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SIGNATURE RANKS OF UNITS IN CYCLOTOMIC EXTENSIONS OF ABELIAN NUMBER FIELDS

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We prove the rank of the group of signatures of the circular units (hence also the full group of units) of $\mathbb{Q}(\zeta_m)^+$ tends to infinity with m. We also show the signature rank of the units differs from its maximum possible value by a bounded amount for all the real subfields of the composite of an abelian field with finitely many odd prime-power cyclotomic towers. In particular, for any prime p the signature rank of the units of $\mathbb{Q}(\zeta_{p^n})^+$ differs from $\varphi(p^n)/2$ by an amount that is bounded independent of n. Finally, we show conditionally that for general cyclotomic fields the unit signature rank can differ from its maximum possible value by an arbitrarily large amount.

1. Introduction

For any positive integer m with m odd or m divisible by 4, let ζ_m be a primitive m-th root of unity, $\mathbb{Q}(\zeta_m)$ the corresponding cyclotomic field and $\mathbb{Q}(\zeta_m)^+ = \mathbb{Q}(\zeta_m + \zeta_m^{-1})$ its maximal totally real subfield. The units in $\mathbb{Q}(\zeta_m)^+$ together with ζ_m are a subgroup of finite index in the group of units of $\mathbb{Q}(\zeta_m)$ (this index is 1 when m is a prime power and 2 otherwise; see [Washington 1982, Corollary 4.13]).

Under the various $\varphi(m)/2$ real embeddings of $\mathbb{Q}(\zeta_m)^+$, each unit ε of $\mathbb{Q}(\zeta_m)^+$ has a sign, and the collection of these signs is called the *signature* of ε. The collection of all such unit signatures is an elementary abelian 2-group, whose rank is called the *unit signature rank* of $\mathbb{Q}(\zeta_m)^+$ (or, by abuse, of $\mathbb{Q}(\zeta_m)$). The signature rank measures how many different possible signs arise from the units and determines the difference between the class number and the strict class number.

The purpose of this paper is to prove that the unit signature rank of $\mathbb{Q}(\zeta_m)^+$ tends to infinity with m. We do this by demonstrating an explicit lower bound for the signature rank of the subgroup of *cyclotomic units* of $\mathbb{Q}(\zeta_m)^+$. We then show that the difference between the signature rank of the units in the maximal real subfield of the cyclotomic field of $mp_1^{n_1}\cdots p_s^{n_s}$-th roots of unity (with all p_i odd) and its maximum possible value is bounded independent of $n_1, \ldots, n_s$ (and is constant if all the n_i are sufficiently large). This in particular provides infinitely many families

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of cyclotomic fields whose unit signature ranks are "nearly maximal". Finally, we show this difference can be arbitrarily large, conditional on the existence of infinitely many cyclic cubic fields with totally positive fundamental units.

2. Signatures

For any totally real field F of degree n over $\mathbb{Q}$ let $F_{\mathbb{R}} = F \otimes_{\mathbb{Q}} \mathbb{R} \simeq \prod_{v \text{ real}} \mathbb{R}$ where the product is taken over the real embeddings v of F. Define the *archimedean signature space* $V_{\infty,F}$ of F to be

$$V_{\infty,F} = F_{\mathbb{R}}^*/F_{\mathbb{R}}^{*2} \simeq \prod_{v \text{ real}} \{\pm 1\} \simeq \mathbb{F}_2^n,$$

where by identifying $\{\pm 1\}$ with the finite field $\mathbb{F}_2$ of two elements, we view the multiplicative group $V_{\infty,F}$ as a vector space over $\mathbb{F}_2$, written additively.

For any $\alpha \in F^*$ and $v : F \hookrightarrow \mathbb{R}$ a real place of F, let $\alpha_v = v(\alpha)$ and define the sign of α_v as usual by $\operatorname{sign}(\alpha_v) = \alpha_v/|\alpha_v| \in \{\pm 1\}$. Write $\operatorname{sgn}(\alpha_v) \in \mathbb{F}_2$ for $\operatorname{sign}(\alpha_v)$ when viewed in the additive group $\mathbb{F}_2$, i.e., $\operatorname{sgn}(\alpha_v) = 0$ if $\alpha_v > 0$ and $\operatorname{sgn}(\alpha_v) = 1$ if $\alpha_v < 0$. The (*archimedean*) *signature map* of F is the homomorphism

$$\operatorname{sgn}_{\infty,F} : F^* \to V_{\infty,F}, \qquad \alpha \mapsto (\operatorname{sgn}(\alpha_v))_v.$$

In the case when $F/\mathbb{Q}$ is Galois and one real embedding of F is fixed, we can index the real embeddings of F by the elements σ in $\operatorname{Gal}(F/\mathbb{Q})$, and

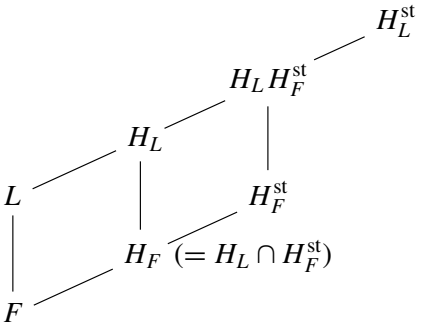
$$\operatorname{sgn}_{\infty,F}(\alpha) = (\operatorname{sgn}(\sigma(\alpha)))_{\sigma \in \operatorname{Gal}(F/\mathbb{Q})},$$

where sgn is the sign (viewed as an element of $\mathbb{F}_2$) in the fixed real embedding. The element $\operatorname{sgn}_{\infty,F}(\alpha)$ is called the *signature* of α.

The collection of all the signatures $\operatorname{sgn}_{\infty,F}(\varepsilon)$ where ε varies over the units of F is called the *unit signature group* of F; the rank of this subspace of $V_{\infty,F}$ is called the (*unit*) *signature rank* of F and, as previously mentioned, is a measure of how many different possible sign configurations arise from the units of F.

Define the (*unit signature rank*) *"deficiency" of* F, denoted $\delta(F)$, to be the corank of the unit signature group of F in $V_{\infty,F}$, i.e., $[F : \mathbb{Q}]$ minus the signature rank of the units of F. The deficiency of F is just the nonnegative difference between the unit signature rank of F and its maximum possible value — the deficiency is 0 if and only if there are units of every possible signature type. The deficiency is also the rank of the group of totally positive units of F modulo squares.

Remark 1. For any finite extension L/F of totally real fields, we have $\delta(F) \le \delta(L)$, a result due to Edgar, Mollin and Peterson [Edgar et al. 1986, Theorem 2.1]. We briefly recall the reason: the intersection of the Hilbert class field H_L of L with the strict (or narrow) Hilbert class field H_F^{st} of F is easily seen to be the Hilbert class



field H_F of F since L is totally real. The composite field $H_L H_F^{\mathrm{st}}$ is a subfield of the strict Hilbert class field, H_L^{st}, of L and has degree over H_L equal to $[H_F^{\mathrm{st}} : H_F]$ because $H_F = H_L \cap H_F^{\mathrm{st}}$. Since $[H_F^{\mathrm{st}} : H_F] = 2^{\delta(F)}$ and $[H_L^{\mathrm{st}} : H_L] = 2^{\delta(L)}$ (see [Dummit and Voight 2018, §2] for details), the result follows.

3. Circular units and signatures in cyclotomic fields

Suppose now that $F = \mathbb{Q}(\zeta_m)^+$ is the maximal (totally) real subfield of the cyclotomic field $\mathbb{Q}(\zeta_m)$ of m-th roots of unity (with m odd or divisible by 4), which is of degree $\varphi(m)/2$ over $\mathbb{Q}$.

We can fix an embedding of $\mathbb{Q}(\zeta_m)$ into $\mathbb{C}$ by mapping ζ_m to $e^{2\pi i/m}$, which also fixes an embedding of $\mathbb{Q}(\zeta_m)^+$ into $\mathbb{R}$.

The Galois group $\mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})$ consists of the automorphisms σ_a that map ζ_m to ζ_m^a for integers a, $1 \le a < m$, relatively prime to m. We identify $\mathrm{Gal}(\mathbb{Q}(\zeta_m)^+/\mathbb{Q})$ with $\mathrm{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})/\{1, \sigma_{-1}\}$ and take the elements $\bar{\sigma}_a \in \mathrm{Gal}(\mathbb{Q}(\zeta_m)^+/\mathbb{Q})$ with $1 \le a < m/2$ relatively prime to m as representatives.

To get information on the unit signature rank of $\mathbb{Q}(\zeta_m)^+$, we consider the signatures of the subgroup of *circular* (or *cyclotomic*) *units*, denoted $C_{\mathbb{Q}(\zeta_m)}$, which when m is a prime power has a set of generators given by -1 and the $\varphi(m)/2 - 1$ independent elements

$$\zeta_m^{(1-a)/2} \frac{1 - \zeta_m^a}{1 - \zeta_m} \tag{1}$$

for $1 < a < m/2$ and a relatively prime to m [Washington 1982, Lemma 8.1] (for m not a prime power the definition of $C_{\mathbb{Q}(\zeta_m)}$ is more complicated; see [Washington 1982, Chapter 8]).

The group $C_{\mathbb{Q}(\zeta_m)}$ is contained in $\mathbb{Q}(\zeta_m)^+$ and, when m is a prime power, is isomorphic (as an additive abelian group) to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}^{\varphi(m)/2-1}$, with -1 and the elements in (1) serving as independent generators over $\mathbb{Z}$.

If we choose an ordering of the elements of $\mathrm{Gal}(\mathbb{Q}(\zeta_m)^+/\mathbb{Q})$ and an ordering of a set of generators for the cyclotomic units, their corresponding signatures in

$\mathbb{F}_2^{\varphi(m)/2}$ define the rows of a $\varphi(m)/2 \times \varphi(m)/2$ matrix over $\mathbb{F}_2$, whose rank is the rank of the subgroup $\mathrm{sgn}_{\infty,\mathbb{Q}(\zeta_m)^+}(C_{\mathbb{Q}(\zeta_m)})$ of signatures of the cyclotomic units.

We first consider the case where m is an odd prime power.

The case $m = p^n$ for an odd prime p. When $m = p^n$ for an odd prime p, every primitive m-th root of unity is the square of another primitive m-th root of unity, so the circular units in (1) can be written in the form

$$\frac{\zeta_m^a - \zeta_m^{-a}}{\zeta_m - \zeta_m^{-1}} \tag{2}$$

for $1 < a < m/2$ and a relatively prime to m. If we order the elements in $\mathrm{Gal}(\mathbb{Q}(\zeta_m)^+/\mathbb{Q})$ by $\bar{\sigma}_b$ with $1 \le b < m/2$ relatively prime to m, then the signature of the element in (2) is given by the signs of the elements

$$\bar{\sigma}_b\left(\frac{\zeta_m^a - \zeta_m^{-a}}{\zeta_m - \zeta_m^{-1}}\right) = \frac{\zeta_m^{ab} - \zeta_m^{-ab}}{\zeta_m^b - \zeta_m^{-b}}, \quad 1 \le b < m/2, \ \ (b,m) = 1.$$

Under the embedding $\zeta_m = e^{2\pi i/m}$ we have

$$\frac{\zeta_m^{ab} - \zeta_m^{-ab}}{\zeta_m^b - \zeta_m^{-b}} = \frac{\sin(2\pi ab/m)}{\sin(2\pi b/m)}. \tag{3}$$

Since $1 \le b < m/2$, the denominator $\sin(2\pi b/m)$ is positive. Hence,

$$\bar{\sigma}_b((\zeta_m^a - \zeta_m^{-a})/(\zeta_m - \zeta_m^{-1}))$$

is positive if and only if $\sin(2\pi ab/m)$ is positive, which happens precisely when the least positive residue of ab modulo m is in $(0, m/2)$.

It follows that the rank of the subspace $\mathrm{sgn}_{\infty,\mathbb{Q}(\zeta_m)^+}(C_{\mathbb{Q}(\zeta_m)})$ of $V_{\infty,\mathbb{Q}(\zeta_m)^+}$ is equal to the rank of a $\varphi(m)/2 \times \varphi(m)/2$ matrix $C = (c_{a,b})$ over $\mathbb{F}_2$ (referred to as the *circular unit signature matrix*) whose rows are indexed by the elements a relatively prime to m with $1 \le a < m/2$ and whose columns are indexed by the elements b relatively prime to m with $1 \le b < m/2$, as follows. The first row of C is $(1, 1, \ldots, 1)$, corresponding to the signs $(-1, -1, \ldots, -1)$ of the element -1, viewed in the additive group $\mathbb{F}_2$, so

$$c_{1,b} = 1,$$

for $1 \le b < m/2$, with b relatively prime to m. For $2 \le a < m/2$, $1 \le b < m/2$, with a and b relatively prime to m, we have

$$c_{a,b} = \begin{cases} 0 & \text{if } ab \pmod m \in (0, m/2), \\ 1 & \text{if } ab \pmod m \in (m/2, m), \end{cases}$$

where $ab \pmod m$ is taken to be the least positive residue of ab modulo m.

For computational purposes, it is useful to note the row indexed by a is the i-th row of the matrix where $i = a - \lfloor (a-1)/p \rfloor$ and the i-th row is indexed by $a = i + \lfloor (i-1)/(p-1) \rfloor$ (and similarly for the numbering of the columns).

If we add the first row of C to the remaining rows (which does not affect the rank of the matrix), we obtain the *modified circular unit signature matrix* $M = (c'_{a,b})$ with

$$c'_{a,b} = \begin{cases} 1 & \text{if } ab \pmod m \in (0, m/2), \\ 0 & \text{if } ab \pmod m \in (m/2, m), \end{cases}$$

for $1 \le a < m/2$ and $1 \le b < m/2$, with a and b relatively prime to m and as before $ab \pmod m$ is taken to be the least positive residue of ab modulo m (this matrix appears in [Davis 1969] in the case when m is an odd prime).

The entry $c'_{2^d,b}$ of the matrix M in the column indexed by b ($b = 1, \ldots, m/2$, with b prime to m) and the row indexed by 2^d ($1 \le 2^d < m/2$) is 1 if the least positive residue of $2^d b$ modulo m lies in $(0, m/2)$ and is 0 if the least positive residue lies in $(m/2, m)$. Writing $2^d b = Am + r$ with an integer A and least positive remainder r with $0 \le r < m$, i.e., $2^{d+1} b/m = 2A + (2r/m)$, it follows that $r \in (0, m/2)$ implies $\lfloor 2^{d+1} b/m \rfloor = 2A$ and $r \in (m/2, m)$ implies $\lfloor 2^{d+1} b/m \rfloor = 2A+1$. As a consequence, the entry of M in the row indexed by 2^d and column indexed by b is 0 if $\lfloor 2^{d+1} b/m \rfloor$ is odd and is 1 if $\lfloor 2^{d+1} b/m \rfloor$ is even.

Lemma. *Suppose $m = p^n$ where p is an odd prime and $n \ge 1$. Let $k \ge 1$ be any integer with $m > 2^{k+2}$. If*

$$b_0(k) = \left\lfloor \frac{(2^k - 2)m}{2^{k+1}} \right\rfloor + 1 \quad \text{and} \quad b_1(k) = \left\lfloor \frac{(2^k - 2)m}{2^{k+1}} \right\rfloor + 2,$$

then $\lfloor 2^{d+1} b_0(k)/m \rfloor$ and $\lfloor 2^{d+1} b_1(k)/m \rfloor$ are both odd for $d = 1, 2, \ldots, k-1$ and both even for $d = k$.

Proof. Write

$$\frac{(2^k - 2)m}{2^{k+1}} = \left\lfloor \frac{(2^k - 2)m}{2^{k+1}} \right\rfloor + \theta$$

where $0 \le \theta < 1$. Then

$$b_0(k) = \frac{2^k - 2}{2^{k+1}} m + (1 - \theta) \quad \text{and} \quad b_1(k) = \frac{2^k - 2}{2^{k+1}} m + (2 - \theta),$$

so

$$\frac{2^{d+1} b_0(k)}{m} = 2^d - \frac{2}{2^{k-d}} + \frac{2^{d+1}}{m}(1-\theta) \text{ and } \frac{2^{d+1} b_1(k)}{m} = 2^d - \frac{2}{2^{k-d}} + \frac{2^{d+1}}{m}(2-\theta).$$

Since $0 < 1 - \theta \le 1$, $1 < 2 - \theta \le 2$, and $m > 2^{k+2}$, when $d = k$ this gives

$$\lfloor 2^{k+1} b_0(k)/m \rfloor = \lfloor 2^{k+1} b_1(k)/m \rfloor = 2^k - 2,$$

so $\lfloor 2^{k+1}b_0(k)/m\rfloor$ and $\lfloor 2^{k+1}b_1(k)/m\rfloor$ are both even. For $1 \le d < k$, we have

$$2^d - 1 < 2^d - \frac{2}{2^{k-d}} + \frac{2^{d+1}}{m}(1-\theta) < 2^d - \frac{2}{2^{k-d}} + \frac{2^{d+1}}{m}(2-\theta) < 2^d,$$

so that

$$\lfloor 2^{d+1}b_0(k)/m\rfloor = \lfloor 2^{d+1}b_1(k)/m\rfloor = 2^d - 1,$$

and $\lfloor 2^{d+1}b_0(k)/m\rfloor$ and $\lfloor 2^{d+1}b_1(k)/m\rfloor$ are both odd, completing the proof of the lemma. □

We can use the lemma to give the following lower bound for the number of independent signatures for the circular units in this case.

Proposition 2. *Suppose p is an odd prime and $n \ge 1$. Then the rank of the group of signatures of the circular units in $\mathbb{Q}(\zeta_{p^n})^+$ is at least $\lfloor \log_2(p^n)\rfloor - 2$.*

Proof. Since $b_0(k)$ and $b_1(k)$ in the lemma differ by 1 and $m = p^n$, at least one is relatively prime to m. It then follows from the lemma that for each $k \ge 1$ with $2^{k+2} < m$ there is a $B(k)$, relatively prime to m and satisfying $1 \le B(k) < m/2$, such that $\lfloor 2^{d+1}B(k)/m\rfloor$ is odd for $d = 1, 2, \ldots, k-1$ and even for $d = k$.

By the remarks before the lemma, it follows that for each $k \ge 1$ with $2^{k+2} < m$, the entries of the matrix M in the column indexed by $B(k)$ and belonging to the rows indexed by $2, 4, 8, \ldots, 2^k$ are $0, 0, \ldots, 0, 1$, respectively. In particular, this shows that the row of M indexed by 2^k is not in the span of the rows indexed by $2, 4, \ldots, 2^{k-1}$. Applying this successively for $k = 1, 2, \ldots, \lfloor \log_2 m\rfloor - 2$ shows that all these rows are linearly independent, implying that the rank of M is at least $\lfloor \log_2 m\rfloor - 2$, which proves the proposition. □

The case $m = 2^n$, $n \ge 2$. In this case let $\zeta_{2^{n+1}}$ be a 2^{n+1}-st root of unity with $\zeta_{2^n} = \zeta_{2^{n+1}}^2$. Fix an embedding into $\mathbb{C}$ mapping $\zeta_{2^{n+1}}$ to $e^{2\pi i/2^{n+1}}$ and order the elements of $\mathrm{Gal}(\mathbb{Q}(\zeta_m)^+/\mathbb{Q})$ as $\bar{\sigma}_b$ with odd $b = 3, 5, \ldots, 2^{n-1} - 1$. Then the conjugates of the elements in (1) are given by

$$\frac{\zeta_{2^{n+1}}^{ab} - \zeta_{2^{n+1}}^{-ab}}{\zeta_{2^{n+1}}^{b} - \zeta_{2^{n+1}}^{-b}} = \frac{\sin(\pi ab/2^n)}{\sin(\pi b/2^n)} \tag{4}$$

for $1 < a < 2^{n-1}$, $1 \le b < 2^{n-1}$, with a and b both odd. The sign of the unit in (4) is $+1$ if the least positive residue of ab modulo 2^{n+1} lies in $(0, 2^n)$ and is -1 if the least positive residue lies in $(2^n, 2^{n+1})$.

In this case the circular unit signature matrix has first row consisting of all 1's and the entry in the row indexed by a and column indexed by b is given by

$$c_{1,b} = 1 \tag{5}$$

for b odd, $1 < b < 2^{n-1}$, and

$$c_{a,b} = \begin{cases} 0 & \text{if } ab \pmod{2^{n+1}} \in (0, 2^n), \\ 1 & \text{if } ab \pmod{2^{n+1}} \in (2^n, 2^{n+1}), \end{cases} \tag{6}$$

for a and b odd, $1 < a < 2^{n-1}$ and $1 \le b < 2^{n-2}$, where $ab \pmod{2^{n+1}}$ is taken to be the least positive residue of ab modulo 2^{n+1}.

An argument similar to that for odd prime powers (here for the rows indexed by $2^d - 1$ and column indexed by $2^n - 2^{n-k+1} + 1$) shows the rank of the circular unit signature matrix is at least $n - 2 = \lfloor \log_2 m \rfloor - 2$, but for this case Weber proved the following definitive result.

Proposition 3 [Weber 1899, B, p. 821]. *Suppose $n \ge 2$. Then the rank of the group of signatures of the circular units in the maximal totally real subfield of the cyclotomic field of 2^n-th roots of unity is maximal, i.e., equal to 2^{n-2}.*

This result was generalized by Hasse [1952], whose simpler and more conceptual proof used the fact that the circular unit $(\zeta_{2^{n+1}}^5 - \zeta_{2^{n+1}}^{-5})/(\zeta_{2^{n+1}} - \zeta_{2^{n+1}}^{-1})$ in $\mathbb{Q}(\zeta_{2^n})^+$ has norm -1 and showed the signatures of its Galois conjugates generate a group of maximal signature rank (see [Hasse 1952, pp. 28–29]). A nice proof of this, using the fact that over $\mathbb{F}_2$ the only irreducible representation of a 2-group is the trivial representation, can be found in [Garbanati 1976]. Another nice proof of Weber's result (due to Iwasawa) can be found in [Dummit 2018].

Remark 4. Unlike the situation for the 2-power cyclotomic fields, not every possible signature type occurs for the circular units in general cyclotomic fields, even for $m = p$ an odd prime (for example, in the case $p = 29$ the circular unit signature group has rank 11, not the maximal possible rank of 14 [Davis 1969, Appendix I, p. 70]).

4. Signatures in composites of extensions

Propositions 2 and 3 already imply that the signature rank of the units of $\mathbb{Q}(\zeta_m)^+$ tends to infinity with m (since $\mathbb{Q}(\zeta_{p^n})^+ \subset \mathbb{Q}(\zeta_m)^+$ if p^n divides m and the largest prime power divisor of m tends to infinity as m tends to infinity), but this can be made more precise using the following result, which may be of independent interest.

Suppose $F/\mathbb{Q}$ and $F'/\mathbb{Q}$ are two totally real Galois extensions of $\mathbb{Q}$ with $F \cap F' = \mathbb{Q}$. Then the composite field FF' has Galois group

$$\mathrm{Gal}(FF'/\mathbb{Q}) = \mathrm{Gal}(F/\mathbb{Q}) \times \mathrm{Gal}(F'/\mathbb{Q}),$$

and independent signatures in F and F' produce essentially independent signatures in FF':

Proposition 5. *With F and F' as above, suppose $\alpha_1, \ldots, \alpha_r$ are nonzero elements of F whose signatures $\mathrm{sgn}_{\infty,F}(\alpha_i)$, $i = 1, \ldots, r$, are linearly independent in the*

$\mathbb{F}_2$-vector space $V_{\infty,F}$. Suppose similarly that $\beta_1, \ldots, \beta_s$ are nonzero elements of F' whose signatures $\operatorname{sgn}_{\infty,F'}(\beta_j)$, $j = 1, \ldots, s$, are linearly independent in the $\mathbb{F}_2$-vector space $V_{\infty,F'}$. Then the dimension of the space generated by the signatures of $\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_s$ in the $\mathbb{F}_2$-vector space $V_{\infty,FF'}$ is $r+s$ unless the subgroups generated by $\alpha_1, \ldots, \alpha_r$ and by $\beta_1, \ldots, \beta_s$ both contain totally negative elements, in which case the dimension is $r+s-1$.

Proof. If the signatures in $V_{\infty,FF'}$ of $\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_s$ are linearly dependent, then there is an element ${\alpha_1}^{a_1} \cdots {\alpha_r}^{a_r} {\beta_1}^{b_1} \cdots {\beta_s}^{b_s}$ in FF' with $a_1, \ldots, a_r, b_1, \ldots, b_s \in \{0, 1\}$, not all 0, that is totally positive. Assume without loss that at least one of $a_1, \ldots, a_r$ is not 0, and let $\alpha = {\alpha_1}^{a_1} \cdots {\alpha_r}^{a_r}$ and $\beta = {\beta_1}^{b_1} \cdots {\beta_s}^{b_s}$. Then $\sigma\tau(\alpha\beta) = \sigma(\alpha)\tau(\beta)$ is positive for every $\sigma \in \operatorname{Gal}(F/\mathbb{Q})$ and every $\tau \in \operatorname{Gal}(F'/\mathbb{Q})$. Since the signatures $\operatorname{sgn}_{\infty,F}(\alpha_i)$, $i = 1, \ldots, r$, are linearly independent and some a_i is nonzero, there exists a σ_0 such that $\sigma_0(\alpha)$ is negative. This implies $\tau(\beta)$ is negative for every τ, i.e., β is totally negative. Then, since there is a τ_0 with $\tau_0(\beta)$ negative, it follows that also $\sigma(\alpha)$ is negative for every σ, i.e., α is totally negative. Hence there is at most one nontrivial relation among the signatures of $\alpha_1, \ldots, \alpha_r, \beta_1, \ldots, \beta_s$ in $V_{\infty,FF'}$ — this nontrivial relation occurs if and only if $(1, 1, \ldots, 1) \in V_{\infty,F}$ is in the $\mathbb{F}_2$-space spanned by $\operatorname{sgn}_{\infty,F}(\alpha_i)$, $i = 1, \ldots, r$, and $(1, 1, \ldots, 1) \in V_{\infty,F'}$ is in the $\mathbb{F}_2$-space spanned by $\operatorname{sgn}_{\infty,F'}(\beta_j)$, $j = 1, \ldots, s$, completing the proof. □

5. The signature rank of the units in $\mathbb{Q}(\zeta_m)$ for general m

If we combine Propositions 2 and 3 on the signature ranks in the prime power case with Proposition 5 we obtain the following result.

Theorem 6. *Suppose the positive integer m is odd or is divisible by 4. Then the rank of the group of signatures of the group of circular units in $\mathbb{Q}(\zeta_m)^+$ is at least $\log_2(m) - 4\omega(m) + 1$, where $\omega(m)$ is the number of distinct prime factors of m. In particular, the signature rank of the units in $\mathbb{Q}(\zeta_m)^+$ tends to infinity with m.*

Proof. Write $m = p_1^{a_1} \ldots p_k^{a_k}$. Then $\mathbb{Q}(\zeta_m)^+$ contains the composite of the totally real fields $\mathbb{Q}(\zeta_{{p_i}^{a_i}})^+$, $i = 1, \ldots, k$. By Propositions 2 and 3, the signature rank of the circular units in these latter fields is at least $\lfloor \log_2({p_i}^{a_i}) \rfloor - 2$ (and much better when $p = 2$). Applying the previous proposition repeatedly shows that the signature rank of the group generated by the circular units is at least

$$\sum_{i=1}^{k} (\lfloor \log_2({p_i}^{a_i}) \rfloor - 2) - (k-1).$$

Since $\lfloor \log_2({p_i}^{a_i}) \rfloor > \log_2({p_i}^{a_i}) - 1$, the lower bound in the theorem follows. The final statement in the theorem follows from standard bounds on the growth of $\omega(m)$ (see, for example, [Hardy and Wright 2008, Section 22.10]). □

Corollary 7. *With the exception of $m = 12$, no maximal real subfield of any cyclotomic field of m-th roots of unity has a fundamental system of units that are all totally positive.*

Proof. The signature rank of the circular units is at least 2 for $m = 2^3, 3^2, 5, 7, 11, 13$ by direct computation and for all $m = p$ for primes $p \geq 17$ by Proposition 2. It follows that the signature rank of the circular units is at least 2 for all m divisible by 2^3, 3^2 or any odd prime $p \geq 5$. The only remaining possible values for m are $m = 3, 4, 12$. The first two have no units of infinite order, and the third has maximal real subfield $\mathbb{Q}(\sqrt{3})$ with totally positive fundamental unit $2 + \sqrt{3}$. □

6. Signatures in cyclotomic towers over cyclotomic fields

Computations suggest that the signature rank of the units in the real subfield of the cyclotomic field of m-th roots of unity is in fact always close to the maximal possible rank of $\varphi(m)/2$ (equivalently, the unit signature rank deficiency for these fields should be close to 0), i.e., nearly all possible signature types arise for units. This is in keeping with the heuristics in [Dummit and Voight 2018] suggesting that "most" totally real fields have nearly maximal unit signature rank (although these abelian extensions are hardly "typical").

In this section we prove that for infinitely many different families of cyclotomic fields the units do indeed have nearly maximal signature rank. We do this by showing the unit signature rank deficiency is bounded in (finitely many composites of) prime power cyclotomic towers over cyclotomic fields.

Theorem 8. *Suppose $p_1, \ldots, p_s$ ($s \geq 1$) are distinct odd primes and suppose m is any positive integer that is either odd or divisible by 4 and that is relatively prime to $p_1, \ldots, p_s$.*

Let $\delta(m; n_1, \ldots, n_s) = \delta(\mathbb{Q}(\zeta_{mp_1^{n_1}\cdots p_s^{n_s}})^+) \geq 0$ denote the unit signature rank deficiency of the maximal real subfield of the cyclotomic field of $mp_1^{n_1}\cdots p_s^{n_s}$-th roots of unity defined in Section 2, i.e., the nonnegative difference between the signature rank of the units of $\mathbb{Q}(\zeta_{mp_1^{n_1}\cdots p_s^{n_s}})^+$ and its maximum possible value $\varphi(mp_1^{n_1}\cdots p_s^{n_s})/2$. Then:

(a) *$\delta(m; n_1, \ldots, n_s) \leq \delta(m; n_1', \ldots, n_s')$ if $n_i \leq n_i'$ for all $1 \leq i \leq s$.*

(b) *$\delta(m; n_1, \ldots, n_s)$ is bounded independent of $n_1, \ldots, n_s$.*

(c) *$\delta(m; n_1, \ldots, n_s)$ is constant (depending on m) if $n_1, \ldots, n_s$ are all large enough.*

Proof. The maximal real subfield of the cyclotomic field of $mp_1^{n_1}\cdots p_s^{n_s}$-th roots of unity is a subfield of the maximal real subfield of the cyclotomic field of $mp_1^{n_1'}\cdots p_s^{n_s'}$-th roots of unity if $n_i \leq n_i'$, $i = 1, \ldots, s$, so (a) follows immediately from Remark 1.

Suppose that $n_i \geq 1$ for $1 \leq i \leq s$ and let L denote the cyclotomic field

$\mathbb{Q}(\zeta_{mp_1^{n_1}\cdots p_s^{n_s}})$, with maximal real subfield $L^+ = \mathbb{Q}(\zeta_{mp_1^{n_1}\cdots p_s^{n_s}})^+$. Then the strict class number of L^+ divides twice the class number of L, as follows. The quadratic extension L/L^+ is ramified at a finite prime (if $m = s = 1$) or is ramified only at infinity (otherwise). Hence, if $H^{\mathrm{st}}_{L^+}$ is the strict Hilbert class field of L^+, then the degree over L of the composite $LH^{\mathrm{st}}_{L^+}$ is either the strict class number of L^+ (if $m = s = 1$) or half that (otherwise). Since $LH^{\mathrm{st}}_{L^+}$ is an abelian extension of L that is unramified at finite primes, it is contained in the Hilbert class field of the complex field L, so $[LH^{\mathrm{st}}_{L^+} : L]$ divides the class number of L, which gives the desired divisibility.

Next observe that the cyclotomic fields $\mathbb{Q}(\zeta_{mp_1^{n_1}\cdots p_s^{n_s}})$ with $n_i \geq 1$ for $1 \leq i \leq s$ are the subfields of the composite of the cyclotomic $\mathbb{Z}_{p_i}$-extensions of $\mathbb{Q}(\zeta_{mp_1\cdots p_s})$ for $1 \leq i \leq s$. Hence the 2-primary part of the class number of $\mathbb{Q}(\zeta_{mp_1^{n_1}\cdots p_s^{n_s}})$ is bounded for all s-tuples $(n_1, \ldots, n_s)$ and is constant if $n_1, \ldots, n_s$ are all sufficiently large by a theorem of Friedman [1981/82] extending a result of Washington. By the previous observation, the strict class number of $L^+ = \mathbb{Q}(\zeta_{mp_1^{n_1}\cdots p_s^{n_s}})^+$ is therefore bounded for all s-tuples $(n_1, \ldots, n_s)$ (and is in fact constant if $n_1, \ldots, n_s$ are all sufficiently large).

Finally, since the strict class number of L^+ is the product of the usual class number of L^+ with $2^{\delta(L^+)}$, we obtain (b). Then by (a), we obtain (c). □

Corollary 9. *With notation as in Theorem 8, the unit signature rank deficiencies for all totally real abelian fields F whose conductor is a product of a divisor of m with an integer whose prime divisors are among the set $\{p_1, \ldots, p_s\}$, are uniformly bounded.*

Proof. Any totally real abelian field F having conductor $dp_1^{n_1}\cdots p_s^{n_s}$ where d is a divisor of m and with $n_i \geq 0$ for $1 \leq i \leq s$ is contained in $\mathbb{Q}(\zeta_{mp_1^{n_1}\cdots p_s^{n_s}})^+$. Hence, $\delta(F) \leq \delta(m; n_1, \ldots, n_s)$ by Remark 1, and the result follows immediately from Theorem 8. □

Remark 10. Corollary 9 shows that among all the abelian fields whose conductors are supported in a fixed finite set of primes, almost all have nearly maximal unit signature rank (in the precise sense that the signature rank deficiencies are uniformly bounded by a constant depending only on the set of primes chosen).

We highlight some particular special cases:

Corollary 11. *Let k be a finite totally real abelian extension of $\mathbb{Q}$. For p an odd prime, let $k^{p,\infty}$ denote the cyclotomic $\mathbb{Z}_p$-extension of k. If k_n is the subfield of $k^{p,\infty}$ of degree p^n over k, then the signature rank of the units of k_n differs from $[k_n : \mathbb{Q}]$ by a constant amount for k_n sufficiently far up the tower.*

Applying Corollary 11 to $k = \mathbb{Q}(\zeta_p)^+$ for p odd, together with Weber's result in Proposition 3, gives the following.

Corollary 12. *For any prime p, the difference between the signature rank of the units of $\mathbb{Q}(\zeta_{p^n})^+$ and $\varphi(p^n)/2$ is constant for n sufficiently large (the constant depending on p).*

Corollary 12 gives another proof of the results in Section 3 that the signature ranks of the units in the fields $\mathbb{Q}(\zeta_{p^n})^+$ tend to infinity as n tends to infinity. Corollary 12 is far superior, asymptotically, to the results in Section 3 for the cyclotomic fields of odd prime power conductor since it shows the signature rank is "nearly" maximal. The result yields relatively little information for any specific $\mathbb{Q}(\zeta_{p^N})^+$, however, since the unit signature rank deficiency for this field could conceivably be close to $\varphi(p^N)/2$ (although, as mentioned, this is not expected to happen). The only explicit lower bounds for the unit signature rank for $\mathbb{Q}(\zeta_{p^n})$ for odd p and, more significantly, for general $\mathbb{Q}(\zeta_m)$ (for example if m is the product of distinct primes, for which the results in this section have little to say), are those in Section 3.

Remark 13. We have done some computations of the signature ranks for the subgroup of circular units in towers of prime power cyclotomic fields. While the computations are somewhat modest (since $\varphi(p^n)$ grows rapidly with n), these computations have exhibited the following behavior: if the signature rank of the circular units in $\mathbb{Q}(\zeta_p)^+$ is $\frac{1}{2}\varphi(p) - \delta$, $(\delta \geq 0)$, then the signature rank of the circular units in the fields $\mathbb{Q}(\zeta_{p^n})^+$ is $\frac{1}{2}\varphi(p^n) - \delta$, i.e., the circular unit signature rank deficiency is constant and equal to its value in the first layer. Whether this behavior persists in general is an extremely interesting question.

We also note that the deficiency of the circular units is at least the deficiency for the full group of units, but may be strictly larger: for the field $\mathbb{Q}(\zeta_{163})^+$ the circular unit deficiency is 2, while the deficiency for the full group of units is 0 (see [Dummit 2018]).

7. Unit signature rank deficiencies in cyclotomic fields

In this section we show that the signature rank deficiency in the maximal real subfields of cyclotomic fields can be arbitrarily large, under the assumption that there exist infinitely many cyclic cubic extensions having a system of totally positive fundamental units.

Suppose k is a cyclic cubic extension of $\mathbb{Q}$ with totally positive fundamental units $\varepsilon_1, \varepsilon_2$. If E_k is the unit group of k, then $E_k = \{\pm 1\} \times \langle \varepsilon_1, \varepsilon_2 \rangle$ and the subgroup $\langle \varepsilon_1, \varepsilon_2 \rangle$ consists of the totally positive units in k.

If the Galois group G of k is generated by σ, then $\langle \varepsilon_1, \varepsilon_2 \rangle$ is a module for the quotient $\mathbb{Z}[G]/(\sigma^2 + \sigma + 1)$ of the group ring $\mathbb{Z}[G]$ of G since ε_1 and ε_2 both have norm $+1$. This quotient of the group ring is isomorphic to the ring of integers in $\mathbb{Q}(\sqrt{-3})$, which is a principal ideal domain, and it follows that $\langle \varepsilon_1, \varepsilon_2 \rangle \simeq \mathbb{Z}[G]/(\sigma^2 + \sigma + 1)$ as G-modules (and not just as abelian groups).

Modulo squares, $\langle \varepsilon_1, \varepsilon_2 \rangle$ is therefore isomorphic to $\mathbb{F}_2[G]/(\sigma^2 + \sigma + 1)$ as a module over the group ring $\mathbb{F}_2[G]$, and hence affords the unique irreducible 2-dimensional representation of $\mathbb{F}_2[G]$. In particular, G acts irreducibly and with no nontrivial fixed points on $\langle \varepsilon_1, \varepsilon_2 \rangle$ modulo squares.

With these preliminaries, we consider the composite of cyclic cubic fields having a system of totally positive fundamental units:

Proposition 14. *Suppose $k_1, \ldots, k_n$ are linearly disjoint cyclic cubic fields, each with a totally positive system of fundamental units, i.e., with unit signature rank deficiency $\delta(k_i)$ equal to 2, $i = 1, \ldots, n$. Then the unit signature rank deficiency $\delta(k_1 \cdots k_n)$ for the composite field $k_1 \cdots k_n$ is at least $2n$.*

Proof. We need to prove there are at least $2n$ totally positive units in $k_1 \cdots k_n$ that are multiplicatively independent modulo squares (in $k_1 \cdots k_n$). We proceed by induction on n, the case $n = 1$ being trivial. Suppose by induction that the composite $k_1 \cdots k_s$ contains $2s$ totally positive units $\varepsilon'_1, \ldots, \varepsilon'_{2s}$ that are multiplicatively independent modulo squares in $k_1 \cdots k_s$. By assumption, the field k_{s+1} contains two totally positive units $\varepsilon_1, \varepsilon_2$ that are multiplicatively independent modulo squares in k_{s+1}.

Suppose $\varepsilon \in \langle \varepsilon_1, \varepsilon_2 \rangle$ and $\varepsilon' \in \langle \varepsilon'_1, \ldots, \varepsilon'_{2s} \rangle$ with $\varepsilon\varepsilon' = \alpha^2$ for some α in the composite field $F = k_1 \cdots k_s k_{s+1}$.

Let $\sigma \in \mathrm{Gal}(F/\mathbb{Q})$ be a lift of a generator for the cyclic group $\mathrm{Gal}(k_{s+1}/\mathbb{Q})$ that is the identity on $k_1 \cdots k_s$. Then $\sigma(\varepsilon)\varepsilon' = \sigma(\alpha)^2$, so $\sigma(\varepsilon)/\varepsilon = (\sigma(\alpha)/\alpha)^2$ is a square in F, hence $k_{s+1}(\sqrt{\sigma(\varepsilon)/\varepsilon})$ is a subfield of F. Since F has degree 3^s over k_{s+1}, $k_{s+1}(\sqrt{\sigma(\varepsilon)/\varepsilon})$ cannot be a quadratic extension, so $\sigma(\varepsilon)/\varepsilon$ is in fact a square in k_{s+1}. Since σ acting on $\langle \varepsilon_1, \varepsilon_2 \rangle$ modulo squares in k_{s+1} has no nontrivial fixed point, it follows that ε is the square of a unit in k_{s+1}. Then $\varepsilon' = \alpha^2/\varepsilon$ would be a square in F, a cubic extension of $k_1 \cdots k_s$, and, as before, this implies that ε' would be a square in $k_1 \cdots k_s$.

This shows that the totally positive units $\varepsilon_1, \varepsilon_2, \varepsilon'_1, \ldots, \varepsilon'_{2s}$ are multiplicatively independent modulo squares in F, completing the proof by induction. □

For a cyclic cubic field, either there is a totally positive system of fundamental units or the units have all possible signatures. Heuristics, supported by computations, in [Breen et al. ≥ 2019] suggest that, when counted by discriminant, there is a positive proportion of cyclic cubic fields of either type. Roughly 3% of cyclic cubic fields (see [Breen et al. ≥ 2019] for the precise value) are predicted to have unit signature rank deficiency 2, so in particular there should exist infinitely many such cubic fields that are linearly disjoint.

Theorem 15. *Suppose, as expected, that there exist infinitely many cyclic cubic fields having a totally positive system of fundamental units. Then the difference between $\varphi(m)/2$ and the unit signature rank of $\mathbb{Q}(\zeta_m)^+$ can be arbitrarily large.*

Proof. By Proposition 14 and Remark 1, to obtain a unit signature rank deficiency at least $2n$ it suffices to take the cyclotomic field whose conductor is the product of the distinct primes (which are congruent to 1 mod 3) dividing the conductors of n linearly disjoint cyclic cubic fields having totally positive fundamental units. □

Remark 16. The same sort of arguments could be applied to composites of other abelian fields of, for example, odd prime degree. As in the case for cubic fields, it is expected that there are infinitely many such cyclic extensions of $\mathbb{Q}$ with nonzero deficiencies (see [Breen et al. ≥ 2019] for specific predictions). This suggests that the unit signature rank deficiency can increase without bound as one moves "horizontally" among cyclotomic fields, that is, over fields $\mathbb{Q}(\zeta_m)^+$ where m is the product of an increasing number of distinct primes, as opposed to the results of Section 6 which show the deficiency is bounded as one moves "vertically" among cyclotomic fields.

8. Remarks on 2-adic unit signature rank deficiencies

The results above have implications for analogous deficiencies for the "2-adic signatures" of units in the sense of [Dummit and Voight 2018, Section 4], which we now very briefly outline. We use the notation of [Dummit and Voight 2018].

If F is a totally real field with $[F:\mathbb{Q}] = n$ then there is a structure theorem for the image of the 2-Selmer group, $\mathrm{Sel}_2(F)$, under the 2-Selmer signature map φ [Dummit and Voight 2018, Theorem A.13]. The space $\varphi(\mathrm{Sel}_2(F))$ is n-dimensional over $\mathbb{F}_2$ and is an orthogonal direct sum $U \perp S \perp U'$, where U is the subspace of elements whose 2-adic signature is trivial, U' is the subspace of elements whose archimedean signature is trivial, and S is a diagonal subspace. Since F is totally real, the dimension of U' is the same as the dimension of U [Dummit and Voight 2018, Theorem A.13(a)] and so S has dimension $n - 2\dim(U)$.

Suppose now that the unit signature rank deficiency of F is $\delta(F)$. Then the set of signatures of units is a subspace of dimension $n - \delta(F)$. It follows that the dimension of U is at most $\delta(F)$ (so S has dimension at least $n - 2\delta(F)$) and that the dimension of the image $\varphi(E_F)$ of the units E_F of F is at least $n - \delta(F)$. Then $\dim(\varphi(E_F) + S)$ is at most $\dim \varphi(\mathrm{Sel}_2(F)) = n$ and

$$\begin{aligned}\dim(\varphi(E_F) \cap S) &= \dim \varphi(E_F) + \dim S - \dim(\varphi(E_F) + S) \\ &\geq (n - \delta(F)) + (n - 2\delta(F)) - n = n - 3\delta(F).\end{aligned}$$

Since S is a diagonal subspace, it follows that the dimension of the subspace of 2-adic signatures of the units of F is at least $n - 3\delta(F)$. Hence the 2-adic signature deficiency of the units of F is at most $3\delta(F)$.

As a consequence, Theorem 8(b) and Corollaries 9, 11, and 12 remain true if the (archimedean) signature rank of the units is replaced by the 2-adic signature rank of the units.

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DAVID S. DUMMIT
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF VERMONT
BURLINGTON, VT
UNITED STATES
dummit@math.uvm.edu

EVAN P. DUMMIT
SCHOOL OF MATHEMATICAL AND STATISTICAL SCIENCES
ARIZONA STATE UNIVERSITY
TEMPE, AZ
UNITED STATES
evan.dummit@asu.edu

HERSHY KISILEVSKY
DEPARTMENT OF MATHEMATICS AND STATISTICS
CONCORDIA UNIVERSITY
MONTREAL, QC
CANADA
hershy.kisilevsky@concordia.ca

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