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Recently, it was shown that Einstein solvmanifolds have maximal symmetry in the sense that their isometry groups contain the isometry groups of any other left-invariant metric on the given Lie group. Such a solvable Lie group is necessarily nonunimodular. In this work we consider unimodular solvable Lie groups and prove that there is always some metric with maximal symmetry. Further, if the group at hand admits a Ricci soliton, then it is the isometry group of the Ricci soliton which is maximal.

1. Introduction

In this work, we restrict ourselves to the setting of Lie groups with left-invariant metrics.

Definition 1.1. Let G be a Lie group. A left-invariant metric g on G is said to be maximally symmetric if given any other left-invariant metric g', there exists a diffeomorphism $\phi \in \mathfrak{Diff}(G)$ such that

$$\mathrm{Isom}(M, g') \subset \mathrm{Isom}(M, \phi^* g) = \phi\,\mathrm{Isom}(M, g)\phi^{-1}.$$

We say G is a maximal symmetry space if it admits a metric of maximal symmetry.

Although our primary interest is in solvable Lie groups with left-invariant metrics, we briefly discuss the more general setting of Lie groups. For G compact and simple, we have that $\mathrm{Isom}(G)_0$, the connected component of the identity, for any left-invariant metric, can be embedded into the isometry group of the bi-invariant metric [Ochiai and Takahashi 1976]. This does not quite say that compact simple Lie groups are maximal symmetry spaces, but it is close.

In the setting of noncompact semisimple groups, one does not have a bi-invariant metric, but there is a natural choice which plays the role of the bi-invariant metric and similar results are known, see [Gordon 1980]; note the work of Gordon actually

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goes beyond the Lie group setting and considers a larger class of homogeneous spaces with transitive reductive Lie group and studies their isometry groups.

If S is a nilpotent or completely solvable unimodular group, then it is a maximal symmetry space. Although not stated in this language, this is a result of Gordon and Wilson [1988]; see Section 2 below for more details. Furthermore, when such a Lie group admits a Ricci soliton, the soliton metric has the maximal isometry group [Jablonski 2011].

The nonunimodular setting for completely solvable groups is not as clean. In special circumstances these groups can and do have maximal symmetry, e.g., if a solvable group admits an Einstein metric, then it is a maximal symmetry space and the Einstein metric actually has the largest isometry group; see [Gordon and Jablonski 2015] for more details. However, it is known that not all nonunimodular, completely solvable groups can be maximal symmetry spaces, see Example 1.6 of [Gordon and Jablonski 2015]. For more on the subtleties of the maximal symmetry question in the nonunimodular setting, see the forthcoming work [Epstein and Jablonski 2018].

Our main result is for unimodular solvable Lie groups.

Theorem 1.2. *Let R be a simply connected, unimodular solvable Lie group. Then R is a maximal symmetry space.*

Corollary 1.3. *Let R be a simply connected, unimodular solvable Lie group which admits a Ricci soliton metric. Then said Ricci soliton has maximal symmetry among R-invariant metrics.*

The strategy for proving both results is to reduce to the setting of completely solvable groups, where the answer is immediate. To do this, we start with a solvable group R, we modify a given initial metric until we obtain a metric whose isometry group contains a transitive solvable S which is completely solvable. Our main contribution, then, is to prove a uniqueness result for which S can appear; up to isomorphism only one can and does appear. This uniqueness result is a consequence of the following, which is of independent interest; see Lemma 4.3. (Here we use the language of [Gordon and Wilson 1988].)

Lemma. *Any modification of a completely solvable group is necessarily a normal modification.*

It seems noteworthy to point out that our work actually shows that any solvable Lie group is associated to a unique completely solvable group (Theorem 4.7) in the same way that type R groups have a well defined, unique nilshadow, cf. [Auslander and Green 1966].

In the last section we give a concrete description of the completely solvable group associated to any solvable group S in terms of S and the derivations of its Lie algebra.

Finally, we observe that the choices made throughout our process allow us to choose our diffeomorphism ϕ from Definition 1.1 to be a composition of an automorphism of R together with an automorphism of its associated completely solvable group S. In the case that $R = S$ is completely solvable, the diffeomorphism which conjugates the isometry groups can be chosen to be an automorphism. It would be interesting to know whether or not this is true in general.

2. Preliminaries

In this section, we recall the basics on isometry groups for (unimodular) solvmanifolds from the foundational work of Gordon and Wilson [1988]. Throughout, our standing assumption is that our solvable groups are simply connected. We begin with a general result for Lie algebras.

Recall that every Lie algebra has a unique, maximal solvable ideal, called the radical. A (solvable) Lie algebra $\mathfrak{g}$ is called completely solvable if $\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}$ has only real eigenvalues for all $X \in \mathfrak{g}$. We have the following.

Proposition 2.1. *Given any Lie algebra* $\mathfrak{g}$ *there exists a unique maximal ideal* $\mathfrak{s}$ *which is completely solvable.*

Remark 2.2. This completely solvable ideal is contained in the radical, but generally they are not equal. Notice that the nilradical of $\mathfrak{g}$ is contained in $\mathfrak{s}$ and so, as with the radical, $\mathfrak{s}$ is trivial precisely when $\mathfrak{g}$ is semisimple.

Proof of proposition. As any solvable ideal is a subalgebra of the radical of $\mathfrak{g}$, it suffices to prove the result in the special case that $\mathfrak{g}$ is solvable. The result follows upon showing that the sum of two such ideals is again an ideal of the same type. As the sum of ideals is again an ideal, we only need to check the condition of complete solvability.

Let $\mathfrak{g}$ be a solvable Lie algebra and $\mathfrak{s}_1$ and $\mathfrak{s}_2$ be completely solvable ideals of $\mathfrak{g}$. We will show that $\mathfrak{s}_1 + \mathfrak{s}_2$ is again completely solvable. Observe that for any ideal $\mathfrak{s}$, the eigenvalues of $\mathrm{ad}\, X|_{\mathfrak{s}}$ are real if and only if the eigenvalues of $\mathrm{ad}\, X|_{\mathfrak{g}}$ are real, as we have only introduced extra zero eigenvalues.

The eigenvalues of $\mathrm{ad}\, X : \mathfrak{s} \to \mathfrak{s}$ do not change if we extend $\mathrm{ad}\, X$ to a map on $\mathfrak{s} \otimes \mathbb{C}$. By Lie's theorem, we may realize $\mathrm{ad}\, \mathfrak{s}$ as a subalgebra of upper triangular matrices and so the eigenvalues of $\mathrm{ad}(X_1 + X_2)$ are sums of eigenvalues of $\mathrm{ad}\, X_1$ and $\mathrm{ad}\, X_2$. Taking $X_1 \in \mathfrak{s}_1$ and $X_2 \in \mathfrak{s}_2$, we see that $\mathfrak{s}_1 + \mathfrak{s}_2$ is completely solvable. □

Isometry groups and modifications. In [Gordon and Wilson 1988], the authors set about the job of giving a description of the full isometry group of any solvmanifold. Given any Lie group R with left-invariant metric, one can build a group of isometries as follows: let C denote the set of orthogonal automorphisms of $\mathfrak{r}$, then $R \rtimes C$ is a subgroup of the isometry group. We call this group the algebraic isometry group and denote it by $\mathrm{AlgIsom}(R, g)$.

For R nilpotent, this gives the full isometry group [Gordon and Wilson 1988, Corollary 4.4]. However, in general, the isometry group $\mathrm{Isom}(R, g)$ will be much more. A good example of this is to look at a symmetric space of noncompact type.

So to understand the general setting, Gordon and Wilson detail a process of modifying the initial solvable group R to one with a "better" presentation R' called a standard modification of R — this is another solvable group of isometries which acts transitively. The modification process ends after (at most) two normal modifications with the solvable group R'' in so-called standard position. See [loc. cit., Section 3] for details.

To illustrate why this process is nice, we present the following result in the case of unimodular, solvable Lie groups.

Lemma 2.3. *Let R be a unimodular solvable Lie group with left-invariant metric g. Then $\mathrm{Isom}(R, g) = \mathrm{AlgIsom}(R'', g) = C \ltimes R''$ where R'' is the solvable group in standard position inside $\mathrm{Isom}(R, g)$ and C consists of orthogonal automorphisms of $\mathfrak{r}''$.*

This follows from the following facts proven in Theorems 3.1, 4.2, and 4.3 of [Gordon and Wilson 1988].

Proposition 2.4. *If there is one transitive solvable Lie group of isometries which is unimodular, then all transitive solvable groups of isometries are unimodular.*

Proposition 2.5. *If R is solvable, unimodular, and in standard position, then the isometry group is the algebraic isometry group.*

Proposition 2.6. *Any almost simply-transitive solvable group of isometries is a modification of one in standard position. Completely solvable groups are always in standard position.*

Regarding normal modifications, we record the following useful facts here.

Lemma 2.7. *For solvable Lie groups R and S in a common isometry group, R being a normal modification of S implies S is a normal modification of R.*

This follows immediately from the description of normal modifications given in Proposition 2.4 of [Gordon and Wilson 1988]. This will be used in the sequel when S is completely solvable. Such S are in standard position in the isometry group and any modification R is a normal modification (see Lemma 4.3), so we see that there exists an abelian subalgebra $\mathfrak{t}$ of the stabilizer subalgebra which normalizes both $\mathfrak{r}$ and $\mathfrak{s}$ such that $\mathfrak{s} \subset \mathfrak{r} \rtimes \mathfrak{t}$ and $\mathfrak{r} \subset \mathfrak{s} \rtimes \mathfrak{t}$; cf. Theorem 3.1 of [Gordon and Wilson 1988]. As such, we have the following.

Lemma 2.8. *For $\mathfrak{s}$ a completely solvable algebra in the isometry algebra and $\mathfrak{r}$ a modification of $\mathfrak{s}$, we have $[\mathfrak{s}, \mathfrak{r}] \subset \mathfrak{s} \cap \mathfrak{r}$.*

Transitive groups of isometries. The following technical lemma will be needed later.

Lemma 2.9. *Consider a solvable Lie group with left-invariant metric* (R_1, g). *Let* K *denote the orthogonal automorphisms of* R_1; *this is a subgroup of the isotropy group which fixes* $e \in R_1$. *Let* R_2 *be a subgroup of isometries satisfying* $\mathfrak{k} + \mathfrak{r}_2 \supset \mathfrak{r}_1$, *then* R_2 *acts transitively.*

Proof. As K fixes $e \in R_1$ and $\mathfrak{k} + \mathfrak{r}_2 \supset \mathfrak{r}_1$, we see that the orbit $R_2 \cdot e$ has the same dimension as $R_1 = R_1 \cdot e$. But the orbit $R_2 \cdot e$ is then an open, complete submanifold of the connected manifold R_1, hence R_2 acts transitively; cf. [Jablonski 2015b, Lemma 3.8]. □

3. Proof of main result in the special case of completely solvable and unimodular groups

Our general strategy is to reduce to the case where the group is completely solvable and so we begin here. Let S be unimodular and completely solvable. For the sake of consistency throughout the later sections, we write $R = S$ in this section.

Theorem 3.1. *Let* S *be a simply connected*, *unimodular*, *completely solvable Lie group. Then* S *is a maximal symmetry space.*

This theorem is an immediate consequence of the following result of Gordon and Wilson, as we see below.

Theorem 3.2 (Gordon and Wilson). *Let* S *be a simply connected*, *unimodular*, *completely solvable Lie group with left-invariant metric* g. *Then* $\mathrm{Isom}(S, g) = S \rtimes C$, *where* $C = \mathrm{Aut}(\mathfrak{s}) \cap O(g)$.

Remark. In the above, we have abused notation as we are viewing C as a subgroup of $\mathrm{Aut}(S)$. This is okay as S being simply connected gives that the action of C on $\mathfrak{s}$ lifts to an action on S.

As $C < \mathrm{Aut}(S)$ is a closed subgroup of $O(g)$, it is compact. Choose any maximal compact subgroup K of $\mathrm{Aut}(S)$ containing C and an inner product g' on $\mathfrak{s}$ so that K acts orthogonally. Now we have

$$\mathrm{Isom}(S, g) < \mathrm{Isom}(S, g')$$

To see that S is indeed a homogeneous maximal symmetry space, we only need to compare isometry groups where $C = K$ is a maximal compact subgroup of $\mathrm{Aut}(S)$.

Let g_1 and g_2 be two left-invariant metrics with isometry groups $S \rtimes K_1$ and $S \rtimes K_2$, respectively, such that K_1 and K_2 are maximal compact subgroups of automorphisms. As maximal compact subgroups are all conjugate, there exists $\phi \in \mathrm{Aut}(S)$ such that $K_1 = \phi K_2 \phi^{-1}$ and hence

$$\mathrm{Isom}(S, g_1) = \phi\, \mathrm{Isom}(S, g_2)\phi^{-1} = \mathrm{Isom}(S, \phi^* g_2).$$

This shows that unimodular, completely solvable Lie groups are indeed homogeneous maximal symmetry spaces.

4. Proof of main result for general solvable, unimodular groups

To prove this result, we start by adjusting our metric so as to enlarge the isometry group to one which is the isometry group of a left-invariant metric on a completely solvable, unimodular group. Then we show that the completely solvable group obtained is unique up to conjugation and use this to prove that there is one largest isometry group for R up to conjugation.

Enlarging the isometry group to find some completely solvable group.

Proposition 4.1. *Let R be a simply connected, unimodular solvmanifold with left-invariant metric g. There exists another left-invariant metric g' such that*

(i) $\mathrm{Isom}(R, g) < \mathrm{Isom}(R, g')$, *and*

(ii) $\mathrm{Isom}(R, g')$ *contains a transitive, completely solvable group S.*

Proof of Proposition 4.1. From the work of Gordon and Wilson (see Lemma 2.3), we have the existence of a transitive, solvable subgroup $R'' < \mathrm{Isom}(R, g)$ such that

$$\mathrm{Isom}(R, g) = \mathrm{AlgIsom}(R'', g) = C \ltimes R'',$$

where C consists of orthogonal automorphisms of $\mathfrak{r}''$. Here R'' is the group in standard position in $\mathrm{Isom}(R, g)$.

There exists a maximal compact subgroup K of $\mathrm{Aut}(R'')$ containing C. Choose any inner product g' on $\mathfrak{r}''$ so that K consists of orthogonal automorphisms. Then we immediately see that $\mathrm{Isom}(R'', g') > K \ltimes R''$. Applying Lemma 2.9, we see that R acts transitively by isometries on (R'', g') and so this new left-invariant metric g' on R'' gives rise to a left-invariant metric on R. This choice of g' satisfies part (i).

To finish, we show that $\mathrm{Isom}(R, g')$ contains a completely solvable group S which acts transitively. Consider the group $\mathrm{Ad}(R'')$ as a subgroup of $\mathrm{Aut}(R'')$. This group is a normal, solvable subgroup and so is a subgroup of the radical $\mathrm{Rad}(\mathrm{Aut}(R''))$ of $\mathrm{Aut}(R'')$.

As $\mathrm{Rad}(\mathrm{Aut}(R''))$ is an algebraic group, it has an algebraic Levi decomposition

$$\mathrm{Rad}(\mathrm{Aut}(R'')) = M \ltimes N,$$

where M is a maximal reductive subgroup and N is the unipotent radical (see [Mostow 1956]). Furthermore, the group M is abelian and decomposes as $M = M_K M_P$, where M_K is a compact torus and M_P is a split torus. As maximal compact subgroups are all conjugate and $\mathrm{Rad}(\mathrm{Aut}(R''))$ does not change under conjugation, we may assume, after possibly changing M, that $M_K < K$. So given $X \in \mathfrak{r}''$, we may write

$$\operatorname{ad} X = K_X + P_X + N_X,$$

where $K_X \in \operatorname{Lie} M_K$, $P_X \in \operatorname{Lie} M_P$, and $N_X \in \mathfrak{n} = \operatorname{Lie} N$. Note, K_X has purely imaginary eigenvalues while P_X has real eigenvalues.

Now define the set $\mathfrak{s} \subset \mathfrak{r}'' \rtimes \operatorname{Lie} M_K \subset \operatorname{Lie} \operatorname{Isom}(R'', g')$ as

$$\mathfrak{s} := \{X - K_X \mid X \in \mathfrak{r}''\}.$$

Since the K_X all commute, the nilradical of $\mathfrak{r}''$ is contained in $\mathfrak{s}$, and derivations of $\mathfrak{r}''$ are valued in the nilradical, we see that $\mathfrak{s}$ is a solvable Lie algebra.

Note that $\mathfrak{s}$ is completely solvable (this follows as in the proof of Proposition 2.1) and S acts transitively (via Lemma 2.9). □

As completely solvable groups are always in standard position (Proposition 2.6), we see that R is a modification of the group S and that

$$\operatorname{Isom}(R, g) < \operatorname{Isom}(R, g') = \operatorname{Isom}(S, g') = S \rtimes C,$$

where C is the compact group of orthogonal automorphisms of $\mathfrak{s}$, relative to g'. Let K denote a maximal compact group of automorphisms of $\mathfrak{s}$ and g'' an inner product on $\mathfrak{s}$ such that $\operatorname{Isom}(S, g'') = S \rtimes K$. As $R \subset S \ltimes C \subset S \ltimes K$ acts transitively and isometrically on (S, g''), it picks up a left-invariant metric g'' such that

$$\operatorname{Isom}(R, g) < \operatorname{Isom}(R, g'').$$

In this way, we have found an isometry group $\operatorname{Isom}(R, g'')$ which is a maximal isometry group for S and so by Theorem 3.1 cannot be any larger.

This is a reasonable candidate for maximal isometry group for R; we verify this in the sequel.

The uniqueness of S. The group S, constructed above, depends on several choices made based on various initial and chosen metrics. More precisely, one starts with metric g, makes two modifications to obtain the group R'', then changes the metric to some g' to extract the group S.

If one were to start with a different metric h on R, then R'' would certainly be different and so it is unclear, a priori, how the resulting S for h would compare to the group S built from the other metric g. Surprisingly, they must be conjugate via $\operatorname{Aut}(R)$.

Proposition 4.2. *There exists a maximal compact subalgebra $\mathfrak{k}$ of* $\operatorname{Der}(\mathfrak{r})$ *such that $\mathfrak{s}$ is the maximal completely solvable ideal of* $\mathfrak{r} \rtimes \mathfrak{k}$.

Before proving this proposition, we use it to show that any two groups S constructed from R are conjugate via $\operatorname{Aut}(R)$.

Let g and h be two different metrics on R with associated completely solvable algebras $\mathfrak{s}_g$ and $\mathfrak{s}_h$, respectively. Let $\mathfrak{k}_g$ and $\mathfrak{k}_h$ be the compact algebras as in Proposition 4.2 for g and h, respectively. As the maximal compact subgroup of a group is unique up to conjugation, we have some $\phi \in \mathrm{Aut}(R)$ such that $\mathfrak{k}_g = \phi \mathfrak{k}_h \phi^{-1}$. This implies

$$\phi \mathfrak{s}_h \phi^{-1} \subset \mathfrak{r} \rtimes \phi \mathfrak{k}_h \phi^{-1} = \mathfrak{r} \rtimes \mathfrak{k}_g.$$

As $\phi \mathfrak{s}_h \phi^{-1}$ is completely solvable and of the same dimension as the maximal completely solvable $\mathfrak{s}_g$, they must be equal; cf. Proposition 2.1.

We now prove Proposition 4.2.

Lemma 4.3. *Let $\mathfrak{s}$ be a completely solvable Lie algebra with inner product. Any modification of $\mathfrak{s}$ (in its isometry algebra) is a normal modification.*

Remark 4.4. In the special case of nilpotent Lie algebras, this result was already known [Gordon and Wilson 1988, Theorem 2.5]. Building on that result, we extend it to all completely solvable groups.

Proof. Let $\mathfrak{r} = (\mathrm{id} + \phi)\mathfrak{s}$ be a modification of $\mathfrak{s}$ with modification map $\phi : \mathfrak{s} \to N_l(\mathfrak{s})$, where $N_l(\mathfrak{s})$ is the set of skew-symmetric derivations of $\mathfrak{s}$; cf. [Gordon and Wilson 1988, Proposition 3.3]. To show this is a normal modification, it suffices to show $[\mathfrak{s}, \mathfrak{s}] \subset \mathrm{Ker}\, \phi$ by Proposition 2.4 of [Gordon and Wilson 1988].

Denote the nilradical of $\mathfrak{s}$ by $\mathfrak{n}(\mathfrak{s})$. As every derivation of $\mathfrak{s}$ takes its value in $\mathfrak{n}(\mathfrak{s})$, we can decompose $\mathfrak{s} = \mathfrak{a} + \mathfrak{n}(\mathfrak{s})$ where $\mathfrak{a}$ is annihilated by $N_l(\mathfrak{s})$. As ϕ is linear, to show $[\mathfrak{s}, \mathfrak{s}] \subset \mathrm{Ker}\, \phi$, it suffices to show $[\mathfrak{a}, \mathfrak{a}]$, $[\mathfrak{a}, \mathfrak{n}(\mathfrak{s})]$, $[\mathfrak{n}(\mathfrak{s}), \mathfrak{n}(\mathfrak{s})] \subset \mathrm{Ker}\, \phi$.

Take $X, Y \in \mathfrak{a}$. By the construction of $\mathfrak{a}$ and Proposition 2.4 (i) of [Gordon and Wilson 1988], we have

$$[X, Y] = \phi(X)Y - \phi(Y)X + [X, Y] = [\phi(X) + X, \phi(Y) + Y] \in \mathrm{Ker}\, \phi.$$

Now consider $X \in \mathfrak{a}$ and $Y \in \mathfrak{n}(\mathfrak{s})$. As above, the following is contained in $\mathrm{Ker}\, \phi$:

$$[\phi(X) + X, \phi(Y) + Y] = \phi(X)Y + [X, Y],$$

that is, $\mathrm{ad}(\phi(X) + X) : \mathfrak{n}(\mathfrak{s}) \to \mathrm{Ker}\, \phi$.

Since every derivation of $\mathfrak{s}$ takes its image in $\mathfrak{n}(\mathfrak{s})$, and $\mathfrak{r} \subset N_l(\mathfrak{s}) \ltimes \mathfrak{s}$, we see that $\mathfrak{n}(\mathfrak{s})$ is stable under $D = \mathrm{ad}(\phi(X) + X) = \phi(X) + \mathrm{ad}\, X$. Denoting the generalized eigenspaces of D on $\mathfrak{n}(\mathfrak{s})^{\mathbb{C}}$ by V_λ, we have

$$\mathfrak{n}(\mathfrak{s}) = \bigoplus (V_\lambda \oplus V_{\bar{\lambda}}) \cap \mathfrak{n}(\mathfrak{s}).$$

Each summand is invariant under both $\phi(X)$ and $\mathrm{ad}\, X$ as these commute. Further, if $\lambda = a + bi$, then on V_λ we have $\phi(X)^2 = -b^2\, \mathrm{Id}$ and $\mathrm{ad}\, X$ can be realized as an upper triangular matrix whose diagonal is $a\, \mathrm{Id}$. Observe that $\mathrm{Ker}\, D = \mathrm{Ker}\, \phi(X) \cap \mathrm{Ker}\, \mathrm{ad}\, X$, so if $b \neq 0$, then we see that D is nonsingular on V_λ and $V_{\bar{\lambda}}$. This implies $V_\lambda = D(V_\lambda)$ and $V_{\bar{\lambda}} = D(V_{\bar{\lambda}})$, which implies

$$\operatorname{Im}(\operatorname{ad} X|_{(V_\lambda \oplus V_{\bar{\lambda}}) \cap \mathfrak{n}(\mathfrak{s})}) \subset (V_\lambda \oplus V_{\bar{\lambda}}) \cap \mathfrak{n}(\mathfrak{s}) \subset \operatorname{Im}(D|_{\mathfrak{n}(\mathfrak{s})}) \subset \operatorname{Ker} \phi.$$

If $b = 0$, then $V_\lambda = V_{\bar{\lambda}}$ and $D|_{V_\lambda} = \operatorname{ad} X|_{V_\lambda}$, which implies

$$\operatorname{ad} X|_{V_\lambda} \subset \operatorname{Ker} \phi.$$

All together, this proves $[\mathfrak{a}, \mathfrak{n}(\mathfrak{s})] \subset \operatorname{Ker} \phi$.

To finish, one must show $[\mathfrak{n}(\mathfrak{s}), \mathfrak{n}(\mathfrak{s})] \subset \operatorname{Ker} \phi$. However, as every derivation of $\mathfrak{s}$ preserves $\mathfrak{n}(\mathfrak{s})$, we may restrict our modification to $\mathfrak{n}(\mathfrak{s})$ and we have a modification

$$\mathfrak{n}' = (\mathrm{id} + \phi)\mathfrak{n}(\mathfrak{s}) \subset N_l(\mathfrak{s}) \ltimes \mathfrak{n}(\mathfrak{s}).$$

Theorem 2.5 of [Gordon and Wilson 1988] shows that any modification of a nilpotent subalgebra must be a normal modification. Now [loc. cit., Proposition 2.4 (ii d)] implies $[\mathfrak{n}(\mathfrak{s}), \mathfrak{n}(\mathfrak{s})] \subset \operatorname{Ker} \phi$. This completes the proof of our lemma. □

Remark 4.5. Not all modifications are normal modifications, even in the case of starting with an algebra in standard position. An example of this can be found in Example 3.9 of [Gordon and Wilson 1988]. We warn the reader that there are some typos in that example, the block diagonal matrices of A and V_1 should be interchanged. And then one should replace $A - V_1$ with $A + V_1$ throughout the example.

As explained in the discussion surrounding Lemmas 2.7 and 2.8, $\mathfrak{r}$ and $\mathfrak{s}$ are normal modifications of each other and there is an abelian subalgebra $\mathfrak{t}$ of the stabilizer subalgebra which normalizes both $\mathfrak{r}$ and $\mathfrak{s}$ satisfying $\mathfrak{s} \subset \mathfrak{r} \rtimes \mathfrak{t}$. Here $\mathfrak{s}$ is an ideal.

The proposition follows immediately from the following lemma.

Lemma 4.6. *Let $\mathfrak{k}$ be any maximal compact subalgebra of* $\operatorname{Der}(\mathfrak{r})$ *containing* $\mathfrak{t}$. *Then $\mathfrak{s}$ is a maximal completely solvable ideal of $\mathfrak{r} \rtimes \mathfrak{k}$ (cf. Proposition 2.1).*

Proof. By the construction of $\mathfrak{s}$, it is clearly a complement of $\mathfrak{k}$ in $\mathfrak{r} \rtimes \mathfrak{k}$. Further, every element of $\operatorname{ad} \mathfrak{k}$ has purely imaginary eigenvalues on $\mathfrak{r} \rtimes \mathfrak{k}$, and so $\mathfrak{s}$ will be a maximal completely solvable ideal as soon as we show that it is ideal.

As $[\mathfrak{s}, \mathfrak{r}] \subset \mathfrak{s}$ by Lemma 2.8, it suffices to show that $\mathfrak{s}$ is stable under $\mathfrak{k}$. However, as every derivation of $\mathfrak{r}$ takes its image in the nilradical, it suffices to show that $\mathfrak{s}$ contains the nilradical of $\mathfrak{r}$.

As stated above, $\mathfrak{r}$ being a normal modification of $\mathfrak{s}$ gives $\mathfrak{r} \subset \mathfrak{s} \rtimes \mathfrak{t}$ where $\mathfrak{t}$ consists of skew-symmetric derivations of $\mathfrak{s}$ and so every element of $\mathfrak{r}$ may be written as $X + K$ where $X \in \mathfrak{s}$ and $K \in \mathfrak{t} \subset \operatorname{Der}(s) \cap \mathfrak{so}(\mathfrak{s})$. One can quickly see, as in the proof of Proposition 2.1, that the eigenvalues of $\operatorname{ad}(X + K)$ are sums of eigenvalues of $\operatorname{ad} X$ and K. Since $\operatorname{ad} X$ has real eigenvalues and K has purely imaginary eigenvalues, we see that $\mathrm{ad}(X+K)$ having only the zero eigenvalue implies $K=0$. That is, the nilradical of $\mathfrak{r}$ is contained in $\mathfrak{s}$. □

Before moving on with the rest of the proof of our main result, we record a consequence of the work done above.

Theorem 4.7. *Let R be a solvable Lie group. Up to isomorphism, there is a single completely solvable group S which can be realized as a modification of R.*

Maximal symmetry for R. We are now in a position to complete the proof that for a simply connected, unimodular solvable Lie group, there is a single largest isometry group up to conjugation by diffeomorphisms. In fact, we will see that the diffeomorphism can be chosen to be a composition of an automorphism of R together with an automorphism of S.

Starting with a metric g on R, we first construct another metric g'' such that

$$\mathrm{Isom}(R,g) < \mathrm{Isom}(R,g'') = S_g \rtimes K,$$

where $S=S_g$ is a completely solvable group (depending on g) and K is some maximal compact subgroup of $\mathrm{Aut}(S)$. Let h be another metric on R with corresponding group S_h. From the above, we may replace h with ϕ^*h for some $\phi\in\mathrm{Aut}(R)$ to assume that $S_h=S_g=S$.

Now, as K is unique up to conjugation in $\mathrm{Aut}(S)$, we have the desired result.

Proof of Corollary 1.3. As in the above, the strategy is to reduce to the setting of completely solvable groups. We briefly sketch the argument for doing this.

By Theorem 8.2 of [Jablonski 2015a], any solvable Lie group R admitting a Ricci soliton metric must be a modification of a completely solvable group S which admits a Ricci soliton. (In fact, those Ricci soliton metrics are isometric.) From our work above, the modification is a normal modification and so the group S is the same as the group we constructed above. Now the problem is reduced to proving that Ricci soliton metrics on completely solvable Lie groups are maximally symmetric, but this has been resolved — see Theorem 4.1 of [Jablonski 2011].

5. Constructing *S* from algebraic data of *R*

In the above work, we started with a solvable Lie group R and built an associated completely solvable Lie group S. The group S was unique, up to conjugation by $\mathrm{Aut}(R)$, but it was built by starting with a metric on R, making modifications to R, changing the metric, making more modifications and then extracting information from the modification R''. We now give a straightforward description of the group S.

Let K be some choice of maximal compact subgroup of $\mathrm{Aut}(R)$. The group S is the simply connected Lie group whose Lie algebra is the "orthogonal complement" of $\mathfrak{k}$ in $\mathfrak{r} \rtimes \mathfrak{k}$ relative to the Killing form of $\mathfrak{r} \rtimes \mathfrak{k}$, i.e.,

$$\mathfrak{s} = \{X \in \mathfrak{r} \rtimes \mathfrak{k} \mid B(X, Y) = 0 \text{ for all } Y \in \mathfrak{k}\}, \tag{5-1}$$

where B is the Killing form of $\mathfrak{r} \rtimes \mathfrak{k}$.

One can see this quickly by showing that the algebra described in (5-1) is also a maximal completely solvable ideal and then Proposition 2.1 shows that it must be $\mathfrak{s}$. The details of the proof are similar to work done above and so we leave them to the diligent reader.

References

[Auslander and Green 1966] L. Auslander and L. W. Green, "G-induced flows", *Amer. J. Math.* **88** (1966), 43–60. MR Zbl

[Epstein and Jablonski 2018] J. Epstein and M. Jablonski, "Lack of maximal symmetry in non-unimodular solvmanifolds", preprint, 2018.

[Gordon 1980] C. Gordon, "Riemannian isometry groups containing transitive reductive subgroups", *Math. Ann.* **248**:2 (1980), 185–192. MR Zbl

[Gordon and Jablonski 2015] C. S. Gordon and M. R. Jablonski, "Einstein solvmanifolds have maximal symmetry", 2015. To appear in *J. Differential Geom.* arXiv

[Gordon and Wilson 1988] C. S. Gordon and E. N. Wilson, "Isometry groups of Riemannian solvmanifolds", *Trans. Amer. Math. Soc.* **307**:1 (1988), 245–269. MR Zbl

[Jablonski 2011] M. Jablonski, "Concerning the existence of Einstein and Ricci soliton metrics on solvable Lie groups", *Geom. Topol.* **15**:2 (2011), 735–764. MR Zbl

[Jablonski 2015a] M. Jablonski, "Homogeneous Ricci solitons", *J. Reine Angew. Math.* **699** (2015), 159–182. MR Zbl

[Jablonski 2015b] M. Jablonski, "Strongly solvable spaces", *Duke Math. J.* **164**:2 (2015), 361–402. MR Zbl

[Mostow 1956] G. D. Mostow, "Fully reducible subgroups of algebraic groups", *Amer. J. Math.* **78** (1956), 200–221. MR Zbl

[Ochiai and Takahashi 1976] T. Ochiai and T. Takahashi, "The group of isometries of a left invariant Riemannian metric on a Lie group", *Math. Ann.* **223**:1 (1976), 91–96. MR Zbl

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