_k + a_t + \cdots + a_{k+2} + a_{k+1} - a_k, \quad 3 \le k \le t.
\end{aligned}$$

If $r = 2t+1$, and therefore $g = 4t+2, 4t+3$, then $\phi_n(B_{2t+1}^{2t+1}) = B_{2t+1}^{2t+1}$.

Proof. We use the following well-known formula for the action of the N-th power of the Dehn twist along a simple closed curve c on $H_1(\Sigma_g)$ repeatedly (see [Farb and Margalit 2012]):

$$t_c^K(d) = d - Ni(d,c)c,$$

for an element d in $H_1(\Sigma_g)$. Recall that $i(a_i, a_i) = i(b_i, b_i) = 0$, $i(a_i, b_j) = 0$ for $i \ne j$ and $i(a_i, b_i) = 1$.

First, we show the equation of $\phi_n(B_{2k-1}^r)$ for $1 \le k \le t$. From Figures 1–3, we see that for $1 \le k \le t$,

$$i(B_{2k-1}^r, a_i) = \begin{cases} 0 & \text{if } 1 \le i \le k-1,\\ -1 & \text{if } k \le i \le t,\end{cases}$$

$$i(B_{2k-1}^r, b_i) = \begin{cases} 1 & \text{if } i = k,\\ 0 & \text{if } i \ne k.\end{cases}$$

Using the above mentioned formula, we get

$$\begin{aligned}
\phi_n(B_{2k-1}^r) &= t_{a_2}^n t_{a_3} t_{a_4} \cdots t_{a_t} t_{b_1} t_{b_2} \cdots t_{b_t}(B_{2k-1}^r)\\
&= t_{a_2}^n t_{a_3} t_{a_4} \cdots t_{a_t} t_{b_1} t_{b_2} \cdots t_{b_{k-1}}(B_{2k-1}^r - b_k)\\
&= t_{a_2}^n t_{a_3} t_{a_4} \cdots t_{a_t} t_{b_1} t_{b_2} \cdots t_{b_{k-1}}(B_{2k-1}^r) - t_{a_2}^n t_{a_3} t_{a_4} \cdots t_{a_t} t_{b_1} t_{b_2} \cdots t_{b_{k-1}}(b_k)\\
&= t_{a_2}^n t_{a_3} t_{a_4} \cdots t_{a_t}(B_{2k-1}^r) - t_{a_2}^n t_{a_3} t_{a_4} \cdots t_{a_t}(b_k).
\end{aligned}$$

Therefore, from $i(b_k, a_k) = -1$ (by $i(a_k, b_k) = 1$), we have

$$\begin{aligned}\phi_n(B^r_{2k-1}) &= t^n_{a_2} t_{a_3} t_{a_4} \cdots t_{a_t}(B^r_{2k-1}) - t^n_{a_2} t_{a_3} t_{a_4} \cdots t_{a_t}(b_k)\\ &= B^r_{2k-1} + a_t + \cdots + a_{k+1} + a_k - (b_k + a_k)\end{aligned}$$

if $3 \le k$, and

$$\begin{aligned}\phi_n(B^r_3) &= t^n_{a_2} t_{a_3} t_{a_4} \cdots t_{a_t}(B^r_3) - t^n_{a_2} t_{a_3} t_{a_4} \cdots t_{a_t}(b_2)\\ &= B^r_3 + a_t + \cdots + a_4 + a_3 + na_2 - (b_2 + na_2),\\ \phi_n(B^r_1) &= t^n_{a_2} t_{a_3} t_{a_4} \cdots t_{a_t}(B^r_1) - t^n_{a_2} t_{a_3} t_{a_4} \cdots t_{a_t}(b_1)\\ &= B^r_1 + a_t + \cdots + a_4 + a_3 + na_2 - b_1.\end{aligned}$$

Therefore, we obtain the required formula of $\phi_n(B^r_{2k-1})$ for $1 \le k \le t$.

Next, we show the equation of $\phi_n(B^r_{2k})$ for $1 \le k \le t$. From Figures 1–3, we see that for $1 \le k \le t$,

$$i(B^{2t}_{2k}, a_i) = \begin{cases} 0 & \text{if } 1 \le i \le k,\\ -1 & (k+1 \le i \le t),\end{cases}$$

$$i(B^{2t}_{2k}, b_i) = \begin{cases} 1 & \text{if } i = k,\\ 0 & \text{if } i \ne k,\end{cases}$$

where $1 \le i$. Using this, a similar argument to $\phi_n(B^r_{2k-1})$ gives

$$\begin{aligned}\phi_n(B^r_{2k}) &= t^n_{a_2} t_{a_3} t_{a_4} \cdots t_{a_t}(B^r_{2k}) - t^n_{a_2} t_{a_3} t_{a_4} \cdots t_{a_t}(b_k)\\ &= B^r_{2k} + a_t + \cdots a_{k+2} + a_{k+1} - (b_k + a_k)\end{aligned}$$

if $3 \le k$ and

$$\begin{aligned}\phi_n(B^r_4) &= t^n_{a_2} t_{a_3} t_{a_4} \cdots t_{a_t}(B^r_4) - t^n_{a_2} t_{a_3} t_{a_4} \cdots t_{a_t}(b_2)\\ &= B^r_4 + a_t + \cdots a_4 + a_3 - (b_2 + na_1),\\ \phi_n(B^r_2) &= t^n_{a_2} t_{a_3} t_{a_4} \cdots t_{a_t}(B^r_2) - t^n_{a_2} t_{a_3} t_{a_4} \cdots t_{a_t}(b_1)\\ &= B^r_2 + a_t + \cdots a_4 + a_3 + na_2 - b_1.\end{aligned}$$

Therefore, we obtain the required formula of $\phi_n(B^r_{2k})$ for $1 \le k \le t$.

Finally, we show the equation for $\phi_n(B^r_{0,j})$. From Figures 1–3, we see that $i(B^{2t}_{0,j}, a_i) = -1$ and $i(B^{2t}_{0,j}, b_i) = 0$. Therefore,

$$\phi_n(B^r_{0,j}) = t^n_{a_2} t_{a_3} t_{a_4} \cdots t_{a_t} t_{b_1} t_{b_2} \cdots t_{b_t}(B^r_{0,j}) = B^r_{0,j} + a_t + \cdots + a_4 + a_3 + na_2.$$

Since $i(B^{2t+1}_{2t+1}, a_i) = (B^{2t+1}_{2t+1}, b_i) = 0$ for $i = 1, 2, \ldots, t$, we have $\phi_n(B^{2t+1}_{2t+1}) = B^{2t+1}_{2t+1}$, and this finishes the proof. □

Proof of Theorem 1.2. It is sufficient to show that $H_1(X_{\widehat{U}_n}) = \mathbb{Z} \oplus \mathbb{Z}_n$ from Theorem 3.10. For a set S, we denote by $\mathbb{Z}\langle S\rangle$ the $\mathbb{Z}$-module generated by S. Let

$$S^h_{0,j} := \{B^h_{0,j}, B^h_1, B^h_2, \dots, B^h_h\},$$
$$T^h_{0,j} := \{\phi_n(B^h_{0,j}), \phi_n(B^h_1), \phi_n(B^h_2), \dots, \phi_n(B^h_h)\}.$$

Recall that $\phi_n(c_t) = c_t$ for $r = 2t$ and $\phi_n(a_{t+1}) = a_{t+1}$ and $\phi_n(a'_{t+1}) = a'_{t+1}$ for $r = 2t+1$. By this fact, $c_t = 0$, $a'_{t+1} = a_{t+1}$ and $a'_{r+1} = a_{r+1}$ in $H_1(\Sigma_g)$ and Lemma 3.4, we have

$$H_1(X_{\widehat{U}_n}) = \begin{cases} H_1(\Sigma_g)/\mathbb{Z}\langle S^{4t}_{0,2} \cup S^{2t}_{0,1} \cup T^{2t}_{0,1}\rangle & \text{if } g = 4t, \\ H_1(\Sigma_g)/\mathbb{Z}\langle S^{4t+1}_{0,2} \cup \{a_{2t+1}\} \cup S^{2t}_{0,2} \cup T^{2t}_{0,2}\rangle & \text{if } g = 4t+1, \\ H_1(\Sigma_g)/\mathbb{Z}\langle S^{4t+2}_{0,2} \cup S^{2t+1}_{0,1} \cup \{a_{t+1}\} \cup T^{2t+1}_{0,1}\rangle & \text{if } g = 4t+2, \\ H_1(\Sigma_g)/\mathbb{Z}\langle S^{4t+3}_{0,2} \cup \{a_{2t+2}\} \cup S^{2t+1}_{0,2} \cup \{a_{t+1}\} \cup T^{2t+1}_{0,2}\rangle & \text{if } g = 4t+3. \end{cases}$$

By $\phi_n(B^r_{2k-1}) = \phi(B^r_{2k}) = 0$ and $B_{2k-1} = B_{2k} = 0$ in $H_1(X_{\widehat{U}_n})$ for $2 \le k \le t$, Lemma 5.1 gives

$$na_2 = a_3 = a_4 = \cdots = a_t = 0. \tag{5}$$

Using this, $B^r_{2k-1} = 0$ and $\phi(B^r_{2k-1}) = 0$ for $1 \le k \le t$ and Lemma 5.1, we have

$$b_1 = b_2 = \cdots = b_t = 0. \tag{6}$$

By (5), the equation $B^r_{0,j} = 0$, and Lemma 5.1, we can remove the relation $\phi_n(B^r_{0,j}) = 0$. Moreover, if $r = 2t+1$, and therefore $g = 4t+2, 4t+3$, then by Lemma 5.1 and $B^{2t+1}_{2t+1} = 0$, we can delete the relation $\phi_n(B^{2t+1}_{2t+1}) = 0$.

Suppose that $r = 2t$ (i.e., $g = 4t, 4t+1$). Let us consider the equations (1)–(3) for $h = 2t$. By $B^{2t}_{2k-1} = B^{2t}_{2k} = 0$ in $H_1(X_{\widehat{U}_n})$, we get

$$b_k + b_{2t+1-k} = 0, \quad 1 \le k \le t.$$

By (6), we have

$$b_1 = b_2 = \cdots b_{2t} = 0. \tag{7}$$

Using this and $B^{2t}_{2k-1} = 0$ for $1 \le k \le t$, we have

$$a_k + a_{2t+1-k} = 0, \quad 1 \le k \le t.$$

Therefore, by (5), we have

$$a_1 + a_{2t} = a_2 + a_{2t-1} = 0; \tag{8}$$

$$a_3 = a_4 = \cdots = a_{2t-2} = 0. \tag{9}$$

Note that $B^{2t}_{0,1} = b_1 + b_2 + \cdots + b_{2t}$ and $B^{2t}_{0,2} = b_1 + b_2 + \cdots + b_{2t} + a_{2t+1}$. By (7) (and $a_{2t+1}=0$ if $g=4t+1$), we can delete $B^{2t}_{0,j}=0$ for $j=1,2$.

Suppose that $r=2t+1$ (i.e., $g=4t+2, 4t+3$). Consider the equations (1)–(3) for $h=2t+1$. By $B^{2t+1}_{2k-1}=B^{2t+1}_{2k}=0$ in $H_1(X_{\widehat{U}_n})$, we get

$$b_k+b_{2t+2-k}=0, \quad 1\le k\le t.$$

In particular, by $B^{2t+1}_{2t+1}=a_{t+1}+b_{t+1}+a_{t+1}=0$ and $a_{t+1}=0$, we have $b_{t+1}=0$. By combining this with (6), we have

$$b_1=b_2=\cdots b_{2t+1}=0. \tag{10}$$

Using this and $B^{2t+1}_{2k-1}=0$ for $1\le k\le t$, we have

$$a_k+a_{2t+2-k}=0, \quad 1\le k\le t.$$

Therefore, by (5) and the relation $a_{t+1}=0$, we have

$$a_1+a_{2t+1}=a_2+a_{2t}=0; \tag{11}$$

$$a_3=a_4=\cdots=a_{2t-1}=0. \tag{12}$$

For a similar reason to the case $r=2t$, we can remove $B^{2t+1}_{0,j}=0$ for $j=1,2$.

Suppose that $g=2r$ (i.e., $g=4t, r=2t$ or $g=4t+2, r=2t+1$). Consider the equations (1)–(3) for $h=2r$. By $B^{2r}_{2k-1}=B^{2r}_{2k}=0$ in $H_1(X_{\widehat{U}_n})$, we obtain

$$b_k+b_{2r+1-k}=0, \quad 1\le k\le r.$$

If $g=4t$ (resp. $g=4t+2$), then the relation (7) (resp. the relation (10)) gives

$$b_1=b_2=\cdots=b_{2r}=0. \tag{13}$$

Using this and $B^{2r}_{2k-1}=0$ for $1\le k\le r$, we have

$$a_k+a_{2r+1-k}=0, \quad 1\le k\le r.$$

By this equations and the equations (8) and (9) (resp. the equations (11) and (12)) if $g=4t$ (resp. $g=4t+2$), we obtain

$$a_1+a_r=a_2+a_{r-1}=0; \tag{14}$$

$$a_1+a_{2r}=a_2+a_{2r-1}=a_{r-1}+a_{r+2}=a_r+a_{r+1}=0; \tag{15}$$

$$a_3=a_4=\cdots=a_{r-2}=a_{r+3}=\cdots=a_{2r-3}=a_{2r-2}=0, \tag{16}$$

and we can delete the relation $B^{2r}_{0,j}=0$ for a similar reason to the case $r=2t$. Since (14) and (15) give $a_{2r}=a_r=-a_1$, $a_{r+1}=a_1$, $a_{2r-1}=a_{r-1}=-a_2$ and $a_{r+2}=a_2$, by (5), (13) and (16), we obtain

$$H_1(X_{\widehat{U}_n})=\mathbb{Z}\langle\{a_1,a_2\}\rangle/\mathbb{Z}\langle\{na_2\}\rangle=\mathbb{Z}\oplus\mathbb{Z}_n,$$

and the proof of Theorem 1.2 for $g=2r$ is complete.

Suppose that $g = 2r + 1$ (i.e., $g = 4t + 1,\ r = 2t$, or $g = 4t + 3,\ r = 2t + 1$). Consider the equations (1)–(3) for $h = 2r + 1$. By $B^{2r+1}_{2k-1} = B^{2r}_{2k} = 0$ in $H_1(X_{\widehat{U}_n})$,

$$b_k + b_{2r+2-k} = 0, \quad 1 \le k \le r.$$

By $B^{2r+1}_{2r+1} = a_{r+1} + b_{r+1} + a_{r+1} = 0$ and $a_{r+1} = 0$, we have $b_{r+1} = 0$. Therefore, if $g = 4t + 1$ (resp. $g = 4t + 3$), then the relation (7) (resp. the relation (10)) gives

$$b_1 = b_2 = \cdots = b_{2r+1} = 0. \tag{17}$$

Using this and $B^{2r+1}_{2k-1} = 0$ for $1 \le k \le r$, we have

$$a_k + a_{2r+2-k} = 0, \quad 1 \le k \le r.$$

By this equations, the equation $a_{r+1} = 0$ and the equations (8) and (9) (resp. the equations (11) and (12)) if $g = 4t + 1$ (resp. $g = 4t + 3$), we obtain

(18) $a_1 + a_r = a_2 + a_{r-1} = 0;$

(19) $a_1 + a_{2r+1} = a_2 + a_{2r} = a_{r-1} + a_{r+3} = a_r + a_{r+2} = 0;$

(20) $a_3 = a_4 = \cdots = a_{r-2} = a_{r+1} = a_{r+4} = a_{r+5} = \cdots = a_{2r-2} = a_{2r-1} = 0,$

and we can delete the relation $B^{2r+1}_{0,j} = 0$ for a similar reason to the case $r = 2t$. Since the equations (18) and (19) give $a_{2r+1} = a_r = -a_1,\ a_{r+2} = a_1,\ a_{2r} = a_{r-1} = -a_2$ and $a_{r+3} = a_2$, by (5), (17) and (20), we obtain

$$H_1(X_{\widehat{U}_n}) = \mathbb{Z}\langle\{a_1, a_2\}\rangle / \mathbb{Z}\langle\{na_2\}\rangle = \mathbb{Z} \oplus \mathbb{Z}_n,$$

and the proof of Theorem 1.2 for $g = 2r + 1$ is complete. □

Acknowledgements

The authors would like to thank R. Inanc Baykur for comments on this paper and for pointing out that there are more various examples of nonholomorphic Lefschetz fibrations than were mentioned in the first version of the manuscript. They are also grateful to Anar Akhmedov for comments. Finally, they are greatly indebted to the referee for invaluable comments, especially, about the proof of Theorem 1.2. Monden was supported by Grant-in-Aid for Young Scientists (B) (No. 16K17601), Japan Society for the Promotion of Science.

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Received December 25, 2016. Revised April 12, 2018.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 298, No. 2, 2019
dx.doi.org/10.2140/pjm.2019.298.399

TILTING MODULES OVER AUSLANDER–GORENSTEIN ALGEBRAS

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For a finite-dimensional algebra Λ and a nonnegative integer n, we characterize when the set $\mathrm{tilt}_n\Lambda$ of additive equivalence classes of tilting modules with projective dimension at most n has a minimal (or equivalently, minimum) element. This generalizes results of Happel and Unger. Moreover, for an n-Gorenstein algebra Λ with $n \geq 1$, we construct a minimal element in $\mathrm{tilt}_n\Lambda$. As a result, we give equivalent conditions for a k-Gorenstein algebra to be Iwanaga–Gorenstein. Moreover, for a 1-Gorenstein algebra Λ and its factor algebra $\Gamma = \Lambda/(e)$, we show that there is a bijection between $\mathrm{tilt}_1\Lambda$ and the set $\mathrm{s}\tau\text{-tilt}\,\Gamma$ of additive equivalence classes of basic support τ-tilting Γ-modules, where e is an idempotent such that $e\Lambda$ is the additive generator of the category of projective-injective Λ-modules.

1. Introduction

Tilting theory is essential in the representation theory of algebras. There are many works (see [Assem et al. 2006; Angeleri Hügel et al. 2007; Happel 1988]) which made the theory fruitful. One interesting topic in tilting theory is to classify tilting modules for some given algebras. Among these, tilting modules over algebras of large dominant dimension have gained more and more attention. For more details, we refer to [Chen and Xi 2016; Crawley-Boevey and Sauter 2017; Nguyen et al. 2018; Iyama and Zhang 2016; Pressland and Sauter 2017; Kajita 2008].

For an algebra Λ, denote by $\mathrm{mod}\,\Lambda$ the category of finitely generated right Λ-modules. Recall that a Λ-module T in $\mathrm{mod}\,\Lambda$ is called a *tilting module* of finite projective dimension if the projective dimension of T is $n < \infty$, $\mathrm{Ext}^i_\Lambda(T, T) = 0$ holds for $i \geq 1$, and there is an exact sequence $0 \to \Lambda \to T_0 \to \cdots \to T_n \to 0$ with $T_i \in \mathrm{add}\,T$, where we use $\mathrm{add}\,T$ to denote the subcategory of $\mathrm{mod}\,\Lambda$ consisting of direct summands of finite direct sums of T. We say that $M, N \in \mathrm{mod}\,\Lambda$ are

Iyama is supported by JSPS Grant-in-Aid for Scientific Research (B) 24340004, (B) 16H03923, (C) 23540045 and (S) 15H05738. Zhang is supported by NSFC (Nos. 11401488, 11571164 and 11671174), Jiangsu Government Scholarship for Overseas Studies (JS-2014-352) and NSF for Jiangsu Province (BK20130983).
MSC2010: primary 16G10; secondary 16E10.
Keywords: n-Gorenstein algebra, tilting module, support τ-tilting module.

additively equivalent if $\mathsf{add}\, M = \mathsf{add}\, N$. For a nonnegative integer n, let $\operatorname{tilt}_n \Lambda$ be the set consisting of additive equivalence classes of tilting modules with projective dimension at most n, and $\operatorname{tilt}\Lambda = \operatorname{tilt}_\infty \Lambda := \bigcup_{n\geq 0} \operatorname{tilt}_n \Lambda$. There is a natural partial order on the set $\operatorname{tilt}\Lambda$ defined as follows [Riedtmann and Schofield 1991; Happel and Unger 2005; Aihara and Iyama 2012]: For $T, U \in \operatorname{tilt}\Lambda$, $T \geq U$ if $\operatorname{Ext}^i_\Lambda(T, U) = 0$ for all $i > 0$. This is equivalent to saying that $T^\perp \supseteq U^\perp$, where $T^\perp$ is the subcategory of $\mathsf{mod}\,\Lambda$ consisting of modules M such that $\operatorname{Ext}^i_\Lambda(T, M) = 0$ for any $i \geq 1$. Clearly Λ is the maximal element in $\operatorname{tilt}\Lambda$, and if Λ is Iwanaga–Gorenstein, then $\mathbb{D}\Lambda$ is the minimal element in $\operatorname{tilt}\Lambda$, where $\mathbb{D}$ is the ordinary duality. However, it is difficult to find the minimal element in $\operatorname{tilt}\Lambda$ for an arbitrary algebra Λ.

For a right Λ-module M, let $0 \to M \to I^0(M) \to I^1(M) \to \cdots$ be a minimal injective resolution of M and $\cdots \to P_1(M) \to P_0(M) \to M \to 0$ a minimal projective resolution of M. Recall that an algebra Λ is called *n-Gorenstein* (resp. *quasi n-Gorenstein*) if the projective dimension of $I^i(\Lambda)$ is less than or equal to i (resp. $i+1$) for $0 \leq i \leq n-1$ [Fossum et al. 1975; Huang 2006]. There are many works on n-Gorenstein algebras [Auslander and Reiten 1994; Auslander and Reiten 1996; Clark 2001; Huang and Iyama 2007; Iwanaga and Sato 1996], but little is known for the tilting modules over this class of algebras. Our first aim is to study the existence of minimal tilting modules over this class of algebras. For a module $M \in \mathsf{mod}\,\Lambda$, we denote by $\Omega^i M$ (resp. $\Omega^{-i} M$) the i-th syzygy (resp. cosyzygy) of M. A special case of our first main theorem, Theorem 3.4, is the following:

Theorem 1.1 (Corollary 3.5). *Let Λ be a quasi n-Gorenstein algebra and* $0 \leq j \leq n$. *Then* $\left(\bigoplus_{i=0}^{j-1} I^i(\Lambda)\right) \oplus \Omega^{-j}\Lambda$ *is the minimum element in* $\operatorname{tilt}_j \Lambda$.

For example, algebras with dominant dimension at least n are n-Gorenstein. In this case, the tilting module given in Theorem 1.1 was studied recently in [Crawley-Boevey and Sauter 2017; Nguyen et al. 2018; Pressland and Sauter 2017] (see Example 3.6).

Recall that a subcategory $\mathscr{C}$ of $\mathsf{mod}\,\Lambda$ is called *contravariantly finite* if for any M in $\mathsf{mod}\,\Lambda$ there is a morphism $C_M \to M$ with $C_M \in \mathscr{C}$ such that the sequence $\operatorname{Hom}_\Lambda(-, C_M) \to \operatorname{Hom}_\Lambda(-, M) \to 0$ is exact over $\mathscr{C}$. Dually, one can define covariantly finite subcategories. For a subcategory $\mathscr{D}$ of $\mathsf{mod}\,\Lambda$, denote by $\mathscr{D}^\perp$ the subcategory consisting of modules N such that $\operatorname{Ext}^i_\Lambda(M, N) = 0$ for $i \geq 1$ and $M \in \mathscr{D}$. We denote this by $M^\perp$ if $\mathscr{D} = \mathsf{add}\, M$. Dually, one can define ${}^\perp\mathscr{D}$ and ${}^\perp M$.

Denote by $\mathcal{P}_\infty(\Lambda)$ the subcategory consisting of Λ-modules with finite projective dimension. Happel and Unger [2005, Theorem 3.3] showed that $\operatorname{tilt}\Lambda$ has a minimal element if and only if $\mathcal{P}_\infty(\Lambda)$ is contravariantly finite. It is natural to ask if there is a similar result for $\operatorname{tilt}_n \Lambda$. We give a positive answer by proving the following result, where we denote by $\mathcal{P}_n(\Lambda)$ the subcategory consisting of modules with projective dimension at most n, where the equivalence of (1) and (5) for $n = \infty$

recovers [Happel and Unger 2005, Theorem 3.3], and the equivalence of (3) and (5) for integer n recovers [Happel and Unger 1996, Corollary 2.3].

Theorem 1.2 (Theorem 3.1). *Let Λ be an algebra, and let n be ∞ or a nonnegative integer. Then the following are equivalent*:

(1) $\mathrm{tilt}_n \Lambda$ *has a minimal element.*

(2) $\mathrm{tilt}_n \Lambda$ *has the minimum element.*

(3) *There exists $T \in \mathrm{tilt}_n \Lambda$ such that ${}^{\perp}T \supseteq \mathcal{P}_n(\Lambda)$.*

(4) *There exists $T \in \mathrm{tilt}_n \Lambda$ such that ${}^{\perp}(T^{\perp}) = \mathcal{P}_n(\Lambda)$.*

(5) *The subcategory $\mathcal{P}_n(\Lambda)$ is contravariantly finite.*

For any $M \in \mathrm{mod}\,\Lambda$, denote by $\mathrm{id}_\Lambda M$ (resp. $\mathrm{pd}_\Lambda M$) the injective (resp. projective) dimension of M. An algebra is called *Iwanaga–Gorenstein* if both $\mathrm{id}_\Lambda \Lambda$ and $\mathrm{id}_{\Lambda^{\mathrm{op}}} \Lambda$ are finite. Auslander and Reiten [1994] posed a question which asks whether Λ must be Iwanaga–Gorenstein if it is n-Gorenstein for all positive integers n. This is a generalization of the Nakayama conjecture which says that an algebra with infinite dominant dimension is self-injective. Moreover, Auslander and Reiten [1994, p. 25] studied the question of whether the $\mathcal{P}_\infty(\Lambda)$ is contravariantly finite if Λ is n-Gorenstein for all positive integers n.

As a result of Theorems 1.1 and 1.2, we connect the two questions of Auslander and Reiten above and show the following corollary which covers [Auslander and Reiten 1994, Corollary 5.5].

Corollary 1.3 (Corollary 3.7). *Let Λ be a k-Gorenstein algebra for all positive integers k, and n a nonnegative integer. Then the following are equivalent*:

(1) *Λ is Iwanaga–Gorenstein with* $\mathrm{id}_\Lambda \Lambda = \mathrm{id}_{\Lambda^{\mathrm{op}}} \Lambda \leq n$.

(2) $\mathrm{id}_\Lambda \Lambda \leq n$.

(3) $\mathrm{id}_{\Lambda^{\mathrm{op}}} \Lambda \leq n$.

(4) $\mathrm{tilt}\,\Lambda$ *has the minimum element T with* $\mathrm{pd}_\Lambda T \leq n$.

(5) $\mathrm{tilt}\,\Lambda^{\mathrm{op}}$ *has the minimum element T with* $\mathrm{pd}_\Lambda T \leq n$.

(6) *The subcategory $\mathcal{P}_\infty(\Lambda)$ is contravariantly finite and $\mathcal{P}_\infty(\Lambda) = \mathcal{P}_n(\Lambda)$.*

(7) *The subcategory $\mathcal{P}_\infty(\Lambda^{\mathrm{op}})$ is contravariantly finite and $\mathcal{P}_\infty(\Lambda^{\mathrm{op}}) = \mathcal{P}_n(\Lambda^{\mathrm{op}})$.*

It may be interesting to ask the following question for a finite-dimensional algebra Λ: Does the existence of the minimum element of $\mathrm{tilt}\,\Lambda$ imply the existence of a minimum element of $\mathrm{tilt}\,\Lambda^{\mathrm{op}}$?

Now we turn to the classical tilting modules over 1-Gorenstein algebras and study the connections with τ-tilting theory.

In 2014, Adachi, Iyama and Reiten introduced τ-tilting modules, see Definition 4.1, which are generalizations of classical tilting modules from the viewpoint of mutation.

For general details of τ-tilting theory, we refer to [Adachi et al. 2014; Demonet et al. 2019; Iyama et al. 2014; Jasso 2015; Wei 2014; Zhang 2017b].

For an algebra Λ, denote by $\mathrm{s}\tau\text{-tilt}\,\Lambda$ the set of the additive equivalence classes of support τ-tilting Λ-modules (see Definition 4.1). In [Demonet et al. 2017; Iyama and Zhang 2016] it is shown that the functor $-\otimes_\Lambda \Gamma$ induces a map from $\mathrm{s}\tau\text{-tilt}\,\Lambda$ to $\mathrm{s}\tau\text{-tilt}\,\Gamma$, where Γ is a factor algebra of Λ. Recall that $\mathrm{tilt}_1\Lambda$ is the set of additive equivalence classes of classical tilting Λ-modules. Our third main result is the following.

Theorem 1.4 (Theorem 4.5). *Let Λ be a 1-Gorenstein algebra and let Γ be the factor algebra $\Lambda/(e)$, where e is an idempotent such that $\mathsf{add}\, e\Lambda = \mathsf{add}\, I^0(\Lambda)$. Then $-\otimes_\Lambda \Gamma$ induces a bijection from $\mathrm{tilt}_1\Lambda$ to $\mathrm{s}\tau\text{-tilt}\,\Gamma$.*

For an algebra Λ, denote by $\#\mathrm{s}\tau\text{-tilt}\,\Lambda$ the number of elements in the set $\mathrm{s}\tau\text{-tilt}\,\Lambda$. As an immediate consequence, we have the following corollary. Recall from [Demonet et al. 2019] that an algebra Λ is called *τ-tilting finite* if there are a finite number of basic τ-tilting modules up to isomorphism.

Corollary 1.5 (Corollaries 4.7, 4.8 and 4.9). *For each case, let e be the idempotent such that $\mathsf{add}\, e\Lambda_n = \mathsf{add}\, I^0(\Lambda_n)$.*

(1) *Let $\Lambda_n = KQ$ be the hereditary Nakayama algebra with $Q = A_n$. Then there are bijections*

$$\mathrm{tilt}_1\Lambda_n \simeq \mathrm{s}\tau\text{-tilt}\,\Lambda_{n-1} \simeq \{\textit{clusters of the cluster algebra of type } A_{n-1}\}.$$

Thus $\#\mathrm{s}\tau\text{-tilt}\,\Lambda_n = (2(n+1))!/((n+2)!(n+1)!)$.

(2) *Let Λ_n be the Auslander algebra of $K[x]/(x^n)$, let Γ_{n-1} be the preprojective algebra of $Q = A_{n-1}$ and let $\mathfrak{S}_n$ be the symmetric group. Then there are bijections*

$$\mathrm{tilt}_1\Lambda_n \simeq \mathrm{s}\tau\text{-tilt}\,\Gamma_{n-1} \simeq \mathfrak{S}_n.$$

Thus $\#\mathrm{s}\tau\text{-tilt}\,\Lambda_n = n!$.

(3) *Let Λ_n be the Auslander algebra of the hereditary Nakayama algebra KQ with $Q = A_n$. Then there is a bijection $\mathrm{tilt}_1\Lambda_n \simeq \mathrm{s}\tau\text{-tilt}\,\Lambda_{n-1}$. Thus Λ is τ-tilting finite if and only if $n \le 4$.*

The organization of this paper is as follows: In Section 2, we recall some preliminaries. In Section 3, we give some equivalent conditions to the existence of minimal elements in $\mathrm{tilt}_n\Lambda$ and show a Happel–Unger type theorem. Moreover, we construct minimal tilting modules for n-Gorenstein algebras and show Theorem 1.1. In Section 4, we build a connection between classical tilting modules over 1-Gorenstein algebras and support τ-tilting modules over factor algebras and we show Theorem 1.4 and Corollary 1.5.

Throughout this paper, we denote by K an algebraically closed field. All algebras are basic connected finite-dimensional K-algebras and all modules are finitely generated right modules. For an algebra A, we denote by $\mathsf{mod}\,A$ the category of finitely generated right A-modules. The composition of homomorphisms $f : X \to Y$ and $g : Y \to Z$ is denoted by $gf : X \to Z$.

2. Preliminaries

We start with the following fundamental theorem due to Auslander and Reiten.

Theorem 2.1 [Auslander and Reiten 1991, Theorem 5.5]. *Let $n \geq 0$ and let Λ be a finite-dimensional algebra. Then there exist bijections between the following objects given by $T \mapsto \mathcal{X} = {}^{\perp}(T^{\perp})$ and $T \mapsto \mathcal{Y} = T^{\perp}$.*

(1) $T \in \mathrm{tilt}_n \Lambda$.

(2) *Contravariantly finite resolving subcategories $\mathcal{X}$ of $\mathsf{mod}\,\Lambda$ contained in $\mathcal{P}_n(\Lambda)$.*

(3) *Covariantly finite coresolving subcategories $\mathcal{Y}$ of $\mathsf{mod}\,\Lambda$ containing $\Omega^{-n}(\Lambda)$.*

Moreover, in this case, $(\mathcal{X}, \mathcal{Y})$ is a cotorsion pair such that $\mathcal{X} \cap \mathcal{Y} = \mathsf{add}\,T$, and $\mathcal{X}$ consists of all $X \in \mathsf{mod}\,\Lambda$ such that there exists an exact sequence $0 \to X \to T^0 \to \cdots \to T^n \to 0$ with $T^i \in \mathsf{add}\,T$.

In the rest, a *subcategory* is always assumed to be full and closed under direct sums and direct summands unless stated otherwise. For later application, we prepare the following observation, which is a relative version of a well-known observation (e.g., [Auslander and Smalø 1980]).

Lemma 2.2. *Let $\mathcal{C}$ be a subcategory of $\mathsf{mod}\,\Lambda$ which is closed under extensions, and $\mathcal{A}$ a subcategory of $\mathcal{C}$. Then the following conditions are equivalent.*

(1) *Any exact sequence $0 \to A \to A' \to C \to 0$ with $A, A' \in \mathcal{A}$ and $C \in \mathcal{C}$ splits.*

(2) *There is no exact sequence $0 \to A \to A' \to C \to 0$ with $A \in \mathsf{ind}\,\mathcal{A}$, $A' \in \mathcal{A}$, $C \in \mathcal{C}$ and $A \notin \mathsf{add}\,A'$.*

Proof. It suffices to prove (2) ⇒ (1). Assume that there exists a nonsplit exact sequence $0 \to A \xrightarrow{f} A' \to C \to 0$ with $A, A' \in \mathcal{A}$ and $C \in \mathcal{C}$. Without loss of generality, we can assume that f is in the radical of $\mathsf{mod}\,\Lambda$.

Take an indecomposable direct summand X of A. Let $\iota : X \to A$ be the inclusion and $g = f\iota$. Then we have the following commutative diagram of nonsplit exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{g} & A' & \longrightarrow & C' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \iota} & & \| & & \downarrow & & \\ 0 & \longrightarrow & A & \xrightarrow{f} & A' & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

Since $0 \to X \xrightarrow{\iota} A \to C' \to C \to 0$ is an exact sequence, C' belongs to $\mathcal{C}$.

Decompose $A' = X^{\oplus \ell} \oplus U$ with $X \notin \mathsf{add}\, U$, and write $g : X \to A' = X^{\oplus \ell} \oplus U$. Let $Y^0 = X$,

$$Y^i = X^{\oplus \ell^i} \oplus U^{\oplus(1+\ell+\cdots+\ell^{i-1})},$$

and

$$g^i = g^{\oplus \ell^i} \oplus 1_U^{\oplus(1+\ell+\cdots+\ell^{i-1})} : Y^i \to Y^{i+1}.$$

Clearly $\operatorname{Ker} g^i = 0$ and $\operatorname{Coker} g^i \in \mathcal{C}$ for any $i \geq 0$.

Take $m > 0$ such that $\operatorname{rad}^m \operatorname{End}_\Lambda(X) = 0$. Then the composition

$$h = g^{m-1} \cdots g^1 g^0 : X \to Y^m = X^{\oplus \ell^m} \oplus U^{\oplus(1+\ell+\cdots+\ell^{m-1})}$$

is a direct sum of morphisms $0 \to X^{\oplus \ell^m}$ and $h' : X \to U^{\oplus(1+\ell+\cdots+\ell^{m-1})}$. Thus we have an exact sequence

$$0 \to X \to U^{\oplus(1+\ell+\cdots+\ell^{m-1})} \to \operatorname{Coker} h' \to 0. \tag{2-1}$$

Since $\operatorname{Coker} h$ is an extension of $\operatorname{Coker} g^i$'s, it belongs to $\mathcal{C}$. Thus $\operatorname{Coker} h'$ also belongs to $\mathcal{C}$. This is contradiction to the condition (2). □

Let $\mathcal{C}$ be a subcategory of $\mathsf{mod}\,\Lambda$ which is closed under extensions. A *cogenerator* for $\mathcal{C}$ is a subcategory $\mathcal{A}$ of $\mathcal{C}$ such that, for any $C \in \mathcal{C}$, there exists an exact sequence

$$0 \to C \to A \to C' \to 0$$

with $A \in \mathcal{A}$ and $C' \in \mathcal{C}$. A cogenerator $\mathcal{A}$ of $\mathcal{C}$ is called *minimal* if no proper subcategory of $\mathcal{A}$ is a cogenerator of $\mathcal{C}$.

The following observation is a relative version of a well-known result in the theory of (co)covers in [Auslander and Smalø 1980].

Proposition 2.3. *Let $\mathcal{C}$ be a subcategory of* $\mathsf{mod}\,\Lambda$ *which is closed under extensions. If $\mathcal{A}$ is a minimal cogenerator for $\mathcal{C}$, then* $\operatorname{Ext}^1_\Lambda(\mathcal{C}, \mathcal{A}) = 0$ *holds.*

Proof. Since $\mathcal{A}$ is a minimal cogenerator for $\mathcal{C}$, the conditions (1) and (2) in Lemma 2.2 are satisfied. Otherwise, there is an exact sequence

$$0 \to A \to A' \to C \to 0$$

with $A \in \mathsf{ind}\mathcal{A}$, $A' \in \mathcal{A}$, $C \in \mathcal{C}$ and $A \notin \mathsf{add}\, A'$. Then the subcategory $\mathcal{A}'$ of $\mathcal{A}$ defined by

$$\mathsf{ind}\mathcal{A}' = (\mathsf{ind}\mathcal{A}) \setminus \{A\}$$

is a cogenerator for $\mathcal{C}$, which is a contradiction to the minimality of $\mathcal{A}$.

Let $A \in \mathcal{A}$ be indecomposable. To prove $\operatorname{Ext}^1_\Lambda(\mathcal{C}, A) = 0$, we take an exact sequence

$$0 \to A \to C' \to C \to 0$$

with $C \in \mathcal{C}$. Since $C' \in \mathcal{C}$, there exists an exact sequence

$$0 \to C' \to A' \to C'' \to 0$$

with $C'' \in \mathcal{C}$ and $A' \in \mathcal{A}$. We have the following commutative diagram of exact sequences:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & C' & \longrightarrow & C & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & A' & \longrightarrow & X & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & C'' & = & C'' & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

By the right vertical sequence, X belongs to $\mathcal{C}$. Thus the middle horizontal sequence splits by Lemma 2.2(1), and so the upper horizontal sequence splits, as desired. □

We need the following result on mutation of tilting modules.

Proposition 2.4 [Happel and Unger 2005; Coelho et al. 1994; Aihara and Iyama 2012]. *For a basic tilting module $T = X \oplus U$ with X indecomposable, if there is an exact sequence $0 \to X \xrightarrow{f} U' \to Y \to 0$ such that $U' \in \mathsf{add}\, U$ and f is a minimal left $(\mathsf{add}\, U)$-approximation of X, then $V = Y \oplus U$ is a basic tilting module such that $V < T$.*

3. Minimal tilting modules and the category $\mathcal{P}_n(\Lambda)$

Characterizations of existence of minimal tilting modules. Throughout this section, let Λ be an arbitrary algebra. We focus on the properties of tilting modules in $\mathrm{tilt}_n \Lambda$ and give some equivalent conditions to the existence of minimal elements in $\mathrm{tilt}_n \Lambda$. More precisely, we generalize the Happel–Unger theorem stating that $\mathrm{tilt}\,\Lambda$ has a minimal element if and only if $\mathcal{P}_\infty(\Lambda)$ is contravariantly finite (see [Happel and Unger 2005, Theorem 3.3]). Now we connect the existence of a minimal element in $\mathrm{tilt}_n \Lambda$ with the contravariantly finiteness of $\mathcal{P}_n(\Lambda)$ and show our main result below. Note that the equivalence of (3) and (5) for integer n recovers [Happel and Unger 1996, Corollary 2.3].

Theorem 3.1. *Let Λ be an algebra, and let n be ∞ or a nonnegative integer. Then the following are equivalent*:

(1) $\mathrm{tilt}_n \Lambda$ *has a minimal element.*

(2) $\mathrm{tilt}_n \Lambda$ *has the minimum element.*

(3) *There exists $T \in \mathrm{tilt}_n \Lambda$ such that ${}^{\perp}T \supseteq \mathcal{P}_n(\Lambda)$.*

(4) *There exists* $T \in \operatorname{tilt}_n \Lambda$ *such that* ${}^{\perp}(T^{\perp}) = \mathcal{P}_n(\Lambda)$.

(5) *The subcategory* $\mathcal{P}_n(\Lambda)$ *is contravariantly finite.*

To prove Theorem 3.1, we need the following result.

Proposition 3.2. *Let* Λ *be an algebra, and let* $\mathcal{C}$ *be a resolving subcategory of* $\operatorname{mod} \Lambda$ *contained in* $\mathcal{P}_\infty(\Lambda)$. *For* $T \in \mathcal{C} \cap \operatorname{tilt} \Lambda$, *the following conditions are equivalent.*

(1) T *is a minimal element in* $\mathcal{C} \cap \operatorname{tilt} \Lambda$.

(2) T *is the minimum element in* $\mathcal{C} \cap \operatorname{tilt} \Lambda$.

(3) ${}^{\perp}(T^{\perp}) = \mathcal{C}$.

(4) ${}^{\perp}T \supseteq \mathcal{C}$.

(5) $\mathcal{C} \cap T^{\perp} = \operatorname{add} T$.

(6) *Every exact sequence* $0 \to T_1 \to T_0 \to C \to 0$ *with* $T_i \in \operatorname{add} T$ *and* $C \in \mathcal{C}$ *splits.*

(7) *There is no monomorphism* $f : X \to T'$ *such that* X *is an indecomposable direct summand of* T, $T' \in \operatorname{add}(T/X)$ *and* $\operatorname{Coker} f \in \mathcal{C}$.

Proof. By the last part of Theorem 2.1, for any $U \in \mathcal{C} \cap \operatorname{tilt} \Lambda$, we have

$$ {}^{\perp}(U^{\perp}) \subseteq \mathcal{C}. \tag{3-1} $$

(1) $\Rightarrow$ (7) Assume that there exists such $f : X \to T'$. Let $g : X \to U'$ be a minimal left $(\operatorname{add} T/X)$-approximation and $Y = \operatorname{Coker} g$. Then f factors through g, and we have a commutative diagram of exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{f} & T' & \longrightarrow & \operatorname{Coker} f & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow & & \\ & & X & \xrightarrow{g} & U' & \longrightarrow & Y & \longrightarrow & 0 \end{array}$$

Thus $\operatorname{Ker} g = 0$, and we have an exact sequence $0 \to U' \to T' \oplus Y \to \operatorname{Coker} f \to 0$. Thus $Y \in \mathcal{C}$. This means that $U = (T/X) \oplus Y$ is a mutation of T (Proposition 2.4) and gives an element of $\mathcal{C} \cap \operatorname{tilt} \Lambda$ such that $T > U$, a contradiction.

(7) $\Rightarrow$ (6) We only need to apply Lemma 2.2 for $\mathcal{A} = \operatorname{add} T$.

(6) $\Rightarrow$ (5) It suffices to show $\mathcal{C} \cap T^{\perp} \subseteq \operatorname{add} T$. Let $C_0 \in \mathcal{C} \cap T^{\perp}$ and $n = \operatorname{pd}_\Lambda C_0$. Since $T^{\perp}$ is an exact category with enough projectives $\operatorname{add} T$, there exists an exact sequence

$$ \cdots \xrightarrow{f_2} T_1 \xrightarrow{f_1} T_0 \xrightarrow{f_0} C_0 \to 0 $$

such that $T_i \in \operatorname{add} T$ and $C_i = \operatorname{Im} f_i$ belongs to $T^{\perp}$. Since $\mathcal{C}$ is resolving, each C_i belongs to $\mathcal{C}$. For every $i > 0$, we have

$$ \operatorname{Ext}^i_\Lambda(C_n, T^{\perp}) \simeq \operatorname{Ext}^{i+1}_\Lambda(C_{n-1}, T^{\perp}) \simeq \cdots \simeq \operatorname{Ext}^{i+n}_\Lambda(C_0, T^{\perp}) = 0. $$

Thus C_n belongs to $T^\perp \cap {}^\perp(T^\perp) = \mathsf{add}\, T$ by Theorem 2.1.

Since $C_{n-1} \in \mathcal{C}$, the exact sequence $0 \to C_n \to T_{n-1} \to C_{n-1} \to 0$ splits by our assumption, and hence C_{n-1} belongs to $\mathsf{add}\, T$. Repeating the same argument, we have $C_0 \in \mathsf{add}\, T$, as desired.

(5) $\Rightarrow$ (1) Assume $U \in \mathcal{C} \cap \operatorname{tilt}\Lambda$ satisfies $T \geq U$. Then $U \in \mathcal{C} \cap T^\perp = \mathsf{add}\, T$ and hence $U = T$.

(5) + (7) $\Rightarrow$ (4) By Proposition 2.3, it suffices to show that $\mathsf{add}\, T$ is a minimal cogenerator for $\mathcal{C}$. Let $C \in \mathcal{C}$. Since $({}^\perp(T^\perp), T^\perp)$ is a cotorsion pair by Theorem 2.1, there exists an exact sequence $0 \to C \to Y \to X \to 0$ with $Y \in T^\perp$ and $X \in {}^\perp(T^\perp)$. By (3-1), we have $X \in \mathcal{C}$. Since $\mathcal{C}$ is extension closed, we have $Y \in \mathcal{C} \cap T^\perp = \mathsf{add}\, T$ by (5). Thus $\mathsf{add}\, T$ is a cogenerator for $\mathcal{C}$. Moreover, it is minimal by (7).

(4) $\Rightarrow$ (3) Thanks to (3-1), it suffices to show ${}^\perp(T^\perp) \supseteq \mathcal{C}$, i.e., any $X \in \mathcal{C}$ and $Y_0 \in T^\perp$ satisfy $\operatorname{Ext}^i_\Lambda(X, Y_0) = 0$ for all $i > 0$. Since $T^\perp$ is an exact category with enough projectives $\mathsf{add}\, T$, there exists an exact sequence

$$\cdots \xrightarrow{f_2} T_1 \xrightarrow{f_1} T_0 \xrightarrow{f_0} Y_0 \to 0$$

such that $T_i \in \mathsf{add}\, T$ and $Y_i = \operatorname{Im} f_i$ belongs to $T^\perp$. For $n = \operatorname{pd}_\Lambda X$, by using (4),

$$\operatorname{Ext}^i_\Lambda(X, Y_0) \simeq \operatorname{Ext}^{i+1}_\Lambda(X, Y_1) \simeq \cdots \simeq \operatorname{Ext}^{i+n}_\Lambda(X, Y_n) = 0$$

as desired.

(3) $\Rightarrow$ (2) Let $U \in \mathcal{C} \cap \operatorname{tilt}\Lambda$. By (3-1), we have ${}^\perp(U^\perp) \subseteq \mathcal{C} = {}^\perp(T^\perp)$. Thus $U^\perp \supseteq T^\perp$ by Theorem 2.1.

(2) $\Rightarrow$ (1) This is clear. □

Now we prove the following theorem, where the equivalence of (3) and (5) recovers [Happel and Unger 1996, Theorems 2.1 and 2.2].

Theorem 3.3. *Let Λ be an algebra, and let $\mathcal{C}$ be a resolving subcategory of* $\mathsf{mod}\, \Lambda$ *contained in $\mathcal{P}_\infty(\Lambda)$. Then the following are equivalent*:

(1) $\mathcal{C} \cap \operatorname{tilt}\Lambda$ *has a minimal element.*

(2) $\mathcal{C} \cap \operatorname{tilt}\Lambda$ *has the minimum element.*

(3) *There exists $T \in \mathcal{C} \cap \operatorname{tilt}\Lambda$ such that ${}^\perp T \supseteq \mathcal{C}$.*

(4) *There exists $T \in \mathcal{C} \cap \operatorname{tilt}\Lambda$ such that ${}^\perp(T^\perp) = \mathcal{C}$.*

(5) *The subcategory $\mathcal{C}$ is contravariantly finite.*

Proof. (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) These are shown in Proposition 3.2.

(4) $\Leftrightarrow$ (5) This is well known (see Theorem 2.1). □

We are ready to prove Theorem 3.1.

Proof of Theorem 3.1. One can obtain the assertion by putting $\mathcal{C} = \mathcal{P}_n(\Lambda)$ in Theorem 3.3. □

Minimal tilting modules of Auslander–Gorenstein algebras. In this section, we construct a class of minimal tilting modules for n-Gorenstein algebras. As a result, we show some equivalent conditions for an n-Gorenstein algebra to be Iwanaga–Gorenstein, which gives a partial answer to a question of Auslander and Reiten mentioned before.

Now we have the following result which gives a method in constructing minimal tilting modules with finite projective dimension.

Theorem 3.4. *For an algebra Λ and a fixed integer $n \geq 0$, assume* $\mathrm{pd}_\Lambda I^i(\Lambda) \leq n$ *for any i, $0 \leq i \leq n-1$ and* $\mathrm{pd}_\Lambda \Omega^{-n}\Lambda \leq n$. *Let* $T = \left(\bigoplus_{i=0}^{n-1} I^i(\Lambda)\right) \oplus \Omega^{-n}\Lambda$. *Then we have the following*:

(1) *T is a tilting module with projective dimension at most n.*

(2) *T is the minimum element in* $\mathrm{tilt}_n \Lambda$.

(3) *$\mathcal{P}_n(\Lambda)$ is contravariantly finite and* ${}^\perp(T^\perp) = \mathcal{P}_n(\Lambda)$.

Proof. We show the assertion (1) step by step.
• By our assumptions, $\mathrm{pd}_\Lambda T \leq n$ holds.
• We prove $\mathrm{Ext}^i_\Lambda(X, T) = 0$ for any $i \geq 1$ and $X \in \mathcal{P}_n(\Lambda)$. This implies $\mathrm{Ext}^i_\Lambda(T, T) = 0$ for $i \geq 1$.

Clearly $\mathrm{Ext}^i_\Lambda(X, I^j(\Lambda)) = 0$ holds for $i \geq 1$ since $I^j(\Lambda)$ is injective. Moreover $\mathrm{Ext}^i_\Lambda(X, \Omega^{-n}\Lambda) = \mathrm{Ext}^{i+n}_\Lambda(X, \Lambda) = 0$ holds for $i \geq 1$ since $X \in \mathcal{P}_n(\Lambda)$.
• The following exact sequence gives the desired sequence in the definition of tilting modules:

$$0 \to \Lambda \to I^0(\Lambda) \to \cdots \to I^{n-1}(\Lambda) \to \Omega^{-n}\Lambda \to 0.$$

(2) It suffices to show that $T \in U^\perp$ holds for any tilting module U with $\mathrm{pd}_\Lambda U \leq n$. This is a special case of the statment above.

(3) This follows from Theorem 3.1. □

Immediately, we have the following corollary.

Corollary 3.5. *Let Λ be a quasi n-Gorenstein algebra with $n \geq 0$. Then* $\mathrm{tilt}_n \Lambda$ *has the minimum element* $\left(\bigoplus_{i=0}^{n-1} I^i(\Lambda)\right) \oplus \Omega^{-n}\Lambda$.

Proof. This is immediate from Theorem 3.4 □

Recall that an algebra Λ is called of *dominant dimension at least n* if $I^i(\Lambda)$ is projective for $0 \leq i \leq n-1$. Then we have the following immediate from Theorem 3.4.

Example 3.6. Let Λ be an algebra with dominant dimension at least $n \geq 0$ and let $T = I^0(\Lambda) \oplus \Omega^{-n}\Lambda$. Then T is the minimum element in $\mathrm{tilt}_n \Lambda$, which was studied recently in [Crawley-Boevey and Sauter 2017; Marczinzik 2018; Nguyen et al. 2018; Pressland and Sauter 2017]. The equality ${}^{\perp}(T^{\perp}) = \mathcal{P}_n(\Lambda)$ in Theorem 3.4(3) was observed in [Marczinzik 2018, 2.4].

Now we give some applications to a question of Auslander and Reiten which says that if Λ is n-Gorenstein for all nonnegative integer n then Λ is Iwanaga–Gorenstein. This is a generalization of the famous Nakayama conjecture. We have the following:

Corollary 3.7. *Let Λ be a k-Gorenstein algebra for all positive integers k, and n a nonnegative integer. Then the following are equivalent*:

(1) Λ *is Iwanaga–Gorenstein with* $\mathrm{id}_\Lambda \Lambda = \mathrm{id}_{\Lambda^{\mathrm{op}}} \Lambda \leq n$.

(2) $\mathrm{id}_\Lambda \Lambda \leq n$.

(3) $\mathrm{id}_{\Lambda^{\mathrm{op}}} \Lambda \leq n$.

(4) $\mathrm{tilt}\,\Lambda$ *has the minimum element T with* $\mathrm{pd}_\Lambda T \leq n$.

(5) $\mathrm{tilt}\,\Lambda^{\mathrm{op}}$ *has the minimum element T with* $\mathrm{pd}_{\Lambda^{\mathrm{op}}} T \leq n$.

(6) *The subcategory $\mathcal{P}_\infty(\Lambda)$ is contravariantly finite, and $\mathcal{P}_\infty(\Lambda) = \mathcal{P}_n(\Lambda)$.*

(7) *The subcategory $\mathcal{P}_\infty(\Lambda^{\mathrm{op}})$ is contravariantly finite, and $\mathcal{P}_\infty(\Lambda^{\mathrm{op}}) = \mathcal{P}_n(\Lambda^{\mathrm{op}})$.*

Proof. (1) $\Rightarrow$ (2) is clear, (2) $\Leftrightarrow$ (3) follows from [Auslander and Reiten 1994, Corollary 5.5]. Hence (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) holds. (2) $\Leftrightarrow$ (4) follows from Corollary 3.5. (4) $\Leftrightarrow$ (6) follows from [Happel and Unger 2005, Theorem 3.3] (see Theorem 3.1). Dually, (3) $\Leftrightarrow$ (5) $\Leftrightarrow$ (7) holds. □

We give an example to show the existence and the constructing of minimal tilting modules.

Example 3.8. Let $\Lambda = KQ/I$ be an algebra with the quiver Q:

$$1 \xrightarrow{a_1} 2 \xrightarrow{a_2} \cdots \xrightarrow{a_{n-1}} n \xrightarrow{a_n} n+1$$

and $I = \mathrm{Rad}^2 KQ$. Then:

(1) The global dimension of Λ is n and $P(i) = I(i+1)$ for $1 \leq i \leq n$

(2) The minimal injective resolution of Λ is

$$0 \to \Lambda \to \Big(\bigoplus_{i=2}^{n+1} I(i)\Big) \oplus I(n+1) \to I(n) \to \cdots \to I(1) \to 0.$$

Hence Λ is n-Gorenstein.

(3) The tilting module $T_j = \big(\bigoplus_{i=2}^{n+1} I(i)\big) \oplus S(n-j+1)$ is of projective dimension j for $0 \leq j \leq n$ and T_j is a minimal element in the set $\mathrm{tilt}_j \Lambda$.

The category $\mathcal{P}_n(\Lambda)^\perp$. Since $\mathcal{P}_n(\Lambda)$ is a resolving subcategory of $\operatorname{mod}\Lambda$, it is very natural to study the corresponding coresolving subcategory $\mathcal{P}_n(\Lambda)^\perp$. Clearly the subcategory

$$\mathcal{Y}_n^1(\Lambda) = \operatorname{add}\Omega^{-n}(\operatorname{mod}\Lambda)$$

is contained in $\mathcal{P}_n(\Lambda)^\perp$ and satisfies ${}^\perp\mathcal{Y}_n^1(\Lambda) = \mathcal{P}_n(\Lambda)$. Auslander and Reiten [1994, Theorem 1.2] proved that $\mathcal{Y}_n^1(\Lambda)$ is always covariantly finite. Moreover they proved that $\mathcal{Y}_i^1(\Lambda)$ is extension closed (or equivalently, coresolving) for every $1 \le i \le n$ if and only if Λ is quasi n-Gorenstein [Auslander and Reiten 1994, Theorem 2.1]. For a more general class of algebras, it is natural to consider the extension closure of $\mathcal{Y}_n^1(\Lambda)$.

For a subcategory $\mathscr{C}$ of $\operatorname{mod}\Lambda$, denote by $\mathscr{C}^{*m}$ the subcategory of $\operatorname{mod}\Lambda$ consisting of all $X \in \operatorname{mod}\Lambda$ having a filtration $X = X_0 \supseteq X_1 \supseteq \cdots \supseteq X_m = 0$ such that $X_i/X_{i+1} \in \mathscr{C}$. Denote by $\Omega^{-n}(\operatorname{mod}\Lambda)$ the subcategory consisting of the modules of the form $\Omega^{-n}C \oplus I$ for all C in $\operatorname{mod}\Lambda$ and injective modules I. (Note that they are not necessarily closed under direct summands.) Now we define subcategories by

$$\mathcal{Y}_n^m(\Lambda) = \operatorname{add}(\Omega^{-n}(\operatorname{mod}\Lambda)^{*m}) \quad \text{and} \quad \mathcal{Y}_n(\Lambda) = \bigcup_{m\ge 0} \mathcal{Y}_n^m(\Lambda).$$

We have the following observations.

Proposition 3.9. *Let Λ be an algebra and n a nonnegative integer.*

(1) *$\mathcal{Y}_n^m(\Lambda)$ is a covariantly finite subcategory for every $m > 0$ such that ${}^\perp\mathcal{Y}_n^m(\Lambda) = \mathcal{P}_n(\Lambda)$.*

(2) *$\mathcal{Y}_n(\Lambda)$ is a coresolving subcategory such that ${}^\perp\mathcal{Y}_n(\Lambda) = \mathcal{P}_n(\Lambda)$.*

Proof. Clearly ${}^\perp\mathcal{Y}_n(\Lambda)$ and ${}^\perp\mathcal{Y}_n^m(\Lambda)$ coincide with ${}^\perp\mathcal{Y}_n^1(\Lambda) = \mathcal{P}_n(\Lambda)$. For (1), we refer to [Chen 2009]. The assertion (2) is clear. □

As a consequence, we have the following observations.

Proposition 3.10. *For an algebra Λ, consider the following five conditions*:

(1) *The subcategory $\mathcal{P}_n(\Lambda)$ is contravariantly finite.*

(2) *$\mathcal{Y}_n(\Lambda)$ is covariantly finite.*

(3) *$\mathcal{Y}_n^m(\Lambda)$ is closed under extensions for some $m > 0$.*

(4) *$\mathcal{Y}_n(\Lambda) = \mathcal{Y}_n^m(\Lambda)$ holds for some $m > 0$.*

(5) *$\mathcal{Y}_n^1(\Lambda)$ is closed under extensions.*

Then we have (5) $\Rightarrow$ (4) $\Leftrightarrow$ (3) $\Leftrightarrow$ (2) $\Rightarrow$ (1).

Proof. (5) $\Rightarrow$ (3) We may choose $m = 1$.

(3) $\Rightarrow$ (2) Assume that $\mathcal{Y}_n^m(\Lambda)$ is extension closed. Then $\mathcal{Y}_n(\Lambda) = \mathcal{Y}_n^m(\Lambda)$. This is covariantly finite by Proposition 3.9(1).

(2) ⇒ (4) Let ℓ be the Loewy length of Λ, $S = \Lambda/\operatorname{rad}\Lambda$ and $S \to Y$ be a left $\mathcal{Y}_n(\Lambda)$-approximation of S. Then Y belongs to $\mathcal{Y}_n^m(\Lambda)$ for some $m > 0$. We claim $\mathcal{Y}_n(\Lambda) = \mathcal{Y}_n^{m\ell}(\Lambda)$.

Since any $X \in \operatorname{mod}\Lambda$ belongs to $(\operatorname{add} S)^{*\ell}$, the Horseshoe-type Lemma [Auslander and Reiten 1991, Proposition 3.6] shows that X has a left $\mathcal{Y}_n(\Lambda)$-approximation $X \to Y$ with $Y \in (\operatorname{add} Y)^{*\ell} \subseteq \mathcal{Y}_n^{m\ell}(\Lambda)$. If $X \in \mathcal{Y}_n(\Lambda)$, then f is a split monomorphism. Thus $X \in \mathcal{Y}_n^{m\ell}(\Lambda)$ holds.

(4) ⇒ (3) This is clear since $\mathcal{Y}_n(\Lambda)$ is extension closed.

(2) ⇒ (1) By Proposition 3.9(2), $\mathcal{Y}_n(\Lambda)$ is a covariantly finite coresolving subcategory such that ${}^{\perp}\mathcal{Y}_n(\Lambda) = \mathcal{P}_n(\Lambda)$. By [Auslander and Reiten 1992, Lemma 3.3(a)], $\mathcal{P}_n(\Lambda)$ is contravariantly finite. □

We should remark that (5) is not equivalent to (2) in Proposition 3.10. We give an example to show this. However, we do not know whether (2) is equivalent to (1).

Example 3.11. Let Λ be a local algebra with Loewy length 2. Then $\mathcal{Y}_1^1(\Lambda) = \operatorname{add}\{K, \mathbb{D}\Lambda\}$. Thus it is closed under extensions if and only if Λ is self-injective.

On the other hand, $\mathcal{Y}_1^2(\Lambda) = \operatorname{mod}\Lambda$ holds, and hence Proposition 3.10(3) is satisfied. Moreover, $\operatorname{tilt}_1\Lambda$ has a minimal element Λ, and $\mathcal{P}_1(\Lambda) = \operatorname{add}\Lambda$ is contravariantly finite.

4. A bijection between classical tilting modules and support τ-tilting modules

Throughout this section, Λ is a 1-Gorenstein algebra and e is an idempotent such that $\operatorname{add} e\Lambda = \operatorname{add} I^0(\Lambda)$ unless stated otherwise. Denote by $\Gamma = \Lambda/(e)$ the factor algebra of Λ. We mainly focus on the bijection between classical tilting modules over a 1-Gorenstein algebra Λ and support τ-tilting modules over the factor algebra Γ.

Now let Λ be an arbitrary algebra. Denote by τ the AR-translation and denote by $|N|$ the number of nonisomorphic indecomposable direct summands of a Λ-module N. Firstly, we recall the definition of support τ-tilting modules in [Adachi et al. 2014].

Definition 4.1. (1) We call $N \in \operatorname{mod}\Lambda$ *τ-rigid* if $\operatorname{Hom}_\Lambda(N, \tau N) = 0$.

(2) We call $N \in \operatorname{mod}\Lambda$ *τ-tilting* if N is τ-rigid and $|N| = |\Lambda|$.

(3) We call $N \in \operatorname{mod}\Lambda$ *support τ-tilting* if there exists an idempotent e of Λ such that N is a τ-tilting $(\Lambda/(e))$-module.

The following property is also needed for the main result in this section.

Lemma 4.2 [Adachi et al. 2014]. *For an algebra Λ, classical tilting Λ-modules are precisely faithful support τ-tilting Λ-modules.*

Now we are in a position to state the following properties of tilting modules over 1-Gorenstein algebras.

Lemma 4.3. *Let Λ be a* 1*-Gorenstein algebra, and let e be an idempotent such that* $\operatorname{add} e\Lambda = \operatorname{add} I^0(\Lambda)$.

(1) *Every classical tilting Λ-module T satisfies $e\Lambda \in \operatorname{add} T$.*

(2) *Every support τ-tilting Λ-module M satisfying $e\Lambda \in \operatorname{add} M$ is a classical tilting module.*

Proof. (1) Since T is a classical tilting module, by Lemma 4.2 T is faithful, and hence any projective module is cogenerated T. Then we get that $e\Lambda$ is a direct summand of T since it is injective.

(2) Since Λ is 1-Gorenstein, then Λ can be embedded in $\operatorname{add}_\Lambda e\Lambda$, and hence Λ can be embedded in $\operatorname{add}_\Lambda M$. Then M is faithful, and by Lemma 4.2, M is a classical tilting module. □

For an algebra Λ and $U \in \operatorname{mod}\Lambda$, denote by $\mathrm{s}\tau\text{-tilt}_U\Lambda$ the set of all $M \in \mathrm{s}\tau\text{-tilt}\,\Lambda$ satisfying $U \in \operatorname{add} M$. Denote by $U^{\perp_0}$ the subcategory consisting of modules M such that $\mathrm{Hom}_\Lambda(U, M) = 0$. The following theorem from [Jasso 2015] is essential to the main result in this section.

Theorem 4.4. *Let A be an algebra and let U be a basic τ-rigid A-module. Let T be the Bongartz completion of U, $B = \mathrm{End}_A(T)$ and $C = B/(e_U)$, where e_U is the idempotent corresponding to the projective B-module $\mathrm{Hom}_A(T, U)$. Then there is a bijection $\phi : \mathrm{s}\tau\text{-tilt}_U A \to \mathrm{s}\tau\text{-tilt}\,C$ via $M \to \mathrm{Hom}_A(T, fM)$, where $0 \to tM \to M \to fM \to 0$ is the canonical sequence according to the torsion pair* $(\operatorname{Fac} U, U^{\perp_0})$.

Recall that $\mathrm{tilt}_1\Lambda$ is the set of additive equivalence classes of classical tilting Λ-modules. Now we are in a position to show our main result in this section.

Theorem 4.5. *Let Λ be a* 1*-Gorenstein algebra and $\Gamma = \Lambda/(e)$, where e is an idempotent such that $\operatorname{add} e\Lambda = \operatorname{add} I^0(\Lambda)$. Then the tensor functor $-\otimes_\Lambda \Gamma$ induces a bijection from* $\mathrm{tilt}_1\Lambda$ *to* $\mathrm{s}\tau\text{-tilt}\,\Gamma$.

Proof. Let $U = e\Lambda$. Then $\mathrm{tilt}_1\Lambda = \mathrm{s}\tau\text{-tilt}_U\Lambda$ holds by Lemma 4.3. On the other hand, the Bongartz completion of U is nothing but $T = \Lambda$. Then $\mathrm{End}_\Lambda(T) = \Lambda$ and $e = e_U$ hold in Theorem 4.4, and we get a bijection ϕ from $\mathrm{tilt}_1\Lambda$ to $\mathrm{s}\tau\text{-tilt}\,\Gamma$.

In the following we show $-\otimes_\Lambda \Gamma = \phi$ as a map. Note that $U = e\Lambda$, so the canonical sequence of $T' \in \mathrm{tilt}_1\Lambda$ according to the torsion pair $(\operatorname{Fac} U, U^{\perp_0})$ is

$$0 \to T'(e) \to T' \to T'/T'(e) \to 0.$$

Then by Theorem 4.4, $\phi(T') = \mathrm{Hom}_\Lambda(\Lambda, T'/T'(e)) \simeq T'/T'(e) \simeq T' \otimes_\Lambda \Gamma$, so $\phi = -\otimes_\Lambda \Gamma$ as a map. □

To give some applications of Theorem 4.5, we fix the following notation.

Notation 4.6. Let Λ_n be the Auslander algebra of $K[x]/(x^n)$ and let Γ_{n-1} be the preprojective algebra of type A_{n-1}. Then we have the following:

(1) Λ_n is given by the quiver

$$1 \underset{b_2}{\overset{a_1}{\rightleftarrows}} 2 \underset{b_3}{\overset{a_2}{\rightleftarrows}} 3 \underset{b_4}{\overset{a_3}{\rightleftarrows}} \cdots \underset{b_{n-1}}{\overset{a_{n-2}}{\rightleftarrows}} n-1 \underset{b_n}{\overset{a_{n-1}}{\rightleftarrows}} n$$

with relations $a_1b_2 = 0$ and $a_ib_{i+1} = b_ia_{i-1}$ for any $2 \leq i \leq n-1$.

(2) Γ_{n-1} is given by the quiver

$$1 \underset{b_2}{\overset{a_1}{\rightleftarrows}} 2 \underset{b_3}{\overset{a_2}{\rightleftarrows}} 3 \underset{b_4}{\overset{a_3}{\rightleftarrows}} \cdots \underset{b_{n-1}}{\overset{a_{n-2}}{\rightleftarrows}} n-1$$

with relations $a_1b_2 = 0$, $a_{n-2}b_{n-1} = 0$ and $a_ib_{i+1} = b_ia_{i-1}$ for any $2 \leq i \leq n-2$.

In the rest of this section, for an algebra Λ_n, we always assume that e is the idempotent such that $\mathsf{add}\, e\Lambda_n = \mathsf{add}\, I^0(\Lambda_n)$. Applying Theorem 4.5 to the Auslander algebras of $K[x]/(x^n)$, we get the following corollary which recovers the results in [Mizuno 2014; Iyama and Zhang 2016].

Corollary 4.7. *Let Λ_n be the Auslander algebra of $K[x]/(x^n)$, let Γ_{n-1} be the preprojective algebra of $Q = A_{n-1}$ and let $\mathfrak{S}_n$ be the symmetric group. Then there are bijections*

$$\operatorname{tilt}_1 \Lambda_n \simeq \operatorname{s\tau-tilt} \Gamma_{n-1} \simeq \mathfrak{S}_n.$$

Thus $\#\operatorname{s\tau-tilt} \Lambda_n = n!$.

Proof. It is not difficult to show that $\Gamma_{n-1} \simeq \Lambda_n/(e)$ by Notation 4.6. Then by Theorem 4.5, $\operatorname{tilt}_1 \Lambda_n \simeq \operatorname{s\tau-tilt} \Gamma_{n-1}$ holds. On the other hand, $\operatorname{s\tau-tilt} \Gamma_{n-1} \simeq \mathfrak{S}_n$ holds by [Mizuno 2014, Theorem 0.1]. We are done. □

Recall that an algebra is called *τ-tilting finite* if there are a finite number of basic τ-tilting modules up to isomorphism. Applying Theorem 4.5 to the Auslander algebra of a Nakayama hereditary algebra, we have the following corollary.

Corollary 4.8. *Let Λ_n be the Auslander algebra of the Nakayama hereditary algebra KQ with $Q = A_n$. Then there is a bijection* $\operatorname{tilt}_1 \Lambda_n \simeq \operatorname{s\tau-tilt} \Lambda_{n-1}$. *Thus Λ is τ-tilting finite if and only if $n \leq 4$.*

Proof. It is not difficult to show that $\Lambda_n/(e) \simeq \Lambda_{n-1}$. Then by Theorem 4.5, $\operatorname{tilt} \Lambda_n \simeq \operatorname{s\tau-tilt} \Lambda_{n-1}$. On the other hand, it was shown in [Kajita 2008] that Λ_n has a finite number of classical tilting modules if and only if $n \leq 5$. Then the assertion holds. □

For more details of τ-rigid modules over Auslander algebras, we refer to [Zhang 2017a]. Recall from [Iyama 2011, Proposition 1.17] that a hereditary algebra is 1-Gorenstein if and only if it is a Nakayama algebra. Now we have the following corollary.

Corollary 4.9. *Let $\Lambda_n = KQ$ be the Nakayama hereditary algebra with $Q = A_n$. Then there are bijections*

$$\mathrm{tilt}_1 \Lambda_n \simeq \mathrm{s}\tau\text{-tilt}\,\Lambda_{n-1} \simeq \{\textit{clusters of the cluster algebra of type } A_{n-1}\}.$$

Thus $\#\mathrm{s}\tau\text{-tilt}\,\Lambda_n = (2(n+1))!/((n+2)!(n+1)!)$.

Proof. A straight calculation shows that $\Lambda_{n-1} \simeq \Lambda_n/(e)$. Then by Theorem 4.5 and [Kajita 2008, Theorem 1],

$$\mathrm{tilt}_1 \Lambda_n \simeq \mathrm{s}\tau\text{-tilt}\,\Lambda_{n-1} \quad \text{and} \quad \#\mathrm{s}\tau\text{-tilt}\,\Lambda_n = (2(n+1))!/((n+2)!(n+1)!).$$

By [Adachi et al. 2014, Theorem 0.5] and [Buan et al. 2006, Theorem 4.5], one gets the second bijection. □

Acknowledgements

Parts of the work were done while Zhang visited Nagoya in 2015. Zhang thanks his co-author for the hospitality received during his stay there. He is also grateful to other people in Nagoya for their kind help. Part of the results were presented in USTC, Tsinghua University, Anhui University and Capital Normal University. Zhang would like to thank all the hosts there for their hospitality. The authors thank the referee for useful suggestions to improve this paper.

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Received March 19, 2018. Revised July 8, 2018.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 298, No. 2, 2019
dx.doi.org/10.2140/pjm.2019.298.417

MAXIMAL SYMMETRY AND UNIMODULAR SOLVMANIFOLDS

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Recently, it was shown that Einstein solvmanifolds have maximal symmetry in the sense that their isometry groups contain the isometry groups of any other left-invariant metric on the given Lie group. Such a solvable Lie group is necessarily nonunimodular. In this work we consider unimodular solvable Lie groups and prove that there is always some metric with maximal symmetry. Further, if the group at hand admits a Ricci soliton, then it is the isometry group of the Ricci soliton which is maximal.

1. Introduction

In this work, we restrict ourselves to the setting of Lie groups with left-invariant metrics.

Definition 1.1. Let G be a Lie group. A left-invariant metric g on G is said to be maximally symmetric if given any other left-invariant metric g', there exists a diffeomorphism $\phi \in \mathfrak{Diff}(G)$ such that

$$\mathrm{Isom}(M, g') \subset \mathrm{Isom}(M, \phi^* g) = \phi \, \mathrm{Isom}(M, g)\phi^{-1}.$$

We say G is a maximal symmetry space if it admits a metric of maximal symmetry.

Although our primary interest is in solvable Lie groups with left-invariant metrics, we briefly discuss the more general setting of Lie groups. For G compact and simple, we have that $\mathrm{Isom}(G)_0$, the connected component of the identity, for any left-invariant metric, can be embedded into the isometry group of the bi-invariant metric [Ochiai and Takahashi 1976]. This does not quite say that compact simple Lie groups are maximal symmetry spaces, but it is close.

In the setting of noncompact semisimple groups, one does not have a bi-invariant metric, but there is a natural choice which plays the role of the bi-invariant metric and similar results are known, see [Gordon 1980]; note the work of Gordon actually

This work was supported by a grant from the Simons Foundation (#360562, MJ) and NSF grant DMS-1612357.
MSC2010: 22E25, 53C25, 53C30.
Keywords: solvmanifold, solvable, Lie group, maximal symmetry, Ricci soliton, unimodular.

goes beyond the Lie group setting and considers a larger class of homogeneous spaces with transitive reductive Lie group and studies their isometry groups.

If S is a nilpotent or completely solvable unimodular group, then it is a maximal symmetry space. Although not stated in this language, this is a result of Gordon and Wilson [1988]; see Section 2 below for more details. Furthermore, when such a Lie group admits a Ricci soliton, the soliton metric has the maximal isometry group [Jablonski 2011].

The nonunimodular setting for completely solvable groups is not as clean. In special circumstances these groups can and do have maximal symmetry, e.g., if a solvable group admits an Einstein metric, then it is a maximal symmetry space and the Einstein metric actually has the largest isometry group; see [Gordon and Jablonski 2015] for more details. However, it is known that not all nonunimodular, completely solvable groups can be maximal symmetry spaces, see Example 1.6 of [Gordon and Jablonski 2015]. For more on the subtleties of the maximal symmetry question in the nonunimodular setting, see the forthcoming work [Epstein and Jablonski 2018].

Our main result is for unimodular solvable Lie groups.

Theorem 1.2. *Let R be a simply connected, unimodular solvable Lie group. Then R is a maximal symmetry space.*

Corollary 1.3. *Let R be a simply connected, unimodular solvable Lie group which admits a Ricci soliton metric. Then said Ricci soliton has maximal symmetry among R-invariant metrics.*

The strategy for proving both results is to reduce to the setting of completely solvable groups, where the answer is immediate. To do this, we start with a solvable group R, we modify a given initial metric until we obtain a metric whose isometry group contains a transitive solvable S which is completely solvable. Our main contribution, then, is to prove a uniqueness result for which S can appear; up to isomorphism only one can and does appear. This uniqueness result is a consequence of the following, which is of independent interest; see Lemma 4.3. (Here we use the language of [Gordon and Wilson 1988].)

Lemma. *Any modification of a completely solvable group is necessarily a normal modification.*

It seems noteworthy to point out that our work actually shows that any solvable Lie group is associated to a unique completely solvable group (Theorem 4.7) in the same way that type R groups have a well defined, unique nilshadow, cf. [Auslander and Green 1966].

In the last section we give a concrete description of the completely solvable group associated to any solvable group S in terms of S and the derivations of its Lie algebra.

Finally, we observe that the choices made throughout our process allow us to choose our diffeomorphism ϕ from Definition 1.1 to be a composition of an automorphism of R together with an automorphism of its associated completely solvable group S. In the case that $R = S$ is completely solvable, the diffeomorphism which conjugates the isometry groups can be chosen to be an automorphism. It would be interesting to know whether or not this is true in general.

2. Preliminaries

In this section, we recall the basics on isometry groups for (unimodular) solvmanifolds from the foundational work of Gordon and Wilson [1988]. Throughout, our standing assumption is that our solvable groups are simply connected. We begin with a general result for Lie algebras.

Recall that every Lie algebra has a unique, maximal solvable ideal, called the radical. A (solvable) Lie algebra $\mathfrak{g}$ is called completely solvable if $\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}$ has only real eigenvalues for all $X \in \mathfrak{g}$. We have the following.

Proposition 2.1. *Given any Lie algebra* $\mathfrak{g}$ *there exists a unique maximal ideal* $\mathfrak{s}$ *which is completely solvable.*

Remark 2.2. This completely solvable ideal is contained in the radical, but generally they are not equal. Notice that the nilradical of $\mathfrak{g}$ is contained in $\mathfrak{s}$ and so, as with the radical, $\mathfrak{s}$ is trivial precisely when $\mathfrak{g}$ is semisimple.

Proof of proposition. As any solvable ideal is a subalgebra of the radical of $\mathfrak{g}$, it suffices to prove the result in the special case that $\mathfrak{g}$ is solvable. The result follows upon showing that the sum of two such ideals is again an ideal of the same type. As the sum of ideals is again an ideal, we only need to check the condition of complete solvability.

Let $\mathfrak{g}$ be a solvable Lie algebra and $\mathfrak{s}_1$ and $\mathfrak{s}_2$ be completely solvable ideals of $\mathfrak{g}$. We will show that $\mathfrak{s}_1 + \mathfrak{s}_2$ is again completely solvable. Observe that for any ideal $\mathfrak{s}$, the eigenvalues of $\mathrm{ad}\, X|_{\mathfrak{s}}$ are real if and only if the eigenvalues of $\mathrm{ad}\, X|_{\mathfrak{g}}$ are real, as we have only introduced extra zero eigenvalues.

The eigenvalues of $\mathrm{ad}\, X : \mathfrak{s} \to \mathfrak{s}$ do not change if we extend $\mathrm{ad}\, X$ to a map on $\mathfrak{s} \otimes \mathbb{C}$. By Lie's theorem, we may realize $\mathrm{ad}\, \mathfrak{s}$ as a subalgebra of upper triangular matrices and so the eigenvalues of $\mathrm{ad}(X_1 + X_2)$ are sums of eigenvalues of $\mathrm{ad}\, X_1$ and $\mathrm{ad}\, X_2$. Taking $X_1 \in \mathfrak{s}_1$ and $X_2 \in \mathfrak{s}_2$, we see that $\mathfrak{s}_1 + \mathfrak{s}_2$ is completely solvable. □

Isometry groups and modifications. In [Gordon and Wilson 1988], the authors set about the job of giving a description of the full isometry group of any solvmanifold. Given any Lie group R with left-invariant metric, one can build a group of isometries as follows: let C denote the set of orthogonal automorphisms of $\mathfrak{r}$, then $R \rtimes C$ is a subgroup of the isometry group. We call this group the algebraic isometry group and denote it by $\mathrm{AlgIsom}(R, g)$.

For R nilpotent, this gives the full isometry group [Gordon and Wilson 1988, Corollary 4.4]. However, in general, the isometry group $\mathrm{Isom}(R, g)$ will be much more. A good example of this is to look at a symmetric space of noncompact type.

So to understand the general setting, Gordon and Wilson detail a process of modifying the initial solvable group R to one with a "better" presentation R' called a standard modification of R — this is another solvable group of isometries which acts transitively. The modification process ends after (at most) two normal modifications with the solvable group R'' in so-called standard position. See [loc. cit., Section 3] for details.

To illustrate why this process is nice, we present the following result in the case of unimodular, solvable Lie groups.

Lemma 2.3. *Let R be a unimodular solvable Lie group with left-invariant metric g. Then* $\mathrm{Isom}(R, g) = \mathrm{AlgIsom}(R'', g) = C \ltimes R''$ *where R'' is the solvable group in standard position inside* $\mathrm{Isom}(R, g)$ *and C consists of orthogonal automorphisms of $\mathfrak{r}''$.*

This follows from the following facts proven in Theorems 3.1, 4.2, and 4.3 of [Gordon and Wilson 1988].

Proposition 2.4. *If there is one transitive solvable Lie group of isometries which is unimodular, then all transitive solvable groups of isometries are unimodular.*

Proposition 2.5. *If R is solvable, unimodular, and in standard position, then the isometry group is the algebraic isometry group.*

Proposition 2.6. *Any almost simply-transitive solvable group of isometries is a modification of one in standard position. Completely solvable groups are always in standard position.*

Regarding normal modifications, we record the following useful facts here.

Lemma 2.7. *For solvable Lie groups R and S in a common isometry group, R being a normal modification of S implies S is a normal modification of R.*

This follows immediately from the description of normal modifications given in Proposition 2.4 of [Gordon and Wilson 1988]. This will be used in the sequel when S is completely solvable. Such S are in standard position in the isometry group and any modification R is a normal modification (see Lemma 4.3), so we see that there exists an abelian subalgebra $\mathfrak{t}$ of the stabilizer subalgebra which normalizes both $\mathfrak{r}$ and $\mathfrak{s}$ such that $\mathfrak{s} \subset \mathfrak{r} \rtimes \mathfrak{t}$ and $\mathfrak{r} \subset \mathfrak{s} \rtimes \mathfrak{t}$; cf. Theorem 3.1 of [Gordon and Wilson 1988]. As such, we have the following.

Lemma 2.8. *For $\mathfrak{s}$ a completely solvable algebra in the isometry algebra and $\mathfrak{r}$ a modification of $\mathfrak{s}$, we have* $[\mathfrak{s}, \mathfrak{r}] \subset \mathfrak{s} \cap \mathfrak{r}$.

Transitive groups of isometries. The following technical lemma will be needed later.

Lemma 2.9. *Consider a solvable Lie group with left-invariant metric* (R_1, g). *Let* K *denote the orthogonal automorphisms of* R_1; *this is a subgroup of the isotropy group which fixes* $e \in R_1$. *Let* R_2 *be a subgroup of isometries satisfying* $\mathfrak{k} + \mathfrak{r}_2 \supset \mathfrak{r}_1$, *then* R_2 *acts transitively.*

Proof. As K fixes $e \in R_1$ and $\mathfrak{k} + \mathfrak{r}_2 \supset \mathfrak{r}_1$, we see that the orbit $R_2 \cdot e$ has the same dimension as $R_1 = R_1 \cdot e$. But the orbit $R_2 \cdot e$ is then an open, complete submanifold of the connected manifold R_1, hence R_2 acts transitively; cf. [Jablonski 2015b, Lemma 3.8]. □

3. Proof of main result in the special case of completely solvable and unimodular groups

Our general strategy is to reduce to the case where the group is completely solvable and so we begin here. Let S be unimodular and completely solvable. For the sake of consistency throughout the later sections, we write $R = S$ in this section.

Theorem 3.1. *Let* S *be a simply connected*, *unimodular*, *completely solvable Lie group. Then* S *is a maximal symmetry space.*

This theorem is an immediate consequence of the following result of Gordon and Wilson, as we see below.

Theorem 3.2 (Gordon and Wilson). *Let* S *be a simply connected*, *unimodular*, *completely solvable Lie group with left-invariant metric* g. *Then* $\mathrm{Isom}(S, g) = S \rtimes C$, *where* $C = \mathrm{Aut}(\mathfrak{s}) \cap O(g)$.

Remark. In the above, we have abused notation as we are viewing C as a subgroup of $\mathrm{Aut}(S)$. This is okay as S being simply connected gives that the action of C on $\mathfrak{s}$ lifts to an action on S.

As $C < \mathrm{Aut}(S)$ is a closed subgroup of $O(g)$, it is compact. Choose any maximal compact subgroup K of $\mathrm{Aut}(S)$ containing C and an inner product g' on $\mathfrak{s}$ so that K acts orthogonally. Now we have

$$\mathrm{Isom}(S, g) < \mathrm{Isom}(S, g')$$

To see that S is indeed a homogeneous maximal symmetry space, we only need to compare isometry groups where $C = K$ is a maximal compact subgroup of $\mathrm{Aut}(S)$.

Let g_1 and g_2 be two left-invariant metrics with isometry groups $S \rtimes K_1$ and $S \rtimes K_2$, respectively, such that K_1 and K_2 are maximal compact subgroups of automorphisms. As maximal compact subgroups are all conjugate, there exists $\phi \in \mathrm{Aut}(S)$ such that $K_1 = \phi K_2 \phi^{-1}$ and hence

$$\mathrm{Isom}(S, g_1) = \phi\, \mathrm{Isom}(S, g_2)\phi^{-1} = \mathrm{Isom}(S, \phi^* g_2).$$

This shows that unimodular, completely solvable Lie groups are indeed homogeneous maximal symmetry spaces.

4. Proof of main result for general solvable, unimodular groups

To prove this result, we start by adjusting our metric so as to enlarge the isometry group to one which is the isometry group of a left-invariant metric on a completely solvable, unimodular group. Then we show that the completely solvable group obtained is unique up to conjugation and use this to prove that there is one largest isometry group for R up to conjugation.

Enlarging the isometry group to find some completely solvable group.

Proposition 4.1. *Let R be a simply connected, unimodular solvmanifold with left-invariant metric g. There exists another left-invariant metric g' such that*

(i) $\mathrm{Isom}(R, g) < \mathrm{Isom}(R, g')$, *and*

(ii) $\mathrm{Isom}(R, g')$ *contains a transitive, completely solvable group S.*

Proof of Proposition 4.1. From the work of Gordon and Wilson (see Lemma 2.3), we have the existence of a transitive, solvable subgroup $R'' < \mathrm{Isom}(R, g)$ such that

$$\mathrm{Isom}(R, g) = \mathrm{AlgIsom}(R'', g) = C \ltimes R'',$$

where C consists of orthogonal automorphisms of $\mathfrak{r}''$. Here R'' is the group in standard position in $\mathrm{Isom}(R, g)$.

There exists a maximal compact subgroup K of $\mathrm{Aut}(R'')$ containing C. Choose any inner product g' on $\mathfrak{r}''$ so that K consists of orthogonal automorphisms. Then we immediately see that $\mathrm{Isom}(R'', g') > K \ltimes R''$. Applying Lemma 2.9, we see that R acts transitively by isometries on (R'', g') and so this new left-invariant metric g' on R'' gives rise to a left-invariant metric on R. This choice of g' satisfies part (i).

To finish, we show that $\mathrm{Isom}(R, g')$ contains a completely solvable group S which acts transitively. Consider the group $\mathrm{Ad}(R'')$ as a subgroup of $\mathrm{Aut}(R'')$. This group is a normal, solvable subgroup and so is a subgroup of the radical $\mathrm{Rad}(\mathrm{Aut}(R''))$ of $\mathrm{Aut}(R'')$.

As $\mathrm{Rad}(\mathrm{Aut}(R''))$ is an algebraic group, it has an algebraic Levi decomposition

$$\mathrm{Rad}(\mathrm{Aut}(R'')) = M \ltimes N,$$

where M is a maximal reductive subgroup and N is the unipotent radical (see [Mostow 1956]). Furthermore, the group M is abelian and decomposes as $M = M_K M_P$, where M_K is a compact torus and M_P is a split torus. As maximal compact subgroups are all conjugate and $\mathrm{Rad}(\mathrm{Aut}(R''))$ does not change under conjugation, we may assume, after possibly changing M, that $M_K < K$. So given $X \in \mathfrak{r}''$, we may write

$$\operatorname{ad} X = K_X + P_X + N_X,$$

where $K_X \in \operatorname{Lie} M_K$, $P_X \in \operatorname{Lie} M_P$, and $N_X \in \mathfrak{n} = \operatorname{Lie} N$. Note, K_X has purely imaginary eigenvalues while P_X has real eigenvalues.

Now define the set $\mathfrak{s} \subset \mathfrak{r}'' \rtimes \operatorname{Lie} M_K \subset \operatorname{Lie} \operatorname{Isom}(R'', g')$ as

$$\mathfrak{s} := \{X - K_X \mid X \in \mathfrak{r}''\}.$$

Since the K_X all commute, the nilradical of $\mathfrak{r}''$ is contained in $\mathfrak{s}$, and derivations of $\mathfrak{r}''$ are valued in the nilradical, we see that $\mathfrak{s}$ is a solvable Lie algebra.

Note that $\mathfrak{s}$ is completely solvable (this follows as in the proof of Proposition 2.1) and S acts transitively (via Lemma 2.9). □

As completely solvable groups are always in standard position (Proposition 2.6), we see that R is a modification of the group S and that

$$\operatorname{Isom}(R, g) < \operatorname{Isom}(R, g') = \operatorname{Isom}(S, g') = S \rtimes C,$$

where C is the compact group of orthogonal automorphisms of $\mathfrak{s}$, relative to g'. Let K denote a maximal compact group of automorphisms of $\mathfrak{s}$ and g'' an inner product on $\mathfrak{s}$ such that $\operatorname{Isom}(S, g'') = S \rtimes K$. As $R \subset S \ltimes C \subset S \ltimes K$ acts transitively and isometrically on (S, g''), it picks up a left-invariant metric g'' such that

$$\operatorname{Isom}(R, g) < \operatorname{Isom}(R, g'').$$

In this way, we have found an isometry group $\operatorname{Isom}(R, g'')$ which is a maximal isometry group for S and so by Theorem 3.1 cannot be any larger.

This is a reasonable candidate for maximal isometry group for R; we verify this in the sequel.

The uniqueness of S. The group S, constructed above, depends on several choices made based on various initial and chosen metrics. More precisely, one starts with metric g, makes two modifications to obtain the group R'', then changes the metric to some g' to extract the group S.

If one were to start with a different metric h on R, then R'' would certainly be different and so it is unclear, a priori, how the resulting S for h would compare to the group S built from the other metric g. Surprisingly, they must be conjugate via $\operatorname{Aut}(R)$.

Proposition 4.2. *There exists a maximal compact subalgebra $\mathfrak{k}$ of* $\operatorname{Der}(\mathfrak{r})$ *such that $\mathfrak{s}$ is the maximal completely solvable ideal of $\mathfrak{r} \rtimes \mathfrak{k}$.*

Before proving this proposition, we use it to show that any two groups S constructed from R are conjugate via $\operatorname{Aut}(R)$.

Let g and h be two different metrics on R with associated completely solvable algebras $\mathfrak{s}_g$ and $\mathfrak{s}_h$, respectively. Let $\mathfrak{k}_g$ and $\mathfrak{k}_h$ be the compact algebras as in Proposition 4.2 for g and h, respectively. As the maximal compact subgroup of a group is unique up to conjugation, we have some $\phi \in \operatorname{Aut}(R)$ such that $\mathfrak{k}_g = \phi\mathfrak{k}_h\phi^{-1}$. This implies

$$\phi\mathfrak{s}_h\phi^{-1} \subset \mathfrak{r} \rtimes \phi\mathfrak{k}_h\phi^{-1} = \mathfrak{r} \rtimes \mathfrak{k}_g.$$

As $\phi\mathfrak{s}_h\phi^{-1}$ is completely solvable and of the same dimension as the maximal completely solvable $\mathfrak{s}_g$, they must be equal; cf. Proposition 2.1.

We now prove Proposition 4.2.

Lemma 4.3. *Let $\mathfrak{s}$ be a completely solvable Lie algebra with inner product. Any modification of $\mathfrak{s}$ (in its isometry algebra) is a normal modification.*

Remark 4.4. In the special case of nilpotent Lie algebras, this result was already known [Gordon and Wilson 1988, Theorem 2.5]. Building on that result, we extend it to all completely solvable groups.

Proof. Let $\mathfrak{r} = (\mathrm{id} + \phi)\mathfrak{s}$ be a modification of $\mathfrak{s}$ with modification map $\phi : \mathfrak{s} \to N_l(\mathfrak{s})$, where $N_l(\mathfrak{s})$ is the set of skew-symmetric derivations of $\mathfrak{s}$; cf. [Gordon and Wilson 1988, Proposition 3.3]. To show this is a normal modification, it suffices to show $[\mathfrak{s}, \mathfrak{s}] \subset \operatorname{Ker}\phi$ by Proposition 2.4 of [Gordon and Wilson 1988].

Denote the nilradical of $\mathfrak{s}$ by $\mathfrak{n}(\mathfrak{s})$. As every derivation of $\mathfrak{s}$ takes its value in $\mathfrak{n}(\mathfrak{s})$, we can decompose $\mathfrak{s} = \mathfrak{a} + \mathfrak{n}(\mathfrak{s})$ where $\mathfrak{a}$ is annihilated by $N_l(\mathfrak{s})$. As ϕ is linear, to show $[\mathfrak{s}, \mathfrak{s}] \subset \operatorname{Ker}\phi$, it suffices to show $[\mathfrak{a}, \mathfrak{a}]$, $[\mathfrak{a}, \mathfrak{n}(\mathfrak{s})]$, $[\mathfrak{n}(\mathfrak{s}), \mathfrak{n}(\mathfrak{s})] \subset \operatorname{Ker}\phi$.

Take $X, Y \in \mathfrak{a}$. By the construction of $\mathfrak{a}$ and Proposition 2.4 (i) of [Gordon and Wilson 1988], we have

$$[X, Y] = \phi(X)Y - \phi(Y)X + [X, Y] = [\phi(X) + X, \phi(Y) + Y] \in \operatorname{Ker}\phi.$$

Now consider $X \in \mathfrak{a}$ and $Y \in \mathfrak{n}(\mathfrak{s})$. As above, the following is contained in $\operatorname{Ker}\phi$:

$$[\phi(X) + X, \phi(Y) + Y] = \phi(X)Y + [X, Y],$$

that is, $\operatorname{ad}(\phi(X) + X) : \mathfrak{n}(\mathfrak{s}) \to \operatorname{Ker}\phi$.

Since every derivation of $\mathfrak{s}$ takes its image in $\mathfrak{n}(\mathfrak{s})$, and $\mathfrak{r} \subset N_l(\mathfrak{s}) \ltimes \mathfrak{s}$, we see that $\mathfrak{n}(\mathfrak{s})$ is stable under $D = \operatorname{ad}(\phi(X)+X) = \phi(X) + \operatorname{ad} X$. Denoting the generalized eigenspaces of D on $\mathfrak{n}(\mathfrak{s})^{\mathbb{C}}$ by V_λ, we have

$$\mathfrak{n}(\mathfrak{s}) = \bigoplus (V_\lambda \oplus V_{\bar{\lambda}}) \cap \mathfrak{n}(\mathfrak{s}).$$

Each summand is invariant under both $\phi(X)$ and $\operatorname{ad} X$ as these commute. Further, if $\lambda = a + bi$, then on V_λ we have $\phi(X)^2 = -b^2 \operatorname{Id}$ and $\operatorname{ad} X$ can be realized as an upper triangular matrix whose diagonal is $a \operatorname{Id}$. Observe that $\operatorname{Ker} D = \operatorname{Ker}\phi(X) \cap \operatorname{Ker} \operatorname{ad} X$, so if $b \neq 0$, then we see that D is nonsingular on V_λ and $V_{\bar\lambda}$. This implies $V_\lambda = D(V_\lambda)$ and $V_{\bar\lambda} = D(V_{\bar\lambda})$, which implies

$$\operatorname{Im}(\operatorname{ad} X|_{(V_\lambda \oplus V_{\bar\lambda}) \cap \mathfrak{n}(\mathfrak{s})}) \subset (V_\lambda \oplus V_{\bar\lambda}) \cap \mathfrak{n}(\mathfrak{s}) \subset \operatorname{Im}(D|_{\mathfrak{n}(\mathfrak{s})}) \subset \operatorname{Ker}\phi.$$

If $b = 0$, then $V_\lambda = V_{\bar\lambda}$ and $D|_{V_\lambda} = \operatorname{ad} X|_{V_\lambda}$, which implies

$$\operatorname{ad} X|_{V_\lambda} \subset \operatorname{Ker}\phi.$$

All together, this proves $[\mathfrak{a}, \mathfrak{n}(\mathfrak{s})] \subset \operatorname{Ker}\phi$.

To finish, one must show $[\mathfrak{n}(\mathfrak{s}), \mathfrak{n}(\mathfrak{s})] \subset \operatorname{Ker}\phi$. However, as every derivation of $\mathfrak{s}$ preserves $\mathfrak{n}(\mathfrak{s})$, we may restrict our modification to $\mathfrak{n}(\mathfrak{s})$ and we have a modification

$$\mathfrak{n}' = (\operatorname{id} + \phi)\mathfrak{n}(\mathfrak{s}) \subset N_l(\mathfrak{s}) \ltimes \mathfrak{n}(\mathfrak{s}).$$

Theorem 2.5 of [Gordon and Wilson 1988] shows that any modification of a nilpotent subalgebra must be a normal modification. Now [loc. cit., Proposition 2.4 (ii d)] implies $[\mathfrak{n}(\mathfrak{s}), \mathfrak{n}(\mathfrak{s})] \subset \operatorname{Ker}\phi$. This completes the proof of our lemma. □

Remark 4.5. Not all modifications are normal modifications, even in the case of starting with an algebra in standard position. An example of this can be found in Example 3.9 of [Gordon and Wilson 1988]. We warn the reader that there are some typos in that example, the block diagonal matrices of A and V_1 should be interchanged. And then one should replace $A - V_1$ with $A + V_1$ throughout the example.

As explained in the discussion surrounding Lemmas 2.7 and 2.8, $\mathfrak{r}$ and $\mathfrak{s}$ are normal modifications of each other and there is an abelian subalgebra $\mathfrak{t}$ of the stabilizer subalgebra which normalizes both $\mathfrak{r}$ and $\mathfrak{s}$ satisfying $\mathfrak{s} \subset \mathfrak{r} \rtimes \mathfrak{t}$. Here $\mathfrak{s}$ is an ideal.

The proposition follows immediately from the following lemma.

Lemma 4.6. *Let $\mathfrak{k}$ be any maximal compact subalgebra of* Der($\mathfrak{r}$) *containing* $\mathfrak{t}$. *Then $\mathfrak{s}$ is a maximal completely solvable ideal of $\mathfrak{r} \rtimes \mathfrak{k}$ (cf. Proposition 2.1).*

Proof. By the construction of $\mathfrak{s}$, it is clearly a complement of $\mathfrak{k}$ in $\mathfrak{r} \rtimes \mathfrak{k}$. Further, every element of $\operatorname{ad}\mathfrak{k}$ has purely imaginary eigenvalues on $\mathfrak{r} \rtimes \mathfrak{k}$, and so $\mathfrak{s}$ will be a maximal completely solvable ideal as soon as we show that it is ideal.

As $[\mathfrak{s}, \mathfrak{r}] \subset \mathfrak{s}$ by Lemma 2.8, it suffices to show that $\mathfrak{s}$ is stable under $\mathfrak{k}$. However, as every derivation of $\mathfrak{r}$ takes its image in the nilradical, it suffices to show that $\mathfrak{s}$ contains the nilradical of $\mathfrak{r}$.

As stated above, $\mathfrak{r}$ being a normal modification of $\mathfrak{s}$ gives $\mathfrak{r} \subset \mathfrak{s} \rtimes \mathfrak{t}$ where $\mathfrak{t}$ consists of skew-symmetric derivations of $\mathfrak{s}$ and so every element of $\mathfrak{r}$ may be written as $X + K$ where $X \in \mathfrak{s}$ and $K \in \mathfrak{t} \subset \operatorname{Der}(s) \cap \mathfrak{so}(\mathfrak{s})$. One can quickly see, as in the proof of Proposition 2.1, that the eigenvalues of $\operatorname{ad}(X + K)$ are sums of eigenvalues of $\operatorname{ad} X$ and K. Since $\operatorname{ad} X$ has real eigenvalues and K has purely imaginary eigenvalues, we see that $\mathrm{ad}(X+K)$ having only the zero eigenvalue implies $K=0$. That is, the nilradical of $\mathfrak{r}$ is contained in $\mathfrak{s}$. □

Before moving on with the rest of the proof of our main result, we record a consequence of the work done above.

Theorem 4.7. *Let R be a solvable Lie group. Up to isomorphism, there is a single completely solvable group S which can be realized as a modification of R.*

Maximal symmetry for R. We are now in a position to complete the proof that for a simply connected, unimodular solvable Lie group, there is a single largest isometry group up to conjugation by diffeomorphisms. In fact, we will see that the diffeomorphism can be chosen to be a composition of an automorphism of R together with an automorphism of S.

Starting with a metric g on R, we first construct another metric g'' such that

$$\mathrm{Isom}(R, g) < \mathrm{Isom}(R, g'') = S_g \rtimes K,$$

where $S = S_g$ is a completely solvable group (depending on g) and K is some maximal compact subgroup of $\mathrm{Aut}(S)$. Let h be another metric on R with corresponding group S_h. From the above, we may replace h with ϕ^*h for some $\phi \in \mathrm{Aut}(R)$ to assume that $S_h = S_g = S$.

Now, as K is unique up to conjugation in $\mathrm{Aut}(S)$, we have the desired result.

Proof of Corollary 1.3. As in the above, the strategy is to reduce to the setting of completely solvable groups. We briefly sketch the argument for doing this.

By Theorem 8.2 of [Jablonski 2015a], any solvable Lie group R admitting a Ricci soliton metric must be a modification of a completely solvable group S which admits a Ricci soliton. (In fact, those Ricci soliton metrics are isometric.) From our work above, the modification is a normal modification and so the group S is the same as the group we constructed above. Now the problem is reduced to proving that Ricci soliton metrics on completely solvable Lie groups are maximally symmetric, but this has been resolved — see Theorem 4.1 of [Jablonski 2011].

5. Constructing *S* from algebraic data of *R*

In the above work, we started with a solvable Lie group R and built an associated completely solvable Lie group S. The group S was unique, up to conjugation by $\mathrm{Aut}(R)$, but it was built by starting with a metric on R, making modifications to R, changing the metric, making more modifications and then extracting information from the modification R''. We now give a straightforward description of the group S.

Let K be some choice of maximal compact subgroup of $\mathrm{Aut}(R)$. The group S is the simply connected Lie group whose Lie algebra is the "orthogonal complement" of $\mathfrak{k}$ in $\mathfrak{r} \rtimes \mathfrak{k}$ relative to the Killing form of $\mathfrak{r} \rtimes \mathfrak{k}$, i.e.,

$$\mathfrak{s} = \{X \in \mathfrak{r} \rtimes \mathfrak{k} \mid B(X, Y) = 0 \text{ for all } Y \in \mathfrak{k}\}, \tag{5-1}$$

where B is the Killing form of $\mathfrak{r} \rtimes \mathfrak{k}$.

One can see this quickly by showing that the algebra described in (5-1) is also a maximal completely solvable ideal and then Proposition 2.1 shows that it must be $\mathfrak{s}$. The details of the proof are similar to work done above and so we leave them to the diligent reader.

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Received March 19, 2018. Revised July 25, 2018.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 298, No. 2, 2019
dx.doi.org/10.2140/pjm.2019.298.429

CONCORDANCE OF SEIFERT SURFACES

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We prove that every oriented nondisk Seifert surface F for an oriented knot K in S^3 is smoothly concordant to a Seifert surface F' for a hyperbolic knot K' of arbitrarily large volume. This gives a new and simpler proof of the result of Friedl and of Kawauchi that every knot is S-equivalent to a hyperbolic knot of arbitrarily large volume. The construction also gives a new and simpler proof of the result of Silver and Whitten and of Kawauchi that for every knot K there is a hyperbolic knot K' of arbitrarily large volume and a map of pairs $f : (S^3, K') \to (S^3, K)$ which induces an epimorphism on the knot groups. An example is given which shows that knot Floer homology is not an invariant of Seifert surface concordance. We also prove that a set of finite volume hyperbolic 3-manifolds with unbounded Haken numbers has unbounded volumes.

1. Introduction

In what follows, the smooth category will always be assumed. This paper concerns two equivalence relations on oriented knots in S^3, concordance and S-equivalence. Knots K and K' are *concordant* if there is a properly embedded oriented annulus A in $S^3 \times [0, 1]$ with $A \cap (S^3 \times \{0\}) = K$ and $A \cap (S^3 \times \{1\}) = K'$ such that $\partial A = K - K'$. Knots K and K' are *S-equivalent* if they have Seifert surfaces F and F' with associated Seifert matrices which are equivalent under integral congruence and elementary expansions and contractions [Trotter 1973].

Concordant knots need not be S-equivalent, e.g., the trivial knot and a slice knot with nontrivial Alexander polynomial, such as the stevedore's knot 6_1. S-equivalent knots need not be concordant; Kearton [2004] has shown that every algebraically slice knot is S-equivalent to a slice knot, but by Casson and Gordon [1978], there are algebraically slice knots which are not slice knots.

The author [Myers 1983] proved that every knot is concordant to a hyperbolic knot, generalizing the result of Kirby and Lickorish [1979] that every knot is concordant to a prime knot. Friedl [2009] and Kawauchi [1989a; 1989b; 1994]

MSC2010: 57M25.
Keywords: knot, Seifert surface, concordance.

have given proofs that every knot is S-equivalent to a hyperbolic knot, generalizing the result of Kearton [2004] that every knot is S-equivalent to a prime knot.

Friedl noted, citing [Kearton 1975; 2004; Levine 1970; 1977; Trotter 1973], that two knots are S-equivalent if and only if they have isometric Blanchfield pairings. He then noted that by combining two results of Kawauchi's imitation theory of knots ([1989a, Theorem 1.1] and [1989b, Properties I and V]) one gets that for every knot K there is a hyperbolic knot K' of arbitrarily large volume and a map $f : (S^3, K') \to (S^3, K)$ which induces isomorphisms on every quotient of the knot groups by their derived subgroups. Friedl then showed that this result implies that the Blanchfield pairings are isometric. He also added a note in proof that one can combine the existence of such knots and maps with another result of Kawauchi ([1994, Theorem 2.2]) to show S-equivalence.

It is natural to ask whether for every knot K there is a hyperbolic knot K' of arbitrarily large volume to which K is both S-equivalent and concordant. It turns out that an affirmative answer is implicit in Kawauchi's construction [1989a]. One can see the concordance by looking at Figure 7 of that paper for the time interval $0 \le t \le 1$.

Silver and Whitten [2006] proved that given any triple (G, μ, λ) where G is a knot group and (μ, λ) is a meridian-longitude pair for G there are infinitely many triples $(\widetilde{G}, \widetilde{\mu}, \widetilde{\lambda})$ where $\widetilde{G}$ is the group of a prime knot and there is an epimorphism $\phi : (\widetilde{G}, \widetilde{\mu}, \widetilde{\lambda}) \longrightarrow (G, \mu, \lambda)$. In [Silver and Whitten 2005], they strengthened this to the $\widetilde{G}$ being the groups of hyperbolic knots of arbitrarily large volume. In a note in proof they added the comment that Kawauchi informed them that many of the results in the paper can be found in [Kawauchi 1989a; Kawauchi 1992].

The constructions of Kawauchi and of Silver and Whitten mentioned above are rather intricate and the proofs somewhat lengthy. In the present paper, the author gives a simpler and shorter construction and proof. Recall that a *Seifert surface* F for a knot K in S^3 is a compact, oriented surface with boundary K. Both F and K will be assumed to be oriented, with $\partial F = K$. A Seifert surface F' for a knot K' will be said to be *concordant* to F if there is an embedding $h : F \times [0, 1] \to S^3 \times [0, 1]$ such that

$$h(F \times \{0\}) = h(F \times [0, 1]) \cap (S^3 \times \{0\}) = F,$$
$$h(F \times \{1\}) = h(F \times [0, 1]) \cap (S^3 \times \{1\}) = F',$$

and there is an orientation of $h(F \times [0, 1])$ such that $(S^3 \times \{0\}) \cap \partial h(F \times [0, 1]) = F$ and $(S^3 \times \{1\}) \cap \partial h(F \times [0, 1]) = -F'$. In this case K and K' are clearly concordant.

They are also S-equivalent, which can be seen as follows.

Let N be a regular neighborhood of $h(K \times [0, 1])$ in $S^3 \times [0, 1]$. Let P be the closure of $h(F \times [0, 1]) - N$. Let $Q = h(F \times [0, 1]) \cap P$. Finally let R be a regular neighborhood of Q in P. Then R is homeomorphic to $Q \times [-1, +1]$

and hence to $F \times [-1, +1]$. Thus one gets a product structure of the form $k : F \times [0, 1] \times [-1, 1] \to R$ on R. By abuse of notation this will be denoted by $R = F \times [0, 1] \times [-1, 1]$. Thus we identify F with $F \times \{0\} \times \{0\}$ and F' with $F \times \{1\} \times \{0\}$.

Now choose a collection of oriented simple closed curves a_i on F which represents a basis for $H_1(F)$. Identify a_i with $a_i \times \{0\} \times \{0\}$. Let $a_i^+ = a_i \times \{0\} \times \{+1\}$; this is regarded as a_i pushed off F in the positive normal direction. Let $v_{i,j}$ be the linking number of a_i and a_j^+. The resulting matrix is a Seifert matrix V for K. Now let $b_i = a_i \times \{1\} \times \{0\}$ and $b_i^+ = a_i \times \{1\} \times \{+1\}$. Letting $v'_{i,j}$ be the linking number of b_i and b_i^+ one gets a Seifert matrix V' for K'. Let A_i be the annulus $a_i \times [0, 1] \times \{0\}$. Let A_i^+ be the annulus $a_i \times [0, 1] \times \{+1\}$. Then A_i joins a_i to b_i and A_i^+ joins a_i^+ to b_i^+.

Recall that if J and J^+ are disjoint oriented simple closed curves in S^3 then they bound properly embedded oriented surfaces G and G^+ in the 4-ball B^4 which can be chosen to meet in a finite number of points. The linking number of J and J^+ is then equal to the algebraic intersection number of G and G^+. See [Rolfsen 1976, page 136].

Regard $S^3 \times \{1\}$ as ∂B^4. Choose surfaces G_i and G_i^+ in B^4 with boundaries b_i and b_i^+, respectively. Let $\widehat{G}_i = A_i \cup G_i$ and $\widehat{G}_i^+ = A_i^+ \cup G_i^+$. Since $A_i \cap A_j^+ = \varnothing$, the intersection number of $\widehat{G}_i$ and $\widehat{G}_j^+$ is equal to that of G_i and G_j^+. It follows that the Seifert matrices V of F and V' of F' with respect to the given bases are the same.

The main result of this paper will now be stated. The notation $S^3 \backslash\backslash K'$ means the compact manifold obtained from S^3 by removing the interior of a regular neighborhood of the knot K'. Recall that the *Haken number* [1968] of a compact 3-manifold M is the maximum number of compact, connected, properly embedded, incompressible, boundary incompressible, pairwise nonparallel surfaces in M.

Theorem 1.1. *Let F be a Seifert surface for a knot K in S^3. Assume that F is not a disk. Then F is concordant to a Seifert surface F' for a knot K' such that*

(a) *K' is hyperbolic*,

(b) *$S^3 \backslash\backslash K'$ has arbitrarily large Haken number*,

(c) *$S^3 - K'$ has arbitrarily large volume*, and

(d) *there is a map of pairs $f : (S^3, K') \to (S^3, K)$ which induces an epimorphism $f_* : \pi_1(S^3 - K') \to \pi_1(S^3 - K)$.*

The paper is organized as follows. Section 2 reviews some basic material. Section 3 proves that every nondisk Seifert surface for a knot can be put in a certain standard position. Section 4 uses standard position to prove (a). Section 5 proves (b). The proof that (b) implies (c) follows from a more general result, that a set of finite volume hyperbolic 3-manifolds with unbounded Haken numbers has unbounded volumes. This fact appears to be "quasi-known" but the author does not know a precise reference in the literature and so gives a simple proof in the Appendix. Section 6 proves (d). Section 7 considers the question of whether Seifert surface concordance implies the invariance of more than just the union of the sets of invariants of concordance and S-equivalence. It shows by example that although the Alexander polynomial is an invariant of Seifert surface concordance, its categorification, knot Floer homology, is not.

2. Preliminaries

As general references on knot theory and on 3-manifolds see [Lickorish 1997] and [Jaco 1980]. A compact, connected, orientable 3-manifold M will be called *excellent* if it is irreducible, boundary-irreducible, anannular, atoroidal, and is not a 3-ball. M will be called *Haken* if it contains a two-sided incompressible surface. By Thurston's uniformization theorem (see, e.g., [Morgan 1984]), excellent Haken manifolds are hyperbolic.

The following standard technical result will be used to build more complicated hyperbolic 3-manifolds out of simpler pieces. A proof can be found in Section 2 of [Myers 1993].

Lemma 2.1 (gluing lemma). *Let X be a compact, connected 3-manifold. Suppose F is a compact, properly embedded, two-sided 2-manifold in X. It is not assumed that F is connected. Let Y be the 3-manifold obtained by splitting X along F. Denote by F_1 and F_2 the two copies of F in ∂Y which are identified to obtain X. If each component of Y is excellent, $F_1 \cup F_2$ and the closure of $Y - (F_1 \cup F_2)$ is incompressible in Y, and each component of $F_1 \cup F_2$ has negative Euler characteristic, then X is excellent.*

An *n-tangle* is the disjoint union $\lambda = \lambda_1 \cup \cdots \cup \lambda_n$ of properly embedded arcs in a 3-ball B. This is sometimes denoted by the pair (B, λ). It will always be assumed that $n \geq 2$. When the specific number n of arcs is not at issue or is clear from the context, λ will just be called a *tangle*.

This paper will assume that B is given a product structure of the form $[a, b] \times [c, d] \times [e, f]$ with each component λ_i of the tangle (B, λ) joining a point of $(a, b) \times (c, d) \times \{f\}$ to a point of $(a, b) \times (c, d) \times \{e\}$. This is done so that one may compose tangles. The product of the tangles (B, λ) and (B, μ) will be obtained by setting (B, λ) on top of (B, μ) so that the lower endpoint of each λ_i equals the upper endpoint of each μ_i.

The *exterior* of a submanifold of a 3-manifold is the closure of the complement of a regular neighborhood of the submanifold. A knot or tangle will be called *excellent* if its exterior is excellent. In this case by a slight abuse of language the knot or tangle will be called hyperbolic.

3. Standard position for Seifert surfaces

Let F be a nondisk Seifert surface for a knot K. This section defines a way of presenting F by a diagram in the plane, standard position. This presentation will be used to build concordances.

The first step is to define an analogue for certain graphs of a plat presentation of a knot.

Recall the idea of a plat presentation for a knot or link. Regard S^3 as the union of a 3-ball B^+, a copy B^0 of $S^2 \times [-1, 1]$, and a 3-ball B^- with $B^+ \cap B^0 = S^2 \times \{1\}$ and $B^- \cap B^0 = S^2 \times \{-1\}$. Choose M unknotted, unlinked, properly embedded arcs in each of B^+ and B^-. Arrange them so that the projections $p^\pm : B^\pm \to D^\pm$ onto equatorial disks have no crossings and are unnested. Let $p^0 : B^0 \to A$ be a projection of B^0 onto an annulus A. Join the endpoints of the arcs in B^+ to the endpoints of the arcs in B^- by arcs in B^0 such that projection on A is given by a braid β. In this paper the braid will be chosen so that it is an element of the braid group of the plane, not the braid group of the sphere. The knot projections will be drawn in $\mathbb{R}^2$ with regions U, C and L representing the projections of regions near the knot in B^+, B^0, and B^-, respectively. The map $p : \mathbb{R}^3 \to \mathbb{R}^2$ denotes this local projection. The vertical coordinate in $\mathbb{R}^2$ is denoted by y. The braid is represented by a box labeled β. The arcs in B^+ and B^- are represented by arcs on the top and bottom of the box, respectively.

Now suppose that $M \geq 2$. Choose g such that $2 \leq 2g \leq M$, and let $m = M - 2g$. Replace $2g$ adjacent arcs in L with the cone X on their boundary points having vertex v in the interior of L; require that X be disjoint from the remaining arcs in L. This gives a 1-complex in $\mathbb{R}^3$ consisting of a wedge W of $2g$ circles with possibly some additional simple closed curve components. Restrict attention to those β for which there are no such additional components. By changing β one may obtain an isotopic embedding of W such that X is to the left of the arcs in L. This will be called a *plat presentation of a wedge of* $2g$ *circles*. See Figure 1.

One may add words to the top and bottom of β to obtain an isotopic embedding of W so that the arcs in each of U and L are concentrically nested. This will be called a *standard presentation of a wedge of* $2g$ *circles*. See Figure 2.

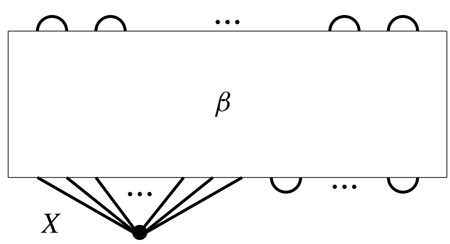


Figure 1. Plat presentation of a wedge of circles.

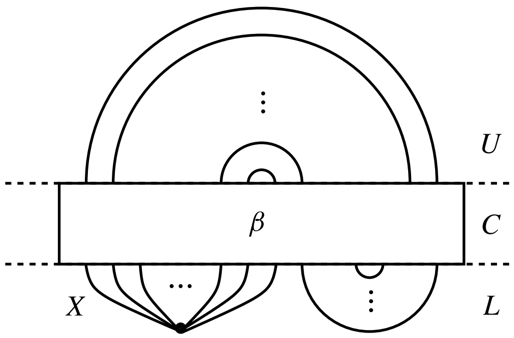


Figure 2. Standard presentation of a wedge of circles.

Now take regular neighborhoods in $U \cup L$ of the arcs and X and widen the arcs in β to bands which cross in the same manner as the arcs. One gets a *standard presentation of a surface with boundary*. The special case of interest is that of an orientable surface with connected boundary.

Lemma 3.1. *Every nondisk Seifert surface for a knot in S^3 has a standard presentation.*

Proof. Let F be a genus $g \geq 1$ Seifert surface for a knot. Choose a wedge of circles W in the interior of F such that the boundary of a regular neighborhood of W in F is parallel in F to K. Let v be the vertex of W. The number of circles is $2g$, where g is the genus of F. Isotop F so that the projection of W onto $\mathbb{R}^2$ has only transverse double point singularities. In particular, $p^{-1}(p(v))$ consists of a single point. Isotop F so that $p(v)$ has y coordinate less than or equal to that of any other point of $p(W)$. We may assume that for each edge of W all the critical values of the function $y \circ p$ are local maxima and minima. Isotop F so as to move all the minima into L and all the maxima into U. One now has a plat presentation of W.

The only obstruction to widening W into a plat presentation for F is that some of the bands may be twisted. Since F is orientable, the twisting in each band consists of a number of full twists. Each full twist is isotopic to a curl, as illustrated in Figure 3. Isotop the local maxima and local minima of the curls into U and L, respectively. The isotopy from plat to standard presentation then preserves the fact that the bands are untwisted. □

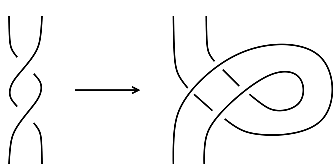


Figure 3. Replacing a twist by a curl.

4. Constructing a concordance

The previous section showed that every nondisk Seifert surface F for a knot in S^3 has a standard presentation which is obtained by widening a standard presentation of a wedge of circles W. The basic idea for constructing a concordance from F to a surface F' is to construct a concordance from W to some W' and widen W' to get F'. However, the obvious surface one gets from a projection of W' might not be concordant to F. As an example of this phenomenon let P be a zero crossing projection of the trivial knot and P' a one crossing projection of the trivial knot. Let A and A' be annuli obtained by widening P and P' into bands. These annuli cannot be concordant because the components of their boundaries have different linking numbers. The way to fix this is to require that any isotopies of a diagram be *regular*, i.e., they are composed of Reidemeister moves of types II and III together with planar isotopies.

Lemma 4.1. *Given a standard presentation of a nondisk Seifert surface F, there is a concordance of F with a Seifert surface F' such that $S^3 - K'$ is hyperbolic, where $K' = \partial F'$.*

Proof. By the gluing lemma, it will be sufficient to show that $S^3 - F'$ is excellent.

Let W be the wedge of circles of which F is a regular neighborhood. Choose a 3-ball E^+ in U and a 3-ball E^- in L as shown in Figure 4. E^+ meets C in a disk and meets W in M straight arcs; these arcs meet C in the leftmost M points of $W \cap U \cap C$. E^- meets C in a disk and meets W in all but the first endpoint of X and in the leftmost $M - 2g$ endpoints of the arcs in $(W \cap L) - X$.

The trivial tangles in E^+ and in E^- are then replaced by concordant hyperbolic tangles α and γ, respectively. The process starts with a hyperbolic n-tangle

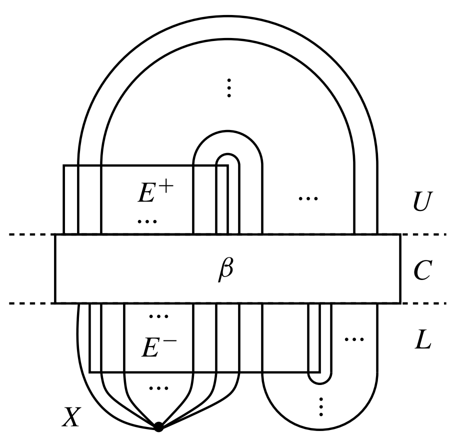


Figure 4. W before surgery.

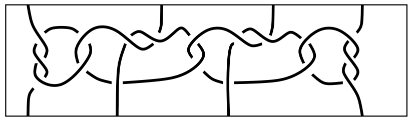


Figure 5. A hyperbolic n-tangle, $n = 4$.

constructed in [Myers 1983]. Figure 5 shows the case $n = 4$.

This tangle is then composed with its mirror image as in Figure 6, left. By the gluing lemma, the new tangle has an excellent Haken exterior and is thus hyperbolic.

A concordance to the trivial tangle is then constructed by attaching 1-handles, performing type II Reidemeister moves, and then attaching 2-handles. See Figure 6, right, for the first stage.

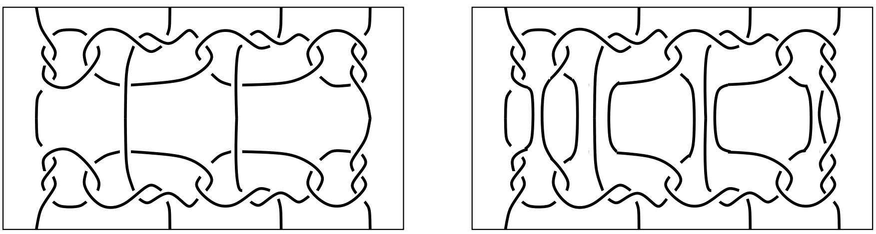


Figure 6. A ribbon n-tangle (left) and a ribbon concordance (right), n=4.

For future reference the general version of this result is stated below.

Lemma 4.2. *Let D be a disk. Let $\tau = \tau_1 \cup \cdots \cup \tau_n$ be an n-tangle in $D \times [0, 1]$ such that each τ_i has one boundary point in* $\text{int}(D) \times \{0\}$ *and the other in* $\text{int}(D) \times \{1\}$. *Let $\delta(\tau)$ be the tangle in $D \times [-1, 1]$ obtained by taking the union of τ and its mirror image in $D \times [-1, 0]$. Then $\delta(\tau)$ is concordant to the trivial tangle ε.* □

Continuing with the proof of Lemma 4.1, one next constructs the Seifert surface F' by replacing the disjoint union of untwisted bands $F \cap (E^+)$ with the disjoint union of untwisted bands in E^+ whose centerlines form the tangle α. These bands are chosen so that their intersections with ∂E^+ are the same as those of F with ∂E^+. A similar construction in ∂E^- then completes the construction of F'.

One now shows that $S^3 - F'$ is hyperbolic. Figure 7 shows the new wedge of circles W'. The exterior of $W' \cap U$ is homeomorphic to the

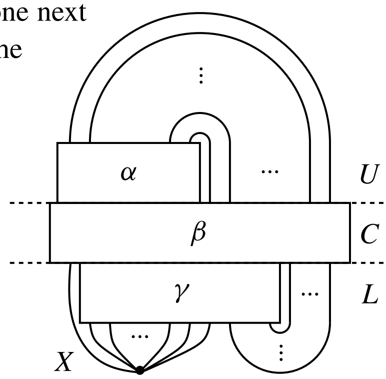


Figure 7. W' after surgery.

exterior of the tangle α and is therefore hyperbolic. The exterior of $W' \cap L$ is homeomorphic to the exterior of the tangle γ. This can be seen as follows. Denote the arcs in X by $\gamma_1, \ldots, \gamma_k$, numbered from left to right. Slide the lower endpoint of γ_2 along γ_1 and into $C \cap L$. Continue with γ_3 through γ_{2g-1}. The complement in L of the new set of arcs is homeomorphic to the complement of γ and is therefore hyperbolic. Since the exterior of β in C is a product, the result follows. $\square$

5. Raising the Haken number and the volume

Lemma 5.1. *Given a standard presentation of a nondisk Seifert surface F, and given an $N > 0$, there is a concordance of F with a Seifert surface F' such that K' is hyperbolic and $S^3 - K'$ has Haken number at least N.*

Proof. In the construction of the previous section, replace each of the tangles α and γ by N copies of itself stacked one on top of the other. This gives N disjoint copies of the exteriors of α and γ. The boundaries of the exteriors of these tangles are all incompressible and nonparallel in $S^3 - K'$. By the gluing lemma, the exterior of the new knot is an excellent Haken manifold and it follows that K' is hyperbolic. $\square$

Lemma 5.2. *Given a standard presentation of a nondisk Seifert surface F, and given a $V > 0$ there is a concordance of F with a Seifert surface F' such that K' is hyperbolic and $S^3 - K'$ has volume at least N.*

Proof. This follows from Corollary A.3 in the Appendix and Lemma 5.1. $\square$

6. Making a map

Lemma 6.1. *Given a standard presentation of a nondisk Seifert surface F, there is a concordance of F with a Seifert surface F' satisfying* (*a*), (*b*), (*c*), *and* (*d*).

Proof. The proof follows from Lemmas 6.2 and 6.3 below by defining f to be the identity outside the tangles involved. $\square$

A tangle $(B^3, \tau_1 \cup \cdots \cup \tau_n)$ is a *boundary tangle* if there are disjoint arcs $\sigma_1, \ldots, \sigma_n$ in ∂B^3 such that $\partial \sigma_i = \partial \tau_i$ and disjoint, compact, orientable surfaces G_i in B^3 such that $\partial G_i = \tau_i \cup \sigma_i$. Let $\tau = \tau_1 \cup \ldots \cup \tau_n$ and $G = G_1 \cup \ldots \cup G_n$.

Let $\tau^* = \tau_1^* \cup \ldots \tau_n^*$ be a trivial tangle in B^3 with $\partial \tau_i^* = \partial \tau_i$ for all i. Choose disjoint disks G_i^* in B^3 with $\partial G_i^* = \partial G_i$.

Lemma 6.2. *There is a map $g : (B^3, \tau, B^3 - \tau) \to (B^3, \tau^*, B^3 - \tau^*)$ which is the identity on ∂B^3 and a homeomorphism from τ to τ^*. In particular, g induces an epimorphism $\pi_1(B^3 - \tau) \to \pi_1(B^3 - \tau^*)$ which carries the meridians of τ to the meridians of τ^*.*

Proof. Let Y be the exterior of τ in B^3. Let $H_i = G_i \cap Y$. Let $N_i = H_i \times [-1, 1]$ be a regular neighborhood of H_i in Y. Let N be the union of the N_i. Let C be a collar on ∂Y whose intersection with each N_i has the form $A_i \times [-1, 1]$, where A_i is a collar on ∂H_i in H_i.

In a similar fashion let Y^* be the exterior of τ^* in B^3, let $H_i^* = G_i^* \cap Y^*$, $N_i^* = H_i^* \times [-1, 1]$, $N^* = \bigcup N_i^*$, $C^* = C$, and $A_i^* = A_i$.

For each $t \in [-1, 1]$ define a map from N_i to N_i^* by crushing $(H_i - C) \times \{t\}$ to a point in H_i^*. This defines the restriction of g to $H_i \times [-1, 1]$.

One next defines g on the closure W of the complement of N in B^3 by crushing $W - C$ to a point. This gives a quotient map onto a 3-ball which may be identified with the closure of $B^3 - N^*$.

Putting the two quotient maps together gives a quotient map from Y to Y^* which extends to a map $g : B^3 \to B^3$ with the required properties. □

The existence of hyperbolic boundary n-tangles was proven by Cochran and Orr in a more abstract general setting. In keeping with the desire to make the constructions in this paper as explicit as possible, their procedure is implemented in the following specific construction.

Lemma 6.3 [Cochran and Orr 1998, Lemma 7.3]. *Hyperbolic boundary n-tangles exist.*

Proof. First choose a hyperbolic $2n$-tangle λ. Configure it so that the ambient 3-cell is a rectangular box with each component of the tangle joining the interior of the top of the box to the interior of the bottom of the box. Place it in the interior of a larger box with which it is concentric. Connect the endpoints of λ to the boundary of the larger box by straight arcs as in the first diagram in Figure 8. Then slide the endpoints of every second arc onto the arc preceding it as in the second diagram. Then slide the bottom endpoints of each resulting graph across the front of the larger box onto the top arc as in the third diagram. This does not change the homeomorphism type of the exterior of the graph.

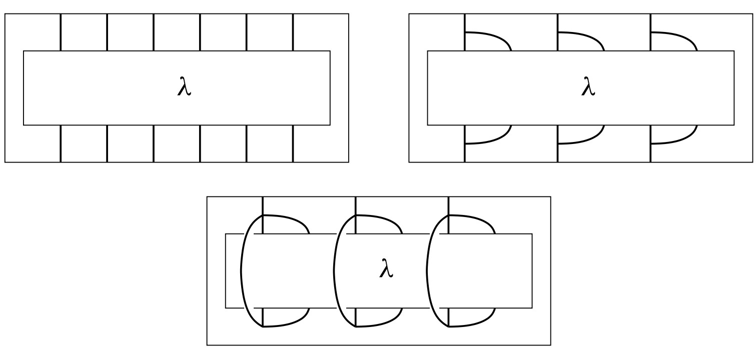


Figure 8. Sliding endpoints to obtain a graph.

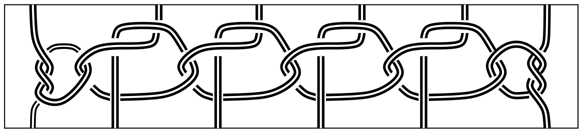


Figure 9. Double $\Delta(\lambda_n)$ of λ_n, $n = 6$.

Then one constructs the $2n$-tangle $\Delta(\lambda)$ by replacing each arc of λ by two parallel copies as in Figure 9.

Next one modifies the last diagram of Figure 8 in the following ways to obtain Figure 10. First, one replaces λ in the inner box by $\Delta(\lambda)$. Second, one widens the graphs which join the inner box to the boundary of the outer box to obtain surfaces whose unions with the bands inside the inner box are punctured tori. Note the three half-twists inserted into the lower portion of each component of Figure 10 to achieve this. By the gluing lemma, the exterior of this new tangle is hyperbolic since it is obtained from a 3-manifold homeomorphic to the exterior of λ by identifying pairs of incompressible once-punctured tori in its boundary.

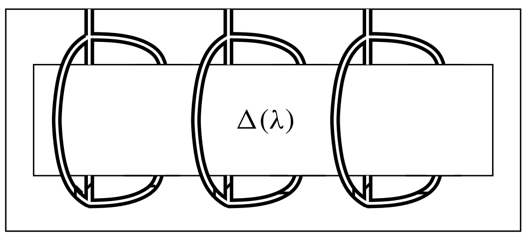


Figure 10. A hyperbolic boundary tangle.

Finally one slides the left endpoints of each arc across the front of the larger box (dotted lines) to obtain the final hyperbolic boundary tangle as in Figure 11. The union of this tangle with its mirror image is still a boundary tangle and the rest of the proof proceeds as before. □

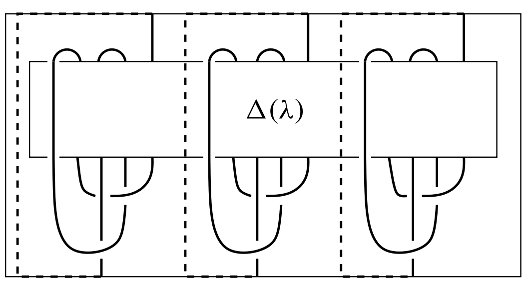


Figure 11. The hyperbolic boundary tangle ready for use.

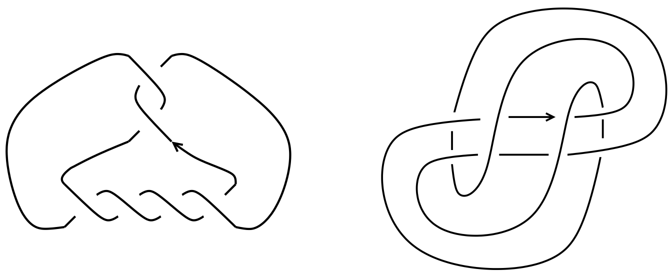


Figure 12. Two projections of the stevedore's knot 6_1.

7. Noninvariance of knot Floer homology

This section gives an example of a knot J with Seifert surface F and a knot J' with Seifert surface F' such that F and F' are concordant, but J and J' have different knot Floer homology.

J is the trefoil knot and F a genus one Seifert surface. J' is a certain twisted double of a copy K of the stevedore's knot 6_1 and F' is a genus one surface contained in a solid torus V whose core is K.

Figure 12 shows two projections of K. The second will be used since it more clearly displays the fact that K is a ribbon knot. See [Mizuma 2005] for an isotopy between them.

Figure 13 shows an annulus A embedded in S^3 with one boundary component being K. The orientations of K and the other boundary component K^* are chosen so that the two curves are homologous in A. From the diagram one computes that the linking number $\text{lk}(K, K^*) = 0$. Let $\widetilde{K} = K \cup K^*$.

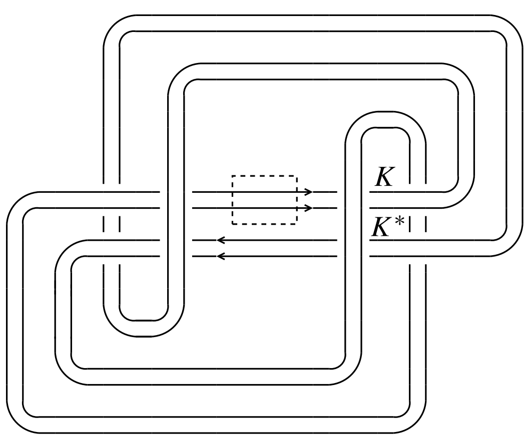
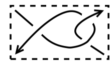


Figure 13. A 2-strand cable link $\widetilde{K} = K \cup K^*$ of $K = 6_1$.

The knot J' is obtained by reversing the orientation on K^* to get a new oriented link $\widehat{K} = K \cup (-K^*)$ and replacing the trivial tangle in the box in Figure 13 by



This results in the annulus being replaced by a genus one surface F'. A saddle move on the new diagram followed by a pair of 2-handle additions shows that F and F' are concordant.

By a result of Ni [2007], knot Floer homology detects fibered knots. J is fibered. If J' were fibered then its companion K would also be fibered (Proposition 9.11 of [Myers 1980]), but the Alexander polynomial $2t^{-1} - 5 + 2t$ of K is not monic, and so K is not fibered (see, e.g., [Rolfsen 1976, Corollary 10.8]); thus J is not fibered, and so the knot Floer homologies of J and J' must be different.

Appendix: Pumping up the volume

There are several results in the literature to the effect that a topologically complicated hyperbolic 3-manifold has high volume. See, for example, [Lackenby 2004; Purcell 2007; Shalen 2007].

One measure of the complexity of a compact 3-manifold M is the *Haken number* $h(M)$ [Haken 1968; Jaco 1980], the maximum number of compact, connected, properly embedded, incompressible, boundary incompressible, pairwise nonparallel surfaces in M. Call the union of such a maximal collection of surfaces a *Haken system* for M.

Theorem A.1. *Let M_n be a sequence of compact, connected, orientable 3-manifolds with complete, finite volume, hyperbolic interiors N_n.*

$$\text{If } \lim_{n\to\infty} h(M_n) = \infty, \text{ then } \lim_{n\to\infty} \mathrm{Vol}(N_n) = \infty.$$

Proof. If not, then by passing to a subsequence we may assume that $\mathrm{Vol}(M_n)$ is bounded above by a positive constant V. It then follows from Jørgensen's theorem [Thurston 1980, Theorem 5.12.1, p. 119] that there is a finite set $\{X_1, \ldots, X_r\}$ of compact, connected, orientable 3-manifolds with finite volume, hyperbolic interiors such that for each M_n there is an X_j such that M_n is homeomorphic to the result of Dehn filling along some of the components of ∂X_j. Let H be the maximum of the $h(X_j)$. Choose an n such that $h(M_n) > H$. By the following lemma, which is stated in greater generality than needed here, we have $h(M_n) \leq H$. □

Lemma A.2. *Let Q and Q^* be compact, connected, orientable, irreducible, ∂-irreducible 3-manifolds. Suppose Q is obtained by Dehn filling along some of the boundary components of Q^*. Then $h(Q) \leq h(Q^*)$.*

Proof. Let $V = V_1 \cup \cdots \cup V_p$ be the union of the solid tori attached to Q^* in order to get Q. Let $S = S_1 \cup \cdots \cup S_{h(Q)}$ be a Haken system for Q. Isotop S so that it meets V in a collection of meridinal disks and the number of such disks is minimal. Let $S^* = S \cap Q^*$. Then S^* is properly embedded in Q^* and has $h(Q)$ components. It suffices to show that they are incompressible, ∂-incompressible, and pairwise nonparallel in Q^*.

S^* is incompressible in Q^*: Suppose D^* is a compressing disk for S^* in Q^*. Then $\partial D^* = \partial D$ for a disk D in S, and $D \cup D^*$ bounds a 3-ball B in Q. Isotoping D across B and off D^* reduces the number of components of $S \cap V$, contradicting minimality.

S^* is ∂-incompressible in Q^*: Suppose Δ is a ∂-compressing bigon for S^* in Q^*. Then $\partial\Delta = \alpha \cup \beta$, where α is a properly embedded arc in S^* and β is a properly embedded spanning arc in an annulus A in ∂V such that $\partial A = \partial D_0 \cup \partial D_1$, where D_0 and D_1 are components of $S \cap V$. Let E be the 3-ball in V bounded by $A \cup D_0 \cup D_1$. A regular neighborhood of $\Delta \cup E$ in Q is a 3-ball across which one can isotop S to remove D_0 and D_1 from $S \cap V$, again contradicting minimality. (Alternatively, one can show from this configuration that S^* is compressible in Q^*.)

The components of S^* are pairwise nonparallel in Q^*: Suppose components S_0^* and S_1^* are parallel in Q^*. These surfaces are the intersections with Q^* of components S_0 and S_1 of S. There is an embedding of $W^* = S_0^* \times [0, 1]$ in Q^* with $S_0^* = S_0^* \times \{0\}$, $S_1^* = S_0^* \times \{0\}$, and $\partial S_0^* \times [0, 1]$ contained in ∂Q^*. Each component A of $W^* \cap \partial V$ is an annulus for which there exists a 3-ball B in V with $A = B \cap W^*$ such that the closure of $\partial B - A$ consists of components of $S \cap V$. These B allow one to extend the product structure on W^* to a product structure $W = S_0 \times [0, 1]$ in Q with $S_0 \times \{0\} = S_0$, $S_0 \times \{1\} = S_1$, and $\partial S_0 \times [0, 1]$ contained in ∂Q, contradicting the fact that S is a Haken system in Q. □

Corollary A.3. *Let Y_n be a sequence of compact, connected, orientable 3-manifolds such that the complement U_n of the torus boundary components of Y_n is a finite volume hyperbolic manifold with totally geodesic boundary.*

$$\text{If } \lim_{n\to\infty} h(Y_n) = \infty, \text{ then } \lim_{n\to\infty} \mathrm{Vol}(U_n) = \infty.$$

Proof. Let M_n be the double of Y_n along the union F of its nontorus boundary components. Suppose S is a Haken system in Y_n. Let $\widehat{S}$ be the double of S along $S \cap F$. The incompressibility of $\widehat{S}$ in M_n follows from the incompressibility of F and S in Y_n and the ∂-incompressibility of S in Y_n. The fact that ∂M_n consists of tori then implies that $\widehat{S}$ is ∂-incompressible in M_n. Suppose two components $\widehat{S}_0$ and $\widehat{S}_1$ of $\widehat{S}$ are parallel in M_n via a product $\widehat{S}_0 \times [0, 1]$. This product is invariant under the involution which interchanges the two copies of Y_n. By [Kim and Tollefson 1977, Theorem A] the restriction of the involution is equivalent to a product involution. Hence the fixed point set consists of annuli. By [Waldhausen 1968, Lemma 3.4] they are isotopic to product annuli. It follows that S_0 and S_1 are parallel in Y_n. Thus $h(M_n) \geq h(Y_n)$. The result then follows from Theorem A.1 and the fact that the interior N_n of M_n is a complete, finite volume hyperbolic 3-manifold with $\mathrm{Vol}(N_n) = 2\,\mathrm{Vol}(U_n)$. □

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Received January 4, 2017. Revised May 18, 2018.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 298, No. 2, 2019
dx.doi.org/10.2140/pjm.2019.298.445

RESOLUTIONS FOR TWISTED TENSOR PRODUCTS

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We build resolutions for general twisted tensor products of algebras. These bimodule and module resolutions unify many constructions in the literature and are suitable for computing Hochschild (co)homology and more generally Ext and Tor for (bi)modules. We analyze in detail the case of Ore extensions, consequently obtaining Chevalley–Eilenberg resolutions for universal enveloping algebras of Lie algebras (defining the cohomology of Lie groups and Lie algebras). Other examples include semidirect products, crossed products, Weyl algebras, Sridharan enveloping algebras, and Koszul pairs.

1. Introduction

Motivated by questions in noncommutative geometry, Čap, Schichl, and Vanžura [Čap et al. 1995] introduced a very general *twisted tensor product* of algebras to replace the (commutative) tensor product. Their examples included noncommutative 2-tori and crossed products of C^*-algebras with groups. Many other algebras of interest arise as twisted tensor product algebras: crossed products with Hopf algebras, algebras with triangular decomposition (e.g., universal enveloping algebras of Lie algebras and quantum groups), braided tensor products defined by R-matrices, and other biproduct constructions. In fact, twisted tensor product algebras are abundant: If an algebra is isomorphic to $A \otimes B$ as a vector space for two of its subalgebras A and B under the canonical inclusion maps, then it must be isomorphic to a twisted tensor product $A \otimes_\tau B$ for some twisting map $\tau : B \otimes A \to A \otimes B$ (see [Čap et al. 1995]).

Modules over a twisted tensor product algebra arise from tensoring together modules for the individual algebras: If M and N are modules over algebras A and B, respectively, compatible with a twisting map τ, then $M \otimes N$ adopts the structure of a module over $A \otimes_\tau B$. We describe in this note a general method to twist together

Shepler was partially supported by Simons grant 429539. Witherspoon was partially supported by NSF grant DMS-1401016.
MSC2010: primary 16E05, 16E30, 16E40; secondary 16S30, 57T10.
Keywords: twisted tensor product, projective resolutions, Hochschild (co)homology, augmented algebras.

resolutions of A-modules and B-modules in order to construct resolutions for the corresponding modules over the twisted tensor product $A \otimes_\tau B$. A similar method works for bimodules. In particular, we twist together resolutions of algebras over a field to obtain a resolution for a twisted tensor product algebra as a bimodule over itself.

We are motivated by a desire to understand deformations of twisted tensor products and to describe the homology theory in terms of the homology of the original factor algebras. For example, under some finiteness assumptions, the Hochschild cohomology of a tensor product of algebras is the tensor product of their Hochschild cohomology rings. A similar statement is true of the cohomology of augmented algebras. Both results hold because the tensor product of projective resolutions for the factor algebras is a projective resolution for the tensor product of the algebras.

In some particular settings, similar homological constructions have appeared for modified versions of the tensor product of algebras. We mention just a few examples. Gopalakrishnan and Sridharan [1966] constructed resolutions for modules of Ore extensions. Bergh and Oppermann [2008] twisted resolutions when the twisting arises from a bicharacter on grading groups. Jara, López Peña, and Ştefan [Jara et al. 2017] worked with Koszul pairs. Guccione and Guccione [1999; 2002] built resolutions for twisted tensor products, in particular crossed products with Hopf algebras, out of bar and Koszul resolutions of the factor algebras. We adapted this last construction in [Shepler and Witherspoon 2014] to handle more general resolutions for the case of skew group algebras in order to understand deformations. Walton and the second author generalized these resolutions to smash products with Hopf algebras in [Walton and Witherspoon 2014].

In this paper, we unify many of these previous constructions and provide methods useful in new settings for finding resolutions of modules over twisted tensor product algebras: We show very generally that projective resolutions for bimodules of two factor algebras can be twisted together to construct a projective resolution for the resulting bimodule for the twisted tensor product given some compatibility conditions. This twisting of resolutions provides an efficient means for computing and handling Hochschild (co)homology in particular. A similar construction applies to projective (left) module resolutions used, for example, to compute (co)homology of augmented algebras.

We verify that many known resolutions may be viewed as twisted resolutions in this way, including some of those mentioned above. We give details in the case of Ore extensions. In particular, the bimodule Koszul resolution of a universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ is a twisted resolution when $\mathfrak{g}$ is a finite-dimensional supersolvable Lie algebra. More general Lie algebras can be handled via triangular decomposition. Our method also leads to standard resolutions for Weyl algebras and some Sridharan enveloping algebras. For an Ore extension, we adapt results of

Gopalakrishnan and Sridharan [1966] to construct twisted product resolutions of modules. We thus regard the Chevalley–Eilenberg complex of $\mathcal{U}(\mathfrak{g})$ as a twisted product resolution. This defines Lie algebra and Lie group cohomology in terms of an iterative twisting of resolutions.

In Section 2, we give definitions and some preliminary results. Bimodule twisted tensor product complexes are constructed in Section 3 and we show they give projective resolutions in Theorem 3.10. Section 4 gives applications to some types of Ore extensions. We construct twisted tensor product complexes for resolving modules in Section 5, and we show these complexes are projective resolutions in Theorem 5.12. Applications to Ore extensions appear in Section 6.

We fix a field k of arbitrary characteristic throughout. All tensor products are over k unless otherwise indicated, i.e., $\otimes = \otimes_k$, and all algebras are k-algebras. Modules are left modules unless otherwise described.

2. Twisted tensor product algebras and compatible resolutions

In this section, we recall twisted tensor product algebras from [Čap et al. 1995] and define a compatibility condition necessary for twisting resolutions together. Examples include skew group algebras and crossed products with Hopf algebras [Montgomery 1993], twisted tensor products given by bicharacters of grading groups [Bergh and Oppermann 2008], braided products arising from R-matrices [Manin 1988], two-cocycle twists of Hopf algebras [Radford and Schneider 2008], and more.

Let A and B be associative algebras over k with multiplication maps $m_A : A\otimes A \to A$ and $m_B : B\otimes B \to B$ and multiplicative identities 1_A and 1_B, respectively. We write 1 for the identity map on any set.

Twisted tensor products. A *twisting map*

$$\tau : B \otimes A \to A \otimes B$$

is a bijective k-linear map for which $\tau(1_B \otimes a) = a \otimes 1_B$ and $\tau(b \otimes 1_A) = 1_A \otimes b$ for all $a \in A$ and $b \in B$, and

$$\tau \circ (m_B \otimes m_A) = (m_A \otimes m_B) \circ (1 \otimes \tau \otimes 1) \circ (\tau \otimes \tau) \circ (1 \otimes \tau \otimes 1) \tag{2.1}$$

as maps $B \otimes B \otimes A \otimes A \to A \otimes B$. The *twisted tensor product algebra* $A \otimes_\tau B$ is the vector space $A \otimes B$ together with multiplication m_τ given by such a twisting map τ. By [Čap et al. 1995, Proposition/Definition 2.3], the algebra $A \otimes_\tau B$ is associative.

Note that the left-right distinction in a twisted tensor product algebra is artificial since $A \otimes_\tau B \cong B \otimes_{\tau^{-1}} A$. Indeed, one might identify $A \otimes_\tau B$ with the algebra generated by A and B (so that A and B are subalgebras) with relations given by (2.1).

If A and B are $\mathbb{N}$-graded algebras, we take the standard $\mathbb{N}$-grading on $A \otimes B$ and $B \otimes A$ and say a twisting map τ is *strongly graded* if it takes $B_j \otimes A_i$ to $A_i \otimes B_j$ for all i, j following Conner and Goetz [2018]. (Note that [Jara et al. 2017] leave off the adjective *strongly*.) In this case, the twisted tensor product algebra $A \otimes_\tau B$ is $\mathbb{N}$-graded.

Example 2.2. The Weyl algebra $\mathcal{W} = k\langle x, y\rangle/(xy - yx - 1)$ is isomorphic to the twisted tensor product $A \otimes_\tau B$ of $A = k[x]$ and $B = k[y]$ with twisting map $\tau : B \otimes A \to A \otimes B$ defined by $\tau(y \otimes x) = x \otimes y - 1 \otimes 1$. Likewise, the Weyl algebra $\mathcal{W}_n$ on $2n$ indeterminates, which is equal to

$$k\langle x_1, \dots, x_n, y_1, \dots, y_n\rangle/(x_i x_j - x_j x_i,\ y_i y_j - y_j y_i,\ x_i y_j - y_j x_i - \delta_{i,j} : 1 \le i, j \le n),$$

is isomorphic to a twisted tensor product. These are examples of (iterated) Ore extensions, which we consider in detail in Section 4.

Example 2.3. A skew group algebra $S \rtimes G$ for a finite group G acting on an algebra S by automorphisms is isomorphic to the twisted tensor product $kG \otimes_\tau S$ of the group algebra kG and of S. The twisting map τ is defined by $\tau(s \otimes g) = g \otimes g^{-1}(s)$ for $s \in S$ and $g \in G$. We consider the special case where S is a Koszul algebra at the end of Section 3.

Bimodules over twisted tensor products. We fix a twisting map $\tau : B \otimes A \to A \otimes B$ for k-algebras A and B.

Definition 2.4. An A-bimodule M is *compatible with* τ if there is a bijective k-linear map $\tau_{B,M} : B \otimes M \to M \otimes B$ commuting with the bimodule structure of M and multiplication in B, i.e., as maps on $B \otimes B \otimes M$ and on $B \otimes A \otimes M \otimes A$, respectively,

$$\tau_{B,M}(m_B \otimes 1) = (1 \otimes m_B)(\tau_{B,M} \otimes 1)(1 \otimes \tau_{B,M}), \quad \text{and} \tag{2.5}$$

$$\tau_{B,M}(1 \otimes \rho_{A,M}) = (\rho_{A,M} \otimes 1)(1 \otimes 1 \otimes \tau)(1 \otimes \tau_{B,M} \otimes 1)(\tau \otimes 1 \otimes 1), \tag{2.6}$$

where $\rho_{A,M} : A \otimes M \otimes A \to M$ is the bimodule structure map. If A is graded and M is a graded A-bimodule, we say that M is *compatible with a strongly graded twisting map* τ if there is a map $\tau_{B,M}$ as above that takes $B_i \otimes M_j$ to $M_j \otimes B_i$ for all i, j.

Remark 2.7. Note that the above definition applies to B-bimodules as well as A-bimodules by reversing the roles of A and B. Indeed, we apply the definition to the algebra B, the twisted tensor product $B \otimes_{\tau^{-1}} A$, and the twisting map τ^{-1} to obtain conditions for a B-bimodule N to be compatible with τ^{-1}. We may rewrite these conditions using the convenient notation $\tau_{N,A} = (\tau^{-1}_{A,N})^{-1}$. We obtain an equivalent right version of the above definition: A given B-bimodule N is *compatible* with τ^{-1} when there is some bijective k-linear map $\tau_{N,A} : N \otimes A \to A \otimes N$ satisfying

$$\tau_{N,A}(1 \otimes m_A) = (m_A \otimes 1)(1 \otimes \tau_{N,A})(\tau_{N,A} \otimes 1) \quad \text{and} \tag{2.8}$$

$$\tau_{N,A}(\rho_{B,N} \otimes 1) = (1 \otimes \rho_{B,N})(\tau \otimes 1 \otimes 1)(1 \otimes \tau_{N,A} \otimes 1)(1 \otimes 1 \otimes \tau), \tag{2.9}$$

as maps on $N \otimes A \otimes A$ and on $B \otimes N \otimes B \otimes A$, respectively, where

$$\rho_{B,N} : B \otimes N \otimes B \to N$$

is the bimodule structure map.

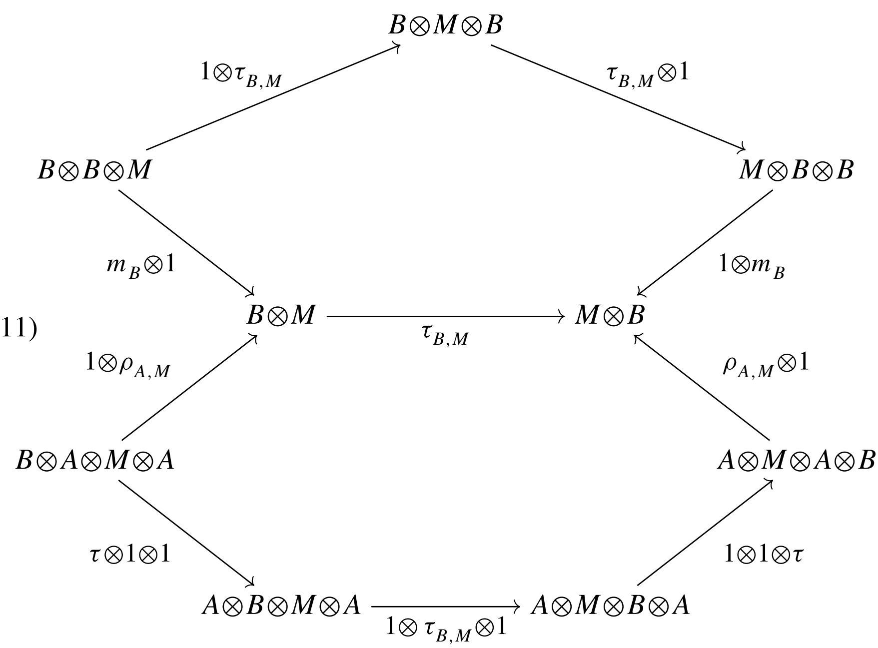
In light of the last remark, we will say a bimodule is *compatible with* τ when it is either an A-bimodule compatible with τ or a B-bimodule compatible with τ^{-1}, since one often identifies $A \otimes_\tau B$ and the isomorphic algebra $B \otimes_{\tau^{-1}} A$ in practice.

Remark 2.10. An A-bimodule M is compatible with the twisting map τ exactly when there is a bijective k-linear map

$$\tau_{B,M} : B \otimes M \to M \otimes B$$

making the following diagram commute:

(2.11)



A similar diagram expresses compatibility of a B-bimodule N with τ.

Example 2.12. Let $M = A$, an A-bimodule via multiplication. Then A is compatible with τ via $\tau_{B,A} = \tau$. Similarly $N = B$ is compatible with τ.

Bimodule structure. When M and N are compatible with τ, the tensor product $M \otimes N$ is naturally an $A \otimes_\tau B$-bimodule via the following composition of maps:

$$A\otimes_\tau B\otimes M\otimes N\otimes A\otimes_\tau B \xrightarrow{1\otimes\tau_{B,M}\otimes\tau_{N,A}\otimes 1} A\otimes M\otimes B\otimes A\otimes N\otimes B \xrightarrow{1\otimes 1\otimes\tau\otimes 1\otimes 1} A\otimes M\otimes A\otimes B\otimes N\otimes B \xrightarrow{\rho_{A,M}\otimes\rho_{B,N}} M\otimes N. \tag{2.13}$$

Bimodule resolutions. For any k-algebra A, let $A^e = A \otimes A^{\mathrm{op}}$ be its enveloping algebra, with A^{op} the opposite algebra to A. We view an A-bimodule M as a left A^e-module. In Lemma 3.1 below, we will construct a projective resolution of an $(A \otimes_\tau B)^e$-module $M \otimes N$ from individual resolutions of M and N using some compatibility conditions. Let $P_\bullet(M)$ be an A^e-projective resolution of M and let $P_\bullet(N)$ be a B^e-projective resolution of N:

$$\cdots \to P_2(M) \to P_1(M) \to P_0(M) \to M \to 0, \tag{2.14}$$

$$\cdots \to P_2(N) \to P_1(N) \to P_0(N) \to N \to 0. \tag{2.15}$$

Bar and reduced bar resolutions. For example, M could be A itself and $P_\bullet(A)$ could be the *bar resolution* $\mathrm{Bar}_\bullet(A)$ given by $\mathrm{Bar}_n(A) = A^{\otimes(n+2)}$ with differential

$$a_0 \otimes a_1 \otimes \cdots \otimes a_{n+1} \mapsto \sum_{i=0}^{n} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}$$

for all $n \geq 0$ and $a_0, a_1, \ldots, a_{n+1} \in A$. We also use the *reduced bar resolution* $\overline{\mathrm{Bar}}_\bullet(A)$ with $\overline{\mathrm{Bar}}_n(A) = A \otimes \bar{A}^{\otimes n} \otimes A$ for $\bar{A} = A/k1_A$ and differential given by the same formula.

Compatibility conditions. We now define what it means for resolutions to be compatible with the twisting map τ. We tensor arbitrary resolutions (2.14) and (2.15) with A and B on the right and left to obtain complexes

$$P_\bullet(N) \otimes A,\ A \otimes P_\bullet(N),\ P_\bullet(M) \otimes B, \text{ and } B \otimes P_\bullet(M).$$

Viewing these simply as exact sequences of vector spaces, we note that any k-linear maps $\tau_{N,A} : N \otimes A \to A \otimes N$ and $\tau_{B,M} : B \otimes M \to M \otimes B$ can be lifted to k-linear chain maps

$$\tau_{P_\bullet(N),A} : P_\bullet(N)\otimes A \to A\otimes P_\bullet(N) \text{ and } \tau_{B,P_\bullet(M)} : B\otimes P_\bullet(M) \to P_\bullet(M)\otimes B. \tag{2.16}$$

For simplicity in the sequel, we will write $\tau_{i,A} = \tau_{P_i(N),A}$ and $\tau_{B,i} = \tau_{B,P_i(M)}$, for each i, when no confusion will arise. We will use such maps to glue the two resolutions together provided they satisfy the following compatibility conditions. These conditions just state that the chain maps commute with multiplication and with bimodule structure maps. There are many settings in which compatible chain maps do exist, as we will see.

Definition 2.17. Let M be an A-bimodule that is compatible with τ. A projective A-bimodule resolution $P_\bullet(M)$ is *compatible with the twisting map* τ if each $P_i(M)$ is compatible with τ via a map

$$\tau_{B,i} : B \otimes P_i(M) \to P_i(M) \otimes B,$$

with $\tau_{B,\bullet}$ a chain map lifting $\tau_{B,M}$. Suppose A is graded, M is a graded A-bimodule, and $P_\bullet(M)$ is a graded projective A-bimodule resolution; we say that $P_\bullet(M)$ is *compatible with a strongly graded twisting map* τ if there are maps $\tau_{B,i}$ as above taking $B_j \otimes (P_i(M))_l$ to $(P_i(M))_l \otimes B_j$ for all j, l.

Remark 2.18. The above definition applies to B-bimodule resolutions as well as A-bimodule resolutions by reversing the roles of A and B in the definition, again as $A \otimes_\tau B = B \otimes_{\tau^{-1}} A$. For a B-bimodule N that is compatible with τ, the definition implies that a projective B-bimodule resolution $P_\bullet(N)$ of N is *compatible with the twisting map* τ when each $P_i(N)$ is compatible with τ via a map

$$\tau_{i,A} : P_i(N) \otimes A \to A \otimes P_i(N),$$

with $\tau_{\bullet,A}$ a chain map lifting $\tau_{N,A}$. Thus we say a resolution is compatible with τ if it is either an A-bimodule resolution or a B-bimodule resolution compatible with τ.

We provide some small examples later: Example 2.21 (Weyl algebra) and Example 3.13 (skew group algebra). First, a remark on embedding resolutions and some general results.

Remark 2.19. Note that compatibility is preserved under embedding of resolutions so long as the extensions of the twisting map τ preserve the embedding. Specifically, assume

$$\phi_\bullet : Q_\bullet(A) \hookrightarrow P_\bullet(A)$$

is an embedding of resolutions of the algebra A, and $P_\bullet(A)$ is compatible with a twisting map $\tau : B \otimes A \to A \otimes B$ via chain maps

$$\tau_{B,i} : B \otimes P_i(A) \to P_i(A) \otimes B.$$

If the maps $\tau_{B,i}$ preserve the embedding in the obvious sense that each $\tau_{B,i}$ restricts to a surjective map $B \otimes \operatorname{Im}(\phi) \twoheadrightarrow \operatorname{Im}(\phi) \otimes B$, then $Q_\bullet(A)$ is compatible with τ via these restrictions.

Compatibility of bar and Koszul resolutions. If A and B are both Koszul algebras and τ is a strongly graded twisting map, then the algebra $A \otimes_\tau B$ is known to be Koszul (see [Polishchuk and Positselski 2005, Example 4.7.3], [Jara et al. 2017, Corollary 4.1.9], or [Walton and Witherspoon 2018, Proposition 1.8]). Conner and Goetz [2018] examine the situation when τ is not strongly graded. We show next that both bar and Koszul resolutions are compatible with twisting maps. We always assume our Koszul algebras are connected graded algebras, so that they are quotients of tensor algebras on generating vector spaces in degree 1. Note that the roles of A and B may be exchanged in the next proposition.

Proposition 2.20. *Let τ be a twisting map for some k-algebras A and B.*

(1) *The bar resolution* $\mathrm{Bar}_\bullet(A)$ *is compatible with* τ.

(2) *The reduced bar resolution* $\overline{\mathrm{Bar}}_\bullet(A)$ *is compatible with* τ.

(3) *If A is a Koszul algebra, B is a graded algebra, and τ is strongly graded, then the Koszul resolution* $\mathrm{Kos}_\bullet(A)$ *is compatible with* τ.

Proof. (i) The bar resolution of A may be twisted by repeated application of the map τ, i.e., define

$$\tau_{B,i} : B \otimes A^{\otimes(i+2)} \to A^{\otimes(i+2)} \otimes B$$

by applying τ to the first two tensor factors on the left, then applying τ to next two tensor factors, and so on:

$$\tau_{B,i} = (1\otimes\cdots\otimes 1\otimes\tau)(1\otimes\cdots\otimes 1\otimes\tau\otimes 1)\cdots(1\otimes\tau\otimes 1\otimes\cdots\otimes 1)(\tau\otimes 1\otimes\cdots\otimes 1).$$

Then $\mathrm{Bar}_\bullet(A)$ is compatible with τ via $\tau_{B,i}$, as may be verified directly by repeated use of (2.1).

(ii) Write the terms in the bar complex $\mathrm{Bar}_\bullet(A)$ as $P_i = A^{\otimes(i+2)}$ for each i, and define the terms in the reduced bar complex $\overline{\mathrm{Bar}}_\bullet(A)$ by $\bar{P}_i = A\otimes\bar{A}^{\otimes i}\otimes A$. For each i, let T_i be the kernel of the quotient map $\mathrm{Bar}_i(A)\to\overline{\mathrm{Bar}}_i(A)$. Then $T_\bullet$ is a subcomplex of $\mathrm{Bar}_\bullet(A)$ and $\overline{\mathrm{Bar}}_\bullet(A)\cong\mathrm{Bar}_\bullet(A)/T_\bullet$. By definition of the twisting map τ, the multiplicative identity 1_A commutes with elements of B under τ, implying that $\tau_{B,i}$ of part (i) takes $B\otimes T_i$ onto $T_i\otimes B$ for each i. Let $\bar{\tau}_{B,i} : B\otimes\overline{\mathrm{Bar}}_i(A)\to\overline{\mathrm{Bar}}_i(A)\otimes B$ be the corresponding map on quotients. Then $\overline{\mathrm{Bar}}_\bullet(A)$ is compatible with τ via the maps $\bar{\tau}_{B,i}$.

(iii) The proof of [Walton and Witherspoon 2018, Proposition 1.8] shows that the embedding $\mathrm{Kos}_\bullet(A)\hookrightarrow\mathrm{Bar}_\bullet(A)$ of bimodule resolutions is preserved by the iterated twisting in part (i) above (see Remark 2.19). Thus $\mathrm{Kos}_\bullet(A)$ satisfies compatibility. □

We next give an example showing how Proposition 2.20 can be used for Koszul resolutions even when the twisting map τ is not strongly graded.

Example 2.21. As in Example 2.2, let $\mathcal{W}$ be the Weyl algebra on x, y with $A = k[x]$ and $B = k[y]$. Let $\mathrm{Kos}_\bullet(A)$ be the Koszul resolution of A as an A-bimodule,

$$0\to A\otimes V\otimes A\xrightarrow{d_1} A\otimes A\xrightarrow{m} A\to 0,$$

where $V = \mathrm{Span}_k\{x\} \subset A$, $d_1(1 \otimes x \otimes 1) = x \otimes 1 - 1 \otimes x$, and m is multiplication. Let $\bar{\tau} : B \otimes V \to V \otimes B$ be the swap map $b \otimes v \mapsto v \otimes b$ for all b in B and v in V, and define

$$\bar{\tau}_{B,\bullet} : B \otimes \mathrm{Kos}_\bullet(A) \to \mathrm{Kos}_\bullet(A) \otimes B$$

by iterations of τ and $\bar{\tau}$:

$$\bar{\tau}_{B,0} : B \otimes A \otimes A \xrightarrow{\tau \otimes 1} A \otimes B \otimes A \xrightarrow{1 \otimes \tau} A \otimes A \otimes B, \quad \text{and}$$

$$\begin{aligned} \bar{\tau}_{B,1} : B \otimes A \otimes V \otimes A &\xrightarrow{\tau \otimes 1 \otimes 1} A \otimes B \otimes V \otimes A \\ &\xrightarrow{1 \otimes \bar{\tau} \otimes 1} A \otimes V \otimes B \otimes A \xrightarrow{1 \otimes 1 \otimes \tau} A \otimes V \otimes A \otimes B. \end{aligned}$$

Define $\bar{\tau}_{\bullet,A} : \mathrm{Kos}_\bullet(B) \otimes A \to A \otimes \mathrm{Kos}_\bullet(B)$ similarly for the Koszul resolution $\mathrm{Kos}_\bullet(B)$ of B. Note that τ is not strongly graded, so part (iii) of Proposition 2.20 does not apply even though both A and B are Koszul algebras. Instead, we appeal to part (ii) and Remark 2.19 after taking canonical embeddings $\mathrm{Kos}_\bullet(A) \hookrightarrow \overline{\mathrm{Bar}}_\bullet(A)$ and $\mathrm{Kos}_\bullet(B) \hookrightarrow \overline{\mathrm{Bar}}_\bullet(B)$. (For example, view $A \otimes V \otimes A$ as a subspace of $A \otimes \bar{A} \otimes A$; the terms in other degrees are either 0 or the same as in the bar resolution.) The maps $\bar{\tau}_{B,\bullet}$ and $\bar{\tau}_{\bullet,A}$ above are the restrictions to $B \otimes \mathrm{Kos}_\bullet(A)$ and $\mathrm{Kos}_\bullet(B) \otimes A$ of the maps of the same name in the proof of Proposition 2.20(ii) (after identifying V with its image under the quotient map $A \to \bar{A}$). In this way, we see that the Koszul resolutions $\mathrm{Kos}_\bullet(A)$ and $\mathrm{Kos}_\bullet(B)$ are compatible with the twisting map τ via $\bar{\tau}_{B,\bullet}$ and $\bar{\tau}_{\bullet,A}$. We extend these ideas in Theorem 4.2.

3. Twisted product resolutions for Bimodules

Again, we fix k-algebras A and B with a twisting map $\tau : B \otimes A \to A \otimes B$ and consider an A-bimodule M and B-bimodule N. We build a projective resolution of $M \otimes N$ as a bimodule over $A \otimes_\tau B$ from resolutions $P_\bullet(M)$ and $P_\bullet(N)$ under our compatibility assumptions. We give the construction in Lemma 3.1, prove exactness in Lemma 3.5, and show in Lemma 3.9 that the modules in the construction are indeed projective under an additional assumption.

Lemma 3.1. *Let M be an A-bimodule and let N be a B-bimodule, both compatible with a twisting map τ. Let $P_\bullet(M)$ and $P_\bullet(N)$ be projective A- and B-bimodule resolutions of M and N, respectively, that are compatible with τ. For each $i, j \geq 0$, let*

$$X_{i,j} = P_i(M) \otimes P_j(N), \tag{3.2}$$

an $A \otimes_\tau B$-bimodule via diagram (2.13). *Then $X_{\bullet,\bullet}$ is a bicomplex of $A \otimes_\tau B$-bimodules with horizontal and vertical differentials given by $d^h_{i,j} = d_i \otimes 1$ and*

$d^v_{i,j} = (-1)^i \otimes d_j$, *where* d_i *and* d_j *denote the differentials of the appropriate resolutions*:

$$
\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\downarrow & & \downarrow & & \downarrow & & \\
X_{0,2} & \xleftarrow{d^h_{1,2}} & X_{1,2} & \xleftarrow{d^h_{2,2}} & X_{2,2} & \longleftarrow & \cdots \\
\downarrow{\scriptstyle d^v_{0,2}} & & \downarrow{\scriptstyle d^v_{1,2}} & & \downarrow{\scriptstyle d^v_{2,2}} & & \\
X_{0,1} & \xleftarrow{d^h_{1,1}} & X_{1,1} & \xleftarrow{d^h_{2,1}} & X_{2,1} & \longleftarrow & \cdots \\
\downarrow{\scriptstyle d^v_{0,1}} & & \downarrow{\scriptstyle d^v_{1,1}} & & \downarrow{\scriptstyle d^v_{2,1}} & & \\
X_{0,0} & \xleftarrow{d^h_{1,0}} & X_{1,0} & \xleftarrow{d^h_{2,0}} & X_{2,0} & \longleftarrow & \cdots
\end{array}
$$

Proof. The k-vector spaces $X_{i,j}$ form a tensor product bicomplex with differentials as stated. The bimodule action of $A \otimes_\tau B$ on $X_{i,j}$ commutes with the horizontal and vertical differentials since $\tau_{\bullet,B}$ and $\tau_{A,\bullet}$ are chain maps. Therefore this is an $A \otimes_\tau B$-bimodule bicomplex. □

Definition 3.3. The *twisted product complex* $X_\bullet$ is the total complex of $X_{\bullet,\bullet}$, i.e., when augmented by $M \otimes N$, it is the complex

$$\cdots \to X_2 \to X_1 \to X_0 \to M \otimes N \to 0 \tag{3.4}$$

with $X_n = \bigoplus_{i+j=n} X_{i,j}$, and n-th differential $\sum_{i+j=n} d_{i,j}$ where

$$d_{i,j} = d_i \otimes 1 + (-1)^i \otimes d_j.$$

Lemma 3.5. *The twisted product complex* (3.4) *is exact.*

Proof. By the Künneth theorem [Weibel 1994, Theorem 3.6.3], for each n there is an exact sequence

$$
\begin{aligned}
0 \to \bigoplus_{i+j=n} \mathrm{H}_i(P_\bullet(M)) \otimes \mathrm{H}_j(P_\bullet(N)) &\to \mathrm{H}_n(P_\bullet(M) \otimes P_\bullet(N)) \\
\to \bigoplus_{i+j=n-1} \mathrm{Tor}^k_1\big(\mathrm{H}_i(P_\bullet(M)), \mathrm{H}_j(P_\bullet(N))\big) &\to 0.
\end{aligned}
$$

Now $P_\bullet(M)$ and $P_\bullet(N)$ are exact other than in degree 0, where they have homology M and N, respectively. Therefore

$$\mathrm{H}_i(P_\bullet(M)) = 0 \text{ for all } i > 0 \quad\text{and}\quad \mathrm{H}_j(P(N)) = 0 \text{ for all } j > 0.$$

The Tor term is 0 since k is a field. Thus for all $n > 0$, $\mathrm{H}_n(P_\bullet(M) \otimes P_\bullet(N)) = 0$, and

$$\mathrm{H}_0(P_\bullet(M) \otimes P_\bullet(N)) \cong \mathrm{H}_0(P_\bullet(M)) \otimes \mathrm{H}_0(P_\bullet(N)) \cong M \otimes N$$

as vector spaces. Thus the complex (3.4) is exact. □

In practice, one often can show directly that each $X_{i,j}$ is projective as an $A \otimes_\tau B$-bimodule, for example, when working with bar resolutions and/or Koszul resolutions. For the general case, we need an extra compatibility assumption, which we explain next. As each $P_i(N)$ is a projective B-bimodule, it embeds into a free B^e-module $(B^e)^{\oplus J}$ for some indexing set J. In the following definition, we use the map $(\tau \otimes 1)(1 \otimes \tau) : B^e \otimes A \to A \otimes B^e$.

Definition 3.6. A chain map $\tau_{i,A} : P_i(N) \otimes A \to A \otimes P_i(N)$ is *compatible with a chosen embedding* $P_i(N) \hookrightarrow (B^e)^{\oplus J}$ (for some indexing set J) if the corresponding diagram is commutative:

$$\begin{array}{ccc} P_i(N)\otimes A & \hookrightarrow & (B^e)^{\oplus J}\otimes A \\ \downarrow{\scriptstyle \tau_{i,A}} & & \downarrow{\scriptstyle ((\tau\otimes 1)(1\otimes\tau))^{\oplus J}} \\ A\otimes P_i(N) & \hookrightarrow & A\otimes (B^e)^{\oplus J} \end{array}$$

Similarly, the map $\tau_{B,i}$ of (2.16) is *compatible with a chosen embedding* of $P_i(M)$ into a free A^e-module $(A^e)^{\oplus I}$ (for some indexing set I) if the corresponding diagram is commutative, i.e., if $\tau_{B,i}$ is the restriction of the map $((1 \otimes \tau)(\tau \otimes 1))^{\oplus I}$ to $B \otimes P_i(M)$.

Remark 3.7. In many settings, one sees directly that each $X_{i,j}$ is projective, in which case one need not consider this extra compatibility condition, as the next lemma is not needed. This is the case, for example, when twisting by a bicharacter on grading groups (see [Bergh and Oppermann 2008, Lemma 3.3]). In other settings, $\tau_{i,A}$ and $\tau_{B,i}$ are automatically compatible with chosen embeddings into free modules, for example if A and B are Koszul algebras and the embeddings are standard embeddings into bar resolutions (see [Walton and Witherspoon 2018, Proposition 1.8]).

Example 3.8. As in Examples 2.2 and 2.21, let $\mathcal{W} \cong A \otimes_\tau B$ be the Weyl algebra on $x, y,$ $A = k[x]$, and $B = k[y]$. By construction, each map $\bar{\tau}_{i,A}$ is compatible with the canonical embedding $\mathrm{Kos}_i(A) \hookrightarrow \overline{\mathrm{Bar}}_i(A)$ and likewise $\bar{\tau}_{B,i}$ is compatible with $\mathrm{Kos}_i(B) \hookrightarrow \overline{\mathrm{Bar}}_i(B)$.

Lemma 3.9. *If $\tau_{B,i}$ and $\tau_{j,A}$ are compatible with chosen embeddings of $P_i(M)$ and $P_j(N)$ into free modules, then $X_{i,j} = P_i(M) \otimes P_j(N)$ is a projective $A \otimes_\tau B$-bimodule.*

Proof. First we verify the lemma in the case where $P_i(M) = A^e,$ $P_j(N) = B^e$, and the chosen embeddings are the identity maps. In this case,

$$X_{i,j} = A^e \otimes B^e = A \otimes A^{\mathrm{op}} \otimes B \otimes B^{\mathrm{op}}.$$

One checks that the map

$$1 \otimes \tau \otimes 1 : \; A \otimes B \otimes (A \otimes B)^{\mathrm{op}} \to A \otimes A^{\mathrm{op}} \otimes B \otimes B^{\mathrm{op}}$$

is an isomorphism of $(A \otimes_\tau B)^e$-modules by (2.1) and the definition of the action given in the proof of Lemma 3.1. If $P_i(M)$ and $P_j(N)$ are arbitrary free modules, and the chosen embeddings are identity maps, we apply the above map to each summand $A^e \otimes B^e$ of $P_i(M) \otimes P_j(N)$ to see that $X_{i,j}$ is a free $(A \otimes_\tau B)^e$-module.

Now we consider the general case, including the possibility that at least one of $P_i(M)$, $P_j(N)$ is free but the corresponding chosen embedding into a (possibly different) free module is not the identity map. The first part of the proof together with the compatibility hypothesis implies that the embedding of k-vector spaces $P_i(M) \otimes P_j(N) \hookrightarrow (A^e)^{\oplus I} \otimes (B^e)^{\oplus J}$ given by the tensor product of the two embedding maps is a map of $(A \otimes_\tau B)^e$-modules. □

We combine the lemmas to obtain the following theorem.

Theorem 3.10. *Let A and B be k-algebras, and let $\tau : B \otimes A \to A \otimes B$ be a twisting map. Let M be an A-bimodule and N a B-bimodule with projective A- and B-bimodule resolutions $P_\bullet(M)$ and $P_\bullet(N)$, respectively. Assume that M, N, $P_\bullet(M)$, and $P_\bullet(N)$ are compatible with τ and the corresponding maps $\tau_{B,i}$ and $\tau_{j,A}$ are compatible with chosen embeddings of $P_i(M)$ and $P_j(N)$ into free modules. Then the twisted product complex with*

$$X_n = \bigoplus_{i+j=n} X_{i,j} \quad \text{for } X_{i,j} = P_i(M) \otimes P_j(N)$$

gives a projective resolution of $M \otimes N$ as a $A \otimes_\tau B$-bimodule:

$$\cdots \to X_2 \to X_1 \to X_0 \to M \otimes N \to 0.$$

Proof. The result follows from Lemmas 3.1, 3.5, and 3.9. □

Remark 3.11. The theorem generally unifies known constructions of resolutions in several different contexts, for example, twisted tensor products given by bicharacters of grading groups [Bergh and Oppermann 2008], crossed products [Guccione and Guccione 2002], skew group algebras (semidirect products) of Koszul algebras and finite groups [Shepler and Witherspoon 2014], and smash products of Koszul algebras with Hopf algebras [Walton and Witherspoon 2014].

Theorem 3.10 combined with Proposition 2.20 and Remark 3.7 implies that a twisted product resolution of $A \otimes_\tau B$ as a bimodule always exists, since bar resolutions may always be twisted (and likewise Koszul resolutions, when one or both of the algebras is Koszul; see also [Jara et al. 2017; Polishchuk and Positselski 2005; Walton and Witherspoon 2018]):

Corollary 3.12. *Let A and B be k-algebras with twisting map $\tau : B \otimes A \to B \otimes A$. The following are projective resolutions of $A \otimes_\tau B$ as a bimodule over itself.*

- *The twisted product complex of two bar resolutions.*
- *The twisted product complex of two Koszul resolutions when A and B are Koszul algebras and τ is strongly graded.*
- *The twisted product complex of one bar resolution and one Koszul resolution in the case where one of A or B is Koszul and the other is graded, for τ strongly graded.*

Moreover, bar resolutions may be replaced by reduced bar resolutions in the above statements.

Examples: skew group algebras. We give some details for a class of examples introduced in Example 2.3. The resolutions in [Shepler and Witherspoon 2014] for $S \rtimes G$, where G is a finite group acting by graded automorphisms on a Koszul algebra S, appear different from but are equivalent to (3.4) when $M = kG$ (the group algebra) and $N = S$. Note that $kG \otimes S$ is isomorphic to $S \rtimes G$ as an $(S \rtimes G)$-bimodule via the twisting map τ. In [Shepler and Witherspoon 2014], the modules $X_{i,j}$ are given as

$$(S \rtimes G) \otimes C'_i \otimes D'_j \otimes (S \rtimes G),$$

where $P_i(kG) = kG \otimes C'_i \otimes kG$, $P_j(S) = S \otimes D'_j \otimes S$ are free $(kG)^e$- and S^e-modules determined by vector spaces C'_i, D'_j, respectively. We assume $P_i(kG)$ is G-graded and the grading is compatible with the kG-bimodule action. We assume $P_j(S)$ is a kG-module in such a way that the differentials are kG-module homomorphisms, and this action is compatible with that of S, so that $P_j(S)$ becomes an $S \rtimes G$-module. Compatibility with τ follows from these assumptions. There is an isomorphism of $S \rtimes G$-bimodules,

$$(kG \otimes C'_i \otimes kG) \otimes (S \otimes D'_j \otimes S) \xrightarrow{\sim} (S \rtimes G) \otimes C'_i \otimes D'_j \otimes (S \rtimes G),$$

similar to that used in the proof of [Shepler and Witherspoon 2014, Theorem 4.3], given by

$$g \otimes x \otimes g' \otimes s \otimes y \otimes s' \mapsto g((hg')s) \otimes x \otimes (g'y) \otimes g's'$$

for all $g, g' \in G$, $s, s' \in S$, x in the h-component of C'_i, and $y \in D'_j$.

Example 3.13. In particular, [Shepler and Witherspoon 2014, Example 4.6] involves a resolution that is neither a Koszul resolution nor a bar resolution and yet satisfies compatibility. In that example, k is a field of positive characteristic p, $S = k[x, y]$, and $G = \langle g \rangle$ is a group of order p acting on S by $g \cdot x = x$, $g \cdot y = x + y$. The resolution $P_\bullet(S)$ is the Koszul resolution $\mathrm{Kos}_\bullet(S)$ of S,

$$0 \to S \otimes \textstyle\bigwedge^2 V \otimes S \to S \otimes \bigwedge^1 V \otimes S \to S \otimes S \to S \to 0,$$

where $V = \mathrm{Span}_k\{x, y\}$. The resolution $P_\bullet(kG)$ is the bimodule resolution of kG,

$$(3.14)\quad \cdots \xrightarrow{\eta\cdot} kG \otimes kG \xrightarrow{\gamma\cdot} kG \otimes kG \xrightarrow{\eta\cdot} kG \otimes kG \xrightarrow{\gamma\cdot} kG \otimes kG \xrightarrow{m} kG \longrightarrow 0,$$

where $\gamma = g \otimes 1 - 1 \otimes g$, $\eta = g^{p-1} \otimes 1 + g^{p-2} \otimes g + \cdots + 1 \otimes g^{p-1}$, and m is multiplication. Compatibility follows from Proposition 2.20(i) using Remark 2.19 after taking the standard embedding $\mathrm{Kos}_\bullet(S) \hookrightarrow \mathrm{Bar}_\bullet(S)$ and embedding (3.14) into $\mathrm{Bar}_\bullet(kG)$ (see, e.g., [Guccione et al. 1991]).

4. Bimodule resolutions of Ore extensions

Many algebras of interest are Ore extensions of other algebras. We show how to twist bimodule resolutions for such extensions in this section.

Ore extensions as twisted tensor products. Let R be a k-algebra and fix a k-algebra automorphism σ of R. Let $\delta : R \to R$ be a left σ-derivation of R, that is,

$$(4.1)\qquad \delta(rs) = \delta(r)s + \sigma(r)\delta(s) \quad \text{for all } r, s \in R.$$

The *Ore extension* $R[x; \sigma, \delta]$ is the algebra with underlying vector space $R[x]$ and multiplication determined by that of R and of $k[x]$ and the additional Ore relation

$$xr = \sigma(r)x + \delta(r) \quad \text{for all } r \in R.$$

An Ore extension $R[x; \sigma, \delta]$ is thus isomorphic to a twisted tensor product $A \otimes_\tau B$ where $A = R$, $B = k[x]$, and the twisting map $\tau : B \otimes A \to A \otimes B$ satisfies

$$\tau(x \otimes r) = \sigma(r) \otimes x + \delta(r) \otimes 1 \quad \text{for all } r \in R.$$

Free resolutions for iterated Ore extensions. We will work with general Ore extensions in Section 6. Here for simplicity we restrict to the case where the automorphism on R is the identity, $\sigma = 1_R$, so the Ore relation sets commutators $xr - rx$ equal to elements in R. In this case, the Ore extension is also known as a *ring of formal differential operators*. We consider an *iterated Ore extension* $S = (\cdots(k[x_1][x_2; 1, \delta_2])\cdots)[x_t; 1, \delta_t]$, which we abbreviate as

$$S = k[x_1, \ldots, x_t; \delta_2, \ldots, \delta_t] = k\langle x_1, \ldots, x_t\rangle / (x_j x_i - x_i x_j - \delta_j(x_i) : 1 \le i < j \le t)$$

with $S \cong k[x_1, \ldots, x_t]$ as a k-vector space. We assume that S is a filtered algebra with $\deg(x_i) = 1$ for all i. Then each δ_j is a filtered map, i.e.,

$$\delta_j(x_i) \in k \oplus \mathrm{Span}_k\{x_1, \ldots, x_{j-1}\}$$

for $i < j$. This setting includes Weyl algebras and universal enveloping algebras of supersolvable Lie algebras.

Theorem 4.2. *Consider an iterated Ore extension* $S = k[x_1, \dots, x_t; \delta_2, \dots, \delta_t]$ *with identity automorphisms* $\sigma_i = 1$ *and filtered derivations* δ_i. *There is an iterated twisted product resolution* $K_\bullet$ *that is a free resolution of* S *as a bimodule over itself*:

$$K_n = S \otimes \textstyle\bigwedge^n V \otimes S$$

for $V = \mathrm{Span}_k\{x_1, \dots, x_t\}$ *with differentials given by* (*for* $1 \le l_1 < \cdots < l_n \le t$)

$$\begin{aligned} &d_n(1\otimes x_{l_1}\wedge\cdots\wedge x_{l_n}\otimes 1)\\ &= \sum_{1\le i\le n} (-1)^{i+1}\big(x_{l_i}\otimes x_{l_1}\wedge\cdots\wedge\hat{x}_{l_i}\wedge\cdots\wedge x_{l_n}\otimes 1 - 1\otimes x_{l_1}\wedge\cdots\wedge\hat{x}_{l_i}\wedge\cdots\wedge x_{l_n}\otimes x_{l_i}\big)\\ &\qquad + \sum_{1\le i<j\le n} (-1)^j \otimes x_{l_1}\wedge\cdots\wedge x_{l_{i-1}}\wedge\bar{\delta}_{l_j}(x_{l_i})\wedge x_{l_{i+1}}\wedge\cdots\wedge\hat{x}_{l_j}\wedge\cdots\wedge x_{l_n}\otimes 1, \end{aligned}$$

where $\bar{\delta}_{l_j}(x_{l_i})$ *is the image of* $\delta_{l_j}(x_{l_i})$ *under the projection* $k \oplus V \twoheadrightarrow V$.

Proof. We induct on t. For each i, the Koszul resolution of $k[x_i]$ is embedded in the (reduced) bar resolution of $k[x_i]$ as

$$0 \to k[x_i] \otimes \mathrm{Span}_k\{x_i\} \otimes k[x_i] \xrightarrow{d_1} k[x_i] \otimes k[x_i] \xrightarrow{m} k[x_i] \to 0, \tag{4.3}$$

where $d_1(1 \otimes x_i \otimes 1) = x_i \otimes 1 - 1 \otimes x_i$ and m is multiplication. For $t = i = 1$, the complex (4.3) is a resolution of S satisfying the statement of the theorem.

Now assume $t \ge 2$ and that the iterated Ore extension

$$A = k[x_1, \dots, x_{t-1}; \delta_2, \dots, \delta_{t-1}]$$

has a free bimodule resolution $P_\bullet(A)$ as in the theorem. Let $B = k[x_t]$ and let $P_\bullet(B)$ be the Koszul resolution (4.3) for $i = t$. Then $S = A \otimes_\tau B$ where

$$\tau(x_t \otimes a) = a \otimes x_t + \delta_t(a) \otimes 1 \quad \text{for all } a \in A.$$

Embedding into the reduced bar resolution. We embed $P_\bullet(A)$ into the reduced bar resolution $\overline{\mathrm{Bar}}_\bullet(A)$ and then define twisting maps for $P_\bullet(A)$ via this embedding: Let $\phi_n : P_n(A) \to A^{\otimes(n+2)}$ be the standard antisymmetrization map defined by

$$\phi_n(1 \otimes x_{l_1} \wedge \cdots \wedge x_{l_n} \otimes 1) = \sum_{\sigma \in \mathrm{Sym}_n} \operatorname{sgn}\sigma \otimes x_{l_{\sigma(1)}} \otimes \cdots \otimes x_{l_{\sigma(n)}} \otimes 1$$

for all $1 \le l_1 < \cdots < l_n \le t-1$. This is a chain map from $P_\bullet(A)$ to $\mathrm{Bar}_\bullet(A)$. Compose with the quotient map $\mathrm{Bar}_\bullet(A) \to \overline{\mathrm{Bar}}_\bullet(A)$ to obtain a chain map

$$\bar{\phi}_\bullet : P_\bullet(A) \to \overline{\mathrm{Bar}}_\bullet(A).$$

Note that the image of $P_\bullet(A)$ in the bar resolution $\mathrm{Bar}_\bullet(A)$, under $\phi_\bullet$, intersects the kernel of this quotient map trivially. Thus the induced map $\bar{\phi}_\bullet$ is injective.

Iterated twisting. The reduced bar resolution is compatible with τ via the map

$$\bar{\tau}_{B,\bullet} : B \otimes \overline{\mathrm{Bar}}_\bullet(A) \to \overline{\mathrm{Bar}}_\bullet(A) \otimes B$$

from the proof of Proposition 2.20(ii). We argue that $\bar{\tau}_{B,\bullet}$ restricts to a surjective map

$$\tilde{\tau}_{B,\bullet} : B \otimes P_\bullet(A) \to P_\bullet(A) \otimes B$$

by verifying that it preserves the image of $\bar{\phi}_\bullet$, i.e., $\bar{\tau}_{B,n}$ takes $B \otimes \mathrm{Im}(\bar{\phi}_n)$ onto $\mathrm{Im}(\bar{\phi}_n) \otimes B$ for all n. We apply $\bar{\tau}_{B,n}$ to

$$x_t \otimes \bar{\phi}_n(a_0 \otimes y_1 \wedge \cdots \wedge y_n \otimes a_{n+1}) = \sum_{\pi \in \mathrm{Sym}_n} \mathrm{sgn}\,\pi\ (x_t \otimes a_0 \otimes y_{\pi(1)} \otimes \cdots \otimes y_{\pi(n)} \otimes a_{n+1}),$$

for some a_0, a_{n+1} in A, in order to move x_t to the far right, obtaining

$$\Bigg(\sum_{\pi \in \mathrm{Sym}_n} (\mathrm{sgn}\,\pi)\, a_0 \otimes y_{\pi(1)} \otimes \cdots \otimes y_{\pi(n)} \otimes a_{n+1} \Bigg) \otimes x_t \in \mathrm{Im}(\bar{\phi}_n) \otimes B$$

plus additional terms that arise from the relation

$$\tau(x_t \otimes y_{\pi(i)}) = y_{\pi(i)} \otimes x_t + \delta_t(y_{\pi(i)}) \otimes 1.$$

(We use the same notation for elements of A and their images under the quotient map $A \to \bar{A}$ in cases where no confusion can arise.) Since $\tau(1 \otimes y_j) = y_j \otimes 1$ for all j, these additional terms sum to

$$\begin{aligned}
&\sum_{\pi \in \mathrm{Sym}_n} (\mathrm{sgn}\,\pi)\ \delta_t(a_0) \otimes y_{\pi(1)} \otimes \cdots \otimes y_{\pi(n)} \otimes a_{n+1} \otimes 1 \\
&\qquad + \sum_{\pi \in \mathrm{Sym}_n} \sum_{1 \le i \le n} (\mathrm{sgn}\,\pi)\ a_0 \otimes y_{\pi(1)} \otimes \cdots \otimes \bar{\delta}_t(y_{\pi(i)}) \otimes y_{\pi(i+1)} \otimes \cdots \otimes y_{\pi(n)} \otimes a_{n+1} \otimes 1 \\
&\qquad + \sum_{\pi \in \mathrm{Sym}_n} (\mathrm{sgn}\,\pi) a_0 \otimes y_{\pi(1)} \otimes \cdots \otimes y_{\pi(n)} \otimes \delta_t(a_{n+1}) \otimes 1 \\
&= \bar{\phi}_n(\delta_t(a_0) \otimes y_1 \wedge \cdots \wedge y_n \otimes a_{n+1}) \otimes 1 + \bar{\phi}_n\big(a_0 \otimes y_1 \wedge \cdots \wedge y_n \otimes \delta_t(a_{n+1})\big) \otimes 1 \\
&\qquad\qquad + \sum_{1 \le i \le n} \bar{\phi}_n\big(a_0 \otimes y_1 \wedge \cdots \wedge \bar{\delta}_t(y_i) \wedge y_{i+1} \wedge \cdots \wedge y_n \otimes a_{n+1}\big) \otimes 1 \in \mathrm{Im}(\bar{\phi}_n) \otimes B.
\end{aligned}$$

We may replace x_t by x_t^m in the above computation using induction after noting that $\tau(x_t^m \otimes x_i) = (1 \otimes m_B)\tau(x_t \otimes (\tau(x_t^{m-1} \otimes x_i))$ for $i < t$. The above arguments can be modified to apply to $\bar{\tau}_{B,i}^{-1}$ as well. Thus the chain map $\bar{\tau}_{B,\bullet}$ preserves the image of $\bar{\phi}_\bullet$ and restricts to a surjective chain map $\tilde{\tau}_{B,\bullet} : B \otimes P_\bullet(A) \to P_\bullet(A) \otimes B$ as claimed.

Compatibility on one side. The complex $P_\bullet(A)$ inherits compatibility with τ from the compatibility of the reduced bar complex $\overline{\mathrm{Bar}}_\bullet(A)$ with τ. Indeed, since $\overline{\mathrm{Bar}}_\bullet(A)$ is compatible with τ via a map $\bar{\tau}_{B,\bullet}$ which preserves the embedding

$$\bar{\phi}_\bullet : P_\bullet(A) \hookrightarrow \overline{\mathrm{Bar}}_\bullet(A),$$

the complex $P_\bullet(A)$ is compatible with τ via the restriction $\tilde{\tau}_{B,\bullet}$ of $\bar{\tau}_{B,\bullet}$ to $B \otimes P_\bullet(A)$. (See Proposition 2.20(ii) and its proof and Remark 2.19.)

Compatibility on the other side. Define a chain map

$$\tau_{\bullet,A} : P_\bullet(B) \otimes A \to A \otimes P_\bullet(B)$$

by setting $\tau_{0,A} = (\tau \otimes 1)(1 \otimes \tau)$ and

$$\tau_{1,A}((1 \otimes x_t \otimes 1) \otimes x_i) = x_i \otimes (1 \otimes x_t \otimes 1)$$

and then extending (uniquely) to $P_1(B) \otimes A$ by requiring that compatibility conditions (2.8) and (2.9) hold. A calculation shows that $\tau_{\bullet,A}$ is a chain map and that $P_\bullet(B)$ is compatible with τ. By their definitions, $\tau_{0,A}$ and $\tau_{1,A}$ are compatible with the embeddings of $P_0(B)$ and $P_1(B)$ into corresponding terms of the (reduced) bar resolution.

Twisted product resolution. By construction, the twisted product resolution $K_\bullet$ arising from $P_\bullet(A)$ and $P_\bullet(B)$ in degree n is isomorphic to

$$S \otimes \textstyle\bigwedge^n V \otimes S$$

as an S-bimodule via the isomorphisms

$$A \otimes \textstyle\bigwedge^i \mathrm{Span}_k\{x_1, \dots, x_{t-1}\} \otimes A \otimes B \otimes \bigwedge^j \mathrm{Span}_k\{x_t\} \otimes B$$
$$\xrightarrow{\sim} \; A \otimes B \otimes \textstyle\bigwedge^i \mathrm{Span}_k\{x_1, \dots, x_{t-1}\} \otimes \bigwedge^j \mathrm{Span}_k\{x_t\} \otimes A \otimes B,$$

for $j = 0, 1$, given by applying τ^{-1} (properly interpreted for each factor) to the innermost tensor factors A and B. We check the differentials: On $X_{n,0}$, the differential is just that arising from the factor $P_n(A)$. Now consider the differential on $X_{n-1,1}$, again writing $x_{l_i} = y_i$ for some indices $1 \le l_1 < \cdots < l_n \le t-1$:

$$\begin{aligned} &d_n(1\otimes y_1\wedge\cdots\wedge y_{n-1}\otimes 1\otimes 1\otimes x_t\otimes 1)\\ &\quad=\Bigg(\sum_{1\le i\le n-1}(-1)^{i+1}\big(y_i\otimes y_1\wedge\cdots\hat{y}_i\wedge\cdots\wedge y_{n-1}\otimes 1-1\otimes y_1\wedge\cdots\wedge\hat{y}_i\wedge\cdots\wedge y_{n-1}\otimes y_i\big)\\ &\qquad+\sum_{1\le i<j\le n-1}(-1)^j\otimes y_1\wedge\cdots\wedge\bar{\delta}_j(y_i)\wedge\cdots\wedge\hat{y}_j\wedge\cdots\wedge y_{n-1}\otimes 1\Bigg)\otimes(1\otimes x_t\otimes 1)\\ &\qquad+(-1)^{n-1}(1\otimes y_1\wedge\cdots\wedge y_{n-1}\otimes 1)\otimes(x_t\otimes 1-1\otimes x_t),\end{aligned}$$

which may be rewritten, under the above isomorphism, as

$$\sum_{1\le i\le n-1}(-1)^{i+1}y_i\otimes y_1\wedge\cdots\wedge\hat{y}_i\wedge\cdots\wedge y_{n-1}\otimes x_t\otimes 1$$
$$-\sum_{1\le i\le n-1}(-1)^{i+1}\otimes y_1\wedge\cdots\wedge\hat{y}_i\wedge\cdots\wedge y_{n-1}\otimes x_t\otimes y_i$$
$$+\sum_{1\le i<j\le n-1}(-1)^{j}\otimes y_1\wedge\cdots\wedge\bar{\delta}_j(y_i)\wedge\cdots\wedge\hat{y}_j\wedge\cdots\wedge y_{n-1}\otimes x_t\otimes 1$$
$$+(-1)^{n-1}x_t\otimes y_1\wedge\cdots\wedge y_{n-1}\otimes 1+(-1)^{n}\otimes y_1\wedge\cdots\wedge y_{n-1}\otimes x_t$$
$$+(-1)^{n}\sum_{1\le i\le n-1}1\otimes y_1\wedge\cdots\wedge\bar{\delta}_t(y_i)\wedge\cdots\wedge y_{n-1}\otimes 1.$$

Once we set $y_n=x_t$, identify $y_1\wedge\cdots\wedge y_{n-1}\otimes x_t$ with $y_1\wedge\cdots\wedge y_{n-1}\wedge x_t$, and make other similar identifications, this agrees with the differential in the statement. □

Examples. The theorem applies in particular to the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of a finite-dimensional solvable Lie algebra $\mathfrak{g}$. Here, we assume the underlying field k is algebraically closed, else $\mathfrak{g}$ should be supersolvable; see [Dixmier 1977, 1.3.14] and [Brown et al. 2015, Section 3]. The theorem gives a bimodule Koszul resolution of $\mathcal{U}(\mathfrak{g})$. Semisimple Lie algebras can then be handled via triangular decomposition. Other examples include Weyl algebras and Sridharan enveloping algebras [Sridharan 1961].

5. Twisted product resolutions for (left) modules

We now consider a twisted product resolution of left modules instead of bimodules. We give the one-sided version of bimodule constructions in Sections 2 and 3. Again, we fix k-algebras A and B with a twisting map $\tau: B\otimes A\to A\otimes B$. In the constructions below, we consider compatible A-modules, but note that we as easily could have started with compatible B-modules instead of A-modules using the inverse twisting map τ^{-1} instead of τ in order to lift (left) modules of A and B to (left) modules of $A\otimes_\tau B\cong B\otimes_{\tau^{-1}} A$.

Let M be an A-module with module structure map $\rho_{A,M}: A\otimes M\to M$ and recall the multiplication map $m_B: B\otimes B\to B$.

Definition 5.1. The A-module M is *compatible with the twisting map* τ if there is a bijective k-linear map $\tau_{B,M}: B\otimes M\to M\otimes B$ such that

$$\tau_{B,M}(m_B\otimes 1)=(1\otimes m_B)(\tau_{B,M}\otimes 1)(1\otimes\tau_{B,M})\quad\text{and}\tag{5.2}$$
$$\tau_{B,M}(1\otimes\rho_{A,M})=(\rho_{A,M}\otimes 1)(1\otimes\tau_{B,M})(\tau\otimes 1)\tag{5.3}$$

as maps on $B\otimes B\otimes M$ and on $B\otimes A\otimes M$, respectively.

Note that this definition is equivalent to the commutativity of a diagram similar to (2.11), where $\rho_{A,M}$ is replaced by a one-sided module structure map.

Let N be a B-module with module structure map $\rho_{B,N} : B \otimes N \to N$. If M is compatible with τ, the tensor product $M \otimes N$ may be given the structure of an $A \otimes_\tau B$-module via the following composition of maps:

$$A \otimes_\tau B \otimes M \otimes N \xrightarrow{1 \otimes \tau_{B,M} \otimes 1} A \otimes M \otimes B \otimes N \xrightarrow{\rho_{A,M} \otimes \rho_{B,N}} M \otimes N. \tag{5.4}$$

Let $P_\bullet(M)$ be an A-projective resolution of M and $P_\bullet(N)$ be a B-projective resolution of N:

$$\cdots \to P_2(M) \to P_1(M) \to P_0(M) \to k \to 0,$$
$$\cdots \to P_2(N) \to P_1(N) \to P_0(N) \to k \to 0.$$

Definition 5.5. Let M be an A-module that is compatible with τ. The projective module resolution $P_\bullet(M)$ of the A-module M is *compatible with the twisting map* τ if each $P_i(M)$ is compatible with τ via maps $\tau_{B,i}$ for which

$$\tau_{B,\bullet} : B \otimes P_\bullet(M) \to P_\bullet(M) \otimes B$$

is a k-linear chain map lifting $\tau_{B,M} : B \otimes M \to M \otimes B$.

Under the assumption of compatibility, we make the following definition.

Definition 5.6. Let M be an A-module compatible with τ and $P_\bullet(M)$ a projective resolution of M that is compatible with τ. Let N be a B-module. The *twisted product complex* $Y_\bullet$ is the total complex of the bicomplex $Y_{\bullet,\bullet}$ defined by

$$Y_{i,j} = P_i(M) \otimes P_j(N), \tag{5.7}$$

with $A \otimes_\tau B$-module structure given by the maps $\tau_{B,\bullet}$ as in (5.4) and with vertical and horizontal differentials given by $d^h_{i,j} = d_i \otimes 1$ and $d^v_{i,j} = (-1)^i \otimes d_j$. In other words, $Y_n = \bigoplus_{i+j=n} Y_{i,j}$ with $d_n = \sum_{i+j=n} d_{i,j}$ where $d_{i,j} = d^h_{i,j} + d^v_{i,j}$.

Lemma 5.8. *Assume M and $P_\bullet(M)$ are compatible with τ. Then the twisted product complex $Y_\bullet$ is a complex of $A \otimes_\tau B$-modules.*

Proof. Each space $Y_{i,j}$ is given the structure of an $A \otimes_\tau B$-module via (5.4). The differentials are module homomorphisms since $\tau_{B,\bullet}$ is a chain map. □

Lemma 5.9. *The twisted product complex $\cdots \to Y_2 \to Y_1 \to Y_0 \to M \otimes N \to 0$ is exact.*

Proof. As in the proof of Lemma 3.5, apply the Künneth theorem to obtain $\mathrm{H}_n(Y_\bullet) = 0$ for all $n > 0$ and $\mathrm{H}_0(Y_\bullet) \cong M \otimes N$. □

We wish to prove in general that the modules $Y_{i,j}$ are projective, so we make an additional assumption in the next lemma. Since $P_\bullet(M)$ is a projective resolution of M as an A-module, each $P_i(M)$ embeds in a free A-module $A^{\oplus I}$.

Definition 5.10. For each $i \geq 0$, the map $\tau_{B,i}$ is *compatible with a chosen embedding* $P_i(M) \hookrightarrow A^{\oplus I}$ (for some indexing set I) if the corresponding diagram is commutative:

$$\begin{array}{ccc} B \otimes P_i(M) & \hookrightarrow & B \otimes A^{\oplus I} \\ \downarrow \tau_{B,i} & & \downarrow \tau^{\oplus I} \\ P_i(M) \otimes B & \hookrightarrow & A^{\oplus I} \otimes B \end{array}$$

In many settings, one proves directly that the modules $Y_{i,j}$ are projective (e.g., the Ore extensions in the next section) and so one does not need this additional compatibility assumption, nor the next lemma.

Lemma 5.11. *For $i \geq 0$, if $\tau_{B,i}$ is compatible with a chosen embedding of $P_i(M)$ into a free A-module, then $Y_{i,j} = P_i(M) \otimes P_j(N)$ is a projective $A \otimes_\tau B$-module.*

Proof. By the hypothesis, it suffices to prove the lemma when $P_i(A) = A$ and $P_j(B) = B$. In that case, $A \otimes B$ is the right regular module $A \otimes_\tau B$ by definition, and so is free. □

Combining Lemmas 5.8, 5.9, and 5.11, we obtain the following theorem.

Theorem 5.12. *Let A and B be k-algebras with twisting map $\tau : B \otimes A \to A \otimes B$. Let $P_\bullet(M)$ and $P_\bullet(N)$ be projective A- and B-module resolutions of M and N, respectively. Assume M and $P_\bullet(M)$ are compatible with τ and that the corresponding maps $\tau_{B,i}$ are compatible with chosen embeddings of $P_i(M)$ into free A-modules. Then the twisted product complex with*

$$Y_n = \bigoplus_{i+j=n} Y_{i,j} \quad \text{for } Y_{i,j} = P_i(M) \otimes P_j(N)$$

gives a projective resolution of $M \otimes N$ as a module over the twisted tensor product $A \otimes_\tau B$:

$$\cdots \to Y_2 \to Y_1 \to Y_0 \to M \otimes N \to 0.$$

Examples. Resolutions that may be constructed in this way include the Koszul resolution of k for a twisted tensor product of two Koszul algebras (see the proof of [Walton and Witherspoon 2018, Proposition 1.8]) and a resolution for a twisted tensor product of algebras whose twisting map is given by a bicharacter on grading groups (see [Bergh and Oppermann 2008]). We give another class of examples in the next section.

6. Resolutions for Ore extensions

In Section 4, we considered resolutions of an Ore extension algebra as a bimodule over itself. Here, we consider (left) modules over an Ore extension and show how to construct projective resolutions of these modules by regarding the Ore extension as a twisted tensor product. Gopalakrishnan and Sridharan [1966] studied Ore extensions $R[x; \sigma, \delta]$ in the case where σ is the identity automorphism. They showed that if M is a (left) module over $R[x; 1, \delta]$, then an R-projective resolution of M lifts to an $R[x; 1, \delta]$-projective resolution. Here we allow arbitrary automorphisms σ of R and give conditions under which an R-projective resolution of an $R[x; \sigma, \delta]$-module M lifts to an $R[x; \sigma, \delta]$-projective resolution.

Again, let R be a k-algebra and σ be a k-algebra automorphism of R. Let δ be a left σ-derivation of R (see (4.1)) and consider the Ore extension $R[x; \sigma, \delta]$. Let $A = R$, $B = k[x]$, and $\tau : B \otimes A \to A \otimes B$ be the twisting map determined by $\tau(x \otimes r) = \sigma(r) \otimes x + \delta(r) \otimes 1$ for all $r \in R$, as in Section 4, so that $R[x; \sigma, \delta]$ is the twisted tensor product $A \otimes_\tau B$.

Modules over Ore extensions. Consider an $R[x; \sigma, \delta]$-module M. Assume that on restriction to R, there is an isomorphism of R-modules, $\phi : M \xrightarrow{\sim} M^\sigma$, where M^σ is the vector space M with R-module action given by $r \cdot_\sigma m = \sigma(r) \cdot m$ for all $r \in R$ and $m \in M$. Then M is compatible with τ: We define $\tau_{B,M} : B \otimes M \to M \otimes B$ by setting, for all $m \in M$,

$$\tau_{B,M}(1 \otimes m) = m \otimes 1,$$
$$\tau_{B,M}(x \otimes m) = \phi(m) \otimes x + xm \otimes 1$$

and extending by applying compatibility condition (5.2). That is, since the algebra $B = k[x]$ is free on the generator x, for each element m of M, we may define $\tau_{B,M}(x^n \otimes m)$ by applying (5.2) to $x \otimes x^{n-1} \otimes m$. We check that (5.3) holds for elements of the form $x \otimes r \otimes m$, where $r \in R$ and $m \in M$. Then a careful induction on the power of x shows that (5.3) holds for all elements of the form $x^n \otimes r \otimes m$.

For example, if $R[x; \sigma, \delta]$ is an augmented algebra with augmentation $\varepsilon : R[x; \sigma, \delta] \to k$ for which $\varepsilon\sigma = \varepsilon$, then $\varepsilon\delta = 0$ and the field k as a module over $R[x; \sigma, \delta]$ via ε has the property that $k \cong k^\sigma$, and so k is compatible with τ.

Projective resolutions. Let $P_\bullet(M)$ be a projective resolution of M as an R-module:

$$\cdots \xrightarrow{d_2} P_1(M) \xrightarrow{d_1} P_0(M) \xrightarrow{\mu} M \to 0.$$

For each i, set $P_i^\sigma(M) = (P_i(M))^\sigma$. Then

$$\cdots \xrightarrow{d_2} P_1^\sigma(M) \xrightarrow{d_2} P_0^\sigma(M) \xrightarrow{\phi^{-1}\mu} M \to 0$$

is also a projective resolution of M as an R-module. By the comparison theorem, there is an R-module chain map from $P_\bullet(M)$ to $P_\bullet^\sigma(M)$ lifting the identity map $M \to M$, which we view as a k-linear chain map

$$\sigma_\bullet : P_\bullet(M) \to P_\bullet(M) \tag{6.1}$$

with $\sigma_i(rz) = \sigma(r)\sigma_i(z)$ for all $i \geq 0$, $r \in R$, and $z \in P_i(M)$. We will assume for Theorem 6.6 below that each σ_i is bijective. Let $P_{\bullet}(B)$ be the Koszul resolution of k for $B = k[x]$,

$$0 \to k[x] \xrightarrow{x\cdot} k[x] \xrightarrow{\epsilon} k \to 0, \tag{6.2}$$

where $\epsilon(x) = 0$. The following two lemmas are proven as in [Gopalakrishnan and Sridharan 1966] (where the authors proved the special case $\sigma = 1$). We include details for completeness.

Lemma 6.3. *Let P be a projective R-module. There is an $R[x; \sigma, \delta]$-module structure on P that extends the action of R.*

Proof. First consider the case that $P = R$, the left regular module. Let x act on R by $x \cdot r = \delta(r)$ for all $r \in R$. One checks that the action of xr in $R[x; \sigma, \delta]$ agrees with that of $\sigma(r)x + \delta(r)$ on P, for all $r \in R$. Next, if P is a free module, it is a direct sum of copies of R, and x acts on each copy in this way. Finally, in general, P is a direct summand of a free R-module F. Let $\iota : P \to F$ and $\pi : F \to P$ be R-module homomorphisms for which $\pi\iota$ is the identity map. Define $x \cdot p = \pi(x \cdot \iota(p))$ for all $p \in P$, where the action of x on $\iota(p)$ is as given previously for a free module. Again one checks that the actions of xr and of $\sigma(r)x + \delta(r)$ agree, and so P is an $R[x; \sigma, \delta]$-module as claimed. □

Compatibility requirements. We will use the next lemma to show that the resolution $P_{\bullet}(M)$ of M as an R-module is compatible with the twisting map τ (see Lemma 6.5). Let $f : M \to M$ be the function given by the action of x on the $R[x; \sigma, \delta]$-module M.

Lemma 6.4. *There is a k-linear chain map $\delta_{\bullet} : P_{\bullet}(M) \to P_{\bullet}(M)$ lifting $f : M \to M$ such that for each $i \geq 0$, $\delta_i(rz) = \sigma(r)\delta_i(z) + \delta(r)z$ for all $r \in R$ and $z \in P_i(M)$.*

Proof. If $i = 0$, let δ_0' be the action of x on $P_0(M)$ given by Lemma 6.3. Then

$$\delta_0'(rz) - \sigma(r)\delta_0'(z) = \delta(r)z$$

for $r \in R$, $z \in P_0(M)$. One checks that $\mu\delta_0' - f\mu : P_0(M) \to M^\sigma$ is an R-module homomorphism. As $P_0(M)$ is a projective R-module, there is an R-module homomorphism $\delta_0'' : P_0(M) \to P_0^\sigma(M)$ such that $\mu\delta_0' - f\mu = \mu\delta_0''$. Let $\delta_0 = \delta_0' - \delta_0''$. One may check this satisfies the equation in the lemma.

Now fix $i > 0$ and assume there are k-linear maps $\delta_j : P_j(M) \to P_j(M)$ such that $\delta_j(rz) = \sigma(r)\delta_j(z) + \delta(r)z$ and $d_j\delta_j = \delta_{j-1}d_j$ for all j, $0 \leq j < i$, and $r \in R$, $z \in P_j(M)$. Let $\delta_i' : P_i(M) \to P_i(M)$ be the action of x on $P_i(M)$ given in Lemma 6.3, so that $\delta_i'(rz) = \sigma(r)\delta_i'(z) + \delta(r)z$ for all $r \in R$, $z \in P_i(M)$. Consider the map

$$d_i\delta_i' - \delta_{i-1}d_i : P_i(M) \to P_{i-1}^\sigma(M).$$

A calculation shows that it is an R-module homomorphism. Since δ_{i-1} is a chain map,

$$d_{i-1}(d_i\delta'_i - \delta_{i-1}d_i) = 0,$$

and so the image of $d_i\delta'_i - \delta_{i-1}d_i$ lies in $\operatorname{Ker}(d_{i-1}) = \operatorname{Im}(d_i)$. Since $P_i(M)$ is projective as an R-module, there is an R-homomorphism $\delta''_i : P_i(M) \to P_i^{\sigma}(M)$ such that $d_i\delta'_i - \delta_{i-1}d_i = d_i\delta''_i$. Let $\delta_i = \delta'_i - \delta''_i$, so that $d_i\delta_i = \delta_{i-1}d_i$ by construction. One checks that for all $r \in R$ and $z \in P_i(M)$,

$$\delta_i(rz) = \delta'_i(rz) - \delta''_i(rz) = \sigma(r)\delta'_i(z) + \delta(r)z - \sigma(r)\delta''_i(z) = \sigma(r)\delta_i(z) + \delta(r)z. \quad \square$$

Lemma 6.5. *The resolution $P_{\bullet}(M)$ is compatible with the twisting map τ.*

Proof. Define $\tau_{B,i} : B \otimes P_i(M) \to P_i(M) \otimes B$ by

$$\tau_{B,i}(1 \otimes z) = z \otimes 1,$$
$$\tau_{B,i}(x \otimes z) = \sigma_i(z) \otimes x + \delta_i(z) \otimes 1$$

for all $z \in P_i(M)$, where $\sigma_{\bullet}$ is the chain map of (6.1), $\delta_{\bullet}$ is the chain map of Lemma 6.4, and we extend $\tau_{B,i}$ to $B \otimes P_i(M)$ as before by requiring that compatibility conditions (5.2) and (5.3) hold. We check condition (5.3) in one case as an example:

$$\tau_{B,i}(x \otimes rz) = \sigma_i(rz) \otimes x + \delta_i(rz) \otimes 1 = \sigma(r)\sigma_i(z) \otimes x + \sigma(r)\delta_i(z) \otimes 1 + \delta(r)z \otimes 1,$$

for all $r \in R$, and $z \in P_i(M)$, while on the other hand,

$$\begin{aligned}
(\rho_{A,i} \otimes 1)&(1 \otimes \tau_{B,i})(\tau \otimes 1)(x \otimes r \otimes z)\\
&= (\rho_{A,i} \otimes 1)(1 \otimes \tau_{B,i})(\sigma(r) \otimes x \otimes z + \delta(r) \otimes 1 \otimes z)\\
&= (\rho_{A,i} \otimes 1)(\sigma(r) \otimes \sigma_i(z) \otimes x + \sigma(r) \otimes \delta_i(z) \otimes 1 + \delta(r) \otimes z \otimes 1)\\
&= \sigma(r)\sigma_i(z) \otimes x + \sigma(r)\delta_i(z) \otimes 1 + \delta(r)z \otimes 1.
\end{aligned}$$

Condition (5.3) holds for all $x^n \otimes rz$ by induction on n. $\square$

Twisting resolutions for an Ore extension. We now construct a projective resolution of M as an $R[x; \sigma, \delta]$-module from a projective resolution of M as an R-module. We take the twisted product of two resolutions: the R-projective resolution of M and the Koszul resolution (6.2) of k as a module over $B = k[x]$.

Theorem 6.6. *Let $R[x; \sigma, \delta]$ be an Ore extension. Let M be an $R[x; \sigma, \delta]$-module for which $M^{\sigma} \cong M$ as R-modules. Consider a projective resolution $P_{\bullet}(M)$ of M as an R-module and suppose that each map $\sigma_i : P_i(M) \to P_i(M)$ of* (6.1) *is bijective. For each $i \geq 0$, set*

$$Y_{i,0} = Y_{i,1} = P_i(M) \otimes k[x] \quad \text{and} \quad Y_{i,j} = 0 \ \text{ for all } j > 1$$

as in Lemma 5.8. Then $Y_{\bullet}$ is a projective resolution of M as an $R[x; \sigma, \delta]$-module.

Proof. By Lemma 6.5, $P_\bullet(M)$ is compatible with τ, and so by Lemmas 5.8 and 5.9, the complex $\cdots \to Y_1 \to Y_0 \to M \to 0$ is an exact complex of $R[x;\sigma,\delta]$-modules. We verify directly that each $Y_{i,j}$ is a projective module: For each $i \geq 0$ and $j = 0, 1$,

$$Y_{i,j} \cong R[x;\sigma,\delta] \otimes_R P_i(M) \tag{6.7}$$

via the $R[x;\sigma,\delta]$-homomorphism given by

$$R[x;\sigma,\delta] \otimes_R P_i(M) \to Y_{i,j}, \qquad x \otimes z \mapsto \sigma_i(z) \otimes x + \delta_i(z) \otimes 1,$$

with inverse map given by

$$z \otimes x \mapsto x \otimes \sigma_i^{-1}(z) - 1 \otimes \delta_i(\sigma_i^{-1}(z)) .$$

Then $R[x;\sigma,\delta] \otimes_R P_i(M)$ is projective since it is a tensor-induced module and $R[x;\sigma,\delta]$ is flat over R. □

Remark 6.8. When σ is the identity, the complex $Y_\bullet$ is precisely that of Gopalakrishnan and Sridharan [1966, Theorem 1], under the isomorphism (6.7) above. As a specific class of examples, we obtain in this way, via iterated Ore extension, the Chevalley–Eilenberg resolution of the $\mathcal{U}(\mathfrak{g})$-module k for a finite-dimensional supersolvable Lie algebra $\mathfrak{g}$.

Acknowledgements

The authors thank Andrew Conner and Peter Goetz for helpful comments on an earlier version of this paper, and a referee for suggesting improvements.

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Received January 10, 2017. Revised July 24, 2018.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 298, No. 2, 2019
dx.doi.org/10.2140/pjm.2019.298.471

ITERATED AUTOMORPHISM ORBITS OF BOUNDED CONVEX DOMAINS IN $\mathbb{C}^n$

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The classification of bounded domains in $\mathbb{C}^n$, with $n > 1$, is related to the geometric properties of the boundary. A conjecture of Greene and Krantz relates the geometry of the boundary with the group of biholomorphic self mappings of the domain. The Greene–Krantz conjecture, if true, can tell us much about the classification of smoothly bounded domains in $\mathbb{C}^n$. Much work has been done to attempt to solve this conjecture, though it has yet to be proved or disproved. However, there are numerous partial results which support the conjecture. In this paper, we prove a special case of the conjecture:

Theorem: Suppose $\Omega \subset \mathbb{C}^n$ is a bounded convex domain with C^∞ boundary. Suppose there exists $\varphi \in \mathrm{Aut}(\Omega)$ and $p \in \Omega$ such that for the sequence of iterates $\{\varphi^j\} \subset \mathrm{Aut}(\Omega)$ we have $\varphi^j(p) \to x \in \partial\Omega$ nontangentially. Then x is of finite type.

1. Introduction

When studying domains in $\mathbb{C}^n$, we care about equivalence under biholomorphism. That is, two domains in $\mathbb{C}^n$ are equivalent if there is a biholomorphism between them. This equivalence is especially useful when our domains are endowed with the Kobayashi or Carathéodory metrics, for under these metrics, any biholomorphism preserves the distance between any two points. So no matter how much their Euclidean distances may differ, they are still the same distance apart in the Kobayashi metric. The Kobayashi metric will be an essential tool in whats follows. Certain properties of the automorphism group (biholomorphic self mappings) of the domain and the type (order of contact with a variety) of the boundary can be used to classify some bounded domains. It is a conjecture of Greene and Krantz that a smoothly bounded domain with a noncompact automorphism group is of finite type at any boundary orbit accumulation point. If this conjecture were true, it would classify all smoothly bounded domains in $\mathbb{C}^2$ with a noncompact automorphism group, for

MSC2010: 32-02.
Keywords: several complex variables, Greene Krantz conjecture, finite type, convex domains, automorphism orbit accumulation points, iterated automorphisms.

they would be, up to biholomorphism, the ball or a complex ellipsoid. The purpose of this paper is to prove the following:

Theorem 1.1. *Suppose $\Omega \subset \mathbb{C}^n$ is a bounded convex domain with C^∞ boundary. Suppose there exists $\varphi \in \mathrm{Aut}(\Omega)$ and $p \in \Omega$ such that for the sequence of iterates $\{\varphi^j\} \subset \mathrm{Aut}(\Omega)$, we have $\varphi^j(p) \to x \in \partial\Omega$ nontangentially. Then x is of finite type.*

Our main theorem gives us a classification of domains with the properties of the hypothesis. Berteloot and Cœuré [1991] showed that if a smoothly bounded domain $\Omega \subset \mathbb{C}^2$ admits an automorphism accumulation point which is of finite type, then Ω is biholomorphic to a domain of the form $E_m = \{(z, w) \in \mathbb{C}^2 : |z|^{2m} + |w|^2 < 1\}$ for some $m > 0$.

Corollary 1.2. *Suppose $\Omega \subset \mathbb{C}^2$ is a bounded convex domain with C^∞ boundary. Suppose there exist $\varphi \in \mathrm{Aut}(\Omega)$ and $p \in \Omega$ such that for the sequence of iterates $\{\varphi^j(p)\} \subset \mathrm{Aut}(\Omega)$, we have $\varphi^j(p) \to x \in \partial\Omega$ nontangentially. Then Ω is biholomorphic to an egg domain, E_m.*

For arbitrary dimension, n, Zimmer [2017] also showed that if a finite type boundary point of a smoothly bounded convex domain is also a nontangential automorphism orbit accumulation point, then the entire boundary is of finite type. Furthermore, Bedford and Pinchuk [1994] showed that smoothly bounded convex domains with finite type boundary and a noncompact automorphism group are biholomorphic to a certain class of polynomial domains. Therefore we have the following, which we state in the language of [Bedford and Pinchuk 1994].

Corollary 1.3. *Suppose $\Omega \subset \mathbb{C}^{n+1}$ is a bounded convex domain with C^∞ boundary. Suppose there exist $\varphi \in \mathrm{Aut}(\Omega)$ and $p \in \Omega$ such that for the sequence of iterates $\{\varphi^j(p)\} \subset \mathrm{Aut}(\Omega)$, we have $\varphi^j(p) \to x \in \partial\Omega$ nontangentially. Then Ω is biholomorphic to a domain of the form*

$$\Big\{(w, z_1, \dots, z_n) \in \mathbb{C} \times \mathbb{C}^n : |w|^2 + \sum\nolimits_{\mathrm{wt}\, J = \mathrm{wt}\, K = \frac{1}{2}} a_{JK} z^J \bar{z}^K < 1\Big\},$$

where $J = (j_1, \dots, j_n)$ and $K = (k_1, \dots k_n)$ are multi-indices, $a_{JK} = \bar{a}_{KJ}$, and $\mathrm{wt}\, J = j_1\delta_1 + \cdots + j_n\delta_n$ for some fixed $\delta_\ell = (2m_\ell)^{-1}$, with m_ℓ a positive integer.

In Section 2 we discuss some notation and definitions to be used throughout. Sections 3, 4, and 5 cover the main ideas of the hypothesis of our result. Finally, in Section 6 we give a proof of the main result. This paper is part of the author's doctoral thesis. The author would like to thank Professor Bun Wong for all his help and guidance as well as the referees for their thoughtful revisions.

2. Preliminaries

For two open subsets $W, V \subset \mathbb{C}^n$, a function $f : W \to V$ is said to be a biholomorphism if f is holomorphic and admits a holomorphic inverse $f^{-1} : V \to W$. We will denote by $\mathrm{Hol}(U, V)$ the collection of holomorphic maps from U to V. The unit disk in $\mathbb{C}$ is given by

$$\Delta = \{z \in \mathbb{C} : |z| < 1\},$$

the upper half plane in $\mathbb{C}$ by

$$\mathcal{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\},$$

and the unit polydisk in $\mathbb{C}^n$ by

$$\Delta^n = \Delta \times \cdots \times \Delta = \{(z_1, \ldots, z_n) \in \mathbb{C}^n : |z_j| < 1 \text{ for all } j = 1, \ldots, n\}.$$

Finally the unit ball in $\mathbb{C}^n$ is denoted

$$B^n = \{(z_1, \ldots, z_n) \in \mathbb{C}^n : |z_1|^2 + \cdots + |z_n|^2 < 1\}.$$

Definition 2.1. Let $\Omega \subset \mathbb{C}^n$ be an open set with C^k boundary. A function $\rho : \mathbb{C}^n \to \mathbb{R}$ is said to be a defining function for Ω if ρ is C^k and

(1) $\rho(x) < 0$ for all $x \in \Omega$,

(2) $\rho(x) > 0$ for all $x \notin \Omega$, and

(3) $\nabla\rho(x) \neq 0$ for all $x \in \partial\Omega$.

Definition 2.2. Let $\Omega \subset \mathbb{R}^n$ have a C^1 defining function ρ. Let $p \in \partial\Omega$. Then $w = (w_1, \ldots, w_n)$ is a tangent vector to $\partial\Omega$ at p if

$$\sum_{k=1}^{n} \frac{\partial\rho}{\partial x_k}\bigg|_p w_k = 0.$$

In this case we write $w \in T_p(\partial\Omega)$.

Definition 2.3. Let $\Omega \subset \mathbb{C}^n$ be a domain with C^2 boundary, $p \in \partial\Omega$, and ρ be a defining function for Ω. We say that p is a point of Levi pseudoconvexity if

$$\sum_{j,k=1}^{n} \frac{\partial^2\rho}{\partial z_j \partial \bar{z}_k}\bigg|_p w_j \bar{w}_k \geq 0$$

for all $w \in \mathbb{C}^n$ such that

$$\sum_{j=1}^{n} \frac{\partial\rho}{\partial z_j}\bigg|_p w_j = 0.$$

If instead, we have a strict inequality for all nonzero w satisfying the second equation, we say that x is a point of strict (Levi) pseudoconvexity. In general, when we say a boundary point is pseudoconvex we mean that it is weakly pseudoconvex.

The vectors satisfying the second equation in the above definition are called complex tangent vectors. We shall denote the complex tangent space by $T_p^{(1,0)}(\partial\Omega)$.

Note that $T_p^{(1,0)}(\partial\Omega) \subset T_p(\partial\Omega)$. We call

$$\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}\bigg|_p w_j \bar{w}_k$$

the Levi form of ρ at p. So $p \in \partial\Omega$ is a point of weak (respectively, strong) pseudoconvexity if its Levi form is positive semidefinite (respectively, positive definite).

Definition 2.4. Given $\Omega \subset \mathbb{C}^n$, $p \in \Omega$, and $v \in \mathbb{C}^n$, the Kobayashi pseudometric is given by

$$K_\Omega(p, v) = \inf\{|\zeta| : f \in \mathrm{Hol}(\Delta, \Omega),\, f(0) = p,\, f'(\zeta) = v\}.$$

The Poincaré metric coincides with this metric on Δ and $\mathcal{H}$. We will use Royden's integral formula [1971] for the Kobayashi pseudodistance.

Definition 2.5. The Kobayashi pseudodistance is given by

$$d_\Omega(z, w) = \inf_{\gamma} \int_0^1 K_\Omega(\gamma(t), \gamma'(t))\, dt,$$

where $z, w \in \Omega$ and $\gamma : [0, 1] \to \Omega$ is any piecewise C^1 curve such that $\gamma(0) = z$, $\gamma(1) = w$.

Proposition 2.6. *Let U, V be domains in $\mathbb{C}^n$ and $f : U \to V$ be a holomorphic map. Then*

$$K_V(f(p), f'(v)) \le K_U(p, v)$$

and

$$d_V(f(z), f(w)) \le d_U(z, w).$$

For convex domains that do not contain any complex lines, Barth [1980] showed that the Kobayashi pseudodistance is an actual distance in the sense that $d_\Omega(z, w) > 0$ if $z \ne w$.

Definition 2.7. We say that a subset $\Omega \subset \mathbb{C}^n$ is $\mathbb{C}$-proper if Ω does not contain any nontrivial complex affine lines.

Definition 2.8. Let $\Omega \subset \mathbb{C}^n$ be a $\mathbb{C}$-proper open set. For $z \in \Omega$ and $v \in \mathbb{C}^n$, let $L(z, v) \subset \mathbb{C}^n$ be the complex line passing through z in the direction of v. We set

$$\delta_\Omega(z, v) = d_{\mathrm{Euc}}(z, \partial\Omega \cap L(z, v))$$

and

$$\delta_\Omega(z) = d_{\mathrm{Euc}}(z, \partial\Omega).$$

That is, $\delta_\Omega(z, v)$ is the Euclidean distance from z to $\partial\Omega$ in the complex direction of v and $\delta_\Omega(z)$ is the overall Euclidean distance from z to $\partial\Omega$.

Proposition 2.9 is a consequence of the definition of the Kobayashi metric.

Proposition 2.9. *Let $\Omega \subset \mathbb{C}^n$ be a domain, $z \in \Omega$, and $v \in \mathbb{C}^n$. Then*

$$K_\Omega(z, v) \leq \frac{\|v\|}{\delta_\Omega(z, v)}.$$

3. Variety type

Definition 3.1. Let $U \subset \mathbb{C}^n$. A subset $V \subset U$ is called a holomorphic variety if it is composed of the roots of a finite number of holomorphic functions. That is

$$V = \{z \in U : f_1(z) = f_2(z) = \cdots = f_k(z) = 0\}$$

where f_i are holomorphic functions on U.

When a variety, V, is (complex) one-dimensional, then it can be parametrized. See [Gunning 1990] for a precise statement of the local parametrization theorem. We state only what is necessary for our purposes.

Proposition 3.2. *If $V \subset \mathbb{C}^n$ is a one-dimensional holomorphic variety and $p \in V$, then there is a neighborhood U of p and a nonconstant holomorphic function, $f : \Delta \to \mathbb{C}^n$ with $f(0) = p$ and $f(\Delta) \subset U \cap V$.*

We often refer to a one-dimensional holomorphic variety as a holomorphic disk or curve. When appropriate, we will refer to the image, $f(\Delta)$, as the holomorphic disk.

Given a smooth function $f : \mathbb{C} \to \mathbb{C}$ with $f(0) = 0$, we let $\nu(f)$ denote the order of vanishing of f at 0. If $g : \mathbb{C} \to \mathbb{C}^n$ is a smooth function with $g(0) = 0$ we let $\nu(g) = \min_i \nu(g_i)$, where $g = (g_1, \ldots, g_d)$.

Definition 3.3. Let Ω be a smooth domain in $\mathbb{C}^n$ with $q = 0 \in \partial\Omega$. Let $\rho(z)$ be a defining function for Ω in a neighborhood of q. We say that $\partial\Omega$ is of finite type C in the sense of D'Angelo if

$$\sup_f \left\{ \frac{\nu(\rho \circ f)}{\nu(f)} \right\} = C < \infty,$$

where f ranges through nonconstant holomorphic parametrizations of one-dimensional holomorphic subvarieties of $\mathbb{C}^n$ with $f(0) = q$. We say that $\partial\Omega$ is of finite line type L if

$$\sup_\ell \{\nu(\rho \circ \ell)\} = L < \infty,$$

where ℓ ranges through complex lines in $\mathbb{C}^n$ with $\ell(0) = q$.

Note that $\nu(\rho \circ \ell) \geq 2$ if and only if the image of ℓ is tangent to $\partial\Omega$ at q. So if we have a domain $\Omega \subset \mathbb{C}^n$ and a point $q \in \partial\Omega$ such that there is a holomorphic disk V passing through q, the D'Angelo (or variety) type of q is essentially a measurement of "how close" V is to actually lying in $\partial\Omega$. Now if $V \subset \partial\Omega$ then q would be a point of infinite type. When working with geometrically convex domains, one need only consider the line type rather than the more general variety type. This is due to McNeal, who gave the following proposition.

Proposition 3.4 [McNeal 1992]. *Let $\Omega \subset \mathbb{C}^n$ be a convex domain with $q \in \partial\Omega$. Then q is a point of finite variety type if and only if it is of finite line type.*

4. Automorphism orbits

For a domain $\Omega \subset \mathbb{C}^n$, the group of automorphisms will be denoted by $\mathrm{Aut}(\Omega)$. That is, $\mathrm{Aut}(\Omega)$ is the collection of biholomorphic self mappings of Ω.

Definition 4.1. Let $\Omega \subset \mathbb{C}^n$ be a domain. We say $p \in \overline{\Omega}$ is an orbit accumulation point of $\mathrm{Aut}(\Omega)$ if there is a sequence $\{\varphi_k\} \subset \mathrm{Aut}(\Omega)$ and a point $q \in \Omega$ such that $\varphi_k(q) \to p$. If $p \in \partial\Omega$ then we say p is a boundary orbit accumulation point for $\{\varphi_k\}$.

H. Cartan showed that for a bounded domain, Ω, $\mathrm{Aut}(\Omega)$ is a Lie group which acts properly on Ω; see [Narasimhan 1971]. We can determine the compactness of $\mathrm{Aut}(\Omega)$ by examining the orbits.

Proposition 4.2. *Suppose $\Omega \subset \mathbb{C}^n$ is a bounded domain.* $\mathrm{Aut}(\Omega)$ *admits a boundary orbit accumulation point if and only if* $\mathrm{Aut}(\Omega)$ *is noncompact.*

The well-known ball characterization theorem of Bun Wong classifies all bounded strongly pseudoconvex domains with a noncompact automorphism group.

Theorem 4.3 [Wong 1977]. *If $\Omega \subset \mathbb{C}^n$ is a strongly pseudoconvex bounded domain with a noncompact automorphism group, then Ω is biholomorphic to the unit ball B^n.*

We now have all the necessary machinery to state the Greene–Krantz conjecture.

Conjecture 4.4 (Greene and Krantz). Let $\Omega \subset \mathbb{C}^n$ be a bounded domain with smooth C^∞ boundary. If $p \in \partial\Omega$ is a boundary orbit accumulation point for $\mathrm{Aut}(\Omega)$, then $\partial\Omega$ is of finite type at p.

In [Krantz 2016], the conjecture above is shown for convex domains in $\mathbb{C}^2$. The proof involves subelliptic estimates for the $\bar\partial$ problem.

One partial result to this conjecture shows that if there is a boundary orbit accumulation point, x, for a smoothly bounded convex domain, then there is no nontrivial holomorphic disk contained in the boundary and passing through x. However, this does not guarantee that x is a point of finite type.

Theorem 4.5 (Lee, Thomas and Wong [Lee et al. 2014]). *Let $\Omega \subset \mathbb{C}^n$ be a smoothly bounded convex domain. Suppose there is a sequence $\{\varphi_j\} \subset$* $\mathrm{Aut}(\Omega)$ *such that $\varphi_j(z)$ converges nontangentially to some boundary point for all $z \in \Omega$. If $p \in \partial\Omega$*

is an orbit accumulation point, then there does not exist any nontrivial complex analytic variety passing through p and lying in $\partial\Omega$.

In $\mathbb{C}^2$, Hamann and Wong showed that we can remove the nontangential condition in the above theorem.

Theorem 4.6 [Hamann and Wong 2017]. *Let D be a bounded convex domain in $\mathbb{C}^2$ with C^2 boundary. If $p \in \partial D$ is an orbit accumulation point, then ∂D contains no nontrivial analytic variety passing through p.*

5. Nontangential convergence

The direction of travel of an automorphism orbit can yield certain conclusions. Nontangential convergence provides us with useful properties.

Definition 5.1. For a domain $\Omega \subset \mathbb{C}^n$ with C^1 boundary, a sequence $\{q_j\} \subset \Omega$, and a point $q \in \partial\Omega$, we say that $q_j \to q$ *nontangentially* if, for all j large enough,

$$q_j \in \Gamma_\alpha(q) = \{z \in \Omega : \|z - q\| \le \alpha\delta_\Omega(z)\}$$

for some $\alpha > 1$. We say that $q_j \to q$ normally if the q_j's approach q along the real normal line to $\partial\Omega$ at q.

Let n_q denote the inward-pointing normal vector to a C^1 domain $\Omega \in \mathbb{C}^n$ at boundary point $q \in \partial\Omega$.

Lemma 5.2. *Let $\Omega \subset \mathbb{C}^n$ be a convex domain with C^1 boundary. Let $z \in \Omega$ and $q' = q + tn_q$ for some $t > 0$. Then,*

$$\Gamma_\alpha(q) \subset \left\{z \in \Omega : 0 \le \angle zqq' \le \arccos\left(\frac{1}{\alpha}\right)\right\}.$$

Proof. Put $H = \{z \in \mathbb{C}^n : \mathrm{Im}(z_1) > 0\}$. We may assume $q = 0,\ n_q = (i, 0, \dots, 0)$, and $\Omega \subset H$. Then $\delta_\Omega(z) \le \delta_H(z) = \mathrm{Im}(z_1)$ which implies that $\|z - q\| \le \alpha \mathrm{Im}(z_1) = \alpha\|(\mathrm{Im}(z_1), 0, \dots, 0)\|$. Then, since

$$\cos(\angle zqq') = \frac{\|(\mathrm{Im}(z_1), 0, \dots, 0)\|}{\|z - q\|},$$

we have $\angle zqq' \le \arccos(1/\alpha)$. □

When $\partial\Omega$ admits a nontangential orbit accumulation point, Lee et al. [2014] showed that there is a sequence of points $\{p_j\} \subset \Omega$, within some fixed Kobayashi distance from $p \in \Omega$, such that the action of the sequence of automorphisms $\{\varphi_j\} \subset \mathrm{Aut}(\Omega)$ on the respective p_j's approaches the accumulation point $q \in \partial\Omega$ along the real normal line to the boundary at q. To be precise:

Lemma 5.3. *Let $\Omega \subset \mathbb{C}^n$ be a convex domain with C^1 boundary. Suppose $\{\varphi_j\} \subset \mathrm{Aut}(\Omega)$ and $\varphi_j(p) \to q \in \partial\Omega$ nontangentially for some $p \in \Omega$. Then for sufficiently* *large j there exists $\{p_j\} \subset \Omega$ such that $\varphi_j(p_j) \to q$ normally and $d_\Omega(p, p_j) \le r$ for some $r > 0$.*

Proof. Let $\ell_q = \{q + tn_q : t \in \mathbb{R}\}$ and define $\pi : \mathbb{C}^n \to \ell_q$ as the projection mapping onto ℓ_q. Put $q_j = \varphi_j(p)$, $\tilde{q}_j = \pi(q_j)$, and $p_j = \varphi^{-1}(\tilde{q}_j)$. Then $\tilde{q}_j \to q$ normally and $\|\tilde{q}_j - q_j\| \le \|q_j - q\|$. Fix j sufficiently large so that by Lemma 5.2

$$\frac{1}{\alpha} \le \cos(\angle zqq') = \frac{\|\tilde{q}_j - q\|}{\|q_j - q\|}.$$

Let $\gamma(t) = (1-t)q_j + t\tilde{q}_j$. Then

$$\begin{aligned} d_\Omega(p, p_j) = d_\Omega(q_j, \tilde{q}_j) &\le \int_0^1 K_\Omega(\gamma(t), \gamma'(t))\,dt \le \int_0^1 \frac{\|\gamma'(t)\|}{\delta_\Omega(\gamma(t), \gamma'(t))} dt \\ &\le \int_0^1 \frac{\|\gamma'(t)\|}{\delta_\Omega(\gamma(t))} dt \le \int_0^1 \frac{\|\gamma'(t)\|\alpha}{\|\gamma(t) - q\|} dt \le \frac{\|\tilde{q}_j - q_j\|\alpha}{\|\tilde{q}_j - q\|} \le \frac{\|q_j - q\|\alpha}{\|\tilde{q}_j - q\|} \le \alpha^2. \end{aligned}$$

Finally, we let $r = \alpha^2$. □

Essentially, this gives us that the Kobayashi distance from each $\varphi_j(p)$ to the real normal line of the boundary at $q \in \partial\Omega$ remains bounded by a fixed constant.

6. Finite type

We will now be able to showcase a condition that guarantees finite type for some boundary point of a smoothly bounded convex domain.

Definition 6.1. For a domain $\Omega \subset \mathbb{C}^n$ denote by $B_\Omega(o, M)$ the closed ball centered at $o \in \Omega$ with Kobayashi radius M. That is,

$$B_\Omega(o, M) = \{z \in \Omega : d_\Omega(o, z) \le M\}.$$

Theorem 6.2 [Zimmer 2017]. *Suppose $\Omega \subset \mathbb{C}^n$ is a bounded convex open set with C^∞ boundary. If there exist $o \in \Omega$, $x \in \partial\Omega$, $M \ge 0$, and $T \in \mathbb{R}$ so that*

$$\{x + e^{-t} n_x : t > T\} \subset \operatorname{Aut}(\Omega) B_\Omega(o, M),$$

then x is of finite type in the sense of D'Angelo.

We now state the main result.

Theorem 6.3. *Suppose $\Omega \subset \mathbb{C}^n$ is a bounded convex domain with C^∞ boundary. Suppose there exist $\varphi \in \operatorname{Aut}(\Omega)$ and $p \in \Omega$ such that for the sequence of iterates $\{\varphi^j\} \subset \operatorname{Aut}(\Omega)$ we have $\varphi^j(p) \to x \in \partial\Omega$ nontangentially. Then x is of finite type.*

Proof. Since the set of automorphisms of Ω forms a group under composition of functions, we have $\varphi^j \in \operatorname{Aut}(\Omega)$ for all $j \in \mathbb{N}$. Put $M = d_\Omega(p, \varphi(p))$. We may assume $M > 0$, since otherwise, φ would fix p. Then for every consecutive pair of iterates we have

$$\begin{aligned} d_\Omega(\varphi^j(p), \varphi^{j+1}(p)) &= d_\Omega(\varphi^j(p), \varphi^j(\varphi(p)) \\ &= d_\Omega(p, \varphi(p)) = M. \end{aligned}$$

By Lemma 5.3, there exists $\{p_j\} \subset \Omega$ such that $d_\Omega(p, p_j) \leq r$ for some $r > 0$, and $\varphi^j(p_j) \to x$ normally. So we have

$$\begin{aligned} &d_\Omega(\varphi^j(p_j), \varphi^{j+1}(p_{j+1})) \\ &\qquad \leq d_\Omega(\varphi^j(p_j), \varphi^j(p)) + d_\Omega(\varphi^j(p), \varphi^{j+1}(p)) + d_\Omega(\varphi^{j+1}(p), \varphi^{j+1}(p_{j+1})) \\ &\qquad \leq r + d_\Omega(\varphi^j(p)), \varphi^{j+1}(p)) + r = 2r + M, \end{aligned}$$

for all $j \in \mathbb{N}$. By convexity, we may assume $x = 0$ and $n_x = (i, 0, \ldots, 0)$. Also, since $\varphi^j(p_j) \to 0$ as $j \to \infty$, there are an infinitely many j such that

$$|\varphi^j(p_j)| > |\varphi^{j+1}(p_{j+1})|.$$

Then in the argument that follows, we will see that there exists a fixed radius, K, such that for any j with $|\varphi^j(p_j)| > |\varphi^{j+1}(p_{j+1})|$, the ball $B_\Omega(\varphi_j(p_j), K)$ contains both $\varphi^{j+1}(p_{j+1})$ and the real line segment connecting $\varphi^j(p_j)$ and $\varphi^{j+1}(p_{j+1})$. Now consider some $z, y, w \in \Omega$ that lie on the real normal line to $\partial\Omega$ at x such that $|w| < |y| < |z|$. We claim that, for sufficiently small $|z|$, if $w \in B_\Omega(z, R)$ for some $R > 0$, then either $y \in B_\Omega(z, R)$ or $y \in B_\Omega(z, 1)$. If z is sufficiently small, then there is a (complex) one-dimensional affine disk, D, centered at z, such that $D \subset \Omega \cap \{\zeta \in \Omega : \operatorname{Im}(\zeta_1) > 0\}$, $0 \in \partial D$, and ∂D is tangent to $\partial\Omega$ at 0. Note that D is essentially a copy of the unit disk under a translation and dilation and so D is biholomorphic to Δ. Now any geodesic under the Poincaré (equivalently, Kobayashi) metric passing through z in D is a straight line. Thus,

$$d_D(z, w) = d_D(z, y) + d_D(y, w).$$

Let $\pi : \mathbb{C}^n \to \mathbb{C}$ be the projection onto the first coordinate so $\pi(\Omega) \subset \mathcal{H}$. Then

$$d_\Omega(z, w) \geq d_{\pi(\Omega)}(\pi(z), \pi(w)) \geq d_{\mathcal{H}}(\pi(z), \pi(w)).$$

For simplicity, we may assume that D has radius 1, so $z = (i, 0, \ldots, 0)$. Then the mapping

$$\zeta \mapsto \left(i\frac{\zeta - i}{\zeta + i} + i, 0, \ldots, 0\right)$$

is a biholomorphism from $\mathcal{H}$ to D where $z \mapsto z$ and any purely imaginary $ib \in \mathcal{H}$ is mapped to some $(ic, 0, \ldots, 0) \in D$ where ic is also purely imaginary. Now

$$d_{\mathcal{H}}(\pi(z), \pi(w)) = d_D\Big(z, w\frac{2}{|w|+1}\Big).$$

Note that since $|w| < 1$, then $\left|w\frac{2}{|w|+1}\right| > |w|$. So if $|y| > \left|w\frac{2}{|w|+1}\right|$, then

$$\begin{aligned} d_D\left(z, w\frac{2}{|w|+1}\right) &= d_D(z, y) + d_D\left(y, w\frac{2}{|w|+1}\right) \\ &\geq d_D(z, y) \\ &\geq d_\Omega(z, y) \end{aligned}$$

where the last inequality is given by the inclusion map from D into Ω. Otherwise, if $|y| \leq \left|w\frac{2}{|w|+1}\right|$, then

$$d_\Omega(y, w) \leq \int_0^1 \frac{|y-w|}{\delta_\Omega(w)}\,dt \leq \frac{\left|w\frac{2}{|w|+1}\right| - |w|}{|w|} = \frac{1-|w|}{1+|w|} < 1.$$

Therefore, if $w \in B_\Omega(z, R)$ for $z, w \in \{x + e^{-t}n_x : t > T\}$ with T sufficiently large, then for all $y \in \{x + e^{-t}n_x : t > T\}$ with $|w| < |y| < |z|$, either $y \in B_\Omega(z, R)$ or $y \in B_\Omega(w, 1)$. Then since $d_\Omega(\varphi^j(p_j), \varphi^{j+1}(p_{j+1})) \leq 2r + M$, there is a $T > 0$ and some $K > 0$ such that

$$\{x + e^{-t}n_x : t > T\} \subset \bigcup_{j\in\mathbb{N}} B_\Omega(\varphi^j p_j, K),$$

and so

$$\{x + e^{-t}n_x : t > T\} \subset \mathrm{Aut}(\Omega) B_\Omega(p, K).$$

Thus, by Theorem 6.2, $x \in \partial\Omega$ is of finite type. □

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Received July 5, 2017. Revised July 16, 2018.

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PACIFIC JOURNAL OF MATHEMATICS
Vol. 298, No. 2, 2019
dx.doi.org/10.2140/pjm.2019.298.483

SHARP LOGARITHMIC SOBOLEV INEQUALITIES ALONG AN EXTENDED RICCI FLOW AND APPLICATIONS

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We prove a sharp Logarithmic Sobolev inequality along an extended Ricci flow. As applications, we derive an integral bound for the conjugate heat kernel and also obtain Lipschitz continuity of the pointed Nash entropy. Finally, based on these results, we prove an ε-regularity theorem for this extended Ricci flow.

1. Introduction

In this paper we study an extended Ricci flow as follows:

$$\begin{cases} \frac{\partial}{\partial t} g(t) = -2\mathrm{Ric}(g(t)) + 2d\phi(t)\otimes d\phi(t), \\ \frac{\partial}{\partial t}\phi(t) = \Delta_{g(t)}\phi(t), \\ g(0) = g_0,\ \phi(0) = \phi_0, \end{cases} \tag{1-1}$$

where $t \in [-T, 0]$, $g(t)$ are metrics, and $\phi(t) : (M, g) \to \mathbb{R}$ are smooth functions. This flow was introduced in [List 2008], where the author proved short time existence and long time existence if ϕ is a smooth function from M to $\mathbb{R}$. Later, R. Müller [2012] considered ϕ as a smooth map from (M, g) to (N, h) and proved some fundamental results for flow equation (1-1). The flow equations (1-1) come from static Einstein vacuum equations arising in the general relativity, and also arise as dimensional reductions of Ricci flow in higher dimensions. For more work on this flow, see [Fang and Zheng 2016b; 2016a; Guo et al. 2015a; 2015b; 2013; Li 2018; Liu and Wang 2017; Yang and Shen 2012]. Before stating our main results, we want to introduce some notation. Suppose $(M^n, g(t), \phi(t))|_{t\in[-T,0]}$ is an extended Ricci flow, fix $x, y \in M$; we use $d_t(x, y)$ to denote the distance between x and y at time t. We use $B_r(x, t)$ to denote the geodesic ball with radius r centered at x. We use Rm to denote the Riemannian curvature operator of the metric g, Ric the Ricci curvature, and R the scalar curvature. For the extended Ricci flow, we denote $\mathrm{Sic}(g(t)) = \mathrm{Ric}(g(t)) - d\phi(t)\otimes d\phi(t)$ and $S(g(t)) = R(g(t)) - |\nabla\phi(t)|^2_{g(t)}$.

MSC2010: primary 53C44; secondary 53C21.
Keywords: conjugate heat kernel, logarithmic Sobolev inequalities, ε-regularity.

Given $(x_0, 0) \in M \times [-T, 0]$, we denote

$$H_{x_0}(y, s) = H(x_0, 0; y, s) = (4\pi|s|)^{-\frac{n}{2}} \exp(-f_{x_0}(y, s))$$

as the conjugate heat kernel based at $(x_0, 0)$, and

$$d\nu_{x_0}(y, s) = H_{x_0}(y, s)\mathrm{dvol}_{g(s)}(y)$$

as the associated probability measure. See Definition 2.2.

Definition 1.1. Let (M, g) be a smooth Riemannian manifold and ϕ and f be smooth functions. Given $\tau > 0$, we define the associated W entropy as

$$W(g, \phi, f, \tau) = \int_M [\tau(S + |\nabla f|^2) + f - n](4\pi\tau)^{-\frac{n}{2}} e^{-f} \,\mathrm{dvol}_g,$$

where $S = R - |\nabla\phi|^2$. Moreover, the μ entropy can also be defined:

$$\mu(g, \phi, \tau) = \inf\left\{ W(g, \phi, f, \tau) \,\Big|\, \int_M (4\pi\tau)^{-\frac{n}{2}} e^{-f} \,\mathrm{dvol}_g = 1 \right\}.$$

Next we state the Poincaré inequality and log-Sobolev inequality along the extended Ricci flow (1-1); previous work on Ricci flow was given in Hein and Naber [2014].

Theorem 1.2. *Let* $(M^n, g(t), \phi(t))|_{t\in[-T,0]}$ *be an extended Ricci flow* (1-1). *Fix a point* $x_0 \in M$ *in the final time slice and let* $s \in [-T, 0]$.

(1) *For all* $u \in C_0^\infty(M)$ *with* $\int_M u \, d\nu_{x_0}(s) = 0$,

$$\int_M u^2 \, d\nu_{x_0}(s) \le 2|s| \int_M |\nabla u|^2_{g(s)} \, d\nu_{x_0}(s). \tag{1-2}$$

Equality holds if and only if $u \equiv 0$.

(2) *For all* $u \in C_0^\infty(M)$ *with* $\int_M u^2 \, d\nu_{x_0}(s) = 1$,

$$\int_M u^2 \log u^2 \, d\nu_{x_0}(s) \le 4|s| \int_M |\nabla u|^2_{g(s)} \, d\nu_{x_0}(s). \tag{1-3}$$

Equality holds if and only if $u \equiv 1$.

Remark. Following [Perelman 2002], Ni [2004] defined entropy for the linear heat equation on a complete Riemannian manifold. Under the Ricci nonnegativity assumption, he proved the monotonicity, and as an application, he characterized the Euclidean space using the sharp log-Sobolev inequality.

Next we state one important application of Theorem 1.2.

Theorem 1.3. *Let $(M^n, g(t), \phi(t))|_{t\in[-T,0]}$ be an extended Ricci flow and $d\nu = d\nu_{x_0}(s)$ be a heat kernel measure. Then the Gaussian concentration inequality*

$$\nu(A)\nu(B) \le \exp\left(-\frac{d_{g(s)}(A,B)^2}{8|s|}\right)$$

holds for $A, B \subseteq M$.

Corollary 1.4. *For any $C > 0$ there exists a $C' = C'(n, C) > 0$ such that the following holds: Let $(M^n, g(t), \phi(t))$ be an extended Ricci flow such that*

$$\sup_{t\in[s,0]} \|S(g(t))\|_\infty \le \frac{C}{|s|} \quad \text{and} \quad \inf_{\tau\in(0,2|s|)} \nu(g(s), \phi(s), \tau) \ge -C,$$

for some $s \in [-T, 0)$. Let $x_1, x_2 \in M$ and $r^2 = |s|$, then

$$\frac{1}{\mathrm{Vol}(B_r(x_1,0))}\int_{B_r(x_1,0)} H(x_2,0;y,s)\,\mathrm{dvol}_{g(s)}(y) \le C'\exp\left(-\frac{-d_{g(s)}(B_r(x_1,0),B_r(x_2,0))^2}{C'|s|}\right). \tag{1-4}$$

Moreover, we have the following distance distortion estimate:

$$d_{g(s)}(B_r(x_1,s), B_r(x_2,s)) \le C' d_{g(0)}(x_1,x_2). \tag{1-5}$$

Definition 1.5. The pointed W entropy at scale $|s|$ based at x_0 is defined by

$$W_{x_0}(s) = W(g(s), \phi(s), f_{x_0}(s), |s|).$$

The pointed Nash entropy at $(x_0, s) \in M \times [-T, 0]$ is defined as

$$N_{x_0}(s) = \frac{1}{|s|}\int_s^0 W_{x_0}(r)\,dr = \int_M f_{x_0}(s)\,d\nu_{x_0}(s) - \frac{n}{2}.$$

Now we can state the Lipschitz continuity of the pointed Nash entropy.

Theorem 1.6. *For each $C > 0$, there exists a $C' = C'(n, C) > 0$ such that the following holds: Let $(M, g(t), \phi(t))$ be an extended Ricci flow* (1-1) *such that*

$$S(g(s)) \ge -\frac{C}{|s|} \quad \text{and} \quad \inf_{\tau\in(0,2|s|)} \mu(g(s), \phi(s), \tau) \ge -C \tag{1-6}$$

for some $s \in [-T, 0]$, then the map

$$x \in (M, g(0)) \to f_x(s)H_x(s) \in L^1(M, \mathrm{dvol}_{g(s)})$$

is globally $C'|s|^{-\frac{1}{2}}$ Lipschitz. In particular, this means that

$$|N_{x_1}(s) - N_{x_2}(s)| \le C'|s|^{-\frac{1}{2}} d_{g(0)}(x_1, x_2).$$

Definition 1.7. For an extended Ricci flow (1-1):

(1) Given $(x, t) \in M \times [-T, 0]$ and $r > 0$, we define the parabolic ball

$$P_r(x, t) = B_r(x, t) \times [t - r^2, t].$$

(2) Given $(x, t) \in M \times [-T, 0]$ and $r > 0$, we define the regularity scale

$$r_{|\mathrm{Rm}|}(x, t) = \sup\Big\{ r > 0 : \sup_{P_r(x,t)} |\mathrm{Rm}| \le r^{-2} \Big\}.$$

Now we can state our main ε-regularity theorem.

Theorem 1.8. *For each $C > 0$, there exists $\varepsilon = \varepsilon(n, C) > 0$ such that the following holds: Let $(M, g(t), \phi(t))$ be an extended Ricci flow* (1-1) *such that*

$$S(g(s)) \ge -\frac{C}{|s|}, \qquad \inf_{\tau \in (0, 2|s|)} \mu(g(s), \phi(s), \tau) \ge -C, \qquad |\phi| \le C, \tag{1-7}$$

for some $s \in [-T, 0]$. If the pointed entropy satisfies

$$W_{x_0}(s) \ge -\varepsilon$$

for some point x_0 in the zero time slice, then we have

$$r_{|\mathrm{Rm}|}(x_0, 0)^2 \ge \varepsilon |s|.$$

Remark. Xu [2017] considered the short time asymptotics of Nash entropy on a complete Riemannian manifold with Ricci lower bound and gave interesting applications.

The paper is organized as follows. In Section 2 we review some background and preliminaries for the conjugate heat kernel, and we also prove a Bochner formula for any space-time function along the extended Ricci flow. In Section 3 we define the W entropy and obtain its monotonicity. As an application, we derive the κ noncollapsing property for the extended Ricci flow. We also clarify the relation between the pointed W entropy and the pointed Nash entropy. In Section 4 we derive various estimates. First, we derive a gradient estimate for the positive solution to the extended Ricci flow. Together with the monotonicity of W entropy, we prove the upper bound for the heat kernel. Second, we generalize Perelman's Harnack inequality to the extended Ricci flow, and based on this we prove the lower bound for the conjugate heat kernel. In Section 5, based on the results from previous sections, we prove the Poincaré inequality and the log-Sobolev inequality along the extended Ricci flow. As one application, we prove the Gaussian concentration inequality and then obtain an integral bound for the conjugate heat kernel. In Section 6, using the Poincaré inequality, we prove the Lipschitz continuity of the pointed Nash entropy. In Section 7, we derive the ε- regularity theorem; the key ingredients are the point picking argument and the Lipschitz continuity of the pointed Nash entropy.

2. Background and preliminary

Letting $(M^n, g(t), \phi(t))$ be an extended Ricci flow (1-1), we give the following definitions.

Definition 2.1. The heat operator and its conjugate are defined by

$$\Box = \frac{\partial}{\partial t} - \Delta \quad \text{and} \quad \Box^* = -\frac{\partial}{\partial t} - \Delta + S. \tag{2-8}$$

Definition 2.2. For $x, y \in M$ and $s < t$ in $[-T, 0]$, we let $H(x, t; y, s)$ denote the conjugate heat kernel based at (x, t), i.e, the unique minimal positive solution with $\lim_{s\to t} H(x, t; y, s) = \delta_x(y)$ of the conjugate heat equation

$$\Box^*_{y,s} H(x, t; y, s) = \Big(-\frac{\partial}{\partial s} - \Delta_{y,g(s)} + S(y, s)\Big) H(x, t; y, s) = 0. \tag{2-9}$$

Lemma 2.3. *The conjugate heat equation satisfies the following properties*:

(1) $\int_M H(x, t; y, s)\,\mathrm{dvol}_{g(s)}(y) = 1$.

(2) $\int_M H(x, t; y, s)\,\mathrm{dvol}_{g(t)}(x) \le \exp(\rho(t - s))$, *where* $\rho = \|S(g(-T))^-\|_\infty$.

Proof. (1) Taking the derivative with respect to s, we get

$$\begin{aligned}\frac{d}{ds}\int_M & H(x, t; y, s)\,\mathrm{dvol}_{g(s)}(y)\\ &= \int_M \Big(\frac{\partial}{\partial s} H(x, t; y, s) - H(x, t; y, s) S(y, s)\Big)\,\mathrm{dvol}_{g(s)}(y)\\ &= \int_M \Big(\frac{\partial}{\partial s} + \Delta_{g(s),y} - S\Big) H(x, t; y, s)\,\mathrm{dvol}_{g(s)}(y) = 0.\end{aligned}$$

Due to $\lim_{s\to t} H(x, t; y, s) = \delta_x(y)$, we have $\int_M H(x, t; y, s)\,\mathrm{dvol}_{g(s)}(y) = 1$.

(2) Taking the derivative with respect to t,

$$\begin{aligned}\frac{d}{dt}\int_M & H(x, t; y, s)\,\mathrm{dvol}_{g(t)}(x)\\ &= \int_M \Big(\frac{\partial}{\partial t} H(x, t; y, s) - H(x, t; y, s) S(x, t)\Big)\,\mathrm{dvol}_{g(t)}(x)\\ &= \int_M (\Delta_{g(t),x} H(x, t; y, s) - H(x, t; y, s) S(x, t))\,\mathrm{dvol}_{g(t)}(x)\\ &\le \rho \int_M H(x, t; y, s)\,\mathrm{dvol}_{g(t)}(x).\end{aligned}$$

In the last inequality we need to use the evolution equation of S along the extended Ricci flow $\frac{\partial}{\partial t} S = \Delta S + 2|\operatorname{Sic}|^2 + 2(\Delta\phi)^2$. Applying the maximum principle, we know the minimum of S is increasing along the flow. Due to

$$\lim_{t\to s}\int_M H(x,t;y,s)\,\mathrm{dvol}_{g(t)}(x)=1,$$

we have

$$\int_M H(x,t;y,s)\,\mathrm{dvol}_{g(t)}(x)\le\exp(\rho(t-s)).\qquad\square$$

Lemma 2.4. *We have the following Bochner formula for all space-time functions u*:

$$\tfrac{1}{2}\square|\nabla u|^2=-|\nabla^2 u|^2+\langle\nabla\square u,\nabla u\rangle-\langle\nabla u,\nabla\phi\rangle^2. \tag{2-10}$$

Proof. Using the extended Ricci flow equation and the Bochner formula for function,

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}|\nabla u|^2&=\frac{\partial}{\partial t}\Big(\frac{1}{2}g^{ij}\nabla_i u\nabla_j u\Big)\\&=-\frac{1}{2}g^{ik}g^{jl}\frac{\partial}{\partial t}g_{kl}\nabla_i u\nabla_j u+g^{ij}\nabla_i\frac{\partial u}{\partial t}\cdot\nabla u\\&=\mathrm{Sic}(\nabla u,\nabla u)+\nabla\frac{\partial u}{\partial t}\cdot\nabla u,\end{aligned}$$

and

$$\frac{1}{2}\Delta|\nabla u|^2=|\nabla^2 u|^2+\langle\nabla\Delta u,\nabla u\rangle+\mathrm{Ric}(\nabla u,\nabla u),$$

so

$$\begin{aligned}\frac{1}{2}\square|\nabla u|^2&=-|\nabla^2 u|^2+\langle\nabla\square u,\nabla u\rangle+\mathrm{Sic}(\nabla u,\nabla u)-\mathrm{Ric}(\nabla u,\nabla u)\\&=-|\nabla^2 u|^2+\langle\nabla\square u,\nabla u\rangle-\langle\nabla u,\nabla\phi\rangle^2.\end{aligned}\qquad\square$$

3. Monotonicity of entropy

Theorem 3.1. *Along the extended Ricci flow*

$$\begin{cases}\dfrac{\partial}{\partial t}g(t)=-2\,\mathrm{Sic}(g(t))=-2\mathrm{Ric}(g(t))+2d\phi(t)\otimes d\phi(t),\\ \dfrac{\partial}{\partial t}\phi(t)=\Delta_{g(t)}\phi(t),\\ \dfrac{\partial f}{\partial t}=-\Delta f+|\nabla f|^2-S(g(t))+\frac{n}{2\tau},\\ \dfrac{d\tau}{dt}=-1,\quad t\in[-T,0],\end{cases}$$

we have

$$\begin{aligned}&\frac{d}{dt}W(g(t),\phi(t),f(t),\tau(t))\\&\quad=2\tau\int_M\Big(|\,\mathrm{Sic}+\nabla^2 f-\frac{1}{2\tau}g|^2+(\Delta\phi-\langle\nabla f,\nabla\phi\rangle)^2\Big)(4\pi\tau)^{-\frac{n}{2}}e^{-f}\,\mathrm{dvol}_{g(t)}.\end{aligned}$$

Define $\mu(g, \phi, \tau) = \inf\{W(g, \phi, f, \tau) \mid \int_M (4\pi\tau)^{-\frac{n}{2}} e^{-f} \, \mathrm{dvol}_g = 1\}$; by the above theorem, we have the following monotonicity fact.

Corollary 3.2. *For any fixed $t_0 \in \mathbb{R}$, the quantity $\mu(g(t), \phi(t), t_0 - t)$ is nondecreasing in t. It is constant if and only if the flow is isometric to the gradient Ricci harmonic soliton with potential $f(t)$.*

Corollary 3.3. *Given $t_0 \in \mathbb{R}$, put $\mu_0 = \mu(g(-T), \phi(-T), t_0 + T)$ and $\tau = t_0 - t$. Then,*

$$\int_M u^2 \log u^2 \, \mathrm{dvol}_{g(t)} \leq \tau \int_M (4|\nabla u|^2 + S u^2) \, \mathrm{dvol}_{g(t)} - \frac{n}{2} \log(4\pi\tau) - n - \mu_0$$

for any $u \in C_0^\infty(M)$ with $\int_M u^2 \, \mathrm{dvol}_{g(t)} = 1$.

Proof. Recall $W(g, \phi, f, \tau) = \int_M [\tau(S + |\nabla f|^2) + f - n](4\pi\tau)^{-\frac{n}{2}} e^{-f} \, \mathrm{dvol}_g$, let $u^2 = (4\pi\tau)^{-\frac{n}{2}} e^{-f}$, then

$$W(g, \phi, f, \tau) = \int_M [\tau(S u^2 + 4|\nabla u|^2) - u^2 \log u^2] \, \mathrm{dvol}_g - \frac{n}{2} \log(4\pi\tau) - n.$$

By Theorem 3.1, we have

$$W(g(-T), \phi(-T), f(-T), \tau(-T)) \leq W(g(t), \phi(t), f(t), \tau(t))$$

for $t \in [-T, t_0]$. Hence

$$\mu_0 \leq \int_M [\tau(S u^2 + 4|\nabla u|^2) - u^2 \log u^2] \, \mathrm{dvol}_{g(t)} - \frac{n}{2} \log(4\pi\tau) - n. \qquad \square$$

Next we prove the κ noncollapsed property for the extended Ricci flow (1-1).

Theorem 3.4. *Fix $t_0 \in [-T, 0]$, $x \in M$ and $r > 0$, and assume*

$$\inf_{\rho \in (0, r)} \mu(g(-T), \phi(-T), t_0 + T + \rho^2) \geq -C \quad \textit{and} \quad \sup_{B_r(x,t)} S(g(t_0)) \leq C r^{-2}.$$

Define $\kappa = \exp(-(2^{n+4} + 2C))$, then $\mathrm{Vol}(B_r(x, t_0)) \geq \kappa r^n$.

Proof. Given $\rho \in (0, r)$, define the function ψ as follows,

$$\begin{cases} \psi = 1 & \text{on } B_{\frac{\rho}{2}}(x, t_0), \\ \psi = 0 & \text{outside } B_\rho(x, t_0), \\ \psi \text{ is linear} & \text{on } B_\rho(x, t_0) \setminus B_{\frac{\rho}{2}}(x, t_0). \end{cases}$$

Denoting $\mu_0 = \mu(g(-T), \phi(-T), t_0 + T + \rho^2)$ and $\tau(t) = t_0 - t + \rho^2$, applying the previous corollary to $u = \psi/\|\psi\|_2$, we have

$$\begin{aligned}\rho^2 \int_M (4|\nabla u|^2 + Su^2)\, \mathrm{dvol}_{g(t_0)} - \frac{n}{2}\log(4\pi\rho^2) - n - \mu_0 \\ \leq \rho^2 \int_M \left(\frac{4|\nabla\psi|^2}{\|\psi\|_2^2} + S\frac{\psi^2}{\|\psi\|_2^2}\right) \mathrm{dvol}_{g(t_0)} - \frac{n}{2}\log(4\pi\rho^2) - n + C \\ \leq \frac{16}{\mathrm{Vol}(B_{\frac{\rho}{2}}(x,t_0))}\big(\mathrm{Vol}(B_\rho(x,t_0)) - \mathrm{Vol}(B_{\frac{\rho}{2}}(x,t_0))\big) + 2C - \frac{n}{2}\log(4\pi\rho^2) - n.\end{aligned}$$

In the above calculation we use $|\nabla\psi| \leq \frac{2}{\rho}$ on $B_\rho(x,t_0) \setminus B_{\frac{\rho}{2}}(x,t_0)$ and $\|\psi\|_2^2 \geq \mathrm{Vol}(B_{\frac{\rho}{2}}(x,t_0))$.

On the other hand,

$$\begin{aligned}\int_M u^2 \log u^2\, \mathrm{dvol}_{g(t_0)} &= \int_{B_{t_0}(x,\rho)} \frac{\psi^2}{\|\psi\|_2^2} \log\frac{\psi^2}{\|\psi\|_2^2}\, \mathrm{dvol}_{g(t_0)} \\ &\geq \log \int_{B_\rho(x,t_0)} \left(\frac{\psi^2}{\|\psi\|_2^2}\right)^2 \mathrm{dvol}_{g(t_0)} \geq \log\frac{1}{\mathrm{Vol}(B_\rho(x,t_0))},\end{aligned}$$

where in the first inequality we use the following Cauchy–Schwarz inequality:

$$\left(\int_{B_\rho(x,t_0)} \psi^2\, \mathrm{dvol}_{g(t_0)}\right)^2 \leq \left(\int_{B_\rho(x,t_0)} \psi^4\, \mathrm{dvol}_{g(t_0)}\right) \mathrm{Vol}(B_\rho(x,t_0)).$$

So

$$\log\frac{1}{\mathrm{Vol}(B_\rho(x,t_0))} \leq 16\left(\frac{\mathrm{Vol}(B_\rho(x,t_0))}{\mathrm{Vol}(B_{\frac{\rho}{2}}(x,t_0))} - 1\right) + 2C - \frac{n}{2}\log(4\pi\rho^2) - n.$$

It is easy to get the implication

$$\mathrm{Vol}(B_{\frac{\rho}{2}}(x,t_0)) \geq \kappa\left(\frac{\rho}{2}\right)^n \Rightarrow \mathrm{Vol}(B_\rho(x,t_0)) \geq \kappa\rho^n$$

for $\kappa = \exp(-(2^{n+4} + 2C))$. Then the claim follows by induction on ρ. □

Remark. Müller [2010] considered more general geometric flows which include Ricci flow, extended Ricci flow and Harmonic Ricci flow as special cases. In the same paper, he introduced more general reduced volume in analogy to Perelman's [2002] and proved its monotonicity. In [Müller 2012], he systematically studied the harmonic Ricci flow and proved some important fundamental estimates, in particular, he proved the κ noncollapsing result along harmonic Ricci flow. The W entropy for general geometric flow was discussed by Guo, Philipowski and Thalmaier [Guo et al. 2013], and they also proved the monotonicity of W entropy.

Proposition 3.5. *The following hold for $x \in M$ and $s \in [-T, 0]$.*

(1) $\lim_{s\to 0} W_x(s) = 0$.

(2) $\mu(g(-T), \phi(-T), T) \le W_x(s) \le 0$.

(3) $W_x(s) = -\int_s^0 2|r| \big(\int_M |\operatorname{Sic} + \nabla^2 f_x - \frac{g}{2|r|}|^2 + (\Delta\phi - \langle \nabla\phi, \nabla f_x \rangle)^2 \big)\, d\nu_x(r)\, dr$.

Proof. Recall $W_x(s) = W(g(s), \phi(s), f_x(x), |s|)$; (1) follows from the asymptotic expansion of the heat kernel at x. For (2) and (3),

$$\frac{d}{ds} W_x(s) = 2|s| \int_M \Big(|\operatorname{Sic} + \nabla^2 f_x - \frac{g}{2|s|}|^2 + (\Delta\phi - \langle \nabla\phi, \nabla f_x \rangle)^2 \Big)\, d\nu_x(s),$$

so

$$\begin{aligned} \mu(g(-T), \phi(-T), T) &\le W(g(-T), \phi(-T), f_x(-T), T) \\ &\le W(g(s), \phi(s), f_x(s), |s|) = W_x(s). \end{aligned}$$

After integrating, we get

$$W_x(s) = -\int_s^0 2|r| \int_M \left(\left| \operatorname{Sic} + \nabla^2 f_x - \frac{g}{2|r|} \right|^2 + (\Delta\phi - \langle \nabla\phi, \nabla f_x \rangle)^2 \right) d\nu_x(r)\, dr. \quad \square$$

Proposition 3.6. *The following hold for $x \in M$ and $s \in [-T, 0]$.*

(1) $W_x(s) \le N_x(s) \le 0$.

(2) $\dfrac{d}{ds} N_x(s) = \dfrac{1}{|s|}(N_x(s) - W_x(s)) \ge 0$.

(3) $N_x(s) = -\displaystyle\int_M \log H_x(s)\, d\nu_x(s) - \frac{n}{2}(\log(4\pi|s|) + 1) = \int_M f_x(s)\, d\nu_x(s) - \frac{n}{2}$.

(4) $N_x(s) = -\displaystyle\int_s^0 2|r| \Big(1 - \frac{r}{s}\Big) \int_M \left(\left| \operatorname{Sic} + \nabla^2 f_x - \frac{g}{2|r|} \right|^2 + (\Delta\phi - \langle \nabla\phi, \nabla f_x \rangle)^2 \right) d\nu_x(r) dr$.

Proof. (1) By the definition of $N_x(s)$ and the monotonicity of $W_x(s)$,

$$N_x(s) - W_x(s) = \frac{1}{|s|} \int_s^0 (W_x(r) - W_x(s))\, dr \ge 0.$$

(2) By direct calculation,

$$\begin{aligned} \frac{d}{ds} N_x(s) &= \frac{d}{ds} \left(\frac{1}{|s|} \int_s^0 W_x(r)\, dr \right) \\ &= \frac{1}{s^2} \int_s^0 W_x(r)\, dr + \frac{1}{s} W_x(s) = \frac{1}{|s|}(N_x(s) - W_x(s)). \end{aligned}$$

(3) Suppose $u(y, l) = H(x, 0; y, l)$, then u solves the conjugate heat equation $\frac{\partial u}{\partial l} + \Delta_{g(l)} u(y, l) - S(y, l) u(y, l) = 0$. Let $\tau(l) = -l$; by direct calculation,

$$\frac{d}{dl} \left(-l \int_M u \log u \operatorname{dvol} \right) = W(g(l), \phi(l), f_x(l), |l|) + n + \frac{n}{2} \log(4\pi|l|).$$

Integrating from s to 0,

$$s\int_M H_x(s)\log H_x(s)\,\mathrm{dvol}=\int_s^0 W_x(l)\,dl+\int_s^0\Big(n+\frac{n}{2}\log(4\pi|l|)\Big)\,dl,$$

hence

$$N_x(s)=-\int_M \log H_x(s)\,d\nu_x(s)-\frac{n}{2}(\log(4\pi|s|)+1)=\int_M f_x(s)\,d\nu_x(s)-\frac{n}{2},$$

where in the last equality we use $H_x(s)=(4\pi|s|)^{-\frac{n}{2}}e^{-f_x}$.
(4) Due to $\frac{d}{dl}N_x(l)=\frac{1}{|l|}(N_x(l)-W_x(l))$, we have $\frac{d}{dl}(lN_x(l))=W_x(l)$. Integrating from s to 0, we get

$$\begin{aligned}&-sN_x(s)\\&\quad=\int_s^0 W_x(r)\,dr\\&\quad=-\int_s^0\int_r^0 2|\tau|\int_M\Big(\Big|\mathrm{Sic}+\nabla^2 f_x-\frac{g}{2|\tau|}\Big|^2+(\Delta\phi-\langle\nabla f_x,\nabla\phi\rangle)^2\Big)\,d\nu_x(\tau)d\tau dr\\&\quad=-\int_s^0 2|r|(r-s)\int_M\Big(\Big|\mathrm{Sic}+\nabla^2 f_x-\frac{g}{2|r|}\Big|^2+(\Delta\phi-\langle\nabla f_x,\nabla\phi\rangle)^2\Big)\,d\nu_x(r)dr\end{aligned}$$

so

$$\begin{aligned}&N_x(s)=\\&\quad-\int_s^0 2|r|\Big(1-\frac{r}{s}\Big)\int_M\Big(\Big|\mathrm{Sic}+\nabla^2 f_x-\frac{g}{2|r|}\Big|^2+(\Delta\phi-\langle\nabla f_x,\nabla\phi\rangle)^2\Big)\,d\nu_x(r)dr.\quad\square\end{aligned}$$

4. Heat kernel estimate

At first we prove a gradient estimate for the heat equation along the extended Ricci flow (1-1).

Lemma 4.1. *Suppose u is a positive solution to the forward heat equation with a family of metrics evolving under the extended Ricci flow on* $[0, T]$, *then*

$$\frac{|\nabla u(x,t)|}{u(x,t)}\le\sqrt{\frac{1}{t}}\sqrt{\log\frac{A}{u(x,t)}}$$

for $A=\sup_{M\times[0,T]}u$ *and* $(x,t)\in M\times[0,T]$.

Proof. By direct calculation,

$$\frac{\partial}{\partial t}\Big(u\log\frac{A}{u}\Big)=\frac{\partial u}{\partial t}\log\frac{A}{u}-\frac{\partial u}{\partial t},$$
$$\Delta\Big(u\log\frac{A}{u}\Big)=\Delta u\log\frac{A}{u}-\Delta u-\frac{|\nabla u|^2}{u},$$

which, combined with the heat equation, gives

$$\Box\Big(u\log\frac{A}{u}\Big)=\frac{|\nabla u|^2}{u}.$$

Using the flow equation (1-1), we get

$$\frac{\partial}{\partial t}\frac{|\nabla u|^2}{u}=\frac{2\,\mathrm{Sic}(\nabla u,\nabla u)+2\nabla\frac{\partial u}{\partial t}\cdot\nabla u}{u}-\frac{\frac{\partial u}{\partial t}\cdot|\nabla u|^2}{u^2},$$
$$\Delta\frac{|\nabla u|^2}{u}=\frac{\Delta|\nabla u|^2}{u}-\frac{4\nabla^2u(\nabla u,\nabla u)}{u^2}+|\nabla u|^2\cdot\frac{2|\nabla u|^2-u\cdot\Delta u}{u^3}.$$

Combined with the Bochner formula, this gives

$$\Box\frac{|\nabla u|^2}{u}=\frac{-2\langle\nabla u,\nabla\phi\rangle^2}{u}-\frac{2}{u}|\nabla^2u-\frac{du\otimes du}{u}|^2.$$

Consider the quantity $t\frac{|\nabla u|^2}{u}-u\log\frac{A}{u}$,

$$\Box\Big(t\frac{|\nabla u|^2}{u}-u\log\frac{A}{u}\Big)=t\Big(\frac{-2\langle\nabla\phi,\nabla u\rangle^2}{u}-\frac{2}{u}|\nabla^2u-\frac{du\otimes du}{u}|^2\Big)\le 0.$$

By the maximum principle,

$$t\frac{|\nabla u|^2}{u}-u\log\frac{A}{u}\le 0,$$

so $|\nabla u|^2/u^2\le\frac{1}{t}\log\frac{A}{u}$. □

Now based on Corollary 3.3, we can use Davies's method to derive the L^∞ estimate for the heat kernel.

Theorem 4.2. *Define* $\rho=\|S(g(-T))^-\|_\infty$ *and*

$$\mu=\inf_{\tau\in(0,2T)}\mu(g(-T),\phi(-T),\tau).$$

Suppose $u:M\times[t_1,t_2]\to\mathbb{R}^+$ *with* $[t_1,t_2]\subseteq[-T,0]$ *is a positive solution to* $\frac{\partial u}{\partial s}=\Delta_{g(s)}u$, *then we have*

$$\|u(s)\|_\infty\le(4\pi(s-t_1))^{-\frac{n}{2}}\exp(\rho(s-t_1)-\mu)\|u(t_1)\|_1.$$

Proof. Given the flow and heat equation

$$\begin{cases}\dfrac{\partial g}{\partial t}=-2\,\mathrm{Sic}(g(t))=-2\mathrm{Ric}(g(t))+2d\phi(t)\otimes d\phi(t),\\ \dfrac{\partial}{\partial t}\phi(t)=\Delta_{g(t)}\phi(t),\\ \dfrac{\partial}{\partial t}u(t)=\Delta_{g(t)}u(t),\end{cases}$$

and letting $p(t)=(s-t_1)/(s-t)$, $t\in[t_1,s]$, with $p(t_1)=1$ and $p(s)=\infty$,

$$
\begin{aligned}
&\frac{d}{dt}\|u(t)\|_{p(t)}\\
&\quad=\frac{d}{dt}\left(\int_M u(t)^{p(t)}\,\mathrm{dvol}_{g(t)}\right)^{\frac{1}{p(t)}}\\
&\quad=-\frac{p'(t)}{p(t)^2}\|u(t)\|_{p(t)}\log\int_M u(t)^{p(t)}\,\mathrm{dvol}_{g(t)}+\frac{1}{p(t)}\left(\int_M u(t)^{p(t)}\,\mathrm{dvol}_{g(t)}\right)^{\frac{1}{p(t)}-1}\\
&\qquad\times\left[\int_M u(t)^{p(t)}p'(t)\log u(t)\,\mathrm{dvol}_{g(t)}+\int_M u(t)^{p(t)-1}(p(t)\Delta u-Su)\,\mathrm{dvol}_{g(t)}\right].
\end{aligned}
$$

Integrating by parts and multiplying by $p(t)^2\|u(t)\|_{p(t)}^{p(t)}$ gives

$$
\begin{aligned}
&p(t)^2\|u(t)\|_{p(t)}^{p(t)}\cdot\frac{\partial}{\partial t}\|u(t)\|_{p(t)}\\
&\quad=-p'(t)\|u(t)\|_{p(t)}^{p(t)+1}\log\int_M u(t)^{p(t)}\,\mathrm{dvol}_{g(t)}\\
&\qquad+p(t)\|u(t)\|_{p(t)}p'(t)\int_M u(t)^{p(t)}\log u(t)\,\mathrm{dvol}_{g(t)}\\
&\qquad-p(t)^2(p(t)-1)\|u(t)\|_{p(t)}\int_M u(t)^{p(t)-2}|\nabla u|^2\,\mathrm{dvol}_{g(t)}\\
&\qquad-p(t)\|u(t)\|_{p(t)}\int_M Su(t)^{p(t)}\,\mathrm{dvol}_{g(t)}.
\end{aligned}
$$

Dividing by $\|u(t)\|_{p(t)}$ on both sides,

$$
\begin{aligned}
&p(t)^2\|u(t)\|_{p(t)}^{p(t)}\cdot\frac{\partial}{\partial t}\log\|u(t)\|_{p(t)}\\
&\quad=-p'(t)\|u(t)\|_{p(t)}^{p(t)}\log\int_M u(t)^{p(t)}\,\mathrm{dvol}_{g(t)}\\
&\qquad+p(t)p'(t)\int_M u(t)^{p(t)}\log u(t)\,\mathrm{dvol}_{g(t)}\\
&\qquad-4(p(t)-1)\int_M|\nabla u^{p(t)/2}|^2\,\mathrm{dvol}_{g(t)}-p(t)\int_M S(u^{p(t)/2})^2\,\mathrm{dvol}_{g(t)}.
\end{aligned}
$$

Define $v=u^{p(t)/2}/\|u^{p(t)/2}\|_2$, then $\|v\|_2=1$ and $v^2\log v^2=p(t)v^2\log u-2v^2\log\|u^p(t)/2\|_2$. So

$$
\begin{aligned}
&p(t)^2\frac{\partial}{\partial t}\log\|u(t)\|_{p(t)}\\
&\quad=p'(t)\int_M v^2\log v^2\,\mathrm{dvol}_{g(t)}-4(p(t)-1)\int_M\left(|\nabla v|^2+\tfrac{1}{4}Sv^2\right)\mathrm{dvol}_{g(t)}\\
&\qquad-\int_M Sv^2\,\mathrm{dvol}_{g(t)}
\end{aligned}
$$

$$= p'(t)\left[\int_M v^2 \log v^2 \,\mathrm{dvol}_{g(t)} - \frac{p(t)-1}{p'(t)} \int_M (4|\nabla v|^2 + Sv^2)\,\mathrm{dvol}_{g(t)}\right] - \int_M Sv^2 \,\mathrm{dvol}_{g(t)}$$

$$\le p'(t)\left[-\frac{n}{2}\log(4\pi)\frac{(t-t_1)(s-t)}{s-t_1} - n - \mu\right] + \rho.$$

Observe $p'(t)/p(t)^2 = 1/(s-t_1)$, hence

$$\frac{\partial}{\partial t}\log\|u(t)\|_{p(t)} \le \frac{1}{s-t_1}\left[-\frac{n}{2}\log(4\pi)\frac{(t-t_1)(s-t)}{s-t_1} - n - \mu\right] + \rho.$$

Integrating from t_1 to s with respect to t, we get

$$\log\frac{\|u(s)\|_\infty}{\|u(t_1)\|_1} \le -\frac{n}{2}\log(4\pi(s-t_1)) - \mu + (s-t_1)\rho,$$

so

$$\|u(s)\|_\infty \le (4\pi(s-t_1))^{-\frac{n}{2}}\exp(\rho(s-t_1)-\mu)\|u(t_1)\|_1. \qquad \square$$

Corollary 4.3. *Given any $C > 0$, there exists a $C' = C'(n, C) > 0$ such that if $S(g(-s)) \ge -\frac{C}{|s|}$ and $\inf_{\tau\in(0,2|s|)}\mu(g(s), \phi(s), \tau) \ge C$, if we denote $H(x, 0; y, s) = (4\pi|s|)^{-\frac{n}{2}}\exp(-f_x(y, s))$, then*

$$|\nabla_x f_x|^2 \le \frac{C'}{|s|}(C' + f_x)$$

at $(x, 0)$.

Proof. Fix y, s and let $u(x, t) = H(x, t; y, s)$, then u satisfies $\frac{\partial u}{\partial t} = \Delta_{g(t)}u$. Applying Theorem 4.2 we get

$$A = \sup_{[\frac{s}{2},0]\times M} u \le C'|s|^{-\frac{n}{2}},$$

then by Lemma 4.1 with $[t_1, t_2] = \left[\frac{s}{2}, 0\right]$,

$$|\nabla_x f_x|^2 = \frac{|\nabla u|^2}{u^2} \le \frac{1}{|s|/2}\log\frac{A}{u} = \frac{2}{|s|}(\log A - \log u)$$

$$\le \frac{2}{|s|}\left(\log C' - \frac{n}{2}\log|s| + \frac{n}{2}\log(4\pi|s|) + f_x\right) \le \frac{C'}{|s|}(C' + f_x). \qquad \square$$

Based on Perelman's Harnack inequality [Perelman 2002], Zhang [2012] obtained the lower bound for the heat kernel along Ricci flow, which can be used to derive the κ noninflating property for Ricci flow. Next we generalize Perelman's Harnack inequality to the extended Ricci flow.

Theorem 4.4. *Let* $u = u(y, s) = H(x, t; y, s),\ s < t$, *and* f *be defined by* $u = (4\pi(t-s))^{-\frac{n}{2}} e^{-f}$. *Denote* $\tau = t - s$ *and let* $P = P(u)$ *be defined as*

$$\begin{aligned} P &= [\tau(2\Delta f - |\nabla f|^2 + S) + f - n]u \\ &= \tau\Big(-2\Delta u + \frac{|\nabla u|^2}{u} + Su\Big) - u\log u - \frac{n}{2}\log(4\pi\tau) - nu; \end{aligned}$$

then $P \le 0$. *Moreover, for any smooth curve* $c = c(s)$ *on* M,

$$-\frac{d}{ds} f(c(s), s) \le \frac{1}{2}(S(c(s), s) + |c'(s)|^2) - \frac{1}{2(t-s)} f(c(s), s).$$

Proof. By Lemma 6 in [Guo et al. 2015a], we know $P \le 0$, so

$$(t-s)(2\Delta f - |\nabla f|^2 + S) + f - n \le 0.$$

Since u solves the conjugate heat equation, we have

$$\frac{\partial f}{\partial s} = -\Delta f + |\nabla f|^2 - S + \frac{n}{2(t-s)}.$$

Combining the above two equations, we get

$$\frac{\partial f}{\partial s} + \frac{1}{2}S - \frac{1}{2}|\nabla f|^2 - \frac{f}{2(t-s)} \ge 0.$$

On the other hand,

$$-\frac{d}{ds} f(c(s), s) = -\frac{\partial f}{\partial s} - \langle \nabla f, c'(s)\rangle \le -\frac{\partial f}{\partial s} + \frac{1}{2}|\nabla f|^2 + \frac{1}{2}|c'(s)|^2.$$

The desired inequality follows from adding the last two inequalities. □

Remark. In [Cao et al. 2015], the authors considered the Harnack estimate for the conjugate heat kernel of general geometric flow, and our Theorem 4.4 is a special case of [Cao et al. 2015, Theorem 1.2]. This kind of estimate is used to derive smooth convergence of the conjugate heat kernel for our particular flow; for general geometric flow in [Cao et al. 2015], we don't even know the convergence of the flow.

Next we introduce the reduced length and prove a bound of the heat kernel which will be used in the following sections. Let $x, y \in M$, $0 \le s < t < T$ and consider a smooth curve $\gamma : [s, t] \to M$ connecting (y, s) and (x, t), i.e., $\gamma(s) = y$ and $\gamma(t) = x$. Its L length is defined as

$$L(\gamma) = \int_s^t \sqrt{t-\sigma}\,(|\gamma'(\sigma)|^2_{g(\sigma)} + S(\gamma(\sigma), \sigma))\, d\sigma.$$

The reduced distance between (x, t) and (y, s) is defined as

$$l_{(x,t)}(y, s) = \frac{1}{2\sqrt{t-s}} \inf\{L(\gamma) : \gamma : [s, t] \to M \text{ between } (y, s) \text{ and } (x, t)\}.$$

Choose $\gamma(\sigma):[s,t]\to M$ to be the L geodesic between (y,s) and (x,t); from Theorem 4.4 we know

$$-\frac{d}{d\sigma}((t-\sigma)^{\frac{1}{2}}f(\gamma(\sigma),\sigma))\le\frac{1}{2}(t-\sigma)^{\frac{1}{2}}(S(\gamma(\sigma),\sigma)+|\gamma'(\sigma)|^2).$$

Integrating from s to t, we have

$$(t-s)^{\frac{1}{2}}f(\gamma(s),s)\le\frac{1}{2}\int_s^t\sqrt{t-\sigma}(S(\gamma(\sigma),\sigma)+|\gamma'(\sigma)|^2)\,d\sigma,$$

hence

$$f(y,s)\le l_{(x,t)}(y,s),$$

i.e.,

$$H(x,t;y,s)\ge(4\pi(t-s))^{-\frac{n}{2}}e^{-l_{(x,t)}(y,s)}. \tag{4-11}$$

Now we are in a position to prove the lower bound of the heat kernel.

Theorem 4.5. *Define* $\rho=\|S(g(-T))^-\|_\infty,\ \mu=\inf_{\tau\in(0,2T)}\mu(g(-T),\phi(-T),\tau)$. *Denote* $\tau=t-s$ *for* $s<t$ *in* $[-T,0]$, *then we have*

$$H(x,t;y,s)\ge(8\pi\tau)^{-\frac{n}{2}}\exp\biggl(-\frac{4d(x,y,t)^2}{\tau}-\frac{1}{\sqrt{\tau}}\int_s^t\sqrt{t-\sigma}S(y,\sigma)\,d\sigma-\rho\tau+\mu\biggr).$$

Proof. Let $u(y,s)=H(y,t;y,s),\ s<t$, then u solves the conjugate heat equation $-\frac{\partial u}{\partial s}-\Delta_{g(s)}u+Su=0$. Define a function f by $u(y,s)=(4\pi\tau)^{-\frac{n}{2}}e^{-f(y,s)}$; we need to use Theorem 4.4. Picking the curve $c(s)$ to be the fixed point, we have

$$-\frac{\partial f}{\partial s}\le\frac{1}{2}S(y,s)-\frac{1}{2\tau}f(y,s).$$

For any $s_2<s_1<t$, we integrate the above inequality to get

$$f(y,s_2)\sqrt{t-s_2}\le f(y,s_1)\sqrt{t-s_1}+\frac{1}{2}\int_{s_2}^{s_1}\sqrt{t-\sigma}S(y,\sigma)\,d\sigma.$$

When s_1 approaches t, $f(y,s_1)$ stays bounded because $H(y,t;y,s)(t-s)^{\frac{n}{2}}$ is bounded between two positive constants. Hence for $s\le t$, we have

$$f(y,s)\le\frac{1}{2\sqrt{\tau}}\int_s^t\sqrt{t-\sigma}S(y,\sigma)\,d\sigma,$$

so,

$$H(y,t;y,s)\ge\frac{1}{(4\pi\tau)^{\frac{n}{2}}}e^{-\frac{1}{2\sqrt{\tau}}\int_s^t\sqrt{t-\sigma}S(y,\sigma)\,d\sigma}.$$

Next we will use the gradient estimate from Lemma 4.1 to get the lower bound for $H(x,t;y,s)$. Define $v(x,l)=H(x,l;y,s)$, then v satisfies $\frac{\partial v}{\partial l}=\Delta_{g(l)}v(l)$. On the interval $\left[\frac{t+s}{2}, t\right]$, applying Theorem 4.2 , we get

$$\|v(l)\|_\infty \le \left(4\pi \cdot \frac{t-s}{2}\right)^{-\frac{n}{2}} e^{\rho(t-s)-\mu} := A.$$

By Lemma 4.1, we have

$$\left|\nabla\sqrt{\log\frac{A}{v(x,t)}}\right| \le \frac{1}{\sqrt{\frac{1}{2}t-s}},$$

hence

$$\sqrt{\log\frac{A}{v(x,t)}} \le \sqrt{\log\frac{A}{v(y,t)}} + \frac{d(x,y,t)}{\sqrt{\frac{1}{2}t-s}}.$$

Using Cauchy–Schwarz,

$$\log\frac{A}{v(x,t)} \le \log\left(\frac{A}{v(y,t)}\right)^2 + \frac{4d(x,y,t)^2}{t-s}.$$

So

$$\begin{aligned} v(x,t) &\ge A^{-1}v(y,t)^2 e^{-\frac{4d(x,y,t)^2}{t-s}} \\ &\ge (2\pi(t-s))^{\frac{n}{2}} e^{-\rho(t-s)+\mu}(4\pi\tau)^{-n} e^{-\frac{1}{\sqrt{t-s}}\int_s^t \sqrt{t-\sigma}\,S(y,\sigma)\,d\sigma} e^{-\frac{4d(x,y,t)^2}{t-s}} \\ &= (8\pi\tau)^{-\frac{n}{2}} \exp\left(-\frac{4d(x,y,t)^2}{t-s} - \frac{1}{\sqrt{t-s}}\int_s^t \sqrt{t-\sigma}\,S(y,\sigma)\,d\sigma - \rho\tau + \mu\right). \quad \square \end{aligned}$$

5. Log-Sobolev inequality and Gaussian concentration

Consider a smooth metric probability space $(M, g, d\nu)$, where $d\nu = e^{-h}dv_g$. If the so-called Bakry–Émery condition,

$$\mathrm{Ric} + \nabla^2 h \ge \tfrac{1}{2}g,$$

is satisfied, a celebrated theorem [Bakry and Émery 1985] asserts that $(M, g, d\nu)$ satisfies a logarithmic Sobolev inequality with a definite constant. More precisely, this means that for every smooth function v with compact support and $\int_M v^2\,d\nu = 1$,

$$\int_M v^2 \log v^2\,d\nu \le 4\int_M |\nabla v|^2\,d\nu.$$

Since the work of [Bakry and Émery 1985] there has been plenty of work on the characterization of the Ricci curvature bound using the log-Sobolev inequality; see [Bakry and Ledoux 2006; Cheng and Thalmaier 2018; Naber 2013]. The above log-Sobolev inequality has many important applications, for example, see [Carrillo and Ni 2009; Munteanu and Wang 2012; Wu and Zhang 2017].

With the same spirit, in this section we will prove Theorem 1.2, i.e., the Poincaré inequality (1-2) and the log-Sobolev inequality (1-3). As applications, we obtain Theorem 1.3 and Corollary 1.4.

In the following argument, for simplicity, we use $d\nu$ to denote $d\nu_{x_0}(y,s)$. We can rewrite the Poincaré inequality (1-2) and the log-Sobolev inequality (1-3) in the following way. For any $u \in C_0^\infty(M)$ with $u \geq 0$ in the second case,

$$\int_M u^2\,d\nu - \left(\int_M u\,d\nu\right)^2 \leq 2|s| \int_M |\nabla u|^2\,d\nu, \tag{5-12}$$

$$\int_M u \log u\,d\nu - \left(\int_M u\,d\nu\right) \log\left(\int_M u\,d\nu\right) \leq |s| \int_M \frac{|\nabla u|^2}{u}\,d\nu. \tag{5-13}$$

Theorem 1.2 can be derived using a similar idea to that in [Hein and Naber 2014], where the gradient estimate can be obtained by applying the "heat kernel homotopy" principle [Bakry and Ledoux 2006]. Given $s \leq t$ in $[-T, 0]$, we define $P_{st}u$ as

$$P_{st}u(x) = \int_M u(y) H(x,t;y,s)\,\mathrm{dvol}_{g(s)}(y).$$

Note that when s is fixed, $P_{st}u$ satisfies the heat equation.

Lemma 5.1. *If $U(t)$ are smooth functions on $M \times [-T, 0]$, then*

$$\frac{d}{dt} P_{t0}U(t) = P_{t0}\Box_t U(t).$$

Proof. When x, t are fixed, $H(x,t;y,s)$ satisfies the conjugate heat equation.

$$\begin{aligned}
\frac{d}{dt} P_{t0}U(t) &= \int_M \frac{\partial}{\partial t} U(y,t) H(x,0;y,t)\,\mathrm{dvol}_{g(t)}(y) \\
&\qquad + \int_M U(y,t)\left(\frac{\partial}{\partial t} - S(y,t)\right) H(x,0;y,t)\,\mathrm{dvol}_{g(t)}(y) \\
&= \int_M \frac{\partial}{\partial t} U(y,t) H(x,0;y,t)\,\mathrm{dvol}_{g(t)}(y) \\
&\qquad - \int_M U(y,t)\Delta_y H(x,0;y,t)\,\mathrm{dvol}_{g(t)}(y) \\
&= \int_M \left(\frac{\partial}{\partial t} U(y,t) - \Delta_y U(y,t)\right) H(x,0;y,t)\,\mathrm{dvol}_{g(t)}(y) \\
&= \int_M \Box_t U(y,t) H(x,0;y,t)\,\mathrm{dvol}_{g(t)}(y) = P_{t0}\Box_t U(t).
\end{aligned}$$

□

Lemma 5.2. *Let $u \in C_0^\infty(M)$ and $u(t) = P_{st}u$ so that $\Box_t u(t) = 0$. Suppose h and ψ are two smooth functions from $\mathbb{R}$ to $\mathbb{R}$.*

(1) *If $U(t) = h(u(t))$, then $\Box_t U(t) = -h''(u)|\nabla u(t)|^2_{g(t)}$.*

(2) *If* $U(t)=\psi(u(t))|\nabla u(t)|^2_{g(t)}$, *then*

$$\Box_t U(t)=-2\psi(u)|\nabla^2 u|^2-2\psi(u)\langle\nabla\phi,\nabla u\rangle^2-\psi''(u)|\nabla u|^4-4\psi'(u)\nabla^2 u(\nabla u,\nabla u).$$

Proof. (1) $\Box_t U(t)=(\frac{\partial}{\partial t}-\Delta)h(u(t))=h'\frac{\partial u}{\partial t}-(h''|\nabla u|^2+h'\Delta u)=-h''(u)|\nabla u|^2$.

(2) Using the Bochner formula (2-10), we get

$$\Box_t|\nabla u|^2=-2|\nabla^2 u|^2-2\langle\nabla u,\nabla\phi\rangle^2.$$

So

$$\begin{aligned}\Box_t(\psi(u)|\nabla u|^2)&=\Box_t\psi(u)|\nabla u|^2+\psi(u)\Box_t|\nabla u|^2-2\langle\nabla\psi(u),\nabla|\nabla u|^2\rangle\\&=-2\psi(u)|\nabla^2 u|^2-2\psi(u)\langle\nabla\phi,\nabla u\rangle^2\\&\qquad-\psi''(u)|\nabla u|^4-4\psi'(u)\nabla^2 u(\nabla u,\nabla u).\quad\Box\end{aligned}$$

Now we are going to prove the Poincaré inequality and the log-Sobolev inequality. Note that

$$\text{(5-14)}\qquad \begin{aligned}\int_M h(u)\,d\nu-h\Big(\int_M u\,d\nu\Big)&=-\int_s^0\frac{d}{dt}P_{t0}(h(P_{st}u))\,dt\\&=\int_s^0 P_{t0}(h''(P_{st}u)|\nabla P_{st}u|^2_{g(t)})\,dt.\end{aligned}$$

***Proof of the Poincaré inequality* (5-12).** Pick $h=x^2$; by (5-14) we have

$$\int_M u^2\,d\nu-\Big(\int_M u\,d\nu\Big)^2=2\int_s^0 P_{t0}(|\nabla P_{st}u|^2_{g(t)})\,dt.$$

Using Lemma 5.1 and Lemma 5.2,

$$\begin{aligned}\frac{\partial}{\partial r}P_{rt}(|\nabla P_{sr}u|^2_{g(r)})&=P_{rt}\Box_r(|\nabla P_{sr}u|^2_{g(r)})\\&=P_{rt}\big(-2|\nabla^2 P_{sr}u|^2_{g(r)}-2\langle\nabla\phi,\nabla P_{sr}u\rangle^2\big).\end{aligned}$$

Integrating from s to t with respect to r,

$$|\nabla P_{st}u|^2_{g(t)}=P_{st}(|\nabla u|^2_{g(s)})-2\int_s^t P_{rt}(|\nabla^2 P_{sr}u|^2_{g(r)})\,dr-2\int_s^t P_{rt}(\langle\nabla\phi,\nabla P_{sr}u\rangle^2)\,dr,$$

so

$$\begin{aligned}\int_M u^2\,d\nu-\Big(\int_M u\,d\nu\Big)^2&=2\int_s^0 P_{t0}(|\nabla P_{st}u|^2_{g(t)})\,dt\le 2\int_s^0 P_{t0}P_{st}(|\nabla u|^2_{g(s)})\,dt\\&=2\int_s^0 P_{s0}|\nabla u|^2_{g(s)}\,dt=2|s|\int_M|\nabla u|^2_{g(s)}\,d\nu_{x_0}(s).\end{aligned}$$

It is easy to see that equality holds if and only if $\nabla^2 P_{sr}u\equiv 0$, i.e., u is constant. $\Box$

***Proof of the log-Sobolev inequality* (5-13).** Pick $h = x\log x$; by (5-14), we obtain

$$\int_M u\log u\,d\nu - \left(\int_M u\,d\nu\right)\log\left(\int_M u\,d\nu\right) = \int_s^0 P_{t0}\left(\frac{|\nabla P_{st}u|^2_{g(t)}}{P_{st}u}\right)dt.$$

Using Lemmas 5.1 and 5.2,

$$\begin{aligned}\frac{\partial}{\partial r}P_{rt}(P_{sr}u|\nabla\log P_{sr}u|^2_{g(r)}) &= P_{rt}\Box_r(P_{sr}u|\nabla\log P_{sr}u|^2_{g(r)})\\ &= P_{rt}\Box_r\left(\frac{|\nabla P_{sr}u|^2}{P_{sr}u}\right) = -2P_{rt}\left[P_{sr}u\left(|\nabla^2\log P_{sr}u|^2 + \frac{\langle\nabla\phi,\nabla P_{sr}u\rangle^2}{(P_{sr}u)^2}\right)\right],\end{aligned}$$

and integrating from s to t with respect to r,

$$\frac{|\nabla P_{st}u|^2_{g(t)}}{P_{st}u} = P_{st}\left(\frac{|\nabla u|^2_{g(s)}}{u}\right) - 2\int_s^t P_{rt}\left[P_{sr}u\left(|\nabla^2\log P_{sr}u|^2 + \frac{\langle\nabla\phi,\nabla P_{sr}u\rangle^2}{(P_{sr}u)^2}\right)\right]dr.$$

So

$$\begin{aligned}\int_M u\log u\,d\nu - \left(\int_M u\,d\nu\right)\log\left(\int_M u\,d\nu\right) &= \int_s^0 P_{t0}\left(\frac{|\nabla P_{st}u|^2_{g(t)}}{P_{st}u}\right)dt\\ &\le \int_s^0 P_{t0}P_{st}\left(\frac{|\nabla u|^2_{g(s)}}{u}\right)dt = \int_s^0 P_{s0}\left(\frac{|\nabla u|^2_{g(s)}}{u}\right)dt = |s|\int_M \frac{|\nabla u|^2_{g(s)}}{u}\,d\nu_{x_0}(s).\end{aligned}$$

One sees that equality holds if and only if $\nabla^2\log P_{sr}u \equiv 0$, i.e., u is constant. □

Next we will use the log-Sobolev inequality to derive Theorem 1.3, where the proof follows from standard theory in the metric measure space.

Proof of Theorem 1.3. Choose any $F \in C^\infty(M)$ with

(5-15)
$$\int_M F\,d\nu = 0, \quad |\nabla F| \le 1.$$

Define $U(\lambda) = \frac{1}{\lambda}\log\int_M e^{\lambda F}\,d\nu$, then

$$\lim_{\lambda\to 0} U(\lambda) = \lim_{\lambda\to 0}\frac{\int_M e^{\lambda F}F\,d\nu}{\int_M e^{\lambda F}\,d\nu} = \int_M F\,d\nu = 0.$$

Applying the log-Sobolev inequality to $u^2 = \frac{e^{\lambda F}}{\int_M e^{\lambda F}\,d\nu}$,

$$\int_M\left(\frac{e^{\lambda F}}{\int_M e^{\lambda F}\,d\nu}\log\frac{e^{\lambda F}}{\int_M e^{\lambda F}\,d\nu}\right)d\nu \le 4|s|\int_M \frac{\frac{\lambda^2}{4}e^{\lambda F}|\nabla F|^2}{\left(\int_M e^{\lambda F}\,d\nu\right)^2}d\nu,$$

so

$$\int_M e^{\lambda F}\,d\nu\int_M e^{\lambda F}\left(\lambda F - \log\int_M e^{\lambda F}\right)d\nu \le |s|\lambda^2\int_M e^{\lambda F}|\nabla F|^2\,d\nu \le |s|\lambda^2\int_M e^{\lambda F}\,d\nu,$$

i.e.,

$$\int_M e^{\lambda F}\left(\lambda F - \log \int_M e^{\lambda F}\right) d\nu \leq |s|\lambda^2.$$

Hence,

$$\begin{aligned}\frac{d}{d\lambda}U = \frac{d}{d\lambda}\left(\frac{1}{\lambda}\log \int_M e^{\lambda F}\,d\nu\right) &= \frac{-1}{\lambda^2}\log \int_M e^{\lambda F}\,d\nu + \frac{1}{\lambda}\frac{\int_M e^{\lambda F} F\,d\nu}{\int_M e^{\lambda F}\,d\nu}\\ &= \frac{1}{\lambda^2}\frac{1}{\int_M e^{\lambda F}\,d\nu}\left(-\int_M e^{\lambda F}\,d\nu \log \int_M e^{\lambda F}\,d\nu + \lambda \int_M e^{\lambda F} F\,d\nu\right) \leq |s|.\end{aligned}$$

In the last inequality we use $\int_M e^{\lambda F}\,d\nu \geq 1$ because $\log \int_M e^{\lambda F}\,d\nu \geq \int_M \lambda F\,d\nu = 0$.

Combining $\frac{d}{d\lambda}U \leq |s|$ and $\lim_{\lambda\to 0} U(\lambda) = 0$, we obtain

$$\int_M e^{\lambda F}\,d\nu \leq e^{|s|\lambda^2}$$

because any F satisfies (5-15).

Define $G(y) = d_{g(s)}(y, B)$ and $F = G - \int_M G\,d\nu$, then

$$\int_A e^{\lambda F(y_1)}\,d\nu(y_1) \leq \int_M e^{\lambda F(y_1)}\,d\nu \leq e^{|s|\lambda^2},$$

and

$$\int_B e^{-\lambda F(y_2)}\,d\nu(y_2) \leq \int_M e^{-\lambda F(y_2)}\,d\nu \leq e^{|s|\lambda^2}.$$

So

$$e^{\lambda d_{g(s)}(A,B)}\nu(A)\nu(B) \leq \int_B \int_A e^{\lambda(F(y_1)-F(y_2))}\,d\nu(y_1)d\nu(y_2) \leq e^{2|s|\lambda^2},$$

i.e.,

$$\nu(A)\nu(B) \leq e^{2|s|\lambda^2 - \lambda d_{g(s)}(A,B)}.$$

Because

$$2|s|\lambda^2 - \lambda d_{g(s)}(A, B) = 2|s|\left(\lambda - \frac{d_{g(s)}(A, B)}{4|s|}\right)^2 - \frac{d_{g(s)}(A, B)^2}{8|s|} \geq -\frac{d_{g(s)}(A, B)^2}{8|s|},$$

we get

$$\nu(A)\nu(B) \leq \exp\left(-\frac{d_{g(s)}(A, B)^2}{8|s|}\right). \qquad \square$$

Remark. Given $x_1,\ x_2 \in M$, take $A = B_r(x_1, s)$ and $B = B_r(x_2, s)$, where $r^2 = |s|$. Applying the above theorem to $d\nu = d\nu_{x_2}$,

$$\text{(5-16)}\quad \int_{B_r(x_1,s)} H(x_2, 0; y, s)\,\mathrm{dvol}_{g(s)}(y) \le \frac{1}{\nu_{x_2}(B_r(x_2, s))} \exp\biggl(-\frac{d_{g(s)}(B_r(x_1, s),\, B_r(x_2, s))^2}{8|s|}\biggr),$$

due to

$$d_{g(s)}(x_1, x_2) \le d_{g(s)}(B_r(x_1, s), (B_r(x_2, s))) + 2r,$$

hence

$$\tfrac{1}{2} d_{g(s)}(x_1, x_2)^2 \le d_{g(s)}(B_r(x_1, s),\, B_r(x_2, s))^2 + 4|s|.$$

So

$$\int_{B_r(x_1,s)} H(x_2, 0; y, s)\,\mathrm{dvol}_{g(s)}(y) \le \frac{C}{\nu_{x_2}(B_r(x_2, s))} \exp\biggl(-\frac{d_{g(s)}(x_1, x_2)^2}{C|s|}\biggr).$$

Together with Perelman's κ noncollapsing property, this can be used to derive certain upper bounds of the heat kernel [Wu $\ge$ 2019].

Proof of Corollary 1.4. Apply Theorem 1.3 with $x_0 = x_2,\ A = B_r(x_1, s)$ and $B = B_r(x_2, s)$. Using Theorem 4.5, we obtain

$$\text{(5-17)}\quad \inf_{B_r(x_2,s)} H(x_2, 0; y, s) \ge \frac{1}{C'} |s|^{-\frac{n}{2}}.$$

Due to the evolution equation of volume along (1-1),

$$\frac{d}{dt} \mathrm{Vol}_{g(t)}(B_r(x_2, 0)) = -\int_{B_r(x_2,0)} S(y, t)\,\mathrm{dvol}_{g(t)}(y) \le \frac{C}{|s|} \mathrm{Vol}_{g(t)}(B_r(x_2, 0)),$$

and integrating from s to 0 with respect to t, by Theorem 3.4 we have

$$\text{(5-18)}\quad \mathrm{Vol}_{g(s)}(B_r(x_2, 0)) \ge \frac{1}{C'} \mathrm{Vol}_{g(0)}(B_r(x_2, 0)) \ge \frac{1}{C'} r^n.$$

Combining (5-17), (5-18) and $\mathrm{Vol}_{g(s)}(B_r(x_1, 0)) \ge \frac{1}{C'} r^n$, we get

$$\frac{1}{\mathrm{Vol}_{g(s)}(B_r(x_1, 0))} \int_{B_r(x_1,0)} H(x_2, 0, y, s)\,\mathrm{dvol}_{g(s)}(y) \le C'|s|^{-\frac{n}{2}} \exp\biggl(-\frac{d_{g(s)}(B_r(x_1, 0),\, B_r(x_2, 0))}{C'|s|}\biggr).$$

From Theorem 4.5 again, we have

$$\inf_{B_r(x_1,0)} H(x_2, 0; y, s) \ge \frac{1}{C'} |s|^{-\frac{n}{2}} \exp\biggl(-\frac{d_{g(0)}(x_1, x_2)^2}{C'|s|}\biggr),$$

and combining this with (1-4), (1-5) follows.

6. Lipschitz continuity of pointed Nash entropy

Recall the pointed Nash entropy at $(x_0, s) \in M \times [-T, 0]$ is defined as

$$N_{x_0}(s) = \frac{1}{|s|}\int_s^0 W_{x_0}(r)\,dr = \int_M f_{x_0}(s)\,d\nu_{x_0}(s) - \frac{n}{2}.$$

Based on the Poincaré inequality (1-2) in Theorem 1.2, we can prove the Lipschitz continuity of the pointed Nash entropy.

Proof of Theorem 1.6. Define $F(x) = f_x(s)H_x(s)$, then

$$\begin{aligned}\|F(x_1) - F(x_2)\| &= \|f_{x_1}(s)H_{x_1}(s) - f_{x_2}(s)H_{x_2}(s)\|\\ &= \int_M |f_{x_1}(s)H_{x_1}(s) - f_{x_2}(s)H_{x_2}(s)|\,\mathrm{dvol}_{g(s)}(y)\\ &\le \int_M \int_0^{d_{g(0)}(x_1,x_2)} |\nabla_{\gamma(t)}(f_{\gamma(t)}H_{\gamma(t)})|\,dt\,\mathrm{dvol}_{g(s)}(y)\\ &= \int_0^{d_{g(0)}(x_1,x_2)} \int_M |\nabla_{\gamma(t)}(f_{\gamma(t)}H_{\gamma(t)})|\,\mathrm{dvol}_{g(s)}(y)dt\\ &\le \sup_{x\in M} \int_M |\nabla_x(f_xH_x)|\,\mathrm{dvol}_{g(s)} \cdot d_{g(0)}(x_1, x_2),\end{aligned}$$

where $\gamma(t)$ is a unit speed geodesic connecting x_1 and x_2 with respect to g(0). All we need to do is to estimate the integral,

$$\begin{aligned}\int_M |\nabla_x(f_xH_x)|\,\mathrm{dvol}_{g(s)}(y) &= \int_M |\nabla_x f_x H_x - f_x H_x \nabla_x f_x|\,\mathrm{dvol}_{g(s)}(y)\\ &= \int_M |\nabla_x f_x - f_x\nabla_x f_x|\,d\nu_x(s) \le \|\nabla_x f_x\|_2(1 + \|f_x\|_2).\end{aligned}$$

From the gradient estimate in Corollary 4.3, we know

$$|\nabla_x f_x|^2 \le \frac{C'}{|s|}(C' + f_x).$$

Hence

$$\int_M |\nabla_x(f_xH_x)|\,\mathrm{dvol}_{g(s)}(y) \le C'|s|^{-\frac{1}{2}}(1 + \|f_x\|_2^2) \le C'|s|^{-\frac{1}{2}},$$

where in the last inequality we use (3) from Theorem 6.1. □

Theorem 6.1. *Under the assumption* (1-6), *the following holds for* $f_x(s)$.

(1) $\int_M f_x\,d\nu \in [\frac{n}{2} - C, \frac{n}{2}]$.

(2) $\int_M |\nabla f_x|^2\,d\nu \le (\frac{n}{2} + C)\frac{1}{|s|}$.

(3) $\int_M |f_x|^2\,d\nu \le (n + 2 + C)^2$. *here we use* $d\nu$ *to denote* $d\nu_x(y, s)$.

Proof. (1) Applying Propositions 3.5 and 3.6, we have

$$-C \le \mu(g(s), \phi(s), |s|) \le W_x(s) \le N_x(s),$$

so $\int_M f_x \, dv \in \left[\frac{n}{2} - C, \frac{n}{2}\right]$.

(2) Recall $W_x(s) = \int_M (|s|(S + |\nabla f_x|^2) + f_x - n) \, dv$ and $N_x(s) = \int_M f_x(s) \, dv - \frac{n}{2}$, so

$$W_x(s) - N_x(s) = \int_M |s|(S + |\nabla f_x|^2) \, dv - \frac{n}{2} \le 0,$$

hence $\int_M (S + |\nabla f_x|^2) \, dv \le \frac{n}{2|s|}$.

(3) Applying the Poincaré inequality (1-2), we have

$$\begin{aligned}\int_M f_x^2 \, dv &\le 2|s| \int_M |\nabla f_x|^2 \, dv + \left(\int_M f_x \, dv\right)^2 \\ &\le 2|s|\left(\frac{n}{2} + c\right)\frac{1}{|s|} + \max\left\{\left(\frac{n}{2} - C\right)^2, \left(\frac{n}{2}\right)^2\right\} \le (n + C + 2)^2. \quad \square\end{aligned}$$

7. Proof of ε-regularity theorem

In this section we prove the ε-regularity theorem. In order to do that, we quote the derivative estimate to be used.

Lemma 7.1 [List 2008]. *Let* $(M^n, g(t), \phi(t))$ *be an extended Ricci flow* (1-1) *on* $M \times [0, T)$ *with initial data* (g_0, ϕ_0), *and assume* $\sup |\phi_0| \le C$, *then for all* $t > 0$,

$$\inf_{x \in M} \phi_0(x) \le \phi(x, t) \le \sup_{x \in M} \phi_0(x),$$
$$\sup_{x \in M} |\nabla \phi|^2(x, t) \le C^2 t^{-1}.$$

Proposition 7.2 [List 2008]. *Let* $(M^n, g(t), \phi(t))$ *be an extended Ricci flow* (1-1). *Fix* $x_0 \in M$ *and* $r > 0$, *if*

$$\sup_{B_r(x_0, s)} r^2 |\operatorname{Rm}| \le \widetilde{C}.$$

Denote $\Phi = (\operatorname{Rm}, \nabla^2 \phi)$, *then the derivatives of* Φ *satisfy the inequality for all* $m \ge 0$, *and for all* $t \in (0, s]$ *the estimate*

$$\sup_{B_{r/2}(x_0, t)} |\nabla^m \Phi|^2 \le C(n, m)\widetilde{C}(r^{-2} + t^{-1})^{m+2}$$

holds, where $C = C(n, m)$ *is a constant only depending on n and m.*

Next we prove a more restricted version of Theorem 1.8, whose proof is based on the point picking argument as in [Anderson 1990]. Once we have this, Theorem 1.8 can be derived using the Lipschitz continuity of the pointed Nash entropy in

Theorem 1.6. In the following argument, for simplicity, we define

$$t(y) = -\min\{T, r_{|\mathrm{Rm}|}(y,0)^2\}.$$

Theorem 7.3. *There exists an* $\varepsilon = \varepsilon(n, C) > 0$ *such that if*

$$N_y(t(y)) \geq -\varepsilon, \quad \textit{for all } y \in B_\delta(x,0),$$

where $0 < \delta \leq \sqrt{T}$, *then*

$$r_{|\mathrm{Rm}|}(y,0) \geq \varepsilon \cdot d_{g(0)}(y, \partial B_\delta(x,0)), \quad \textit{for all } y \in B_\delta(x,0).$$

Proof. Without loss of generality, we assume $\delta = 1 \leq T$. Suppose the contrary, then we have a sequence of the extended Ricci flow $(M_i, g_i(t), \phi_i(t))$ satisfying (1-7) and $x_i \in M_i$ such that

$$N_y(t(y)) \geq -\frac{1}{i}, \quad \text{for all } y \in B_1(x_i,0), \tag{7-19}$$

but any point y_i minimizing the quantity

$$w(y) = \frac{r_{|\mathrm{Rm}|}(y,0)}{d_{g_i(0)}(y, \partial B_1(x_i,0))}$$

satisfies $0 < w(y_i) \leq \frac{1}{i}$.

Choose any such y_i and denote $r_i = r_{|\mathrm{Rm}|}(y_i, 0)$. Consider the rescaled extended Ricci flow $(M_i, \widetilde{g}_i(t), \widetilde{\phi}_i(t))$, where

$$\widetilde{g}_i(t) = \frac{1}{r_i^2} g_i(r_i^2 t), \quad \widetilde{\phi}_i(t) = \phi_i(r_i^2 t), \quad t \in \left[-\frac{1}{r_i^2}, 0\right].$$

Clearly $r_{|\mathrm{Rm}|}(y_i, \widetilde{g}_i(0)) = 1$ and

$$d_i = \frac{1}{2} d_{\widetilde{g}_i(0)}(y_i, \partial B_{\frac{1}{r_i}}(x_i, \widetilde{g}_i(0))) \geq \frac{i}{2}.$$

Because y_i minimizes w,

$$r_{|\mathrm{Rm}|}(y, \widetilde{g}_i(0)) \geq \frac{1}{2}, \quad \text{for all } y \in B_{d_i}(y_i, \widetilde{g}_i(0)). \tag{7-20}$$

This curvature bound, together with the assumption above and the κ noncollapsing property, implies that

$$\mathrm{Vol}_{\widetilde{g}_i(0)}(B_1(y,0)) \geq \kappa(n,C).$$

So we have a uniform curvature bound on $P_{1/4}(y, \widetilde{g}_i(0))$ for any $y \in B_{d_i}(y_i, \widetilde{g}_i(0))$. Then we have the smooth convergence

$$(M_i, \widetilde{g}_i(t), (y_i, 0)) \to (M_\infty, g_\infty(t), (y_\infty, 0)).$$

The limit is completely defined on $[-\frac{1}{16}, 0]$ and is of bounded curvature.

Now we have the heat kernel bound

$$(4\pi|t|)^{-\frac{n}{2}}\exp(-l_{(\widetilde{y}_i,0)}(y,t))\le H(\widetilde{y}_i,0;y,t)\le C'(n,C)|t|^{-\frac{n}{2}}.$$

The lower bound is due to (4-11) and the upper bound is due to Theorem 4.2.

As before, we write $H(\widetilde{y}_i,0;y,t)=(4\pi|t|)^{-\frac{n}{2}}\exp(-f_i(y,t))$. By Lemma 4.1, the gradient of $H(\widetilde{y}_i,0;y,t)$ is uniformly bounded on any compact domain; (7-20) implies higher order derivatives of $H(\widetilde{y}_i,0;y,t)$ are also bounded, so the $f_i(y,t)$ converge to $f_\infty(y,t)$ smoothly on any compact subset. Because $|\widetilde{\phi}_i|\le C$, by Lemma 7.1 and Proposition 7.2 we know the various order derivatives of $\widetilde{\phi}_i$ are uniformly bounded. Equation (7-19) together with (4) in Proposition 3.6 gives

$$\int_{-\frac{1}{16}}^{0}2|t|(1-16|t|)\int_{M_i}\left(\left|\operatorname{Sic}(\widetilde{g}_i)+\nabla^2 f_i-\frac{\widetilde{g}_i}{2|t|}\right|^2+(\Delta\widetilde{\phi}_i-\langle\nabla f_i,\nabla\widetilde{\phi}_i\rangle)^2\right)d\nu_{\widetilde{y}_i(t)}dt\le\frac{1}{i}.$$

Letting $i\to\infty$, we see f_∞ satisfies

$$\begin{cases}\operatorname{Sic}(g_\infty)+\nabla^2 f_\infty-\frac{g_\infty}{2|t|}=0,\\ \Delta\phi_\infty=\langle\nabla\phi_\infty,\nabla f_\infty\rangle.\end{cases}\tag{7-21}$$

Because $|\widetilde{\phi}_i|\le C$, after blowing up, $\widetilde{\phi}_i\to\phi_\infty=const$, so (7-21) can be simplified to

$$\operatorname{Ric}(g_\infty)+\nabla^2 f_\infty-\frac{g_\infty}{2|t|}=0,$$

which is nonflat and of bounded curvature on $\left[-\frac{1}{16},0\right]$. This is impossible, because the curvature at time t is $\frac{1}{16|t|}$ times the curvature at time $-\frac{1}{16}$, and hence tends to infinity as $t\to 0$. □

Proof of Theorem 1.8. Define $\varepsilon=\min\{\frac{1}{2}\varepsilon_{7.3},\frac{\varepsilon_{7.3}^2}{2C'}\}$ and $\delta=\frac{|s|^{\frac{1}{2}}\varepsilon_{7.3}}{2C'}$, where C' is the constant from Theorem 1.6 and $\varepsilon_{7.3}$ is the constant from Theorem 7.3. Assume $W_{x_0}(s)\ge-\varepsilon$, so $N_{x_0}(s)\ge-\varepsilon$. By Theorem 1.6, we have for any $x\in B_\delta(x_0,0)$,

$$N_x(s)\ge N_{x_0}(s)-C'|s|^{-\frac{1}{2}}d(x_0,x)\ge-\varepsilon_{7.3}.$$

Then we can apply Theorem 7.3 to get $r_{|\operatorname{Rm}|}(x_0,0)\ge\varepsilon_{7.3}\delta\ge\varepsilon|s|^{\frac{1}{2}}$. □

Acknowledgments

Wu would like to thank Professor Xianzhe Dai, Professor Guofang Wei and Professor Xi-nan Ma for their constant encouragement. Wu is supported by the National Natural Science Foundation of China (grant no. 11701516) and the Scientific Research Foundation of Zhejiang Sci-Tech University (grant no. 17062066-Y). Zheng is supported by National Natural Science Foundation of China (grant no. 11671141).

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Received January 20, 2018. Revised May 15, 2018.

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PACIFIC JOURNAL OF MATHEMATICS

Volume 298 No. 2 February 2019

0030-8730(201902)298:2;1-Q