Pacific Journal of Mathematics

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Volume 298 No. 2 February 2019

PACIFIC JOURNAL OF MATHEMATICS
Vol. 298, No. 2, 2019
dx.doi.org/10.2140/pjm.2019.298.375

NONHOLOMORPHIC LEFSCHETZ FIBRATIONS WITH (−1)-SECTIONS

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We construct two types of nonholomorphic Lefschetz fibrations over S^2 with (-1)-sections — hence, they are fiber sum indecomposable — by giving the corresponding positive relators. One type of the two does not satisfy the slope inequality (a necessary condition for a fibration to be holomorphic) and has a simply connected total space, and the other has a total space that cannot admit any complex structure in the first place. These give an alternative existence proof for nonholomorphic Lefschetz pencils without Donaldson's theorem.

1. Introduction

The notion of Lefschetz fibrations in the smooth category was introduced from algebraic geometry by Moishezon [1977] to study complex surfaces from a topological viewpoint. It is therefore natural to ask how far smooth (symplectic) Lefschetz fibrations are from holomorphic ones. One approach to this question is to construct various nonholomorphic examples. Motivated by this, we give the following results.

Theorem 1.1. *For each $g \geq 3$, there is a genus-g nonholomorphic Lefschetz fibration $X \to S^2$ with a (-1)-section and $\pi_1(X) = 1$ such that it does not satisfy the "slope inequality".*

Theorem 1.2. *For each $g \geq 4$, there is a family of genus-g nonholomorphic Lefschetz fibrations $X_{\widehat{U}_n} \to S^2$ with two disjoint (-1)-sections (for each positive integer n) such that $X_{\widehat{U}_n}$ does not admit any complex structure with either orientation and is not homotopically equivalent to $X_{\widehat{U}_m}$ when $n \neq m$.*

Here, a nonholomorphic Lefschetz fibration means that it is not isomorphic to any holomorphic one. We would like to emphasize that we are able to give explicit monodromy factorizations of the above fibrations although we only give a procedure to get such factorizations without explicitly showing them. In the rest of this section, we give some background on Theorems 1.1 and 1.2.

MSC2010: 14D06, 55R55, 57R15, 57R20.
Keywords: Lefschetz fibrations, (-1)-sections, slope inequality, complex structure.

1A. ***Lefschetz fibrations with* (-1)*-sections.*** The reason that we focus on Lefschetz fibrations that have (-1)-sections is that they play an important role as follows. Blowing up at the base loci of a genus-g Lefschetz pencil yields a genus-g Lefschetz fibration with (-1)-sections, and conversely, blowing down of (-1)-sections of a genus-g Lefschetz fibration gives a genus-g Lefschetz pencil. Furthermore, a closed 4-manifold admits a symplectic structure if and only if it admits a Lefschetz pencil (Donaldson [1999] proved the "if" part, and the "only if" part was shown in [Gompf and Stipsicz 1999]). On the other hand, a Lefschetz fibration with a (-1)-section is fiber sum indecomposable (see [Stipsicz 2001; Smith 2001a]); hence, such a fibration can be considered "prime" with respect to the fiber sum operation. Therefore, as a corollary of Theorems 1.1 and 1.2, we obtain the following result.

Corollary 1.3. *For arbitrary $g \geq 3$, there exists a genus-g nonholomorphic Lefschetz pencil on a simply connected 4-manifold. For arbitrary $g \geq 4$, there exist infinitely many genus-g nonholomorphic Lefschetz pencils on 4-manifolds that cannot admit any complex structure with either orientation.*

Remark 1.4. Baykur [2015] constructed infinitely many nonholomorphic genus-3 Lefschetz pencils with explicit monodromies. The 4-manifolds obtained as the total spaces are not simply connected and do not admit any complex structure with either orientation.

Remark 1.5. Donaldson's construction of Lefschetz pencils on symplectic 4-manifolds immediately implies the existence of nonholomorphic Lefschetz pencils since there are symplectic 4-manifolds that cannot be complex. Yet this does not tell much about the genera of the resulting pencils. Our result shows the existence of nonholomorphic Lefschetz pencils for *arbitrary genus $g \geq 3$*.

1B. ***The slope inequality and simply connected examples.*** The "slope inequality" derives from the geography problem of relatively minimal holomorphic fibrations. Let us consider a relatively minimal genus-g holomorphic fibration $f : S \to C$ where S and C are a complex surface and a complex curve, respectively. Xiao [1987] defined a certain numerical invariant λ_f, called the "slope" of f, determined by the signature and Euler characteristic of S, the genera of C and a generic fiber. Then he showed that every relatively minimal genus-g holomorphic fibration f satisfies $4 - 4/g \leq \lambda_f$. We call this inequality the slope inequality.

The notion of the slope can be extended for (smooth) Lefschetz fibrations as λ_f is determined by topological invariants (see Section 3D); hence we can also consider the slope inequality in the smooth category. Note that the slope inequality can be rewritten as an inequality giving a lower bound on the signatures of Lefschetz fibrations in terms of the genus of a generic fiber and the number of singular fibers (see Remark 3.11). It is known that the slope inequality holds for any hyperelliptic Lefschetz fibration, especially any genus-2 Lefschetz fibration. Hain conjectured that every Lefschetz fibration over S^2 satisfies the slope inequality as well (see [Amorós et al. 2000; Endo and Nagami 2005]). This conjecture in fact fails; Monden [2014, Theorem 3.1] gave examples violating the slope inequality. In particular, those examples are nonholomorphic by Xiao's result. However, we do not know if they are fiber sum indecomposable. Hence, we ask the following question: *Is there a fiber sum indecomposable Lefschetz fibration violating the slope inequality?* Theorem 1.1 together with the above-mentioned work of Stipsicz [2001] and Smith [2001a] implies that the answer to this question is positive for any $g \geq 3$.

Let us consider a genus-g nonholomorphic Lefschetz fibration $X \to S^2$ with a (-1)-section such that $\pi_1(X) = 1$. To the best of our knowledge, all known such fibrations with explicit monodromy factorizations are Fuller's example $(g = 3)$[1] and those of Endo and Nagami [2005] $(g = 3, 4, 5)$. Theorem 1.1 gives such examples with explicit monodromy factorizations for arbitrary $g \geq 3$.

Remark 1.6. We do not know whether the examples in [Smith 2001b; Endo and Nagami 2005] and Theorem 1.1 have noncomplex total spaces or not. On the other hand, Li [2008] constructed nonholomorphic Lefschetz pencils (fibrations with (-1)-sections) on complex surfaces. However, their genera are implicit.

1C. ***Lefschetz fibrations with noncomplex total spaces.*** Many Lefschetz fibrations with explicit monodromies and noncomplex total spaces have been constructed using the (twisted) fiber sum operation (see for instance [Smith 1998; Ozbagci and Stipsicz 2000; Fintushel and Stern 1998; Korkmaz 2001; Akhmedov and Ozbagci 2017[2] Akhmedov and Monden 2015; Baykur and Korkmaz 2017]). They are nonholomorphic, however, they do not have any (-1)-section since they are decomposable. On the other hand, Stipsicz [2001] and, independently, Smith [2001a] proved that there are infinitely many fiber sum indecomposable Lefschetz fibrations with noncomplex total spaces. Since the constructions of these fibrations are based on Donaldson's theorem [1999], their monodromy factorizations are not explicitly given. Theorem 1.2 gives infinitely many fiber sum indecomposable Lefschetz fibrations with explicit monodromy factorizations and noncomplex total spaces for any $g \geq 4$.

The fundamental group of the total space $X_{\widehat{U}_n}$ of a genus-g Lefschetz fibration in the family in Theorem 1.2 is $H_1(X_{\widehat{U}_n}) = \mathbb{Z} \oplus \mathbb{Z}_n$. By improving the work of [Ozbagci and Stipsicz 2000] (see also [Baykur 2012]) slightly, we see that the 4-manifold $X_{\widehat{U}_n}$ does not carry any complex structure with either orientation. For $g \geq 22$,

[1] It was shown by Smith [2001b] that Fuller's example is nonholomorphic.

[2] Baykur has informed us that the examples in [Akhmedov and Ozbagci 2017] should be fiber sum decomposable from Ozbagci's talk in Turkey a few years ago.

nonholomorphic Lefschetz fibrations with the same property of Theorem 1.2 were constructed in [Kobayashi and Monden 2016] based on the technique of this paper. Theorem 1.2 improves this result.

Remark 1.7. Nonholomorphic genus-2 Lefschetz fibrations with finite cyclic fundamental groups and without any (-1)-sections were constructed in [Akhmedov and Monden 2015] by rationally blowing down a twisted fiber sum of two copies of Matsumoto's fibration. However, we do not know whether these are decomposable.

2. Preliminaries

2A. ***Notation.*** From now on, we use the same letter for a loop and its homotopy class and homology class by abuse of notation. Similarly, we use the same letter for a diffeomorphism and its isotopy class, or for a simple closed curve and its isotopy class. A simple loop and a simple closed curve are even denoted by the same letter. It will cause no confusion as it will be clear from the context which one we mean.

For convenience's sake, we first fix the notation and the symbols for the curves which we use throughout the paper. Let Σ_g be the closed oriented surface of genus g standardly embedded in the 3-space and

$$a_1, b_1, a_2, b_2, \ldots, a_g, b_g$$

be the standard generators of the fundamental group $\pi_1(\Sigma_g)$ of Σ_g as shown in Figure 1. We choose orientations of a_i, b_i so that $i(a_i, b_i) = 1$, where $i(a_i, b_i)$ is the algebraic intersection number of a_i and b_i. For loops a and b in $\pi_1(\Sigma_g)$, the product ab means that we traverse first a then b as usual. Let $c_1, c_2, \ldots, c_g$ and a_{g+1} be the simple closed curves on Σ_g as shown in Figure 1. Note that in $\pi_1(\Sigma_g)$, $c_g = 1$ and $a_{g+1} = 1$. Then, the fundamental group $\pi_1(\Sigma_g)$ has the presentation

$$\pi_1(\Sigma_g) = \langle a_1, b_1, a_2, b_2, \ldots, a_g, b_g \mid c_g \rangle.$$

Let $B^h_{0,1}, B^h_{0,2}, B^h_1, B^h_2, \ldots, B^h_h$ $(h = 1, 2, \ldots, g)$ and $a'_1, a'_2, \ldots, a'_g$ be the simple closed curves on Σ_g as shown in Figures 2 and 3.

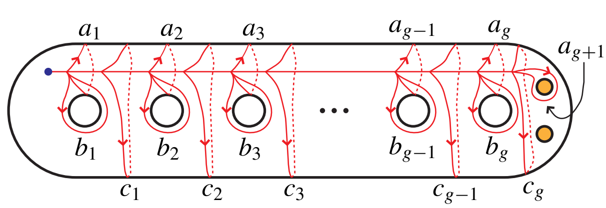

Figure 1. The standardly embedded Σ_g with two indicated disks on the rightmost position and the generators a_j, b_j of the fundamental group and loops c_j.

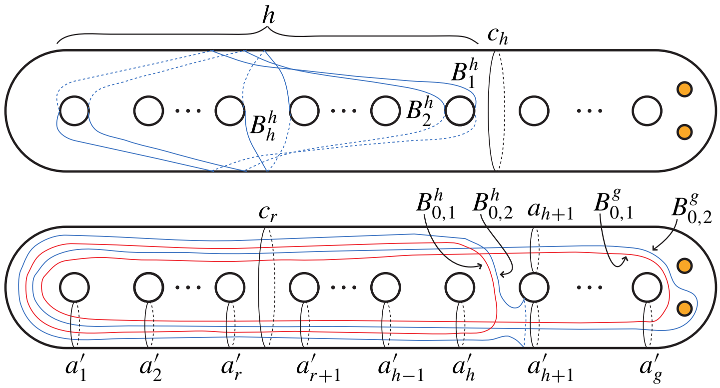

Figure 2. The curves $B_{0,1}^h, B_{0,2}^h, B_1^h, B_2^h, \dots, B_h^h, a_1', a_2', \dots, a_g'$ for $h = 2r$.

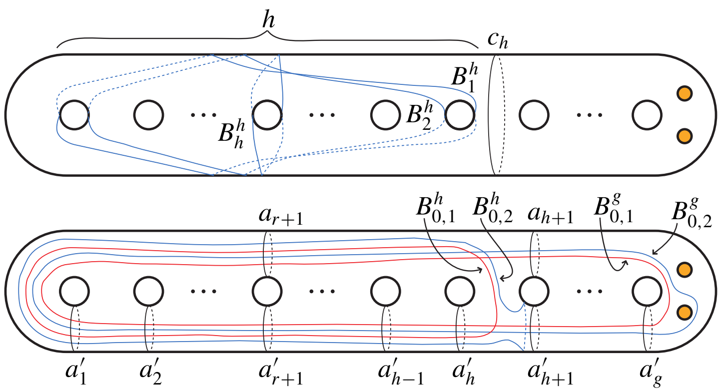

Figure 3. The curves $B_{0,1}^h, B_{0,2}^h, B_1^h, B_2^h, \dots, B_h^h, a_1', a_2', \dots, a_g'$ for $h = 2r+1$.

Suppose $h = 2r$. It is easy to check that the following equalities hold in $H_1(\Sigma_g)$:

(1) $B_{0,1}^h = b_1 + b_2 + \cdots + b_h, \quad B_{0,2}^h = b_1 + b_2 + \cdots + b_h + a_{h+1}, \quad 1 \le h \le g;$

(2) $B_{2k-1}^h = a_k + b_k + b_{k+1} + \cdots + b_{h+1-k} + a_{h+1-k}, \quad 1 \le k \le r, \ 1 \le h \le g;$

(3) $B_{2k}^h = a_k + b_{k+1} + b_{k+2} + \cdots + b_{h-k} + a_{h+1-k}, \quad 1 \le k \le r, \ 1 \le h \le g.$

In the case of $h = 2r+1$, the same equalities (1) and (3) hold without change, while the equality (2) holds for $1 \le k \le r+1, \ 1 \le h \le g$.

2B. ***Substitution technique.*** In this subsection, we introduce key techniques, called a *substitution* and a *partial conjugation*, for constructing a new word in mapping class groups from a word and a relator. We will utilize this technique to construct Lefschetz fibrations with (−1)-section in the later sections.

Let Σ_g^b be a compact oriented surface of genus g with b boundary components.

The *mapping class group* Γ_g^b of Σ_g^b is the group of isotopy classes of orientation-preserving self-diffeomorphisms of Σ_g^b, where all the maps involved are assumed to fix $\partial\Sigma_g^b$ pointwise. For simplicity, we write $\Sigma_g = \Sigma_g^0$ and $\Gamma_g = \Gamma_g^0$. For two elements ϕ_1 and ϕ_2 in Γ_g^b, the product $\phi_2\phi_1$ means that we first apply ϕ_1 then ϕ_2. We denote by t_c the right-handed *Dehn twist* along a simple closed curve c on Σ_g^b.

Definition 2.1. Let $v_1, v_2, \dots, v_n$ be simple closed curves on Σ_g^b. If $t_{v_n}^{\epsilon_n} \cdots t_{v_2}^{\epsilon_2} t_{v_1}^{\epsilon_1} = 1$ in Γ_g^b, where $\epsilon_i = \pm 1$, then this factorization is called a *relator*. In the special case where $\epsilon_i = 1$ for all i, namely, $t_{v_n} \cdots t_{v_2} t_{v_1} = 1$ holds in Γ_g, then this factorization is called a *positive relator*.

We introduce a key technique for constructing a new product of right-handed Dehn twists in Γ_g^b from old ones.

Definition 2.2. Let $v_1, v_2, \dots, v_k$ and $d_1, d_2, \dots, d_l$ be simple closed curves on Σ_g^b such that the following product, denoted by R, is a relator in Γ_g^b:

$$R := \; t_{v_1} t_{v_2} \cdots t_{v_k} t_{d_l}^{-1} \cdots t_{d_2}^{-1} t_{d_1}^{-1},$$

which equals the identity as a mapping class by definition. If a mapping class ϕ in Γ_g^b satisfies $\phi(d_i) = d_i$, then by the relation $t_{\phi(c)} = \phi t_c \phi^{-1}$, we obtain the following relator, denoted by R^ϕ, in Γ_g^b:

$$R^\phi = \; t_{\phi(v_1)} t_{\phi(v_2)} \cdots t_{\phi(v_k)} t_{d_l}^{-1} \cdots t_{d_2}^{-1} t_{d_1}^{-1}.$$

Let W be a product of right-handed Dehn twists including $t_{d_1} t_{d_2} \cdots t_{d_l}$ as a subword:

$$W = U \cdot t_{d_1} t_{d_2} \cdots t_{d_l} \cdot V,$$

where U and V are products of right-handed Dehn twists. Then, we get a new product of right-handed Dehn twists, denoted by W', as follows:

$$U \cdot R^\phi \cdot t_{d_1} t_{d_2} \cdots t_{d_l} \cdot V = U \cdot t_{\phi(v_1)} t_{\phi(v_2)} \cdots t_{\phi(v_k)} \cdot V =: W',$$

where the first equality means the equality as a mapping class. Then, W' is said to be obtained by applying a *R^ϕ-substitution* to W.

Remark 2.3. A R^ϕ-substitution is a combination of a substitution technique and a partial conjugation introduced by Fuller and Auroux [Auroux 2006b; Auroux 2006a], respectively.

2C. *Relators in mapping class groups.* In this subsection, we introduce some well-known relators in mapping class groups, called the braid relator B, the lantern relator L, the chain relators C_k, $\overline{C}_k$ and certain relators W_1^h, W_2^h.

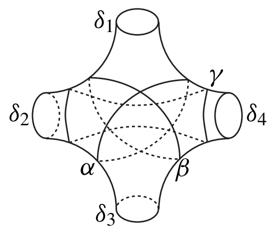

Figure 4. The curves δ_1, δ_2, δ_3, δ_4 and α, β, γ.

Definition 2.4 (braid relator). Let α and β be simple closed curves on Σ_g^b. If the geometric intersection number of α and β is equal to 0 (resp. 1), then we have the *braid relator* B:

$$B := t_\alpha t_\beta t_\alpha^{-1} t_\beta^{-1} \quad (\text{resp. } B := t_\alpha t_\beta t_\alpha t_\beta^{-1} t_\alpha^{-1} t_\beta^{-1}).$$

Definition 2.5 (lantern relator). Let δ_1, δ_2, δ_3 and δ_4 be the four boundary curves of Σ_0^4 and let α, β and γ be the interior curves as shown in Figure 4. Then, we have the *lantern relator* L in Γ_0^4:

$$L := t_\alpha t_\beta t_\gamma t_{\delta_4}^{-1} t_{\delta_3}^{-1} t_{\delta_2}^{-1} t_{\delta_1}^{-1}.$$

The lantern relator was discovered by Dehn [1938] and was rediscovered by Johnson [1979].

Definition 2.6 (chain relator). Suppose $h \geq 1$. Let $\alpha_1, \alpha_2, \ldots, \alpha_{2h+1}$ be simple closed curves on an oriented surface such that α_i and α_{i+1} intersect transversally at exactly one point for $1 \leq i \leq 2h$ and that α_i and α_j are disjoint if $|i-j| \geq 2$. Then, a regular neighborhood of $\alpha_1 \cup \alpha_2 \cup \cdots \cup \alpha_{2h}$ (resp. $\alpha_1 \cup \alpha_2 \cup \cdots \cup \alpha_{2h+1}$) is a subsurface of genus h with one boundary component (resp. two boundary components), say d (resp. d_1 and d_2). We then have the *even chain relator* C_{2h} and the *odd chain relator* C_{2h+1}:

$$C_{2h} := (t_{\alpha_1} t_{\alpha_2} \cdots t_{\alpha_{2h}})^{4h+2} t_d^{-1},$$
$$C_{2h+1} := (t_{\alpha_1} t_{\alpha_2} \cdots t_{\alpha_{2h+1}})^{2h+2} t_{d_2}^{-1} t_{d_1}^{-1}.$$

Definition 2.7. Suppose $g \geq 2$. Let Σ_g^2 be the surface of genus g with two boundary components obtained from Σ_g by removing two disjoint open disks (see Figures 1, 2 and 3). Let a_{g+1} be one of the boundary curves of Σ_g^2 as shown in Figure 1, and let a'_{g+1} be the other boundary curve of Σ_g^2 defined by $a'_{g+1} = c_g a_{g+1}$. We then have the following two relators $W_{1,h}$, $W_{2,h}$ in Γ_g^2 for each $h = 1, 2, \ldots, g$:

$$W_{1,h} := \begin{cases} (t_{B_{0,1}^h} t_{B_1^h} t_{B_2^h} \cdots t_{B_{h-1}^h} t_{B_h^h} t_{c_\ell})^2 t_{c_h}^{-1} & \text{if } h = 2\ell, \\ (t_{B_{0,1}^h} t_{B_1^h} t_{B_2^h} \cdots t_{B_{h-1}^h} t_{B_h^h} t_{a_{\ell+1}}^2 t_{a'_{\ell+1}}^2)^2 t_{c_h}^{-1} & \text{if } h = 2\ell+1, \end{cases}$$

$$W_{2,h} := \begin{cases} (t_{B^h_{0,2}} t_{B^h_1} t_{B^h_2} \cdots t_{B^h_{h-1}} t_{B^h_h} t_{c_\ell})^2 t^{-1}_{a_{h+1}} t^{-1}_{a'_{h+1}} & \text{if } h = 2\ell, \\ (t_{B^h_{0,2}} t_{B^h_1} t_{B^h_2} \cdots t_{B^h_{h-1}} t_{B^h_h} t^2_{a_{\ell+1}} t^2_{a'_{\ell+1}})^2 t^{-1}_{a_{h+1}} t^{-1}_{a'_{h+1}} & \text{if } h = 2\ell+1. \end{cases}$$

Note that in Γ_g, the relator $W_{2,g}$ is a positive relator. Matsumoto [1996] discovered this positive relator for $g = 2$, and Cadavid [1998] and independently Korkmaz [2001] generalized Matsumoto's relator to $g \geq 3$. $W_{1,g}$ was shown to be a relator in Γ^1_g by Ozbagci and Stipsicz [2004]. In [Korkmaz 2009], it was claimed without proof that $W_{2,g}$ is a relator in Γ^2_g. Yet, we can show it to be true by applying the same argument in Section 2 of [Korkmaz 2001] (for example see Section 6 of [Kobayashi and Monden 2016]).

3. Lefschetz fibrations

3A. ***Basics on Lefschetz fibrations.*** We recall the definition and basic properties of Lefschetz fibrations. More details can be found in [Gompf and Stipsicz 1999].

Definition 3.1. Let X be a closed, oriented smooth 4-manifold. A smooth map $f : X \to S^2$ is a *Lefschetz fibration* if for each critical point p of f and $f(p)$, there are complex local coordinate charts agreeing with the orientations of X and S^2 on which f is of the form $f(z_1, z_2) = z_1 z_2$.

It follows that f has finitely many critical points $C = \{p_1, p_2, \ldots, p_n\}$. We can assume that f is injective on C and relatively minimal (i.e., no fiber contains a sphere with self-intersection number -1). Each fiber which contains a critical point, called a *singular fiber*, is obtained by "collapsing" a simple closed curve in the prescribed regular fiber to a point. We call the simple closed curve in the regular fiber the *vanishing cycle*. If the genus of the regular fiber of f is g, then we call f a *genus-g Lefschetz fibration*.

The monodromy of the fibration around a singular fiber $f^{-1}(f(p_i))$ is given by a right-handed Dehn twist along the corresponding vanishing cycle, denoted by v_i. Once we fix an identification of Σ_g with the fiber over a base point of S^2, we can characterize the Lefschetz fibration $f : X \to S^2$ by its *monodromy representation* $\pi_1(S^2 - f(C)) \to \Gamma_g$. Here, this map is indeed an antihomomorphism. Let $\gamma_1, \gamma_2, \ldots, \gamma_n$ be an ordered system of generating loops for $\pi_1(S^2 - f(C))$ such that each γ_i encircles only $f(p_i)$ and $\gamma_1\gamma_2\cdots\gamma_n = 1$ in $\pi_1(S^2 - f(C))$. Thus, the monodromy of f comprises a positive relator

$$t_{v_n} \cdots t_{v_2} t_{v_1} = 1 \ \text{ in } \Gamma_g.$$

Conversely, for any positive relator P in Γ_g, one can construct a genus-g Lefschetz fibration over S^2 whose monodromy is P. Therefore, we denote a genus-g Lefschetz fibration associated with a positive relator P in Γ_g by $f_P : X_P \to S^2$.

Two Lefschetz fibrations $f_{P_i} : X_{P_i} \to S^2$ $(i = 1, 2)$ are said to be *isomorphic* if there exist orientation-preserving diffeomorphisms $H : X_{P_1} \to X_{P_2}$ and $h : S^2 \to S^2$ such that $f_{P_2} \circ H = h \circ f_{P_1}$. According to theorems of Kas [1980] and Matsumoto [1996], if $g \geq 2$, then the isomorphism class of a Lefschetz fibration is determined by a positive relator modulo *simultaneous conjugations*

$$t_{v_n} \cdots t_{v_2} t_{v_1} \sim t_{\phi(v_n)} \cdots t_{\phi(v_2)} t_{\phi(v_1)} \quad \text{for any } \phi \in \Gamma_g$$

and *elementary transformations*

$$t_{v_n} \cdots t_{v_{i+2}} t_{v_{i+1}} t_{v_i} t_{v_{i-1}} t_{v_{i-2}} \cdots t_{v_1} \sim t_{v_n} \cdots t_{v_{i+2}} t_{v_i} t_{t_{v_i}^{-1}(v_{i+1})} t_{v_{i-1}} t_{v_{i-2}} \cdots t_{v_1},$$

$$t_{v_n} \cdots t_{v_{i+2}} t_{v_{i+1}} t_{v_i} t_{v_{i-1}} t_{v_{i-2}} \cdots t_{v_1} \sim t_{v_n} \cdots t_{v_{i+2}} t_{v_{i+1}} t_{t_{v_i}(v_{i-1})} t_{v_i} t_{v_{i-2}} \cdots t_{v_1}.$$

Therefore, if P_2 is obtained by applying a series of elementary transformations and simultaneous conjugations to P_1, then

$$\sigma(X_{P_1}) = \sigma(X_{P_2}) \quad \text{and} \quad e(X_{P_1}) = e(X_{P_2}), \tag{4}$$

where $\sigma(X)$ and $e(X)$ stand for the signature and Euler characteristic of a 4-manifold X, respectively.

3B. *Sections of Lefschetz fibrations.*

Definition 3.2. Let $f : X \to S^2$ be a Lefschetz fibration. A map $\sigma : S^2 \to X$ is called a k*-section* of f if it satisfies $f \circ \sigma = \mathrm{id}_{S^2}$ and the self-intersection number $[\sigma(S^2)]^2 = k$, where $[\sigma(S^2)]$ is the homology class in $H_2(X; \mathbb{Z})$.

If the factorization $P = t_{v_n} \cdots t_{v_2} t_{v_1} (= 1)$ lifts from Γ_g to Γ_g^1 as

$$t_\delta^k = t_{\tilde{v}_n} \cdots t_{\tilde{v}_2} t_{\tilde{v}_1} \quad (\text{i.e., } 1 = t_{\tilde{v}_n} \cdots t_{\tilde{v}_2} t_{\tilde{v}_1} t_\delta^{-k}),$$

then the Lefschetz fibration f_P has a $(-k)$-section. Here, δ is the boundary curve of Σ_g^1 and $t_{\tilde{v}_i}$ is a Dehn twist mapped to t_{v_i} under $\Gamma_g^1 \to \Gamma_g$. Conversely, if a genus-g Lefschetz fibration admits a $(-k)$-section, we obtain a relator of the above type in Γ_g^1. A similar relator holds for b disjoint sections (in which case one has to work in the mapping class group Γ_g^b).

A necessary condition for a Lefschetz fibration to admit a (-1)-section was shown independently by Stipsicz [2001] and Smith [2001a]:

Theorem 3.3 [Stipsicz 2001; Smith 2001a]. *Let $g \geq 1$. If a genus-g Lefschetz fibration $f : X \to S^2$ admits a (-1)-section, then f is fiber sum indecomposable.*

Here, we recall the definition of fiber sum. Let $f_i : X_i \to S^2$ be a genus-g Lefschetz fibration for $i = 1, 2$, and let D_i be an open disk on S^2 which does not contain any critical values. Then, the *fiber sum* $f_1 \#_F f_2 : X_1 \#_F X_2 \to S^2$ is obtained by gluing $X_1 - f_1^{-1}(D_1)$ and $X_2 - f_2^{-1}(D_2)$ along their boundaries via a

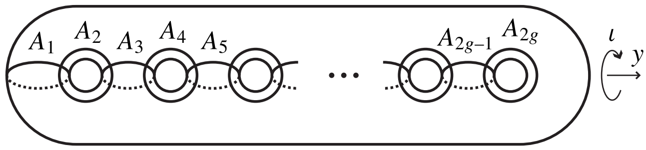

Figure 5. The involution ι of Σ and the curves $A_1, A_2, \ldots, A_{2g}$ on Σ_g.

fiber-preserving orientation-reversing diffeomorphism and extending f_1 and f_2 in a natural way. A Lefschetz fibration is said to be *fiber sum indecomposable* if it cannot be decomposed as a fiber sum of two Lefschetz fibrations each of which has at least one singular point.

For a Lefschetz fibration over S^2 with a positive relator and a section, we can determine the fundamental group of X as follows:

Lemma 3.4 (see [Gompf and Stipsicz 1999]). *Let P be a positive relator $P = t_{v_n} \cdots t_{v_2} t_{v_1}$ in Γ_g. Suppose that the corresponding genus-g Lefschetz fibration $f : X_P \to S^2$ admits a section σ. Then, the fundamental group $\pi_1(X)$ is isomorphic to the quotient of $\pi_1(\Sigma_g)$ by the normal subgroup generated by the vanishing cycles $v_1, v_2, \ldots, v_n$. The same holds for the first homology group $H_1(X)$.*

3C. ***Signatures of Lefschetz fibrations.*** This subsection gives two results about the signatures of Lefschetz fibrations.

Let Δ_g be the *hyperelliptic mapping class group* of genus g, i.e., the subgroup of Γ_g consisting of those mapping classes commuting with the isotopy class of the involution ι shown in Figure 5. Note that $\Delta_g = \Gamma_g$ for $g = 1, 2$ and that t_c is in Δ_g if and only if $\iota(c) = c$.

A genus-g Lefschetz fibration is said to be *hyperelliptic* if it is associated with a positive relator $P = t_{v_1} \cdots t_{v_n}$ such that each t_{v_i} is contained in Δ_g. To compute the signatures of Lefschetz fibrations, we present Matsumoto and Endo's signature formula for hyperelliptic Lefschetz fibrations.

Theorem 3.5 ([Matsumoto 1983; 1996] ($g = 1, 2$), [Endo 2000] ($g \geq 3$)). *Let us consider a genus-g hyperelliptic Lefschetz fibration $f_P : X_P \to S^2$ with n nonseparating and $s = \Sigma_{h=1}^{[g/2]} s_h$ separating vanishing cycles, where s_h is the number of separating vanishing cycles that separate Σ_g into two surfaces, one of which has genus h. Then, we have*

$$\sigma(X_P) = -\frac{g+1}{2g+1} n + \sum_{h=1}^{[g/2]} \left(\frac{4h(g-h)}{2g+1} - 1 \right) s_h.$$

By the work of Endo and Nagami [2005], we see the behavior of signatures of Lefschetz fibrations under a monodromy substitution as follows.

Proposition 3.6 [Endo and Nagami 2005, Theorem 4.3, Definition 3.3, Lemma 3.5 and Propositions 3.9, 3.10 and 3.12]. *Let B, L and C_{2h+1} be the braid relator, the* *lantern relator and the odd chain relator in Definitions 2.4, 2.5 and 2.6, respectively. We assume that those relators are in Σ_g.*

Let $f_{P_i} : X_{P_i} \to S^2$ be a genus-g Lefschetz fibration with a positive relator P_i $(i = 1, 2)$. Suppose that P_2 is obtained by applying an R^ϕ-substitution to P_1, where ϕ is a mapping class and R is a relator in Γ_g.

(1) *If $R = B$, then $\sigma(X_{P_2}) = \sigma(X_{P_1})$.*
(2) *If $R = L$, then $\sigma(X_{P_2}) = \sigma(X_{P_1}) + 1$. Hence, if $R = L^{-1}$, then $\sigma(X_{P_2}) = \sigma(X_{P_1}) - 1$.*
(3) *Assume that both d_1 and d_2 are not nullhomotopic in Σ_g. If $R = C_{2h+1}$, then $\sigma(X_{P_2}) = \sigma(X_{P_1}) + 2h(h + 2)$. Hence, if $R = C_{2h+1}^{-1}$, then $\sigma(X_{P_2}) = \sigma(X_{P_1}) - 2h(h + 2)$.*

3D. *Nonholomorphicity of Lefschetz fibrations.*

Definition 3.7. A Lefschetz fibration $f : X \to S^2$ is said to be *holomorphic* if there are complex structures on both X and S^2 with respect to which f is a holomorphic projection. We say f is *nonholomorphic* if it is not isomorphic to any holomorphic Lefschetz fibration.

Suppose that $g \geq 2$. In order to prove Theorems 1.1 and 1.2, we introduce two sufficient conditions for a Lefschetz fibration to be nonholomorphic.

One comes from the result of Xiao [1987]. For an almost complex 4-manifold X, we set $K^2(X) := 3\sigma(X) + 2e(X)$ and $\chi_h(X) := (\sigma(X) + e(X))/4$. Xiao proved the following theorem, called the *slope inequality*:

Theorem 3.8 [Xiao 1987]. *Every relatively minimal holomorphic genus-g fibration f on a complex surface X over a complex curve C of genus $k \geq 0$ satisfies the inequality*

$$4 - 4/g \leq \lambda_f,$$

where

$$\lambda_f := \frac{K^2(X) - 8(g-1)(k-1)}{\chi_h(X) - (g-1)(k-1)}.$$

As a consequence of Theorem 3.8, we have:

Proposition 3.9. *If a genus-g Lefschetz fibration $f : X \to S^2$ does not satisfy the slope inequality, namely, $\lambda_f < 4 - 4/g$, then f is nonholomorphic.*

The other comes from the result of Ozbagci and Stipsicz [2000]. We present a slightly improved version of their result where we replace π_1 by H_1, but this can be concluded from the proof of Theorem 1.3 in [Ozbagci and Stipsicz 2000]:

Theorem 3.10. *If a Lefschetz fibration $f : X \to S^2$ satisfies $H_1(X) = \mathbb{Z} \oplus \mathbb{Z}_n$ for some positive integer n, then X admits no complex structure with either orientation, so f is nonholomorphic.*

For the convenience of the readers, we give a proof of this theorem, which is merely a simplification of that in [Ozbagci and Stipsicz 2000].

Proof. Assume that X carries a complex structure and let X' be the minimal model of X. By the Enriques–Kodaira classification of complex surfaces, together with the fact that $b_1(X') = 1$ and $b_2^+(X') \geq 1$ (since X admits a symplectic structure and so does X'), we can observe that X' is an elliptic surface. If X' is an elliptic fibration over a Riemann surface Σ, we have $b_1(X') \geq b_1(\Sigma)$. Since $b_1(X') = 1$, Σ must be S^2. Since $b_1(X') = b_3(X') = 1$ and $b_2(X') \neq 0$, the Euler characteristic of X' cannot be 0. Now we suppose that X' is a minimal elliptic surface over S^2 with nonzero Euler characteristic. According to [Gompf 1991], a presentation for the fundamental group of such an elliptic surface is given as

$$\pi_1(X') = \langle x_1, \cdots, x_k \mid x_i^{p_i} = 1, i = 1, \ldots, k;\ x_1 \cdots x_k = 1 \rangle.$$

So it is clear that $H_1(X')$ has only torsion elements, which contradicts the assumption $H_1(X) = \mathbb{Z} \oplus \mathbb{Z}_n$. □

Remark 3.11. If X admits a genus-g Lefschetz fibration $f : X \to S^2$ with n singular fibers, then the Euler characteristic of X is $e(X) = -4(g-1) + n$. Using this fact, the slope λ_f of f can be written as

$$\lambda_f = 12 - \frac{4}{(\sigma(X)/n) + 1},$$

where $\sigma(X)$ is the signature of X. Therefore, we can regard the slope λ_f as the "average signature" $\sigma(X)/n$ per singular fiber. Moreover, the slope inequality $\lambda_f \geq 4 - 4/g$ can be rewritten as

$$\sigma(X) \geq -\frac{g+1}{2g+1} n,$$

that is, it gives a lower bound on σ in terms of g and n.

Remark 3.12. The work of Xiao [1987] was mainly motivated by the so-called Severi inequality, stating that every minimal surface of general type of maximal Albanese dimension satisfies $K^2 \geq 4\chi_h$. This is equivalent to stating that if a minimal complex surface S of general type satisfies $K^2 < 4\chi_h$, then S admits a relatively minimal holomorphic fibration over C of genus $b_1(S)/2$. The Severi inequality was stated in [di Severi 1932] (but the proof was not correct) and independently posed as a conjecture by Reid [1979] and by Catanese [1983]. Xiao proved it when S admits a relatively minimal holomorphic fibration over a curve of positive genus, that is, a complex surface S admitting a holomorphic genus-g fibration f over C of positive genus k with $K^2 < 4\chi_h + 4(g-1)(k-1)$ (i.e., $\lambda_f < 4$) satisfies $k = b_1(S)/2$. The Severi inequality has been studied by many authors (for example [di Severi 1932;

Reid 1979; Catanese 1983; Konno 1996; Manetti 2003; Pardini 2005]) and was proved by Pardini [2005].

Remark 3.13. We denote by $\overline{\mathcal{M}_g}$ the Deligne–Mumford compactified moduli space of stable curves of genus g. We can reformulate the slope inequality for Lefschetz fibrations in terms of $\overline{\mathcal{M}_g}$ as follows. For a genus-g Lefschetz fibration $f : X \to S^2$ with n singular fibers, there is a symplectic structure on X such that for all $x \in S^2$, $f^{-1}(x)$ is a pseudo-holomorphic curve. Since a 2-dimensional almost-complex structure is integrable, $f^{-1}(x)$ determines a point in $\overline{\mathcal{M}_g}$. By defining $\phi_f(x) = [f^{-1}(x)] \in \overline{\mathcal{M}_g}$ for $x \in S^2$, we obtain the moduli map $\phi_f : S^2 \to \overline{\mathcal{M}_g}$. Let $\mathcal{H}_g$ be the Hodge bundle on $\overline{\mathcal{M}_g}$ with fiber the determinant line $\wedge^g H^0(C; K_C)$, where C is the set of critical points of f. Then, by combining the signature formula $\sigma(X) = \langle c_1(\mathcal{H}_g), [\phi_f(S^2)]\rangle - n$ given by Smith [1999] and the slope inequality, we have

$$(2g+1)\langle c_1(\mathcal{H}_g), [\phi_f(S^2)]\rangle - g \cdot n \geq 0.$$

4. Nonholomorphic Lefschetz fibrations admitting (−1)-sections

In this section, we prove Theorem 1.1

Theorem 1.1. *For each $g \geq 3$, there is a genus-g nonholomorphic Lefschetz fibration $X \to S^2$ with a (-1)-section and $\pi_1(X) = 1$ such that it does not satisfy the slope inequality.*

To prove this, we need a lemma. Suppose $g \geq 3$. Let Σ_g^1 be the surface of genus g with one boundary component obtained from Σ_g by removing the open disk whose boundary curve is a_{g+1} (see Figure 1). Let us consider $A_1, A_2, \dots, A_{2g}$ to be the simple closed curves on Σ_g^1 (see Figure 6) defined as follows: $A_1 = a_1$, $A_2 = b_1$, $A_{2h-1} = a_{h-1}a_h^{-1}$ and $A_{2h} = b_h$ for $h = 2, 3, \dots, g$.

Lemma 4.1. $(t_{A_1}t_{A_2}\cdots t_{A_{2g}})^{2g+1} = (t_{A_1}t_{A_2}\cdots t_{A_{2g-1}})^{2g} t_{A_{2g}}\cdots t_{A_2}t_{A_1}t_{A_1}t_{A_2}\cdots t_{A_{2g}}$.

Proof. The proof follows from the braid relations $t_{A_i}t_{A_{i+1}}t_{A_i} = t_{A_{i+1}}t_{A_i}t_{A_{i+1}}$ and $t_{A_i}t_{A_j} = t_{A_j}t_{A_i}$ for $|i-j| > 1$ (i.e., by applying B-substitutions to the left side). □

We now prove Theorem 1.1.

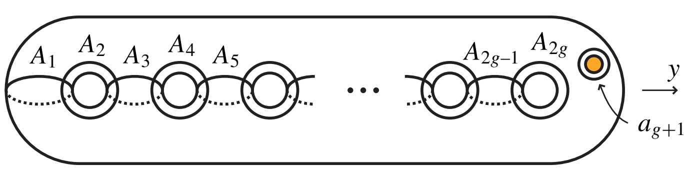

Figure 6. The curves $A_1, A_2, \dots, A_{2g}$ on Σ_g^1.

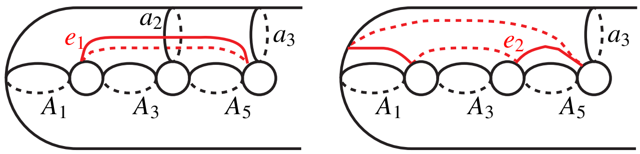

Figure 7. The curves that give a Lantern relator.

Proof of Theorem 1.1. Suppose $g \geq 3$. Let us consider the following chain relators, C_{2g} and C_{2g+1}:

$$C_{2g} = (t_{A_1} t_{A_2} \cdots t_{A_{2g}})^{4g+2} t_{a_{g+1}}^{-1}, \qquad C_{2g-1} = (t_{A_1} t_{A_2} \cdots t_{A_{2g-1}})^{2g} t_{a_g}^{-1} t_{a'_g}^{-1},$$

where a_g and a'_g are the curves as shown in Figures 2 and 3. By Lemma 4.1 and the even chain relator C_{2g}, we obtain the following relator C'_{2g}:

$$C'_{2g} = \{(t_{A_1} t_{A_2} \cdots t_{A_{2g-1}})^{2g} \cdot t_{A_{2g}} \cdots t_{A_2} t_{A_1} t_{A_1} t_{A_2} \cdots t_{A_{2g}}\}^2 t_{a_{g+1}}^{-1}.$$

By applying C_{2g-1}^{-1}-substitution to C'_{2g} twice, we get a new relator H in Γ_g^1:

$$H = (t_{a_g} t_{a'_g} \cdot t_{A_{2g}} \cdots t_{A_2} t_{A_1} t_{A_1} t_{A_2} \cdots t_{A_{2g}})^2 t_{a_{g+1}}^{-1}.$$

Consider the curves on Σ_g^1 in Figure 7. Since A_1, a_2, e_1, and e_2 are nonseparating curves on the subsurface of genus $g-1 (\geq 2)$ with two boundary components a_g and a'_g, there are diffeomorphisms ψ_1, ψ_2 and ψ_3 in Γ_g^1 such that $\psi_1(A_1) = a_2$, $\psi_2(A_1) = e_1$, $\psi_3(A_1) = e_2$, and each ψ_i is identical near a_g and a'_g. Then, we have the following relator H^{ψ_1}:

$$H^{\psi_1} = (t_{a_g} t_{a'_g} \cdot t_{\psi_1(A_{2g})} \cdots t_{\psi_1(A_2)} t_{a_2} t_{a_2} t_{\psi_1(A_2)} \cdots t_{\psi_1(A_{2g})})^2 \cdot t_{a_{g+1}}^{-1}.$$

Applying $C_{2g-1}^{\psi_2}$- and $C_{2g-1}^{\psi_3}$-substitutions to H^{ψ_1}, we get a relator H':

$$\begin{aligned} H' = (t_{e_1} t_{\psi_2(A_2)} \cdots t_{\psi_2(A_{2g-1})})^{2g} t_{\psi_1(A_{2g})} \cdots t_{\psi_1(A_2)} t_{a_2} t_{a_2} t_{\psi_1(A_2)} \cdots t_{\psi_1(A_{2g})} \\ \cdot (t_{e_2} t_{\psi_3(A_2)} \cdots t_{\psi_3(A_{2g-1})})^{2g} t_{\psi_1(A_{2g})} \cdots t_{\psi_1(A_2)} t_{a_2} t_{a_2} t_{\psi_1(A_2)} \cdots t_{\psi_1(A_{2g})} \cdot t_{a_{g+1}}^{-1}. \end{aligned}$$

Here, let us consider a word $t_c \cdot t_{v_1} t_{v_2} \cdots t_{v_k}$. By repeating elementary transformations on this word, we obtain the word $t_{t_c(v_1)} t_{t_c(v_2)} \cdots t_{t_c(v_k)} \cdot t_c$. Therefore, since H' is a positive relator including t_{e_1}, t_{a_2} and t_{e_2} in this order, we can put them together to the right side of the word to obtain a relator in the form

$$H'' = T \cdot t_{e_1} t_{a_2} t_{e_2} \cdot t_{a_{g+1}}^{-1},$$

where T is a product of $8g^2 + 4g - 3$ right-handed Dehn twists. Let L denote the lantern relator $L = t_{e_1} t_{a_2} t_{e_2} t_{A_1}^{-1} t_{a_3}^{-1} t_{A_5}^{-1} t_{A_3}^{-1}$. Finally, we do L^{-1}-substitution to H'', to obtain the following relator I in Γ_g^1:

$$I = T \cdot t_{A_3} t_{A_5} t_{a_3} t_{A_1} \cdot t_{a_{g+1}}^{-1}.$$

The relator I reduces to a positive relator $\widehat{I}$ in Γ_g. Thus, $\widehat{I}$ gives a genus-g Lefschetz fibration $f_{\widehat{I}} : X_{\widehat{I}} \to S^2$ which admits a (-1)-section.

We see that a genus-g Lefschetz fibration $f_{\widehat{I}} : X_{\widehat{I}} \to S^2$ has $2g(4g+2)+1$ singular fibers. Hence, we have

$$e(X_{\widehat{I}}) = 8g^2 + 5.$$

Here, note that C_{2g} is a positive relator in Γ_g. This gives a genus-g Lefschetz fibration $f_{C_{2g}} : X_{C_{2g}} \to S^2$ with $2g(4g+2)$ nonseparating singular fibers. In particular, this fibration is hyperelliptic since $\iota(A_i) = A_i$ for each $i = 1, 2, \dots, 2g$ (see Figure 5). Therefore, we have $\sigma(X_{C_{2g}}) = -4g(g+1)$ by Theorem 3.5. Since I is obtained from C_{2g} by some B-substitutions, two C_{2g-1}^{-1}-substitutions, $C_{2g-1}^{\psi_2}$- and $C_{2g-1}^{\psi_3}$-substitutions, several other B-substitutions, and one L^{-1}-substitution, by (4) and Proposition 3.6, we have

$$\sigma(X_{\widehat{I}}) = \sigma(X_{C_{2g}}) - 1 = -4g(g+1) - 1.$$

This gives $\lambda_{f_{\widehat{I}}} = 4 - 4/g - 1/g^2 < 4 - 4/g$. By Proposition 3.9, this fibration is nonholomorphic.

It is easy to check that $\widehat{I}$ includes the Dehn twist about the curve $t_{e_1}(\psi_1(A_i))$ for $1 \le i \le 2g$. Since $f_{\widehat{I}}$ admits a section, by Lemma 3.4 we have

$$\pi_1(X_{\widehat{I}}) \;\subset\; \pi_1(\Sigma_g)/\langle t_{e_1}(\psi_1(A_1)), \dots, t_{e_1}(\psi_1(A_{2g}))\rangle.$$

On the other hand, it is easy to check that

$$\pi_1(\Sigma_g)/\langle t_{e_1}(\psi_1(A_1)), \dots, t_{e_1}(\psi_1(A_{2g})) = \pi_1(\Sigma_g)/\langle A_1, \dots, A_{2g}\rangle = 1,$$

hence $\pi_1(X_{\widehat{I}}) = 1$. □

Remark 4.2. We do not provide a monodromy factorization of $f_{\widehat{I}}$ explicitly; however, we can obtain it by giving explicit $\psi_j(A_i)$ for $j = 1, 2, 3$ and $i = 1, 2, \dots, 2g$.

Remark 4.3. All vanishing cycles of the Lefschetz fibration $f_{\widehat{I}}$ are nonseparating since all curves of the lantern relator employed in the proof of Theorem 1.1 are nonseparating. For $g \ge 3$, we can consider a lantern relator such that six curves are nonseparating and one curve, denoted by s_h, is separating, which separates Σ_g^1 into two subsurfaces Σ_h^1 and Σ_{g-h}^2 for $h \ge 2$. Then, a similar argument to the proof of Theorem 1.1 gives a genus-g Lefschetz fibration with a (-1)-section on a simply connected total space having s_h as a vanishing cycle and violating the slope inequality, for each $h = 2, 3, \dots, g-1$. Therefore, we can construct at least $g-1$ different genus-g Lefschetz fibrations with the same conditions as in Theorem 1.1.

Remark 4.4. Miyachi and Shiga [2011] produced genus-g Lefschetz fibrations over Σ_{2m} ($m \geq 1$) which do not satisfy the slope inequality.

5. Noncomplex Lefschetz fibrations admitting (-1)-sections

In this section, we prove Theorem 1.2.

Theorem 1.2. *For each $g \geq 4$ and each positive integer n, there is a genus-g nonholomorphic Lefschetz fibration $f_{\widehat{U}_n} : X_{\widehat{U}_n} \to S^2$ with two disjoint (-1)-sections such that $X_{\widehat{U}_n}$ does not admit any complex structure with either orientation.*

We assume $g \geq 4$ and $g = 4t, 4t+1, 4t+2, 4t+3$ throughout this section. To prove Theorem 1.2, we construct a relator U_n in Γ_g^2 by applying substitutions to the relator $W_{2,g}$ in Γ_g^2, which gives the Lefschetz fibration $f_{\widehat{U}_n} : X_{\widehat{U}_n} \to S^2$.

Let a_j, a'_j, b_j and c_j ($j = 1, 2, \ldots, g$) be the simple closed curves on Σ_g^2 in Figures 1, 2 and 3, and let a_{g+1} and a'_{g+1} be the boundary curves of Σ_g^2 as before. For a positive integer n, we define a map ϕ_n to be

$$\phi_n = t_{a_2}^n t_{a_3} t_{a_4} \cdots t_{a_t} t_{b_1} t_{b_2} \cdots t_{b_t}.$$

Note that $\phi_n(c_r) = c_r$ for $g = 2r$, that $\phi_n(a_{r+1}) = a_{r+1}$ and $\phi_n(a'_{r+1}) = a'_{r+1}$ for $g = 2r+1$, that $\phi_n(c_t) = c_t$ for $r = 2t$, and that $\phi_n(a_{t+1}) = a_{t+1}$ and $\phi_n(a'_{t+1}) = a'_{t+1}$ for $r = 2t+1$.

The relator $W_{2,g}$ in Γ_g^2 includes Dehn twist t_{c_r} twice if $g = 2r$ and the product $t_{a_{r+1}} t_{a'_{r+1}}$ of two Dehn twists four times if $g = 2r+1$. Therefore, we can apply $W_{1,r}$- and $W_{1,r}^{\phi_n}$-substitutions to W_2^g if $g = 2r$ and $W_{2,r}$- and $W_{2,r}^{\phi_n}$-substitutions to W_2^g if $g = 2r+1$. Then, for $g = 2r$ (resp. $g = 2r+1$), we denote by

$$U_n$$

a relator which is obtained by applying once trivial and once ϕ_n-twisted $W_{1,r}$- (resp. $W_{2,r}$-) substitutions to $W_{2,g}$. For the convenience of the reader we write the definition of the relator U_n in detail. Let us consider the following word in Γ_g^2 for $j = 1, 2$:

$$V_j := \begin{cases} (t_{B_{0,j}^r} t_{B_1^r} t_{B_2^r} \cdots t_{B_r^r} t_{c_t})^2 & \text{if } r = 2t, \\ (t_{B_{0,j}^r} t_{B_1^r} t_{B_2^r} \cdots t_{B_r^r} t_{a_{t+1}}^2 t_{a'_{t+1}}^2)^2 & \text{if } r = 2t+1. \end{cases}$$

Note that $V_1 = W_{1,r} t_{c_r}^{-1}$ and $V_2 = W_{2,r} t_{a'_{r+1}}^{-1} t_{a_{r+1}}^{-1}$. Then, we can write U_n as follows: If $g = 2r$, and therefore $g = 4t, 4t+2$, then

$$U_n := (t_{B_{0,2}^g} t_{B_1^g} t_{B_2^g} \cdots t_{B_g^g} V_1)(t_{B_{0,2}^g} t_{B_1^g} t_{B_2^g} \cdots t_{B_g^g} V_1^{\phi_n}) t_{a_{g+1}}^{-1} t_{a'_{g+1}}^{-1},$$

and if $g = 2r+1$, and therefore $g = 4t+1, 4t+3$, then

$$U_n := (t_{B_{0,2}^g} t_{B_1^g} t_{B_2^g} \cdots t_{B_g^g} V_2 t_{a_{r+1}} t_{a'_{r+1}})(t_{B_{0,2}^g} t_{B_1^g} t_{B_2^g} \cdots t_{B_g^g} V_2^{\phi_n} t_{a_{r+1}} t_{a'_{r+1}}) t_{a_{g+1}}^{-1} t_{a'_{g+1}}^{-1}.$$

Since the relator U_n in Γ_g^2 is a product of $t_{a_{g+1}}^{-1} t_{a'_{g+1}}^{-1}$ and positive Dehn twists, it reduces to a positive relator of Γ_g, which is denoted by $\widehat{U}_n$. This gives a genus-g Lefschetz fibration $f_{\widehat{U}_n} : X_{\widehat{U}_n} \to S^2$ with two disjoint (-1)-sections.

We prepare the following lemma.

Lemma 5.1. *For $g = 2r, 2r+1$ and $r = 2t, 2t+1$, the following holds in $H_1(\Sigma_g)$:*

$$\begin{aligned}
\phi_n(B^r_{0,j}) &= B^r_{0,j} + a_t \cdots + a_4 + a_3 + na_2.\\
\phi_n(B^r_1) &= B^r_1 - b_1 + a_t + \cdots + a_4 + a_3 + na_2.\\
\phi_n(B^r_2) &= B^r_2 - b_1 + a_t + \cdots + a_4 + a_3 + na_2.\\
\phi_n(B^r_3) &= B^r_3 - b_2 + a_t + \cdots + a_4 + a_3.\\
\phi_n(B^r_4) &= B^r_4 - b_2 + a_t + \cdots + a_4 + a_3 - na_2.\\
\phi_n(B^r_{2k-1}) &= B^r_{2k-1} - b_k + a_t + \cdots + a_{k+2} + a_{k+1}, \qquad 3 \le k \le t.\\
\phi_n(B^r_{2k}) &= B^r_{2k} - b_k + a_t + \cdots + a_{k+2} + a_{k+1} - a_k, \quad 3 \le k \le t.
\end{aligned}$$

If $r = 2t+1$, and therefore $g = 4t+2, 4t+3$, then $\phi_n(B^{2t+1}_{2t+1}) = B^{2t+1}_{2t+1}$.

Proof. We use the following well-known formula for the action of the N-th power of the Dehn twist along a simple closed curve c on $H_1(\Sigma_g)$ repeatedly (see [Farb and Margalit 2012]):

$$t_c^K(d) = d - N i(d,c)c,$$

for an element d in $H_1(\Sigma_g)$. Recall that $i(a_i, a_i) = i(b_i, b_i) = 0$, $i(a_i, b_j) = 0$ for $i \neq j$ and $i(a_i, b_i) = 1$.

First, we show the equation of $\phi_n(B^r_{2k-1})$ for $1 \le k \le t$. From Figures 1–3, we see that for $1 \le k \le t$,

$$i(B^r_{2k-1}, a_i) = \begin{cases} 0 & \text{if } 1 \le i \le k-1,\\ -1 & \text{if } k \le i \le t,\end{cases}$$

$$i(B^r_{2k-1}, b_i) = \begin{cases} 1 & \text{if } i = k,\\ 0 & \text{if } i \neq k.\end{cases}$$

Using the above mentioned formula, we get

$$\begin{aligned}
\phi_n(B^r_{2k-1}) &= t^n_{a_2} t_{a_3} t_{a_4} \cdots t_{a_t} t_{b_1} t_{b_2} \cdots t_{b_t}(B^r_{2k-1})\\
&= t^n_{a_2} t_{a_3} t_{a_4} \cdots t_{a_t} t_{b_1} t_{b_2} \cdots t_{b_{k-1}}(B^r_{2k-1} - b_k)\\
&= t^n_{a_2} t_{a_3} t_{a_4} \cdots t_{a_t} t_{b_1} t_{b_2} \cdots t_{b_{k-1}}(B^r_{2k-1}) - t^n_{a_2} t_{a_3} t_{a_4} \cdots t_{a_t} t_{b_1} t_{b_2} \cdots t_{b_{k-1}}(b_k)\\
&= t^n_{a_2} t_{a_3} t_{a_4} \cdots t_{a_t}(B^r_{2k-1}) - t^n_{a_2} t_{a_3} t_{a_4} \cdots t_{a_t}(b_k).
\end{aligned}$$

Therefore, from $i(b_k, a_k) = -1$ (by $i(a_k, b_k) = 1$), we have

$$\begin{aligned}\phi_n(B^r_{2k-1}) &= t^n_{a_2} t_{a_3} t_{a_4} \cdots t_{a_t}(B^r_{2k-1}) - t^n_{a_2} t_{a_3} t_{a_4} \cdots t_{a_t}(b_k)\\ &= B^r_{2k-1} + a_t + \cdots + a_{k+1} + a_k - (b_k + a_k)\end{aligned}$$

if $3 \leq k$, and

$$\begin{aligned}\phi_n(B^r_3) &= t^n_{a_2} t_{a_3} t_{a_4} \cdots t_{a_t}(B^r_3) - t^n_{a_2} t_{a_3} t_{a_4} \cdots t_{a_t}(b_2)\\ &= B^r_3 + a_t + \cdots + a_4 + a_3 + na_2 - (b_2 + na_2),\\ \phi_n(B^r_1) &= t^n_{a_2} t_{a_3} t_{a_4} \cdots t_{a_t}(B^r_1) - t^n_{a_2} t_{a_3} t_{a_4} \cdots t_{a_t}(b_1)\\ &= B^r_1 + a_t + \cdots + a_4 + a_3 + na_2 - b_1.\end{aligned}$$

Therefore, we obtain the required formula of $\phi_n(B^r_{2k-1})$ for $1 \leq k \leq t$.

Next, we show the equation of $\phi_n(B^r_{2k})$ for $1 \leq k \leq t$. From Figures 1–3, we see that for $1 \leq k \leq t$,

$$i(B^{2t}_{2k}, a_i) = \begin{cases} 0 & \text{if } 1 \leq i \leq k,\\ -1 & (k+1 \leq i \leq t),\end{cases}$$

$$i(B^{2t}_{2k}, b_i) = \begin{cases} 1 & \text{if } i = k,\\ 0 & \text{if } i \neq k,\end{cases}$$

where $1 \leq i$. Using this, a similar argument to $\phi_n(B^r_{2k-1})$ gives

$$\begin{aligned}\phi_n(B^r_{2k}) &= t^n_{a_2} t_{a_3} t_{a_4} \cdots t_{a_t}(B^r_{2k}) - t^n_{a_2} t_{a_3} t_{a_4} \cdots t_{a_t}(b_k)\\ &= B^r_{2k} + a_t + \cdots a_{k+2} + a_{k+1} - (b_k + a_k)\end{aligned}$$

if $3 \leq k$ and

$$\begin{aligned}\phi_n(B^r_4) &= t^n_{a_2} t_{a_3} t_{a_4} \cdots t_{a_t}(B^r_4) - t^n_{a_2} t_{a_3} t_{a_4} \cdots t_{a_t}(b_2)\\ &= B^r_4 + a_t + \cdots a_4 + a_3 - (b_2 + na_1),\\ \phi_n(B^r_2) &= t^n_{a_2} t_{a_3} t_{a_4} \cdots t_{a_t}(B^r_2) - t^n_{a_2} t_{a_3} t_{a_4} \cdots t_{a_t}(b_1)\\ &= B^r_2 + a_t + \cdots a_4 + a_3 + na_2 - b_1.\end{aligned}$$

Therefore, we obtain the required formula of $\phi_n(B^r_{2k})$ for $1 \leq k \leq t$.

Finally, we show the equation for $\phi_n(B^r_{0,j})$. From Figures 1–3, we see that $i(B^{2t}_{0,j}, a_i) = -1$ and $i(B^{2t}_{0,j}, b_i) = 0$. Therefore,

$$\phi_n(B^r_{0,j}) = t^n_{a_2} t_{a_3} t_{a_4} \cdots t_{a_t} t_{b_1} t_{b_2} \cdots t_{b_t}(B^r_{0,j}) = B^r_{0,j} + a_t + \cdots + a_4 + a_3 + na_2.$$

Since $i(B^{2t+1}_{2t+1}, a_i) = (B^{2t+1}_{2t+1}, b_i) = 0$ for $i = 1, 2, \ldots, t$, we have $\phi_n(B^{2t+1}_{2t+1}) = B^{2t+1}_{2t+1}$, and this finishes the proof. □

Proof of Theorem 1.2. It is sufficient to show that $H_1(X_{\widehat{U}_n}) = \mathbb{Z} \oplus \mathbb{Z}_n$ from Theorem 3.10. For a set S, we denote by $\mathbb{Z}\langle S\rangle$ the $\mathbb{Z}$-module generated by S. Let

$$S^h_{0,j} := \{B^h_{0,j}, B^h_1, B^h_2, \dots, B^h_h\},$$
$$T^h_{0,j} := \{\phi_n(B^h_{0,j}), \phi_n(B^h_1), \phi_n(B^h_2), \dots, \phi_n(B^h_h)\}.$$

Recall that $\phi_n(c_t) = c_t$ for $r = 2t$ and $\phi_n(a_{t+1}) = a_{t+1}$ and $\phi_n(a'_{t+1}) = a'_{t+1}$ for $r = 2t+1$. By this fact, $c_t = 0$, $a'_{t+1} = a_{t+1}$ and $a'_{r+1} = a_{r+1}$ in $H_1(\Sigma_g)$ and Lemma 3.4, we have

$$H_1(X_{\widehat{U}_n}) = \begin{cases} H_1(\Sigma_g)/\mathbb{Z}\langle S^{4t}_{0,2} \cup S^{2t}_{0,1} \cup T^{2t}_{0,1}\rangle & \text{if } g = 4t, \\ H_1(\Sigma_g)/\mathbb{Z}\langle S^{4t+1}_{0,2} \cup \{a_{2t+1}\} \cup S^{2t}_{0,2} \cup T^{2t}_{0,2}\rangle & \text{if } g = 4t+1, \\ H_1(\Sigma_g)/\mathbb{Z}\langle S^{4t+2}_{0,2} \cup S^{2t+1}_{0,1} \cup \{a_{t+1}\} \cup T^{2t+1}_{0,1}\rangle & \text{if } g = 4t+2, \\ H_1(\Sigma_g)/\mathbb{Z}\langle S^{4t+3}_{0,2} \cup \{a_{2t+2}\} \cup S^{2t+1}_{0,2} \cup \{a_{t+1}\} \cup T^{2t+1}_{0,2}\rangle & \text{if } g = 4t+3. \end{cases}$$

By $\phi_n(B^r_{2k-1}) = \phi(B^r_{2k}) = 0$ and $B_{2k-1} = B_{2k} = 0$ in $H_1(X_{\widehat{U}_n})$ for $2 \le k \le t$, Lemma 5.1 gives

$$na_2 = a_3 = a_4 = \cdots = a_t = 0. \tag{5}$$

Using this, $B^r_{2k-1} = 0$ and $\phi(B^r_{2k-1}) = 0$ for $1 \le k \le t$ and Lemma 5.1, we have

$$b_1 = b_2 = \cdots = b_t = 0. \tag{6}$$

By (5), the equation $B^r_{0,j} = 0$, and Lemma 5.1, we can remove the relation $\phi_n(B^r_{0,j}) = 0$. Moreover, if $r = 2t+1$, and therefore $g = 4t+2, 4t+3$, then by Lemma 5.1 and $B^{2t+1}_{2t+1} = 0$, we can delete the relation $\phi_n(B^{2t+1}_{2t+1}) = 0$.

Suppose that $r = 2t$ (i.e., $g = 4t, 4t+1$). Let us consider the equations (1)–(3) for $h = 2t$. By $B^{2t}_{2k-1} = B^{2t}_{2k} = 0$ in $H_1(X_{\widehat{U}_n})$, we get

$$b_k + b_{2t+1-k} = 0, \quad 1 \le k \le t.$$

By (6), we have

$$b_1 = b_2 = \cdots b_{2t} = 0. \tag{7}$$

Using this and $B^{2t}_{2k-1} = 0$ for $1 \le k \le t$, we have

$$a_k + a_{2t+1-k} = 0, \quad 1 \le k \le t.$$

Therefore, by (5), we have

$$a_1 + a_{2t} = a_2 + a_{2t-1} = 0; \tag{8}$$

$$a_3 = a_4 = \cdots = a_{2t-2} = 0. \tag{9}$$

Note that $B^{2t}_{0,1} = b_1 + b_2 + \cdots + b_{2t}$ and $B^{2t}_{0,2} = b_1 + b_2 + \cdots + b_{2t} + a_{2t+1}$. By (7) (and $a_{2t+1}=0$ if $g=4t+1$), we can delete $B^{2t}_{0,j}=0$ for $j=1,2$.

Suppose that $r=2t+1$ (i.e., $g=4t+2, 4t+3$). Consider the equations (1)–(3) for $h=2t+1$. By $B^{2t+1}_{2k-1}=B^{2t+1}_{2k}=0$ in $H_1(X_{\widehat{U}_n})$, we get

$$b_k+b_{2t+2-k}=0, \quad 1\le k\le t.$$

In particular, by $B^{2t+1}_{2t+1}=a_{t+1}+b_{t+1}+a_{t+1}=0$ and $a_{t+1}=0$, we have $b_{t+1}=0$. By combining this with (6), we have

$$b_1=b_2=\cdots b_{2t+1}=0. \tag{10}$$

Using this and $B^{2t+1}_{2k-1}=0$ for $1\le k\le t$, we have

$$a_k+a_{2t+2-k}=0, \quad 1\le k\le t.$$

Therefore, by (5) and the relation $a_{t+1}=0$, we have

$$a_1+a_{2t+1}=a_2+a_{2t}=0; \tag{11}$$

$$a_3=a_4=\cdots=a_{2t-1}=0. \tag{12}$$

For a similar reason to the case $r=2t$, we can remove $B^{2t+1}_{0,j}=0$ for $j=1,2$.

Suppose that $g=2r$ (i.e., $g=4t, r=2t$ or $g=4t+2, r=2t+1$). Consider the equations (1)–(3) for $h=2r$. By $B^{2r}_{2k-1}=B^{2r}_{2k}=0$ in $H_1(X_{\widehat{U}_n})$, we obtain

$$b_k+b_{2r+1-k}=0, \quad 1\le k\le r.$$

If $g=4t$ (resp. $g=4t+2$), then the relation (7) (resp. the relation (10)) gives

$$b_1=b_2=\cdots=b_{2r}=0. \tag{13}$$

Using this and $B^{2r}_{2k-1}=0$ for $1\le k\le r$, we have

$$a_k+a_{2r+1-k}=0, \quad 1\le k\le r.$$

By this equations and the equations (8) and (9) (resp. the equations (11) and (12)) if $g=4t$ (resp. $g=4t+2$), we obtain

$$a_1+a_r=a_2+a_{r-1}=0; \tag{14}$$

$$a_1+a_{2r}=a_2+a_{2r-1}=a_{r-1}+a_{r+2}=a_r+a_{r+1}=0; \tag{15}$$

$$a_3=a_4=\cdots=a_{r-2}=a_{r+3}=\cdots=a_{2r-3}=a_{2r-2}=0, \tag{16}$$

and we can delete the relation $B^{2r}_{0,j}=0$ for a similar reason to the case $r=2t$. Since (14) and (15) give $a_{2r}=a_r=-a_1,\ a_{r+1}=a_1,\ a_{2r-1}=a_{r-1}=-a_2$ and $a_{r+2}=a_2$, by (5), (13) and (16), we obtain

$$H_1(X_{\widehat{U}_n})=\mathbb{Z}\langle\{a_1,a_2\}\rangle/\mathbb{Z}\langle\{na_2\}\rangle=\mathbb{Z}\oplus\mathbb{Z}_n,$$

and the proof of Theorem 1.2 for $g=2r$ is complete.

Suppose that $g = 2r+1$ (i.e., $g = 4t+1,\ r = 2t$, or $g = 4t+3,\ r = 2t+1$). Consider the equations (1)–(3) for $h = 2r+1$. By $B^{2r+1}_{2k-1} = B^{2r}_{2k} = 0$ in $H_1(X_{\widehat{U}_n})$,

$$b_k + b_{2r+2-k} = 0, \quad 1 \le k \le r.$$

By $B^{2r+1}_{2r+1} = a_{r+1} + b_{r+1} + a_{r+1} = 0$ and $a_{r+1} = 0$, we have $b_{r+1} = 0$. Therefore, if $g = 4t+1$ (resp. $g = 4t+3$), then the relation (7) (resp. the relation (10)) gives

$$b_1 = b_2 = \cdots = b_{2r+1} = 0. \tag{17}$$

Using this and $B^{2r+1}_{2k-1} = 0$ for $1 \le k \le r$, we have

$$a_k + a_{2r+2-k} = 0, \quad 1 \le k \le r.$$

By this equations, the equation $a_{r+1} = 0$ and the equations (8) and (9) (resp. the equations (11) and (12)) if $g = 4t+1$ (resp. $g = 4t+3$), we obtain

$$a_1 + a_r = a_2 + a_{r-1} = 0; \tag{18}$$

$$a_1 + a_{2r+1} = a_2 + a_{2r} = a_{r-1} + a_{r+3} = a_r + a_{r+2} = 0; \tag{19}$$

$$a_3 = a_4 = \cdots = a_{r-2} = a_{r+1} = a_{r+4} = a_{r+5} = \cdots = a_{2r-2} = a_{2r-1} = 0, \tag{20}$$

and we can delete the relation $B^{2r+1}_{0,j} = 0$ for a similar reason to the case $r = 2t$. Since the equations (18) and (19) give $a_{2r+1} = a_r = -a_1,\ a_{r+2} = a_1,\ a_{2r} = a_{r-1} = -a_2$ and $a_{r+3} = a_2$, by (5), (17) and (20), we obtain

$$H_1(X_{\widehat{U}_n}) = \mathbb{Z}\langle\{a_1, a_2\}\rangle / \mathbb{Z}\langle\{na_2\}\rangle = \mathbb{Z} \oplus \mathbb{Z}_n,$$

and the proof of Theorem 1.2 for $g = 2r+1$ is complete. □

Acknowledgements

The authors would like to thank R. Inanc Baykur for comments on this paper and for pointing out that there are more various examples of nonholomorphic Lefschetz fibrations than were mentioned in the first version of the manuscript. They are also grateful to Anar Akhmedov for comments. Finally, they are greatly indebted to the referee for invaluable comments, especially, about the proof of Theorem 1.2. Monden was supported by Grant-in-Aid for Young Scientists (B) (No. 16K17601), Japan Society for the Promotion of Science.

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Received December 25, 2016. Revised April 12, 2018.

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PACIFIC JOURNAL OF MATHEMATICS

Volume 298 No. 2 February 2019

0030-8730(201902)298:2;1-Q