Pacific Journal of Mathematics

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Volume 298 No. 2 February 2019

PACIFIC JOURNAL OF MATHEMATICS
Vol. 298, No. 2, 2019
dx.doi.org/10.2140/pjm.2019.298.399

TILTING MODULES OVER AUSLANDER–GORENSTEIN ALGEBRAS

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For a finite-dimensional algebra Λ and a nonnegative integer n, we characterize when the set $\mathrm{tilt}_n\Lambda$ of additive equivalence classes of tilting modules with projective dimension at most n has a minimal (or equivalently, minimum) element. This generalizes results of Happel and Unger. Moreover, for an n-Gorenstein algebra Λ with $n \geq 1$, we construct a minimal element in $\mathrm{tilt}_n\Lambda$. As a result, we give equivalent conditions for a k-Gorenstein algebra to be Iwanaga–Gorenstein. Moreover, for a 1-Gorenstein algebra Λ and its factor algebra $\Gamma = \Lambda/(e)$, we show that there is a bijection between $\mathrm{tilt}_1\Lambda$ and the set $\mathrm{s}\tau\text{-tilt}\,\Gamma$ of additive equivalence classes of basic support τ-tilting Γ-modules, where e is an idempotent such that $e\Lambda$ is the additive generator of the category of projective-injective Λ-modules.

1. Introduction

Tilting theory is essential in the representation theory of algebras. There are many works (see [Assem et al. 2006; Angeleri Hügel et al. 2007; Happel 1988]) which made the theory fruitful. One interesting topic in tilting theory is to classify tilting modules for some given algebras. Among these, tilting modules over algebras of large dominant dimension have gained more and more attention. For more details, we refer to [Chen and Xi 2016; Crawley-Boevey and Sauter 2017; Nguyen et al. 2018; Iyama and Zhang 2016; Pressland and Sauter 2017; Kajita 2008].

For an algebra Λ, denote by $\mathrm{mod}\,\Lambda$ the category of finitely generated right Λ-modules. Recall that a Λ-module T in $\mathrm{mod}\,\Lambda$ is called a *tilting module* of finite projective dimension if the projective dimension of T is $n < \infty$, $\mathrm{Ext}^i_\Lambda(T, T) = 0$ holds for $i \geq 1$, and there is an exact sequence $0 \to \Lambda \to T_0 \to \cdots \to T_n \to 0$ with $T_i \in \mathrm{add}\,T$, where we use $\mathrm{add}\,T$ to denote the subcategory of $\mathrm{mod}\,\Lambda$ consisting of direct summands of finite direct sums of T. We say that $M, N \in \mathrm{mod}\,\Lambda$ are

Iyama is supported by JSPS Grant-in-Aid for Scientific Research (B) 24340004, (B) 16H03923, (C) 23540045 and (S) 15H05738. Zhang is supported by NSFC (Nos. 11401488, 11571164 and 11671174), Jiangsu Government Scholarship for Overseas Studies (JS-2014-352) and NSF for Jiangsu Province (BK20130983).
MSC2010: primary 16G10; secondary 16E10.
Keywords: n-Gorenstein algebra, tilting module, support τ-tilting module.

additively equivalent if $\mathsf{add}\,M = \mathsf{add}\,N$. For a nonnegative integer n, let $\mathrm{tilt}_n\Lambda$ be the set consisting of additive equivalence classes of tilting modules with projective dimension at most n, and $\mathrm{tilt}\,\Lambda = \mathrm{tilt}_\infty\Lambda := \bigcup_{n\geq 0}\mathrm{tilt}_n\Lambda$. There is a natural partial order on the set $\mathrm{tilt}\,\Lambda$ defined as follows [Riedtmann and Schofield 1991; Happel and Unger 2005; Aihara and Iyama 2012]: For $T, U \in \mathrm{tilt}\,\Lambda$, $T \geq U$ if $\mathrm{Ext}^i_\Lambda(T, U) = 0$ for all $i > 0$. This is equivalent to saying that $T^\perp \supseteq U^\perp$, where $T^\perp$ is the subcategory of $\mathsf{mod}\,\Lambda$ consisting of modules M such that $\mathrm{Ext}^i_\Lambda(T, M) = 0$ for any $i \geq 1$. Clearly Λ is the maximal element in $\mathrm{tilt}\,\Lambda$, and if Λ is Iwanaga–Gorenstein, then $\mathbb{D}\Lambda$ is the minimal element in $\mathrm{tilt}\,\Lambda$, where $\mathbb{D}$ is the ordinary duality. However, it is difficult to find the minimal element in $\mathrm{tilt}\,\Lambda$ for an arbitrary algebra Λ.

For a right Λ-module M, let $0 \to M \to I^0(M) \to I^1(M) \to \cdots$ be a minimal injective resolution of M and $\cdots \to P_1(M) \to P_0(M) \to M \to 0$ a minimal projective resolution of M. Recall that an algebra Λ is called *n-Gorenstein* (resp. *quasi n-Gorenstein*) if the projective dimension of $I^i(\Lambda)$ is less than or equal to i (resp. $i+1$) for $0 \leq i \leq n-1$ [Fossum et al. 1975; Huang 2006]. There are many works on n-Gorenstein algebras [Auslander and Reiten 1994; Auslander and Reiten 1996; Clark 2001; Huang and Iyama 2007; Iwanaga and Sato 1996], but little is known for the tilting modules over this class of algebras. Our first aim is to study the existence of minimal tilting modules over this class of algebras. For a module $M \in \mathsf{mod}\,\Lambda$, we denote by $\Omega^i M$ (resp. $\Omega^{-i}M$) the i-th syzygy (resp. cosyzygy) of M. A special case of our first main theorem, Theorem 3.4, is the following:

Theorem 1.1 (Corollary 3.5). *Let Λ be a quasi n-Gorenstein algebra and $0 \leq j \leq n$. Then $\left(\bigoplus_{i=0}^{j-1} I^i(\Lambda)\right) \oplus \Omega^{-j}\Lambda$ is the minimum element in* $\mathrm{tilt}_j\Lambda$.

For example, algebras with dominant dimension at least n are n-Gorenstein. In this case, the tilting module given in Theorem 1.1 was studied recently in [Crawley-Boevey and Sauter 2017; Nguyen et al. 2018; Pressland and Sauter 2017] (see Example 3.6).

Recall that a subcategory $\mathscr{C}$ of $\mathsf{mod}\,\Lambda$ is called *contravariantly finite* if for any M in $\mathsf{mod}\,\Lambda$ there is a morphism $C_M \to M$ with $C_M \in \mathscr{C}$ such that the sequence $\mathrm{Hom}_\Lambda(-, C_M) \to \mathrm{Hom}_\Lambda(-, M) \to 0$ is exact over $\mathscr{C}$. Dually, one can define covariantly finite subcategories. For a subcategory $\mathscr{D}$ of $\mathsf{mod}\,\Lambda$, denote by $\mathscr{D}^\perp$ the subcategory consisting of modules N such that $\mathrm{Ext}^i_\Lambda(M, N) = 0$ for $i \geq 1$ and $M \in \mathscr{D}$. We denote this by $M^\perp$ if $\mathscr{D} = \mathsf{add}\,M$. Dually, one can define ${}^\perp\mathscr{D}$ and ${}^\perp M$.

Denote by $\mathcal{P}_\infty(\Lambda)$ the subcategory consisting of Λ-modules with finite projective dimension. Happel and Unger [2005, Theorem 3.3] showed that $\mathrm{tilt}\,\Lambda$ has a minimal element if and only if $\mathcal{P}_\infty(\Lambda)$ is contravariantly finite. It is natural to ask if there is a similar result for $\mathrm{tilt}_n\Lambda$. We give a positive answer by proving the following result, where we denote by $\mathcal{P}_n(\Lambda)$ the subcategory consisting of modules with projective dimension at most n, where the equivalence of (1) and (5) for $n = \infty$

recovers [Happel and Unger 2005, Theorem 3.3], and the equivalence of (3) and (5) for integer n recovers [Happel and Unger 1996, Corollary 2.3].

Theorem 1.2 (Theorem 3.1). *Let Λ be an algebra, and let n be ∞ or a nonnegative integer. Then the following are equivalent*:

(1) $\operatorname{tilt}_n \Lambda$ *has a minimal element.*

(2) $\operatorname{tilt}_n \Lambda$ *has the minimum element.*

(3) *There exists* $T \in \operatorname{tilt}_n \Lambda$ *such that* ${}^{\perp}T \supseteq \mathcal{P}_n(\Lambda)$.

(4) *There exists* $T \in \operatorname{tilt}_n \Lambda$ *such that* ${}^{\perp}(T^{\perp}) = \mathcal{P}_n(\Lambda)$.

(5) *The subcategory* $\mathcal{P}_n(\Lambda)$ *is contravariantly finite.*

For any $M \in \operatorname{mod} \Lambda$, denote by $\operatorname{id}_\Lambda M$ (resp. $\operatorname{pd}_\Lambda M$) the injective (resp. projective) dimension of M. An algebra is called *Iwanaga–Gorenstein* if both $\operatorname{id}_\Lambda \Lambda$ and $\operatorname{id}_{\Lambda^{\mathrm{op}}} \Lambda$ are finite. Auslander and Reiten [1994] posed a question which asks whether Λ must be Iwanaga–Gorenstein if it is n-Gorenstein for all positive integers n. This is a generalization of the Nakayama conjecture which says that an algebra with infinite dominant dimension is self-injective. Moreover, Auslander and Reiten [1994, p. 25] studied the question of whether the $\mathcal{P}_\infty(\Lambda)$ is contravariantly finite if Λ is n-Gorenstein for all positive integers n.

As a result of Theorems 1.1 and 1.2, we connect the two questions of Auslander and Reiten above and show the following corollary which covers [Auslander and Reiten 1994, Corollary 5.5].

Corollary 1.3 (Corollary 3.7). *Let Λ be a k-Gorenstein algebra for all positive integers k, and n a nonnegative integer. Then the following are equivalent*:

(1) Λ *is Iwanaga–Gorenstein with* $\operatorname{id}_\Lambda \Lambda = \operatorname{id}_{\Lambda^{\mathrm{op}}} \Lambda \leq n$.

(2) $\operatorname{id}_\Lambda \Lambda \leq n$.

(3) $\operatorname{id}_{\Lambda^{\mathrm{op}}} \Lambda \leq n$.

(4) $\operatorname{tilt} \Lambda$ *has the minimum element T with* $\operatorname{pd}_\Lambda T \leq n$.

(5) $\operatorname{tilt} \Lambda^{\mathrm{op}}$ *has the minimum element T with* $\operatorname{pd}_\Lambda T \leq n$.

(6) *The subcategory* $\mathcal{P}_\infty(\Lambda)$ *is contravariantly finite and* $\mathcal{P}_\infty(\Lambda) = \mathcal{P}_n(\Lambda)$.

(7) *The subcategory* $\mathcal{P}_\infty(\Lambda^{\mathrm{op}})$ *is contravariantly finite and* $\mathcal{P}_\infty(\Lambda^{\mathrm{op}}) = \mathcal{P}_n(\Lambda^{\mathrm{op}})$.

It may be interesting to ask the following question for a finite-dimensional algebra Λ: Does the existence of the minimum element of $\operatorname{tilt} \Lambda$ imply the existence of a minimum element of $\operatorname{tilt} \Lambda^{\mathrm{op}}$?

Now we turn to the classical tilting modules over 1-Gorenstein algebras and study the connections with τ-tilting theory.

In 2014, Adachi, Iyama and Reiten introduced τ-tilting modules, see Definition 4.1, which are generalizations of classical tilting modules from the viewpoint of mutation.

For general details of τ-tilting theory, we refer to [Adachi et al. 2014; Demonet et al. 2019; Iyama et al. 2014; Jasso 2015; Wei 2014; Zhang 2017b].

For an algebra Λ, denote by $\mathrm{s}\tau\text{-tilt}\,\Lambda$ the set of the additive equivalence classes of support τ-tilting Λ-modules (see Definition 4.1). In [Demonet et al. 2017; Iyama and Zhang 2016] it is shown that the functor $-\otimes_\Lambda \Gamma$ induces a map from $\mathrm{s}\tau\text{-tilt}\,\Lambda$ to $\mathrm{s}\tau\text{-tilt}\,\Gamma$, where Γ is a factor algebra of Λ. Recall that $\mathrm{tilt}_1\Lambda$ is the set of additive equivalence classes of classical tilting Λ-modules. Our third main result is the following.

Theorem 1.4 (Theorem 4.5). *Let Λ be a 1-Gorenstein algebra and let Γ be the factor algebra $\Lambda/(e)$, where e is an idempotent such that $\mathsf{add}\, e\Lambda = \mathsf{add}\, I^0(\Lambda)$. Then $-\otimes_\Lambda \Gamma$ induces a bijection from $\mathrm{tilt}_1\Lambda$ to $\mathrm{s}\tau\text{-tilt}\,\Gamma$.*

For an algebra Λ, denote by $\#\mathrm{s}\tau\text{-tilt}\,\Lambda$ the number of elements in the set $\mathrm{s}\tau\text{-tilt}\,\Lambda$. As an immediate consequence, we have the following corollary. Recall from [Demonet et al. 2019] that an algebra Λ is called *τ-tilting finite* if there are a finite number of basic τ-tilting modules up to isomorphism.

Corollary 1.5 (Corollaries 4.7, 4.8 and 4.9). *For each case, let e be the idempotent such that $\mathsf{add}\, e\Lambda_n = \mathsf{add}\, I^0(\Lambda_n)$.*

(1) *Let $\Lambda_n = KQ$ be the hereditary Nakayama algebra with $Q = A_n$. Then there are bijections*

$$\mathrm{tilt}_1\Lambda_n \simeq \mathrm{s}\tau\text{-tilt}\,\Lambda_{n-1} \simeq \{\textit{clusters of the cluster algebra of type } A_{n-1}\}.$$

Thus $\#\mathrm{s}\tau\text{-tilt}\,\Lambda_n = (2(n+1))!/((n+2)!(n+1)!)$.

(2) *Let Λ_n be the Auslander algebra of $K[x]/(x^n)$, let Γ_{n-1} be the preprojective algebra of $Q = A_{n-1}$ and let $\mathfrak{S}_n$ be the symmetric group. Then there are bijections*

$$\mathrm{tilt}_1\Lambda_n \simeq \mathrm{s}\tau\text{-tilt}\,\Gamma_{n-1} \simeq \mathfrak{S}_n.$$

Thus $\#\mathrm{s}\tau\text{-tilt}\,\Lambda_n = n!$.

(3) *Let Λ_n be the Auslander algebra of the hereditary Nakayama algebra KQ with $Q = A_n$. Then there is a bijection* $\mathrm{tilt}_1\Lambda_n \simeq \mathrm{s}\tau\text{-tilt}\,\Lambda_{n-1}$. *Thus Λ is τ-tilting finite if and only if $n \leq 4$.*

The organization of this paper is as follows: In Section 2, we recall some preliminaries. In Section 3, we give some equivalent conditions to the existence of minimal elements in $\mathrm{tilt}_n\Lambda$ and show a Happel–Unger type theorem. Moreover, we construct minimal tilting modules for n-Gorenstein algebras and show Theorem 1.1. In Section 4, we build a connection between classical tilting modules over 1-Gorenstein algebras and support τ-tilting modules over factor algebras and we show Theorem 1.4 and Corollary 1.5.

Throughout this paper, we denote by K an algebraically closed field. All algebras are basic connected finite-dimensional K-algebras and all modules are finitely generated right modules. For an algebra A, we denote by $\mathsf{mod}\,A$ the category of finitely generated right A-modules. The composition of homomorphisms $f : X \to Y$ and $g : Y \to Z$ is denoted by $gf : X \to Z$.

2. Preliminaries

We start with the following fundamental theorem due to Auslander and Reiten.

Theorem 2.1 [Auslander and Reiten 1991, Theorem 5.5]. *Let $n \geq 0$ and let Λ be a finite-dimensional algebra. Then there exist bijections between the following objects given by $T \mapsto \mathcal{X} = {}^{\perp}(T^{\perp})$ and $T \mapsto \mathcal{Y} = T^{\perp}$.*

(1) $T \in \mathrm{tilt}_n \Lambda$.

(2) *Contravariantly finite resolving subcategories $\mathcal{X}$ of $\mathsf{mod}\,\Lambda$ contained in $\mathcal{P}_n(\Lambda)$.*

(3) *Covariantly finite coresolving subcategories $\mathcal{Y}$ of $\mathsf{mod}\,\Lambda$ containing $\Omega^{-n}(\Lambda)$.*

Moreover, in this case, $(\mathcal{X}, \mathcal{Y})$ is a cotorsion pair such that $\mathcal{X} \cap \mathcal{Y} = \mathsf{add}\,T$, and $\mathcal{X}$ consists of all $X \in \mathsf{mod}\,\Lambda$ such that there exists an exact sequence $0 \to X \to T^0 \to \cdots \to T^n \to 0$ with $T^i \in \mathsf{add}\,T$.

In the rest, a *subcategory* is always assumed to be full and closed under direct sums and direct summands unless stated otherwise. For later application, we prepare the following observation, which is a relative version of a well-known observation (e.g., [Auslander and Smalø 1980]).

Lemma 2.2. *Let $\mathcal{C}$ be a subcategory of $\mathsf{mod}\,\Lambda$ which is closed under extensions, and $\mathcal{A}$ a subcategory of $\mathcal{C}$. Then the following conditions are equivalent.*

(1) *Any exact sequence $0 \to A \to A' \to C \to 0$ with $A, A' \in \mathcal{A}$ and $C \in \mathcal{C}$ splits.*

(2) *There is no exact sequence $0 \to A \to A' \to C \to 0$ with $A \in \mathsf{ind}\,\mathcal{A}$, $A' \in \mathcal{A}$, $C \in \mathcal{C}$ and $A \notin \mathsf{add}\,A'$.*

Proof. It suffices to prove (2) $\Rightarrow$ (1). Assume that there exists a nonsplit exact sequence $0 \to A \xrightarrow{f} A' \to C \to 0$ with $A, A' \in \mathcal{A}$ and $C \in \mathcal{C}$. Without loss of generality, we can assume that f is in the radical of $\mathsf{mod}\,\Lambda$.

Take an indecomposable direct summand X of A. Let $\iota : X \to A$ be the inclusion and $g = f\iota$. Then we have the following commutative diagram of nonsplit exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{g} & A' & \longrightarrow & C' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\iota} & & \| & & \downarrow & & \\ 0 & \longrightarrow & A & \xrightarrow{f} & A' & \longrightarrow & C & \longrightarrow & 0 \end{array}$$

Since $0 \to X \xrightarrow{\iota} A \to C' \to C \to 0$ is an exact sequence, C' belongs to $\mathcal{C}$.

Decompose $A' = X^{\oplus \ell} \oplus U$ with $X \notin \mathsf{add}\, U$, and write $g : X \to A' = X^{\oplus \ell} \oplus U$. Let $Y^0 = X$,

$$Y^i = X^{\oplus \ell^i} \oplus U^{\oplus(1+\ell+\cdots+\ell^{i-1})},$$

and

$$g^i = g^{\oplus \ell^i} \oplus 1_U^{\oplus(1+\ell+\cdots+\ell^{i-1})} : Y^i \to Y^{i+1}.$$

Clearly $\operatorname{Ker} g^i = 0$ and $\operatorname{Coker} g^i \in \mathcal{C}$ for any $i \geq 0$.

Take $m > 0$ such that $\operatorname{rad}^m \operatorname{End}_\Lambda(X) = 0$. Then the composition

$$h = g^{m-1} \cdots g^1 g^0 : X \to Y^m = X^{\oplus \ell^m} \oplus U^{\oplus(1+\ell+\cdots+\ell^{m-1})}$$

is a direct sum of morphisms $0 \to X^{\oplus \ell^m}$ and $h' : X \to U^{\oplus(1+\ell+\cdots+\ell^{m-1})}$. Thus we have an exact sequence

$$0 \to X \to U^{\oplus(1+\ell+\cdots+\ell^{m-1})} \to \operatorname{Coker} h' \to 0. \tag{2-1}$$

Since $\operatorname{Coker} h$ is an extension of $\operatorname{Coker} g^i$'s, it belongs to $\mathcal{C}$. Thus $\operatorname{Coker} h'$ also belongs to $\mathcal{C}$. This is contradiction to the condition (2). □

Let $\mathcal{C}$ be a subcategory of $\mathsf{mod}\,\Lambda$ which is closed under extensions. A *cogenerator* for $\mathcal{C}$ is a subcategory $\mathcal{A}$ of $\mathcal{C}$ such that, for any $C \in \mathcal{C}$, there exists an exact sequence

$$0 \to C \to A \to C' \to 0$$

with $A \in \mathcal{A}$ and $C' \in \mathcal{C}$. A cogenerator $\mathcal{A}$ of $\mathcal{C}$ is called *minimal* if no proper subcategory of $\mathcal{A}$ is a cogenerator of $\mathcal{C}$.

The following observation is a relative version of a well-known result in the theory of (co)covers in [Auslander and Smalø 1980].

Proposition 2.3. *Let $\mathcal{C}$ be a subcategory of $\mathsf{mod}\,\Lambda$ which is closed under extensions. If $\mathcal{A}$ is a minimal cogenerator for $\mathcal{C}$, then $\operatorname{Ext}^1_\Lambda(\mathcal{C}, \mathcal{A}) = 0$ holds.*

Proof. Since $\mathcal{A}$ is a minimal cogenerator for $\mathcal{C}$, the conditions (1) and (2) in Lemma 2.2 are satisfied. Otherwise, there is an exact sequence

$$0 \to A \to A' \to C \to 0$$

with $A \in \mathsf{ind}\,\mathcal{A}$, $A' \in \mathcal{A}$, $C \in \mathcal{C}$ and $A \notin \mathsf{add}\, A'$. Then the subcategory $\mathcal{A}'$ of $\mathcal{A}$ defined by

$$\mathsf{ind}\,\mathcal{A}' = (\mathsf{ind}\,\mathcal{A}) \setminus \{A\}$$

is a cogenerator for $\mathcal{C}$, which is a contradiction to the minimality of $\mathcal{A}$.

Let $A \in \mathcal{A}$ be indecomposable. To prove $\operatorname{Ext}^1_\Lambda(\mathcal{C}, A) = 0$, we take an exact sequence

$$0 \to A \to C' \to C \to 0$$

with $C \in \mathcal{C}$. Since $C' \in \mathcal{C}$, there exists an exact sequence

$$0 \to C' \to A' \to C'' \to 0$$

with $C'' \in \mathcal{C}$ and $A' \in \mathcal{A}$. We have the following commutative diagram of exact sequences:

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A & \longrightarrow & C' & \longrightarrow & C & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A & \longrightarrow & A' & \longrightarrow & X & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & C'' & = & C'' & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

By the right vertical sequence, X belongs to $\mathcal{C}$. Thus the middle horizontal sequence splits by Lemma 2.2(1), and so the upper horizontal sequence splits, as desired. □

We need the following result on mutation of tilting modules.

Proposition 2.4 [Happel and Unger 2005; Coelho et al. 1994; Aihara and Iyama 2012]. *For a basic tilting module $T = X \oplus U$ with X indecomposable, if there is an exact sequence $0 \to X \xrightarrow{f} U' \to Y \to 0$ such that $U' \in \mathsf{add}\, U$ and f is a minimal left $(\mathsf{add}\, U)$-approximation of X, then $V = Y \oplus U$ is a basic tilting module such that $V < T$.*

3. Minimal tilting modules and the category $\mathcal{P}_n(\Lambda)$

Characterizations of existence of minimal tilting modules. Throughout this section, let Λ be an arbitrary algebra. We focus on the properties of tilting modules in $\operatorname{tilt}_n \Lambda$ and give some equivalent conditions to the existence of minimal elements in $\operatorname{tilt}_n \Lambda$. More precisely, we generalize the Happel–Unger theorem stating that $\operatorname{tilt} \Lambda$ has a minimal element if and only if $\mathcal{P}_\infty(\Lambda)$ is contravariantly finite (see [Happel and Unger 2005, Theorem 3.3]). Now we connect the existence of a minimal element in $\operatorname{tilt}_n \Lambda$ with the contravariantly finiteness of $\mathcal{P}_n(\Lambda)$ and show our main result below. Note that the equivalence of (3) and (5) for integer n recovers [Happel and Unger 1996, Corollary 2.3].

Theorem 3.1. *Let Λ be an algebra, and let n be ∞ or a nonnegative integer. Then the following are equivalent*:

(1) $\operatorname{tilt}_n \Lambda$ *has a minimal element.*

(2) $\operatorname{tilt}_n \Lambda$ *has the minimum element.*

(3) *There exists $T \in \operatorname{tilt}_n \Lambda$ such that ${}^\perp T \supseteq \mathcal{P}_n(\Lambda)$.*

(4) *There exists* $T \in \operatorname{tilt}_n \Lambda$ *such that* ${}^{\perp}(T^{\perp}) = \mathcal{P}_n(\Lambda)$.

(5) *The subcategory* $\mathcal{P}_n(\Lambda)$ *is contravariantly finite.*

To prove Theorem 3.1, we need the following result.

Proposition 3.2. *Let* Λ *be an algebra, and let* $\mathcal{C}$ *be a resolving subcategory of* $\operatorname{mod}\Lambda$ *contained in* $\mathcal{P}_\infty(\Lambda)$. *For* $T \in \mathcal{C} \cap \operatorname{tilt}\Lambda$, *the following conditions are equivalent.*

(1) T *is a minimal element in* $\mathcal{C} \cap \operatorname{tilt}\Lambda$.

(2) T *is the minimum element in* $\mathcal{C} \cap \operatorname{tilt}\Lambda$.

(3) ${}^{\perp}(T^{\perp}) = \mathcal{C}$.

(4) ${}^{\perp}T \supseteq \mathcal{C}$.

(5) $\mathcal{C} \cap T^{\perp} = \operatorname{add} T$.

(6) *Every exact sequence* $0 \to T_1 \to T_0 \to C \to 0$ *with* $T_i \in \operatorname{add} T$ *and* $C \in \mathcal{C}$ *splits.*

(7) *There is no monomorphism* $f : X \to T'$ *such that* X *is an indecomposable direct summand of* T, $T' \in \operatorname{add}(T/X)$ *and* $\operatorname{Coker} f \in \mathcal{C}$.

Proof. By the last part of Theorem 2.1, for any $U \in \mathcal{C} \cap \operatorname{tilt}\Lambda$, we have

$$ {}^{\perp}(U^{\perp}) \subseteq \mathcal{C}. \tag{3-1}$$

(1) $\Rightarrow$ (7) Assume that there exists such $f : X \to T'$. Let $g : X \to U'$ be a minimal left $(\operatorname{add} T/X)$-approximation and $Y = \operatorname{Coker} g$. Then f factors through g, and we have a commutative diagram of exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{f} & T' & \longrightarrow & \operatorname{Coker} f & \longrightarrow & 0 \\ & & \| & & \uparrow & & \uparrow & & \\ & & X & \xrightarrow{g} & U' & \longrightarrow & Y & \longrightarrow & 0 \end{array}$$

Thus $\operatorname{Ker} g = 0$, and we have an exact sequence $0 \to U' \to T' \oplus Y \to \operatorname{Coker} f \to 0$. Thus $Y \in \mathcal{C}$. This means that $U = (T/X) \oplus Y$ is a mutation of T (Proposition 2.4) and gives an element of $\mathcal{C} \cap \operatorname{tilt}\Lambda$ such that $T > U$, a contradiction.

(7) $\Rightarrow$ (6) We only need to apply Lemma 2.2 for $\mathcal{A} = \operatorname{add} T$.

(6) $\Rightarrow$ (5) It suffices to show $\mathcal{C} \cap T^{\perp} \subseteq \operatorname{add} T$. Let $C_0 \in \mathcal{C} \cap T^{\perp}$ and $n = \operatorname{pd}_\Lambda C_0$. Since $T^{\perp}$ is an exact category with enough projectives $\operatorname{add} T$, there exists an exact sequence

$$\cdots \xrightarrow{f_2} T_1 \xrightarrow{f_1} T_0 \xrightarrow{f_0} C_0 \to 0$$

such that $T_i \in \operatorname{add} T$ and $C_i = \operatorname{Im} f_i$ belongs to $T^{\perp}$. Since $\mathcal{C}$ is resolving, each C_i belongs to $\mathcal{C}$. For every $i > 0$, we have

$$\operatorname{Ext}^i_\Lambda(C_n, T^{\perp}) \simeq \operatorname{Ext}^{i+1}_\Lambda(C_{n-1}, T^{\perp}) \simeq \cdots \simeq \operatorname{Ext}^{i+n}_\Lambda(C_0, T^{\perp}) = 0.$$

Thus C_n belongs to $T^{\perp} \cap {}^{\perp}(T^{\perp}) = \operatorname{add} T$ by Theorem 2.1.

Since $C_{n-1} \in \mathcal{C}$, the exact sequence $0 \to C_n \to T_{n-1} \to C_{n-1} \to 0$ splits by our assumption, and hence C_{n-1} belongs to $\operatorname{add} T$. Repeating the same argument, we have $C_0 \in \operatorname{add} T$, as desired.

(5) ⇒ (1) Assume $U \in \mathcal{C} \cap \operatorname{tilt} \Lambda$ satisfies $T \geq U$. Then $U \in \mathcal{C} \cap T^{\perp} = \operatorname{add} T$ and hence $U = T$.

(5) + (7) ⇒ (4) By Proposition 2.3, it suffices to show that $\operatorname{add} T$ is a minimal cogenerator for $\mathcal{C}$. Let $C \in \mathcal{C}$. Since $({}^{\perp}(T^{\perp}), T^{\perp})$ is a cotorsion pair by Theorem 2.1, there exists an exact sequence $0 \to C \to Y \to X \to 0$ with $Y \in T^{\perp}$ and $X \in {}^{\perp}(T^{\perp})$. By (3-1), we have $X \in \mathcal{C}$. Since $\mathcal{C}$ is extension closed, we have $Y \in \mathcal{C} \cap T^{\perp} = \operatorname{add} T$ by (5). Thus $\operatorname{add} T$ is a cogenerator for $\mathcal{C}$. Moreover, it is minimal by (7).

(4) ⇒ (3) Thanks to (3-1), it suffices to show ${}^{\perp}(T^{\perp}) \supseteq \mathcal{C}$, i.e., any $X \in \mathcal{C}$ and $Y_0 \in T^{\perp}$ satisfy $\operatorname{Ext}^i_{\Lambda}(X, Y_0) = 0$ for all $i > 0$. Since $T^{\perp}$ is an exact category with enough projectives $\operatorname{add} T$, there exists an exact sequence

$$\cdots \xrightarrow{f_2} T_1 \xrightarrow{f_1} T_0 \xrightarrow{f_0} Y_0 \to 0$$

such that $T_i \in \operatorname{add} T$ and $Y_i = \operatorname{Im} f_i$ belongs to $T^{\perp}$. For $n = \operatorname{pd}_{\Lambda} X$, by using (4),

$$\operatorname{Ext}^i_{\Lambda}(X, Y_0) \simeq \operatorname{Ext}^{i+1}_{\Lambda}(X, Y_1) \simeq \cdots \simeq \operatorname{Ext}^{i+n}_{\Lambda}(X, Y_n) = 0$$

as desired.

(3) ⇒ (2) Let $U \in \mathcal{C} \cap \operatorname{tilt} \Lambda$. By (3-1), we have ${}^{\perp}(U^{\perp}) \subseteq \mathcal{C} = {}^{\perp}(T^{\perp})$. Thus $U^{\perp} \supseteq T^{\perp}$ by Theorem 2.1.

(2) ⇒ (1) This is clear. □

Now we prove the following theorem, where the equivalence of (3) and (5) recovers [Happel and Unger 1996, Theorems 2.1 and 2.2].

Theorem 3.3. *Let* Λ *be an algebra, and let* $\mathcal{C}$ *be a resolving subcategory of* $\operatorname{mod} \Lambda$ *contained in* $\mathcal{P}_{\infty}(\Lambda)$. *Then the following are equivalent*:

(1) $\mathcal{C} \cap \operatorname{tilt} \Lambda$ *has a minimal element.*

(2) $\mathcal{C} \cap \operatorname{tilt} \Lambda$ *has the minimum element.*

(3) *There exists* $T \in \mathcal{C} \cap \operatorname{tilt} \Lambda$ *such that* ${}^{\perp}T \supseteq \mathcal{C}$.

(4) *There exists* $T \in \mathcal{C} \cap \operatorname{tilt} \Lambda$ *such that* ${}^{\perp}(T^{\perp}) = \mathcal{C}$.

(5) *The subcategory* $\mathcal{C}$ *is contravariantly finite.*

Proof. (1) ⇔ (2) ⇔ (3) ⇔ (4) These are shown in Proposition 3.2.

(4) ⇔ (5) This is well known (see Theorem 2.1). □

We are ready to prove Theorem 3.1.

Proof of Theorem 3.1. One can obtain the assertion by putting $\mathcal{C} = \mathcal{P}_n(\Lambda)$ in Theorem 3.3. □

Minimal tilting modules of Auslander–Gorenstein algebras. In this section, we construct a class of minimal tilting modules for n-Gorenstein algebras. As a result, we show some equivalent conditions for an n-Gorenstein algebra to be Iwanaga–Gorenstein, which gives a partial answer to a question of Auslander and Reiten mentioned before.

Now we have the following result which gives a method in constructing minimal tilting modules with finite projective dimension.

Theorem 3.4. *For an algebra Λ and a fixed integer $n \geq 0$, assume* $\operatorname{pd}_\Lambda I^i(\Lambda) \leq n$ *for any i, $0 \leq i \leq n-1$ and* $\operatorname{pd}_\Lambda \Omega^{-n}\Lambda \leq n$. *Let $T = \left(\bigoplus_{i=0}^{n-1} I^i(\Lambda)\right) \oplus \Omega^{-n}\Lambda$. Then we have the following*:

(1) *T is a tilting module with projective dimension at most n.*

(2) *T is the minimum element in* $\operatorname{tilt}_n \Lambda$.

(3) *$\mathcal{P}_n(\Lambda)$ is contravariantly finite and ${}^\perp(T^\perp) = \mathcal{P}_n(\Lambda)$.*

Proof. We show the assertion (1) step by step.
• By our assumptions, $\operatorname{pd}_\Lambda T \leq n$ holds.
• We prove $\operatorname{Ext}^i_\Lambda(X, T) = 0$ for any $i \geq 1$ and $X \in \mathcal{P}_n(\Lambda)$. This implies $\operatorname{Ext}^i_\Lambda(T, T) = 0$ for $i \geq 1$.

Clearly $\operatorname{Ext}^i_\Lambda(X, I^j(\Lambda)) = 0$ holds for $i \geq 1$ since $I^j(\Lambda)$ is injective. Moreover $\operatorname{Ext}^i_\Lambda(X, \Omega^{-n}\Lambda) = \operatorname{Ext}^{i+n}_\Lambda(X, \Lambda) = 0$ holds for $i \geq 1$ since $X \in \mathcal{P}_n(\Lambda)$.
• The following exact sequence gives the desired sequence in the definition of tilting modules:

$$0 \to \Lambda \to I^0(\Lambda) \to \cdots \to I^{n-1}(\Lambda) \to \Omega^{-n}\Lambda \to 0.$$

(2) It suffices to show that $T \in U^\perp$ holds for any tilting module U with $\operatorname{pd}_\Lambda U \leq n$. This is a special case of the statment above.

(3) This follows from Theorem 3.1. □

Immediately, we have the following corollary.

Corollary 3.5. *Let Λ be a quasi n-Gorenstein algebra with $n \geq 0$. Then* $\operatorname{tilt}_n \Lambda$ *has the minimum element $\left(\bigoplus_{i=0}^{n-1} I^i(\Lambda)\right) \oplus \Omega^{-n}\Lambda$.*

Proof. This is immediate from Theorem 3.4 □

Recall that an algebra Λ is called of *dominant dimension at least n* if $I^i(\Lambda)$ is projective for $0 \leq i \leq n-1$. Then we have the following immediate from Theorem 3.4.

Example 3.6. Let Λ be an algebra with dominant dimension at least $n \geq 0$ and let $T = I^0(\Lambda) \oplus \Omega^{-n}\Lambda$. Then T is the minimum element in $\operatorname{tilt}_n \Lambda$, which was studied recently in [Crawley-Boevey and Sauter 2017; Marczinzik 2018; Nguyen et al. 2018; Pressland and Sauter 2017]. The equality ${}^{\perp}(T^{\perp}) = \mathcal{P}_n(\Lambda)$ in Theorem 3.4(3) was observed in [Marczinzik 2018, 2.4].

Now we give some applications to a question of Auslander and Reiten which says that if Λ is n-Gorenstein for all nonnegative integer n then Λ is Iwanaga–Gorenstein. This is a generalization of the famous Nakayama conjecture. We have the following:

Corollary 3.7. *Let Λ be a k-Gorenstein algebra for all positive integers k, and n a nonnegative integer. Then the following are equivalent*:

(1) *Λ is Iwanaga–Gorenstein with* $\operatorname{id}_\Lambda \Lambda = \operatorname{id}_{\Lambda^{\mathrm{op}}} \Lambda \leq n$.

(2) $\operatorname{id}_\Lambda \Lambda \leq n$.

(3) $\operatorname{id}_{\Lambda^{\mathrm{op}}} \Lambda \leq n$.

(4) $\operatorname{tilt} \Lambda$ *has the minimum element T with* $\operatorname{pd}_\Lambda T \leq n$.

(5) $\operatorname{tilt} \Lambda^{\mathrm{op}}$ *has the minimum element T with* $\operatorname{pd}_{\Lambda^{\mathrm{op}}} T \leq n$.

(6) *The subcategory $\mathcal{P}_\infty(\Lambda)$ is contravariantly finite, and $\mathcal{P}_\infty(\Lambda) = \mathcal{P}_n(\Lambda)$.*

(7) *The subcategory $\mathcal{P}_\infty(\Lambda^{\mathrm{op}})$ is contravariantly finite, and $\mathcal{P}_\infty(\Lambda^{\mathrm{op}}) = \mathcal{P}_n(\Lambda^{\mathrm{op}})$.*

Proof. (1) $\Rightarrow$ (2) is clear, (2) $\Leftrightarrow$ (3) follows from [Auslander and Reiten 1994, Corollary 5.5]. Hence (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) holds. (2) $\Leftrightarrow$ (4) follows from Corollary 3.5. (4) $\Leftrightarrow$ (6) follows from [Happel and Unger 2005, Theorem 3.3] (see Theorem 3.1). Dually, (3) $\Leftrightarrow$ (5) $\Leftrightarrow$ (7) holds. □

We give an example to show the existence and the constructing of minimal tilting modules.

Example 3.8. Let $\Lambda = KQ/I$ be an algebra with the quiver Q:

$$1 \xrightarrow{a_1} 2 \xrightarrow{a_2} \cdots \xrightarrow{a_{n-1}} n \xrightarrow{a_n} n+1$$

and $I = \operatorname{Rad}^2 KQ$. Then:

(1) The global dimension of Λ is n and $P(i) = I(i+1)$ for $1 \leq i \leq n$

(2) The minimal injective resolution of Λ is

$$0 \to \Lambda \to \Big(\bigoplus_{i=2}^{n+1} I(i)\Big) \oplus I(n+1) \to I(n) \to \cdots \to I(1) \to 0.$$

Hence Λ is n-Gorenstein.

(3) The tilting module $T_j = \big(\bigoplus_{i=2}^{n+1} I(i)\big) \oplus S(n-j+1)$ is of projective dimension j for $0 \leq j \leq n$ and T_j is a minimal element in the set $\operatorname{tilt}_j \Lambda$.

The category $\mathcal{P}_n(\Lambda)^{\perp}$. Since $\mathcal{P}_n(\Lambda)$ is a resolving subcategory of $\mathsf{mod}\,\Lambda$, it is very natural to study the corresponding coresolving subcategory $\mathcal{P}_n(\Lambda)^{\perp}$. Clearly the subcategory

$$\mathcal{Y}_n^1(\Lambda) = \mathsf{add}\,\Omega^{-n}(\mathsf{mod}\,\Lambda)$$

is contained in $\mathcal{P}_n(\Lambda)^{\perp}$ and satisfies ${}^{\perp}\mathcal{Y}_n^1(\Lambda) = \mathcal{P}_n(\Lambda)$. Auslander and Reiten [1994, Theorem 1.2] proved that $\mathcal{Y}_n^1(\Lambda)$ is always covariantly finite. Moreover they proved that $\mathcal{Y}_i^1(\Lambda)$ is extension closed (or equivalently, coresolving) for every $1 \le i \le n$ if and only if Λ is quasi n-Gorenstein [Auslander and Reiten 1994, Theorem 2.1]. For a more general class of algebras, it is natural to consider the extension closure of $\mathcal{Y}_n^1(\Lambda)$.

For a subcategory $\mathscr{C}$ of $\mathsf{mod}\,\Lambda$, denote by $\mathscr{C}^{*m}$ the subcategory of $\mathsf{mod}\,\Lambda$ consisting of all $X \in \mathsf{mod}\,\Lambda$ having a filtration $X = X_0 \supseteq X_1 \supseteq \cdots \supseteq X_m = 0$ such that $X_i/X_{i+1} \in \mathscr{C}$. Denote by $\Omega^{-n}(\mathsf{mod}\,\Lambda)$ the subcategory consisting of the modules of the form $\Omega^{-n}C \oplus I$ for all C in $\mathsf{mod}\,\Lambda$ and injective modules I. (Note that they are not necessarily closed under direct summands.) Now we define subcategories by

$$\mathcal{Y}_n^m(\Lambda) = \mathsf{add}\,(\Omega^{-n}(\mathsf{mod}\,\Lambda)^{*m}) \quad \text{and} \quad \mathcal{Y}_n(\Lambda) = \bigcup_{m \ge 0} \mathcal{Y}_n^m(\Lambda).$$

We have the following observations.

Proposition 3.9. *Let Λ be an algebra and n a nonnegative integer.*

(1) $\mathcal{Y}_n^m(\Lambda)$ *is a covariantly finite subcategory for every $m > 0$ such that* ${}^{\perp}\mathcal{Y}_n^m(\Lambda) = \mathcal{P}_n(\Lambda)$.

(2) $\mathcal{Y}_n(\Lambda)$ *is a coresolving subcategory such that* ${}^{\perp}\mathcal{Y}_n(\Lambda) = \mathcal{P}_n(\Lambda)$.

Proof. Clearly ${}^{\perp}\mathcal{Y}_n(\Lambda)$ and ${}^{\perp}\mathcal{Y}_n^m(\Lambda)$ coincide with ${}^{\perp}\mathcal{Y}_n^1(\Lambda) = \mathcal{P}_n(\Lambda)$. For (1), we refer to [Chen 2009]. The assertion (2) is clear. □

As a consequence, we have the following observations.

Proposition 3.10. *For an algebra Λ, consider the following five conditions*:

(1) *The subcategory $\mathcal{P}_n(\Lambda)$ is contravariantly finite.*

(2) $\mathcal{Y}_n(\Lambda)$ *is covariantly finite.*

(3) $\mathcal{Y}_n^m(\Lambda)$ *is closed under extensions for some $m > 0$.*

(4) $\mathcal{Y}_n(\Lambda) = \mathcal{Y}_n^m(\Lambda)$ *holds for some $m > 0$.*

(5) $\mathcal{Y}_n^1(\Lambda)$ *is closed under extensions.*

Then we have (5) ⇒ (4) ⇔ (3) ⇔ (2) ⇒ (1).

Proof. (5) ⇒ (3) We may choose $m = 1$.

(3) ⇒ (2) Assume that $\mathcal{Y}_n^m(\Lambda)$ is extension closed. Then $\mathcal{Y}_n(\Lambda) = \mathcal{Y}_n^m(\Lambda)$. This is covariantly finite by Proposition 3.9(1).

(2) ⇒ (4) Let ℓ be the Loewy length of Λ, $S = \Lambda/\operatorname{rad}\Lambda$ and $S \to Y$ be a left $\mathcal{Y}_n(\Lambda)$-approximation of S. Then Y belongs to $\mathcal{Y}_n^m(\Lambda)$ for some $m > 0$. We claim $\mathcal{Y}_n(\Lambda) = \mathcal{Y}_n^{m\ell}(\Lambda)$.

Since any $X \in \operatorname{mod}\Lambda$ belongs to $(\operatorname{add} S)^{*\ell}$, the Horseshoe-type Lemma [Auslander and Reiten 1991, Proposition 3.6] shows that X has a left $\mathcal{Y}_n(\Lambda)$-approximation $X \to Y$ with $Y \in (\operatorname{add} Y)^{*\ell} \subseteq \mathcal{Y}_n^{m\ell}(\Lambda)$. If $X \in \mathcal{Y}_n(\Lambda)$, then f is a split monomorphism. Thus $X \in \mathcal{Y}_n^{m\ell}(\Lambda)$ holds.

(4) ⇒ (3) This is clear since $\mathcal{Y}_n(\Lambda)$ is extension closed.

(2) ⇒ (1) By Proposition 3.9(2), $\mathcal{Y}_n(\Lambda)$ is a covariantly finite coresolving subcategory such that ${}^{\perp}\mathcal{Y}_n(\Lambda) = \mathcal{P}_n(\Lambda)$. By [Auslander and Reiten 1992, Lemma 3.3(a)], $\mathcal{P}_n(\Lambda)$ is contravariantly finite. □

We should remark that (5) is not equivalent to (2) in Proposition 3.10. We give an example to show this. However, we do not know whether (2) is equivalent to (1).

Example 3.11. Let Λ be a local algebra with Loewy length 2. Then $\mathcal{Y}_1^1(\Lambda) = \operatorname{add}\{K, \mathbb{D}\Lambda\}$. Thus it is closed under extensions if and only if Λ is self-injective.

On the other hand, $\mathcal{Y}_1^2(\Lambda) = \operatorname{mod}\Lambda$ holds, and hence Proposition 3.10(3) is satisfied. Moreover, $\operatorname{tilt}_1\Lambda$ has a minimal element Λ, and $\mathcal{P}_1(\Lambda) = \operatorname{add}\Lambda$ is contravariantly finite.

4. A bijection between classical tilting modules and support τ-tilting modules

Throughout this section, Λ is a 1-Gorenstein algebra and e is an idempotent such that $\operatorname{add} e\Lambda = \operatorname{add} I^0(\Lambda)$ unless stated otherwise. Denote by $\Gamma = \Lambda/(e)$ the factor algebra of Λ. We mainly focus on the bijection between classical tilting modules over a 1-Gorenstein algebra Λ and support τ-tilting modules over the factor algebra Γ.

Now let Λ be an arbitrary algebra. Denote by τ the AR-translation and denote by $|N|$ the number of nonisomorphic indecomposable direct summands of a Λ-module N. Firstly, we recall the definition of support τ-tilting modules in [Adachi et al. 2014].

Definition 4.1. (1) We call $N \in \operatorname{mod}\Lambda$ *τ-rigid* if $\operatorname{Hom}_\Lambda(N, \tau N) = 0$.

(2) We call $N \in \operatorname{mod}\Lambda$ *τ-tilting* if N is τ-rigid and $|N| = |\Lambda|$.

(3) We call $N \in \operatorname{mod}\Lambda$ *support τ-tilting* if there exists an idempotent e of Λ such that N is a τ-tilting $(\Lambda/(e))$-module.

The following property is also needed for the main result in this section.

Lemma 4.2 [Adachi et al. 2014]. *For an algebra Λ, classical tilting Λ-modules are precisely faithful support τ-tilting Λ-modules.*

Now we are in a position to state the following properties of tilting modules over 1-Gorenstein algebras.

Lemma 4.3. *Let Λ be a 1-Gorenstein algebra, and let e be an idempotent such that* $\operatorname{add} e\Lambda = \operatorname{add} I^0(\Lambda)$.

(1) *Every classical tilting Λ-module T satisfies $e\Lambda \in \operatorname{add} T$.*

(2) *Every support τ-tilting Λ-module M satisfying $e\Lambda \in \operatorname{add} M$ is a classical tilting module.*

Proof. (1) Since T is a classical tilting module, by Lemma 4.2 T is faithful, and hence any projective module is cogenerated T. Then we get that $e\Lambda$ is a direct summand of T since it is injective.

(2) Since Λ is 1-Gorenstein, then Λ can be embedded in $\operatorname{add}_\Lambda e\Lambda$, and hence Λ can be embedded in $\operatorname{add}_\Lambda M$. Then M is faithful, and by Lemma 4.2, M is a classical tilting module. □

For an algebra Λ and $U \in \operatorname{mod}\Lambda$, denote by $\mathrm{s}\tau\text{-tilt}_U\Lambda$ the set of all $M \in \mathrm{s}\tau\text{-tilt}\,\Lambda$ satisfying $U \in \operatorname{add} M$. Denote by $U^{\perp_0}$ the subcategory consisting of modules M such that $\operatorname{Hom}_\Lambda(U, M) = 0$. The following theorem from [Jasso 2015] is essential to the main result in this section.

Theorem 4.4. *Let A be an algebra and let U be a basic τ-rigid A-module. Let T be the Bongartz completion of U, $B = \operatorname{End}_A(T)$ and $C = B/(e_U)$, where e_U is the idempotent corresponding to the projective B-module $\operatorname{Hom}_A(T, U)$. Then there is a bijection $\phi : \mathrm{s}\tau\text{-tilt}_U A \to \mathrm{s}\tau\text{-tilt}\,C$ via $M \to \operatorname{Hom}_A(T, fM)$, where $0 \to tM \to M \to fM \to 0$ is the canonical sequence according to the torsion pair* $(\operatorname{Fac} U, U^{\perp_0})$.

Recall that $\operatorname{tilt}_1\Lambda$ is the set of additive equivalence classes of classical tilting Λ-modules. Now we are in a position to show our main result in this section.

Theorem 4.5. *Let Λ be a 1-Gorenstein algebra and $\Gamma = \Lambda/(e)$, where e is an idempotent such that $\operatorname{add} e\Lambda = \operatorname{add} I^0(\Lambda)$. Then the tensor functor $-\otimes_\Lambda \Gamma$ induces a bijection from* $\operatorname{tilt}_1\Lambda$ *to* $\mathrm{s}\tau\text{-tilt}\,\Gamma$.

Proof. Let $U = e\Lambda$. Then $\operatorname{tilt}_1\Lambda = \mathrm{s}\tau\text{-tilt}_U\Lambda$ holds by Lemma 4.3. On the other hand, the Bongartz completion of U is nothing but $T = \Lambda$. Then $\operatorname{End}_\Lambda(T) = \Lambda$ and $e = e_U$ hold in Theorem 4.4, and we get a bijection ϕ from $\operatorname{tilt}_1\Lambda$ to $\mathrm{s}\tau\text{-tilt}\,\Gamma$.

In the following we show $-\otimes_\Lambda \Gamma = \phi$ as a map. Note that $U = e\Lambda$, so the canonical sequence of $T' \in \operatorname{tilt}_1\Lambda$ according to the torsion pair $(\operatorname{Fac} U, U^{\perp_0})$ is

$$0 \to T'(e) \to T' \to T'/T'(e) \to 0.$$

Then by Theorem 4.4, $\phi(T') = \operatorname{Hom}_\Lambda(\Lambda, T'/T'(e)) \simeq T'/T'(e) \simeq T' \otimes_\Lambda \Gamma$, so $\phi = -\otimes_\Lambda \Gamma$ as a map. □

To give some applications of Theorem 4.5, we fix the following notation.

Notation 4.6. Let Λ_n be the Auslander algebra of $K[x]/(x^n)$ and let Γ_{n-1} be the preprojective algebra of type A_{n-1}. Then we have the following:

(1) Λ_n is given by the quiver

$$1 \underset{b_2}{\overset{a_1}{\rightleftarrows}} 2 \underset{b_3}{\overset{a_2}{\rightleftarrows}} 3 \underset{b_4}{\overset{a_3}{\rightleftarrows}} \cdots \underset{b_{n-1}}{\overset{a_{n-2}}{\rightleftarrows}} n-1 \underset{b_n}{\overset{a_{n-1}}{\rightleftarrows}} n$$

with relations $a_1b_2 = 0$ and $a_ib_{i+1} = b_ia_{i-1}$ for any $2 \leq i \leq n-1$.

(2) Γ_{n-1} is given by the quiver

$$1 \underset{b_2}{\overset{a_1}{\rightleftarrows}} 2 \underset{b_3}{\overset{a_2}{\rightleftarrows}} 3 \underset{b_4}{\overset{a_3}{\rightleftarrows}} \cdots \underset{b_{n-1}}{\overset{a_{n-2}}{\rightleftarrows}} n-1$$

with relations $a_1b_2 = 0$, $a_{n-2}b_{n-1} = 0$ and $a_ib_{i+1} = b_ia_{i-1}$ for any $2 \leq i \leq n-2$.

In the rest of this section, for an algebra Λ_n, we always assume that e is the idempotent such that $\mathsf{add}\, e\Lambda_n = \mathsf{add}\, I^0(\Lambda_n)$. Applying Theorem 4.5 to the Auslander algebras of $K[x]/(x^n)$, we get the following corollary which recovers the results in [Mizuno 2014; Iyama and Zhang 2016].

Corollary 4.7. *Let Λ_n be the Auslander algebra of $K[x]/(x^n)$, let Γ_{n-1} be the preprojective algebra of $Q = A_{n-1}$ and let $\mathfrak{S}_n$ be the symmetric group. Then there are bijections*

$$\operatorname{tilt}_1 \Lambda_n \simeq \operatorname{s\tau-tilt} \Gamma_{n-1} \simeq \mathfrak{S}_n.$$

Thus $\#\operatorname{s\tau-tilt} \Lambda_n = n!$.

Proof. It is not difficult to show that $\Gamma_{n-1} \simeq \Lambda_n/(e)$ by Notation 4.6. Then by Theorem 4.5, $\operatorname{tilt}_1 \Lambda_n \simeq \operatorname{s\tau-tilt} \Gamma_{n-1}$ holds. On the other hand, $\operatorname{s\tau-tilt} \Gamma_{n-1} \simeq \mathfrak{S}_n$ holds by [Mizuno 2014, Theorem 0.1]. We are done. □

Recall that an algebra is called *τ-tilting finite* if there are a finite number of basic τ-tilting modules up to isomorphism. Applying Theorem 4.5 to the Auslander algebra of a Nakayama hereditary algebra, we have the following corollary.

Corollary 4.8. *Let Λ_n be the Auslander algebra of the Nakayama hereditary algebra KQ with $Q = A_n$. Then there is a bijection* $\operatorname{tilt}_1 \Lambda_n \simeq \operatorname{s\tau-tilt} \Lambda_{n-1}$. *Thus Λ is τ-tilting finite if and only if $n \leq 4$.*

Proof. It is not difficult to show that $\Lambda_n/(e) \simeq \Lambda_{n-1}$. Then by Theorem 4.5, $\operatorname{tilt} \Lambda_n \simeq \operatorname{s\tau-tilt} \Lambda_{n-1}$. On the other hand, it was shown in [Kajita 2008] that Λ_n has a finite number of classical tilting modules if and only if $n \leq 5$. Then the assertion holds. □

For more details of τ-rigid modules over Auslander algebras, we refer to [Zhang 2017a]. Recall from [Iyama 2011, Proposition 1.17] that a hereditary algebra is 1-Gorenstein if and only if it is a Nakayama algebra. Now we have the following corollary.

Corollary 4.9. *Let $\Lambda_n = KQ$ be the Nakayama hereditary algebra with $Q = A_n$. Then there are bijections*

$$\mathrm{tilt}_1\Lambda_n \simeq \mathrm{s}\tau\text{-tilt}\,\Lambda_{n-1} \simeq \{\textit{clusters of the cluster algebra of type } A_{n-1}\}.$$

Thus $\#\mathrm{s}\tau\text{-tilt}\,\Lambda_n = (2(n+1))!/((n+2)!(n+1)!)$.

Proof. A straight calculation shows that $\Lambda_{n-1} \simeq \Lambda_n/(e)$. Then by Theorem 4.5 and [Kajita 2008, Theorem 1],

$$\mathrm{tilt}_1\Lambda_n \simeq \mathrm{s}\tau\text{-tilt}\,\Lambda_{n-1} \quad \text{and} \quad \#\mathrm{s}\tau\text{-tilt}\,\Lambda_n = (2(n+1))!/((n+2)!(n+1)!).$$

By [Adachi et al. 2014, Theorem 0.5] and [Buan et al. 2006, Theorem 4.5], one gets the second bijection. □

Acknowledgements

Parts of the work were done while Zhang visited Nagoya in 2015. Zhang thanks his co-author for the hospitality received during his stay there. He is also grateful to other people in Nagoya for their kind help. Part of the results were presented in USTC, Tsinghua University, Anhui University and Capital Normal University. Zhang would like to thank all the hosts there for their hospitality. The authors thank the referee for useful suggestions to improve this paper.

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Received March 19, 2018. Revised July 8, 2018.

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Subscriptions, requests for back issues and changes of subscriber address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFlow® from Mathematical Sciences Publishers.

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