Pacific Journal of Mathematics

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Volume 298 No. 2 February 2019

PACIFIC JOURNAL OF MATHEMATICS
Vol. 298, No. 2, 2019
dx.doi.org/10.2140/pjm.2019.298.257

UNIQUENESS QUESTIONS FOR C^*-NORMS ON GROUP RINGS

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We provide a large class of discrete amenable groups for which the complex group ring has several C*-completions, thus providing partial evidence towards a positive answer to a question raised by Rostislav Grigorchuk, Magdalena Musat and Mikael Rørdam.

1. Introduction

The interplay between group theory and operator algebras dates back to the seminal papers by Murray and von Neumann [1936] and by choosing different completions of a discrete countable group Γ one obtains interesting analytic objects; for instance the Banach algebra $\ell^1(\Gamma)$, the full and reduced C^*-algebras $\mathrm{C}^*(\Gamma)$ and $\mathrm{C}^*_{\mathrm{r}}(\Gamma)$, and the group von Neumann algebra $L\Gamma$. In general there are many norms on, say, $\ell^1(\Gamma)$ such that the completion with respect to this norm gives a C^*-algebra, and the question of when the C^*-completion is unique (in which case Γ is said to be C^*-*unique*) has been studied by various authors [Leung and Ng 2004; Boidol 1984; Barnes 1983]. A C^*-unique discrete group is evidently amenable and it is, to the best of the authors' knowledge, an open question whether the converse is true, although it is known to be false in the more general context of locally compact groups [Leung and Ng 2004]. More recently, the paper [Grigorchuk et al. 2018] put emphasis on the question of when the complex group algebra $\mathbb{C}\Gamma$ has a unique C^*-completion. As is easily seen [Grigorchuk et al. 2018, Proposition 6.7], if Γ is locally finite (i.e., if every finitely generated subgroup is finite) then $\mathbb{C}\Gamma$ has a unique C^*-completion, and [Grigorchuk et al. 2018, Question 6.8] asks if the converse is true. The present paper provides partial evidence towards a positive answer to this, in that we prove that the following classes of nonlocally finite groups have several C^*-completions.

Theorem A (see Proposition 2.4 and Corollary 3.7). *The class of countable groups Γ for which $\mathbb{C}\Gamma$ does not have a unique C^*-norm includes the following*:

(i) *Infinite groups of polynomial growth.*

MSC2010: 16S34, 46L05, 46L10.
Keywords: group rings, C*-norms, the Atiyah conjecture.

(ii) *Torsion free, elementary amenable groups with a nontrivial, finite conjugacy class.*

(iii) *Groups with a central element of infinite order.*

The key to the proof of (i) and (ii) is the so-called strong Atiyah conjecture (see Section 3A), which predicts a concrete restriction on the von Neumann dimension of kernels of elements in the complex group algebra under the left regular representation — notably these are predicted to be either zero or one if the group in question is torsion free.

2. Basic results on C*-uniqueness

In what follows, all discrete groups are implicitly assumed to be at most countable. We will use several operator algebras associated to a discrete group Γ: the maximal C*-algebra $\mathrm{C}^*(\Gamma)$, the reduced C*-algebra $\mathrm{C}^*_{\mathrm{r}}(\Gamma)$ and the von Neumann algebra $L\Gamma$. For more information on these, we refer to [Brown and Ozawa 2008, §2.5]. We recall that $L\Gamma = (\lambda(\mathbb{C}\Gamma))'' \subset \mathbb{B}(\ell^2\Gamma)$ is generated by the left regular representation $\lambda : \mathbb{C}\Gamma \to \mathbb{B}(\ell^2\Gamma)$ and carries a canonical, faithful, normal trace given by $\tau(x) = \langle x\delta_e, \delta_e\rangle$. In what follows, tr will denote the normalized trace on $\mathbb{M}_n(\mathbb{C})$ while Tr will denote the nonnormalized trace.

We begin by formally introducing the notion of C*-uniqueness. In order to avoid a notational conflict with the already existing notions studied in [Leung and Ng 2004; Boidol 1984], we emphasize that we are investigating the uniqueness of C*-norms on the complex group algebra in contrast to the ℓ^1-algebra.

Definition 2.1. Let Γ be a discrete group. $\mathbb{C}\Gamma$ is said to be

(i) C*-unique if it carries a unique C*-norm;

(ii) $\mathrm{C}^*_{\mathrm{r}}$-unique if no C*-norm on $\mathbb{C}\Gamma$ is properly majorised by the reduced C*-norm.

Γ is said to be algebraically C*-unique if $\mathbb{C}\Gamma$ is C*-unique, and it is said to be $\mathrm{C}^*_{\mathrm{r}}$-unique if $\mathbb{C}\Gamma$ is $\mathrm{C}^*_{\mathrm{r}}$-unique.

Amenable groups are characterized by the property that the maximal and reduced C*-algebras coincide, and thus a nonamenable group is never algebraically C*-unique; on the other hand, for amenable groups the above notions coincide. Note also that the class of C*-simple groups, which has recently received a lot of attention [Breuillard et al. 2017; Le Boudec 2017], falls within the class of algebraically $\mathrm{C}^*_{\mathrm{r}}$-unique groups. As already mentioned in the introduction, algebraic C*-uniqueness appeared in the recent paper [Grigorchuk et al. 2018] in which the authors observed that locally finite groups have this property and asked if this characterizes the class of locally finite groups. Below we prove a few basic permanence results regarding algebraic C*-uniqueness, but before doing so we give an alternative characterization,

which is a straightforward algebraic adaptation of the similar result for ℓ^1-algebras [Barnes 1983, Proposition 2.4].

Lemma 2.2. *Let Γ be a discrete group. Then $\mathbb{C}\Gamma$ is C^*-unique (respectively, C^*_r-unique) if and only if every nontrivial closed, two-sided ideal in $\mathrm{C}^*(\Gamma)$ (respectively, $\mathrm{C}^*_\mathrm{r}(\Gamma)$) intersects $\mathbb{C}\Gamma$ nontrivially.*

Proof. We give the proof for the statement about algebraic C^*-uniqueness; the other case is obtained by replacing $\mathrm{C}^*(\Gamma)$ by $\mathrm{C}^*_\mathrm{r}(\Gamma)$ throughout the proof. Assume that there is a nontrivial ideal $J \trianglelefteq \mathrm{C}^*(\Gamma)$ intersecting $\mathbb{C}\Gamma$ trivially and denote by $q : \mathrm{C}^*(\Gamma) \to \mathrm{C}^*(\Gamma)/J$ the quotient map. Composing q with the inclusion $\mathbb{C}\Gamma \hookrightarrow \mathrm{C}^*(\Gamma)$ yields a faithful representation of $\mathbb{C}\Gamma$ and it defines a C^*-norm on it that is properly majorised by the maximal norm by nontriviality of J. Conversely, if there is a C^*-norm on $\mathbb{C}\Gamma$ which is properly majorised by the norm coming from $\mathrm{C}^*(\Gamma)$, then $\mathrm{C}^*(\Gamma)$ surjects onto the corresponding quotient, and the kernel of this surjection is a nontrivial ideal intersecting $\mathbb{C}\Gamma$ trivially. □

Corollary 2.3. *Let Γ and Λ be discrete groups. If $\mathbb{C}(\Gamma \times \Lambda)$ is C^*-unique (respectively, C^*_r-unique), then so are $\mathbb{C}\Gamma$ and $\mathbb{C}\Lambda$.*

Proof. Let $J \trianglelefteq \mathrm{C}^*_\mathrm{r}(\Gamma)$ be a nontrivial ideal intersecting $\mathbb{C}\Gamma$ trivially. Then

$$J \otimes_{\max} \mathrm{C}^*_\mathrm{r}(\Lambda) \trianglelefteq \mathrm{C}^*(\Gamma) \otimes_{\max} \mathrm{C}^*(\Lambda) = \mathrm{C}^*(\Gamma \times \Lambda)$$

is a nontrivial ideal intersecting $\mathbb{C}(\Gamma \times \Lambda) = \mathbb{C}\Gamma \otimes_{\mathrm{alg}} \mathbb{C}\Lambda$ trivially. The same proof with C^* replaced by C^*_r and $\otimes_{\max}$ replaced by $\otimes_{\min}$ works for the reduced case. □

Proposition 2.4. *If Γ is a discrete group with a central element of infinite order then $\mathbb{C}\Gamma$ is not C^*_r-unique.*

Proof. Denote by Z the subgroup in Γ generated by a central element of infinite order. Then $\mathrm{C}^*_\mathrm{r}(Z) \cong C(S^1)$ and $LZ \cong L^\infty(S^1)$ via the Fourier transform and we denote by $p \in LZ$ the projection corresponding to the characteristic function of the upper half-circle $\{e^{i\theta} : \theta \in [0, \pi]\}$. Define $\pi := \lambda_\Gamma p$; i.e., the left regular representation of Γ restricted to the invariant subspace $p\ell^2(\Gamma)$. Choosing a nonzero function $f \in C(S^1)$ supported in the *lower* half-circle we obtain a nonzero element $x \in \mathrm{C}^*_\mathrm{r}(Z) \subset \mathrm{C}^*_\mathrm{r}(\Gamma)$ with $xp = 0$ and hence the norm on $\mathrm{C}^*_\mathrm{r}(\Gamma)$ induced by π is not the one induced by λ_Γ. We now only need to show that π is faithful on $\mathbb{C}\Gamma$. To this end, consider the trace-preserving conditional expectation $E : L\Gamma \to LZ$ [Brown and Ozawa 2008, Lemma 1.5.11] and assume that $a \in \mathbb{C}\Gamma$ is in the kernel of π. Then a^*a is also in the kernel of π and since E is an LZ-bimodule map [Brown and Ozawa 2008, Proposition 1.5.7], we get

$$0 = E(\lambda_\Gamma(a^*a)p) = E(\lambda_\Gamma(a^*a))p.$$

However, $E(\mathbb{C}\Gamma) \subset \mathbb{C}Z \simeq \mathrm{Pol}(z, \bar{z}) \subset C(S^1)$ and therefore $E(\lambda_\Gamma(a^*a)) = 0$ and

since E is trace-preserving and the trace on $L\Gamma$ is faithful we conclude that a^*a, and hence a, is zero. □

Corollary 2.5. *An abelian group is algebraically* C*-*unique if and only if it is locally finite* (*i.e.*, *pure torsion*).

Remark 2.6. The result in Corollary 2.5 was also observed, independently and with different proofs, by Rostislav Grigorchuk, Magdalena Musat and Mikael Rørdam (unpublished).

Remark 2.7. The class of locally finite groups has many stability properties — for instance it is closed under subgroups, quotients and extensions and, moreover, being virtually locally finite is the same as being locally finite. However, verifying these properties for the class of C*-unique groups seems to be a much bigger challenge.

3. The strong Atiyah conjecture and C^*_r-uniqueness

3A. ***The strong Atiyah conjecture.*** The key to our main result is the so-called strong Atiyah conjecture which is briefly described in the following. A good general reference is [Lück 2002, Chapter 10] where all of the results below can be found, and to which we also refer for the original references. Let Γ be a discrete group and denote by $1/|\operatorname{FIN}(\Gamma)|\,\mathbb{Z}$ the additive subgroup in $\mathbb{Q}$ generated by the set

$$\left\{\frac{1}{|\Lambda|} : \Lambda \leqslant \Gamma \text{ a finite subgroup}\right\}.$$

Given a matrix $A \in \mathbb{M}_n(\mathbb{C}\Gamma)$ we denote by $L_A \in L\Gamma \otimes \mathbb{M}_n(\mathbb{C}) \subset \mathbb{B}(\ell^2(\Gamma)^n)$ the bounded operator given by left multiplication with A (via the left regular representation of Γ). The strong Atiyah conjecture then predicts that

$$\dim_{L\Gamma} \ker(L_A) := (\tau \otimes \operatorname{Tr})(P_{\ker L_A}) \in \frac{1}{|\operatorname{FIN}(\Gamma)|}\mathbb{Z}.$$

Here $\dim_{L\Gamma}(-)$ denotes the von Neumann dimension of the (right) Hilbert $L\Gamma$-module $\ker(L_A)$ defined as the nonnormalized trace of the kernel projection $P_{\ker L_A}$; see [Lück 2002] for details on this. It should be noted that the strong Atiyah conjecture is false in general [Lück 2002, Theorem 10.23], but is known to hold for all groups which have a bound on the order of finite subgroups and belong to Linnell's class $\mathcal{C}$ [Lück 2002, Theorem 10.19], the latter being the smallest class of groups which contain all free groups, and is closed under directed unions and extensions by elementary amenable groups (i.e., if $\Lambda \lhd \Gamma$, $\Lambda \in \mathcal{C}$ and Γ/Λ is elementary amenable, then $\Gamma \in \mathcal{C}$). The above discussion motivates the following notion.

Definition 3.1. Let Γ be a countable group. The *torsion multiplier* of Γ is defined as

$$\theta(\Gamma) = \frac{1}{\operatorname{lcm}\{|H| : H \leqslant \Gamma \text{ finite}\}} \in [0, 1].$$

In this definition, and in what follows, we use the convention that the least common multiple (lcm) of an infinite set of natural numbers is infinity and that $\frac{1}{\infty} = 0$. Note that if Γ has an upper bound on the set of finite subgroups, then

$$\frac{1}{|\mathrm{FIN}(G)|}\mathbb{Z} = \{n\theta(\Gamma) : n \in \mathbb{Z}\},$$

and $1/|\mathrm{FIN}(G)|\,\mathbb{Z}$ has 0 as an accumulation point otherwise. In view of this, the strong Atiyah conjecture for a group Γ with $\theta(\Gamma) > 0$ implies that the possible kernel dimensions are properly quantized in the sense that they can only take values in the discrete set $\{n\theta(\Gamma) : n \in \mathbb{N}\} \subset \mathbb{R}$. Theorem A (i) and (ii) will follow directly from our main technical result, Theorem 3.6. The key idea in the proof is to play the aforementioned "quantization" of the kernel dimensions against an abundance of central projections in $L\Gamma$ with small traces which provide representations of $C_r^*(\Gamma)$ with nontrivial kernels. To quantify this, we need the following definition.

Definition 3.2. The *central granularity* of Γ is defined as

$$\sigma(\Gamma) = \inf\{\tau(p) : p \in \mathrm{Proj}(Z(L\Gamma)),\ p \neq 0\} \in [0, 1].$$

We note that $\sigma(\Gamma) < 1$ if and only if $Z(L\Gamma)$ is nontrivial which is equivalent to Γ not being icc.[1] The next proposition computes the central granularity of Γ in group-theoretic terms. Recall that the FC-*centre* Γ_{fc} is the normal subgroup of Γ consisting of all elements with finite conjugacy classes.

Proposition 3.3. *Let* $\Gamma_{\mathrm{fc}} \trianglelefteq \Gamma$ *be the* FC-*centre of* Γ. *Then*

$$\sigma(\Gamma) = \frac{1}{|\Gamma_{\mathrm{fc}}|},$$

where the right-hand side is interpreted as 0 *if* $|\Gamma_{\mathrm{fc}}| = \infty$.

Proof. Γ_{fc} is an increasing union of a sequence of finitely generated normal subgroups $\Lambda_n \trianglelefteq \Gamma$; to see this, note that Γ_{fc} is clearly an increasing union of a sequence of finitely generated subgroups Λ_n', and defining Λ_n to be generated by the Γ-conjugacy classes of a finite system of generators for Λ_n' yields the desired sequence of finitely generated subgroups which are normal in Γ. We now have two cases to consider:

(i) All Λ_n are finite (equivalently, Γ_{fc} is a torsion group),

(ii) Λ_n is infinite for some n.

In case (i), setting $p_n := 1/|\Lambda_n| \sum_{g\in\Lambda_n} g$, we get a projection $p_n \in L\Gamma_{\mathrm{fc}}$ with $\tau(p_n) = 1/|\Lambda_n|$; moreover, p_n is central in $L\Gamma$ since Λ_n is normal in Γ. This proves that $\sigma(\Gamma) = 0$ if Γ_{fc} is an infinite torsion group (in this case $|\Lambda_n| \to \infty$). If Γ_{fc} is

[1] Recall that a group is *icc* if all nontrivial conjugacy classes are infinite

finite, then the sequence stabilizes, and therefore we get a central projection p in $L\Gamma$ with trace $1/|\Gamma_{\mathrm{fc}}|$. The centre of $L\Gamma$ consists of elements whose associated Fourier series in $\ell^2(\Gamma) = L^2(L\Gamma, \tau)$ are supported only on Γ_{fc} and are constant along conjugacy classes, and is therefore contained in the centre of $L\Gamma_{\mathrm{fc}}$; hence we get $1/|\Gamma_{\mathrm{fc}}| \geqslant \sigma(\Gamma) \geqslant \sigma(\Gamma_{\mathrm{fc}})$. But we also have $L\Gamma_{\mathrm{fc}} = \mathbb{C}\Gamma_{\mathrm{fc}}$ which by representation theory of finite groups is isomorphic to a direct sum of matrix algebras $\bigoplus_\pi \mathbb{M}_{d_\pi}(\mathbb{C})$ with the trace given by $\bigoplus_\pi d_\pi^2/|\Gamma_{\mathrm{fc}}|\,\mathrm{tr}$; thus, the minimal central projection has trace $1/|\Gamma_{\mathrm{fc}}| = \sigma(\Gamma_{\mathrm{fc}})$; this proves the claim.

In case (ii) we fix an $n \in \mathbb{N}$ such that $\Lambda_n =: \Lambda$ is infinite and note that since Λ is generated by a finite number of elements with finite conjugacy classes, its centralizer $C_\Gamma(\Lambda)$ is of finite index in Γ. We now claim that $L\Lambda$ has a diffuse von Neumann subalgebra and thus projections of arbitrarily small trace. This can be seen as follows: if $L\Lambda$ has a direct summand of type II_1, it is clear because such von Neumann algebras are diffuse. Otherwise $L\Lambda$ is of type I, but then Λ is virtually abelian [Lück 1997, Lemma 3.3], and hence, being infinite by assumption and finitely generated by construction, contains a copy of $\mathbb{Z}$ which generates a diffuse von Neumann algebra $L\mathbb{Z} \cong L^\infty(S^1)$. In view of the above, for an arbitrary $\varepsilon > 0$, there is a projection $p \in L\Lambda \subset L\Gamma_{\mathrm{fc}}$ of trace $\tau(p) < \varepsilon/[\Gamma : C_\Gamma(\Lambda)]$. Now let

$$q := \bigvee_{g\in\Gamma} {}^g p,$$

where ${}^g p := gpg^{-1}$. Then q is a central projection in $L\Gamma$. Moreover, p is invariant under the centralizer $C_\Gamma(\Lambda)$ and upon choosing coset representatives $g_1, \ldots, g_{[\Gamma:C_\Gamma(\Lambda)]}$ for $\Gamma/C_\Gamma(\Lambda)$ we obtain

$$q = \bigvee_{i=1}^{[\Gamma:C_\Gamma(\Lambda)]} {}^{g_i} p$$

and hence $\tau(q) \leqslant [\Gamma : C_\Gamma(\Lambda)] \cdot \tau(p) < \varepsilon$. Thus $\sigma(\Gamma) = 0$. □

Lemma 3.4. *Let Γ be a discrete non-icc group. For every $\varepsilon > 0$ there exists a nonzero projection $p \in Z(L\Gamma)$ with $\tau(p) < \sigma(\Gamma) + \varepsilon$ and a nonzero, central element $x \in \mathrm{C}^*_{\mathrm{r}}(\Gamma)$ with $xp^\perp = 0$.*

Proof. Since Γ is non-icc, $Z(L\Gamma) \neq \mathbb{C}1$ so $\sigma(\Gamma) < 1$. Let $\varepsilon > 0$ be given and assume, without loss of generality, that $\sigma(\Gamma) + \varepsilon < 1$. One has $Z(L\Gamma) = Z(\mathrm{C}^*_{\mathrm{r}}(\Gamma))'' = Z(\mathbb{C}\Gamma)''$, as can be seen for instance by using Kaplansky's density theorem together with the centre valued trace, and noting that $Z(\mathbb{C}\Gamma)$ consists of the elements whose coefficients are constant along conjugacy classes. By Gelfand duality, $Z(\mathrm{C}^*_{\mathrm{r}}(\Gamma))$ is isomorphic to the C*-algebra $C(Z)$ of continuous functions on its Gelfand spectrum Z, which is a compact Hausdorff space; it is metrizable because $\mathrm{C}^*_{\mathrm{r}}(\Gamma)$ is separable. The canonical trace τ thus gives a regular Borel

probability measure μ on Z [Rudin 1966, Theorem 2.14] and an isomorphism $Z(L\Gamma) = Z(\mathrm{C}^*_{\mathrm{r}}(\Gamma))'' \cong L^\infty(Z,\mu)$ compatible with the natural inclusions. Projections in $Z(L\Gamma)$ correspond via this isomorphism to measurable subsets of Z (up to null sets), and we therefore obtain a measurable subset $A \subset Z$ such that $0 < \mu(A) < \sigma(\Gamma) + \varepsilon/2$. By regularity of μ, there exists $U \supseteq A$ open such that

$$0 < \mu(A) \leqslant \mu(U) < \sigma(\Gamma) + \varepsilon < 1.$$

Now, there is a nonzero element $x \in C(Z)$ vanishing on the compact set $K := Z \setminus U$ (for instance, the distance function to K); letting p be the projection corresponding to U finishes the proof. □

The following lemma gives a concrete description of the decomposition of the left regular representation of a discrete group Γ over the cosets of a finite index normal subgroup Λ.

Lemma 3.5. *Let $\Lambda \trianglelefteqslant \Gamma$ be a normal subgroup of finite index. For every choice of coset representatives $g_1, \ldots, g_{[\Gamma:\Lambda]} \in \Gamma$ there exists a trace-preserving inclusion of von Neumann algebras $\pi : (L\Gamma, \tau) \hookrightarrow (\mathbb{M}_{[\Gamma:\Lambda]}(L\Lambda), \tau \otimes \mathrm{tr})$ which restricts to corresponding inclusions at the level of reduced C^*-algebras and complex group rings, and which for $x \in L\Lambda$ is given by*

$$\pi(x) = \mathrm{diag}({}^{g_1}x, {}^{g_2}x, \ldots, {}^{g_{[\Gamma:\Lambda]}}x), \tag{3-1}$$

where ${}^g x = gxg^{-1}$ is the conjugation action of $g \in \Gamma$ on $L\Lambda$.

Proof. Choose coset representatives $g_1, g_2, \ldots, g_{[\Gamma:\Lambda]}$ of Γ/Λ and consider the isomorphisms of Hilbert spaces

$$\ell^2(\Gamma) \cong \bigoplus_{i=1}^{[\Gamma:\Lambda]} \ell^2(g_i^{-1}\Lambda) \cong \bigoplus_{i=1}^{[\Gamma:\Lambda]} \ell^2(\Lambda).$$

These induce a $*$-isomorphism $\pi : \mathbb{B}(\ell^2\Gamma) \xrightarrow{\cong} \mathbb{M}_n(\mathbb{B}(\ell^2(\Lambda)))$. It is routine to check that π restricts to a trace-preserving inclusion of $\mathbb{C}\Gamma$ into $\mathbb{M}_{[\Gamma:\Lambda]}(\mathbb{C}\Lambda)$ which automatically implies the corresponding results for the reduced C^*-algebras and von Neumann algebras. Finally, for $h \in \Lambda$ we have

$$\pi(h) = \mathrm{diag}(\lambda(h), \ldots, \lambda(h)) \in \bigoplus_{i=1}^{[\Gamma:\Lambda]} \mathbb{B}(\ell^2(g_i^{-1}\Lambda)),$$

and thus formula (3-1) follows in view of the identity

$$hg_i^{-1}h' = g_i^{-1}({}^{g_i}h)h', \quad h, h' \in \Lambda.$$

□

Theorem 3.6. *Let Λ be a discrete group satisfying the strong Atiyah conjecture and let $\Lambda \leqslant \Gamma$ be a finite index inclusion of Λ into a group Γ as a normal subgroup. If $[\Gamma : \Lambda]^2 \cdot \sigma(\Lambda) < \theta(\Lambda)$ then $\mathbb{C}\Gamma$ is not $\mathrm{C}^*_{\mathrm{r}}$-unique.*

Proof. The assumption $[\Gamma : \Lambda]^2 \cdot \sigma(\Lambda) < \theta(\Lambda)$ forces Λ to be non-icc and applying Lemma 3.4 we get a projection $p \in Z(L\Lambda)$ with $\tau(p) < \theta(\Lambda)/[\Gamma : \Lambda]^2$ and a nonzero central element $x \in \mathrm{C}^*_{\mathrm{r}}(\Lambda)$ with $xp^\perp = 0$. We are going to construct a representation of $\mathrm{C}^*_{\mathrm{r}}(\Gamma)$ which is injective on $\mathbb{C}\Gamma$ but with x in the kernel. To this end, consider a set of coset representatives $g_1, \ldots, g_{[\Gamma:\Lambda]}$ for Γ/Λ and the $*$-homomorphism $\pi : L\Gamma \to \mathbb{M}_{[\Gamma:\Lambda]}(L\Lambda)$ provided by Lemma 3.5. From this we obtain a central projection $q := \bigvee_{i=1}^{[\Gamma:\Lambda]} g_i p \in Z(L\Lambda)$, and cutting π with the complement of $\tilde{q} := \mathrm{diag}(q, \ldots, q) \in Z(\mathbb{M}_{[\Gamma:\Lambda]}(L\Lambda))$, we get a representation

$$\pi_q : \mathrm{C}^*_{\mathrm{r}}(\Gamma) \to \mathbb{B}\big(\ell^2(\Lambda)^{[\Gamma:\Lambda]}\tilde{q}^\perp\big), \qquad a \mapsto \pi(a)\tilde{q}^\perp.$$

As $q^\perp = \bigwedge_{i=1}^{[\Gamma:\Lambda]} g_i(p^\perp)$ and $xp^\perp = 0$, it follows that $x \in \ker \pi_q$ in view of (3-1). Let $a \in \mathbb{C}\Gamma \cap \ker \pi_q$. This means that $\pi(a)\tilde{q}^\perp = 0$, and thus the kernel projection r of $\pi(a)$ satisfies $r \geqslant \tilde{q}^\perp$. Therefore

$$(\tau \otimes \mathrm{Tr})(r) \geqslant (\tau \otimes \mathrm{Tr})(\tilde{q}^\perp) \geqslant [\Gamma : \Lambda](1 - [\Gamma : \Lambda]\tau(p)) > [\Gamma : \Lambda] - \theta(\Lambda).$$

On the other hand, the assumption $[\Gamma : \Lambda]^2 \cdot \sigma(\Lambda) < \theta(\Lambda)$ forces an upper bound on the order of finite subgroups in Λ, i.e., $\theta(\Lambda) > 0$, and since Λ is furthermore assumed to satisfy the strong Atiyah conjecture we obtain (using the notation of Section 3A) that

$$\dim_{L\Lambda}(\ker(L_A)) = (\tau \otimes \mathrm{Tr})(P_{\ker L_A}) \in \{n\theta(\Lambda) : n \in \mathbb{Z}\}$$

for any matrix $A \in \mathbb{M}_{[\Gamma:\Lambda]}(\mathbb{C}\Lambda)$. Thus $(\tau \otimes \mathrm{Tr})(r) \leqslant [\Gamma : \Lambda] - \theta(\Lambda)$ unless $\pi(a) = 0$. This proves that π_q is injective on $\mathbb{C}\Gamma$ and hence completes the proof. □

As a corollary, we deduce that some important families of groups are not $\mathrm{C}^*_{\mathrm{r}}$-unique. In particular, this includes the groups mentioned in Theorem A (i) and (ii), and together with Proposition 2.4 this completes the proof of Theorem A.

Corollary 3.7. *All groups in following classes are not $\mathrm{C}^*_{\mathrm{r}}$-unique*:

(i) *Torsion free, non-icc groups satisfying the strong Atiyah conjecture*; *in particular all elementary amenable, non-icc, torsion free groups.*

(ii) *Virtually polycyclic groups with infinite* FC*-centre*; *in particular, all infinite groups of polynomial growth.*

Proof. To see (i), note that the existence of a nontrivial finite conjugacy class implies the existence of a nontrivial central element in $\mathbb{C}\Gamma$ (namely the sum of the elements in the finite conjugacy class) and hence a nontrivial projection in $Z(L\Gamma)$; thus $\sigma(\Gamma) < 1$. Moreover, since Γ is torsion free, $\theta(\Gamma) = 1$ and since Γ is assumed to

satisfy the strong Atiyah conjecture it follows that it is $\mathrm{C}^*_{\mathrm{r}}$-unique by Theorem 3.6. The last statement in (i) follows directly from this since the elementary amenable groups are contained in Linnell's class $\mathscr{C}$ (see Section 3A) for which the strong Atiyah conjecture is known to hold in the presence of a bound on the order of finite subgroups [Lück 2002, Theorem 10.19].

To see (ii), let $\Lambda \trianglelefteq \Gamma$ be a normal finite index polycyclic subgroup of Γ. As Γ has infinite FC-centre, so does Λ and the FC-centre of Λ is moreover finitely generated by polycyclicity. A classical result by Hirsch [1946, Theorem 3.21] implies that the orders of finite subgroups of Λ are bounded; thus $\theta(\Lambda) > 0$. On the other hand, $\sigma(\Lambda) = 0$ by Proposition 3.3(ii). Moreover, polycyclic groups, being elementary amenable, satisfy the strong Atiyah conjecture. Thus, it follows that $\mathbb{C}\Gamma$ is non-$\mathrm{C}^*_{\mathrm{r}}$-unique by Theorem 3.6.

Finally, the claim about infinite groups of polynomial growth follows by first observing that by Gromov's theorem [1981], these are exactly finitely generated virtually nilpotent groups. As finitely generated nilpotent groups are polycyclic, the claim follows once we argue that virtually nilpotent groups automatically have infinite FC-centre. To see this, recall that a finitely generated virtually nilpotent group Γ contains a finite index torsion free nilpotent normal subgroup Λ (by polycyclicity and [Hirsch 1946, Theorem 3.21]). Now it follows that the centre of Λ is infinite, and therefore so is the FC-centre Λ_{fc}; but as $\Lambda \trianglelefteq \Gamma$ is a finite index inclusion, $\Lambda_{\mathrm{fc}} \subseteq \Gamma_{\mathrm{fc}}$. Thus, Γ_{fc} is infinite. □

Acknowledgements

The authors would like to thank Mikael Rørdam and Thomas Schick for helpful discussions revolving around the topics of the present paper, and the anonymous referee for valuable remarks improving Corollary 2.3 and Proposition 3.3. Kyed gratefully acknowledges the financial support from the Villum Foundation (grant no. 7423) and from the Independent Research Fund Denmark (grant no. 7014-00145B).

References

[Barnes 1983] B. A. Barnes, "The properties *-regularity and uniqueness of C*-norm in a general *-algebra", *Trans. Amer. Math. Soc.* **279**:2 (1983), 841–859. MR Zbl

[Boidol 1984] J. Boidol, "Group algebras with a unique C*-norm", *J. Funct. Anal.* **56**:2 (1984), 220–232. MR Zbl

[Breuillard et al. 2017] E. Breuillard, M. Kalantar, M. Kennedy, and N. Ozawa, "C*-simplicity and the unique trace property for discrete groups", *Publ. Math. Inst. Hautes Études Sci.* **126** (2017), 35–71. MR Zbl

[Brown and Ozawa 2008] N. P. Brown and N. Ozawa, *C*-algebras and finite-dimensional approximations*, Graduate Studies in Math. **88**, Amer. Math. Soc., Providence, RI, 2008. MR Zbl

[Grigorchuk et al. 2018] R. Grigorchuk, M. Musat, and M. Rørdam, "Just-infinite C*-algebras", *Comment. Math. Helv.* **93**:1 (2018), 157–201. MR Zbl

[Gromov 1981] M. Gromov, "Groups of polynomial growth and expanding maps", *Inst. Hautes Études Sci. Publ. Math.* **53** (1981), 53–73. MR Zbl

[Hirsch 1946] K. A. Hirsch, "On infinite soluble groups, III", *Proc. London Math. Soc.* (2) **49** (1946), 184–194. MR Zbl

[Le Boudec 2017] A. Le Boudec, "C*-simplicity and the amenable radical", *Invent. Math.* **209**:1 (2017), 159–174. MR Zbl

[Leung and Ng 2004] C.-W. Leung and C.-K. Ng, "Some permanence properties of C*-unique groups", *J. Funct. Anal.* **210**:2 (2004), 376–390. MR Zbl

[Lück 1997] W. Lück, "Hilbert modules and modules over finite von Neumann algebras and applications to L^2-invariants", *Math. Ann.* **309**:2 (1997), 247–285. MR Zbl

[Lück 2002] W. Lück, *L^2-invariants: theory and applications to geometry and K-theory*, Ergebnisse der Mathematik (3) **44**, Springer, 2002. MR Zbl

[Murray and von Neumann 1936] F. J. Murray and J. von Neumann, "On rings of operators", *Ann. of Math.* (2) **37**:1 (1936), 116–229. MR Zbl

[Rudin 1966] W. Rudin, *Real and complex analysis*, McGraw-Hil, New York, 1966. MR Zbl

Received August 17, 2018.

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Subscriptions, requests for back issues and changes of subscriber address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

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PACIFIC JOURNAL OF MATHEMATICS

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